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CONCENTRATED, NEARLY MONOTONIC, EPIPERIMETRIC MEASURES IN EUCLIDEAN SPACE

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Abstract

We characterize Hölder continuously differentiable m dimensional submanifolds of Euclidean space among m rectifiable sets S in terms of growth conditions on the m density ratios of the Hausdorff measure $\mathcal{H}^m \llcorner S$.

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1. Foreword

We consider an m dimensional differentiable submanifold $S \subset \mathbf{R}^n$, $0 < m < n$, whose tangent spaces $\operatorname{Tan}(S, x)$ vary Hölder continuously with respect to $x \in S$. In other words there are $0 < \alpha \leq 1$ and $C_1 > 0$ such that

$$\operatorname{dist}(\operatorname{Tan}(S, x_1), \operatorname{Tan}(S, x_2)) \leq C_1 |x_1 - x_2|^\alpha$$

whenever $x_1, x_2 \in S$. Here $\operatorname{dist}(W_1, W_2)$ measures the distance between two m dimensional vector subspaces of $\mathbf{R}^n$; for instance we may set it equal to the Hilbert-Schmidt norm of $P_{W_1} - P_{W_2}$ where P_W denotes the nearest point projection on W. Our purpose is to study the *measure of area* on S, that is the Radon measure $\phi = \mathcal{H}^m \llcorner S$ defined as follows:

$$(\mathcal{H}^m \llcorner S)(A) = \mathcal{H}^m(S \cap A)$$

whenever $A \subset \mathbf{R}^n$. We have denoted by $\mathcal{H}^m$ the m dimensional Hausdorff measure on S, see e.g., [**10**, 2.10.2], so that ϕ "coincides with the Lebesgue measure in coordinate charts" according to the area theorem, [**10**, 3.2.3]. We observe (Proposition 3.6.1) that each $x_0 \in S$ has a neighborhood U with the following property (here $\operatorname{spt}(\phi) = S$).

(A) For every $x \in \operatorname{spt}(\phi) \cap U$ and every $0 < r < R$ such that $\mathbf{B}(x, R) \subset U$ one has:

$$\left| \frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m) r^m} - \frac{\phi(\mathbf{B}(x,R))}{\boldsymbol{\alpha}(m) R^m} \right| \leq C_2 R^{2\alpha}.$$

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(B) For every $x \in U$ and every $0 < r < R$ such that $\mathbf{B}(x,R) \subset U$, one has:

$$\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m} - \frac{\phi(\mathbf{B}(x,R))}{\boldsymbol{\alpha}(m)R^m} \leq C_2 R^\alpha.$$

We notice that condition (B) guarantees the existence of a limit of the density ratios for each $x \in U$:

$$\Theta^m(\phi, x) := \lim_{r \to 0^+} \frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m}.$$

The normalizing constant $\boldsymbol{\alpha}(m)$ is chosen in order that:

(C) $\Theta^m(\phi, x) = 1$ whenever $x \in \operatorname{spt}(\phi) \cap U$.

Inequality (A) is a simple consequence of the area theorem together with the Taylor expansion of the area integrand in coordinates (Lemma 2.3.4). Inequality (B) follows from (A) by comparing $\phi(\mathbf{B}(x,r))$ (where possibly $x \notin S$) to $\phi(\mathbf{B}(x',r'))$ for some $x' \in S$ and $r' > 0$.

Our main result is the reciprocal of the above stated observation.

Theorem. *Assume ϕ is a Radon measure in $\mathbf{R}^n$ verifying conditions* (A), (B) *and* (C) *above in some open set $U \subset \mathbf{R}^n$. Then* $\operatorname{spt}(\phi) \cap U$ *is a Hölder continuously differentiable submanifold of* $\mathbf{R}^n$.

In case $m = n-1$ the same conclusion holds without assuming (B) according to the work of G. David, C. Kenig and T. Toro, [**6**]. For the purpose of proving such results it seems important that the testing balls $\mathbf{B}(x,r)$ be Euclidean. Regarding Lipschitzian regularity implied by the controlled behavior of density ratios computed with respect to (very) non Euclidean balls, see the recent account [**13**] by A. Lorent.

We will refer to condition (A) as the *epiperimetry of ϕ near x_0*, to condition (B) as its *nearly monotonicity near* x_0, and to condition (C) as to its *density* 1 *property near* x_0. These three conditions are met by measures ϕ corresponding to solutions of some variational problems including soap films and soap bubbles. Specifically let $S \subset \mathbf{R}^n$ be $(\mathcal{H}^m, m)$ rectifiable ([**10**, 3.2.14]) and $(\mathbf{M}, \varepsilon, \delta)$ minimal in the sense introduced by F.J. Almgren in [**2**]; then $\phi = \mathcal{H}^m \llcorner S$ meets these three requirements in a neighborhood of $\mathcal{H}^m$ almost every point $x_0 \in S$ (for a definition of $(\mathbf{M}, \varepsilon, \delta)$ almost minimality in the setting of currents see subsection 3.4). The near monotonicity property (near *every* $x_0 \in S$) is a classical consequence of almost minimality (see Proposition 3.4.5 for the analogous result in the setting of currents). Nevertheless, to the author's knowledge it doesn't seem to have been written up so far in the context of sets (see the forthcoming [**9**]). The epiperimetry property near $\mathcal{H}^m$ almost every $x_0 \in S$ has been proved by E.R. Reifenberg [**19**] in the context of minimal sets. The density 1 property near $\mathcal{H}^m$ almost every $x_0 \in S$ follows simply from $(\mathcal{H}^m, m)$ rectifiability, [**10**, 3.2.19], together with the epiperimetry property. Epiperimetry is the most important property needed to prove the regularity theorem. It says that the density ratios $\boldsymbol{\alpha}(m)^{-1}r^{-m}\phi(\mathbf{B}(x,r))$ decrease sufficiently fast to their limit when $r \downarrow 0$ for most $x \in \operatorname{spt}(\phi)$. Assuming that $S \cap \mathbf{B}(x_0, r_0)$ is sufficiently close to being a graph with small Lipschitz constant parametrized on some m dimensional affine subspace $x_0 + W$, the epiperimetry property is proved by comparing the area of S inside $\mathbf{B}(x_0, r_0)$ to that of the graph of the solution of the Dirichlet problem with boundary data close to $S \cap \operatorname{Bdry} \mathbf{B}(x_0, r_0)$ — however, the length of this brief description is no indication of the technical complications that arise when carrying out the actual proof. Adapting E.R. Reifenberg's proof of epiperimetry to the case of almost minimality together with the main result of the present paper yields another proof of F.J. Almgren almost everywhere regularity theorem in [**2**]. Regarding F.J. Almgren's memoir one may also consult E. Bombieri's different proof in the setting of currents [**4**] as well as G. David and S. Semmes' account [**7**] on the uniform rectifiability of the so-called restricted sets introduced in [**2**].

We now turn to giving a general idea of the methods used in this paper (which are inspired in part by D. Preiss' moments computations in [**17**]). Assume ϕ verifies conditions (A), (B) and (C) above. It is first observed (see also [**8**]) that (B) and (C) imply the following. For every $\varepsilon > 0$ there exists $r = r(x_0, \varepsilon) > 0$ and an m dimensional vector subspace W (depending on x_0 and also possibly on $r > 0$ and $\varepsilon > 0$) such that

$$\operatorname{dist}_{\mathcal{H}}\big[\operatorname{spt}(\phi) \cap \mathbf{B}(x_0, \gamma r), (x_0 + W) \cap \mathbf{B}(x_0, \gamma r)\big] \leq \varepsilon\gamma r \tag{1}$$

where $\operatorname{dist}_{\mathcal{H}}$ denotes the Hausdorff distance and $\gamma = \gamma(n, m, \varepsilon) > 0$. This can be seen as follows. There exists $\delta = \delta(n, m, \varepsilon) > 0$ such that if ϕ verifies (B) and (C) and if $r > 0$ is such that

$$\phi(\mathbf{B}(x_0, r)) \leq (1 + \delta)\boldsymbol{\alpha}(m)r^m, \tag{2}$$

then (1) holds for some W (see Lemma 4.5.5). For if this were not true, a compactness argument would yield a weakly converging sequence $\phi_1, \phi_2, \ldots$ with a limit ϕ verifying (C) and (B) with $C_2 = 0$ and $U = \mathbf{R}^n$. Such a measure ϕ would be necessarily of the type $\mathcal{H}^m \llcorner W$ according to a theorem of O. Kowalski and D. Preiss (see e.g., Theorem 4.5.4), therefore contradicting the convergence in (local) Hausdorff distance of $\operatorname{spt}(\phi_j)$, $j = 1, 2, \ldots$, to $\operatorname{spt}(\phi)$ (Corollary 3.3.5). One also notices that inequality (2) is inherited by neighbooring points of x_0 and persists at smaller scales according to condition (B). Therefore E.R. Reifenberg's topological disk theorem (see Theorem 2.5.10) applies, asserting that $\operatorname{spt}(\phi)$ contains a neighborhood of x_0 which is homeomorphic to an m dimensional ball. This consequence of (2) will be used repeatedly in the present work, for instance for finding orthonormal families $e_1, \ldots, e_m$ such that $x_0 + \rho e_j \in \operatorname{spt}(\phi)$, $j = 1, \ldots, m$, for $0 < \rho < r$ small enough.

The remaining part of the proof consists in controlling the distance between m dimensional subspaces W_{x_1,r_1} and W_{x_2,r_2} which approximate $\operatorname{spt}(\phi)$ in the sense of (1) at different points x_i and different scales r_i, $i = 1, 2$. The reason for doing this is twofold. On the one hand we want to estimate $\operatorname{dist}(W_{x,r}, W_{x,2r})$ by a quantity $\varepsilon(r)$ with the property that $\sum_{j=1}^{\infty} \varepsilon(2^{-j}) < \infty$ — this implies that a limit $W_x = \lim_{r \to 0^+} W_{x,r}$ exists, therefore providing us with a tangent space of $\operatorname{spt}(\phi)$ at x. On the other hand, we want to estimate $\operatorname{dist}(W_{x_1,r}, W_{x_2,r})$ by a multiple of r^{β}, for some $0 < \beta \leq 1$, whenever $r = 2|x_1 - x_2|$ — this shows that the tangent spaces W_x vary Hölder continuously with respect to x. In order to derive such estimates from information regarding the growth of ϕ in Euclidean balls, we start by replacing the balls themselves by "smooth" balls; specifically we introduce the quantity

$$\widehat{V}(\phi, x, r) = \int_{\mathbf{B}(x_0+x,r)} \left(r^2 - |x - (y - x_0)|^2\right)^2 d\phi(y).$$

It is a plain consequence of Cavalieri's principle (see e.g., [**15**, 1.15]) that conditions (A) and (B) remain valid with $r^{-4}\widehat{V}(\phi, x, r)$ replacing $\phi(\mathbf{B}(x, r))$ (for condition (A) to hold we obviously assume that $x_0 + x \in \operatorname{spt}(\phi)$). Next we consider the quantity

$$V(\phi, x, r) = \int_{\mathbf{B}(x_0,r)} \left(r^2 - |x - (y - x_0)|^2\right)^2 d\phi(y)$$

and we estimate $|\widehat{V}(\phi, x, r) - V(\phi, x, r)|$ in terms partly of the error term $C_2 r^{2\alpha}$ appearing in (A) (Lemma 4.2.1), whereas $\widehat{V}(\phi, x, r) - \widehat{V}(\phi, x_0, r)$ has a positive part controlled partly in terms of the error term $C_2 r^{\alpha}$ appearing in (B) (Lemma 4.2.3), $x \in U$ arbitrary. The reason for studying $V(\phi, x, r)$ lies in its geometrical significance. To see this we split the integrand into homogeneous polynomials of the variable x:

$$\begin{aligned} V(\phi, x, r) &= \sum_{k=0}^{4} P_k(\phi, x, r) \\ &= \int_{\mathbf{B}(x_0,r)} \left(r^2 - |y - x_0|^2\right)^2 d\phi(y) \\ &\quad + 4 \left\langle x, \int_{\mathbf{B}(x_0,r)} y \left(r^2 - |y - x_0|^2\right) d\phi(y) \right\rangle \\ &\quad + 4 \int_{\mathbf{B}(x_0,r)} \langle x, y - x_0 \rangle^2 d\phi(y) \\ &\quad - 2|x|^2 \int_{\mathbf{B}(x_0,r)} \left(r^2 - |y - x_0|^2\right)^2 d\phi(y) \\ &\quad + O(r^{m+1}|x|^3). \end{aligned}$$

If normalized properly the term of degree 0 can be thought of as a density ratio of ϕ at the point x_0, at scale r, with respect to our "smooth ball". The term of degree 1 is associated with a vector

$$b(\phi, r) = \int_{\mathbf{B}(x_0,r)} y\left(r^2 - |y - x_0|^2\right) d\phi(y)$$

whose proper normalization can be thought of as a "smoothly weighted" center of mass of ϕ in $\mathbf{B}(x_0, r)$. Most importantly we abbreviate

$$Q(\phi, r)(x) = \int_{\mathbf{B}(x_0,r)} \langle x, y - x_0\rangle^2 d\phi(y),$$

which is a positive semi-definite quadratic form appearing in the term of degree 2. To appreciate the meaning of $Q(\phi, r)$ think of $\phi = \mathcal{H}^m \llcorner S$, S being a differentiable submanifold, $x_0 \in S$ and r being sufficiently small. In that case $Q(\phi, r)(x)$ is close to $r^{m+2}|P_{\mathrm{Tan}(S,x_0)}(x)|^2$ (whenever $|x|$ is small). The justification for the computations involving the comparison of $V(\phi, x, r)$ and $\widehat{V}(\phi, x, r)$ is the following. After normalizing properly all the quantities introduced so far (we divide them by some multiple of r^{m+2} in order that the term of degree 2 in the above expansion becomes dimensionless) we show that *if* we are in the "close to flat" situation described in the preceding paragraph *then* the normalized $\mathbf{Q}(\phi, r)$ has a trace close to m and there is an orthogonal family $e_1, \ldots, e_m$ so that $x_0 + e_i \in \mathrm{spt}(\phi)$ and $\mathbf{Q}(\phi, r)(e_i) = |e_i|^2$, $i = 1, \ldots, m$. In other words, $\mathbf{Q}(\phi, r)(x)$ is close to $|P_{W_{x_0,r}}(x)|^2$ where $W_{x_0,r}$ is the subspace generated by $e_1, \ldots, e_m$. In fact we show that the following is small:

$$r^{-m-2} \int_{\mathbf{B}(x_0,r)} \mathrm{dist}(y - x_0, W_{x_0,r})^2 d\phi(y).$$

It then follows that

$$\mathrm{spt}(\phi) \cap \mathbf{B}(x_0, r/2) \subset \mathbf{B}(x_0 + W_{x_0,r}, \varepsilon r)$$

for some small $\varepsilon > 0$. In turn, arguing that $\mathrm{spt}(\phi) \cap \mathbf{B}(x_0, r)$ is essentially a topological disk we obtain that (1) is satisfied for $W_{x_0,r}$. Of course the whole point is that we now have gained information about the error ε: it has been controlled at each stage of the computation by the error terms $C_2 r^\alpha$ appearing in conditions (A) and (B), i.e., $\varepsilon \lesssim r^\beta$ (our estimates give $\beta = \alpha/8(m + 2)$, which is not optimal). Since we can repeat the whole construction at smaller scales r this is enough to complete the proof of the theorem.

2. Notations and preliminaries

Given $x \in \mathbf{R}^n$ and $A \subset \mathbf{R}^n$ nonempty, we let $\mathrm{dist}(x, A) = \inf\{|x - y| : y \in A\}$ where $|\cdot|$ is the Euclidean norm. To a nonempty set $A \subset \mathbf{R}^n$ and $r > 0$ we associate the *closed r neighborhood of A* defined by

$$\mathbf{B}(A, r) = \mathbf{R}^n \cap \{x : \mathrm{dist}(x, A) \le r\}$$

and we often write $\mathbf{B}(a,r)$ instead of $\mathbf{B}(A,r)$ in case $A = \{a\}$ is a singleton. We define *open balls* as $\mathbf{U}(a,r) = \mathbf{R}^n \cap \{x : |x-a| < r\}$. The *closure*, *interior* and *boundary* of $A \subset \mathbf{R}^n$ are denoted respectively by Clos A, Int A and Bdry A.

Linear subspaces and orthogonal maps. In this subsection we are given two integers $0 \leq m \leq n$ and two finite dimensional real Hilbert spaces V and W, their inner products being denoted respectively by $\langle\, .\, ,\, .\, \rangle_V$ and $\langle\, .\, ,\, .\, \rangle_W$. A linear map $L \in \operatorname{Hom}(V,W)$ is an *orthogonal injection* if $\langle L(v_1), L(v_2)\rangle_W = \langle v_1, v_2\rangle_V$ whenever $v_1, v_2 \in V$; notice that L need indeed be injective, and therefore $\dim(W) \geq \dim(V)$. The *adjoint* of a linear map $L \in \operatorname{Hom}(V,W)$ is the only linear map $L^* \in \operatorname{Hom}(W,V)$ such that $\langle v, L^*(w)\rangle_V = \langle L(v), w\rangle_W$, $v \in V$, $w \in W$. We let $\mathbf{O}(n,m)$ be the collection of orthogonal injections $\mathbf{R}^m \to \mathbf{R}^n$. Elements belonging to $\mathbf{O}^*(n,m) := \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^m) \cap \{p : p^* \in \mathbf{O}(n,m)\}$ are called *orthogonal projections*. Furthermore, we denote by $\mathbf{G}(n,m)$ the collection of m dimensional linear subspaces of $\mathbf{R}^n$. Given $W \in \mathbf{G}(n,m)$ we let $i_W \in \operatorname{Hom}(W, \mathbf{R}^n)$ be the orthogonal injection such that $i_W(w) = w$, $w \in W$, and we define $P_W \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by $P_W := i_W \circ i_W^*$. Note also that $i_W^* \circ i_W = \mathbf{id}_W$.

We consider two norms on $\operatorname{Hom}(V,W)$. For $L \in \operatorname{Hom}(V,W)$ we let $\|L\| := \sup\{|L(v)| : v \in V \text{ and } |v| = 1\}$. Next we define the inner product $L_1 \cdot L_2 := \operatorname{trace} L_2^* \circ L_1$, $L_1, L_2 \in \operatorname{Hom}(V,W)$, as well as the corresponding norm $\|L\|_2 := \sqrt{L \cdot L}$, $L \in \operatorname{Hom}(V,W)$. One readily checks that $\|L\|_2^2 = \sum_{i \in I} |L(e_i)|^2$ whenever $\{e_i : i \in I\}$ is an orthonormal base of V. It follows that $\|L\| \leq \|L\|_2 \leq \sqrt{\dim V}\|L\|$. Next we endow $\mathbf{G}(n,m)$ with the distance defined by $\operatorname{dist}(W_1, W_2) := \|P_{W_1} - P_{W_2}\|_2$. We recall that if $W \in \mathbf{G}(n,m)$ then $W^\perp \in \mathbf{G}(n, n-m)$ is defined by $W^\perp := \mathbf{R}^n \cap \{v : \langle v, w\rangle = 0 \text{ for every } w \in W\}$. The following is intended for estimating the "angles" between tangent planes to a graph.

Lemma 2.1.1. *Let $W \in \mathbf{G}(n,m)$, $L_1, L_2 \in \operatorname{Hom}(W, W^\perp)$ and let $W_j \in \mathbf{G}(n,m)$ be such that $W_j := \operatorname{im}(i_W + i_{W^\perp} \circ L_j)$, $j = 1,2$. Then*

(A) $\operatorname{dist}(W_1, W_2) \leq 2m\|L_1 - L_2\|$;
(B) $\|L_1 - L_2\| \leq 2\sqrt{1 + \|L_1\|^2}\sqrt{1 + \|L_2\|^2}\operatorname{dist}(W_1, W_2)$.

Proof. We define $K_j := i_W + i_{W^\perp} \circ L_j$, $j = 1,2$. Fix $e \in W_2$ with $|e| = 1$ and let $w_1, w_2 \in W$ be such that $P_{W_1}(e) = K_1(w_1)$ and $e = P_{W_2}(e) = K_2(w_2)$. Notice that $\langle e - P_{W_1}(e), K_1(w_2)\rangle = 0$. Therefore

$$\begin{aligned} \langle P_{W_1}(e), e\rangle &= 1 - \langle e - P_{W_1}(e), e\rangle \\ &= 1 - \langle P_{W_2}(e) - P_{W_1}(e), K_2(w_2) - K_1(w_2)\rangle. \end{aligned} \tag{3}$$

Moreover one checks that $|w_2| \leq 1$ so that

$$|\langle P_{W_2}(e) - P_{W_1}(e), K_2(w_2) - K_1(w_2)\rangle| \leq \|P_{W_1} - P_{W_2}\|_2\, \|L_1 - L_2\|. \tag{4}$$

On letting $e_1, \dots, e_n$ be an orthonormal base of $\mathbf{R}^n$ such that $e_1, \dots, e_m$ span W_2, we use (3) and (4) to infer that

$$\begin{aligned}\|P_{W_1} - P_{W_2}\|_2^2 &= 2m - 2\sum_{j=1}^{n} \langle P_{W_1}(e_j), P_{W_2}(e_j)\rangle \\ &= 2m - 2\sum_{j=1}^{m} \langle P_{W_1}(e_j), e_j\rangle \\ &\le 2m\, \|P_{W_1} - P_{W_2}\|_2\, \|L_1 - L_2\|,\end{aligned}$$

which proves conclusion (A).

Next we choose $e \in W$ with $|e| = 1$ and $\|L_1 - L_2\| = |L_1(e) - L_2(e)|$. Let $w \in W$ be such that $P_{W_1}(K_2(e)) = K_1(w)$. Observe that

$$\begin{aligned}|e - w|^2 &\le |e - w|^2 + |L_2(e) - L_1(w)|^2 \\ &= |K_2(e) - K_1(w)|^2 \\ &= |P_{W_2}(K_2(e)) - P_{W_1}(K_2(e))|^2 \\ &\le \|P_{W_1} - P_{W_2}\|_2^2\, \|K_2\|^2;\end{aligned}$$

therefore,

$$\begin{aligned}\|L_1 - L_2\| &= |K_1(e) - K_2(e)| \\ &\le |K_1(e) - K_1(w)| + |K_1(w) - K_2(e)| \\ &\le \|K_1\||e - w| + \|P_{W_1} - P_{W_2}\|_2\, \|K_2\| \\ &\le 2\|K_1\|\, \|K_2\|\, \|P_{W_1} - P_{W_2}\|_2\,,\end{aligned}$$

and it remains to observe that $\|K_j\| \le \sqrt{1 + \|L_j\|^2}$, $j = 1, 2$. q.e.d.

$C^{1,\alpha}$ **functions and submanifolds.** Given two metric spaces X and Y, a function $f : X \to Y$, and $0 < \alpha \le 1$, we denote by $\mathbf{h}_\alpha(f)$ the smallest $0 \le C \le \infty$ such that $\operatorname{dist}_Y(f(x_1), f(x_2)) \le C \operatorname{dist}_X(x_1, x_2)^\alpha$ for every $x_1, x_2 \in X$. If $\mathbf{h}_\alpha(f) < \infty$ we say that f is Hölder continuous with exponent α. Given an open set $U \subset \mathbf{R}^m$, a differentiable function $f : U \to \mathbf{R}^n$ and $0 < \alpha \le 1$, we say that f is of class $C^{1,\alpha}$ whenever its derivative $Df : U \to \operatorname{Hom}(\mathbf{R}^m, \mathbf{R}^n)$ is Hölder continuous of exponent α. Therefore it follows from the mean value theorem that $|f(z+h) - f(z) - Df(z)(h)| \le \mathbf{h}_\alpha(Df)|h|^{1+\alpha}$ whenever $z, z+h \in U$, in case U is convex.

Definition 2.2.2. Given $A \subset \mathbf{R}^n$ and $0 < \alpha \le 1$, we say that A is an m dimensional $C^{1,\alpha}$ submanifold of $\mathbf{R}^n$ whenever it is a submanifold of class 1 and

$$A \to \mathbf{G}(n, m) : x \mapsto \operatorname{Tan}(A, x)$$

is locally Hölder continuous with exponent α.

The definition of submanifold of class 1 can be found in [**10**, 3.1.19], whereas tangent spaces are defined in [**10**, 3.1.21]. The following lemma is obtained by applying uniformly the inverse mapping theorem. To state it we need to recall that the graph of a function $u : W \cap B \to W^\perp$, $W \in \mathbf{G}(n, m)$, $B \subset W$, is defined by

$$\operatorname{graph}(u) := \mathbf{R}^n \cap \{i_W(w) + i_{W^\perp}(u(w)) : w \in W\} .$$

Lemma 2.2.3. *Let $A \subset \mathbf{R}^n$ and $0 < \alpha \le 1$. The following conditions are equivalent.*

(A) *A is an m dimensional $C^{1,\alpha}$ submanifold of $\mathbf{R}^n$.*

(B) *For every $a \in A$ there exist $r > 0$, $0 < C < \infty$ and an open neighborhood $U \subset \mathbf{R}^n$ of a with the following property. For each $x \in A \cap U$ there are $W_x \in \mathbf{G}(n, m)$ and $u_x : W_x \cap \mathbf{U}(0, r) \to W_x^\perp$ such that*

(B.1) *u_x is of class $C^{1,\alpha}$ and $\mathbf{h}_\alpha(Du_x) \le C$;*

(B.2) *$u_x(0) = 0$ and $Du_x(0) = 0$;*

(B.3) *$A \cap U = (x + \operatorname{graph}(u_x)) \cap U$.*

Proof. Assume (A) holds true and fix $a \in A$. One infers from the definition of submanifold of class 1 that there exist $0 < \varepsilon_0 < 1$, $W_a \in \mathbf{G}(n, m)$, an open neighborhood $U_a \subset \mathbf{R}^n$ of a, and $u_a : W_a \cap \mathbf{U}(0, \varepsilon_0) \to W_a^\perp$ of class 1 such that $\operatorname{Lip}(u_a) < \infty$ and conclusions (B.2) and (B.3) are verified for $x = a$. We define $f_a : W_a \cap \mathbf{U}(0, \varepsilon_0) \to \mathbf{R}^n$ by $f_a := i_{W_a} + i_{W_a^\perp} \circ u_a$, and $W_x := \operatorname{im} Df_a(P_{W_a}(x))$ for each $x \in A \cap U_a \cap P_{W_a}^{-1}(\mathbf{U}(P_{W_a}(a), \varepsilon_0))$. One readily checks that $W_x = \operatorname{Tan}(A, x)$. Lemma 2.1.1 (B) now implies that

$$\begin{aligned} (5) \qquad & \|Du_a(z_1) - Du_a(z_2)\| \\ & \le 2\left(1 + \operatorname{Lip}(u_a)^2\right) \operatorname{dist}\left(\operatorname{Tan}(A, f_a(z_1)), \operatorname{Tan}(A, f_a(z_2))\right) \\ & \le 2\left(1 + \operatorname{Lip}(u_a)^2\right)^{1+\alpha} \mathbf{h}_\alpha\left(\operatorname{Tan}(A, .\,) \restriction U_a\right) |z_1 - z_2|^\alpha , \end{aligned}$$

for $z_1, z_2 \in W_a \cap \mathbf{U}(0, \varepsilon_0) \cap P_{W_a}(U_a - a)$, so that u_a is $C^{1,\alpha}$ in this domain. For $x \in A \cap U \cap P_{W_a}^{-1}(\mathbf{U}(P_{W_a}(a), \varepsilon_0))$ we now define $h_x : W_a \cap \mathbf{U}(0, \varepsilon_0) \to W_x$ by the formula $h_x := i_{W_x}^* \circ f_a$ so that

$$(6) \qquad Dh_x(z) = i_{W_x}^* \circ i_{W_a} + i_{W_x}^* \circ i_{W_a^\perp} \circ Du_a(z)\,,\ z \in W_a \cap \mathbf{U}(0, \varepsilon_0).$$

It is easy to check that for every $w \in W_a$ one has

$$(7) \qquad \left|\left(i_{W_x}^* \circ i_{W_a}\right)(w)\right| \ge \frac{|w|}{\sqrt{1 + \|Du_a(P_{W_a}(x))\|^2}} .$$

Therefore

$$(8) \qquad 1 \le \left\|Dh_x(0)^{-1}\right\| = \left\|\left(i_{W_x}^* \circ i_{W_a}\right)^{-1}\right\| \le \sqrt{1 + \mathbf{h}_\alpha(Du_a)^2 \varepsilon^{2\alpha}}$$

whenever $|P_{W_a}(x)| \le \varepsilon$, $0 < \varepsilon \le \varepsilon_0$. For such $0 < \varepsilon \le \varepsilon_0$, on setting $\delta(\varepsilon) := \left(\varepsilon \max\{1, \mathbf{h}_\alpha(Du_a)\}^{-1}\right)^{\frac{1}{\alpha}} \le \varepsilon$ it follows from the inverse mapping theorem (see [**10**, 3.1.1], second half of page 210) that

$$h_x^{-1} : W_x \cap \mathbf{B}\left(0, \frac{\delta(\varepsilon)(1+\varepsilon)}{\sqrt{1+\mathbf{h}_\alpha(Du_a)^2\varepsilon^{2\alpha}}}\right) \to W_a \cap \mathbf{B}(0,\delta(\varepsilon))$$

is well-defined and a homeomorphism onto its image. Next, if $z \in W_a \cap \mathbf{B}(0,\delta(\varepsilon))$ and $w \in W_a$ then

$$\begin{aligned} (9) \qquad |Dh_x(z)(w)| &\geq |w| \left(\left(1 + \|Du_a(P_{W_a}(x))\|^2\right)^{-\frac{1}{2}} - \|Du_a(z)\| \right) \\ &\geq |w| \left(\left(1 + \mathbf{h}_\alpha(Du_a)^2\varepsilon^{2\alpha}\right)^{-\frac{1}{2}} - \mathbf{h}_\alpha(Du_a)\delta(\varepsilon)^\alpha \right) \\ &\geq \mu|w| \end{aligned}$$

for some $\mu > 0$ provided $0 < \varepsilon \leq \varepsilon_0$ is chosen small enough. Therefore there exists $r > 0$, independent of x, such that $h_x^{-1} : W_x \cap \mathbf{U}(0,r) \to W_a$ is a diffeomorphism of class 1 onto its image, and $\operatorname{Lip}(h_x^{-1})$ is bounded above uniformly in x. Finally we define $u_x : W_x \cap \mathbf{U}(0,r) \to W_x^\perp$ by the formula

$$u_x := i^*_{W_x^\perp} \circ \boldsymbol{\tau}_{-x} \circ f_a \circ \boldsymbol{\tau}_{P_{W_a}(x)} \circ h_x^{-1}.^*$$

It is obvious that $u_x(0) = 0$. Moreover,

$$Du_x(0) = i^*_{W_x^\perp} \circ Df_a(P_{W_a}(x)) \circ \left(i^*_{W_x} \circ i_W\right)^{-1} = 0$$

because $W_x = \operatorname{im} Df_a(P_{W_a}(x))$. On choosing a small enough neighborhood of a, $U \subset U_a$, conclusion (B.3) is clearly verified. One checks that u_x, $x \in A \cap U$, are uniformly Hölder continuous with a formula analogous to (5). This finishes the proof that (A) implies (B). The proof that (B) implies (A) is similar, shorter, and left to the reader. q.e.d.

Jacobians. Given $W \in \mathbf{G}(n,m)$, a map $f : W \to \mathbf{R}^n$ and $z \in W$ such that f is differentiable at z, we recall that the m *jacobian* of f at z is defined by $J_m f(z) := \sqrt{\det(Df(z)^* \circ Df(z))}$. The following well-known lemma is useful for estimating the measure of a graph.

Lemma 2.3.4. *For each $m = 1, 2, \ldots$ there exists $0 < \mathbf{c}_{2.3.4}(m) < \infty$ with the following property. Whenever*

(A) $1 \leq m < n$, $W \in \mathbf{G}(n,m)$, $u : W \to W^\perp$*;*
(B) $z \in W$*, u is differentiable at z and* $\|Du(z)\| \leq 1$*;*
(C) $f : W \to \mathbf{R}^n$ *and* $f = i_W + i_{W^\perp} \circ u$*;*

there exists $\sigma \in \mathbf{R}$ such that $|\sigma| \leq \mathbf{c}_{2.3.4}(m)$ and

$$J_m f(z) = 1 + \frac{1}{2}\|Du(z)\|_2^2 + \sigma\|Du(z)\|_2^4.$$

*Here and in the remaining part of this paper $\boldsymbol{\tau}_z(v) = z + v$.

Proof. Set $L := Du(z) \in \operatorname{Hom}(W, W^\perp)$ and notice that

$$\begin{aligned} J_m f(z)^2 &= \det\big((i_W + i_{W^\perp} \circ L)^* \circ (i_W + i_{W^\perp} \circ L)\big) \\ &= \det(\mathbf{id}_W + L^* \circ L). \end{aligned}$$

On letting $\lambda_1, \dots, \lambda_m$ be the eigenvalues of $L^* \circ L$, we readily check that $1 + \lambda_1, \dots, 1 + \lambda_m$ are the eigenvalues of $\mathbf{id}_W + L^* \circ L$, whence

$$J_m f(z)^2 = \prod_{i=1}^m (1 + \lambda_i) = 1 + \sum_{i=1}^m \lambda_i + R$$

where R is a sum of products involving at least two eigenvalues. Since $\lambda_j \geq 0$, $j = 1, \dots, m$, we see that $R \geq 0$. Now $\sum_{i=1}^m \lambda_i = \operatorname{trace} L^* \circ L = \|L\|_2^2$; in particular $R \leq c(m)\|L\|_2^4$ because $\lambda_i \leq \|L\|_2^2$, $i = 1, \dots, m$, and $\|L\|_2 \leq 1$. The intermediate value theorem then shows that

$$J_m f(z)^2 = 1 + \|L\|_2^2 + \sigma_0 \|L\|_2^4$$

for some $0 \leq \sigma_0 \leq c(m)$. It now suffices to plug in the Taylor expansion of order 2 of $\sqrt{1+t}$ with $t := \|L\|_2^2 + \sigma_0\|L\|_2^4$. q.e.d.

Radon measures. A Radon measure is a Borel regular measure ϕ on $\mathbf{R}^n$ such that $\phi(C) < \infty$ whenever $C \subset \mathbf{R}^n$ is compact, [**10**, 2.2.5]. According to Riesz's representation theorem they correspond biunivoquely to nonnegative linear functionals on the space $C_c(\mathbf{R}^n)$ of compactly supported continuous real-valued functions on $\mathbf{R}^n$, [**10**, 2.5.13]. The weak convergence of Radon measures $\phi_j \rightharpoonup \phi$ is then defined as the weak convergence of the corresponding linear functionals (see also [**15**, 1.24] for testing weak convergence at the level of sets). Together with the Banach-Alaoglu theorem [**20**, 3.15], Riesz's representation theorem implies de la Vallée Poussin's compactness theorem [**15**, 1.23]. The support of a Radon measure ϕ is the smallest closed set C such that $\phi(\mathbf{R}^n \sim C) = 0$. We will also use the following trivial result.

Lemma 2.4.5. *Let $\phi, \phi_1, \phi_2, \dots$ be Radon measures in an open set $U \subset \mathbf{R}^n$ and assume that $\phi_j \rightharpoonup \phi$ as $j \to \infty$. Then for every compact $K \subset U$ and every $\varepsilon > 0$ there is an integer j_0 such that*

$$\operatorname{spt}(\phi) \cap K \subset U \cap \{x : \operatorname{dist}(x, \operatorname{spt}(\phi_j)) \leq \varepsilon\}$$

whenever $j \geq j_0$.

Proof. Suppose instead that there is a compact set $K \subset U$, $\varepsilon > 0$ and a sequence $k(1), k(2), \dots$ as well as $x_{k(j)} \in \operatorname{spt}(\phi) \cap K$ such that for every integer j one has $\operatorname{dist}(x_{k(j)}, \operatorname{spt}(\phi_{k(j)})) > \varepsilon$. Choose $x \in \operatorname{spt}(\phi) \cap K$ and a subsequence $l(1), l(2), \dots$ of $k(1), k(2), \dots$ such that $x_{l(j)} \to x$ as $j \to \infty$. When j is sufficiently large for $|x_{l(j)} - x| \leq \varepsilon/2$ we have $\mathbf{B}(x, \varepsilon/2) \subset \mathbf{B}(x_{l(j)}, \varepsilon)$ so that $\mathbf{U}(x, \varepsilon/2) \cap \operatorname{spt}(\phi_{l(j)}) = \emptyset$. Therefore $\phi_{l(j)}(\mathbf{U}(x, \varepsilon/2) \cap U) = 0$, hence $\phi(\mathbf{U}(x, \varepsilon/2) \cap U) = 0$ as well, in contradiction with $x \in \operatorname{spt}(\phi)$. q.e.d.

Reifenberg flatness. Given two nonempty bounded sets $A_1, A_2 \subset \mathbf{R}^n$ we define their *Hausdorff distance* as follows:

$$\operatorname{dist}_{\mathcal{H}}(A_1, A_2) = \inf\{r > 0 : A_1 \subset \mathbf{B}(A_2, r) \text{ and } A_2 \subset \mathbf{B}(A_1, r)\}.$$

We readily check that

$$\operatorname{dist}_{\mathcal{H}}(A_1, A_2) = \max\left\{\sup_{x_1 \in A_1} \operatorname{dist}(x_1, A_2), \sup_{x_2 \in A_2} \operatorname{dist}(x_2, A_1)\right\}.$$

The following lemma is proved for instance in [**5**].

Lemma 2.5.6. *Let $0 \le m \le n$ be integers and $W_1, W_2 \in \mathbf{G}(n,m)$. The following holds:*

$$\|P_{W_1} - P_{W_2}\| = \operatorname{dist}_{\mathcal{H}}(W_1 \cap \mathbf{B}(0,1), W_2 \cap \mathbf{B}(0,1))$$

as well as

$$\|P_{W_1} - P_{W_2}\| = \max\{\operatorname{dist}(z, W_2) : z \in W_1 \cap \mathbf{B}(0,1)\}.$$

We now turn to defining *Reifenberg flat sets*.

Definition 2.5.7. Let $0 < m < n$ be integers, $S \subset \mathbf{R}^n$, $x \in S$, $r > 0$ and $\varepsilon > 0$. We say that S is (ε, m) *flat at* (x,r) if there exists $W \in \mathbf{G}(n,m)$ such that

$$d_{\mathcal{H}}\left(S \cap \mathbf{B}(x,r), (x+W) \cap \mathbf{B}(x,r)\right) \le \varepsilon r.$$

Given $\varepsilon > 0$ we also define

$$\mathbf{G}(S,x,r,\varepsilon) = \mathbf{G}(n,m) \cap \{W : d_{\mathcal{H}}(S \cap \mathbf{B}(x,r), (x+W) \cap \mathbf{B}(x,r)) \le \varepsilon r\}.$$

The following two easy lemmas are proved for instance in [**5**].

Lemma 2.5.8 (Same center, different scales)**.** *Assume that*

(A) *$S \subset \mathbf{R}^n$, $x \in S$, $\varepsilon > 0$, $0 < r < R$ and $\varepsilon R \le r$;*
(B) *$W_{x,r} \in \mathbf{G}(S,x,r,\varepsilon)$ and $W_{x,R} \in \mathbf{G}(S,x,R,\varepsilon)$.*

Then $W_{x,R} \in \mathbf{G}(S,x,r,2\varepsilon R r^{-1})$ and

$$\|P_{W_{x,r}} - P_{W_{x,R}}\| \le \varepsilon(1 + 2Rr^{-1}).$$

Lemma 2.5.9 (Different centers, same scale)**.** *Assume that*

(A) *$S \subset \mathbf{R}^n$, $x_1, x_2 \in S$, $\varepsilon > 0$, $\nu > 1$, $R > 0$, $0 < \lambda < 1$, $|x_1 - x_2| \le (1-\lambda)R$;*
(B) *$1 - \lambda + \varepsilon + \nu^{-1} \le 1$;*
(C) *$W_{x_i,R} \in \mathbf{G}(S,x_i,R,\varepsilon)$, $i = 1,2$.*

Then

$$\|P_{W_{x_1,R}} - P_{W_{x_2,R}}\| \le 6\varepsilon\nu.$$

The following is usually referred to as *Reifenberg's topological disk theorem*. Indeed E.R. Reifenberg proved it in [**18**]. Later C.B. Morrey extended the result, replacing the ambient space $\mathbf{R}^n$ by a Riemannian manifold, see [**16**]. For recent developments see [**5**].

Theorem 2.5.10. *Let $n > 0$ be an integer. There exist constants $0 < \boldsymbol{\varepsilon}_{2.5.10}(n) < \infty$ and $0 < \mathbf{c}_{2.5.10}(n) < \infty$ with the following property. Assume that*

(A) $0 < m < n$ *is an integer,* $S \subset \mathbf{R}^n$ *is closed,* $x_0 \in S$*,* $r_0 > 0$;
(C) $0 < \varepsilon \leq \boldsymbol{\varepsilon}_{2.5.10}(n)$;
(D) *for every* $x \in S \cap \mathbf{B}(x_0, 2r_0)$ *and every* $0 < r \leq 2r_0$*,* S *is* (ε, m) *flat at* (x, r);
(E) $W_0 \in \mathbf{G}(S, x_0, r_0, \varepsilon)$.

Then there exists a continuous map

$$\tau : (x_0 + W_0) \cap \mathbf{B}(x_0, r_0) \to S$$

such that:

(F) $|\tau(x) - x| \leq \mathbf{c}_{2.5.10}(n)\varepsilon r_0$ *for every* $x \in (x_0 + W_0) \cap \mathbf{B}(x_0, r_0)$;
(G) τ *is Hölder bicontinuous: for every* $x, y \in (x_0 + W_0) \cap \mathbf{B}(x_0, r_0)$ *one has*

$$\begin{aligned}&[1 - \mathbf{c}_{2.5.10}(n)\varepsilon]|y - x|^{1+\mathbf{c}_{2.5.10}(n)\varepsilon} \\ &\leq |\tau(y) - \tau(x)| \\ &\leq [1 + \mathbf{c}_{2.5.10}(n)\varepsilon]|y - x|^{1-\mathbf{c}_{2.5.10}(n)\varepsilon};\end{aligned}$$

(H) τ *is one-to-one;*
(I) $S \cap \mathbf{B}(x_0, r_0/2) \subset \operatorname{im} \tau$.

3. Monotonicity and epiperimetry

3.1. Spherical excess.

Definition 3.1.1. Given an open set $U \subset \mathbf{R}^n$, a Radon measure ϕ on U, $x \in U$ and $R > 0$ such that $\mathbf{B}(x, R) \subset U$, and an integer $m \in \{0, \ldots, n\}$, we define the *lower spherical excess* and *upper spherical excess* of (ϕ, x, R, m) as follows:

$$\begin{aligned}&\mathbf{exc}_*^m(\phi, x, R) \\ &\quad := \sup\left\{ \left(\frac{\phi(\mathbf{B}(x, \rho_2))}{\boldsymbol{\alpha}(m)\rho_2^m} - \frac{\phi(\mathbf{B}(x, \rho_1))}{\boldsymbol{\alpha}(m)\rho_1^m} \right)^- : 0 < \rho_1 \leq \rho_2 \leq R \right\}\end{aligned}$$

and

$$\begin{aligned}&\mathbf{exc}^{m\,*}(\phi, x, R) \\ &\quad := \sup\left\{ \left(\frac{\phi(\mathbf{B}(x, \rho_2))}{\boldsymbol{\alpha}(m)\rho_2^m} - \frac{\phi(\mathbf{B}(x, \rho_1))}{\boldsymbol{\alpha}(m)\rho_1^m} \right)^+ : 0 < \rho_1 \leq \rho_2 \leq R \right\}.\end{aligned}$$

We also put

$$\|\mathbf{exc}^m\|(\phi, x, R) := \max\{\mathbf{exc}_*^m(\phi, x, R), \mathbf{exc}^{m\,*}(\phi, x, R)\}.$$

Remark 3.1.2. It is clear that $\mathbf{exc}^m_*(\phi, x, R) < \infty$ and $\mathbf{exc}^{m\,*}(\phi, x, R) < \infty$ whenever $\Theta^{*\,m}(\phi, x) < \infty$. The definitions are so that

$$-\mathbf{exc}^m_*(\phi, x, R) \le \frac{\phi(\mathbf{B}(x,R))}{\boldsymbol{\alpha}(m)R^m} - \frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m} \le \mathbf{exc}^{m\,*}(\phi, x, R)$$

whenever $0 < r \le R$.

Lemma 3.1.3. *Asume that*

(A) $U \subset \mathbf{R}^n$ *is open,* $R > 0$, $\mathbf{B}(0, 2R) \subset U$, $0 < \eta \le 1$;

(B) ϕ *is a Radon measure in* U;

then for every $x \in \mathbf{B}(0, \eta R)$ *and every* $0 < r \le R$ *the following holds:*

$$\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m} \le \left(\Theta^{m\,*}(\phi, 0) + \mathbf{exc}^{m\,*}(\phi, 0, 2R)\right)(1+\eta)^m + \mathbf{exc}^m_*(\phi, x, R).$$

Proof. It suffices to observe that

$$\begin{aligned}\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m} &\le \frac{\phi(\mathbf{B}(x,R))}{\boldsymbol{\alpha}(m)R^m} + \mathbf{exc}^m_*(\phi, x, R)\\ &\le \frac{\phi(\mathbf{B}(0,R+|x|))}{\boldsymbol{\alpha}(m)(R+|x|)^m}\left(1+\frac{|x|}{R}\right)^m + \mathbf{exc}^m_*(\phi, x, R)\\ &\le \left(\Theta^{m\,*}(\phi, 0) + \mathbf{exc}^{m\,*}(\phi, 0, R)\right)(1+\eta)^m + \mathbf{exc}^m_*(\phi, x, R).\end{aligned}$$

q.e.d.

3.2. Nearly monotonic measures.

Definition 3.2.1. A *gauge* ξ is a nondecreasing function of $r > 0$, such that $\xi(r) \to 0$ as $r \to 0$.

Definition 3.2.2. Given an open set $U \subset \mathbf{R}^n$, a Radon measure ϕ on U, a gauge ξ and an integer $m \in \{0, \dots, n\}$, we say that ϕ is (ξ, m) *nearly monotonic in* U if for every $x \in U$ and every $0 < r < \operatorname{dist}(x, \operatorname{Bdry} U)$ one has

$$\mathbf{exc}^m_*(\phi, x, r) \le \xi(r).$$

Furthermore, if ϕ is (ξ, m) nearly monotonic in U and ξ vanishes identically we say that ϕ is m *monotonic* in U.

Lemma 3.2.3. *Let* $U \subset \mathbf{R}^n$ *be open, let* ξ *be a gauge and let* ϕ *be a* (ξ, m) *nearly monotonic measure on* U. *Then* $\Theta^m(\phi, x)$ *exists and is finite for every* $x \in U$, *and the function* $U \to \mathbf{R} : x \mapsto \Theta^m(\phi, x)$ *is upper semicontinuous.*

Proof. For $\mathbf{B}(x,r) \subset U$ we abbreviate $\varphi_x(r) := \boldsymbol{\alpha}(m)^{-1} r^{-m} \phi(\mathbf{B}(x,r))$. By definition one has

$$-\xi(R) \le \varphi_x(R) - \varphi_x(r)$$

whenever $\mathbf{B}(x,R) \subset U$ and $0 < r \le R$. Therefore,

$$-\xi(R) \le \liminf_{r \downarrow 0}\left(\varphi_x(R) - \varphi_x(r)\right) = \varphi_x(R) - \limsup_{r \downarrow 0} \varphi_x(r)$$

and, in turn,

$$0 \leq \liminf_{R \downarrow 0} \varphi_x(R) - \limsup_{r \downarrow 0} \varphi_x(r).$$

Next we let $x, x_1, x_2, \ldots \in U$ be such that $x_j \to x$ as $j \to \infty$, and we let $r > 0$ be such that $\mathbf{B}(x_j, 2r) \subset U$, $j = 1, 2, \ldots$, and $\rho \mapsto \phi(\mathbf{B}(x, \rho))$ is continuous at r. The above implies that

$$\begin{aligned}\Theta^m(\phi, x_j) &\leq \varphi_{x_j}(r) + \xi(r) \\ &\leq \varphi_x(r + |x_j - x|)\left(1 + \frac{|x_j - x|}{r}\right)^m + \xi(r)\end{aligned}$$

for each $j = 1, 2, \ldots$. Consequently,

$$\limsup_{j \to \infty} \Theta^m(\phi, x_j) \leq \varphi_x(r) + \xi(r)$$

and the conclusion follows on letting r tend to 0. q.e.d.

Remark 3.2.4. We could have possibly allowed, in the above definition, for any *real* number $m > 0$. However since the density $\Theta^m(\phi, x)$ exists at every $x \in U$ whenever ϕ is a nearly monotonic measure, making the further assumption that $\phi\,(U \cap \{x : 0 < \Theta^m(\phi, x) < \infty\}) \neq 0$ — as we will —forces m to be an integer in the range $0, \ldots, n$ (a theorem of J.M. Marstrand, see [**14**], [**15**, Theorem 14.10] or [**12**]). In the sequel we will in fact assume for m to be within the range $1, \ldots, n-1$: the case $m = 0$ is irrelevant (*every* Radon measure ϕ in U is 0 monotonic in U) and the case $m = n$ and $\xi = 0$ corresponds to $\phi = \mathcal{L}^n \llcorner u$ where $u \in \mathbf{L}^\infty(\mathcal{L}^n \llcorner U)$, $u \geq 0$ and $\triangle\, u \geq 0$ weakly, i.e., u is subharmonic (see [**8**, Example 3.9, Proposition 5.7]).

Lemma 3.2.5. *Let ϕ_j be a (ξ_j, m) nearly monotonic measure in an open set $U \subset \mathbf{R}^n$, $j = 1, 2, \ldots$, and let ϕ be a Radon measure in U and ξ a gauge. If $\phi_j \rightharpoonup \phi$ as $j \to \infty$ and $\xi_j \to \xi$ (pointwise) as $j \to \infty$, then ϕ is (ξ, m) nearly monotonic in U.*

Proof. For $\mathbf{B}(x, R) \subset U$, $0 < \hat{\rho} < \rho_2 < R$ and $j = 1, 2, \ldots$, the following holds:

$$-\xi_j(R) \leq \frac{\phi_j(\mathbf{B}(x, \rho_2))}{\boldsymbol{\alpha}(m)\rho_2^m} - \frac{\phi_j(\mathbf{U}(x, \hat{\rho}))}{\boldsymbol{\alpha}(m)\hat{\rho}^m}.$$

Therefore,

$$\begin{aligned}-\xi(R) &\leq \limsup_{j \to \infty} \frac{\phi_j(\mathbf{B}(x, \rho_2))}{\boldsymbol{\alpha}(m)\rho_2^m} - \liminf_{j \to \infty} \frac{\phi_j(\mathbf{U}(x, \hat{\rho}))}{\boldsymbol{\alpha}(m)\hat{\rho}^m} \\ &\leq \frac{\phi(\mathbf{B}(x, \rho_2))}{\boldsymbol{\alpha}(m)\rho_2^m} - \frac{\phi(\mathbf{U}(x, \hat{\rho}))}{\boldsymbol{\alpha}(m)\hat{\rho}^m}.\end{aligned} \tag{10}$$

Given $0 < \rho_1 < \rho_2 < R$ we pick a decreasing sequence $\rho_1 < \hat{\rho}_k < \rho_2$, $k = 1, 2, \ldots$, such that $\hat{\rho}_k \to \rho_1$ when $k \to \infty$. Since $\mathbf{B}(x, \rho_1) = \cap_{k=1}^\infty \mathbf{U}(x, \hat{\rho}_k)$

it follows readily from (10) that

$$-\xi(R) \le \frac{\phi(\mathbf{B}(x,\rho_2))}{\boldsymbol{\alpha}(m)\rho_2^m} - \frac{\phi(\mathbf{B}(x,\rho_1))}{\boldsymbol{\alpha}(m)\rho_1^m}.$$

The lemma is proved. q.e.d.

3.3. Concentrated measures.

Definition 3.3.1. Let $U \subset \mathbf{R}^n$ be open, $0 \le m \le n$ and let ϕ be a Radon measure in U. We say that ϕ is *m concentrated in U* if $\Theta_*^m(\phi, x) \ge 1$ for ϕ almost every $x \in U$. We also define the *m set* of ϕ as follows:

$$\mathrm{set}_m(\phi) = U \cap \{x : \Theta_*^m(\phi, x) \ge 1\}.$$

Remark 3.3.2. Notice that $\mathrm{set}_m(\phi) \subset \mathrm{spt}(\phi)$ but equality need not hold (it does not for instance if $n = 2$, $m = 1$ and

$$\phi = \sum_{j=1}^{\infty} \mathcal{H}^1 \llcorner \mathrm{Bdry}\,\mathbf{B}(a_j, 2^{-j})$$

where $a_1, a_2, \dots$ is a dense sequence in $\mathbf{R}^2$ – the next lemma shows this measure is not nearly monotonic).

Lemma 3.3.3. *Let $U \subset \mathbf{R}^n$ be open, $0 \le m \le n$, let ξ be a gauge and let ϕ be a Radon measure in U. If ϕ is m concentrated and (ξ, m) nearly monotonic in U then $\Theta^m(\phi, x) \ge 1$ for every $x \in \mathrm{spt}(\phi)$.*

Proof. Since ϕ is m concentrated we have $\phi(\mathrm{spt}(\phi) \sim \mathrm{set}_m(\phi)) = 0$, therefore $\mathrm{set}_m(\phi)$ is relatively dense in $\mathrm{spt}(\phi)$. It then follows from the (ξ, m) near monotonicity of ϕ and Lemma 3.2.3 that $\mathrm{set}_m(\phi) = \mathrm{spt}(\phi)$. q.e.d.

Lemma 3.3.4. *Let ϕ_j be an m concentrated (ξ_j, m) nearly monotonic measure in an open set $U \subset \mathbf{R}^n$, $j = 1, 2, \dots$, and let ϕ be a Radon measure in U and ξ a gauge. If $\phi_j \rightharpoonup \phi$ as $j \to \infty$ and $\xi_j \to \xi$ (pointwise) as $j \to \infty$, then ϕ is m concentrated and (ξ, m) nearly monotonic in U. Furthermore, for every compact $K \subset U$ and every $\varepsilon > 0$, there is an integer j_0 such that*

$$\mathrm{spt}(\phi) \cap K \subset U \cap \{x : \mathrm{dist}(x, \mathrm{spt}(\phi_j)) \le \varepsilon\}$$

as well as

$$\mathrm{spt}(\phi_j) \cap K \subset U \cap \{x : \mathrm{dist}(x, \mathrm{spt}(\phi)) \le \varepsilon\}$$

whenever $j \ge j_0$. In particular, if $x_0 \in \mathrm{spt}(\phi_j)$ for every $j = 1, 2, \dots$ then $x_0 \in \mathrm{spt}(\phi)$.

Proof. That ϕ is (ξ, m) nearly monotonic in U is the conclusion of Lemma 3.2.5. We turn to proving that ϕ is m concentrated in U. Let $x \in \mathrm{spt}\,\phi$. It suffices to show that $\Theta^m(\phi, x) \ge 1$. Suppose instead that

$\Theta^m(\phi, x) < 1 - \delta$ for some $0 < \delta < 1$. We choose $r_0 > 0$ so that $\xi(r_0) < \delta/3$, and $0 < r \leq r_0$ so that $\mathbf{B}(x, r) \subset U$,

$$\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m) r^m} < 1 - \delta \tag{11}$$

as well as $\phi(\operatorname{Bdry} \mathbf{B}(x, r)) = 0$. Next we choose an integer j_0 such that $\xi_j(r) \leq \xi_j(r_0) < \delta/3$ whenever $j \geq j_0$. Referring to (11) we also choose an integer j_1 such that

$$\frac{\phi_j(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m) r^m} < 1 - \delta \tag{12}$$

whenever $j \geq j_1$. We select $\varepsilon > 0$ sufficiently small for

$$(1 - \delta) \left(\frac{r}{r - \varepsilon} \right)^m < 1 - \frac{\delta}{3} \tag{13}$$

and finally we refer to Lemma 2.4.5 (applied for instance with $K = \{x\}$) to choose an integer $j \geq \max\{j_0, j_1\}$ such that $x \in \mathbf{B}(\operatorname{spt}(\phi_j), \varepsilon)$. Therefore there exists $y \in \operatorname{spt}(\phi_j)$ with $|x - y| \leq \varepsilon$. Now we have

$$\frac{\phi_j(\mathbf{B}(y, r - \varepsilon))}{\boldsymbol{\alpha}(m)(r - \varepsilon)^m} \leq \frac{\phi_j(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m) r^m} \left(\frac{r}{r - \varepsilon} \right)^m < 1 - \frac{\delta}{3}$$

according to (12) and (13). Whence

$$\begin{aligned}
\Theta^m(\phi_j, y) &\leq \frac{\phi_j(\mathbf{B}(y, r - \varepsilon))}{\boldsymbol{\alpha}(m)(r - \varepsilon)^m} + \mathbf{exc}^m_*(\phi_j, y, r - \varepsilon) \\
&\leq 1 - \delta/3 + \xi_j(r) \\
&\leq 1 - 2\delta/3,
\end{aligned}$$

in contradiction with Lemma 3.3.3.

In view of Lemma 2.4.5 it remains only to show that for every compact $K \subset U$ and $\varepsilon > 0$ there exists an integer j_0 such that $\operatorname{spt}(\phi_j) \cap K \subset U \cap \{x : \operatorname{dist}(x, \operatorname{spt} \phi_j) \leq \varepsilon\}$ whenever $j \geq j_0$. Suppose instead that there is a compact set $K \subset U$, $\varepsilon > 0$ and a sequence $k(1), k(2), \dots$ as well as $x_{k(j)} \in \operatorname{spt}(\phi_{k(j)}) \cap K$ such that $\operatorname{dist}(x_{k(j)}, \operatorname{spt} \phi) \geq \varepsilon$ for every integer j. Choose $x \in K$ and a subsequence $l(1), l(2), \dots$ of $k(1), k(2), \dots$ such that $x_{l(j)} \to x$ as $j \to \infty$. Let $\varepsilon' > 0$ be sufficiently small for $\varepsilon' < \varepsilon$, $\mathbf{B}(x, \varepsilon') \subset U$ and $\xi(\varepsilon') < 1/2$. If j is large enough for $|x - x_{l(j)}| \leq \varepsilon'/4$ then referring to Lemma 3.3.3 we obtain

$$\begin{aligned}
\phi_{l(j)}(\mathbf{B}(x, \varepsilon'/2)) &\geq \phi_{l(j)}(\mathbf{B}(x_{l(j)}, \varepsilon'/4)) \\
&\geq \boldsymbol{\alpha}(m)(\varepsilon'/4)^m \big(\Theta^m(\phi_{l(j)}, x_{l(j)}) \\
&\qquad - \mathbf{exc}^m_*(\phi_{l(j)}, x_{l(j)}, \varepsilon'/4) \big) \\
&\geq \boldsymbol{\alpha}(m)(\varepsilon'/4)^m \big(1 - \xi_{l(j)}(\varepsilon'/4) \big).
\end{aligned}$$

Letting $j \to \infty$ in the above inequality yields

$$\phi(\mathbf{B}(x, \varepsilon'/2)) \geq \boldsymbol{\alpha}(m)(\varepsilon'/4)^m (1 - \xi(\varepsilon'/4)) > 0$$

in contradiction with $\operatorname{dist}(x, \operatorname{spt}\phi) \geq \varepsilon > \varepsilon'$.

Finally, if $x_0 \in \operatorname{spt}(\phi_j)$ for every $j = 1, 2, \ldots$ we apply the inclusion we just proved to $K = \{x_0\}$ and we infer that $x_0 \in \operatorname{spt}(\phi)$. q.e.d.

Corollary 3.3.5. *Let ϕ_j be an m concentrated (ξ_j, m) nearly monotonic measure in an open set $U \subset \mathbf{R}^n$, $j = 1, 2, \ldots$, and let ϕ be a Radon measure in U and ξ a gauge. Assume there exists a cone $C \subset \mathbf{R}^n$ such that $\operatorname{spt}\phi = C \cap U$. If $\phi_j \rightharpoonup \phi$ as $j \to \infty$ and $\xi_j \to \xi$ (pointwise) as $j \to \infty$, then*

$$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}(\phi) \cap \mathbf{B}(0, r), \operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r)) \to 0 \text{ as } j \to \infty$$

whenever $r > 0$ is such that $\mathbf{B}(0, r) \subset U$.

Proof. Let $0 < \varepsilon < r$. According to Lemma 2.4.5 there is an integer j_0 such that if $j \geq j_0$ then

$$\operatorname{spt}(\phi) \cap \mathbf{B}(0, r) \subset U \cap \{x : \operatorname{dist}(x, \operatorname{spt}(\phi_j)) \leq \varepsilon\}.$$

Fix $j \geq j_0$ and let $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, r)$. If $|x| \leq r - \varepsilon$ there exists $y \in \operatorname{spt}(\phi_j)$ such that $|y - x| \leq \varepsilon$, whence $y \in \operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r)$. If $|x| > r - \varepsilon$ then referring to the coneness of $\operatorname{spt}(\phi)$ we choose $x' \in \operatorname{spt}(\phi)$ such that $|x' - x| \leq \varepsilon$ and $|x'| = r - \varepsilon$. We find $y' \in \operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r)$ such that $|y' - x'| \leq \varepsilon$, so that $|y' - x| \leq 2\varepsilon$. Therefore

$$\sup\{\operatorname{dist}(x, \operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r)) : x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, r)\} \leq 2\varepsilon.$$

According to Lemma 3.3.4 there is an integer j_1 such that

$$\operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r) \subset U \cap \{y : \operatorname{dist}(y, \operatorname{spt}(\phi)) \leq \varepsilon\}$$

whenever $j \geq j_1$. Fix $j \geq j_1$ and let $y \in \operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r)$. There exists $x \in \operatorname{spt}(\phi)$ such that $|y - x| \leq \varepsilon$. Therefore $|x| \leq r + \varepsilon$. Referring to the coneness of $\operatorname{spt}(\phi)$ we find $x' \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, r)$ such that $|x' - x| \leq x$. It follows that $|x' - y| \leq 2\varepsilon$ and, in turn,

$$\sup\{\operatorname{dist}(y, \operatorname{spt}(\phi) \cap \mathbf{B}(0, r)) : y \in \operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r)\} \leq 2\varepsilon.$$

If $j \geq \max\{j_0, j_1\}$ then

$$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}(\phi) \cap \mathbf{B}(0, r), \operatorname{spt}(\phi_j) \cap \mathbf{B}(0, r)) \leq 2\varepsilon$$

and the proof is complete. q.e.d.

3.4. Examples of concentrated nearly monotonic measures. We start by proving a useful criterion for near monotonicity.

Lemma 3.4.1. *Let U_0 and U be bounded open sets such that $\operatorname{Clos} U_0 \subset U$. Let ξ be a gauge and ϕ a Radon measure in U such that for every $x \in U$ the function*

$$(0, \operatorname{dist}(x, \operatorname{Bdry} U)) \to \mathbf{R} : r \mapsto \exp[\xi(r)] \frac{\phi(\mathbf{B}(x, r))}{\boldsymbol{\alpha}(m) r^m}$$

is increasing. Then there exists a constant $0 < \mathbf{c}_{3.4.1}(U_0, U, \phi, \xi, m) < \infty$ such that ϕ is $(\mathbf{c}_{3.4.1}\xi, m)$ nearly monotonic in U_0.

Proof. For $\mathbf{B}(x,r)\subset U$ we abbreviate $\varphi_x(r):=\boldsymbol{\alpha}(m)^{-1}r^{-m}\phi(\mathbf{B}(x,r))$. On letting $r(U_0,U) := \inf\{\operatorname{dist}(x, \operatorname{Bdry} U) : x \in U_0\} > 0$ we observe that

$$\varphi_x(r) \leq \mathbf{c}_1 := \exp[\xi(\operatorname{diam} U)]\frac{\phi(U)}{\boldsymbol{\alpha}(m)r(U_0,U)^m}$$

whenever $x \in U_0$ and $\mathbf{B}(x,r) \subset U$. Next choose $\mathbf{c}_2$ such that $\exp[t] \leq 1 + \mathbf{c}_2 t$ whenever $0 \leq t \leq \xi(\operatorname{diam} U)$. It remains to notice that for $\mathbf{B}(x,R) \subset U_0$ and $0 < \rho_1 < \rho_2 < R$ one has

$$\begin{aligned}\varphi_x(\rho_1) \leq \exp[\xi(\rho_1)]\varphi_x(\rho_1) &\leq \exp[\xi(\rho_2)]\varphi_x(\rho_2)\\ &\leq (1+\mathbf{c}_2\xi(R))\varphi_x(\rho_2)\\ &\leq \varphi_x(\rho_2) + \mathbf{c}_2\mathbf{c}_1\xi(R),\end{aligned}$$

which proves the lemma. q.e.d.

The assumption of the preceding lemma leads to our next definition.

Definition 3.4.2. Given an open set $U \subset \mathbf{R}^n$, a Radon measure ϕ on U, a gauge ξ and an integer $m \in \{0,\dots,n\}$ we say that ϕ is (ξ, m) *almost monotonic in* U if for every $x \in U$ the function

$$(0, \operatorname{dist}(x, \operatorname{Bdry} U)) \to \mathbf{R} : r \mapsto \exp[\xi(r)]\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m}$$

is increasing.

We now turn to giving a family of examples of nearly monotonic measures. Along with the concept of *mass* of an integral current, we will need the notion of *size* of such a current, introduced by F.J. Almgren in [**3**] (see also the work of H. Federer, [**11**]). In the remaining part of this section we will use some specific notations borrowed from [**10**] (see pp. 670–671 Ibid).

Definition 3.4.3. Let $U \subset \mathbf{R}^n$ be open and $T \in \mathbf{I}_m(U)$. The *size* of T is the defined as follows:

$$\mathbf{S}(T) := \mathcal{H}^m(\operatorname{set}_m(\|T\|)).$$

The *almost minimal* currents were introduced by E. Bombieri in [**4**], after F.J. Almgren defined and proved the regularity of *minimal sets* in his memoir [**2**].

Definition 3.4.4. Let $U \subset \mathbf{R}^n$ be open, ε a gauge, $0 < \delta \leq \infty$ and $T \in \mathbf{I}_m(U)$. We say that T is $(\mathbf{M}, \varepsilon, \delta)$ *minimal* (resp. $(\mathbf{S}, \varepsilon, \delta)$ *minimal*) whenever the following holds: for every compact set $C \subset U$ and for every $X \in \mathbf{I}_m(U)$, if

(A) $\operatorname{spt} X \subset C$;
(B) $\partial X = 0$;
(C) $r := \operatorname{diam} \operatorname{spt} X \leq \delta$;

then

$$\mathbf{M}(T \llcorner C) \leq (1 + \varepsilon(r))\mathbf{M}(T \llcorner C + X)$$

(resp. $\mathbf{S}(T \llcorner C) \leq (1 + \varepsilon(r))\mathbf{S}(T \llcorner C + X)$).

The following proposition and its proof are more or less classical (see e.g., [**10**, 5.4.3; 5.4.4]). As a particular case we notice that if $T \in \mathbf{I}_m(U)$ is *absolutely mass minimizing* then $\|T\|$ is m monotonic (take $\varepsilon = 0$ in the proposition). This generalizes to *stationary currents* (see [**10**, 5.4.2]) and for that matter to *stationary varifolds* (see [**1**, 5.1(2)]).

Proposition 3.4.5. *Assume that*

(A) $U_0 \subset U \subset \mathbf{R}^n$ *are open,* ε *is a gauge and* $0 < \delta \leq \infty$;
(B) ξ *is a gauge and*

$$\xi(r) = m \int_0^r \frac{\varepsilon(2\rho)}{\rho} d\mathcal{L}^1(\rho) < \infty$$

for every $0 \leq r \leq \frac{\delta}{2}$;
(C) $T \in \mathbf{I}_m(U)$ *is* $(\mathbf{M}, \varepsilon, \delta)$ *minimal* (*resp.* $(\mathbf{S}, \varepsilon, \delta)$ *minimal*);
(D) $\operatorname{diam} U_0 \leq \delta$ *and* $\operatorname{spt}(\partial T) \cap \operatorname{Clos} U_0 = \emptyset$.

Then the measure $\|T\|$ (*resp.* $\mathcal{H}^m \llcorner \operatorname{set}_m(\|T\|)$) *is* (ξ, m) *almost monotonic in* U_0.

Proof. We let $\phi = \|T\|$ in case T is $(\mathbf{M}, \varepsilon, \delta)$ minimal, whereas in case T is $(\mathbf{S}, \varepsilon, \delta)$ minimal we put $\phi = \mathcal{H}^m \llcorner \operatorname{set}_m(\|T\|)$. For each $\mathbf{B}(x, r) \subset U$ we abbreviate $f_x(r) := \phi(\mathbf{B}(x, r))$. Next we let $C := \operatorname{Clos} U_0$, we fix $\mathbf{B}(x, r) \subset U_0$ and we define

$$X := \boldsymbol{\delta}_x ⨳ \langle T, u, r+\rangle - T \llcorner \mathbf{B}(x, r)$$

where $u(y) = |y - x|$, $y \in \mathbf{R}^n$. We observe that $\operatorname{spt} X \subset C$, $2r = \operatorname{diam} \operatorname{spt} X \leq \delta$, and $\partial X = 0$ (the latter follows from [**10**, 4.1.11; 4.2.1], and the fact that $\operatorname{spt}(\partial T) \cap \mathbf{B}(x, r) = \emptyset$). Therefore

$$\mathbf{M}(T \llcorner \mathbf{B}(x, r)) \leq (1 + \varepsilon(2r))\mathbf{M}(\boldsymbol{\delta}_x ⨳ \langle T, u, r+\rangle) \tag{14}$$

or

$$\mathbf{S}(T \llcorner \mathbf{B}(x, r)) \leq (1 + \varepsilon(2r))\mathbf{S}(\boldsymbol{\delta}_x ⨳ \langle T, u, r+\rangle), \tag{15}$$

according to whether T is $(\mathbf{M}, \varepsilon, \delta)$ minimal or $(\mathbf{S}, \varepsilon, \delta)$ minimal. It follows from [**10**, 4.2.1; 4.1.11] that

$$\begin{aligned} \mathbf{M}(\boldsymbol{\delta}_x ⨳ \langle T, u, r+\rangle) &\leq \frac{r}{m}\mathbf{M}(\langle T, u, r+\rangle) \\ &\leq \frac{r}{m} \limsup_{h \to 0^+} \frac{f_x(r + h) - f_x(r)}{h} \end{aligned} \tag{16}$$

in case T is $(\mathbf{M}, \varepsilon, \delta)$ minimal. In the other case we deduce from [**10**, 4.3.8, 3.2.22] that

$$\begin{aligned}(17)\qquad \mathbf{S}(\boldsymbol{\delta}_x \mathbin{⨳} \langle T, u, r+\rangle) &\leq \frac{r}{m}\mathbf{S}(\langle T, u, r+\rangle) \\ &\leq \frac{r}{m}\limsup_{h\to 0^+}\frac{f_x(r+h)-f_x(r)}{h}.\end{aligned}$$

Plugging (16) into (14) (resp. (17) in (15)) yields

$$(18)\qquad f_x(r) \leq (1+\varepsilon(2r))\frac{r}{m}f_x'(r)$$

whenever f_x is differentiable at r. Since f_x is increasing, so is $\log\circ f_x$ and, according to (18), one has

$$\begin{aligned}(\log\circ f_x)'(r) = \frac{f_x'(r)}{f_x(r)} &\geq \frac{m}{r}\left(\frac{1}{1+\varepsilon(2r)}\right) \\ &\geq \frac{m}{r}(1-\varepsilon(2r))\end{aligned}$$

for $\mathcal{L}^1$ almost every $0 \leq r \leq \frac{\delta}{2}$. From this we infer that the function $r \mapsto \exp[\xi(r)]r^{-m}f_x(r)$, $0 \leq r \leq \frac{\delta}{2}$, is increasing. The conclusion follows at once. q.e.d.

Remark 3.4.6. In case $S \subset \mathbf{R}^n$ is an $(\mathbf{M}, \varepsilon, \delta)$ minimal *set* in $\mathbf{R}^n$ with respect to some closed set $B \subset \mathbf{R}^n$ (in the sense of F.J. Almgren, see [**2**]) and ε verifies the integrability condition (B) of Proposition 3.4.5, the measure $\phi = \mathcal{H}^m \llcorner S$ is (ξ, m) nearly monotonic as well. Proving this is slightly more difficult than proving the above proposition because the cut and paste procedure is no more available: comparison surfaces have to be Lipschitzian deformations of the original surface. For a full proof see [**9**] (notice J. Taylor [**21**, II.1] infers such a monotonicity formula from [**2**] for balls centered on the set S).

3.5. Epiperimetric measures.

Definition 3.5.1. Given an open set $U \subset \mathbf{R}^n$, a Radon measure ϕ on U, a gauge ξ, an integer $m \in \{0, \ldots, n\}$, an open set $U_0 \subset U$ and a Borel set $B \subset U_0$, we say that ϕ is (ξ, m) *epiperimetric in* (B, U_0) whenever for every $x \in B$ and every $r > 0$ such that $\mathbf{B}(x, r) \subset U_0$ the following holds:

$$\mathbf{exc}^{m\,*}(\phi, x, r) \leq \xi(r).$$

In the sequel we will require that a measure ϕ be epiperimetric in (B, U_0) only in case ϕ is m concentrated and $B \subset \operatorname{set}_m \phi$.

Remark 3.5.2. The following comments are in order.

(A) The proof of the so-called *epiperimetric inequality* uses comparison surfaces which are obtained by solving a Dirichlet problem together with a priori estimates on the rate of decrease of the

Dirichlet energy on balls. In case $\varepsilon = 0$ and $\delta = \infty$ this is essentially the content of E.R. Reifenberg's paper [**19**].

(B) The cases of epiperimetry alluded to so far are particular to the following geometrical situation: one assumes that $\operatorname{spt}(T)$ is sufficiently close (in Hausdorff distance and in density) to some "multiplicity one" m dimensional affine subspace. Epiperimetry is also known for mass minimizing currents in case $m = 2$ and $\operatorname{spt}(T)$ is sufficiently close to an integer multiplicity plane (see [**22**]), as well as for $(\mathbf{M}, \varepsilon, \delta)$ minimal sets when $m = 2$ and $n = 3$ (see [**21**]).

3.6. $C^{1,\alpha}$ embedded submanifolds. In this subsection we show that the Hausdorff measures carried by $C^{1,\alpha}$ submanifolds are locally nearly monotonic and epiperimetric (on their support).

Proposition 3.6.1. *Let $0 < \alpha \leq 1$ and let $S \subset \mathbf{R}^n$ be an m dimensional $C^{1,\alpha}$ embedded submanifold of $\mathbf{R}^n$. For each $a \in S$ one has $\Theta^m(\mathcal{H}^m \llcorner S, a) = 1$ and there exists a neighborhood $U \subset \mathbf{R}^n$ of a such that the following holds.*

(A) *The restriction of the measure $\mathcal{H}^m \llcorner S$ to U is (ξ_2, m) epiperimetric in (S, U) with $\xi_2(r) = c_2\, r^{2\alpha}$ for some $c_2 > 0$ depending only upon m, S, a and U.*

(B) *The restriction of the measure $\mathcal{H}^m \llcorner S$ to U is (ξ_1, m) nearly monotonic with $\xi_1(r) = c_1\, r^{\alpha}$ for some $c_1 > 0$, depending only upon m, S, a and U.*

Proof. Given $a \in S$ we choose U and $0 < C < \infty$ as in Lemma 2.2.3(B).

We first proceed to prove (A). Fix $x \in S \cap U$ and set $W := W_x$, $u := u_x$ and $f := i_W + i_{W^\perp} \circ u$ where W_x and u_x are as in Lemma 2.2.3(B). Next fix $r > 0$ such that $\mathbf{B}(x, r) \subset U$. In order to keep the notation short we will assume that $x = 0$. We claim that

$$f\left(\mathbf{B}(0, r_*)\right) \subset S \cap \mathbf{B}(0, r) \subset f\left(\mathbf{B}(0, r)\right) \tag{19}$$

where

$$r_* = r\sqrt{1 - C^2 r^{2\alpha}}. \tag{20}$$

The second inclusion above is obvious. For proving the first one we let $z \in W \cap \mathbf{B}(0, r_*)$ and we observe that

$$\begin{aligned}
|f(z)|^2 &= |z|^2 + |u(z)|^2 \\
&\leq |z|^2\left(1 + C^2|z|^{2\alpha}\right) \\
&\leq r_*^2\left(1 + C^2 r^{2\alpha}\right) \\
&\leq r^2.
\end{aligned}$$

Next we infer from Lemma 2.3.4 that for every $z \in W \cap \mathbf{B}(0,r)$ one has:

$$|J_m f(z) - 1| \leq C' \, r^{2\alpha} \tag{21}$$

where $C' = \frac{1}{2}C^2 + \mathbf{c}_{2.3.4}(m)C^4$. This, together with (19) and the area theorem [**10**, 3.2.3(1)], yields on the one hand

$$\begin{aligned} \mathcal{H}^m \left(S \cap \mathbf{B}(0,r)\right) - \boldsymbol{\alpha}(m) r^m &\leq \int_{W \cap \mathbf{B}(0,r)} \left(J_m f(z) - 1\right) d\,\mathcal{H}^m(z) \\ &\leq C' \, r^{2\alpha} \boldsymbol{\alpha}(m) r^m, \end{aligned} \tag{22}$$

and on the other hand (recalling (20))

$$\begin{aligned} &\mathcal{H}^m \left(S \cap \mathbf{B}(0,r)\right) - \boldsymbol{\alpha}(m) r^m \\ &\geq \int_{W \cap \mathbf{B}(0,r_*)} J_m f(z) d\,\mathcal{H}^m(z) - \boldsymbol{\alpha}(m) r^m \\ &\geq \left(1 - C' \, r^{2\alpha}\right) \boldsymbol{\alpha}(m) r_*^m - \boldsymbol{\alpha}(m) r^m \\ &= \boldsymbol{\alpha}(m) r^m \left(\left(1 - C' \, r^{2\alpha}\right) \left(1 - C^2 \, r^{2\alpha}\right)^{\frac{m}{2}} - 1 \right) \\ &\geq -\boldsymbol{\alpha}(m) r^m \left(1 + \frac{m}{2}\right) \max\{C^2, C'\} r^{2\alpha}. \end{aligned} \tag{23}$$

Conclusion (A) as well as the fact that $\Theta^m(\mathcal{H}^m \llcorner S, x) = 1$ now easily follow from inequalities (22) and (23).

We turn to the proof of conclusion (B). We start by defining $\delta(x_0) := \inf\{|x_0 - x| : x \in S \cap U\}$ for each $x_0 \in U$. Since S is locally compact the set $C(x_0) := S \cap \operatorname{Clos} U \cap \{x : |x_0 - x| = \delta(x_0)\}$ is nonempty. It is easily seen that $\tilde{U} := U \cap \{x_0 : C(x_0) \cap U \neq \emptyset\}$ is a neighborhood of a. We will prove that conclusion (B) holds true when $\mathcal{H}^m \llcorner S$ is restricted to that possibly smaller set $\tilde{U}$. We now fix some $x_0 \in \tilde{U} \sim S$, we set $\delta := \delta(x_0) > 0$ and we choose $x \in C(x_0) \cap U$. We let W, u and f be as before and, in order to keep the notation short, we will assume that $x = 0$ (so that $\delta = |x_0|$). We readily check that $x_0 \in W^\perp$. Let $r > \delta$ be such that $\mathbf{B}(x_0, r) \subset U$. We claim that

$$f\left(\mathbf{B}(0, r_*)\right) \subset S \cap \mathbf{B}(x_0, r) \subset f\left(\mathbf{B}(0, r^*)\right) \tag{24}$$

where

$$r_* = \sqrt{\max\left\{0, r^2 - (\delta + C\, r^{1+\alpha})^2\right\}} \tag{25}$$

$$r^* = \sqrt{r^2 - (\delta - C\, r^{1+\alpha})^2}. \tag{26}$$

In order to prove the first inclusion we let $z \in W \cap \mathbf{B}(0, r_*)$ and we simply observe that (if $r_* > 0$)

$$\begin{aligned}
|f(z) - x_0|^2 &= |f(z)|^2 + |x_0|^2 - 2\langle x_0, f(z)\rangle \\
&= |z|^2 + |u(z)|^2 + \delta^2 - 2\langle x_0, u(z)\rangle \\
&\le r_*^2 + |u(z)|^2 + \delta^2 + 2\delta|u(z)| \\
&\le r_*^2 + \left(\delta + C\, r^{1+\alpha}\right)^2 \\
&\le r^2.
\end{aligned}$$

In order to prove the second inclusion we let $y \in S \cap \mathbf{B}(x_0, r)$, we choose $z \in W$ with $y = f(z)$ and we compute, as above,

$$\begin{aligned}
|z|^2 &\le r^2 - |u(z)|^2 - |x_0|^2 + 2\langle x_0, u(z)\rangle \\
&\le r^2 - |u(z)|^2 - \delta^2 + 2\delta|u(z)| \\
&= r^2 - (\delta - |u(z)|)^2 \\
&\le r^2 - \left(\delta - C\, r^{1+\alpha}\right)^2.
\end{aligned}$$

Next we claim that, on letting $\rho := \sqrt{r^2 - \delta^2}$, the following holds:

$$r_*^m - \rho^m \ge -C_* r^m r^\alpha \tag{27}$$

$$\frac{r_*}{r} \le 1 \tag{28}$$

$$(r^*)^m - \rho^m \le C^* r^m r^\alpha \tag{29}$$

$$\frac{r^*}{r} \le 1, \tag{30}$$

where $0 < C_*, C^* < \infty$ depend only upon m and C. We start by proving (27) in case $r_* = 0$, that is when $r^2 - \left(\delta + C\, r^{1+\alpha}\right)^2 \le 0$. Expanding the square and recalling that $\delta < r \le 1$ we find that

$$\begin{aligned}
\rho^2 = r^2 - \delta^2 &\le 2\delta C r^{1+\alpha} + C^2 r^{2(1+\alpha)} \\
&\le \left(2C + C^2\right) r^{2+\alpha},
\end{aligned}$$

and it suffices to raise this inequality to the power $\frac{m}{2}$ and to recall that $m \ge 2$. In case $r_* > 0$ the analogous computation yields

$$r_*^2 = r^2 - \left(\delta + C\, r^{1+\alpha}\right)^2 = \rho^2 - \varepsilon, \tag{31}$$

where

$$\varepsilon := 2\delta C r^{1+\alpha} + C^2 r^{2(1+\alpha)} \le \left(2C + C^2\right) r^{2+\alpha}. \tag{32}$$

Furthermore, one easily checks that if $b \ge a \ge 0$ then

$$a^m - b^m \ge c(m) b^{m-2}(a^2 - b^2),$$

where $c(m) = \frac{m}{2}$ if m is even, and $c(m) = \frac{m+1}{2} + 2^{\frac{m-3}{2}}$ if m is odd. Applying this inequality to $a = r_*$ and $b = \rho$, and referring to (31) and (32), we obtain

$$\begin{aligned} r_*^m - \rho^m &\geq c(m)\rho^{m-2}(r_*^2 - \rho^2) \\ &\geq -\varepsilon c(m) r^{m-2} \\ &\geq -c(m)(C^2 + 2C) r^{m+\alpha} \end{aligned}$$

which finishes the proof of (27). In order to prove (29) we recall from (26) and the definition of ρ that

$$\begin{aligned} (r^*)^2 &= r^2 - (\delta - Cr^{1+\alpha})^2 \\ &= \rho^2 + \varepsilon \end{aligned} \tag{33}$$

where

$$\varepsilon := 2\delta C r^{1+\alpha} - C^2 r^{2(1+\alpha)} \leq (2C + C^2) r^{2+\alpha}. \tag{34}$$

If $\varepsilon \leq 0$ then $r^* \leq \rho$ and (29) is obvious; therefore, we subsequently assume that $\varepsilon > 0$. It is an easy matter to check that if $a \geq b \geq 0$ then

$$a^m - b^m \leq c(m) a^{m-2}(a^2 - b^2).$$

Applied to $a = r^*$ and $b = \rho$, this inequality together with (33) and (34) yields the following:

$$\begin{aligned} (r^*)^m - \rho^m &\leq c(m)(r^*)^{m-2}((r^*)^2 - \rho^2) \\ &\leq c(m) r^{m-2}\varepsilon \\ &\leq c(m)(2C + C^2) r^{m+\alpha}, \end{aligned}$$

which proves (29). Inequalities (28) and (30) are obvious consequences of (25) and (26). We now use (24), (21), (26), (29), (30) together with the area theorem [**10**, 3.2.3(1)] and we find that

$$\begin{aligned} &\mathcal{H}^m(S \cap \mathbf{B}(x_0, r)) - \boldsymbol{\alpha}(m)\rho^m \\ &\leq \mathcal{H}^m\left(f\left(\mathbf{B}(0, r^*)\right)\right) - \boldsymbol{\alpha}(m)\rho^m \\ &= \int_{W \cap \mathbf{B}(0,r^*)} J_m f(z)\, d\mathcal{H}^m(z) - \boldsymbol{\alpha}(m)\rho^m \\ &\leq \left(1 + C'\left(r^*\right)^{2\alpha}\right)\boldsymbol{\alpha}(m)\left(r^*\right)^m - \boldsymbol{\alpha}(m)\rho^m \\ &= \boldsymbol{\alpha}(m)\big(\left(r^*\right)^m - \rho^m\big) + \boldsymbol{\alpha}(m)C'\left(r^*\right)^m\left(r^*\right)^{2\alpha} \\ &\leq \boldsymbol{\alpha}(m) r^m\left(C^* r^\alpha + C' r^{2\alpha}\right). \end{aligned} \tag{35}$$

On the other hand, using (24), (21), (25), (27), (28) and the area theorem, we obtain

$$
\begin{aligned}
(36)\qquad & \mathcal{H}^m(S\cap \mathbf{B}(x_0,r)) - \boldsymbol{\alpha}(m)\rho^m \\
&\geq \mathcal{H}^m\left(f\left(\mathbf{B}(0,r_*)\right)\right) - \boldsymbol{\alpha}(m)\rho^m \\
&= \int_{W\cap \mathbf{B}(0,r_*)} J_m f(z)\, d\mathcal{H}^m(z) - \boldsymbol{\alpha}(m)\rho^m \\
&\geq \left(1 - C' r_*^{2\alpha}\right)\boldsymbol{\alpha}(m) r_*^m - \boldsymbol{\alpha}(m)\rho^m \\
&= \boldsymbol{\alpha}(m)\left(r_*^m - \rho^m\right) - C'\boldsymbol{\alpha}(m) r_*^m r_*^{2\alpha} \\
&\geq -\boldsymbol{\alpha}(m) r^m \left(C_* r^\alpha + C' r^{2\alpha}\right).
\end{aligned}
$$

Define

$$\xi(r) := \max\left\{C^* + C', C_* + C'\right\} r^\alpha .$$

Now on letting $\varphi(r) := \boldsymbol{\alpha}(m)^{-1} r^{-m}\mathcal{H}^m(S\cap \mathbf{B}(x_0,r))$ and $g(r) := r^{-m}\left(r^2-\delta^2\right)^{\frac{m}{2}}$ whenever $r\geq\delta$ and $\mathbf{B}(x_0,r)\subset U$ we infer from (35) and (36) that

$$
\begin{aligned}
\varphi(R)-\varphi(r) &= (\varphi(R)-g(R)) + (g(R)-g(r)) + (g(r)-\varphi(r)) \\
&\geq -\xi(R) - \xi(r)
\end{aligned}
$$

whenever $R\geq r\geq\delta$ and $\mathbf{B}(x_0,R)\subset U$. This is because g is increasing, as can be checked easily (in fact $g(r)$ can be thought of as $\boldsymbol{\alpha}(m)^{-1} r^{-m}\mathcal{H}^m(W\cap \mathbf{B}(x_0,r))$, and $\mathcal{H}^m \llcorner W$ is an m monotonic measure). Consequently we obtain that $\mathbf{exc}_*(\mathcal{H}^m\llcorner S, x_0, R)\leq 2\xi(R)$, from which conclusion (B) follows at once. q.e.d.

4. First and second moments computations

In this section we are given the following data: *an open set* $U\subset\mathbf{R}^n$ *such that* $0\in U$ *and a Radon measure* ϕ *on* U *such that* $\Theta^m(\phi,0)$ *exists.*

4.1. Definitions and normalization. For $x\in U$ and $r>0$ such that $\mathbf{B}(x,r)\subset U$ we define

$$V(\phi,x,r) := \int_{\mathbf{B}(0,r)} \left(r^2 - |x-y|^2\right)^2 d\phi(y),$$

which we develop into successive homogeneous polynomials of x:

$$V(\phi,x,r) = \sum_{k=0}^{4} P_k(\phi,x,r).$$

It is a simple matter to check that

$$P_0(\phi, x, r) = \int_{\mathbf{B}(0,r)} \left(r^2 - |y|^2\right)^2 \, d\phi(y)$$
$$P_1(\phi, x, r) = 4 \left\langle x, \int_{\mathbf{B}(0,r)} y(r^2 - |y|^2) \, d\phi(y) \right\rangle$$
$$P_2(\phi, x, r) = 4 \int_{\mathbf{B}(0,r)} \langle x, y\rangle^2 \, d\phi(y) - 2|x|^2 \int_{\mathbf{B}(0,r)} (r^2 - |y|^2) \, d\phi(y)$$
$$P_3(\phi, x, r) = -4|x|^2 \int_{\mathbf{B}(0,r)} \langle x, y\rangle \, d\phi(y)$$
$$P_4(\phi, x, r) = |x|^4 \phi(\mathbf{B}(0, r)).$$

We single out two quantities because of their geometric significance:

$$b(\phi, r) := \int_{\mathbf{B}(0,r)} y(r^2 - |y|^2) \, d\phi(y),$$
$$Q(\phi, r)(x) := \int_{\mathbf{B}(0,r)} \langle x, y\rangle^2 \, d\phi(y).$$

If "normalized properly", $b(\phi, r)$ may be thought of as a weighted center of mass of ϕ in $\mathbf{B}(0, r)$, whereas $Q(\phi, r)(x)$ should be thought of (as a function of x) being close to $|P_W(x)|^2$ for some $W \in \mathbf{G}(n, m)$ approximating $\operatorname{spt}(\phi)$ in $\mathbf{B}(0, r)$, provided of course that we make assumptions assuring that $\operatorname{spt}(\phi)$ is sufficiently close to flat in $\mathbf{B}(0, r)$. One trouble is that $b(\phi, r)$ and $Q(\phi, r)$ do not normalize simultaneously to being "dimensionless". We choose to normalize V and each polynomial P_k, $k = 0, \ldots, 4$, dividing them by ar^p where a and p are as follows: p is chosen in order that the normalized version of P_2 be "dimensionless", and a is chosen so that the normalized version of Q has trace close to m.

In order to describe the normalization we introduce the following constants

$$\boldsymbol{\omega}(m, q) := \int_{\mathbf{R}^m \cap \mathbf{B}(0,1)} \left(1 - |y|^2\right)^q \, d\mathcal{L}^m(y), \; q = 0, 1, 2, \ldots$$

For instance $\boldsymbol{\omega}(m, 0) = \boldsymbol{\alpha}(m)$ and $\boldsymbol{\omega}(m, 1) = 2(m + 2)^{-1}\boldsymbol{\alpha}(m)$. The significance of these constants is indicated in the following lemma.

Lemma 4.1.1. *Let $x \in U$ and $r > 0$ be such that $\mathbf{B}(x, r) \subset U$. Let also $\theta > 0$ and $\varepsilon > 0$ be such that*

$$\left| \frac{\phi(\mathbf{B}(x, \rho))}{\boldsymbol{\alpha}(m)\rho^m} - \theta \right| \leq \varepsilon$$

for every $0 < \rho < r$. *Then for each* $q = 0, 1, 2, \ldots$ *one has*

$$\left| \int_{\mathbf{B}(x,r)} \left(r^2 - |x-y|^2\right)^q \, d\phi(y) - \theta \boldsymbol{\omega}(m,q) r^{2q+m} \right| \le \varepsilon \boldsymbol{\omega}(m,q) r^{2q+m}.$$

Proof. It suffices to observe that

$$\begin{aligned}
&\int_{\mathbf{B}(x,r)} \left(r^2 - |x-y|^2\right)^q \, d\phi(y) \\
&= \int_0^r \phi\left(\mathbf{B}\left(x, \sqrt{r^2 - \sqrt[q]{t}}\right)\right) \, d\mathcal{L}^1(t) \\
&\le (\theta + \varepsilon) \int_0^r \mathcal{L}^m \left(\mathbf{B}\left(0, \sqrt{r^2 - \sqrt[q]{t}}\right)\right) \, d\mathcal{L}^1(t) \\
&= (\theta + \varepsilon) \int_{\mathbf{B}(0,r)} \left(r^2 - |y|^2\right)^q \, d\mathcal{L}^m(y) \\
&= (\theta + \varepsilon) \boldsymbol{\omega}(m,q) r^{2q+m}.
\end{aligned}$$

The other inequality is proved exactly the same way. q.e.d.

Finally we define

$$\boldsymbol{\nu}(m) := \frac{\boldsymbol{\omega}(m,0) - \boldsymbol{\omega}(m,1)}{m} = \frac{\boldsymbol{\alpha}(m)}{m+2}$$

for a reason that will be transparent in the next lemma. We are now able to define the normalized versions of the various integrals introduced so far. We adopt the convention that **boldface** indicates that the corresponding quantity is normalized, i.e., divided by $\boldsymbol{\nu}(m) r^{m+2}$:

$$\begin{aligned}
\mathbf{V}(\phi, x, r) &:= \boldsymbol{\nu}(m)^{-1} r^{-m-2} V(\phi, x, r) \\
\mathbf{P}_k(\phi, x, r) &:= \boldsymbol{\nu}(m)^{-1} r^{-m-2} P_k(\phi, x, r), \ k = 0, \ldots, 4 \\
\mathbf{b}(\phi, r) &:= \boldsymbol{\nu}(m)^{-1} r^{-m-2} b(\phi, r) \\
\mathbf{Q}(\phi, r) &:= \boldsymbol{\nu}(m)^{-1} r^{-m-2} Q(\phi, r).
\end{aligned}$$

Lemma 4.1.2. *Let* $r > 0$ *be such that* $\mathbf{B}(0,r) \subset U$, *and* $\varepsilon > 0$ *be such that*

$$\left| \frac{\phi(\mathbf{B}(0,\rho))}{\boldsymbol{\alpha}(m)\rho^m} - \Theta^m(\phi, 0) \right| \le \varepsilon$$

for every $0 < \rho < r$. *Then*

$$|\operatorname{trace} \mathbf{Q}(\phi, r) - m\Theta^m(\phi, 0)| \le \varepsilon(m+4).$$

Proof. If $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbf{R}^n$ then

$$\operatorname{trace} Q(\phi, r) = \sum_{i=1}^n Q(\phi, r)(e_i) = \int_{\mathbf{B}(0,r)} |y|^2 \, d\phi(y).$$

Therefore, on letting $\theta := \Theta^m(\phi, 0)$, we have

$$\begin{aligned}
&\left|\theta m\boldsymbol{\nu}(m) r^{m+2} - \operatorname{trace} Q(\phi, r)\right| \\
&= \left|\theta m\boldsymbol{\nu}(m) r^{m+2} - \int_{\mathbf{B}(0,r)} |y|^2 \, d\phi(y)\right| \\
&\leq \left|\theta \boldsymbol{\omega}(m,1) r^{m+2} - \int_{\mathbf{B}(0,r)} (r^2 - |y|^2) \, d\phi(y)\right| \\
&\quad + \left|\theta \boldsymbol{\omega}(m,0) r^{m+2} - r^2 \phi(\mathbf{B}(0,r))\right| \\
&\leq \varepsilon \boldsymbol{\omega}(m,1) r^{m+2} + \varepsilon \boldsymbol{\omega}(m,0) r^{m+2}
\end{aligned}$$

according to Lemma 4.1.1. The proof is completed on dividing both sides by $\boldsymbol{\nu}(m) r^{m+2}$. q.e.d.

4.2. Two estimates. In order to deduce some relevant geometric information from the moments, we will need two basic estimates. These involve comparing $V(\phi, x, r)$ and $\mathbf{V}(\phi, x, r)$ to the following:

$$\begin{aligned}
\hat{V}(\phi, x, r) &:= \int_{\mathbf{B}(x,r)} \left(r^2 - |x-y|^2\right)^2 \, d\phi(y)\,, \\
\hat{\mathbf{V}}(\phi, x, r) &:= \boldsymbol{\nu}(m)^{-1} r^{-m-2} \hat{V}(\phi, x, r).
\end{aligned}$$

Lemma 4.2.1. *There exists a constant $0 < \mathbf{c}_{4.2.1}(m) < \infty$ with the following property. Whenever*

(A) *$U \subset \mathbf{R}^n$ is open, ϕ is a Radon measure in U, $0 \in U$ and $\Theta^m(\phi, 0)$ exists;*

(B) *$x \in U$, $r > 0$, $\mathbf{B}(0, 2r) \subset U$ and $2|x| < r$;*

one has:

$$\begin{aligned}
\left|V(\phi, x, r) - \hat{V}(\phi, x, r)\right| \leq \mathbf{c}_{4.2.1}(m) \boldsymbol{\nu}(m) r^m \Big(& r|x|^3 \left(r^{-m}\phi(\mathbf{B}(0,r)\right) \\
& + r^2|x|^2 \mathbf{exc}^{m*}(\phi, 0, 2r) \Big).
\end{aligned}$$

Proof. We first claim that if $x \in \mathbf{R}^n$, $r > 0$ and $2|x| < r$ then

(a) $\big(\mathbf{B}(x,r) \sim \mathbf{B}(0,r)\big) \cup \big(\mathbf{B}(0,r) \sim \mathbf{B}(x,r)\big) \subset \mathbf{B}(x, r+|x|) \sim \mathbf{B}(x, r-|x|)$;

(b) if $y \in \big(\mathbf{B}(x,r) \sim \mathbf{B}(0,r)\big) \cup \big(\mathbf{B}(0,r) \sim \mathbf{B}(x,r)\big)$ then one has $\left|r^2 - |x-y|^2\right| \leq 3r|x|$;

(c) $\mathbf{B}(0, r - 2|x|) \subset \mathbf{B}(x, r - |x|)$;

(d) $\mathbf{B}(x, r + |x|) \subset \mathbf{B}(0, r + 2|x|)$.

These are rather obvious statements and we leave the proof of (a), (c) and (d) to the reader. For proving (b), in view of (a) we have that

$$0 \leq r - |x| \leq |x-y| \leq r + |x|$$

whence

$$r^2 - 2r|x| + |x|^2 \leq |x-y|^2 \leq r^2 + 2r|x| + |x|^2$$

and in turn

$$-|x|^2 - 2r|x| \le r^2 - |x-y|^2 \le 2r|x| - |x|^2 \,.$$

Since $|x| \le r$, one has $|x|^2 \le r|x|$ and (b) follows immediately.

Using these four properties we now see that

$$\begin{aligned}
(37) \qquad & \left|V(\phi,x,r) - \hat{V}(\phi,x,r)\right| \\
& \le 9r^2|x|^2\phi\Big[\big(\mathbf{B}(0,r) \sim \mathbf{B}(x,r)\big) \cup \big(\mathbf{B}(x,r) \sim \mathbf{B}(0,r)\big)\Big] \\
& \le 9r^2|x|^2\Big(\phi\big(\mathbf{B}(x,r+|x|)\big) - \phi\big(\mathbf{B}(x,r-|x|)\big)\Big) \\
& \le 9r^2|x|^2\Big(\phi\big(\mathbf{B}(0,r+2|x|)\big) - \phi\big(\mathbf{B}(0,r-2|x|)\big)\Big).
\end{aligned}$$

Since $r + 2|x| < 2r$ we have that

$$\frac{\phi(\mathbf{B}(0,r+2|x|))}{\boldsymbol{\alpha}(m)(r+2|x|)^m} \le \frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m} + \mathbf{exc}^{m\,*}(\phi,0,2r)$$

so that

$$\begin{aligned}
\phi(\mathbf{B}(0,r+2|x|)) \le \left(1 + \frac{2|x|}{r}\right)^m & \phi(\mathbf{B}(0,r)) \\
& + \boldsymbol{\alpha}(m)(r+2|x|)^m \mathbf{exc}^{m\,*}(\phi,0,2r)\,,
\end{aligned}$$

as well as

$$\frac{\phi(\mathbf{B}(0,r-2|x|))}{\boldsymbol{\alpha}(m)(r-2|x|)^m} \ge \frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m} - \mathbf{exc}^{m\,*}(\phi,0,r)$$

whence

$$\begin{aligned}
\phi(\mathbf{B}(0,r-2|x|)) \ge \left(1 - \frac{2|x|}{r}\right)^m & \phi(\mathbf{B}(0,r)) \\
& - \boldsymbol{\alpha}(m)(r-2|x|)^m \mathbf{exc}^{m\,*}(\phi,0,r)\,.
\end{aligned}$$

From this we deduce that

$$\begin{aligned}
(38) \qquad & \phi(\mathbf{B}(0,r+2|x|)) - \phi(\mathbf{B}(0,r-2|x|)) \\
& \le \phi(\mathbf{B}(0,r))\left(\left(1+\frac{2|x|}{r}\right)^m - \left(1-\frac{2|x|}{r}\right)^m\right) \\
& \quad + (1+2^m)\boldsymbol{\alpha}(m)r^m\mathbf{exc}^{m\,*}(\phi,0,2r) \\
& \le \phi(\mathbf{B}(0,r))m2^m\frac{2|x|}{r} \\
& \quad + (1+2^m)\boldsymbol{\alpha}(m)r^m\mathbf{exc}^{m\,*}(\phi,0,2r)
\end{aligned}$$

because on letting $f(t) := (1+t)^m - (1-t)^m$, $0 < t < 1$, it is easily checked that $f(t) \le m2^m t$. Plugging (38) into (37) proves the lemma.

q.e.d.

Definition 4.2.2. Given $x_1, x_2 \in U$ and $r > 0$ such that $\mathbf{B}(x_1, r) \cup \mathbf{B}(x_2, r) \subset U$, as well as an integer $m \in \{0, \dots, n\}$, we define the *deviation* relative to (ϕ, x_1, x_2, r, m) as follows:

$$\mathbf{dev}^m(\phi, x_1, x_2, r) := \frac{\phi(\mathbf{B}(x_1, r)) - \phi(\mathbf{B}(x_2, r))}{r^m}.$$

Lemma 4.2.3. *There exists a constant* $0 < \mathbf{c}_{4.2.3}(m) < \infty$ *with the following property. Whenever*

(A) $U \subset \mathbf{R}^n$ *is open and* ϕ *is a Radon measure in* U*;*
(B) $0, x \in U$*,* $r > 0$*,* $\mathbf{B}(0, r) \cup \mathbf{B}(x, r) \subset U$*;*

one has

$$\begin{aligned}\hat{V}(\phi, x, r) - \hat{V}(\phi, 0, r) \le \mathbf{c}_{4.2.3}(m)\boldsymbol{\nu}(m) r^{m+4}\Big(&\mathbf{dev}^m(\phi, x, 0, r) \\ &+ \mathbf{exc}_*^m(\phi, x, r) + \mathbf{exc}^{m\,*}(\phi, 0, r)\Big),\end{aligned}$$

as well as

$$\begin{aligned}\Big|\hat{V}(\phi, x, r) - \hat{V}(\phi, 0, r)\Big| \le \mathbf{c}_{4.2.3}(m)\boldsymbol{\nu}(m) r^{m+4}\Big(&|\mathbf{dev}^m(\phi, x, 0, r)| \\ &+ \|\mathbf{exc}^m\|(\phi, x, r) + \|\mathbf{exc}^m\|(\phi, 0, r)\Big).\end{aligned}$$

Proof. We observe that

$$\begin{aligned}\hat{V}(\phi, x, r) &= \int_{\mathbf{B}(x,r)} \left(r^2 - |x-y|^2\right)^2 \, d\phi(y) \\ &= \int_0^{r^2} \phi\left(\mathbf{B}\left(x, \sqrt{r^2 - \sqrt{t}}\right)\right) d\mathcal{L}^1(t) \\ &= \int_0^r \phi(\mathbf{B}(x, \rho)) 4\rho(r^2 - \rho^2) \, d\mathcal{L}^1(\rho);\end{aligned}$$

similarly,

$$\hat{V}(\phi, 0, r) = \int_0^r \phi(\mathbf{B}(0, \rho)) 4\rho(r^2 - \rho^2) \, d\mathcal{L}^1(\rho),$$

so that

(39)
$$\hat{V}(\phi, x, r) - \hat{V}(\phi, 0, r) = \int_0^r \big(\phi(\mathbf{B}(x, \rho)) - \phi(\mathbf{B}(0, \rho))\big) 4\rho(r^2 - \rho^2) \, d\mathcal{L}^1(\rho).$$

For $0 < \rho < r$ we have on the one hand

$$\frac{\phi(\mathbf{B}(x, \rho))}{\boldsymbol{\alpha}(m)\rho^m} \le \frac{\phi(\mathbf{B}(x, r))}{\boldsymbol{\alpha}(m) r^m} + \mathbf{exc}_*^m(\phi, x, r)$$

so that

$$\phi(\mathbf{B}(x, \rho)) \le \rho^m \frac{\phi(\mathbf{B}(x, r))}{r^m} + \boldsymbol{\alpha}(m)\rho^m \mathbf{exc}_*^m(\phi, x, r), \tag{40}$$

and on the other hand

$$-\frac{\phi(\mathbf{B}(0,\rho))}{\boldsymbol{\alpha}(m)\rho^m} \leq -\frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m} + \mathbf{exc}^{m\,*}(\phi,0,r),$$

hence

$$-\phi(\mathbf{B}(0,\rho)) \leq -\rho^m \frac{\phi(\mathbf{B}(0,r))}{r^m} + \boldsymbol{\alpha}(m)\rho^m \mathbf{exc}^{m\,*}(\phi,0,r). \tag{41}$$

One also checks that

$$\int_0^r 4\rho^{m+1}(r^2-\rho^2)\, d\mathcal{L}^1(\rho) = 8(m+2)^{-1}(m+4)^{-1}r^{m+4}. \tag{42}$$

Plugging (40) and (41) into (39), and using (42) yields the first required estimate. In order to prove the second conclusion it suffices to apply the first one with 0 and x swapped. q.e.d.

4.3. Controlling the length of the first moment. In this subsection we obtain some estimates about $|\mathbf{b}(\phi,r)|$. The first one is a trivial bound $O(r)$ due to the normalization we have chosen.

Lemma 4.3.1. *Let $U \subset \mathbf{R}^n$ be open, $0 \in U$, and let ϕ be a Radon measure on U so that $\Theta^m(\phi,0)$ exists. For each $r > 0$ such that $\mathbf{B}(0,r) \subset U$ one has*

$$|\mathbf{b}(\phi,r)| \leq 2r\left(\Theta^m(\phi,0) + \|\mathbf{exc}^m\|(\phi,0,r)\right).$$

Proof. It suffices to apply Lemma 4.1.1:

$$\begin{aligned} |b(\phi,r)| &\leq \int_{\mathbf{B}(0,r)} |y|\left(r^2-|y|^2\right)\, d\phi(y) \\ &\leq r\left(\Theta^m(\phi,0) + \|\mathbf{exc}^m\|(\phi,0,r)\right)\boldsymbol{\omega}(m,1)r^{m+2}, \end{aligned}$$

and divide by $\boldsymbol{\nu}(m)r^{m+2}$. q.e.d.

We will also need to control the deviation in the following way.

Lemma 4.3.2. *Assume that*

(A) *$U \subset \mathbf{R}^n$ is open, $0 \in U$ and ϕ is a Radon measure on U so that $\Theta^m(\phi,0)$ exists;*
(B) *$0 < r \leq R$ and $\mathbf{B}(0,2R) \subset U$;*
(C) *ε is a gauge, $\varepsilon(R) \leq 1$, $x \in U$ and $|x| = \varepsilon(R)R$.*

Then,

$$\begin{aligned} \boldsymbol{\alpha}(m)^{-1}\mathbf{dev}^m(\phi,x,0,r) \leq m2^{m-1}\varepsilon(R)\big(\Theta^m(\phi,0) + \|\mathbf{exc}^m\|(\phi,0,2R)\big) \\ + \mathbf{exc}^{m\,*}(\phi,0,2R) + \mathbf{exc}^m_*(\phi,x,R). \end{aligned}$$

Proof. It suffices to compute:

$$
\begin{aligned}
&\boldsymbol{\alpha}(m)^{-1}\mathbf{dev}^m(\phi,x,0,r)\\
&= \frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m} - \frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m}\\
&\leq \frac{\phi(\mathbf{B}(x,R))}{\boldsymbol{\alpha}(m)R^m} + \mathbf{exc}^m_*(\phi,x,R) - \frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m}\\
&\leq \frac{(R+|x|)^m}{R^m}\frac{\phi(\mathbf{B}(0,R+|x|))}{\boldsymbol{\alpha}(m)(R+|x|)^m} - \frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m}\\
&\quad + \mathbf{exc}^m_*(\phi,x,R)\\
&\leq \big((1+\varepsilon(R))^m - 1\big)\frac{\phi(\mathbf{B}(0,R+|x|))}{\boldsymbol{\alpha}(m)(R+|x|)^m}\\
&\quad + \frac{\phi(\mathbf{B}(0,R+|x|))}{\boldsymbol{\alpha}(m)(R+|x|)^m} - \frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m} + \mathbf{exc}^m_*(\phi,x,R)\\
&\leq m2^{m-1}\varepsilon(R)\big(\Theta^m(\phi,0) + \|\mathbf{exc}^m\|(\phi,0,R+|x|)\big)\\
&\quad + \mathbf{exc}^{m\,*}(\phi,0,R+|x|) + \mathbf{exc}^m_*(\phi,x,R).
\end{aligned}
$$

q.e.d.

We are now able to improve on Lemma 4.3.1.

Proposition 4.3.3. *There exists a constant* $0 < \mathbf{c}_{4.3.3}(m) < \infty$ *with the following property. Whenever*

(A) $U \subset \mathbf{R}^n$ *is open,* $0 \in U$*,* ϕ *is a Radon measure on* U *so that* $\Theta^m(\phi,0) = 1$*, and* ξ *is a continuous gauge;*
(B) ϕ *is* (ξ,m) *nearly monotonic in* U *and is* (ξ,m) *epiperimetric in* $(\{0\},U)$*;*
(C) $0 < r \leq 1$*,* $\xi(2\sqrt{r}) \leq 1$*,* $\mathbf{B}(0,2\sqrt{r}) \subset U$*;*

the following holds:

$$|\mathbf{b}(\phi,r)| \leq \mathbf{c}_{4.3.3}(m) r \max\left\{\sqrt[4]{r}, \sqrt{\xi(2\sqrt{r})}\right\}.$$

Proof. We start by choosing $0 < \gamma(m) \leq \frac{1}{8}$ and $\eta(m)$ such that

$$\eta(m) := 4\gamma(m) - \gamma(m)^2\left(2\,\mathbf{c}_{4.2.1}(m) + 8 + 8(m+2)\right) > 0. \tag{43}$$

We define a gauge ε by the formula $\varepsilon(\rho) := \max\{\rho,\xi(\rho)\}$, $\rho > 0$. Now **either** $|\gamma(m)\mathbf{b}(\phi,r)| \leq r\sqrt{\varepsilon(\sqrt{r})}$ **or** $|\gamma(m)\mathbf{b}(\phi,r)| > r\sqrt{\varepsilon(\sqrt{r})}$: we will subsequently derive an estimate for $|\mathbf{b}(\phi,r)|$ in the latter case. We first observe that

$$r\varepsilon(r) \leq r\varepsilon(\sqrt{r}) \leq r\sqrt{\varepsilon(\sqrt{r})} < |\gamma(m)\mathbf{b}(\phi,r)|. \tag{44}$$

Since $|\mathbf{b}(\phi,r)| \le 4r$ (according to Lemma 4.3.1), $|\gamma(m)\mathbf{b}(\phi,r)| \le \frac{r}{2} \le r$ as well, and we see that

$$|\gamma(m)\mathbf{b}(\phi,r)| \le r \le \sqrt{r}\varepsilon(\sqrt{r}). \tag{45}$$

According to (44) and (45), the intermediate value theorem applied to the function

$$[r,\sqrt{r}] \to \mathbf{R} : \rho \mapsto \rho\varepsilon(\rho)$$

ensures that there exists some $R > 0$ with

$$r \le R \le \sqrt{r}$$

and

$$R\varepsilon(R) = |\gamma(m)\mathbf{b}(\phi,r)|\,.$$

Let $x := \gamma(m)\mathbf{b}(\phi,r)$. Since $P_0(\phi,x,r) = V(\phi,0,r) = \hat{V}(\phi,0,r)$ we deduce from Lemma 4.2.1, Lemma 4.2.3, and Lemma 4.3.2 together with (45) that

$$\begin{aligned}
&P_1(\phi,x,r) + P_2(\phi,x,r) + P_3(\phi,x,r) + P_4(\phi,x,r)\\
&= V(\phi,x,r) - P_0(\phi,x,r)\\
&\le \Big|V(\phi,x,r) - \hat{V}(\phi,x,r)\Big| + \hat{V}(\phi,x,r) - \hat{V}(\phi,0,r)\\
&\le \mathbf{c}_{4.2.1}(m)\boldsymbol{\nu}(m)r^m\big(2r|x|^3 + r^2|x|^2\big)\\
&\quad + \mathbf{c}_{4.2.3}(m)\boldsymbol{\nu}(m)r^{m+4}\Big(2\boldsymbol{\alpha}(m)m2^{m-1}\varepsilon(R)\\
&\quad + (\boldsymbol{\alpha}(m)+1)\big(\mathbf{exc}^{m\,*}(\phi,0,2R) + \mathbf{exc}^m_*(\phi,x,R)\big)\Big).
\end{aligned}$$

Dividing by $\boldsymbol{\nu}(m)r^{m+2}$ and recalling the definition of ε, hypothesis (B) and relation (44), we obtain

(46)

$$\begin{aligned}
&\mathbf{P}_1(\phi,x,r) + \mathbf{P}_2(\phi,x,r) + \mathbf{P}_3(\phi,x,r) + \mathbf{P}_4(\phi,x,r)\\
&\le |x|^2\,\mathbf{c}_{4.2.1}(m)\left(\frac{2|x|}{r} + 1\right)\\
&\quad + \mathbf{c}_{4.2.3}(m)3^{-1}\mathbf{c}(m)r^2\Big(\varepsilon(R) + \mathbf{exc}^{m\,*}(\phi,0,2R) + \mathbf{exc}^m_*(\phi,x,R)\Big)\\
&\le |x|^2 2\,\mathbf{c}_{4.2.1}(m) + \mathbf{c}_{4.2.3}(m)\mathbf{c}(m)r^2\varepsilon(2R),
\end{aligned}$$

where

$$\mathbf{c}(m) := 3\max\{\boldsymbol{\alpha}(m)m2^m, \boldsymbol{\alpha}(m)+1\}\,.$$

We further observe that (according to Lemma 4.1.1)

$$\begin{aligned}
\mathbf{P}_2(\phi,x,r) &\ge -2|x|^2\boldsymbol{\nu}(m)^{-1}r^{-m-2}\int_{\mathbf{B}(0,r)}\big(r^2 - |y|^2\big)\,d\phi(y)\\
&\ge -8|x|^2,
\end{aligned}$$

as well as

$$|\mathbf{P}_3(\phi, x, r)| \leq 4|x|^2 \boldsymbol{\nu}(m)^{-1} r^{-m-2} \int_{\mathbf{B}(0,r)} |x|\,|y|\,d\phi(y)$$
$$\leq 8(m+2)|x|^2$$

and

$$\mathbf{P}_4(\phi, x, r) \geq 0,$$

which together with (46) yields
(47)
$\mathbf{P}_1(\phi, x, r) \leq |x|^2 \left(2\,\mathbf{c}_{4.2.1}(m) + 8 + 8(m+2)\right) + \mathbf{c}_{4.2.3}(m)\mathbf{c}(m) r^2 \varepsilon(2R).$

Finally recall that $x = \gamma(m)\mathbf{b}(\phi, r)$ so that

$$\mathbf{P}_1(\phi, x, r) = 4\gamma(m)|\mathbf{b}(\phi, r)|^2,$$
$$|x|^2 = \gamma(m)^2 |\mathbf{b}(\phi, r)|^2.$$

Therefore, by (43), (47) becomes

$$\eta(m)|\mathbf{b}(\phi, r)|^2 \leq \mathbf{c}_{4.2.3}(m)\mathbf{c}(m) r^2 \varepsilon(2R),$$

and in turn:

$$|\mathbf{b}(\phi, r)| \leq \sqrt{\eta(m)^{-1}\mathbf{c}_{4.2.3}(m)\mathbf{c}(m)}\, r \sqrt{\varepsilon(2\sqrt{r})}. \tag{48}$$

We recall that according to the initial dichotomy either (48) holds true or

$$|\mathbf{b}(\phi, r)| \leq \gamma(m)^{-1} r \sqrt{\varepsilon(\sqrt{r})}. \tag{49}$$

This readily proves the proposition. q.e.d.

4.4. Controlling the large eigenvalues of the second moment.

Proposition 4.4.1. *There exists a constant* $0 < \mathbf{c}_{4.4.1}(m) < \infty$ *with the following property. Whenever*

(A) $U \subset \mathbf{R}^n$ *is open,* $0 \in U$*,* ϕ *is a Radon measure in* U *and* ξ *is a continuous gauge;*
(B) $0 < r \leq 1$*,* $\mathbf{B}(0, 2\sqrt{r}) \subset U$*,* $\xi(2\sqrt{r}) \leq 1$*,* $x \in U$ *and*
$$|x| = r \max\left\{ \sqrt[8]{r}, \sqrt[4]{\xi\left(2\sqrt{r}\right)} \right\};$$
(C) ϕ *is* (ξ, m) *nearly monotonic in* U *and* (ξ, m) *epiperimetric in* $(\{0, x\}, U)$*;*
(D) $\Theta^m(\phi, 0) = \Theta^m(\phi, x) = 1$*;*

the following holds true:

$$\left|\mathbf{Q}(\phi, r)(x) - |x|^2\right| \leq \mathbf{c}_{4.4.1}(m)|x|^2 \max\left\{ \sqrt[8]{r}, \sqrt[4]{\xi\left(2\sqrt{r}\right)} \right\}.$$

Proof. We first notice, as in the proof of Proposition 4.3.3, that Lemmas 4.2.1 and 4.2.3 imply the following:

$$\begin{aligned}(50)\quad &\big|P_1(\phi,x,r)+P_2(\phi,x,r)+P_3(\phi,x,r)+P_4(\phi,x,r)\big|\\ &\le \mathbf{c}_{4.2.1}(m)\boldsymbol{\nu}(m)r^m\left(2r|x|^3+r^2|x|^2\mathbf{exc}^{m\,*}(\phi,0,2r)\right)\\ &\quad+\mathbf{c}_{4.2.3}(m)\boldsymbol{\nu}(m)r^{m+4}\big(|\mathbf{dev}^m(\phi,x,0,r)|+\|\mathbf{exc}^m\|(\phi,0,r)\\ &\quad+\|\mathbf{exc}^m\|(\phi,x,r)\big).\end{aligned}$$

Next we estimate $|\mathbf{dev}^m(\phi,x,0,r)|$:

$$\begin{aligned}(51)\quad \big|\boldsymbol{\alpha}(m)^{-1}\mathbf{dev}^m(\phi,x,0,r)\big| &= \left|\frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m}-\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m}\right|\\ &\le \left|\frac{\phi(\mathbf{B}(0,r))}{\boldsymbol{\alpha}(m)r^m}-\Theta^m(\phi,0)\right|\\ &\quad+\left|\Theta^m(\phi,x)-\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m}\right|\\ &\le \|\mathbf{exc}^m\|(\phi,0,r)+\|\mathbf{exc}^m\|(\phi,x,r)\\ &\le 2\,\xi(r).\end{aligned}$$

In order to simplify the writings we introduce the following notation:

$$\eta(r)=\max\left\{\sqrt[8]{r},\,\sqrt[4]{\xi\left(2\sqrt{r}\right)}\right\}.$$

Dividing (50) by $\boldsymbol{\nu}(m)r^{m+2}$ and using (51) and hypothesis (B), we obtain the following:

$$\begin{aligned}(52)\quad &\big|\mathbf{P}_1(\phi,x,r)+\mathbf{P}_2(\phi,x,r)+\mathbf{P}_3(\phi,x,r)+\mathbf{P}_4(\phi,x,r)\big|\\ &\le \mathbf{c}_{4.2.1}(m)|x|^2\left(\frac{|x|}{r}(1+\xi(r))+\xi(2r)\right)\\ &\quad+\mathbf{c}_{4.2.3}(m)r^2\left(|\mathbf{dev}^m(\phi,x,0,r)|+2\,\xi(r)\right)\\ &\le \mathbf{c}_{4.2.1}(m)|x|^2\left(\eta(r)(1+\xi(r))+\xi(2r)\right)\\ &\quad+\mathbf{c}_{4.2.3}(m)|x|^2 4\,\eta(r)^{-2}\xi(r).\end{aligned}$$

According to Proposition 4.3.3 we also also have that

$$\begin{aligned}(53)\quad |\mathbf{P}_1(\phi,x,r)| &= 4\,|\langle x,\mathbf{b}(\phi,r)\rangle|\\ &\le 4\,\mathbf{c}_{4.3.3}(m)|x|r\max\left\{\sqrt[4]{r},\sqrt{\xi\left(2\sqrt{r}\right)}\right\}\\ &= 4\,\mathbf{c}_{4.3.3}(m)|x|^2\eta(r).\end{aligned}$$

Furthermore,

$$
\begin{aligned}
(54) \qquad |\mathbf{P}_3(\phi,x,r)| &\leq 4\,|x|^2\boldsymbol{\nu}(m)^{-1}r^{-m-2}\int_{\mathbf{B}(0,r)}|x|\,|y|\,d\phi(y)\\
&\leq 4\,|x|^2\boldsymbol{\nu}(m)^{-1}r^{-m-2}r\eta(r)r\phi(\mathbf{B}(0,r))\\
&\leq 4(m+2)|x|^2\eta(r)(1+\xi(r)),
\end{aligned}
$$

as well as

$$
\begin{aligned}
(55) \qquad |\mathbf{P}_4(\phi,x,r)| &= |x|^4\boldsymbol{\nu}(m)^{-1}r^{-m-2}\phi(\mathbf{B}(0,r))\\
&\leq |x|^2\eta(r)^2r^2\boldsymbol{\nu}(m)^{-1}r^{-m-2}\phi(\mathbf{B}(0,r))\\
&\leq (m+2)|x|^2\eta(r)^2(1+\xi(r)).
\end{aligned}
$$

Plugging (53), (54), and (55) into (52), and observing that $\eta(r)^{-2}\xi(r) \leq \sqrt{\xi(r)}$, we find that

$$
\begin{aligned}
(56) \qquad |\mathbf{P}_2(\phi,x,r)| \leq\ & \mathbf{c}_{4.2.1}(m)|x|^2\left(\eta(r)(1+\xi(r))+\xi(2r)\right)\\
&+4\,\mathbf{c}_{4.2.3}(m)|x|^2\sqrt{\xi(r)}\\
&+4\,\mathbf{c}_{4.3.3}(m)|x|^2\eta(r)\\
&+5(m+2)|x|^2\eta(r)^2(1+\xi(r))\\
\leq\ & \mathbf{c}(m)|x|^2\eta(r),
\end{aligned}
$$

for some $\mathbf{c}(m)>0$ depending only upon m. Finally, recalling the definition of $P_2(\phi,x,r)$ and referring to Lemma 4.1.1, it is an easy matter to check that

$$
(57) \qquad 4\left|\mathbf{Q}(\phi,r)(x)-|x|^2\right| \leq 4\|\mathbf{exc}\|(\phi,0,r)|x|^2+|\mathbf{P}_2(\phi,x,r)|\,.
$$

Plugging (56) into (57) yields the expected estimate. q.e.d.

4.5. Closeness to flat.

Definition 4.5.1. Let $U\subset\mathbf{R}^n$ be open, let ϕ be a Radon measure in U, $A\subset U$, $R>0$ and let m be a nonnegative integer. We say that ϕ *is* (m,A,R) *uniform* if there exists $C>0$ such that for every $x\in A\cap\operatorname{spt}\phi$ and every $0<r\leq R$ with $\mathbf{B}(x,r)\subset U$ one has $\phi(\mathbf{B}(x,r))=Cr^m$. In case $U=\mathbf{R}^n$ and ϕ is (m,A,R) uniform for every $A\subset\mathbf{R}^n$ and every $R>0$ we simply say that ϕ is m *uniform*.

The following are two easy lemmas.

Lemma 4.5.2. *Assume that:*

(A) *ϕ is an m uniform measure in $\mathbf{R}^n$;*
(B) *$0<q<m$ is an integer, $W\in\mathbf{G}(n,q)$, and $\operatorname{spt}\phi\subset W$.*

Then $\phi=0$.

Proof. Let $U \subset \mathbf{R}^n$ be open and bounded, and $\varepsilon > 0$. Referring to Besicovitch's covering theorem [**10**, 2.8.15], find a disjointed family of closed balls $\mathbf{B}(x_j, r_j)$ contained in U, such that $x_j \in \operatorname{spt}\phi$ and $r_j \leq \varepsilon$, $j = 1, 2, \ldots$, as well as

$$\phi\left(U \cap \operatorname{spt}\phi \sim \cup_{j=1}^{\infty} \mathbf{B}(x_j, r_j)\right) = 0.$$

Notice that

$$\phi(U \cap \operatorname{spt}\phi) = \sum_{j=1}^{\infty} C r_j^m \leq C \boldsymbol{\alpha}(q)^{-1} \varepsilon^{m-q} \mathcal{H}^q(W \cap U).$$

The conclusion follows from the arbitrariness of $\varepsilon > 0$ and U. q.e.d.

Lemma 4.5.3. *Assume that:*

(A) $U \subset \mathbf{R}^n$ *is open,* ϕ *is a nonzero Radon measure in* U*;*
(B) $A \subset U$ *is open,* $R > 0$ *and* ϕ *is* (m, A, R) *uniform;*
(C) $W \in \mathbf{G}(n, m)$ *and* $A \cap \operatorname{spt}\phi \subset W$*.*

Then there exists $C > 0$ *such that* $\phi \llcorner A = C\,\mathcal{H}^m \llcorner W \cap A$*.*

Proof. In order to keep the notations short we put $\psi = \mathcal{H}^m \llcorner W \cap A$. We first observe that Besicovitch's covering theorem [**10**, 2.8.14] implies that $\phi \ll \psi \llcorner \operatorname{spt}\phi$. Indeed let $C \subset W \cap A \cap \operatorname{spt}\phi$ be a compact such that $\psi(C) = 0$, let $\varepsilon > 0$, and select a bounded open set $V \subset U$ containing C such that $\psi(V) < \varepsilon$. Find disjointed families of balls, $\mathcal{B}_1, \ldots, \mathcal{B}_{\Gamma(n)}$, such that $C \subset \cup_{i=1}^{\Gamma(n)} \cup \mathcal{B}_i$, $\cup\mathcal{B}_i \subset V$, $i = 1, \ldots, \Gamma(n)$, and each $B \in \mathcal{B}_i$ is centered in C. Then

$$\phi(C) \leq \sum_{i=1}^{\Gamma(n)} \sum_{B \in \mathcal{B}_i} \phi(B) = \sum_{i=1}^{\Gamma(n)} C\ \boldsymbol{\alpha}(m)^{-1} \sum_{B \in \mathcal{B}_i} \psi(B) \leq \Gamma(n) C\ \boldsymbol{\alpha}(m)^{-1} \varepsilon.$$

The absolute continuity follows from the arbitrariness of $\varepsilon > 0$.

Next we will show that $W \cap A \cap \operatorname{spt}\phi = W \cap A$. Suppose instead that $V = W \cap A \sim \operatorname{spt}\phi \neq \emptyset$ and pick $x \in \operatorname{Bdry} V$ (relative to $W \cap A$). Choose $0 < r \leq R$ such that $\mathbf{B}(x, r) \subset A$ and $\phi(\mathbf{B}(x, r)) = \phi(\mathbf{U}(x, r))$. Again referring to Besicovitch's covering theorem (see e.g., [**15**, Theorem 2.8]), find a finite or countable disjointed family of closed balls $\mathbf{B}(x_j, r_j)$, $j = 1, 2, \ldots$, contained in $\mathbf{U}(x, r)$ and centered in $\operatorname{spt}\phi$, such that

$$(\psi \llcorner \operatorname{spt}\phi)\left(\mathbf{U}(x, r) \sim \cup_{j=1}^{\infty} \mathbf{B}(x_j, r_j)\right) = 0.$$

This implies that

$$\begin{aligned}\psi(\operatorname{spt}\phi\cap\mathbf{U}(x,r)) &= \sum_{j=1}^{\infty}\psi(\mathbf{B}(x_j,r_j))\\ &= \sum_{j=1}^{\infty}C^{-1}\boldsymbol{\alpha}(m)\phi(\mathbf{B}(x_j,r_j))\\ &= C^{-1}\boldsymbol{\alpha}(m)\phi(\mathbf{B}(x,r))\\ &= \psi(\mathbf{U}(x,r)).\end{aligned}$$

The last equality follows from the fact that $x\in A\cap\operatorname{spt}\phi$ and the next to last from that $\phi \ll \psi \llcorner \operatorname{spt}\phi$. The above relation yields in turn $\psi(V\cap\mathbf{U}(x,r))=0$. Since $V\cap\mathbf{U}(x,r)$ is open relative to W, and contained in A, this clearly implies that $V\cap\mathbf{U}(x,r)=\emptyset$, contradicting the fact that $x\in\operatorname{Bdry}V$.

We now know that $W\cap A\cap\operatorname{spt}\phi = W\cap A$ and that $\phi\ll\psi$. The conclusion becomes an easy consequence of the differentiation theory of Radon measures, [**10**, 2.9]. q.e.d.

The following is due to D. Preiss; see the argument starting near the second third of page 541 in [**17**].

Theorem 4.5.4. *Let ϕ be an m uniform and m monotonic Radon measure in $\mathbf{R}^n$ such that $0\in\operatorname{spt}\phi$. Then there exists $W\in\mathbf{G}(n,m)$ such that*

$$\phi=\boldsymbol{\alpha}(m)^{-1}\phi(\mathbf{B}(0,1))\mathcal{H}^m\llcorner W.$$

Proof. We observe that for each $x\in\mathbf{R}^n$ and $r>0$ such that $2|x|<r$ one has

$$\begin{aligned}|\mathbf{P}_3(\phi,x,r)| &\le 4\,|x|^2\boldsymbol{\nu}(m)^{-1}r^{-m-2}\int_{\mathbf{B}(0,r)}|x||y|d\phi(y)\\ &\le 4\,|x|^3r^{-1}\boldsymbol{\nu}(m)^{-1}\phi(\mathbf{B}(0,1))\end{aligned}\tag{58}$$

as well as

$$\begin{aligned}0\le\mathbf{P}_4(\phi,x,r) &\le |x|^4\boldsymbol{\nu}(m)^{-1}r^{-m-2}\phi(\mathbf{B}(0,r))\\ &\le |x|^3r^{-1}\boldsymbol{\nu}(m)^{-1}\phi(\mathbf{B}(0,1)).\end{aligned}\tag{59}$$

We observe next that for each $x\in\mathbf{R}^n$ and $r>0$,

$$\hat{V}(\phi,x,r)-\hat{V}(\phi,0,r)\le 0,$$

with equality if $x \in \operatorname{spt}\phi$. This follows from writing $\hat{V}(\phi, x, r)$ in terms of $\phi(\mathbf{B}(x,\rho))$ (as in the proof of Lemma 4.2.3) and the following computation:

$$\begin{aligned}\frac{\phi(\mathbf{B}(x,\rho))}{\rho^m} &\le \frac{\phi(\mathbf{B}(0,\rho+|x|))}{\rho^m}\\ &\le \limsup_{r\to\infty}\frac{\phi(\mathbf{B}(0,r+|x|))}{r^m}\\ &= \limsup_{r\to\infty}\left(1+\frac{|x|}{r}\right)^m\frac{\phi(\mathbf{B}(0,r+|x|))}{(r+|x|)^m}\\ &= \phi(\mathbf{B}(0,1)) = \frac{\phi(\mathbf{B}(0,\rho))}{\rho^m}.\end{aligned}$$

Therefore, referring to Lemma 4.2.1 we see that for every $x \in \mathbf{R}^n$ and $r > 0$ such that $2|x| < r$ one has

$$\begin{aligned}(60)\quad P_1(\phi,x,r) &+ P_2(\phi,x,r) + P_3(\phi,x,r) + P_4(\phi,x,r)\\ &= V(\phi,x,r) - P_0(\phi,x,r)\\ &\le \left|V(\phi,x,r) - \hat{V}(\phi,x,r)\right| + \hat{V}(\phi,x,r) - \hat{V}(\phi,0,r)\\ &\le \mathbf{c}_{4.2.1}(m)\boldsymbol{\nu}(m)r^{m+1}|x|^3\phi(\mathbf{B}(0,1)).\end{aligned}$$

If moreover $x \in \operatorname{spt}\phi$ then

$$\begin{aligned}(61)\quad &\left|P_1(\phi,x,r) + P_2(\phi,x,r) + P_3(\phi,x,r) + P_4(\phi,x,r)\right|\\ &\le \mathbf{c}_{4.2.1}(m)\boldsymbol{\nu}(m)r^{m+1}|x|^3\phi(\mathbf{B}(0,1)).\end{aligned}$$

Dividing (60) and (61) by $\boldsymbol{\nu}(m)r^{m+2}$ and plugging (58) and (59) into the resulting inequalities yields

$$(62)\quad \langle x, \mathbf{b}(\phi,r)\rangle + \mathbf{Q}(\phi,r)(x) - |x|^2 \le \mathbf{c}(\phi,m)\frac{|x|^3}{r} \text{ whenever } x \in \mathbf{R}^n,$$

and
(63)

$$\left|\langle x, \mathbf{b}(\phi,r)\rangle + \mathbf{Q}(\phi,r)(x) - |x|^2\right| \le \mathbf{c}(\phi,m)\frac{|x|^3}{r} \text{ whenever } x \in \operatorname{spt}\phi,$$

provided that $r > 2|x|$, where $\mathbf{c}(\phi,m) = \phi(\mathbf{B}(0,1))(\mathbf{c}_{4.2.1}(m) + 5\boldsymbol{\nu}(m)^{-1})$. According to Lemma 4.1.2 we also have that

$$0 \le \mathbf{Q}(\phi,r)(x) \le |x|^2 \text{trace } \mathbf{Q}(\phi,r) = m|x|^2,\ x \in \mathbf{R}^n.$$

Therefore there exists a sequence $r_j \to \infty$ as $j \to \infty$ and a quadratic polynomial $\mathbf{Q}(\phi)$ on $\mathbf{R}^n$ such that $\mathbf{Q}(\phi, r_j)(x) \to \mathbf{Q}(\phi)(x)$ as $j \to \infty$, $x \in \mathbf{R}^n$. Clearly $\mathbf{Q}(\phi) \ge 0$ and trace $\mathbf{Q}(\phi) \le m$. It now follows from (63) that

$$\left|\langle x, \mathbf{b}(\phi,r_j)\rangle + \mathbf{Q}(\phi)(x) - |x|^2\right| \to 0 \text{ as } j \to \infty$$

whenever $x \in \operatorname{spt}\phi$. Since clearly $\mathbf{b}(\phi, r_j) \in \operatorname{span}\operatorname{spt}\phi$, $j = 1, 2, \ldots$, we infer that there exists $\mathbf{b}(\phi) \in \operatorname{spt}\phi$ such that $\mathbf{b}(\phi, r_j) \to \mathbf{b}(\phi)$ as $j \to \infty$. Next we infer from (62) that

$$\langle x, \mathbf{b}(\phi)\rangle \leq |x|^2 - \mathbf{Q}(\phi)(x) \leq |x|^2,\ x \in \mathbf{R}^n.$$

Applying this inequality with $x = \frac{1}{2}\mathbf{b}(\phi)$ we obtain $\mathbf{b}(\phi) = 0$. Inequalities (62) and (63) now read as follows:

$$\mathbf{Q}(\phi)(x) \leq |x|^2,\ x \in \mathbf{R}^n \tag{64}$$

and

$$\mathbf{Q}(\phi)(x) = |x|^2,\ x \in \operatorname{spt}\phi. \tag{65}$$

Let $e_1, \ldots, e_n$ be an orthonormal family of eigenvectors of $\mathbf{Q}(\phi)$ and let $\lambda_1, \ldots, \lambda_n$ be their corresponding eigenvalues. Recall that $0 \leq \lambda_i \leq 1$, $i = 1, \ldots, n$, and define $W = \operatorname{span}\{e_i : \lambda_i = 1\}$. Then clearly $\mathbf{Q}(\phi)(x) < |x|^2$ whenever $x \in \mathbf{R}^n \sim W$, whence (65) implies that $\operatorname{spt}\phi \subset W$. Now either $\phi = 0$ and the theorem is obviously verified, or else $\phi \neq 0$ and then Lemma 4.5.2 implies that $\dim W \geq m$. The equation trace $\mathbf{Q}(\phi) = m$ yields in turn $\dim W = m$ and the proof is completed upon reference to Lemma 4.5.3. q.e.d.

The following two lemmas are taken from [**8**].

Lemma 4.5.5. *For every $\varepsilon > 0$ there exists $0 < \boldsymbol{\delta}_{4.5.5}(n, m, \varepsilon) < \infty$ such that whenever*

- (A) *$U \subset \mathbf{R}^n$ is open, ϕ is a Radon measure in U, $0 \in \operatorname{spt}\phi$, $r > 0$, $\mathbf{B}(0, r) \subset U$, ξ is a gauge;*
- (B) *ϕ is m concentrated and (ξ, m) nearly monotonic in U;*
- (C) *$\phi(\mathbf{B}(0, r)) \leq (1 + \boldsymbol{\delta}_{4.5.5}(n, m, \varepsilon))\boldsymbol{\alpha}(m) r^m$;*
- (D) *$\xi(r) \leq \boldsymbol{\delta}_{4.5.5}(n, m, \varepsilon)$;*

there exists $W \in \mathbf{G}(n, m)$ with

$$d_{\mathcal{H}}\left(\operatorname{spt}\phi \cap \mathbf{B}(0, \boldsymbol{\delta}_{4.5.5}(n, m, \varepsilon) r), W \cap \mathbf{B}(0, \boldsymbol{\delta}_{4.5.5}(n, m, \varepsilon) r)\right) \leq \varepsilon \boldsymbol{\delta}_{4.5.5}(n, m, \varepsilon) r.$$

Proof. Assume if possible that there exists $\varepsilon > 0$ and for every $j = 1, 2, \ldots$ an open set $U_j \subset \mathbf{R}^n$, a Radon measure ϕ_j in U_j such that $0 \in \operatorname{spt}(\phi_j)$, a gauge ξ_j and $r_j > 0$ with the following properties: ϕ_j is m concentrated and (ξ_j, m) nearly monotonic in U_j, $\phi_j(\mathbf{B}(0, r_j)) \leq (1 + j^{-1})\boldsymbol{\alpha}(m) r_j^m$, $\xi_j(r_j) \leq j^{-1}$, yet

$$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}(\phi_j) \cap \mathbf{B}(0, j^{-1} r_j), W \cap \mathbf{B}(0, j^{-1} r_j)) \geq \varepsilon j^{-1} r_j$$

for each $W \in \mathbf{G}(n, m)$.

We define $\psi_j = j^m r_j^{-m} \boldsymbol{\mu}_{jr_j^{-1}\,\#}\phi_j$ † as well as $\zeta_j(r) = \xi_j(rj^{-1}r_j)$, $j = 1, 2, \ldots$. We check that: ψ_j is an m concentrated (ζ_j, m) nearly monotonic measure in $\mathbf{U}(0, j)$, $0 \in \operatorname{spt}(\psi_j)$, $\psi_j(\mathbf{U}(0, j)) \leq (1 + j^{-1})\boldsymbol{\alpha}(m)j^m$ and $\zeta_j(j) \leq j^{-1}$, $j = 1, 2, \ldots$, yet

$$\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}(\psi_j) \cap \mathbf{B}(0, 1), W \cap \mathbf{B}(0, 1)) \geq \varepsilon \tag{66}$$

whenever $W \in \mathbf{G}(n, m)$. We notice that if $j \geq k$ then

$$\frac{\psi_j(\mathbf{U}(0, k))}{\boldsymbol{\alpha}(m)k^m} \leq \frac{\psi_j(\mathbf{U}(0, j))}{\boldsymbol{\alpha}(m)j^m} + \mathbf{exc}^m_*(\psi_j, 0, j)$$

so that

$$\psi_j(\mathbf{U}(0, k)) \leq (1 + 2j^{-1})\boldsymbol{\alpha}(m)k^m\,. \tag{67}$$

Set $k_0(j) = j$, $j = 1, 2, \ldots$. Referring to (67) and de la Vallée Poussin's compactness theorem we define inductively for $l \geq 1$ a subsequence $k_l(1), k_l(2), \ldots$ of $k_{l-1}(1), k_{l-1}(2), \ldots$ and a Radon measure ψ^l in $\mathbf{U}(0, l)$ such that $\psi_{k_l(j)} \rightharpoonup \psi^l$ as $j \to \infty$. Observe that $\psi^{l_1} = \psi^{l_2}$ on $\mathbf{U}(0, l_1)$ whenever $l_1 \leq l_2$ so that, according to Riesz's representation theorem, there exists a Radon measure ψ in $\mathbf{R}^n$ with $\psi = \psi^l$ on $\mathbf{U}(0, l)$ for each $l = 1, 2, \ldots$. Since $\zeta_j(r) \to 0$ as $j \to \infty$ for every $r > 0$ we infer from Lemma 3.3.4 that ψ is m concentrated and m monotonic in $\mathbf{R}^n$, and that $0 \in \operatorname{spt}(\psi)$. Notice that (67) also implies that $\psi(\mathbf{U}(0, k)) = \boldsymbol{\alpha}(m)k^m$ for every $k = 1, 2, \ldots$. This shows that ψ is m uniform: if $x \in \operatorname{spt}(\psi)$ and $r > 0$ then

$$\begin{aligned} 1 \leq \frac{\psi(\mathbf{B}(x, r))}{\boldsymbol{\alpha}(m)r^m} &\leq \lim_{R \to \infty} \frac{\psi(\mathbf{B}(x, R))}{\boldsymbol{\alpha}(m)R^m} \\ &\leq \lim_{R \to \infty} \frac{\psi(\mathbf{B}(0, |x| + R))}{\boldsymbol{\alpha}(m)(|x| + R)^m} \left(1 + \frac{|x|}{R}\right)^m \\ &= 1. \end{aligned}$$

Theorem 4.5.4 now implies that $\psi = \mathcal{H}^m \llcorner W$ for some $W \in \mathbf{G}(n, m)$. Finally we see that (67) would be in contradiction with Corollary 3.3.5. q.e.d.

Lemma 4.5.6. *For every $\varepsilon > 0$ there exists $0 < \boldsymbol{\delta}_{4.5.6}(n, m, \varepsilon) < \infty$ such that if*

(A) *$U \subset \mathbf{R}^n$ is open, ϕ is a Radon measure in U, $0 \in \operatorname{spt}\phi$, $R > 0$, $\mathbf{B}(0, R) \subset U$, ξ is a gauge;*
(B) *ϕ is m concentrated and (ξ, m) nearly monotonic in U;*
(C) *$\phi(\mathbf{B}(0, R)) \leq (1 + \boldsymbol{\delta}_{4.5.6}(n, m, \varepsilon))\boldsymbol{\alpha}(m)R^m$;*
(D) *$\xi(R) \leq \boldsymbol{\delta}_{4.5.6}(n, m, \varepsilon)$;*

†Here and in the remaining part of this paper $\boldsymbol{\mu}_\varepsilon(x) = \varepsilon x$.

then for every $x \in \operatorname{spt}\phi \cap \mathbf{B}(0, \boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon)R)$ *and every* $0 < r \leq R/2$ *there exists* $W \in \mathbf{G}(n,m)$ *with*

$$d_{\mathcal{H}}\left(\operatorname{spt}\phi \cap \mathbf{B}(x, \boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon)r), (x+W) \cap \mathbf{B}(x, \boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon)r)\right) \leq \varepsilon \boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon)r.$$

Proof. We first choose $\eta(n,m,\varepsilon) > 0$ sufficiently small for

$$(1+\eta(n,m,\varepsilon))^m (1+\boldsymbol{\delta}_{4.5.5}(n,m,\varepsilon)/2) \leq 1 + \boldsymbol{\delta}_{4.5.5}(n,m,\varepsilon).$$

We claim that the lemma holds with

$$\boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon) = \min\left\{\frac{1}{4}\boldsymbol{\delta}_{4.5.5}(n,m,\varepsilon), \eta(n,m,\varepsilon), \frac{1}{2}\right\}.$$

Indeed let $x \in \mathbf{B}(0, \boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon)R) \cap \operatorname{spt}(\phi)$ and $0 < r \leq R/2$. Then $|x| + r \leq R$ and

$$\begin{aligned}
\frac{\phi(\mathbf{B}(x,r))}{\boldsymbol{\alpha}(m)r^m} &\leq \frac{\phi(\mathbf{B}(0,|x|+r))}{\boldsymbol{\alpha}(m)(|x|+r)^m}\left(1 + \frac{|x|}{r}\right)^m \\
&\leq \left(\frac{\phi(\mathbf{B}(0,R))}{\boldsymbol{\alpha}(m)R^m} + \mathbf{exc}_*^m(\phi,0,R)\right)\left(1+\frac{|x|}{r}\right)^m \\
&\leq (1 + 2\boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon))(1+\boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon))^m \\
&\leq 1 + \boldsymbol{\delta}_{4.5.5}(n,m,\varepsilon),
\end{aligned}$$

so that Lemma 4.5.5 applies to the measure $\boldsymbol{\tau}_{-x\,\#}\phi$ in the open set $\boldsymbol{\tau}_{-x}(U)$, the scale $r > 0$ and the gauge ξ. q.e.d.

4.6. Finding orthogonal families in Reifenberg flat sets.

Lemma 4.6.1. *Let* $g : \mathbf{B}(0,1) \to \mathbf{B}(0,1)$ *and* $0 < \rho < 1$. *Assume that*

(A) g *is continuous;*
(B) $|g(x) - x| < 1 - \rho$ *whenever* $x \in \operatorname{Bdry}\mathbf{B}(0,1)$.

Then $\mathbf{B}(0,\rho) \subset \operatorname{im} g$.

Proof. We let $i : \operatorname{Bdry}\mathbf{B}(0,1) \to \mathbf{B}(0,1)$ be the canonical injection and $P : \mathbf{B}(0,1) \sim \{0\} \to \operatorname{Bdry}\mathbf{B}(0,1)$ be defined by $P(x) = x|x|^{-1}$. We first claim that $P \circ g \circ i$ is homotopic to the identity of $\operatorname{Bdry}\mathbf{B}(0,1)$. It follows indeed from hypothesis (B) that $tx + (1-t)g(x) \in \mathbf{B}(0,1) \sim \{0\}$ whenever $0 \leq t \leq 1$ and $x \in \operatorname{Bdry}\mathbf{B}(0,1)$, whence $H(t,x) = P(tx + (1-t)g(x))$ is a homotopy witnessing our claim. Therefore the induced homomorphism in $m-1$ dimensional homology (m is the dimension of the ball $\mathbf{B}(0,1)$), $H_{m-1}(P \circ g \circ i)$, is the identity of $\mathbf{Z}$.

On the other hand $|g(x) - x| \leq 1 - \rho - \varepsilon$, $x \in \operatorname{Bdry}\mathbf{B}(0,1)$, for some $0 < \varepsilon < 1 - \rho$. If the conclusion were not true there would exist $x_0 \in \mathbf{B}(0,1)$ such that $x_0 \notin \operatorname{im} g$. On letting $r : \mathbf{B}(0,1) \sim \{x_0\} \to \operatorname{Bdry}\mathbf{B}(0,1)$ be a retraction with the property that $r(x) = P(x)$ if $\rho + \varepsilon \leq |x| \leq 1$, one would infer that $r \circ g \circ i = P \circ g \circ i$ and, in turn,

$$\mathbf{id_Z} = H_{m-1}(P \circ g \circ i) = H_{m-1}(r) \circ H_{m-1}(g) \circ H_{m-1}(i)$$

in contradiction with the equation $H_{m-1}(g) = 0$. q.e.d.

Proposition 4.6.2. *Let $n > 0$ be an integer. There exists a constant $0 < \boldsymbol{\varepsilon}_{4.6.2}(n) < \infty$ with the following property. Assume that*

(A) *$0 < m < n$ is an integer, $S \subset \mathbf{R}^n$ is closed, $x_0 \in S$ and $r_0 > 0$;*
(C) *for every $x \in S \cap \mathbf{B}(x_0, 2r_0)$ and every $0 < r \leq 2r_0$, S is $(\boldsymbol{\varepsilon}_{4.6.2}(n), m)$ flat at (x, r);*
(D) *$0 < \rho \leq \frac{r}{2}$.*

Then there exists an orthonormal family $e_1, \ldots, e_m$ of $\mathbf{R}^n$ such that $x_0 + \rho e_i \in S$, $i = 1, \ldots, m$.

Proof. We set

$$\boldsymbol{\varepsilon}_{4.6.2}(n) = \min\left\{\frac{1}{8.3^{n-1}}, \boldsymbol{\varepsilon}_{2.5.10}(n), \frac{1}{100\mathbf{c}_{2.5.10}(n)}\right\}.$$

In order to keep the notation short we abbreviate

$$\eta = \mathbf{c}_{2.5.10}(n)\boldsymbol{\varepsilon}_{4.6.2}(n).$$

It readily suffices to prove the proposition under the additional assumption $x_0 = 0$, $r_0 = 1$. We start with the case $\rho = 1/2$. Let $W_0 \in \mathbf{G}(S, 0, 1, \boldsymbol{\varepsilon}_{4.6.2}(n))$ and let τ be associated with S as in Theorem 2.5.10. First we claim that

$$S \cap \operatorname{Bdry} \mathbf{B}(0, 1/2) \neq \emptyset.$$

We notice indeed that $Z = \mathbf{R} \cap \{|x| : x \in \operatorname{im} \tau\}$ is connected, and that $Z \cap [0, \eta] \neq \emptyset$ as well as $Z \cap [1 - \eta, 1 + \eta] \neq \emptyset$. Since $\eta \leq 1/4$ and since $\operatorname{im} \tau \subset S$ it becomes obvious that $S \cap \operatorname{Bdry} \mathbf{B}(0, 1/2) \neq \emptyset$.

Pick $x_1 \in S \cap \operatorname{Bdry} \mathbf{B}(0, 1/2)$, so that if $m = 1$ the proof of the case $\rho = 1/2$ is completed; we will subsequently assume that $m \geq 2$. Put $x_1^* = P_{W_0}(x_1)$, and observe that $x_1^* \neq 0$. Let $e_2^*, \ldots, e_m^* \in \mathbf{R}^n$ be such that $x_1^*, e_2^*, \ldots, e_m^*$ is an orthogonal family spanning W_0, and define $W_1 = \operatorname{span}\{x_1, e_2^*, \ldots, e_m^*\}$. It is easily checked that

$$d_{\mathcal{H}}(W_0 \cap \mathbf{B}(0, 1), W_1 \cap \mathbf{B}(0, 1)) \leq 2\boldsymbol{\varepsilon}_{4.6.2}(n).$$

We will define inductively a family of pairs $(x_1, W_1), \ldots, (x_m, W_m)$ verifying the following conditions. For each $j = 1, \ldots, m$:

(1) $x_j \in S \cap \operatorname{Bdry} \mathbf{B}(0, 1/2)$;
(2) $\langle x_j, x_i \rangle = 0$ for every $i = 0, \ldots, j - 1$;
(3) $x_1, \ldots, x_j \in W_j \in \mathbf{G}(n, m)$;
(4) $d_{\mathcal{H}}(W_{j-1} \cap \mathbf{B}(0, 1), W_j \cap \mathbf{B}(0, 1)) \leq 2.3^{j-1}\boldsymbol{\varepsilon}_{4.6.2}(n)$.

The pair (x_1, W_1) defined above clearly verifies these conditions for $j = 1$. Assume that $(x_1, W_1), \dots, (x_j, W_j)$ have been defined for some $j = 1, \dots, m-1$. We set $E_j = \operatorname{span}\{x_1, \dots, x_j\}$ and we aim to show that

$$S \cap E_j^{\perp} \cap \operatorname{Bdry} \mathbf{B}(0, 1/2) \neq \emptyset. \tag{68}$$

For this purpose we define $f_j : \mathbf{B}(0,1) \cap \mathbf{B}(W_j, 1/4) \to \mathbf{B}(0,1) \cap W_j$ by the following formula:

$$f_j(x) = \begin{cases} \frac{P_{W_j}(x)}{|P_{W_j}(x)|} \sqrt{\left|P_{W_j}(x)\right|^2 + \left|P_{W_j^{\perp}}(x)\right|^2 \chi\left(\left|P_{W_j}(x)\right|\right)} & \text{if } P_{W_j}(x) \neq 0 \\ 0 & \text{if } P_{W_j}(x) = 0 \end{cases}$$

where $\chi : [0,1] \to [0,1]$ is given by $\chi(t) = 4t/\sqrt{3}$ if $0 \leq t \leq \sqrt{3}/4$ and $\chi(t) = 1$ if $\sqrt{3}/4 \leq t \leq 1$. It is not too hard to check that f_j is continuous and that

$$f_j^{-1}\left(W_j \cap E_j^{\perp} \cap \operatorname{Bdry} \mathbf{B}(0,1/2)\right) = \mathbf{B}(W_j, 1/4) \cap E_j^{\perp} \cap \operatorname{Bdry} \mathbf{B}(0,1/2). \tag{69}$$

It follows from the choice of W_0 and condition (D) above that

$$\begin{aligned} & d_{\mathcal{H}}(S \cap \mathbf{B}(0,1), W_j \cap \mathbf{B}(0,1)) \\ & \leq d_{\mathcal{H}}(S \cap \mathbf{B}(0,1), W_0 \cap \mathbf{B}(0,1)) \\ & \quad + \sum_{k=0}^{j-1} d_{\mathcal{H}}(W_k \cap \mathbf{B}(0,1), W_{k+1} \cap \mathbf{B}(0,1)) \\ & \leq \left(1 + \sum_{k=0}^{j-1} 2.3^k\right) \boldsymbol{\varepsilon}_{4.6.2}(n) \\ & = 3^j \boldsymbol{\varepsilon}_{4.6.2}(n). \end{aligned} \tag{70}$$

Therefore our choice of $\boldsymbol{\varepsilon}_{4.6.2}(n)$ readily implies that $S \cap \mathbf{B}(0,1) \subset \mathbf{B}(W_j, 1/4)$. This means that a map $g_j : W_j \cap \mathbf{B}(0,1) \to W_j \cap \mathbf{B}(0,1)$ is well-defined by the relation $g_j = f_j \circ \tau \circ h \circ P_{W_0}$ where $h : W_0 \cap \mathbf{B}(0,1) \to W_0 \cap \mathbf{B}(0, 1-\eta)$ is given by $h(x) = (1-\eta)x$, $x \in W_0 \cap \mathbf{B}(0,1)$. It is obvious that g_j is continuous. If (68) were not valid then there would exist $z_j \in W_j \cap \operatorname{Bdry} \mathbf{B}(0,1/2)$ such that $z_j \notin \operatorname{im} g_j$. This, however, would be in contradiction with Lemma 4.6.1, provided we show that

$$|g_j(\zeta) - \zeta| < 1/2 \tag{71}$$

whenever $\zeta \in W_j \cap \operatorname{Bdry} \mathbf{B}(0,1)$. We now turn to establishing this inequality for such ζ. We first notice that

$$|\zeta - P_{W_0}(\zeta)| \leq \|P_{W_0} - P_{W_j}\| \leq 1/8$$

(recall (70) and the definition of $\boldsymbol{\varepsilon}_{4.6.2}(m, \kappa)$), that

$$|P_{W_0}(\zeta) - h(P_{W_0}(\zeta))| \leq \eta$$

and that

$$|h(P_{W_0}(\zeta)) - \tau(h(P_{W_0}(\zeta)))| \leq \eta\,.$$

Therefore

$$|\zeta - \tau(h(P_{W_0}(\zeta)))| \leq 1/8 + 2\eta \tag{72}$$

and in turn $|\tau(h(P_{W_0}(\zeta)))| \geq \sqrt{3}/4$. Notice that

$$\sup\{|f_j(z) - z| : z \in W_j \cap \mathbf{B}(W_j, 1/4) \cap \mathbf{B}(0,1) \sim \mathbf{B}(0, \sqrt{3}/4)\} \leq \frac{1}{\sqrt{8}}.$$

Therefore

$$|\tau(h(P_{W_0}(\zeta))) - f_j(\tau(h(P_{W_0}(\zeta))))| \leq \frac{1}{\sqrt{8}}$$

and (72) yields (recall that $\eta \leq 1/100$)

$$|\zeta - g_j(\zeta)| \leq \frac{1}{8} + \frac{1}{50} + \frac{1}{\sqrt{8}} < \frac{1}{2}$$

so that (71) is proved and (68) follows at once from Lemma 4.6.1 as explained above.

Now pick $x_{j+1} \in S \cap E_j^{\perp} \cap \operatorname{Bdry} \mathbf{B}(0, 1/2)$, so that conditions (A) and (B) are verified. In order to define W_{j+1} we put $x^*_{j+1} = P_{W_j}(x_{j+1})$ and we choose $e^*_{j+2}, \ldots, e^*_m$ so that $x_1, \ldots, x_j, x^*_{j+1}, e^*_{j+2}, \ldots, e^*_m$ is an orthogonal family spanning W_j, and we define

$$W_{j+1} = \operatorname{span}\{x_1, \ldots, x_j, x_{j+1}, e^*_{j+2}, \ldots, e^*_m\}.$$

Condition (C) is now trivially verified, whereas (D) is easy to check with help of (70). The validity of (A) and (B) when $j = m$ completes the proof in case $\rho = 1/2$.

If $\rho < 1/2$ we check that the previous case applies to the set $(2\rho)^{-1}S$.

q.e.d.

4.7. Controlling the mean squared distance to flat.

Definition 4.7.1. Let $U \subset \mathbf{R}^n$ be open, let $x \in U$ and $r > 0$ be such that $\mathbf{B}(x,r) \subset U$, let ϕ be a Radon measure in U, let $Z \subset \mathbf{R}^n$ be closed and $1 \leq q < \infty$. We define

$$\boldsymbol{\beta}_q(\phi, x, r, Z) = \sqrt[q]{r^{-m-q} \int_{\mathbf{B}(x,r)} \operatorname{dist}^q(y - x, Z)\, d\phi(y)}$$

as well as

$$\boldsymbol{\beta}_\infty(\phi, x, r, Z) = r^{-1} \sup\left\{\operatorname{dist}(y - x, Z) : y \in \operatorname{spt}(\phi) \cap \mathbf{B}(x,r)\right\}.$$

Lemma 4.7.2. *There exists a constant* $0 < \mathbf{c}_{4.7.2}(m) < \infty$ *with the following property. Assume that*

(A) $U \subset \mathbf{R}^n$ *is open,* $x \in U$, $r > 0$, $\mathbf{B}(x,r) \subset U$, ϕ *is a Radon measure in* U, ξ *is a gauge,* $Z \subset \mathbf{R}^n$ *is closed;*

(B) ϕ *is* m *concentrated and* (ξ, m) *nearly monotonic in* U*;*

(C) $\xi(r) \leq \frac{1}{2}$.

Then

$$\boldsymbol{\beta}_\infty\left(\phi, x, \frac{r}{2}, Z\right) \leq \mathbf{c}_{4.7.2}(m)\boldsymbol{\beta}_q(\phi, x, r, Z)^{\frac{q}{m+q}}.$$

Proof. In order to keep the notations short we set $p_{q,m} = m(m+q)^{-1}$. Notice that $1 - p_{q,m} = q(m+q)^{-1}$. We define

$$\mathbf{c}_1(m) = 1 + \sqrt[m]{4\,\boldsymbol{\alpha}(m)^{-1}}\,,$$

and

$$\boldsymbol{\delta}_{q,m} = \sqrt[1-p_{q,m}]{\frac{1}{2}(\mathbf{c}_1(m) - 1)^{-1}}$$

as well as

$$\mathbf{c}_{4.7.2}(m) = \max\left\{\mathbf{c}_1(m), \boldsymbol{\delta}_{q,m}^{-\frac{q}{m+q}}\right\} = \max\left\{\mathbf{c}_1(m), 2(\mathbf{c}_1(m) - 1)\right\}\,.$$

We abbreviate

$$\delta = \boldsymbol{\beta}_q(\phi, x, r, Z).$$

Avoiding a triviality, we may assume that $\delta > 0$. If $\delta > \boldsymbol{\delta}_{q,m}$ then

$$\boldsymbol{\beta}_\infty(\phi, x, r, Z) \leq 1 \leq \left(\boldsymbol{\delta}_{q,m}^{-1}\delta\right)^{\frac{q}{m+q}} \leq \mathbf{c}_{4.7.2}(m)\delta^{\frac{q}{m+q}};$$

therefore we will subsequently assume that $\delta \leq \boldsymbol{\delta}_{q,m}$. Define

$$B = \operatorname{spt}(\phi) \cap \mathbf{B}(x, r) \cap \left\{y : \operatorname{dist}(y - x, Z) \geq \delta^{1-p_{q,m}} r\right\}$$

and observe that

$$\begin{aligned}\delta^q &= r^{-m-q}\int_{\mathbf{B}(x,r)} \operatorname{dist}^q(y - x, Z)\,d\phi(y) \\ &\geq r^{-m-q}\phi(B)\delta^{q(1-p_{q,m})}r^q\,,\end{aligned}$$

whence

$$\phi(B) \leq r^m \delta^{q p_{q,m}}. \tag{73}$$

Now assume that there exists $y \in \operatorname{spt}(\phi) \cap \mathbf{B}\left(x, \frac{r}{2}\right)$ such that

$$\operatorname{dist}(y - x, Z) \geq \mathbf{c}_1(m)\delta^{1-p_{q,m}} r. \tag{74}$$

Put $\rho = (\mathbf{c}_1(m) - 1)\delta^{1-p_{q,m}} r$ and notice that the choice of $\mathbf{c}_1(m)$, $\boldsymbol{\delta}_m$ and the relation $\delta \leq \boldsymbol{\delta}_m$ implies $\rho \leq \frac{r}{2}$, so that $\mathbf{B}(y, \rho) \subset \mathbf{B}(x, r)$. This in turn implies that

$$\operatorname{spt}(\phi) \cap \mathbf{B}(y, \rho) \subset B$$

and, according to (73),

$$\phi(\mathbf{B}(y, \rho)) \leq \phi(B) \leq \delta^{q p_{q,m}} r^m\,. \tag{75}$$

Hypotheses (B) and (C) yield

$$\frac{\phi(\mathbf{B}(y, \rho))}{\boldsymbol{\alpha}(m)\rho^m} \geq \Theta^m(\phi, y) - \xi(\rho) \geq \frac{1}{2}. \tag{76}$$

Combining (75) and (76) we obtain

$$\frac{\boldsymbol{\alpha}(m)}{2}(\mathbf{c}_1(m)-1)^m\delta^{m(1-p_{q,m})}r^m \leq \delta^{qp_{q,m}}r^m.$$

Since the choice of $p_{q,m}$ is so that $m(1-p_{q,m}) = qp_{q,m}$, this is in contradiction with the definition of $\mathbf{c}_1(m)$. This shows that (74) cannot hold for $y \in \operatorname{spt}(\phi) \cap \mathbf{B}\left(x, \frac{r}{2}\right)$, and therefore

$$\boldsymbol{\beta}_\infty\left(\phi, x, \frac{r}{2}, Z\right) \leq \mathbf{c}_1(m)\boldsymbol{\beta}_q(\phi, x, r, Z)^{1-p_{q,m}}.$$

q.e.d.

Proposition 4.7.3. *There exist constants* $0 < \boldsymbol{\delta}_{4.7.3}(n,m) < 1$ *and* $0 < \mathbf{c}_{4.7.3}(m) < \infty$ *with the following property. Assume that*

(A) $U \subset \mathbf{R}^n$ *is open,* $0 \in U$, $0 < R < 2^{-8}$, $\mathbf{B}(0,R) \subset U$, ϕ *is a Radon measure in* U, $0 \in \operatorname{spt}\phi$, ξ *is a continuous gauge;*
(B) *for every* $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0,R)$, $\Theta^m(\phi, x) = 1$*;*
(C) ϕ *is* (ξ, m) *nearly monotonic in* U *and* (ξ, m) *epiperimetric in* $(\operatorname{spt}(\phi) \cap \mathbf{B}(0,R), U)$*;*
(D) $\phi(\mathbf{B}(0,R)) \leq (1+\boldsymbol{\delta}_{4.7.3}(n,m))\boldsymbol{\alpha}(m)R^m$ *and* $\xi(\sqrt{R}) \leq \boldsymbol{\delta}_{4.7.3}(n,m)$.

Then for every $0 < r \leq \boldsymbol{\delta}_{4.7.3}(n,m)R$ *there exists* $W \in \mathbf{G}(n,m)$ *such that*

$$\boldsymbol{\beta}_2(\phi, 0, r, W)^2 \leq \mathbf{c}_{4.7.3}(m)\max\left\{\sqrt[8]{r}, \sqrt[4]{\xi\left(2\sqrt{r}\right)}\right\}.$$

Proof. We let

$$\boldsymbol{\delta}_{4.7.3}(n,m) = \min\left\{2^{-4}, \boldsymbol{\delta}_{4.5.6}(n, m, \boldsymbol{\varepsilon}_{4.6.2}(n)).\right\}$$

It follows from Lemma 4.5.5 and Proposition 4.6.2 that with each $0 < r \leq \boldsymbol{\delta}_{4.7.3}(n,m)R$ we can associate an orthogonal family $x_1, \ldots, x_m \in \operatorname{spt}\phi$ with $|x_i| = r\eta(r)$, $i = 1, \ldots, m$, where we have put

$$\eta(r) = \max\left\{\sqrt[8]{r}, \sqrt[4]{\xi\left(2\sqrt{r}\right)}\right\} \leq \frac{1}{2}.$$

We define $W = \operatorname{span}\{x_1, \ldots, x_m\}$ and we observe that

$$\begin{aligned}
&\int_{\mathbf{B}(0,r)} \operatorname{dist}^2(y, W)\,d\phi(y) \\
&= \int_{\mathbf{B}(0,r)} |y|^2 d\phi(y) - \int_{\mathbf{B}(0,r)} |P_W(y)|^2\,d\phi(y) \\
&= \int_{\mathbf{B}(0,r)} |y|^2 d\phi(y) - \sum_{i=1}^m |x_i|^{-2}\int_{\mathbf{B}(0,r)} \langle y, x_i\rangle^2 d\phi(y) \\
&= \operatorname{trace} Q(\phi, r) - \sum_{i=1}^m |x_i|^{-2} Q(\phi, r)(x_i).
\end{aligned}$$

Dividing by $\boldsymbol{\nu}(m)r^{m+2}$ and referring to Lemma 4.1.2 and Proposition 4.4.1 we obtain

$$\begin{aligned}
&\boldsymbol{\nu}(m)^{-1}r^{-m-2}\int_{\mathbf{B}(0,r)} \operatorname{dist}^2(y,W)\,d\phi(y)\\
&= \operatorname{trace} \mathbf{Q}(\phi,r) - \sum_{i=1}^{m} |x_i|^{-2}\mathbf{Q}(\phi,r)(x_i)\\
&\leq m + \xi(r)(m+4) - \sum_{i=1}^{m} (1 - \mathbf{c}_{4.4.1}(m)\eta(r))\\
&\leq (m+4+m\mathbf{c}_{4.4.1}(m))\eta(r).
\end{aligned}$$

q.e.d.

4.8. A regularity theorem.

Lemma 4.8.1. *There exist $0 < \boldsymbol{\delta}_{4.8.1}(n,m) < \infty$, $0 < \mathbf{c}_{4.8.1}(n,m) < \infty$ and $0 < \boldsymbol{\gamma}_{4.8.1}(n,m) < \infty$ with the following property. Whenever*

(A) *$U \subset \mathbf{R}^n$ is open, ϕ is a Radon measure in U, $x \in \operatorname{spt}\phi$, $r > 0$, $\mathbf{B}(x,r) \subset U$, ξ is a gauge;*
(B) *ϕ is m concentrated and (ξ, m) nearly monotonic in U;*
(C) *$\phi(\mathbf{B}(x,r)) \leq (1+\boldsymbol{\delta}_{4.8.1}(n,m))\boldsymbol{\alpha}(m)r^m$;*
(D) *$\xi(r) \leq \boldsymbol{\delta}_{4.8.1}(n,m)$;*
(E) *$W \in \mathbf{G}(n,m)$ and $\boldsymbol{\beta}_\infty(\phi,x,r,W) \leq \boldsymbol{\delta}_{4.8.1}(n,m)$;*

the following holds:

$$\begin{aligned}
\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}(\phi) \cap \mathbf{B}(x,\boldsymbol{\gamma}_{4.8.1}(n,m)r), (x+W) \cap \mathbf{B}(x,\boldsymbol{\gamma}_{4.8.1}(n,m)r))&\\
\leq \mathbf{c}_{4.8.1}(n,m)\boldsymbol{\beta}_\infty(\phi,x,r,W)\boldsymbol{\gamma}_{4.8.1}(n,m)r&
\end{aligned}$$

as well as

$$W \cap \mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r) \subset P_W\big[\boldsymbol{\tau}_{-x}(\operatorname{spt}\phi) \cap \mathbf{B}(0,r)\big].$$

Proof. First notice that it suffices to prove the lemma for $x=0$ and $r=1$. We define

$$\varepsilon = \min\left\{\frac{1}{3}\boldsymbol{\varepsilon}_{2.5.10}(n), \frac{1}{24\mathbf{c}_{2.5.10}(n)}, \frac{1}{25}\right\}$$

and

$$\boldsymbol{\delta}_{4.8.1}(n,m) = \frac{\varepsilon}{2}\min\{\boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon),1\}.$$

We also let $r_0 = \frac{1}{2}\min\{\boldsymbol{\delta}_{4.5.6}(n,m,\varepsilon),1\}$. Our first goal is to prove that

$$W \in \mathbf{G}(\operatorname{spt}(\phi),0,2r_0,3\varepsilon). \tag{77}$$

For that purpose we select $W_0 \in \mathbf{G}(\operatorname{spt}(\phi),0,2r_0,\varepsilon)$ (the existence of such W_0 follows from Lemma 4.5.6). Let $w_0 \in W_0 \cap \mathbf{B}(0,1)$. There exists $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0,2r_0)$ such that

$$|x - 2r_0w_0| \leq \varepsilon 2r_0. \tag{78}$$

According to hypothesis (D) there exists $w \in W$ with

$$|x - w| \leq \boldsymbol{\delta}_{4.8.1}(n, m) \leq \varepsilon 2r_0. \tag{79}$$

Now (78) and (79) yield $|w_0 - (2r_0)^{-1}(2r_0 w_0)| \leq 2\varepsilon$ and, in turn,

$$\sup \{\operatorname{dist}(w_0, W) : w_0 \in W_0 \cap \mathbf{B}(0,1)\} \leq 2\varepsilon.$$

According to Lemma 2.5.6 this implies

$$\operatorname{dist}_{\mathcal{H}}(W_0 \cap \mathbf{B}(0,1), W \cap \mathbf{B}(0,1)) \leq 2\varepsilon\,.$$

We infer from the triangle inequality for the Hausdorff distance that

$$\begin{aligned}
&\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}(\phi) \cap \mathbf{B}(0, 2r_0), W \cap \mathbf{B}(0, 2r_0)) \\
&\leq \operatorname{dist}_{\mathcal{H}}(\operatorname{spt}(\phi) \cap \mathbf{B}(0, 2r_0), W_0 \cap \mathbf{B}(0, 2r_0)) \\
&\quad + \operatorname{dist}_{\mathcal{H}}(W_0 \cap \mathbf{B}(0, 2r_0), W \cap \mathbf{B}(0, 2r_0)) \\
&\leq 3\varepsilon 2r_0;
\end{aligned}$$

this completes the proof of (77).

Now we observe that $\mathbf{G}(\operatorname{spt}(\phi), x, r, \varepsilon) \neq \emptyset$ whenever $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, 2r_0)$ and $0 < r \leq 2r_0$, according to Lemma 4.5.6. Therefore Theorem 2.5.10 applies: our choice of ε and relation (77) ensure the existence of a continuous map $\tau : W \cap \mathbf{B}(0, r_0) \to \operatorname{spt}(\phi)$ such that

$$\begin{aligned}
|\tau(y) - y| &\leq \mathbf{c}_{2.5.10}(n) 3\varepsilon r_0 \\
&\leq (1/8) r_0.
\end{aligned} \tag{80}$$

We claim that

$$P_W\left(\operatorname{spt}(\phi) \cap \mathbf{B}(0, r_0)\right) \supset W \cap \mathbf{B}(0, r_0/2). \tag{81}$$

In order to prove this we let $\pi : W \to W \cap \mathbf{B}(0, r_0)$ be the nearest point projection on $W \cap \mathbf{B}(0, r_0)$ and we define a continuous map

$$g : W \cap \mathbf{B}(0, r_0) \to W \cap \mathbf{B}(0, r_0)$$

by the relation $g = \pi \circ P_W \circ \tau$. Let $y \in W \cap \operatorname{Bdry} \mathbf{B}(0, r_0)$. We infer from (80) that

$$|\tau(y)| \leq (1 + 1/8) r_0, \tag{82}$$

therefore $|P_W(\tau(y))| \leq (1 + 1/8) r_0$ as well. Hence

$$|\pi(P_W(\tau(y))) - P_W(\tau(y))| \leq (1/8) r_0. \tag{83}$$

It also follows from (82) that $\tau(y) \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, 2r_0)$, and in turn we deduce from (77) that

$$|P_W(\tau(y)) - \tau(y)| \leq 3\varepsilon 2r_0. \tag{84}$$

Putting together (80), (83) and (84) we obtain

$$\begin{aligned}|g(y)-y| &\leq |\pi(P_W(\tau(y))) - P_W(\tau(y))| \\ &\quad + |P_W(\tau(y)) - \tau(y)| \\ &\quad + |\tau(y)-y| \\ &\leq (1/8)r_0 + 6\varepsilon r_0 + (1/8)r_0 \\ &< r_0/2,\end{aligned}$$

according to our choice of ε. It now follows from Lemma 4.6.1 that $W \cap \mathbf{B}(0,r_0/2) \subset \operatorname{im} g$. Therefore

$$W \cap \mathbf{B}(0,r_0/2) \subset P_W(\operatorname{im}\tau) \subset P_W(\operatorname{spt}(\phi) \cap \mathbf{B}(0,9r_0/8)). \tag{85}$$

Notice our choices of $\boldsymbol{\delta}_{4.8.1}(n,m)$ and ε together with hypothesis (D) imply that $\boldsymbol{\beta}_\infty(\phi,0,1,W) < r_0/2$. Choose $0<s<1$ such that $\boldsymbol{\beta}_\infty(\phi,0,1,W) = sr_0/2$ and define $t=\sqrt{1-s}$. Pick $y \in W \cap \mathbf{B}(0,r_0/2)$ and refer to (85) to choose $x \in \operatorname{spt}(\phi)\cap\mathbf{B}(0,9r_0/8)$ such that $P_W(x) = ty$. Notice that $|x| \leq 1$. We observe that

$$\begin{aligned}\operatorname{dist}(ty,\operatorname{spt}(\phi)) &\leq |ty-x| \\ &= |P_{W^\perp}(x)| \\ &\leq \operatorname{dist}(x,W) \\ &\leq \boldsymbol{\beta}_\infty(\phi,0,1,W),\end{aligned}$$

and

$$\begin{aligned}|x|^2 &= |P_{W^\perp}(x)|^2 + |P_W(x)|^2 \\ &\leq \operatorname{dist}(x,W)^2 + |ty|^2 \\ &\leq \boldsymbol{\beta}_\infty(\phi,0,1,W)^2 + t^2(r_0/2)^2 \\ &= (s^2+t^2)(r_0/2)^2,\end{aligned}$$

so that in fact $x \in \operatorname{spt}(\phi)\cap\mathbf{B}(0,r_0/2)$. On the other hand,

$$|y-ty| \leq (1-t)r_0/2 \leq sr_0/2 = \boldsymbol{\beta}_\infty(\phi,0,1,W)\,.$$

We conclude that

$$\sup\{\operatorname{dist}(y,\operatorname{spt}(\phi)\cap\mathbf{B}(0,r_0/2)) : y \in W\cap\mathbf{B}(0,r_0/2)\} \leq 2\boldsymbol{\beta}_\infty(\phi,0,1,W).$$

We see that the conclusion of the proposition holds with $\boldsymbol{\gamma}_{4.8.1}(n,m) = r_0/2$ and $\mathbf{c}_{4.8.1}(n,m) = r_0$. q.e.d.

Next we recall a (very) classical method for producing Lipschitz graphs.

Definition 4.8.2. Let $S \subset \mathbf{R}^n$, $r>0$, $\sigma>0$ and $W \in \mathbf{G}(n,m)$. We define

$$\mathcal{G}(S,r,\sigma,W) = S \cap \mathbf{B}(0,r/2) \cap \big\{x : S\cap\mathbf{B}(x,\rho) \subset \mathbf{B}(x+W,\sigma\rho) \\ \text{for every } 0<\rho\leq r\big\}.$$

Lemma 4.8.3. *Assume that $S \subset \mathbf{R}^n$, $r > 0$, $0 < \sigma < 1$ and $W \in \mathbf{G}(n,m)$. Then there exists a Lipschitzian map*

$$u : P_W(\mathcal{G}(S, r, \sigma, W)) \to W^\perp$$

such that $\operatorname{Lip} u \le \sigma/\sqrt{1-\sigma^2}$ *and* $\operatorname{graph}(u) = \mathcal{G}(S, r, \sigma, W)$.

Proof. We abbreviate $\mathcal{G} = \mathcal{G}(S, r, \sigma, W)$. Let $x, y \in \mathcal{G}$ and put $\rho = |x - y|$, so that $\rho \le r$. Therefore $y \in S \cap \mathbf{B}(x, \rho) \subset \mathbf{B}(x + W, \sigma\rho)$, which means that

$$(86) \qquad |P_{W^\perp}(y - x)| \le \sigma\rho = \sigma|y - x|.$$

Since also $|y - x|^2 = |P_W(y - x)|^2 + |P_{W^\perp}(y - x)|^2$, (86) becomes

$$(87) \qquad (1 - \sigma^2)\, |P_{W^\perp}(y - x)|^2 \le \sigma^2\, |P_W(y - x)|^2 .$$

Observe that (87) implies that the restriction $P_W : \mathcal{G} \to P_W(\mathcal{G})$ is injective. Let $f : P_W(\mathcal{G}) \to \mathcal{G}$ be its inverse and $u = P_{W^\perp} \circ f$. Clearly $\operatorname{graph}(u) = \mathcal{G}$ and (87) readily yields the claimed estimate of $\operatorname{Lip} u$. q.e.d.

We are now ready to prove the first version of our main result.

Theorem 4.8.4. *For every $0 < \alpha \le 1$ and every $0 < m < n$ there exist $0 < \boldsymbol{\delta}_{4.8.4}(n,m) < \infty$, $0 < \boldsymbol{\delta}^*_{4.8.4}(n,m,\alpha) < \infty$, $0 < \boldsymbol{\gamma}_{4.8.4}(n,m) < \infty$ and $0 < \mathbf{c}_{4.8.4}(n,m,\alpha) < \infty$ with the following property. Assume that*

- (A) *$U \subset \mathbf{R}^n$ is open, ϕ is a Radon measure in U, $0 \in \operatorname{spt}\phi$, $C \ge 1$, $\xi(t) = Ct^\alpha$;*
- (B) *$0 < R < \boldsymbol{\delta}^*_{4.8.4}(n,m,\alpha)C^{-2/\alpha}$, $\mathbf{B}(0,R) \subset U$;*
- (C) *ϕ is (ξ, m) nearly monotonic in U;*
- (D) *ϕ is (ξ, m) epiperimetric in $(U \cap \operatorname{spt}(\phi), U)$;*
- (E) *$\Theta^m(\phi, x) = 1$ for every $x \in \operatorname{spt}(\phi) \cap U$;*
- (F) *$\phi(\mathbf{B}(0,R)) \le (1 + \boldsymbol{\delta}_{4.8.4}(n,m))\boldsymbol{\alpha}(m)R^m$;*
- (G) *$\xi(\sqrt{R}) \le \boldsymbol{\delta}_{4.8.4}(n,m)$;*
- (H) *$W \in \mathbf{G}(n,m)$ and $\boldsymbol{\beta}_\infty(\phi, 0, R, W) \le \boldsymbol{\delta}_{4.8.4}(n,m)$.*

Then there exists a map $u : W \cap \mathbf{B}(0, \boldsymbol{\gamma}_{4.8.4}(n,m)R) \to W^\perp$ such that

- (I) *u is continuously differentiable and*

$$\|Du(z_1) - Du(z_2)\| \le \mathbf{c}_{4.8.4}(n,m,\alpha)C|z_1 - z_2|^{\frac{\alpha}{8(m+2)}}$$

whenever $z_1, z_2 \in W \cap \mathbf{B}(0, \boldsymbol{\gamma}_{4.8.4}(n,m)R)$;

- (J) $\operatorname{spt}(\phi) \cap \mathbf{B}(0, \boldsymbol{\gamma}_{4.8.4}(n,m)R) = \operatorname{graph}(u) \cap \mathbf{B}(0, \boldsymbol{\gamma}_{4.8.4}(n,m)R)$.

Proof. We start by defining all the necessary constants. For further reference in Claims 2 and 3 below we put

$$\mathbf{c}_2(n,m) := 2\mathbf{c}_{4.7.2}(n,m)\mathbf{c}_{4.7.3}(n,m)$$

and

$$\mathbf{c}_3(n,m) := \frac{2\mathbf{c}_2(n,m)\mathbf{c}_{4.8.1}(n,m)}{\boldsymbol{\gamma}_{4.8.1}(n,m)^{\frac{1}{8(m+2)}}}.$$

Next we choose $0 < \boldsymbol{\eta}(n,m) < 2^{-8}$ small enough for

$$\left(1 + \frac{1}{2}\boldsymbol{\delta}_{4.7.3}(n,m)\right)(1 + 2\boldsymbol{\eta}(n,m))^m \leq 1 + \boldsymbol{\delta}_{4.7.3}(n,m)$$

as well as

$$\left(1 + \frac{1}{2}\boldsymbol{\delta}_{4.8.1}(n,m)\right)(1 + 2\boldsymbol{\eta}(n,m))^m \leq 1 + \boldsymbol{\delta}_{4.8.1}(n,m).$$

In order to keep the notations short we also set

$$\beta = \frac{\alpha}{8(m+2)}.$$

We claim that the theorem holds with $\boldsymbol{\delta}_{4.8.4}(n,m)$, $\boldsymbol{\delta}^*_{4.8.4}(n,m,\alpha)$, $\mathbf{c}_{4.8.4}(n,m)$ and $\boldsymbol{\gamma}_{4.8.4}(n,m)$ defined as follows:

$$\begin{aligned}\boldsymbol{\delta}_{4.8.4}(n,m) := \min\Bigg\{ & \frac{1}{4}\boldsymbol{\delta}_{4.7.3}(n,m), \\ & \frac{1}{4}\boldsymbol{\delta}_{4.8.1}(n,m), \\ & \frac{\boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)}{4\mathbf{c}_{4.8.1}(n,m)}, \\ & \frac{1}{16\mathbf{c}_{4.8.1}(n,m)\left(1 + \frac{8}{\boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)}\right)}\Bigg\},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\delta}^*_{4.8.4}(n,m,\alpha) := \min\Bigg\{ & \left(\frac{\boldsymbol{\delta}_{4.8.1}(n,m)}{\mathbf{c}_2(n,m)}\right)^{\beta^{-1}}, \\ & \left(\frac{1}{384\mathbf{c}_3(n,m)}\right)^{\beta^{-1}}, \\ & \left(\frac{1 - 2^{-\beta}}{160\mathbf{c}_3(n,m)}\right)^{\beta^{-1}}, \\ & \left(\frac{\boldsymbol{\delta}_{4.7.3}\boldsymbol{\eta}(n,m)}{4\mathbf{c}_3(n,m)}\right)^{\beta^{-1}}, \\ & \left(\frac{1}{16\mathbf{c}_3(n,m)\left(1 + \frac{8}{\boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)}\right)}\right)^{\beta^{-1}}\Bigg\},\end{aligned}$$

and

$$\mathbf{c}_{4.8.4}(n,m,\alpha) = \frac{256\mathbf{c}_3(n,m)}{1-2^{-\beta}},$$

as well as

$$\boldsymbol{\gamma}_{4.8.4}(n,m) = \frac{\boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)\boldsymbol{\gamma}_{4.8.1}(n,m)^2}{8}.$$

The proof is subdivided into several claims as follows.

Claim 1. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, \boldsymbol{\eta}(n,m)R)$ and every $0 < r \leq \boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)R$ there exists $W^*_{x,r} \in \mathbf{G}(n,m)$ such that

$$\boldsymbol{\beta}_2(\phi, x, r, W^*_{x,r})^2 \leq 2\mathbf{c}_{4.7.3}(n,m)C^{\frac{1}{4}}r^{\frac{\alpha}{8}}.$$

Since

$$\max\left\{\sqrt[8]{r}, \sqrt[4]{\xi\left(2\sqrt{r}\right)}\right\} = 2C^{\frac{1}{4}}r^{\frac{\alpha}{8}}$$

we need only to check that Proposition 4.7.3 applies to $\boldsymbol{\tau}_{-x\,\#}\phi$ at the scale $R/2$. Observe that hypothesis (D) of this proposition is verified:

$$\begin{aligned}
\frac{\phi(\mathbf{B}(x,R/2))}{\boldsymbol{\alpha}(m)(R/2)^m} &\leq \frac{\phi(\mathbf{B}(0,|x|+R/2))}{\boldsymbol{\alpha}(m)(|x|+R/2)^m}\left(1+2\boldsymbol{\eta}(n,m)\right)^m \\
&\leq \left(\frac{\phi(\mathbf{B}(0,R))}{\boldsymbol{\alpha}(m)R^m} + \xi(R)\right)\left(1+2\boldsymbol{\eta}(n,m)\right)^m \\
&\leq 1 + \boldsymbol{\delta}_{4.7.3}(n,m).
\end{aligned}$$

Claim 2. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, \boldsymbol{\eta}(n,m)R)$ and every $0 < r \leq \boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)R$ one has

$$\boldsymbol{\beta}_\infty\left(\phi, x, \frac{r}{2}, W^*_{x,r}\right) \leq \mathbf{c}_2(n,m)C^{\frac{1}{4(m+2)}}\left(\frac{r}{2}\right)^\beta.$$

This is a straightforward consequence of Claim 1 and Lemma 4.7.2.

Claim 3. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, \boldsymbol{\eta}(n,m)R)$ and every $0 < r \leq \boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)R/2$ one has

$$\begin{aligned}
&\operatorname{dist}_{\mathcal{H}}\big[\operatorname{spt}(\phi)\cap\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r/2), (x+W^*_{x,r})\cap\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r/2)\big] \\
&\leq \mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}\left(\frac{\boldsymbol{\gamma}_{4.8.1}(n,m)r}{2}\right)^{1+\beta}.
\end{aligned}$$

This follows from Claim 2 together with Lemma 4.8.1. One checks that hypothesis (C) of this lemma is verified as in the proof of Claim 1. Regarding hypothesis (E) of this lemma we infer from Claim 2 that

$$\boldsymbol{\beta}_\infty(\phi, x, r, W^*_{x,r}) \leq \mathbf{c}_2(n,m)C^{\frac{1}{4(m+2)}}r^\beta \leq \boldsymbol{\delta}_{4.8.1}(n,m);$$

the latter holds because $r \leq R$ is sufficiently small according to hypothesis (B) of the present theorem.

In order to keep the notations short in the remainder of this proof we put

$$r_0 = \frac{\boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)\boldsymbol{\gamma}_{4.8.1}(n,m)R}{4},$$

as well as

$$r_j = 2^{-j} r_0\,,\ j = 1,2,\dots.$$

We also set

$$W_{x,r} = W^*_{x,2r/\boldsymbol{\gamma}_{4.8.1}(n,m)}$$

whenever the right member is defined.

Claim 4. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, \boldsymbol{\eta}(n,m)R)$ and every $0 < r \le r_0$ one has

$$\operatorname{dist}_{\mathcal{H}}\big[\operatorname{spt}(\phi) \cap \mathbf{B}(x,r), (x + W_{x,r}) \cap \mathbf{B}(x,r)\big] \le \mathbf{c}_3(n,m) C^{\frac{1}{4(m+2)}} r^{1+\beta}.$$

This is merely a reformulation of Claim 3.

Claim 5. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, \boldsymbol{\eta}(n,m)R)$, every $j = 0,1,2,\dots$ and every $r_{j+1} \le r \le r_j$ one has

$$\operatorname{dist}(W_{x,r}, W_{x,r_j}) \le 5\mathbf{c}_3(n,m) C^{\frac{1}{4(m+2)}} r_j^{\beta}.$$

This is a consequence of Lemma 2.5.8 applied with R (there) equal to r_j (here) and $\varepsilon = \mathbf{c}_3(n,m) C^{\frac{1}{4(m+2)}} r_j^{\beta}$. One checks that $\varepsilon \le 1/2$ because $r_j \le r_0 \le R$ is sufficiently small.

We infer from Claim 5 that for such x and $k > j$ the following holds:

$$\begin{aligned} \operatorname{dist}(W_{x,r_k}, W_{x,r_j}) &\le 5\mathbf{c}_3(n,m) C^{\frac{1}{4(m+2)}} \sum_{l=j}^{\infty} r_l^{\beta} \\ &\le 5\mathbf{c}_3(n,m) C^{\frac{1}{4(m+2)}} (1 - 2^{-\beta})^{-1} r_j^{\beta}. \end{aligned} \tag{88}$$

Therefore $W_{x,r_0}, W_{x,r_1}, \dots$ is a Cauchy sequence in $\mathbf{G}(n,m)$. We let W_x denote its limit.

Claim 6. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, \boldsymbol{\eta}(n,m)R)$, every $j = 0,1,2,\dots$ and every $r_{j+1} \le r \le r_j$ one has

$$\operatorname{dist}(W_x, W_{x,r}) \le 10\mathbf{c}_3(n,m) C^{\frac{1}{4(m+2)}} (1 - 2^{-\beta})^{-1} r_j^{\beta}.$$

This readily follows from (88) and Claim 5.

Claim 7. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, r_0/2)$ and every $0 < r \le r_0$ one has

$$\operatorname{dist}(W_{x,r}, W_0) \le \frac{1}{4}.$$

In order to prove this we first apply Lemma 2.5.9 to the points 0 and x, with R (there) equal to r_0 (here), $\lambda = 1/2$, $\nu = 4$ and $\varepsilon = \mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}r_0^\beta$ (recall Claim 4). Notice that $r_0 \leq R$ has been chosen so small that $\varepsilon \leq 1/384$. It now follows from Lemma 2.5.9 that

$$\operatorname{dist}(W_{x,r_0}, W_{0,r_0}) \leq \frac{1}{16}.$$

This, together with three applications of Claim 6, yields

$$\begin{aligned}\operatorname{dist}(W_{x,r}, W_0) &\leq \operatorname{dist}(W_{x,r}, W_x) + \operatorname{dist}(W_x, W_{x,r_0}) \\ &\quad + \operatorname{dist}(W_{x,r_0}, W_{0,r_0}) + \operatorname{dist}(W_{0,r_0}, W_0) \\ &\leq \frac{1}{16} + 30\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}(1-2^{-\beta})^{-1}r_0^\beta \\ &\leq \frac{1}{4},\end{aligned}$$

where the last inequality is again a consequence of r_0 being small enough.

Claim 8. One has $\operatorname{dist}(W, W_0) \leq \frac{1}{8}$ (recall W is as in hypothesis (H)).

Hypothesis (H) and Lemma 4.8.1 yield

$$W \in \mathbf{G}(\operatorname{spt}(\phi), 0, \boldsymbol{\gamma}_{4.8.1}(n,m)R, \varepsilon_1)$$

where

$$\varepsilon_1 = \mathbf{c}_{4.8.1}(n,m)\boldsymbol{\delta}_{4.8.4}(n,m).$$

On the other hand we infer from Claim 4 that $W_{0,r_0} \in \mathbf{G}(\operatorname{spt}(\phi), 0, r_0, \varepsilon_2)$ where

$$\varepsilon_2 = \mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}r_0^\beta.$$

Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and apply Lemma 2.5.8 at scale $r_0 < \boldsymbol{\gamma}_{4.8.1}(n,m)R$. Notice that our choice of $\boldsymbol{\delta}_{4.8.4}(n,m)$ and the smallness of r_0 implied by hypothesis (B) guarantee that $\varepsilon\boldsymbol{\gamma}_{4.8.1}(n,m)R \leq r_0$ as well as

$$\varepsilon\left(1 + \frac{8}{\boldsymbol{\delta}_{4.7.3}(n,m)\boldsymbol{\eta}(n,m)}\right) \leq \frac{1}{16}.$$

Therefore it follows from Lemma 2.5.8 that

$$\operatorname{dist}(W, W_{0,r_0}) \leq \frac{1}{16}.$$

On the other hand Claim 6 implies that

$$\operatorname{dist}(W_0, W_{0,r_0}) \leq 10\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}(1-2^{-\beta})^{-1}r_0^\beta \leq \frac{1}{16}$$

again because r_0 is small enough. The proof of Claim 8 is complete.

Claim 9. For every $x \in \operatorname{spt}(\phi) \cap \mathbf{B}(0, r_0/2)$ and every $0 < r \leq r_0$ one has

$$\operatorname{spt}(\phi) \cap \mathbf{B}(x,r) \subset \mathbf{B}(x + W, r/2).$$

We first infer from Claim 4 and the smallness of r_0 that

$$\begin{aligned}\operatorname{dist}_{\mathcal{H}}\big[\operatorname{spt}(\phi)\cap\mathbf{B}(x,r),(x+W_{x,r})\cap\mathbf{B}(x,r)\big] &\leq \mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}r_0^{\beta}r\\ &\leq \frac{r}{8}.\end{aligned}$$

Furthermore, we deduce from Claims 7 and 8 that

$$\operatorname{dist}_{\mathcal{H}}\big[(x+W)\cap\mathbf{B}(x,r),(x+W_{x,r})\cap\mathbf{B}(x,r)\big] \leq \frac{3}{8}r.$$

The conclusion follows from the triangle inequality for the Hausdorff distance.

Claim 10. There exists a Lipschitzian map

$$u : W\cap\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r_0/2)\to W^{\perp}$$

such that $\operatorname{Lip} u\leq 1$ and

$$\operatorname{graph}(u)\cap\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r_0/2)=\operatorname{spt}(\phi)\cap\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r_0/2).$$

On letting $\mathcal{G}=\mathcal{G}(\operatorname{spt}(\phi),r_0,1/2,W)$ we infer from Lemma 4.8.3 that there exists a Lipschitzian map

$$\tilde{u} : P_W(\mathcal{G})\to W^{\perp}$$

so that $\operatorname{graph}(\tilde{u})=\mathcal{G}$. It also follows from Claim 9 that $\mathcal{G}=\operatorname{spt}(\phi)\cap\mathbf{B}(0,r_0/2)$. Next we apply Lemma 4.8.1 at the scale $r_0/2$ and we obtain

$$\begin{aligned}W\cap\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r_0/2) &\subset P_W(\operatorname{spt}(\phi)\cap\mathbf{B}(0,r_0/2))\\ &=P_W(\mathcal{G}).\end{aligned}$$

Therefore, on letting $u=\tilde{u}\restriction\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r_0/2)$ we see our conclusion holds.

Claim 11. Let $z_1,z_2\in W\cap\mathbf{B}(0,\boldsymbol{\gamma}_{4.8.1}(n,m)r_0/2)$ be such that u is differentiable at z_1 and z_2. Then

$$\|Du(z_1)-Du(z_2)\|\leq 256\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}(1-2^{-\beta})^{-1}|z_1-z_2|^{\beta}.$$

Set $x_i=u(z_i)$, $i=1,2$, and $r=2|x_1-x_2|\leq r_0$. We infer from Claim 6 that

$$\operatorname{dist}(W_{x_i},W_{x_i,r})\leq 20\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}(1-2^{-\beta})^{-1}r^{\beta}.$$

Next we apply Lemma 2.5.9 at the scale r with $\lambda=1/2$, $\nu=4$ and $\varepsilon=\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}r^{\beta}<1/4$ (recall Claim 4). We obtain

$$\operatorname{dist}(W_{x_1,r},W_{x_2,r})\leq 24\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}r^{\beta}.$$

Adding these inequalities yields

$$\begin{aligned}\operatorname{dist}(W_{x_1},W_{x_2}) &\leq 64\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}(1-2^{-\beta})^{-1}r^{\beta}\\ &\leq 64\mathbf{c}_3(n,m)C^{\frac{1}{4(m+2)}}(1-2^{-\beta})^{-1}|z_1-z_2|^{\beta}.\end{aligned}$$

It is not hard to check that $W_{x_i} = \mathrm{im}(i_W + i_{W^\perp} \circ Du(z_i))$, $i = 1, 2$, so that our conclusion becomes a consequence of Lemma 2.1.1(B).

According to Rademacher's theorem, [**10**, 3.1.6], u is differentiable $\mathcal{H}^m$ almost everywhere. Therefore the estimate of Claim 11 remains valid with the smoothing $u * \Phi_\varepsilon$ replacing u. This implies in turn that $D(u * \Phi_\varepsilon)$ converges uniformly so that u is differentiable everywhere. The proof of the theorem is now complete. q.e.d.

The following immediate consequence of Theorem 4.8.4 is perhaps more user-friendly.

Corollary 4.8.5. *Let $S \subset \mathbf{R}^n$ and $0 < m < n$ an integer. The following conditions are equivalent.*

(1) *S is an m dimensional Hölder continuously differentiable submanifold.*

(2) *$S = \mathrm{spt}(\mathcal{H}^m \llcorner S)$ and each $x \in S$ has a neighborhood U verifying the following conditions:*
   (A) *$\mathcal{H}^m \llcorner S$ is (ξ, m) epiperimetric in $(U \cap S, U)$;*
   (B) *$\mathcal{H}^m \llcorner S$ is (ξ, m) nearly monotonic in U;*
   for some gauge ξ of the type $\xi(t) = Ct^\alpha$, $0 < \alpha \leq 1$, $C \geq 1$.

Proof. That (1) implies (2) is the content of Proposition 3.6.1. With regard to the reverse implication we infer from Theorem 4.8.4 that one needs only to check that $\Theta^m(\mathcal{H}^m \llcorner S, x) = 1$ for every $x \in S \cap U$. This readily follows from the (ξ, m) epiperimetry of $\mathcal{H}^m \llcorner S$ on $S \cap U$ together with [**10**, 3.2.19]. q.e.d.

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