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Q.E.D. FOR ALGEBRAIC VARIETIES

Fabrizio Catanese

Abstract

We introduce a new equivalence relation for complete algebraic varieties with canonical singularities, generated by birational equivalence, by flat algebraic deformations (of varieties with canonical singularities), and by quasi-étale morphisms, i.e., morphisms which are unramified in codimension 1. We denote the above equivalence by A.Q.E.D. : = Algebraic-Quasi-Étale- Deformation.

A completely similar equivalence relation, denoted by $\mathbb{C}$-Q.E.D., can be considered for compact complex spaces with canonical singularities.

By a recent theorem of Siu, dimension and Kodaira dimension are invariants for $A.Q.E.D.$ of complex varieties.

We address the interesting question whether conversely two algebraic varieties of the same dimension and with the same Kodaira dimension are Q.E.D. - equivalent (A.Q.E.D., or at least $\mathbb{C}$-Q.E.D.), the answer being positive for curves by well known results.

Using Enriques' (resp. Kodaira's) classification we show first that the answer to the $\mathbb{C}$-Q.E.D. question is positive for special algebraic surfaces (those with Kodaira dimension at most 1), resp. for compact complex surfaces with Kodaira dimension $0, 1$ and even first Betti number.

The appendix by Sönke Rollenske shows that the hypothesis of even first Betti number is necessary: he proves that any surface which is $\mathbb{C}$-Q.E.D.-equivalent to a Kodaira surface is itself a Kodaira surface.

We show also that the answer to the A.Q.E.D. question is positive for complex algebraic surfaces of Kodaira dimension ≤ 1.

The answer to the Q.E.D. question is instead negative for surfaces of general type: the other appendix, due to Fritz Grunewald, is devoted to showing that the (rigid) Kuga-Shavel type surfaces of general type obtained as quotients of the bidisk via discrete groups constructed from quaternion algebras belong to countably many distinct Q.E.D. equivalence classes.

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1. Introduction

The purpose of the present article is to define some new and broad equivalence relations, called Q.E.D., in the classification theory of algebraic varieties and compact complex spaces, and to pose some problems concerning invariants for Q.E.D. equivalences.

To briefly explain the prehistory of the question, let me first recall that, in order to make the study of algebraic varieties possible, it is customary to introduce some equivalence relation. The most classical one is the so-called birational equivalence (respectively, one considers the bimeromorphic equivalence for complex spaces)

Since moreover quite often the construction of algebraic varieties depends upon parameters, Kodaira and Spencer introduced the notion of **$\mathbb{C}$-deformation equivalence** for complex manifolds: they ([**K-S**]) defined two complex manifolds X', X to be **directly deformation equivalent** if there is a proper holomorphic submersion $\pi : \Xi \to \Delta$ of a complex manifold Ξ to the unit disk in the complex plane, such that X, X' occur as fibres of π. If we take the equivalence relation generated by direct deformation equivalence, we obtain the relation of complex deformation equivalence, and we say that X is a complex **deformation** of X' **in the large** if X, X' are complex deformation equivalent.

These two notions extend to the case of compact complex manifolds the classical notions of irreducible, resp. connected, components of moduli spaces.

My first main motivation for introducing Q.E.D.-equivalence is the following: to explain the Kodaira classification in the case of algebraic curves, one has to say that a curve has Kodaira dimension 1 iff it has genus $g \geq 2$, and then to recall that all curves of a fixed genus g are deformation equivalent.

The simple but key observation is that for each $g \geq 2$ there is a curve of genus g which is an étale (unramified) covering of a curve of genus 2. Therefore all the curves with Kodaira dimension 1 are equivalent by the equivalence relation generated by deformation and by étale maps (the same holds of course for curves of Kodaira dimension 0, resp. $-\infty$).

More remarkable is the consideration of the (complex) algebraic surfaces of Kodaira dimension 0: the Enriques surfaces have an étale double cover which is a $K3$-surface, and the hyperelliptic surfaces have an étale cover which is a torus (indeed, the product of two elliptic curves). So, in this case, we should link tori and $K3$'s by étale maps and deformations. This is obviously not possible, because tori are $K(\pi, 1)$'s while $K3$'s are simply connected.

That's why the solution is to divide the torus by multiplication by -1, obtaining the (singular) Kummer surfaces, and then take a smoothing of the Kummer surface to obtain a smooth $K3$ surface. The small price to pay is to allow morphisms which are not étale, but only étale in codimension 1, and moreover to allow singularities, ordinary double points in the case of surfaces. This singularity is a very special case of the Rational Double Points, which are the canonical singularities of dimension 2 (cf. [**Reid1**], [**Reid2**]).

This remark helps to justify the following definition:

Definition 1.1. The relation of algebraic quasi-étale deformation is the equivalence relation, for complete algebraic varieties with canonical singularities defined over a fixed algebraically closed field, generated by

(1) birational equivalence,
(2) flat proper algebraic deformation $\pi : \mathcal{X} \to B$ with base B a connected algebraic variety, and with all the fibres having canonical singularities,
(3) quasi-étale morphisms $f : X \to Y$, i.e., surjective morphisms which are étale in codimension 1 on X (there is $Z \subset X$ of codimension ≥ 2 such that f is étale on $X - Z$),

and denoted by $A.Q.E.D.$ ($X \sim_{A.Q.E.D.} Y$).

Note that we have a completely analogous $\mathbb{C} - Q.E.D.$ -equivalence for compact complex spaces with canonical singularities generated by

(1) bimeromorphic equivalence,
(2) flat proper complex deformations $\pi : \mathcal{X} \to B$ with connected base B, and with all the fibres having canonical singularities,
(3) quasi-étale morphisms $f : X \to Y$.

One may of course restrict the latter equivalence relation to algebraic varieties, and to Kähler manifolds and spaces.

Finally, we define a compact complex space to be **standard** if it is $\mathbb{C} - Q.E.D.$ -equivalent to a product of curves, and similarly define the concept of an **A. standard = algebraically standard** algebraic variety.

Remark 1.2. By Siu's recent theorem ([**Siu**]), not only dimension, but also the Kodaira dimension is an invariant for $A.Q.E.D.$ -equivalence if we restrict ourselves to consider projective varieties with canonical singularities (defined over $\mathbb{C}$). It is conjectured (ibidem, cf. also [**Siu2**]) that the deformation invariance of plurigenera should be true more generally for Kähler complex spaces (with canonical singularities).

Question 0: Is Kodaira dimension also an invariant for $\mathbb{C}$-Q.E.D. equivalence of algebraic varieties and compact Kähler manifolds?

We begin with the following two theorems:

Theorem 1.3. *Let S and S' be smooth Kähler surfaces which have the same Kodaira dimension $\mathcal{K} \leq 1$. Then S and S' are $\mathbb{C} - Q.E.D.$-equivalent.*

Corollary 1.4. *Let S and S' be smooth compact complex surfaces with even first Betti numbers and which have the same Kodaira dimension $\mathcal{K} \in \{0, 1\}$. Then S and S' are $\mathbb{C} -$ Q.E.D.-equivalent.*

The next theorem is not a special case of the previous, since we only consider projective deformations:

Theorem 1.5. *Let S and S' be smooth complex algebraic surfaces which have the same Kodaira dimension $\mathcal{K} \leq 1$. Then S and S' are A.Q.E.D.-equivalent.*

The ingredients of the proof of 1.3 are, beyond the Enriques-Kodaira classification and a detailed knowledge of the deformation types of elliptic surfaces, the notion of orbifold fundamental group of a fibration, and the following very simple devise.

Main Observation: Assume that we have two effective actions of a finite group G on algebraic varieties X, resp. Y (effective means that no element $g \in G$ acts trivially). Then the product action of G on the product $X \times Y$ yields a quasi - étale map $X \times Y \rightarrow (X \times Y)/G$: hence, if both $X \times Y$ and $(X \times Y)/G$ have canonical singularities, they are Q.E.D.-equivalent.

A couple of words concerning A.Q.E.D. equivalence: in the case of Kodaira dimension 1 we have to face the problem that algebraic deformation is not completely understood for elliptic surfaces, and, even more, the determination of quasi-étale maps on models with canonical singularities requires a rather deep understanding of the configuration of curves allowed on some deformation of a given elliptic surface. Such a study, as it is the case for K3 surfaces, is related to a systematic investigation of the period map for elliptic surfaces (this investigation was started for Jacobian elliptic surfaces, i.e., elliptic surfaces admitting a section, by Chakiris in [**Chak1**, **Chak2**]).

Our solution to prove the Q.E.D. statement for Kodaira dimension 1 is to try to reduce to the case of no multiple fibres: this is done via the orbifold fundamental group of a fibration and works easily except for elliptic surfaces over $\mathbb{P}^1$ with one or two multiple fibres. For these, the simplest approach (deformation to constant moduli) fails to work. In this case, however, the result follows by showing the existence of an algebraic deformation of such a surface to another one possessing two (resp. : one) $\tilde{D}_4$ fibres: after contracting the non central -2 curves we get a singular surface with a large orbifold fundamental group, and we again reduce to the case of no multiple fibres.

Another purpose of this article is to pose the following

Main Question: Which are the Q.E.D. equivalence classes of surfaces of general type and of special varieties in higher dimension?

Let us try to separate the two issues. Our proof that Kodaira dimension and $\mathbb{C}$-Q.E.D. equivalence coincide for special surfaces is somehow related to the Def = Diff problem (cf. [**F-M1**], but compare also [**F-M**], 205-208). We know that two special surfaces S_1, S_2 are orientedly diffeomorphic if and only if either they are deformation equivalent, or S_1 is deformation equivalent to the complex conjugate of S_2.

It was recently shown ([**Man**], [**K-K**],[**Cat1**], [**C-W**]) that for algebraic surfaces of general type (for these, $\mathbb{C}$-Q.E.D. and A.Q.E.D. equivalence coincide) diffeomorphism or symplectic equivalence are not sufficient criteria to guarantee complex deformation equivalence. Moreover, we observe that almost all (cf. Question 8) the known counterexamples are known to be in the "standard" Q.E.D. equivalence class, i.e., distinct connected components of the moduli space of surfaces of general type are simply obtained via surfaces which are Q.E.D. -equivalent to products of curves.

This observation leads to the second main motivation for introducing Q.E.D. equivalence: quasi- étale maps have for long time been an ace of diamonds in the sleeves of algebraic geometers in order to produce very interesting counterexamples (we shall point out other examples later). Our philosophy here is that quasi- étale maps are a fact of life which in classification theory should be considered more as the daily rule rather than the exception.

The main questions that we want to pose can then be summarized as:

Question 1: Are there more (effectively computable) invariants for Q.E.D. equivalence, than dimension and Kodaira dimension?

Question 2: Is it possible to determine the Q.E.D. equivalence classes inside the class of varieties with fixed dimension n, and with Kodaira dimension k?

For curves and special algebraic surfaces over $\mathbb{C}$ we saw that there is only one A.Q.E.D. class, but in the first appendix Fritz Grunewald shows, considering some Kuga-Shavel type surfaces of general type constructed from quaternion algebras according to general lines suggested by Shimura (cf. [**Shav**]):

Corollary 5.10. *There are infinitely many Q.E.D. -equivalence classes of algebraic surfaces of general type.*

The above surfaces are rigid, but the Q.E.D. -equivalence class contains countably many distinct birational classes. We can then pose a more daring

Question 3: Are there for instance varieties which are isolated in their Q.E.D.-equivalence class (up to birational equivalence, of course)?

Remark 1.6.

- Singularities play an essential role here. Note first of all that (as we will show in the next section), without the restriction on these given in (2), we obtain the trivial equivalence relation for algebraic varieties of the same dimension (does this also hold for compact Kähler manifolds?).
- Assume that a variety X has the following properties of being
  1) rigid,
  2) smooth with ample canonical bundle,
  3) with a trivial algebraic fundamental group,
  4) with a trivial group of automorphisms.

  Then any variety X' birational to X and with canonical singularities has X as canonical model, and since X has no deformations, and there is no non trivial quasi-étale map $Y \to X$ the only possibility, to avoid that X be isolated in its Q.E.D.-equivalence class, would be that there exists a quasi-étale map $f : X \to Y$.

  If f is not birational, however (cf. section 3 for more details) the Galois closure of f yields another quasi-étale map $\phi : Z \to X$. Since ϕ must be birational, it follows that f is Galois and we have a contradiction to $Aut(X) = \{1\}$.

  Is it possible to construct such a variety X with properties 1)–4)?

The analysis of the Q.E.D. equivalence classes for Kuga-Shavel surfaces is based on similar ideas, except that smooth ball quotients, or polydisk quotients, have a residually finite fundamental group. The key result due to Grunewald is then:

Theorem 5.9. *Let k be a real quadratic field, and let $\mathcal{A}$ be the indefinite division quaternion algebra corresponding, by Hasse's theorem, to a choice of $\mathcal{S}$ made as in* 5.7.

Define $\mathcal{F}_{\mathcal{A}}$ to be the family of subgroups $\Delta \subset \mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ commensurable with a subgroup Γ associated to a maximal order $\mathcal{R} \subset \mathcal{A}$.

Each $\Delta \in \mathcal{F}_{\mathcal{A}}$ acts freely on $\mathbb{H}^2$, and denote by $S_\Delta := \mathbb{H}^2/\Delta$ the corresponding algebraic surface.

Then the family of surfaces $\{S_\Delta | \Delta \in \mathcal{F}_{\mathcal{A}}\}$ is a union of Q.E.D. equivalence classes.

The interesting corollary is that, varying k and $\mathcal{A}$, we obtain countably many Q.E.D. equivalence classes.

Question 4: Do there exist for each $n \geq 2$ varieties obtained as ball quotients, and which yield non standard varieties of general type?

In the complex non Kähler world, things get complicated already in dimension 2 for special surfaces. In fact, a compact complex surface with odd first Betti number is non Kähler, and in an appendix Sönke Rollenske shows:

Theorem 6.1. *Let S be a minimal Kodaira surface. Then a smooth surface S' is Q.E.D. equivalent to S if and only if S' is itself a Kodaira surface. Thus Kodaira surfaces constitute a single Q.E.D. equivalence class.*

Recall moreover that Kodaira dimension is known not to be deformation invariant for compact complex manifolds which are not Kählerian (due to some examples originating from the work of Blanchard, cf. [**Ue80**], [**Ue82**] Section 5, and also Section 5 of [**Cat02**] for a more general description).

On the other hand, recently Claire Voisin ([**Voi1**], [**Voi2**]) has given a negative answer to the so-called Kodaira' s question whether a compact Kähler manifold is always a deformation of a projective variety.

Her counterexamples, however, leave open the following more general question (which may in turn also have a negative answer)

Question 5: Is a compact Kähler manifold always $\mathbb{C}$-Q.E.D. equivalent to an algebraic variety?

We leave aside here the study of A.Q.E.D. equivalence classes for algebraic surfaces of Kodaira dimension ≤ 1 defined over an algebraically closed field of positive characteristic; we hope to be able to address this question in the future.

There is perhaps a reason why Q.E.D. equivalence may be more meaningful for algebraic varieties defined over $\mathbb{C}$. One should in fact keep in mind that every projective variety over $\mathbb{C}$ is an algebraic deformation of a projective variety defined over $\bar{\mathbb{Q}}$: this follows since Hilbert schemes are defined over $\mathbb{Z}$. Thus, the study of algebraic varieties defined over $\bar{\mathbb{Q}}$ could play a key role for the Q.E.D. problem (cf. the next sections for more questions).

A final observation is that also the classical questions of unirationality can be seen through a different perspective if we adopt Q.E.D. -equivalence: for instance, the classical counterexamples to the Lüroth problem are (cf. Remark 7.1) Q.E.D. -equivalent to projective space.

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2. The trivial equivalence relation

In this section we show, as already mentioned, the necessity, in defining the Q.E.D. equivalence, to put some restriction on the singularities of the varieties that we consider.

Definition 2.1. The t-equivalence relation for Algebraic Varieties is the one generated by

- birational equivalence,
- flat deformation with connected base.

Theorem 2.2. *Two (irreducible) algebraic varieties are t-equivalent if and only if they have the same dimension.*

Proof. Let Z^n be an irreducible algebraic variety of dimension n: then Z^n is birational to a projective hypersurface $V_d^n \subset \mathbb{P}^{n+1}$ (cf. [**Hart**], I, Prop. 4.9). In turn, by varying the coefficients of the polynomial of degree d defining V_d^n, we see that V_d^n is deformation equivalent to the cone CW_d^{n-1} over a projective variety $W_d^{n-1} \subset \mathbb{P}^n$. Since CW_d^{n-1} is birationally equivalent to $\mathbb{P}^1 \times W_d^{n-1}$, and we can easily show that $X \sim_t X'$,$Y \sim_t Y'$ implies $X \times Y \sim_t X' \times Y'$, we infer by induction that our variety Z is t-equivalent to $\mathbb{P}^{n-1} \times C_d^1$, where $C_d^1 \subset \mathbb{P}^2$ is a plane curve of degree d. Obviously C_d^1 is deformation equivalent to a rational nodal curve ${C'}_d^1 \subset \mathbb{P}^2$, which is birational to $\mathbb{P}^1$. Whence, Z is t-equivalent to $\mathbb{P}^n$. Conversely, it is clear that t-equivalence respects the dimension. q.e.d.

Remark 2.3. Actually, the proof holds more generally if we consider connected algebraic varieties (i.e., reduced and pure dimensional).

3. Elementary properties of quasi-étale morphisms

For the reader's benefit, recall that (cf. [**Reid2**]) a variety X has canonical singularities iff:

1) X is a normal variety of dimension n.
2) K_X is $\mathbb{Q}$-Cartier, i.e., there is a positive integer r (the minimal such integer is called the **index** of X) such that the Weil divisor rK_X is Cartier. This means that the following holds: letting $i : X_0 \to X$ be the inclusion of the nonsingular locus of X, then Zariski defined the canonical sheaf K_X as $i_*(\Omega^n_{X_0})$, and we want that $i_*({\Omega^n_{X_0}}^{\otimes r})$ be invertible on X.
3) If $p : Z \to X$ is a resolution of singularities of X,
$$rK_Z = p^*(rK_X) + \sum_j a_j E_j,$$
where the E_j's are the exceptional divisors, and the a_j's are **non negative** integers ($a_j \geq 0$).

It follows directly from the definition that, $\forall m \geq 0$, there is a natural isomorphism between $H^0(X, mrK_X) := H^0(X_0, (\Omega^n_{X_0})^{\otimes rm})$ and $H^0(Z, mrK_Z)$.

Given $f : X \to Y$ a quasi-étale morphism between varieties with canonical singularities, of degree d, let r be a common multiple of the indices of X, Y.

By definition there is an open set X^0 such that $X^0 \to Y^0$ is étale, and $X - X^0$ has codimension ≥ 2. Without loss of generality we shall assume always that $X^0 \subset X_0$, and obviously then $Y^0 \subset Y_0$. Since $X_0 - X^0$ and $Y_0 - Y^0$ have both codimension ≥ 2, $H^0(X, mrK_X) = H^0(X^0, (\Omega^n_{X^0})^{\otimes rm})$ and the same holds for Y^0. Then X and Y have the same Kodaira dimension since

$$H^0(Y, mrK_Y) \subset H^0(X, mrK_X) \subset H^0(Y, dmrK_Y).$$

Remark 3.1. Assume that Y is smooth: then a quasi-étale morphism $f : X \to Y$ with X normal is étale.

Proof. In fact, $\pi_1(Y^0) \cong \pi_1(Y)$, thus there is an étale covering $W \to Y$ such that X^0 and W^0 are isomorphic. E.g. by Zariski's main theorem, the birational map induces a morphism $f : X \to W$. Moreover, cf. Theorem 7.17 of [**Hart**], and Exercise 7.11 (c), p. 171, X is the blow up of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_W$ such that supp$(\mathcal{O}_W/\mathcal{I})$ has codimension ≥ 2 and contained in $W - W^0$. But then, since the pull back of $\mathcal{I}$ is invertible, we contradict cod$(X - X^0) \geq 2$ unless $\mathcal{I}$ is equal to $\mathcal{O}_W$, that is, $X \cong W$. q.e.d.

Proposition 3.2. *Let $f : X \to Y$ be a quasi-étale morphism (of normal varieties) and let $f \circ g : W \to X \to Y$ be the Galois tower of f. I.e., over $\mathbb{C}$, we let $W^0 \to X^0 \to Y^0$ be the sequence of étale coverings corresponding to the biggest normal subgroup contained in $Im(\pi_1(X^0) \to \pi_1(Y^0))$, and let $g : W \to X$ be the corresponding finite normal ramified covering.*

It is clear that $W \to Y$ is quasi-étale, and we claim that if X, Y have canonical singularities, then also W does.

Proof. K_W is $\mathbb{Q}$-Cartier since $rK_W|W^0 = g^*(rK_X|X^0)$ for each positive integer r, thus $rK_W = g^*(rK_X)$ as Weil divisors, and it suffices to take r such that rK_X is Cartier.

Let moreover $\pi_X : X' \to X, \pi_W : W' \to W$ be respective resolutions such that $g \circ \pi_W$ factors as $\pi_X \circ g'$ for some morphism $g' : W' \to X'$: then we have

$$\begin{aligned}\pi_W^*(rK_W) &= \pi_W^* g^*(rK_X) = (g')^* \pi_X^*(rK_X) \\ &= (g')^*\Big(rK_{X'} - \sum_j a_j E_j\Big) = \Big(r(K_{W'} - R) - \sum_j a_j (g')^* E_j\Big),\end{aligned}$$

where $a_j \geq 0$ and R is the ramification divisor (an effective divisor): whence condition 3 is satisfied. q.e.d.

Corollary 3.3. *Let $f : X \to Y$ be a quasi-étale morphism, with X smooth and Y normal. Then there is an étale covering W of X and an action of a finite group G on W, free in codimension 1, such that Y is a birational image of W/G under a small contraction.*

Moreover, if Y has canonical singularities, then also W/G has canonical singularities.

Proof. Let $g : W \to X$ be as in the previous proposition; thus g is étale.

Letting G be the Galois group of f, we obtain a birational morphism $\phi : W/G \to Y$ such that $f \circ g = \phi \circ p$, $p : W \to W/G$ being the quotient projection.

Now the birational morphism ϕ induces an isomorphism $W^0/G \to Y^0$, i.e., outside an algebraic set of codimension ≥ 2 in W^0/G, which means that the contraction is small.

It is clear that G acts freely in codimension 1, whence it follows that $K_{W/G}$ is $\mathbb{Q}$-Cartier, and since $\phi^*(K_Y) = K_{W/G}$ we obtain that property 3 is satisfied. q.e.d.

The importance of the previous corollary lies in the fact that the conditions that W be smooth, and W/G have canonical singularities, impose restrictions on the actions of the stabilizers G_w, for $w \in W$.

Remark 3.4. By a well known lemma by H. Cartan, if a finite group H acts on a smooth germ $(\mathbb{C}^n, 0)$, then we may assume, up to a biholomorphism of the germ, that the action is linearized, i.e., $H \subset GL(n, \mathbb{C})$.

The assumption that H acts freely in codimension 1 is equivalent to the condition that H contains no pseudoreflections.

The condition that the germ $\mathbb{C}^n/H$ has canonical singularities (cf. [**Reid2**], Exercise 1.10, p. 352) is easily characterized only for $n = 2$: this happens iff $H \subset SL(2, \mathbb{C})$.

The analytic singularities $\mathbb{C}^2/H$, $H \subset SL(2, \mathbb{C})$, are precisely the Rational Double Points.

Concerning the varieties with Kodaira dimension $\mathcal{K} = -\infty$, it is conjectured that they are precisely the **uniruled** varieties (cf. e.g., [**Kol**] 1.12, p. 189), i.e., the varieties X of dimension n such that there exists a dominant (and separable if char $\neq 0$) rational map $Y \times \mathbb{P}^1 \to X$, where $\dim Y = n - 1$. The current status of the conjecture is as follows: the main result of Boucksom, Demailly, Paun and Peternell in [**BDPP**] is that X is uniruled if and only if K_X is not *pseudo-effective* (a divisor is said to be pseudo-effective if it is in the closure of the cone of effective divisors, and the crucial result of [**BDPP**] is that the cone of pseudo-effective divisors is dual to the cone of 'movable' curves, i.e., curves which cover X).

In the context of the Q.E.D. problem, observe first that, by a result of Fujiki and Levine (cf. [**Fuj**], [**Lev**], and cf. also Chapter IV of [**Kol**]), the class of uniruled varieties is stable by deformation, at least over $\mathbb{C}$.

Then the following proposition (whose proof uses a precious suggestion by Thomas Peternell) ensures that the family of complex uniruled varieties is stable by Q.E.D. equivalence:

Proposition 3.5. *Let $f : X \to Y$ be a quasi-étale morphism, with Y uniruled. Then, if X, Y have canonical singularities, also X is uniruled.*

Proof. By the theorems of Miyaoka and Mori, resp. Miyaoka ([**M-M**] and [**Miya2**], cf. also [**Kol**], Thm. 1.16, page 191), if we have a smooth projective variety Z of dimension n, then Z is separably uniruled if and only if there is a covering family of curves $C_t, t \in T$, with $C_t \cdot K_Z < 0$.

Let first $Z = \tilde{Y}$ be a resolution of Y. Then, if Y is uniruled, there is such a family of curves C_t on $Z = \tilde{Y}$: let us push down this family to a covering family of curves D_t on Y. By property 3 of canonical singularities, we obtain that $D_t \cdot K_Y < 0$.

Let Γ_t be the family of proper transforms of the curves D_t on X: since f is quasi- étale, it follows from the projection formula that $\Gamma_t \cdot K_X < 0$.

Let us consider a resolution X' of X; then by [**BDPP**] it suffices to show that $K_{X'}$ is not pseudoeffective.

Otherwise, if L is an ample divisor on X, then for $m, N \gg 0$ the linear system $|N(mK_{X'} + \pi^* L)|$ is effective and big (yields a birational map). In particular, if we denote by Δ_t the proper transform of a general Γ_t, the intersection number $(mK_{X'} + \pi^* L) \cdot \Delta_t \geq 0$.

This is, however, a contradiction, since for m sufficiently divisible and by the projection formula it follows that $|N(mK_{X'} + \pi^* L)| = \pi^* |N(mK_X + L)|$. Whence, by the projection formula, $(mK_{X'} + \pi^* L) \cdot \Delta_t = (mK_X + L) \cdot \Gamma_t$. However, this last number is negative for $m \gg 0$, a contradiction. q.e.d.

4. Proof of the main theorems 1.3 and 1.5

Proof of Theorem 1.3. We must show that, if S and S' are smooth Kähler surfaces which have the same Kodaira dimension $\mathcal{K} \leq 1$, then S and S' are $\mathbb{C} - Q.E.D.$-equivalent.

We proceed distinguishing the several cases, according to the value of the Kodaira dimension $\mathcal{K}$.

<u>$\mathcal{K} = -\infty$.</u>

In this case S is projective algebraic (since by the Kähler assumption $b^+ > 0$, $p_g(S) = 0$ implies that there is a positive line bundle) and it suffices to show that S is A.Q.E.D. - equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$.

But S is birational to a product $C' \times \mathbb{P}^1$, and the curve C' is deformation equivalent to a hyperelliptic curve C. Let ι be the hyperelliptic involution, and let $j : \mathbb{P}^1 \to \mathbb{P}^1$ be the involution such that $j(z) = -z$.

We have an action of $\mathbb{Z}/2$ on $C \times \mathbb{P}^1$ provided by $\iota \times j$, which has only isolated fixed points. Set $X := (C \times \mathbb{P}^1)/(\mathbb{Z}/2)$: there is a fibration $f : X \to C/(\mathbb{Z}/2) \cong \mathbb{P}^1$ with fibres $\cong \mathbb{P}^1$, thus by Noether's theorem X is birational to $\mathbb{P}^1 \times \mathbb{P}^1$. q.e.d.

Remark 4.1. More generally, let C_1, C_2 be hyperelliptic curves, so that we have an action of $(\mathbb{Z}/2)^2$ on $C_1 \times C_2$, and let us consider the diagonal embedding of $(\mathbb{Z}/2) \subset (\mathbb{Z}/2)^2$. Set $X := C_1 \times C_2$, $Y := (C_1 \times C_2)/(\mathbb{Z}/2)$ and observe that $(C_1 \times C_2)/(\mathbb{Z}/2)^2 \cong (\mathbb{P}^1 \times \mathbb{P}^1)$.

We have $f : X \to Y, p : Y \to \mathbb{P}^1 \times \mathbb{P}^1$, and Y has only nodes as singularities, while f is quasi-etále, so that X is A.Q.E.D. equivalent to Y. On the other hand, the branch locus B' consists (if we denote by g_i the genus of C_i) of the union of $(2g_1 + 2)$ vertical lines with $(2g_2 + 2)$ horizontal lines.

Let S be a double cover of $(\mathbb{P}^1 \times \mathbb{P}^1)$ branched on a smooth curve B of bidegree $(2g_1 + 2, 2g_2 + 2)$: since B is a deformation of B', S is a deformation of Y, and hence it is A.Q.E.D. equivalent to X.

Observe that the composition of the double cover $S \to (\mathbb{P}^1 \times \mathbb{P}^1)$ with the second fibration yields a fibration of hyperelliptic curves of genus g_1.

For $g_1 = g_2 = 1$ we get a particular case of the A.Q.E.D. equivalence, via Kummer surfaces, between Abelian surfaces and K3 surfaces (here, K3 surfaces which are a double cover of $(\mathbb{P}^1 \times \mathbb{P}^1)$).

We proceed now with the proof of the next cases:

<u>$\mathcal{K}= 0$.</u>

Recall that a minimal compact complex surface with Kodaira dimension 0 is either

a) a complex torus,
b) a K3 surface,
c) an Enriques surface,
d) a hyperelliptic surface,
e) a Kodaira surface.

Cases a)–d) consist of Kähler surfaces, while in case e), $b_1(S) = 3$ if S is primary, $b_1(S) = 1$ if S is secondary, and a fortiori a Kodaira surface is not Kaḧler.

In cases c) and d) S is projective and it has a finite étale covering $f : S' \to S$ where:

c) If S is Enriques, $\deg(f) = 2$ and S' is a K3 surface.
d) If S is hyperelliptic, $\deg(f) | 12$ and S' is a product of elliptic curves.

By virtue of the previous remark 4.1 the proof is concluded since it is well known that all complex tori are deformation equivalent, and Kodaira proved ([**Kod64**]) that every K3 surface is deformation equivalent to a smooth quartic surface in $\mathbb{P}^3$. q.e.d.

$\mathcal{K}= 1$.

Recall that a complex surface of Kodaira dimension 1 is properly (canonically) elliptic, i.e., it admits a (pluri-) canonical elliptic fibration $f : S \to B$.

Step I) We show first, replacing S by a finite unramified covering, that we may assume that S has an elliptic fibration without multiple fibres, unless we are in the following

EXCEPTIONAL CASE*: $f : S \to \mathbb{P}^1$ has at most two multiple fibres, with coprime multiplicities, and S is simply connected.

Proof of I: We use (cf. [**CKO**], Lemma 3 and Theorem A for a similar idea, and also e.g., [**Cat4**, 4.1, 4.2]) the orbifold fundamental group sequence

$$\pi_1(F) \cong \mathbb{Z}^2 \to \pi_1(S) \to \pi_1^{orb}(f) \to 1$$

where F is a smooth fibre $F_b := f^{-1}(b)$, $F_{b_1}, \dots F_{b_r}$ are the multiple fibres of respective multiplicities $m_1, \dots m_r$, and $\pi_1^{\mathrm{orb}}(f)$ is defined as the quotient $\pi_1(B - \{b_1, \dots b_r\})/ \ll \gamma_1^{m_1}, \dots \gamma_r^{m_r} \gg$ of $\pi_1(B - \{b_1, \dots b_r\})$ by the subgroup normally generated by the respective m_j-th powers of simple geometric loops γ_j around the respective points b_j.

Note that the image of γ_j inside $\pi_1^{\mathrm{orb}}(f)$ has order precisely m_j unless we are in the exceptional case where $B \cong \mathbb{P}^1$ and $r \leq 2$ (for $r = 1$ the group is trivial, else for $r = 2$ it is cyclic of order $= G.C.D.(m_1, m_2)$).

It is also known (cf. [**CKO**], loc. cit.) that, if we are not in the exceptional case, there is a finite quotient G of $\pi_1^{\mathrm{orb}}(f)$ where the image of each γ_j has order precisely m_j. To this surjection corresponds an unramified covering such that the normalization of the pull back of f is an elliptic fibration without multiple fibres.

In the exceptional case with two multiple fibres we may take a cyclic cover $\mathbb{P}^1 \to \mathbb{P}^1$ of order $= G.C.D.(m_1, m_2)$, branched on the two points corresponding to the multiple fibres, so that the normalization S' of the pull-back has two multiple fibres whose multiplicities m'_1, m'_2 are coprime, whence $\pi_1^{\mathrm{orb}}(f')$ is trivial, and $\mathbb{Z}^2 \to \pi_1(S')$ is surjective. Since we assume the first Betti number to be even, and since by [**Dolg**], Theorem on page 137, $\pi_1^{\mathrm{orb}}(f)$ contributes here to the torsion subgroup of $H^1(S, \mathbb{Z})$, we infer that also S' has even $b_1(S')$. But if $b_1(S') = 2$ S' is a trivial fibration, contradicting $Kod(S) = 1$. Thus $\pi_1(S')$ is finite abelian and passing to the universal cover we find ourselves in the EXCEPTIONAL CASE* (actually, one can indeed show that S' is itself simply connected).

Step II) Let q be the irregularity of S: if $q = g(B) + 1$, where $g(B)$ is the genus of B, then S is a product $B \times F$, where F is a smooth fibre, and B is a curve of genus $g \geq 2$. Thus we have only one A.Q.E.D.-equivalence class, to which we shall show that all the other cases are $\mathbb{C}$-Q.E.D. equivalent.

Step III) Consider now any two numbers g, q with $g \geq 2q$: then there is a ramified double covering $j : B \to C$ where B has genus g and C has genus q . Consider then a product $B \times F$ as before, and use the trick of 4.1, to obtain an elliptic surface without multiple fibres $S \to C$ with $q(S) = q$, $p_g(S) = g - q \geq q$.

Step IV) Assume that S is an elliptic surface without multiple fibres and with topological Euler number $e(S) = 0$: then, by the Zeuthen-Segre theorem (cf. [**BPV**], 11.5, p. 97), all the fibres of $S \to C$ are smooth. It follows then that the j-invariant is constant, since $j : C \to \mathbb{C}$ is holomorphic; thus all the fibres are isomorphic and we have a holomorphic fibre bundle.

In this case the Jacobian elliptic surface J associated to S has an étale cover which is a product (cf. e.g., [**BPV**], (2) p. 143, since there is an étale cover of C which pulls back a principal bundle with a section, cf. also [**F-M**], Section 1.5.4) and by Theorems 11.9 and 11.10 of [**Kod63**] (cf. also Theorem 11.1 of [**BPV**]) it follows that, since $b_1(S)$ is even, then S is a complex deformation of J: hence S is in the $\mathbb{C} - Q.E.D.$ class II).

Step V) We may then assume that, if the elliptic surface S has no multiple fibres, then it has topological Euler number $e(S) > 0$: by the Noether formula $12\chi(\mathcal{O}_S) = K_S^2 + e(S)$, and since $K_S^2 = 0$ (S is minimal), this means that $p_g(S) \geq q(S)$.

We use now Theorem 7.6 of [**F-M**], asserting that two complex elliptic surfaces without multiple fibres, with $e(S) > 0$, and with the same q, p_g are complex deformation equivalent.

By Step III, it then follows that such a surface is $\mathbb{C} - Q.E.D.$-equivalent to a product $B \times F$, and we have therefore shown that there is only one $\mathbb{C} - Q.E.D.$ class, unless possibly if we are in the EXCEPTIONAL CASE*, which we treat next.

Step VI) Assume now that S is simply connected, and that $f : S \to \mathbb{P}^1$ has multiple fibres, and at most two, of coprime orders $1 \leq m_1 < m_2$. A further invariant of f is the geometric genus $p_g(S) = \frac{1}{12}e(S) - 1$.

Two such surfaces with the same invariants $p_g(S), m_1, m_2$ are known to be complex deformation equivalent (cf. [**F-M**], Theorem 7.6).

Therefore it suffices to find, for each choice of $p_g(S), m_1, m_2$ as above, one such exceptional elliptic fibration $f : S \to \mathbb{P}^1$ which is QED-equivalent to one without multiple fibres.

To this purpose it suffices to find a divisor D contained in a finite union of fibres and which is a disjoint union of (connected) (-2)-curve configurations $D_1, \dots D_k$, such that the open surface $S^0 := S - D$ has now at least three multiple fibres. In fact there exists then a birational morphism $\pi : S \to X$ contracting the configurations $D_1, \dots D_k$, to Rational Double Points $p_1, \dots p_k$, and a fibration $\tilde{f} : X \to \mathbb{P}^1$ with three multiple fibres.

Consider the orbifold fundamental group exact sequence for the open surface $S^0 := S - D$, as in [**CKO**]: then there exists an unramified covering S'^0 of S^0 which yields an elliptic fibration without multiple fibres.

As S^0 is the complement of a finite set of Rational Double Points in the surface X, similarly S'^0 is the complement of a finite set in a surface Y with Rational Double Points, mapping in a quasi-étale way to X.

We are then able to conclude as in Step I) that S is $\mathbb{C} - QED$-equivalent to the minimal resolution S' of Y, which has an elliptic fibration without multiple fibres.

Let us show the existence, for given $p_g(S), m_1, m_2$, of an exceptional elliptic surface with those invariants and moreover with two singular fibres (only one suffices indeed if $m_1 > 1$) whose extended Dynkin diagram is of type $\tilde{D}_n, n \geq 4$ (also type $\tilde{E}_n, n = 6, 7, 8$ would do).

Observe that the value of $p_g(S)$ is determined by $e(S)$, and that logarithmic transformations do not change $e(S)$.

We are then reduced to show the existence of a simply connected elliptic fibration (over $\mathbb{P}^1$) with at least two singular fibres whose extended Dynkin diagram is of type $\tilde{D}_4$.

As in Remark 4.1, let us consider a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on a divisor B of bidegree $(2g+2, 4)$. If B is smooth we get a simply connected elliptic surface S with $p_g(S) = g$. If $g \geq 1$ it is easy to show that we may obtain a branch curve B' with two ordinary triple points: then the double covering surface S' gets two singular fibres of type $\tilde{D}_4$.

If $g = 0$ we obtain B' as the union of four divisors: L_1, L_2 of bidegree $(0, 1)$ and D_1, D_2 of bidegree $(1, 1)$.

Viewing in fact $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric in $\mathbb{P}^3$, letting L_1, L_2 be two disjoint sections, and fixing $P_1 \in L_1, P_2 \in L_2$, we choose D_1, D_2 as two general plane conic sections through P_1, P_2. The proof of Theorem 1.3 is complete. q.e.d.

Proof of Theorem 1.5. We essentially rerun the proof of 1.3, mutatis mutandis.

For $\mathcal{K} = -\infty$ the Kähler surfaces are algebraic and the proof is already there.

For $\mathcal{K} = 0$ we simply have to observe that:

1) any abelian surface is an algebraic deformation of a product of elliptic curves,
2) two algebraic K3 surfaces are algebraic deformations of each other.

Statement 1) is easy, in any dimension n, since Abelian varieties with a polarization of type $(d_1, d_2, \dots d_n)$ are parametrized by a quotient of the Siegel upper halfspace, and a product of elliptic curves clearly admits such a polarization.

The case of K3 surfaces is similar and requires the Torelli theorem ([**PS-Shaf**], cf. also [**K3**], expose' XIII): again we have an irreducible subvariety parametrizing the projective K3 surfaces with a (primitive) pseudopolarization of degree d, and inside this family we find the special Kummer surfaces, i.e., more precisely, the K3 surfaces obtained as the minimal resolution of the Kummer surface of a product of elliptic curves.

Let's consider now the case $\mathcal{K} = 1$.

Steps I, II, III are identical.

Step IV: Assume $e(S) = 0$; then $f : S \to C$ is a holomorphic bundle and there is (cf. [**BPV**], p. 143) an étale covering of the base C such that the pull back is a principal holomorphic bundle with cocycle ξ whose cohomological invariant $c(\xi) = 0$ (else, by [**Kod60**] Theorem 11.9, cf. also [**BPV**] Prop. 5.3, p. 145, $b_1(S) \equiv 1 (\mathrm{mod}\, 2)$, contradicting the algebraicity of S).

Let F be the fibre of f: then (cf. also [**F-M**] p. 92) there is a finite homomorphism $\pi_1(C) \to F$ classifying f, and taking the associate étale cover C', we obtain an étale covering $S' \to S$ which is indeed a product.

Step V: Recall that a Jacobian elliptic surface is algebraic. As shown by Seiler (cf. [**Sei**], and also [**Kas**] or [**Mir**] for an introduction to the subject) all Jacobian elliptic fibrations which are not a product and have the same invariants $q(S), p_g(S)$ belong to an irreducible algebraic family.

Therefore, any Jacobian elliptic surface is an algebraic deformation of some Jacobian surface with constant invariant j obtained from construction 4.1 as in Step III).

Let us use the fact that the base space of a maximal family of algebraic elliptic surfaces is a finite covering of the corresponding base space of the corresponding family of Jacobian elliptic surfaces (cf. [**Sei**], and also [**F-M**] Prop. 5.30, p. 93). This is derived from Kodaira's theorem 11.5 of [**Kod60**] asserting that if S is an algebraic elliptic fibration without multiple fibres, then the corresponding cohomology class η is torsion, and conversely.

We conclude that an algebraic elliptic fibration without multiple fibres $f : S \to C$ is algebraic deformation of one with constant moduli and with multiplication by ± 1. Whence, a double étale covering of the base yields a double étale covering of S which is a holomorphic bundle. We are done by Step IV.

Step VI): We are in the exceptional case where $f : S \to \mathbb{P}^1$ has multiple fibres, of coprime multiplicities m_1, m_2 with $1 \leq m_1 < m_2$.

We are done once we can show the validity of the following:

Claim. Assume that we have an algebraic exceptional elliptic surface $S \to \mathbb{P}^1$, i.e., with $r \in \{1, 2\}$ multiple fibres, of coprime multiplicities m_1, m_2 with $1 \leq m_1 < m_2$. Then there exists an algebraic deformation of S yielding a surface S' with $3 - r$ singular fibres of type $\tilde{D}_4$.

Proof of the claim. We argue as in Step VI of the previous Theorem 1.3, using the characterization of algebraicity of logarithmic transforms given in [**F-M**], Lemma 6.13, p. 106, and which is a translation in complex geometry of the theory of Ogg-Shafarevich (cf. [**Shaf1**], [**Dolg**]).

Let us then consider an algebraic exceptional elliptic surface $\phi : S \to \mathbb{P}^1$, and let $\psi : X \to \mathbb{P}^1$ be its Jacobian fibration. By doing the inverse of logarithmic transformations, we obtain an elliptic fibration $f : Y \to \mathbb{P}^1$, whose Jacobian fibration is X.

The following is the content of the cited Lemma 6.13 of [**F-M**].

Remark 4.2. Let $f' : Y' \to B$ be another elliptic surface whose Jacobian fibration is X, and let $\phi' : S' \to B$ be obtained from Y' via the same logarithmic transformations as the ones constructing S from Y (i.e., at the fibres over the same points, and with the same associated torsion bundles): then S' is algebraic if and only if the difference of the corresponding elements in the classifying group $H^1(\mathbb{P}^1, \mathcal{O}(X^*))$ ($\mathcal{O}(X^*)$ is Kodaira's sheaf of groups of local holomorphic sections) is torsion.

Arguing as in Step VI of 1.3, there is an algebraic 1-parameter family of Jacobian elliptic surfaces $X_t, t \in T$, containing the given X and a special X_0 which has $3 - r$ singular fibres of type $\tilde{D}_4$.

Let us treat now the case $3 - r = 1$. Let us consider the elliptic surfaces $Y_{t,w}, w \in W_t$, without multiple fibres having some X_t as Jacobian elliptic surface and the family of single logarithmic transforms of the surfaces $Y_{t,w}$: this family is parametrized by a complex variety Z where $Z \to W$ has pure and irreducible 1-dimensional fibres.

Inside this family we consider the subfamily of the algebraic elliptic surfaces: these form a countable union of subvarieties fibred over our irreducible curve T, whence, up to replacing T by an irreducible finite covering of it, we may find, given our initial S, a 1-parameter complex family S_t containing S and such that the corresponding family of Jacobian surfaces is X_t.

This shows that we have such a complex family of algebraic surfaces. In order to show that we have an algebraic family we only need to observe that our elliptic surfaces all have a multisection D of a fixed degree, whence for very large $n, m \in \mathbb{N}$ $|nD + mF|$ is very ample on each surface, and we get a non trivial complex curve in a Hilbert scheme of projective surfaces. We only need to remark that if two points of a Hilbert scheme are joined by a complex curve, they are also joined by an algebraic curve.

The argument for the case $3 - r = 2$ is entirely similar, whence our claim is proven, together with Theorem 1.5. q.e.d.

5. Appendix due to Fritz Grunewald: Q.E.D. classes constructed from quaternion algebras.

As in [**Shav**] (cf. also [**Shim**], Chapter 9) we consider a division quaternion algebra $\mathcal{A}$ with centre a totally real number field k. For simplicity, we may further assume k to be a real quadratic field.

We assume further that $\mathcal{A}$ is totally indefinite: this means that, for each of the two embeddings $j : k \to \mathbb{R}$, $\mathcal{A}$ does not ramify, i.e., $\mathcal{A} \otimes_j \mathbb{R} \cong M(2, \mathbb{R}) := Mat(2 \times 2, \mathbb{R})$.

As usual, denoting by $\mathcal{O}_k$ the ring of integers of k, and by $k_{\mathcal{P}}$ the local field which is the completion of the localization $\mathcal{O}_{\mathcal{P}}$ of the ring $\mathcal{O}_k$ at a prime ideal $\mathcal{P}$, one considers the set of primes where $\mathcal{A}$ ramifies, i.e., the subset

$$\mathcal{S}(\mathcal{A}) := \{\mathcal{P} \in \operatorname{Spec}(\mathcal{O}_k) | \mathcal{A} \otimes_k k_{\mathcal{P}} \text{ is a skew field }\}.$$

By the classical results of Hasse (which are exposed for instance in the book [**Weil**], cf. especially Th. 2 of Chapter XI-2, and Theorem 4, Section 6 of Chapter XIII) we know that (cf. also [**Shim**], Section 9.2, pp. 243-246):

- s-1) The cardinality of $\mathcal{S}(\mathcal{A})$ is finite and even (and nonzero since $\mathcal{A}$ is a division algebra),
- s-2) $\mathcal{A}$ is completely determined by its centre k and by $\mathcal{S}(\mathcal{A})$,
- s-3) for each choice of k and of a set $\mathcal{S} \subset Spec(\mathcal{O}_k)$ with even cardinality, there is a quaternion algebra over k with $\mathcal{S}(\mathcal{A}) = \mathcal{S}$.

Remark 5.1. Usually one also considers inside $\mathcal{S}(\mathcal{A})$ the places at infinity (embeddings of k into $\mathbb{R}$), and the result holds more generally. Since, however, we assume the quaternion algebra to be totally indefinite, there are no ramified places at infinity.

Now let $\mathcal{R} \subset \mathcal{A}$ be a maximal order (an order, cf. [**Weil**], Def. 2, p. 81, is a subring which is a $\mathbb{Q}$-lattice for $\mathcal{A}$) and consider the group

$$\Gamma(1) := \{a \in \mathcal{R} |\ nr(a) = 1\},$$

where nr denotes the reduced norm ([**Weil**], IX-2).

The following facts are also well known (cf. [**Shav**], Section 1, and [**Shim**], 9.2).

Remark 5.2.

1) If k is a quadratic field, and j_1, j_2 are the two embeddings $k \to \mathbb{R}$, then $\Gamma(1) \cong (j_1 \times j_2)(\Gamma(1)) \subset SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.
2) The image Γ of $\Gamma(1)$ in $\mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ is isomorphic to $\Gamma(1)/\{\pm 1\}$.
3) Γ operates properly discontinuously with compact projective quotient on the product $\mathbb{H}^2$ of two upper -halfplanes ($\mathbb{H} := \{z \in \mathbb{C} | Im\ z > 0\}$).
4) The action of Γ on $\mathbb{H}^2$ is irreducible; whence, if the action of Γ is free, then the projective surface $X := \mathbb{H}^2/\Gamma$ is strongly rigid (cf. [**J-Y**], and also [**Cat0**]), i.e., every surface S with the same Euler number as X and with isomorphic fundamental group $\pi_1(S) \cong \pi_1(X)$ is either biholomorphic to X or to the complex conjugate surface $\overline{X}$.
5) Assume the quotient $X := \mathbb{H}^2/\Gamma$ to be smooth: then its first Betti number equals zero (Proposition 2.1 of [**Shav**], which follows by the theorem of Matsushima and Shimura), and by Hirzebruch's proportionality principle, we have $e(X) = 2 + b_2(X) = 4(1 + p_g(X))$.

Lemma 5.3. *Let $\Gamma'' \subset \Gamma(1)$ be a finite index subgroup. Then the $\mathbb{Q}$-linear span of Γ'' equals $\mathcal{A}$.*

Proof. Replacing Γ'' by a subgroup of finite index (since $\Gamma(1)$ is finitely generated), we may assume that Γ'' is a normal subgroup of $\Gamma(1)$ and invariant by the involution of $\mathcal{A}$ sending an element to its conjugate (it has k as set of fixed points).

As a first step, let's now prove that

i) the $\mathbb{Q}$-linear span G of Γ'' contains $k = k \cdot 1$.

In order to show this, let us consider an element $\alpha \in k$ subject to the conditions:

- α is totally positive,
- $k(\sqrt{\alpha}) \otimes_k k_{\mathcal{P}}$ is a field for every $\mathcal{P} \in \mathcal{S}(\mathcal{A})$,
- there is a finite place $\mathcal{Q} \notin \mathcal{S}(\mathcal{A})$ of k such that $k_{\mathcal{Q}}$ is an extension field of degree 2 over the corresponding completion of $\mathbb{Q}$ and such that α is not a square in $k_{\mathcal{Q}}$,

and set

$$L := k(y), \qquad (y^2 = \alpha).$$

The existence of such an α is guaranteed by the weak approximation theorem (cf. 18.3 Exercise 2, p. 351 of [**Pie**]) which says that k is dense when diagonally embedded into the direct product of any finite subset of the set of its completions. Note also that the set of non-squares is open in any completion of k.

The field L has the following properties:

- L is totally real,
- L is isomorphic to a subfield of $\mathcal{A}$,
- the only subfield of L which is of degree 2 over $\mathbb{Q}$ is k.

The second condition follows from the second condition on α, cf. Prop. 4.5 of [**Shav**].

Then we have a chain of degree 2 extensions $\mathcal{A} \supset L \supset k$, and k is the only quadratic subfield of L; in fact the Galois group of the splitting field of L over $\mathbb{Q}$ is either cyclic of order 4 or is the dihedral group D_4.

Now, consider $\Gamma'' \cap L$: we claim that $\Gamma'' \cap L$ contains a nontrivial infinite cyclic subgroup generated by a unit ϵ.

In fact, the maximal order $\mathcal{R}$ of $\mathcal{A}$ intersects L in an order $\mathcal{B}$ of L, and by Dirichlet's Theorem (cf. e.g., Theorem 5 of Section II, 4 of [**B-S**], p. 113) the group of units in $\mathcal{B}$ has rank 2; since by the same theorem the group of units in $\mathcal{B} \cap k$ has rank 1, the group of units of norm 1 in $\mathcal{B}$ contains an infinite cyclic group, which in turn intersects Γ'' in an infinite cyclic group.

Note that $\epsilon \notin k$, since for elements of k the norm is just given by the square, and $\epsilon \neq \pm 1$.

Thus $\epsilon, \overline{\epsilon} \notin k$ but, clearly, $(\epsilon + \overline{\epsilon}) \in G \cap k$ and we claim that $(\epsilon + \overline{\epsilon}) \notin \mathbb{Q}$.

Otherwise, if $(\epsilon + \overline{\epsilon}) \in \mathbb{Q}$, since $\epsilon \cdot \overline{\epsilon} = 1$ it follows that $\epsilon, \overline{\epsilon}$ belong to a quadratic extension of $\mathbb{Q}$. But this quadratic extension, being contained in L, would then equal k, contradicting our previous assertion.

We conclude then that $(\epsilon + \overline{\epsilon}) \in k \setminus \mathbb{Q}$, thus G contains k.

ii) Now that we know that the $\mathbb{Q}$-linear span G of Γ'' contains $k \cdot 1$, we can show that G is a field. In fact we observe that G is a ring, which is invariant by the involution of $\mathcal{A}$: thus if G contains x, it contains also $x^{-1} = \bar{x} \cdot rn(x)^{-1}$.

iii) Set $d := \dim_k(G)$: then $d | 4$, and if $d = 4$ there is nothing to prove. If instead $d \leq 2$, then G is commutative, hence also Γ'' is commutative.

This gives however two contradictions:

1) since we know that there exists a finite index subgroup of Γ'' which is infinite, operates freely on $\mathbb{H}^2$, and has a quotient X having a finite homology group,

2) since we know that Γ'' is Zariski dense in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, by 5.4. q.e.d.

Lemma 5.4. *Any finite index subgroup* Γ'' *of* $\Gamma(1)$ *is Zariski dense in* $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. $\mathcal{A}^* := \mathcal{A} \setminus \{0\}$ *is Zariski dense in* $GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$.

Proof. The first assertion is a special case of a general theorem by Armand Borel (cf. [**Bor**]). In fact Γ'' has the Selberg property since the quotient $\mathbb{H}^2/\Gamma''$ is compact, hence Γ'' is not contained in any proper subgroup having a finite number of components.

The second assertion follows immediately since $\Gamma(1)$ is Zariski dense in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ and $\mathcal{A}\supset k$. We would also like to give an elementary proof of the Zariski density in $GL(2,\mathbb{C})\times GL(2,\mathbb{C})$, as follows.

Define $\mathcal{A}_1^* := \overline{\mathcal{A}^*}\cap(GL(2,\mathbb{C})\times\{1\})$, and $\mathcal{A}_2^*$ similarly (where $\overline{\mathcal{A}^*}$ denotes the Zariski closure).

We observe that if $\mathcal{A}_1^* = GL(2,\mathbb{C})$, then by extending to $\mathbb{C}$ the Galois automorphism of k, we see that also $\mathcal{A}_2^* = GL(2,\mathbb{C})$, thus there is nothing left to prove.

Else, both $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ are proper algebraic subgroups of $GL(2,\mathbb{C})$, thus they are both solvable. Each of the respective projections $p_i(\overline{\mathcal{A}^*})\subset GL(2,\mathbb{C})$ is surjective (by two reasons: if it would yield a proper subgroup it follows that $\overline{\mathcal{A}^*}$ is solvable, a contradiction; or, just use that $\mathcal{A}\otimes_j\mathbb{C} = GL(2,\mathbb{C})$).

Again by extending to $\mathbb{C}$ the Galois automorphism of k we also see that $\overline{k}$ equals the centre $\mathbb{C}^*\times\mathbb{C}^*$, and that $\overline{\mathcal{A}^*}$ contains a commutative subgroup of dimension equal to 4.

Thus, $\overline{\mathcal{A}^*}$ contains the direct product $T_1\times T_2$ of two respective maximal tori of $GL(2,\mathbb{C})$. Since $\overline{\mathcal{A}^*}$ projects onto $GL(2,\mathbb{C})$ by the first projection, it contains also the union of the conjugates of $T_1\times\{1\}$; thus, being closed, it also contains $GL(2,\mathbb{C})\times\{1\}$. Similarly, it contains $\{1\}\times GL(2,\mathbb{C})$ and we are done. q.e.d.

Lemma 5.5. *Let $\Delta\subset\mathbb{P}SL(2,\mathbb{R})\times\mathbb{P}SL(2,\mathbb{R})$ be a subgroup commensurable with Γ: then "$\Delta\subset\mathcal{A}$", more precisely $\Delta\subset(\mathbb{P}j_1\times\mathbb{P}j_2)(\mathcal{A})$.*

Proof.

Consider the inverse image $\Delta(1)$ of Δ inside $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$: then $\Delta(1)$ is commensurable with $\Gamma(1)$ and there is a finite index subgroup $\Gamma''\subset\Gamma(1)$ such that each $\delta\in\Delta(1)$ normalizes Γ''.

Thus δ normalizes the k-linear span of Γ'' inside $M(2,\mathbb{R})\times M(2,\mathbb{R})$. By the previous lemma, the $\mathbb{Q}$-linear span of Γ'' a fortiori equals $\mathcal{A}$.

It follows then that δ normalizes $\mathcal{A}$, and by the Skolem-Noether Theorem (cf. e.g., [**Blan**], Theorem III-4, p. 70) it follows that there is an element $\gamma\in\mathcal{A}$ such that conjugation of $\mathcal{A}$ by δ equals the inner automorphism associated to γ. Therefore we obtain that $\delta\gamma^{-1}$ centralizes $\mathcal{A}$.

Since, however (cf. lemma above), $\mathcal{A}$ is Zariski dense in $M(2,\mathbb{R})\times M(2,\mathbb{R})$, it follows that the element $\delta\gamma^{-1}$ lies in the centre $\{\pm1\}\times\{\pm1\}$ of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, whence the image of δ inside $\mathbb{P}SL(2,\mathbb{R})\times\mathbb{P}SL(2,\mathbb{R})$ lies in the image of $\mathcal{A}$. q.e.d.

Lemma 5.6. *Let $\delta\in\mathcal{A}\setminus\{1\}$ yield a transformation of $\mathbb{H}^2$ which has a fixed point: then the subfield $K_\delta := k[\delta]\subset\mathcal{A}$ is a cyclotomic extension $k[\zeta_m]$ where $m\in\{3,4,5,6,8,10,12\}$.*

The proof of the above lemma is contained in [**Shav**], Prop. 4.6, and in the considerations following it. The main idea is that if δ has a fixed point, then it has finite order, whence K_δ is a cyclotomic extension: but then the degree of the extension $\mathbb{Q} \subset \mathbb{Q}[\zeta_m]$ divides 4, and one concludes calculating the m's for which the Euler function $\phi(m)$ divides 4.

Definition 5.7. Consider the greatest common multiple 120 of the integers appearing in the previous lemma, and let K be the cyclotomic extension $k[\zeta_{120}]$.

For each intermediate field K', $k \subset K' \subset K$, choose a prime ideal $\mathcal{P}' \subset \mathcal{O}_k$ such that $\mathcal{P}'\mathcal{O}_{K'}$ is not primary. Such an ideal exists for each such K' by the density theorem (cf. [**Lang**]. VII, 4, p. 168) and guarantees that $k_{\mathcal{P}'} \otimes_k K'$ is not a field (indeed, it is not an integral domain).

Let $\mathcal{S}' := \{\mathcal{P}'\} \subset \mathrm{Spec}\,(\mathcal{O}_k)$, and take $\mathcal{S} \subset \mathrm{Spec}\,(\mathcal{O}_k)$ as a set of even cardinality containing $\mathcal{S}'$.

Theorem 5.8. *Let k be a real quadratic field, and let $\mathcal{A}$ be the indefinite division quaternion algebra corresponding, by Hasse's theorem, to a choice of $\mathcal{S}$ made as in* 5.7.

Then any subgroup $\Delta \subset \mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ commensurable with the subgroup Γ associated to a maximal order $\mathcal{R} \subset \mathcal{A}$ acts freely on $\mathbb{H}^2$.

Proof. Assume that $\delta \in \Delta$ is a nontrivial element which does not act freely. We have shown that $\delta \in \mathcal{A}$, and that K_δ is an intermediate field K' between k and $K := k[\zeta_{120}]$.

By our choice of $\mathcal{P}'$, it follows that $\mathcal{A} \otimes_k k_{\mathcal{P}'}$ is a division algebra; but on the other hand we have that $\mathcal{A} \otimes_k k_{\mathcal{P}'}$ contains $K_\delta \otimes_k k_{\mathcal{P}'} = K' \otimes_k k_{\mathcal{P}'}$ which is not an integral domain. This is a contradiction. q.e.d.

Hence follows

Theorem 5.9. *Let k be a real quadratic field, and let $\mathcal{A}$ be the indefinite division quaternion algebra corresponding, by Hasse's theorem, to a choice of $\mathcal{S}$ made as in* 5.7.

Define $\mathcal{F}_\mathcal{A}$ to be the family of subgroups $\Delta \subset \mathbb{P}SL(2,\mathbb{R}) \times \mathbb{P}SL(2,\mathbb{R})$ commensurable with a subgroup Γ associated to a maximal order $\mathcal{R} \subset \mathcal{A}$.

Each $\Delta \in \mathcal{F}_\mathcal{A}$ acts freely on $\mathbb{H}^2$, and denote by $S_\Delta := \mathbb{H}^2/\Delta$ the corresponding algebraic surface.

Then the family of surfaces $\{S_\Delta | \Delta \in \mathcal{F}_\mathcal{A}\}$ is a union of Q.E.D. equivalence classes.

Proof. Assume that a surface S is Q.E.D. -equivalent to S_Δ: then it is Q.E.D. -equivalent to S_Γ, whence it corresponds to a subgroup Δ' commensurable with Γ. q.e.d.

Corollary 5.10. *There are infinitely many Q.E.D.–equivalence classes of algebraic surfaces of general type.*

Proof. It suffices to observe that the fundamental group Δ of S_Δ has, by the cited theorem of Jost-Yau, at most two embeddings inside

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ with isomorphic image acting freely and cocompactly.

These two are conjugates of each other, and for both one sees by 5.3 that the $\mathbb{Q}$-linear span of Δ equals the $\mathbb{Q}$-linear span of Γ, which is indeed the embedded quaternion algebra $\mathcal{A}$.

Since $\mathcal{A}$ determines its centre k and its set of primes $\mathcal{S}(\mathcal{A})$, we see that to surfaces of the same Q.E.D. class corresponds the same pair $(k, \mathcal{S})$.

Since there are countably many choices for k and for $\mathcal{S}$, we conclude that there are infinitely (countably) many Q.E.D. classes. q.e.d.

6. Appendix by Sönke Rollenske: Q.E.D. for Kodaira surfaces

The aim of this appendix is to study the *Q.E.D.* equivalence relation for Kodaira surfaces. More precisely we want to prove the following

Theorem 6.1. *Let S be a minimal Kodaira surface. Then a smooth surface S' is Q.E.D. equivalent to S if and only if S' is itself a Kodaira surface. Thus Kodaira surfaces constitute a single Q.E.D. equivalence class.*

The only if part of the theorem is mostly an adaptation of notes of F. Catanese regarding the *Q.E.D.* equivalence for Hopf surfaces. Let us begin with some preliminary considerations.

The surfaces of Kodaira dimension zero which are not Kähler are called Kodaira surfaces. The minimal surfaces fall in the following two classes of which Kodaira gave an explicit description (cf. [**Kod64**], [**Kod66**] and [**BPV**]).

A minimal surface S of Kodaira dimension zero is called a *primary Kodaira surface* if one of the following equivalent conditions holds:

- The first Betti number $b_1 = 3$.
- S is a holomorphic principal bundle of elliptic curves over an elliptic curve, which is not topologically trivial.
- S is isomorphic to a Quotient $\mathbb{C}^2/G$ where G is a group of affine transformations generated by

  $$g_i : (z_1, z_2) \mapsto (z_1 + \alpha_i, z_2 + \bar{\alpha}_i z_1 + \beta_i), \qquad i = 1, \dots, 4$$

  with

  $$(*) \qquad \begin{gathered} \alpha_1, \alpha_2 = 0, \qquad \beta_1\bar{\beta}_2 - \beta_2\bar{\beta}_1 \neq 0, \\ \alpha_3\bar{\alpha}_4 - \alpha_4\bar{\alpha}_3 = m\beta_2 \neq 0, \end{gathered}$$

  where m is a positive integer. The global holomorphic forms on S are given by scalar multiples of (the classes of) dz_1 and $dz_1 \wedge dz_2$.

Sometimes a primary Kodaira surface admits a finite group of fixed point free automorphisms such that the quotient is an elliptic quasi-bundle over $\mathbb{P}^1$. Such a surface has $b_1 = 1$ and is called a *secondary Kodaira surface.*

A smooth surface S of Kodaira dimension zero is bimeromorphically equivalent to a Kodaira surface if and only if $b_1(S)$ is 1 or 3. From this we get immediately the following:

Corollary 6.2. *Assume that we have a flat family $F : \mathcal{X} \to \Delta$ over the unit disk with special fibre a compact complex surface X with canonical singularities, and with another fibre which is a smooth Kodaira surface S. Then X is bimeromorphic to a Kodaira surface.*

Proof. By Tyurina's result ([**Tyu**]) the minimal resolution Z of the singularities of X is a surface diffeomorphic to S and so has uneven Betti number. By Theorem S7 of [**F-M**] (p. 224) it also has Kodaira dimension zero and thus Z is a Kodaira surface. q.e.d.

Now let us analyse the automorphisms of minimal Kodaira surfaces by making use of the above description. Here we closely follow Kodaira: Let Γ be a finite group of automorphisms of $S = \mathbb{C}/G$. By pulling back to the universal covering we get an extension of finite index

$$1 \to G \to \Gamma' \to \Gamma \to 1$$

where Γ' is a group of automorphisms of $\mathbb{C}^2$. Now let $\phi = (\phi_1, \phi_2)$ be in Γ'. The linear action of Γ on $H^0(S, \Omega^p_S)$ becomes

$$\phi^* dz_1 = d\phi_1 = \sigma dz_1,$$
$$\phi^*(dz_1 \wedge dz_2) = d\phi_1 \wedge d\phi_2 = \kappa dz_1 \wedge dz_2 = \sigma dz_1 \wedge d\phi_2$$

with $\sigma, \kappa \in \mathbb{C}$ and consequently there exist a function $h(z_1)$ and a constant h_0 such that

$$\phi_1 = \sigma z_1 + h_0 \text{ and } \phi_2 = \frac{\kappa}{\sigma} z_2 + h(z_1).$$

Since G is a normal subgroup of finite index in Γ', we have $\phi^n \in G$ so σ is a root of unity and for every generator of G there exists an element $\tilde{g}_i(z_1, z_2) = (z_1 + a_i, z_2 + \bar{a}_i z_1 + b_i) \in G$ such that $\phi \circ g_i = \tilde{g}_i \circ \phi$. Calculating both sides we get $\sigma \alpha_i = a_i$ and also - after deriving the second part with respect to z_1:

$$h'(z_1) - h'(z_1 + \alpha_i) = \bar{\alpha}_i \left(\frac{\kappa}{\sigma} - \sigma \bar{\sigma} \right) = \bar{\alpha}_i \left(\frac{\kappa}{\sigma} - 1 \right).$$

Now $h''(z_1)$ is constant because it has two linear independent periods α_3, α_4 and consequently

$$-h''(z_1)\alpha_i = \bar{\alpha}_i \left(\frac{\kappa}{\sigma} - 1 \right), \qquad i = 3, 4$$
$$\Rightarrow h'' = \left(\frac{\kappa}{\sigma} - 1 \right) = 0.$$

Hence we have

$$\phi(z_1, z_2) = \begin{pmatrix} \sigma & h_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} h_0 \\ h_2 \end{pmatrix}. \tag{1}$$

Now assume that an automorphism $\bar{\phi} \in \Gamma$ has fixed points. We take a lift $\phi \in \Gamma'$ and can assume (by multiplying by an element of G if necessary) that ϕ itself has a fixed point. By (1) this is the case iff the equation

$$0 = \left[\begin{pmatrix} \sigma & h_1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \gamma \\ h_2 \end{pmatrix} = \begin{pmatrix} \sigma - 1 & h_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \gamma \\ h_2 \end{pmatrix}$$

has a solution. Clearly the same arguments work for secondary Kodaira surfaces.

Since an automorphism maps rational curves to rational curves and therefore covers a unique automorphism of the corresponding minimal model we have shown the first part of the following

Proposition 6.3. *If an automorphism of finite order of a Kodaira surface has fixed points, it has fixed points in codimension one. In particular if S is a Kodaira surface and $f : S \to X$ is a quasi-étale map where X has canonical singularities, then f is étale and X is in fact a smooth Kodaira surface.*

Proof. By Corollary 3.3 there is a Kodaira surface W and a finite group G acting freely in codimension 1 on W such that X is a birational image of W/G by a small contraction. But the first part implies that the action of G is free, hence the quotient is smooth, $W/G \cong X$ and f is itself étale. q.e.d.

Lemma 6.4. *Assume that $f : X \to Y$ is a quasi étale morphism, where Y has canonical singularities and is bimeromorphic to a smooth Kodaira surface S. Then X is bimeromorphic to a Kodaira surface.*

Proof. Without loss of generality, we may assume that $\pi : S \to Y$ is the minimal resolution of the singularities of Y, and that S is the blow up $p : S \to Z$ of a minimal Kodaira surface Z.

By definition, there are finite sets $\Sigma_Y \subset Y, \Sigma_X \subset X$ such that $X - \Sigma_X \to Y - \Sigma_Y$ is a finite unramified covering.

By pull back, we obtain a finite unramified covering of $S - \pi^{-1}(\Sigma_Y)$. Now, $\pi^{-1}(\Sigma_Y)$ consists of a finite set plus a finite union of smooth rational curves with self intersection -2.

But there is no rational curve on a minimal Kodaira surface Z, the universal covering being $\mathbb{C}^2$, whence $\pi^{-1}(\Sigma_Y)$ maps onto a finite set on Z, and X is bimeromorphic to a quasi étale covering W of Z. Since Z is smooth, W is a finite unramified cover of Z; in particular it is not Kähler.

Every minimal Kodaira surface admits a volume form, which remains invariant under the action of the complex structure, so this is also true for W. Kodaira classified surfaces with volume-preserving complex structure completely in ([**Kod66**], Theorem 39) and by his results W is covered either by a K3 surface, a complex torus or a primary Kodaira surface. Since W covers S only the third case occurs and W is itself a Kodaira surface. q.e.d.

Proof of the Theorem. First let S be a minimal Kodaira surface. We want to show that every surface $Q.E.D.$ equivalent to S is a Kodaira surface. By 6.2 and 6.4 it suffices to show that if we have a quasi - étale morphism $p : X \to Y$ where X, Y have canonical singularities and X is bimeromorphic to S, then Y is also bimeromorphic to a Kodaira surface.

By taking the normal closure and applying 6.4 we may assume that $p : X \to Y$ is the quotient map by the action of a finite group G. Now Y is bimeromorphic to S/G, which is a Kodaira surface by Proposition 6.3.

It remains to show that all Kodaira surfaces are $Q.E.D.$ equivalent. Let $S_0 = \mathbb{C}^2/G_0$ be the primary Kodaira surface given by $(*)$ with $\beta_1 = \alpha_3 = 1$, $\beta_2 = 2\alpha_4 = 2i$ and $\beta_3 = \beta_4 = 0$. We have the relation $\alpha_3\bar{\alpha}_4 - \alpha_4\bar{\alpha}_3 = \beta_2$ and thus the fundamental group of S_0 is isomorphic as an abstract group to F/R where F is the free group on generators $f_1, \ldots, f_4$ and R is the subgroup generated by the relations

$$[f_i, f_j] = \begin{cases} f_2 & i = 3, j = 4 \\ 0 & \text{otherwise} \end{cases} \qquad 1 \le i < j \le 4.$$

It clearly suffices to show that every primary Kodaira surface is $Q.E.D.$ equivalent to S_0. First consider an arbitrary $S = \mathbb{C}^2/G$ with G as in $(*)$. By changing α_3 to $\alpha_3' = \frac{\alpha_3}{m}$ we get another group G', a surface $S' = \mathbb{C}^2/G'$ and finite covering maps

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{C}/\langle \alpha_3, \alpha_4 \rangle \\ \downarrow & & \downarrow \\ S' & \longrightarrow & \mathbb{C}/\langle \alpha_3', \alpha_4 \rangle. \end{array}$$

We have the relation $\alpha_3'\bar{\alpha}_4 - \alpha_4\bar{\alpha}_3' = \beta_2$ and thus $\pi_1(S') = G'$ is isomorphic to F/R. By Corollary II.7.17 of [**F-M**] it follows that S is deformation equivalent to S_0 or to $S_0^{\text{conj}} = \mathbb{C}^2/\bar{G}_0$ with the conjugated complex structure. But an easy calculation shows that $G_0 = \bar{G}_0$, thus $S_0 = S_0^{\text{conj}}$ and consequently S is $Q.E.D.$ equivalent to S_0, which concludes the proof. q.e.d.

The last part of the proof can also be obtained using the description of the moduli space obtained by Borcea in [**Borc**].

7. Open problems and final remarks

We want in this section to add more questions, and some comments regarding some of the questions previously posed in the introduction.

It is not clear whether the following question should have a positive answer:

Question 6: If a variety X (we assume canonical singularities throughout) has Kodaira dimension $\mathcal{K} = -\infty$, does there exist a quasi-étale morphism $f : Z \to X'$ where Z is birationally ruled and X' is a deformation of X?

Remark 7.1. Observe that a smooth cubic threefold $X \subset \mathbb{P}^4$ is unirational but not rational, and that a smooth quartic threefold $Y \subset \mathbb{P}^4$ is unirational but not rational. By Lefschetz' theorem they are both simply connected (as Kollár appropriately reminded me), whence they have no nontrivial quasi-étale cover. However, we can deform X to a cubic threefold X' with a double point, respectively Y to Y' with a triple point. Both X', Y' have canonical singularities and are rational, whence a positive answer to the above question in this special case.

The above two classes of smooth Fano manifolds are stable by deformation only if we restrict ourselves to the condition that the fibres be smooth and projective.

Already for conic bundles it is not clear whether question 6 has a positive answer.

The second remark above shows that the following stronger question has a negative answer: if a variety X has Kodaira dimension $\mathcal{K} = -\infty$, is there a quasi-étale morphism $f : Z \to X'$ where Z is ruled, Z and X' are smooth, and X' is a deformation of X?

This question is somehow related to a stronger version of a well known conjecture by Mumford:

Quasi-étale unirationality question: let X be a smooth projective variety of dimension n. Then does

$$H^0((\Omega^1_X)^{\otimes m}) = 0, \ \forall m > 0,$$

imply that X is quasi-étale equivalent to $\mathbb{P}^n$ and unirational?

Observe however that even the invariance of the condition

$$H^0((\Omega^1_X)^{\otimes m}) = 0, \ \forall m > 0,$$

under deformation of smooth projective varieties is not yet established.

Remark 7.2. Kollár constructed (see e.g., Chapter V, Section 5 of [**Kol**], p. 273 and foll.) examples of complex Fano varieties (these are rationally connected) which are not ruled. Are these counterexamples to Question 6?

Reduction to varieties over $\overline{\mathbb{Q}}$ can be thought of as the distinguishing feature between algebraic and Kähler varieties. In fact, if we have a smooth projective variety $X \subset \mathbb{P}^n_{\mathbb{C}}$, we get a corresponding point $[X]$ of a Hilbert Scheme $\mathcal{H}$. Since $\mathcal{H}$ is defined as a closed algebraic set in an appropriate Grassmannian G by rank equations of certain multiplication by monomials, $\mathcal{H}$ is defined over $\mathbb{Z}$, and an irreducible component containing $[X]$ contains a dense set of points defined over $\overline{\mathbb{Q}}$, whence we obtain a smooth projective variety $Y \subset \mathbb{P}^n_{\mathbb{C}}$, where Y is defined over $\overline{\mathbb{Q}}$ and is an algebraic deformation of X. Assume that Y is defined over a number field K: then the theory developed so far suggests to consider the quasi-étale generalization of Grothendieck's fundamental group, which should play an important role (in case of canonical models of varieties of general type, i.e., of varieties X with canonical singularities with K_X ample, this is exactly the Grothendieck fundamental group of the smooth locus of X).

It would be also interesting to enlarge our equivalence relation as to include, for varieties defined over a number field, also the action of the absolute Galois group (thus for instance considering a variety over $\mathbb{C}$ and its complex conjugate as equivalent).

Question 7: For which classes of algebraic varieties is A.Q.E.D.-equivalence the same as the weaker $\mathbb{C}$-Q.E.D. equivalence?

In the case of uniruled varieties the Q.E.D. question is strictly related to the question of "generic" splitting of normal bundles for the curves of a covering family of rational curves (here, "generic" stands not only for the generic curve of the family, but also for a general deformation of the given variety).

Question 8: What is the t-equivalence of compact complex manifolds? (This is hard since for instance we do not know all the compact complex surfaces.)

Question 9: Assume that $S = B^2/\Gamma$ is a compact minimal smooth surface which is a ball quotient (equivalently, by Yau and Miyaoka's theorem, cf. [**Yau**] and [**Miya1**], $K_S^2 = 9\chi(\mathcal{O}_S)$). Does there exist, as in the case of Kuga-Shavel surfaces, a group Γ such that every group Γ'' commensurable with Γ is either torsion free (it acts freely), or it has a fixed point z where the (finite) stabilizer Γ''_z has a tangent representation not contained in $SL(2, \mathbb{Z})$?

Fritz Grunewald suggested that such examples should indeed exist, more precisely that there are such groups Γ such that every group Γ'' commensurable with Γ equals Γ, and such examples should be found among the ones of Deligne-Mostow (cf. [**D-M**]).

Question 10: (Lucia Caporaso) Which are the Q.E.D. equivalence class of Kodaira fibrations?

Note 1. As remarked by Frederic Campana in the footnote to [**Cam**], the equivalence relation introduced by him is only apparently similar to ours, but indeed quite different, cf. Section 5 of [**Cam**] and our main theorems for special surfaces.

Note 2. Claire Voisin pointed out that the decision to use also the notion of K-equivalence (introduced in [**Voi3**]) might lead to other interesting equivalence relations preserving the Kodaira dimension.

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Lehrstuhl Mathematik VIII
Universität Bayreuth, NWII
D-95440 Bayreuth, Germany
E-mail address: Fabrizio.Catanese@uni-bayreuth.de