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SUSPENSION FLOWS ARE QUASIGEODESIC

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Abstract

A hyperbolic 3-manifold M which fibers over the circle admits a flow called the suspension flow. We show that such a flow can be isotoped to be uniformly quasigeodesic in the hyperbolic metric on M; i.e., the flow lines lifted to hyperbolic space are K-bilipschitz embeddings of $\mathbb{R}$ ($K > 1$ fixed).

A **flow** on a manifold M is a continuous action of R on M; i.e., a continuous map $\Phi : M \times \mathbb{R} \to M$, written $\Phi_t(x) = \Phi(x,t)$, such that for each t, Φ_t is a homeomorphism of M, and such that $\Phi_s \circ \Phi_t = \Phi_{s+t}$ for all $x \in M, s, t \in \mathbb{R}$. Note that we may isotope the flow so that all flow lines are rectifiable. As an example of a flow, a closed orientable 3-manifold has Euler characteristic zero and therefore admits a nowhere zero vector field, which in turn produces a flow given by the integral curves of that vector field.

Let M be a hyperbolic 3-manifold which fibers over the circle. Then M can be represented as a product of a hyperbolic surface (the fiber, F) with the unit interval $[0,1]$, with $F \times \{1\}$ glued to $F \times \{0\}$ using a pseudo-Anosov [**Th1**] monodromy map Ψ:

$$M = F \times [0,1] \;/\; (x,1) \equiv (\Psi x, 0).$$

Thus M is covered by $F \times \mathbb{R}$. The flow on M obtained by projecting the 1-manifolds $\{x\} \times \mathbb{R}$ from $F \times \mathbb{R}$ to M is called the **suspension flow**, and M is referred to as the **suspension** of F. Our main result is

Main Theorem 1. *The suspension flow on a cusped hyperbolic* 3-*manifold which fibers over the circle is* (*can be isotoped to be*) *uniformly quasigeodesic.*

Informally, a flow on a manifold M is uniformly quasigeodesic if the lifts of all sufficiently long flow lines to the universal cover of M are K-bilipschitz embeddings of $\mathbb{R}$ for fixed $K > 1$; a more careful definition is given in Section 1.1.

Zeghib has shown that a closed hyperbolic 3-manifold does not admit any flow where each flow line is geodesic [**Z**]; however, there are examples of hyperbolic 3-manifolds which admit quasigeodesic flows. Cannon and

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Thurston [**CT**] showed that the suspension flow on any compact hyperbolic 3-manifold fibering over the circle is quasigeodesic. Zeghib [**Z**] also gave an elementary proof that if M is any compact 3-manifold fibering over the circle, then any flow transverse to the fibration is quasigeodesic. More recently, combined works of Gabai [**G**], Mosher [**Mo1**], and Fenley [**FM**] show that any closed, oriented, hyperbolic three-manifold with nontrivial second homology admits many quasigeodesic flows. In each of the above cases, the manifold under consideration is compact; in this paper, we consider non-compact hyperbolic 3-manifolds which have finitely many cusps.

To prove the main theorem, we make use of the idea of a path "tracking" [**F**] a geodesic: a path c in $\mathbb{H}^3$ **tracks** a geodesic γ in $\mathbb{H}^3$ if

1) There exists $R > 0$ so that c lies within the neighborhood of radius R of γ (c **is close to** γ).
2) There exists $Q > 0$ such that if the length of a subpath of c with endpoints a and b is at least Q, then the distance between $\pi(a)$ and $\pi(b)$ along γ is at least 1, where π denotes orthogonal projection onto γ (c **makes progress** along γ).

The following three basic facts are used repeatedly throughout this paper:

Basic Fact 1. A path in $\mathbb{H}^n$ is quasigeodesic if and only if it tracks a geodesic.

Basic Fact 2. Geodesic segments in $\mathbb{H}^n$ with nearby ends are close everywhere.

Basic Fact 3. To show that a flow is uniformly quasigeodesic, it suffices to show that all *sufficiently long* flow lines are quasigeodesic, with uniform quasigeodesic constants, as we are only interested in the large scale behavior of the flow.

We will show that the suspension flow on M is uniformly quasigeodesic by showing that there are constants R and Q so that each flow line tracks a geodesic with these tracking constants.

The manifold M has finitely many cusps. Fix an open neighborhood of each cusp in M such that each neighborhood lifts to a collection of horoballs in the universal cover $\widetilde{M}$. We may choose these cusp neighborhoods so that each horoball lies a hyperbolic distance at least 1000 from all the other horoballs. Then the **neutered space** $\widetilde{N}$ is obtained by removing these open horoballs from $\widetilde{M}$. In this paper, we will sometimes use the term neutered space to also refer to the compact submanifold $N = M \setminus \{\text{open neighborhood of cusps}\}$ covered by $\widetilde{N}$.

In Section 1, the dynamics of the flow which arise from pseudo-Anosov monodromy Ψ are used to prove the following 2 crucial lemmas:

Lemma 1. *Every flow line segment inside a horoball is uniformly quasigeodesic.*

Lemma 2. *Each flow line enters a given horoball at most once.*

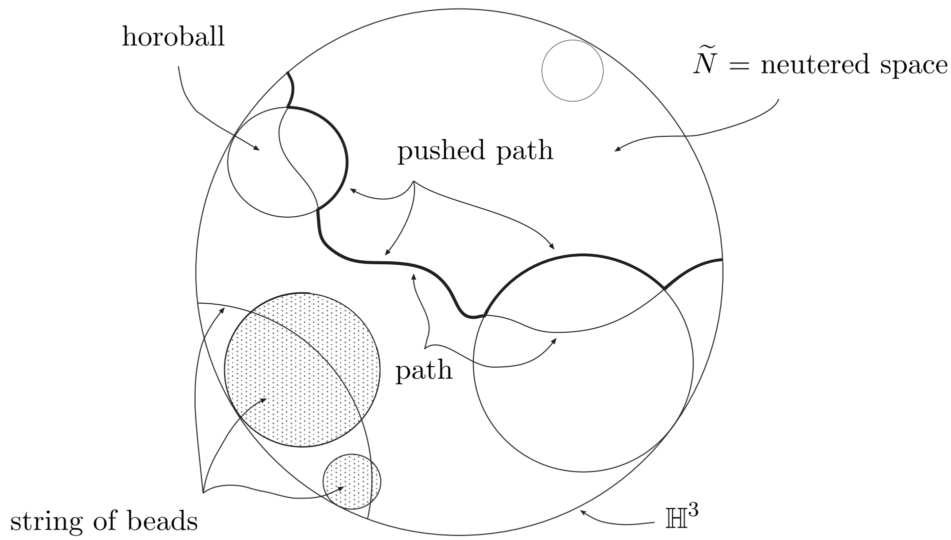

Figure 1. A pushed path and a string of beads.

In Section 2, we show that flow lines are close to geodesics (part 1 of the definition of tracking). Define $r : \widetilde{M} \to \widetilde{N}$ to be the nearest point retraction (in the hyperbolic metric). Given a path c in M, its associated **pushed path** $\bar{c}$ is $r \circ c$ (Figure 1). Define the **string of beads** along a geodesic γ in $\mathbb{H}^3$, denoted γ_+, to be the geodesic γ together with all the horoballs which γ intersects. In Section 2.1 we will show that each *pushed* flow line lies close to the string of beads connecting its endpoints. We prove this by showing

1) Pushed flow lines are uniformly quasigeodesic in the neutered space of M.
2) Any quasigeodesic of the neutered space is close (in the hyperbolic metric) to the string of beads with the same endpoints.

In Section 2.2, we will show that every flow line lies close to a string of beads, as follows: Since a flow line is identical to its corresponding pushed flow line outside the horoballs, the portions of this flow line which lie outside the horoballs are close to the string of beads. The portion of this flow line which lies inside a horoball lies close to the geodesic segment γ' connecting its endpoints by Lemma 1. Since the endpoints of γ' are close to the string of beads, it follows that all of γ' lies close to the string of beads. Thus each flow line is close to a string of beads.

In Section 2.3, we prove that every flow line is close to the *string* in the string of beads. We now know a flow line has segments which lie within a fixed distance, R_{f}, of the string (for which there is nothing to prove) and segments which lie within R_{f} of some bead. We will show that the subsegments of flow line which lie near the bead but do not enter the bead have a bounded length M_{f}, where the bound depends only on R_{f}, and therefore these subsegments are close to the string. On the other hand, since flow lines are quasigeodesic inside cusps, the subsegments of ℓ which lie inside the bead lie within R_{c} of some geodesic segment in the bead. We will show this geodesic lies within an $R_{\mathrm{f}} + M_{\mathrm{f}}$ neighborhood of the string, and therefore the subsegment of ℓ inside the bead lies within $R_{\mathrm{f}} + M_{\mathrm{f}} + R_{\mathrm{c}}$ of the string.

Finally, in Section 3 we show that all flow lines make uniform progress along the geodesic with the same endpoints.

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1. Preliminaries

Cannon and Thurston introduce a useful metric called the singular Solv metric on the suspension X of a closed surface in [**CT**]. This metric is useful because in the singular Solv metric on the universal cover of X, the lifts of the flow lines of the suspension flow on X are geodesics. In this section we will define a new metric based on the singular Solv metric with this same useful property outside cusp neighborhoods, which will help us understand the dynamics of the flow.

Remark 1.1. In this paper, when we write the word metric, we will always mean a path metric arising from an infinitesimal length function.

Most of the time, these infinitesimal length metrics are Riemannian metrics; the only exceptions are the singular Solv metric and the modified metric, which are Riemannian except along a certain 1-manifold called the singular set.

1.1. Definition of Quasi-Isometry and Quasigeodesic.

Let (X, d) be a metric space. If the distance between every pair of points in X is equal to the infimum of the lengths of rectifiable curves joining them, then we call (X, d) a **length space**. We also say that d is a path metric. Furthermore, if the infimum is always attained, we call (X, d) a **geodesic space**.

For example, a Riemannian manifold with boundary is a length space; a complete Riemannian manifold is a geodesic space. Note that every path connected covering $\pi : \widetilde{X} \to X$ of a length space is also a length space, where the length of a path $c : [a, b] \to \widetilde{X}$ is defined to be the length of $\pi \circ c : [a, b] \to X$.

In this paper, all underlying spaces are manifolds, so they always have universal covers. Most of the spaces in this paper have Riemannian metrics as well, or metrics which are Riemannian except at well-behaved singularities. Every space in the paper is length space.

In the following definitions, all paths are rectifiable, I, I' are (possibly infinite) subintervals of $\mathbb{R}$, and X is a length space with universal cover $\pi : \widetilde{X} \to X$.

Definition 1.2. A path $c : I \to X$ is **parametrized by arc length** if for all $a, b \in I$, $\text{length}(c|[a, b]) = |b - a|$.

Definition 1.3. A path $c' : I' \to X$ is a **reparametrization** of $c : I \to X$ if there exists a homeomorphism $h : I \to I'$ such that $c = c' \circ h$.

Lemma 3. *If $c : I \to X$ is rectifiable and injective, then there exists a reparametrization of c by arc length.*

In this paper, when we say a path is geodesic we will mean that its lift to the universal cover is a minimal geodesic, as in the following definition:

Definition 1.4. A path $c : I \to X$ is a **geodesic** if a lift $\widetilde{c} : I \to \widetilde{X}$ of c is an isometry; i.e., for all $a, b \in I$, $d_{\widetilde{X}}(\widetilde{c}(a), \widetilde{c}(b)) = |b - a|$. A path is a **local geodesic** if every point is contained in a subpath of positive length which is a geodesic.

Remark 1.5. Geodesics are parametrized by arc length.

Definition 1.6. A path $c : I \to X$ is a **ugeodesic** or unparametrized geodesic if it can be reparametrized to be a geodesic.

Remark 1.7. Equivalently, an injective $c : I \to X$ is a ugeodesic if for all $a, b \in I$, $d_{\widetilde{X}}(\widetilde{c}(a), \widetilde{c}(b)) = \text{length}(\widetilde{c}|[a, b])$.

Definition 1.8. Let (X, d_X) and (Y, d_Y) be metric spaces. For $K \geq 1$ and $L \geq 0$, a (K, L)**-quasi-isometry** is a map $f : (X, d_X) \to (Y, d_Y)$ such that for all $x, x' \in X$, if $d_X(x, x') \geq L$, then

$$\frac{1}{K} \leq \frac{d_Y(fx, fx')}{d_X(x, x')} \leq K.$$

Remark/Definition 1.9. A K**-bilipschitz** map is a $(K, 0)$-quasi-isometry.

Note that two path metrics on a manifold M are K-bilipschitz if and only if the infinitesimal versions of these path metrics (Finsler metrics) on M are K bilipschitz.

Definition 1.10. A (K, L)**-quasigeodesic** in a length space X with universal cover $\pi : \widetilde{X} \to X$ is a path $c : I \to X$ such that if $\widetilde{c} : I \to \widetilde{X}$ is a lift of c (so $\pi \circ \widetilde{c} = c$), then $\widetilde{c}$ is a (K, L)-quasi-isometry of I (with the usual metric) into $\widetilde{X}$. A path $c : I \to X$ is **quasigeodesic** in X if c is a (K, L)-quasigeodesic for some $K \geq 1$, $L \geq 0$.

Note that it is standard to define a quasigeodesic in the universal cover; the definition given above is slightly nonstandard as it defines a quasigeodesic in the base space to be a path which is covered by a (standard definition) quasigeodesic in the universal cover.

Remark 1.11. A (1,0)-quasigeodesic is simply a geodesic.

The following definition gives a notion of a quasi-fication of ugeodesic; in other words, a criterion for a path to be quasigeodesic without concern for a parametrization.

Definition 1.12. Let M be a manifold and a length space, and $c : I \to M$ be a rectifiable path. For $K \geq 1$ and $L \geq 0$, c is a (K, L)**-quasi-ugeodesic** if for all subintervals $[a, b] \subseteq I$ with $\text{length}(c|[a, b]) \geq L$, we have $\text{length}(c|[a, b]) \leq K \cdot d_{\widetilde{M}}(\widetilde{c}(a), \widetilde{c}(b))$.

Lemma 4. *Let M be a manifold and a length space, and $c : I \to M$ be (K, L)-quasi-ugeodesic. Then there exists a reparametrization c' of c which is a (K, L)-quasigeodesic.*

Proof. Let $c' : I' \to M$ be a reparametrization of c by arc length; then for all $a, b \in I'$, $\text{length}(c'|[a, b]) = b - a$. Since c is a (K, L)-quasi-ugeodesic and the definition of (K, L)-quasi-ugeodesic is independent of parametrization, we know that if $a, b \in I'$ with $\text{length}(c'|[a, b]) \geq L$, then $\text{length}(c'|[a, b]) \leq K \cdot d_{\widetilde{M}}(\widetilde{c'}(a), \widetilde{c'}(b))$. But also, $\text{length}(c|[a, b]) \geq d_{\widetilde{M}}(\widetilde{c}(a), \widetilde{c}(b))$. Therefore, if $\text{length}(c'|[a, b]) \geq L$, then

$$\frac{1}{K} \leq 1 \leq \frac{b - a}{d_{\widetilde{M}}(\widetilde{c}(a), \widetilde{c}(b))} \leq K,$$

so c' is a (K, L)-quasigeodesic. q.e.d.

Many of the proofs in this paper prove directly that certain paths are quasi-ugeodesics (rather than quasigeodesics). In light of Lemma 4, to avoid extra complication, we will refer to these quasi-ugeodesics simply as quasigeodesics.

Finally, a flow on a manifold M is **quasigeodesic** if there exist uniform $K \geq 1, L \geq 0$ such that every flow line lifts to a (K, L)-quasigeodesic in $\widetilde{M}$.

1.2. Definition of Singular Solv Metric.

Definition 1.13. A **Euclidean cone metric** on a surface F is a path metric (actually, a Finsler metric [**Sp**]) on F which is Euclidean except at a discrete set of points called **cone points**. At each cone point p_i there is a cone angle θ_i and a neighborhood U_i of p_i where the infinitesimal metric is given by

$$ds^2 = dr^2 + r^2 \left(\frac{\theta_i}{2\pi}\right)^2 d\theta^2.$$

Definition 1.14. Let F be a (possibly punctured) surface. A homeomorphism $\Psi : F \to F$ is **pseudo-Anosov** if there is a Euclidean cone metric on F such that

1) Ψ permutes the cone points on F, preserving cone angle.
2) There are one dimensional foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ in $F - \{$cone points$\}$, called stable and unstable foliations respectively, whose leaves are local Euclidean geodesics, and the leaves of $\mathcal{F}^+$ are orthogonal to those of $\mathcal{F}^-$.
3) At each nonsingular point p in F, there are x, y coordinates in a neighborhood of p such that the leaves of $\mathcal{F}^+$ are $y =$ constant and the leaves of $\mathcal{F}^-$ are $x =$ constant.
4) Using these local coordinates, Ψ is given locally by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda > 1$.

The closure of a leaf in F that contains a singular point is called a **singular leaf**. The singularity is called n**-pronged** if n leaves of $\mathcal{F}^+$ and n leaves of $\mathcal{F}^-$ meet at the singularity (we will also use the notation $\mathcal{F}^+$ and $\mathcal{F}^-$ to refer to the singular foliations).

Figure 2 is a picture of a 3-pronged singularity. The total angle around the singular point is 3π.

For $\lambda > 1$, the **Solv metric on** $\mathbb{R}^3$ is the path metric arising from the infinitesimal length function

$$ds^2 = \lambda^{2t} dx^2 + \lambda^{-2t} dy^2 + dt^2.$$

Up to isometry, this metric does not depend on λ because rescaling the t axis is equivalent to changing λ. For each integer $n \geq 1$, we will describe a path metric on $\mathbb{R}^3$ called the n-prong singular Solv metric. This metric is Riemannian everywhere except on the t-axis. Using global cylindrical coordinates (r, ϕ, t) on $\mathbb{R}^3$, it is the path metric on $\mathbb{R}^3$ arising from the infinitesimal length function on $\mathbb{R}^3 - t$-axis given by

$$\begin{aligned} ds^2 = \lambda^{2t} &\left(\cos\left(\frac{n\phi}{2}\right) dr - \frac{rn}{2} \sin\left(\frac{n\phi}{2}\right) d\phi\right)^2 \\ &+ \lambda^{-2t} \left(\sin\left(\frac{n\phi}{2}\right) dr + \frac{rn}{2} \cos\left(\frac{n\phi}{2}\right) d\phi\right)^2 + dt^2. \end{aligned}$$

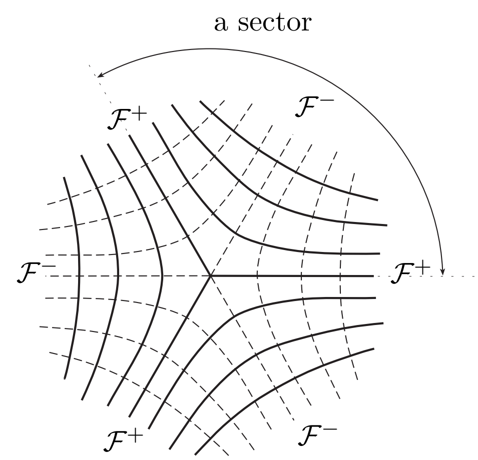

Figure 2. Leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$ near a 3-prong singularity.

The metric completion of this is the Finsler metric called the n**-prong singular Solv metric** on $\mathbb{R}^3$.

A **sector** of $\mathbb{R}^3$ is a subset of this metric space with $\frac{2\pi k}{n} \leq \phi \leq \frac{2\pi(k+1)}{n}$ for an integer $k \in [0, n-1]$ (Figure 2). It is isometric to the subset of $\mathbb{R}^3$ with the Solv metric given by $y \geq 0$. Later on we will use the local parameter $\theta = \frac{n}{2}\phi$ in each sector, so that a sector corresponds to θ between $k\pi$ and $(k+1)\pi$ for k an integer in $[0, n-1]$. We refer to the θ parameter as angle. The restriction of the infinitesimal version of the singular Solv metric to a horizontal plane $t =$ constant is a Euclidean cone metric with the origin a singular point with cone angle $n\pi$.

Definition 1.15. A **singular Solv metric** on a 3-manifold M is a path metric which is locally isometric to some **model**; i.e., an open set in some n-prong singular Solv metric on $\mathbb{R}^3$.

We allow different values of n for different points in M. The subset of M corresponding to cone points on the t-axis in some model with $n \neq 2$ is called the **singular set** of M, written ΣM; it is a 1-submanifold of M. For each component C of ΣM the cone angle, $n\pi$, is constant.

There is a foliation on $\mathbb{R}^3$ given by lines parallel to the t-axis. These are geodesics for every n-prong singular Solv metric on $\mathbb{R}^3$. This gives a foliation by geodesics on any manifold M with a singular Solv metric.

Definition 1.16. A set V in a manifold M with a singular Solv metric is **vertical** if it is a union of leaves of the foliation by geodesics described above. A surface H in M is **horizontal** if for every point in H there is a neighborhood U which is locally isometric to an open set in $\mathbb{R}^3$ with the singular Solv metric and $H \cap U$ is mapped into the horizontal plane given by $t =$ constant.

The following theorem describes the connection between the singular Solv metric and surface bundles over the circle with pseudo-Anosov monodromy.

Theorem 1.17. [**Th1**][**CT**] *Suppose $\Psi : F \to F$ is pseudo-Anosov with stretch factor λ, and M is the F-bundle over S^1 with monodromy Ψ. Then there is a submersion $p : M \to S^1$ such that for all $x \in S^1$, $p^{-1}(x) \equiv F_x$ is homeomorphic to F, and*

1) *M is hyperbolic*
2) *M has a singular Solv metric (with the given λ)*
3) *Each fiber F_x is horizontal in the singular Solv metric; i.e., it is given locally by $t =$ constant. Thus the induced infinitesimal metric on F_x is a Euclidean cone metric.*

A covering translation in the universal cover associated to the monodromy is given locally by $(x, y, t) \mapsto (\lambda^{-1}x, \lambda y, t + 1)$.

In such a 3-manifold, the stable and unstable singular foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ on F give rise to 2-dimensional singular stable and unstable foliations on M, where each 2-dimensional leaf is the suspension of a one-dimensional leaf in $\mathcal{F}^+$ or $\mathcal{F}^-$. The flow lines of the suspension flow on M consist of the 1-manifolds which are the intersections of the leaves of these 2-dimensional foliations. Note that in an unstable leaf, every pair of flow lines move closer together as t increases, and in a stable leaf, every pair of flow lines move farther apart as you flow forward.

1.3. Cylinders in $\widehat{M}$. We are studying a hyperbolic 3-manifold M which fibers over the circle with fiber a surface F. This surface F is homeomorphic to a closed surface F^+ with finitely many points, corresponding to **punctures** of F, deleted. The monodromy $\Psi : F \to F$ extends to a map $\Psi^+ : F^+ \to F^+$. Since Ψ is pseudo-Anosov, it follows that Ψ^+ is also pseudo-Anosov. The mapping cylinder M^+ of Ψ^+ is a closed 3-manifold such that $M^+ - M$ is a finite union of circles. We call these circles **cusp lines**.

From the previous discussion we have a singular Solv metric on M^+. The restriction of this metric to M is not quasi-isometric to the hyperbolic metric on M, because the diameter of M in this metric is finite. Below we construct, for each cusp line c_i of M^+, a certain tubular neighborhood $N(c_i)$ in M^+. The submanifold $C_i = N(c_i) - c_i$ is called a **cusp** of M; it has boundary a torus. We will construct a new metric on C_i which equals the infinitesimal singular Solv metric on ∂C_i. This will give a new path metric on M which is quasi-isometric to the hyperbolic metric on M.

The bundle map $p : M \to S^1$ extends to a bundle map $p^+ : M^+ \to S^1$ such that for all $x \in S^1$, $(p^+)^{-1}(x) \equiv F_x^+$ is homeomorphic to F^+. Fix a cusp line c in M^+. Each point q on c lies in some fiber F_x^+ of M^+. For each $q \in c$ on each F_x^+, take the topological disk consisting of all points on F_x^+ whose distance to q in the induced Euclidean cone metric on F_x^+ (coming from restricting the infinitesimal singlar Solv metric) is at most $\frac{1}{e}$. Then define $N(c)$ to be the union of these disks. (Scale the metric on M, if necessary, so that each neighborhood is a solid torus, neighborhoods around different cusp lines do not intersect, and $N(c)-c$ contains no singularity of the flow.)

Let $\widehat{F}$ be the cover of F corresponding to the kernel of the inclusion map $i_* : \pi_1 F \to \pi_1 F^+$. Then $\widehat{F}$ is a punctured plane with a singular Euclidean metric, with cone points where the cone angle is a multiple of π. Let $\widehat{M} = \widehat{F} \times \mathbb{R}$, which is a cover of M; let $\rho : \widehat{M} \to M$ be this cover. The manifold $\widehat{M}$ is a subset of $\widetilde{M^+}$, the universal cover of M^+. We use the path metric on $\widehat{M}$ given by restricting the singular Solv metric on $\widetilde{M^+}$.

Definition 1.18. A **cylinder** in $\widehat{M}$ is a component of $\rho^{-1}(C)$ where C is a cusp of M; it is homeomorphic to $(D^2 - \{\text{pt}\}) \times \mathbb{R}^1$.

Note that by the way we have chosen C, the cylinders in $\widehat{M}$ do not intersect one another and do not contain any singularity of the flow.

When we draw pictures of the Solv metric, we will always draw flow lines as vertical straight lines. In this model, a cylinder becomes more eccentrically star shaped, lengthened along the unstable singular foliation and shortened along the stable singular foliation, as t tends to infinity, as in Figure 3. The elongating and shortening effect can be more easily seen if we "square off" a 2-pronged cylinder so that it has 4 sides, two of which are parallel to the x-axis and the other two parallel to the y-axis (Figure 4).

Choose a cylinder T in $\widehat{M}$ with an n-pronged singularity. Choose local coordinates (r, θ, t) for T with $\theta \in [0, n\pi)$, $t \in \mathbb{R}$ in such a way that $\theta = k\pi$ corresponds to the unstable eigendirections and $\theta = k\pi + \frac{\pi}{2}$ corresponds to the stable eigendirections, with the rest of the values of θ increasing monotonically when traveling counterclockwise from the $\theta = 0$ eigendirection. This lamentable choice leads to local coordinates

$$x = r\sin\theta, \qquad y = r\cos\theta;$$

under these coordinates T is described by

$$0 < r\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta} \le \frac{1}{e}. \tag{1}$$

Note that the puncture in each constant t cross section of T occurs on the local t-axis. The universal cover of T is obtained by unwrapping T about the t-axis.

Flow lines are vertical in the singular Solv metric on T. Since r, θ, and t are local singular Solv coordinates on T, each segment of flow line contained in T is determined by some fixed choice of r and θ, and is parameterized by the coordinate t. If the segment has θ coordinate an

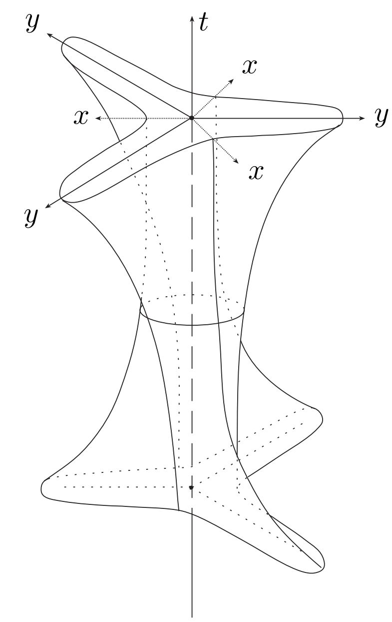

Figure 3. A portion of a 3-pronged cylinder of $\widehat{M}$ showing constant singular Solv distance.

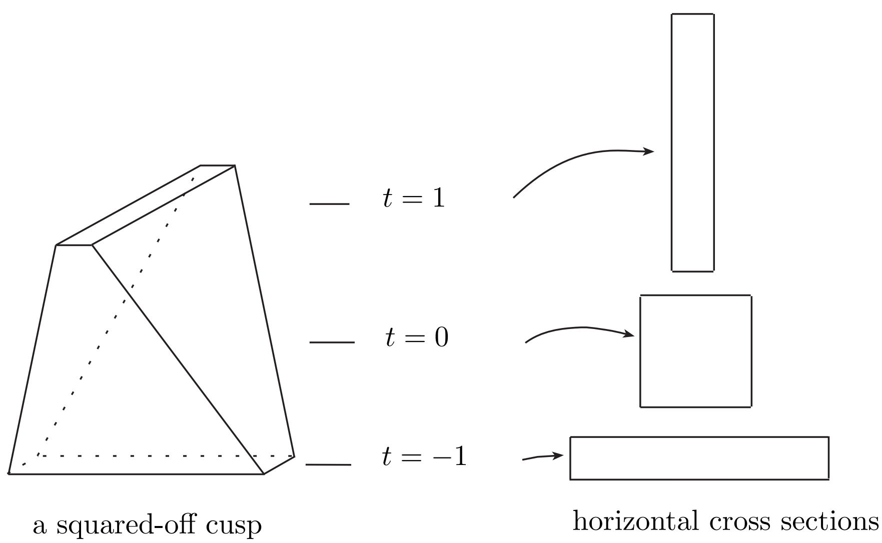

Figure 4. A portion of a "squared-off" 2-pronged cusp of $\widehat{M}$.

integer multiple of $\frac{\pi}{2}$, then it lies in a non-compact leaf of either the stable or unstable 2-dimensional foliation which contains the cusp line. In this case, the flow line segment approaches the cusp line asymptotically in either forwards (i.e. upwards) time (in an unstable leaf) or backwards time (in a stable leaf). We will refer to this type of flow line as a **trapped** flow line, for one end of the flow line becomes trapped in the cusp forever. If θ is not an integer multiple of $\frac{\pi}{2}$, it is easy to check that this type of flow line will leave the cusp through its boundary in both finite forwards and backwards time. As these flow lines will enter the cusp and then return to the compact part of the manifold, we will call them **returning** flow lines.

1.4. Modifying the Singular Solv Metric. We now construct a new path metric on M called the modified metric as follows. The infinitesimal version of this metric is the infinitesimal singular Solv metric except in the cusps of M. In the cusps of M we use a standard hyperbolic cusp metric. Near the boundary of each cusp we interpolate with a Riemannian metric between the infinitesimal singular Solv and the hyperbolic metrics.

Let $\widetilde{C} \to C$ be the universal cover of a cusp C. To define the standard hyperbolic metric on C we construct an embedding $f : \widetilde{C} \to \mathbb{H}^3$ so that the covering transformations $\pi_1(C)$ of C correspond to isometries of $\mathbb{H}^3$. Then using the pull back metric on $\widetilde{C}$ we get a hyperbolic metric on $C = f(\widetilde{C})/\pi_1(C)$. The map is chosen so that the images of flow lines are quasigeodesics (see Section 1.6).

We use the upper half space model of $\mathbb{H}^3$, namely, $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ with Riemannian metric $(dx^2 + dy^2 + dz^2)/z^2$. The image of f is the horoball $|z| \geq 1$. The universal cover, $\widetilde{C}$, is the union of infinitely many sectors $\widetilde{T}_i$ such that $\widetilde{T}_n$ intersects $\widetilde{T}_m$ if and only if $n = m, m \pm 1$, and $\widetilde{T}_n \cap \widetilde{T}_{n+1} =$ vertical half plane.

The cylindrical coordinates on each sector of $\widetilde{C}$ fit together to give global coordinates (r, θ, t) in the subset of $(0, \infty) \times \mathbb{R} \times \mathbb{R}$ given by equation (1), such that $\widetilde{T}_n$ is given by $(0, \infty) \times [(n-1)\pi, n\pi] \times \mathbb{R}$. Then $f : \widetilde{C} \to \mathbb{H}^3$ is given by

$$(2) \quad f(r, \theta + n\pi, t) = \Big(t,\, n\pi + \tan^{-1}(\lambda^{2t} \tan\theta), \\ -\log(r\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta})\Big)$$

for $0 \leq \theta < \pi$. To a first approximation, the map sends (r, θ, t) to $(x, y, z) =$ rectangular coordinates $(t, \theta, -\log r)$ in $\mathbb{H}^3$. However, this map has been adjusted so that the boundary of T is mapped to the plane $z = 1$.

An easy calculation shows that f is equivariant under the covering translation sending (x, y, z) to $(x+1, y, z)$ in the upper half space model of $\mathbb{H}^3$, which corresponds to $(x, y, t) \mapsto (\lambda^{-1}x, \lambda y, t+1)$ the singular Solv coordinates in a sector of a cylinder in $\widehat{M}$. (Note that the x and y coordinates in the upper half space model of $\mathbb{H}^3$ are not the same as the local x and y coordinates in the singular Solv metric on $\mathbb{R}^3$.)

Further, f is equivariant under the covering translation corresponding to $(r, \theta, t) \mapsto (r, \theta + n\pi, t)$. Therefore the pull-back to $\widetilde{T}$ of the hyperbolic metric on $\mathbb{H}^3$ under f covers a hyperbolic metric on T.

Under f, a given flow line is specified by a fixed choice of r and θ, and is parameterized by t. By symmetry it suffices to consider the case $0 < \theta < \frac{\pi}{2}$. By differentiating the function $f(r, \theta, t)$ in the case where θ is not a multiple of $\frac{\pi}{2}$ one can show that the image of a returning flow line in $\mathbb{H}^3$ travels monotonically in the x and y directions while it is inside the horoball, and increases monotonically in the z direction until it reaches a peak, then decreases monotonically until it exits the horoball again. Further, one can calculate that the critical point of the z-coordinate of each flow line is at $t_c = \frac{1}{2}\log_\lambda \cot\theta$. We define a reparametrization F of f, by composing with translations in the t direction, which puts the critical point of each flow line at $t = 0$:

$$\text{(3)} \quad F(r, \theta, t) = \left(t + t_c, \tan^{-1}(\lambda^{2t}), -\log\left(r\sqrt{\frac{\sin 2\theta}{2}} \right) - \frac{1}{2}\log(\lambda^{-2t} + \lambda^{2t}) \right).$$

Using this new map F one can easily see that the foliation in the cusp is invariant under translations in both the x and z directions.

Recall that pushing is defined in the upper half space model by projecting in the z direction onto the horosphere $z = 1$. Since flow lines in the upper half space model are mapped to other flow lines by vertical translation, the image of the flow lines under the pushing map gives a foliation on the horosphere $z = 1$, as seen in Figure 5. In fact, the pushing map identifies all flow lines with the same local θ coordinate. The lines of this foliation on the horosphere are preserved by the parabolic subgroup and therefore project to a foliation on the boundary of N. The projection of a leaf of the foliation in the cusp into the xz plane is given in Figure 6.

Definition 1.19. The **modified metric** on M is a path metric coming from the following infinitesimal metric:

1) On $M - \cup C_i$, use the infinitesimal singular Solv metric, where each C_i is a cusp of M.
2) Put a Riemannian metric on each cusp C_i as follows:
   a) Choose a diffeomorphism to identify $C_i = \mathbb{T}^2 \times [1, \infty)$.
   b) On each sector of $\mathbb{T}^2 \times [2, \infty)$, use the pullback of the infinitesimal hyperbolic metric under f induced in a cusp of M.
   c) Let ds_h and ds_s denote the hyperbolic and singular Solv infinitesimal metrics restricted to $\mathbb{T}^2 \times [1, 2]$. Then on $\mathbb{T}^2 \times [1, 2]$, use the **blended** infinitesimal metric given by $(t-1)ds_h + (2-t)ds_s$, where t is the coordinate in $[1, 2]$.

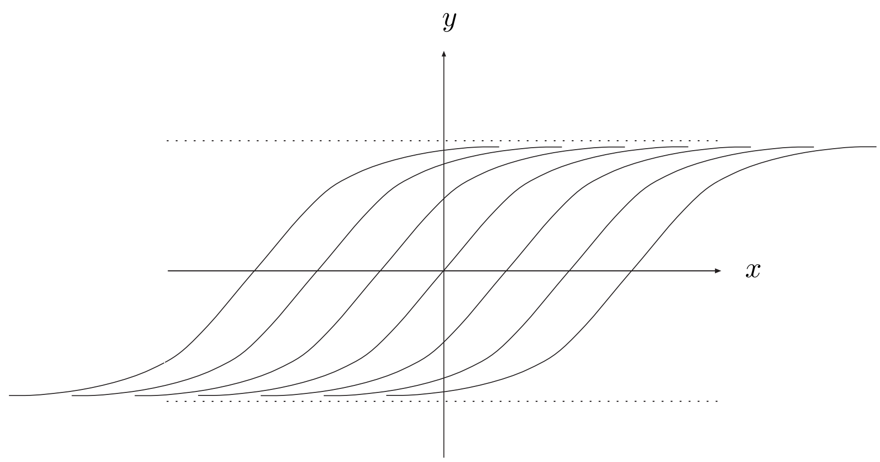

Figure 5. Projection of the foliation in the cusp to the xy plane.

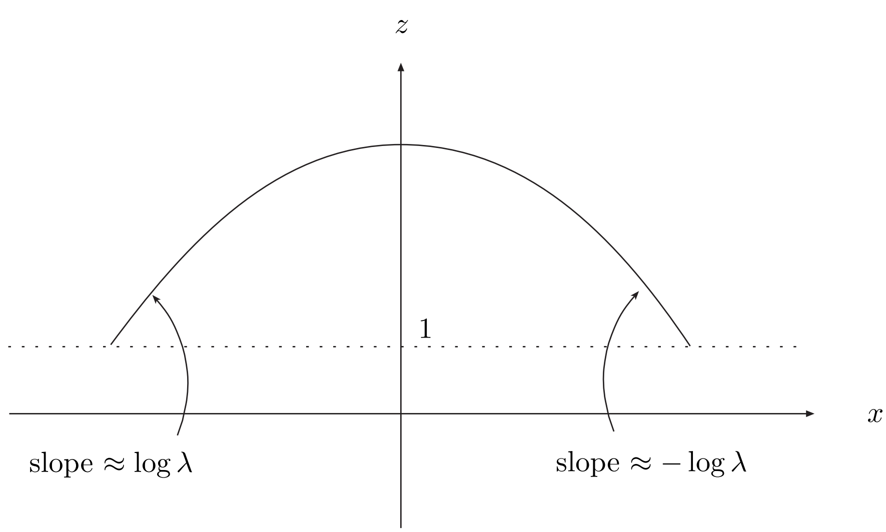

Figure 6. Projection of the foliation in the cusp to the xz plane.

Proposition 1.20. *The modified metric on* $\widetilde{M}$ *is quasi-isometric to the hyperbolic metric on* $\widetilde{M}$.

Proof. Any two Finsler metrics on a compact manifold are bilipschitz. The infinitesimal version of the singular Solv metric is a Finsler metric. Recall that the neutered space N is a compact submanifold of M. It follows that the infinitesimal versions of the hyperbolic metric and singular Solv metric are bilipschitz on N (see Figure 7). Furthermore, a cusp under the modified metric is a hyperbolic cusp by the way we construct the modified metric. This cusp will have a different shape than the hyperbolic cusp coming from the actual hyperbolic metric on M. However, it is easy to see that any two hyperbolic cusps are bilipschitz. Hence the hyperbolic and modified metrics on M are bilipschitz. Therefore, the hyperbolic (path) metric and the modified (path) metric on $\widetilde{M}$ are K-bilipschitz for some K, and thus are $(K, 0)$-quasi-isometric.

q.e.d.

Therefore the distance in $\widetilde{M}$ between two points measured in the modified metric is within a bounded factor of distance between the same points in the hyperbolic metric, so a quasigeodesic in one metric will necessarily be quasigeodesic (with different quasigeodesic constants) in the other metric. To avoid extra complication we will forget about this quasi-isometry factor, and pretend that distances are the same in two metrics which are quasi-isometric.

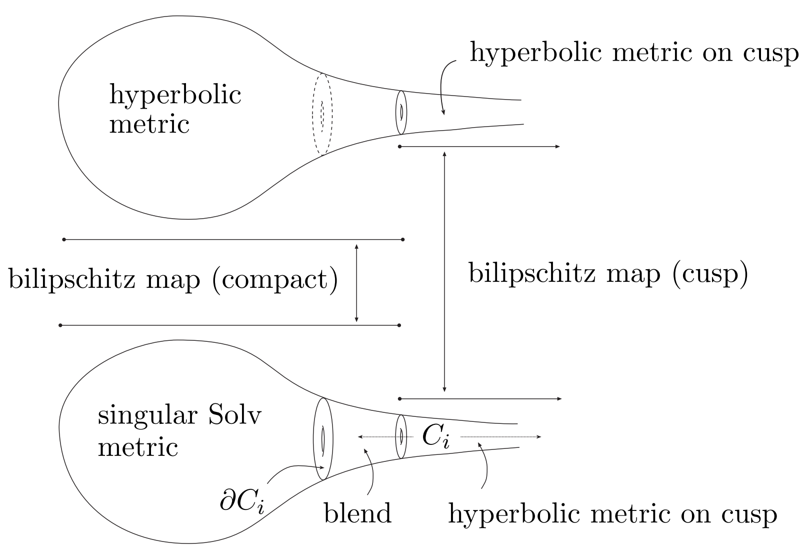

Figure 7. Bilipschitz map between modified and hyperbolic metrics.

1.5. The Effect of Pushing a Flow Line. Each flow line travels monotonically in the $\mathbb{R}$ direction in $\widehat{M}$, typically entering, passing through, and leaving several of the cylinders in $\widehat{M}$ which cover the boundary of cusps in M.

Proposition 1.21. *Let ℓ be a flow line which passes through a cylinder T. Then the portion of ℓ that lies entirely inside T stays inside a single sector of T.*

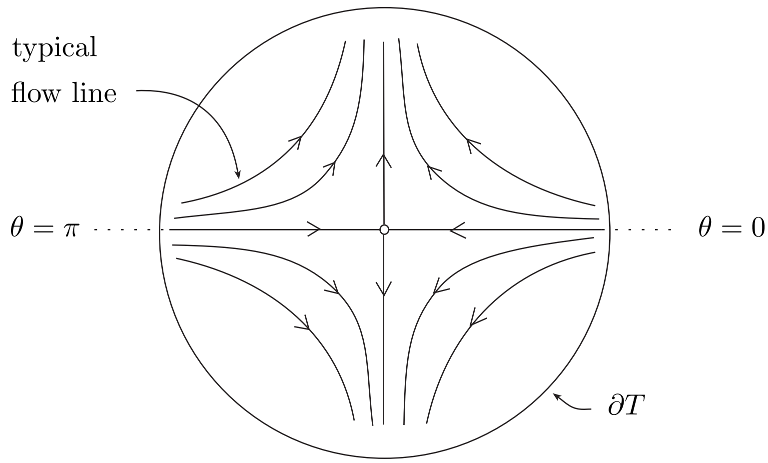

Figure 8. Flow lines passing through T, projected onto the plane $t = 0$.

Proof. Study the dynamics of the flow (Figure 8). A flow line moves from the unstable direction to the stable direction, but never moves from one sector to another. q.e.d.

We will push each flow line out of all the cylinders in $\widehat{M}$ to the boundary of these cylinders (while fixing portions of the flow line which are already outside the cylinders) to produce a pushed flow line in the neutered space. The pushing is realized as follows:

Proposition 1.22. *Let ℓ be a flow line segment which lies inside a cylinder T. In the local cylindrical coordinates in the sector of T containing ℓ, r and θ are constant along ℓ. Then, in the singular Solv metric, a point (r, θ, t) on ℓ will be pushed to the point*

$$\left(\frac{1}{e\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta}}, \theta, t \right)$$

on the boundary of T.

Thus, in the local cylindrical coordinates of the sector of T containing a flow line segment, a pushed flow line segment has constant θ coordinate, the same as that of the unpushed flow line.

Proof. We push a flow line out of the cusp neighborhoods using nearest point retraction in the hyperbolic metric. In the upper half space model, nearest point retraction sends a point (x, y, z) with $z > 1$ to the point $(x, y, 1)$.

The modified singular Solv metric on the cusp neighborhoods is given by the pullback of the map f (Equation 2). Therefore, the nearest point retraction $(x, y, z) \mapsto (x, y, 1)$ will fix $x = t$ and $y = \tan^{-1}(\lambda^{2t} \tan\theta)$, and will set $z = -\log(r\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta})$ to 1. Fixing x and y forces t and θ to be fixed, and setting $z = 1$ results in r changing to $1/(e\sqrt{\lambda^{-2t}\cos^2\theta + \lambda^{2t}\sin^2\theta})$. q.e.d.

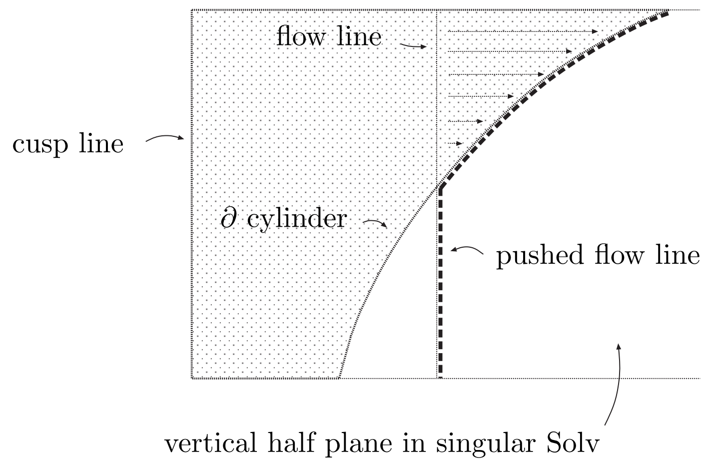

Figure 9. Pushing a flow line out of a cusp in the singular Solv metric.

To visualize this pushing, note that a flow line ℓ and a cusp flow line determine a vertical (Euclidean) plane in $\widehat{M}$. Then pushing ℓ out of the cylinder containing w is realized by projecting ℓ horizontally across this half plane to the boundary of the cylinder (see Figure 9).

1.6. Proof of Lemma 1. In this section we will prove that every (sufficiently long) segment of flow line inside a horoball is quasigeodesic. Since the modified metric and the hyperbolic metric on $\widetilde{M}$ are quasi-isometric, we will show the portions of the flow lines which lie in the horoballs are quasigeodesic in the modified metric by proving that their images are quasigeodesic in $\mathbb{H}^3$ under the map f.

Under this map, the trapped flow lines are Euclidean straight lines going off to infinity in the upper half space model of $\mathbb{H}^3$; these trapped flow lines are shown to be quasigeodesic by a fairly simple calculation. The returning flow line segments are somewhat more complicated to handle: using a direct calculation we will show that if a returning flow segment is sufficiently long, then its length is bounded above by a uniform constant multiple of the hyperbolic distance between its endpoints.

Proposition 1.23. *There is a* $K_t > 0$ *such that all trapped flow line segments are uniformly* $(K_t, 0)$*-quasigeodesic.*

Proof. Suppose ℓ is the image under f (Equation (2)) of a trapped flow line segment for which $\theta = k\pi$, $k \in \mathbb{Z}$ (the argument in the case

$\theta = k\pi + \frac{\pi}{2}$ is similar). Then f simplifies to

$$f(r, \theta, t) = (t, 0, t \log \lambda - \log r);$$

therefore ℓ is a Euclidean straight line segment in the upper half space model of $\mathbb{H}^3$ with slope $\frac{dz}{dx} = \log \lambda$.

If the endpoints of ℓ are $(x_1, 0, z_1)$ and $(x_2, 0, z_2)$, then an easy calculation shows

$$\begin{aligned}\text{length}(\ell) &= K_{\text{t}} \cdot (\text{hyp length of vertical segment} \\ &\qquad \text{between } (x_1, 0, z_1) \ \& \ (x_1, 0, z_2)) \\ &\leq K_{\text{t}} \cdot d_{\mathbb{H}^3}((x_1, 0, z_1), (x_2, 0, z_2)),\end{aligned}$$

where $K_{\text{t}} = \frac{\sqrt{1+(\log\lambda)^2}}{\log \lambda}$. Thus such flow line segments are $(K_{\text{t}}, 0)$-quasigeodesic. q.e.d.

We now consider a segment of a returning flow line inside a horoball.

Proposition 1.24. *There exist constants K_r, L_r depending only on λ such that all returning leaf segments are uniformly (K_r, L_r)-quasigeodesic.*

Proof. The segments of flow lines in the cusp have been mapped into the standard hyperbolic cusp using a carefully defined map F (Equation (3)). Recall that under F, a given flow line is specified by a fixed choice of r and θ, and is parameterized by t. By symmetry it suffices to consider the case $0 < \theta < \frac{\pi}{2}$.

Let ℓ be a segment of returning flow line inside a horoball with endpoints a and b on the boundary of the horoball. We will show that ℓ is quasigeodesic by showing that the hyperbolic length of ℓ is no more than a constant multiple of the hyperbolic distance between a and b. Let E be the Euclidean distance between a and b. Then the hyperbolic distance $d_{\mathbb{H}^3}(a, b)$ between a and b is given by

$$d_{\mathbb{H}^3}(a, b) = \cosh^{-1}\left(1 + \frac{E^2}{2}\right).$$

Now let's consider the hyperbolic length of ℓ. Let $D = \dfrac{\cosh^{-1}(2)}{2 \log \lambda}$. Divide ℓ into three pieces: one piece where $|t| \leq D$, called the **crown**, and two pieces having $|t| > D$, called **legs**. It is easy to check that the hyperbolic length of the crown is globally bounded above by a constant $B(\lambda)$ depending only on λ (the uniform boundedness follows from the invariance of the flow under vertical translation and the horizontal translations described on page 226).

Next we will show that slopes of the legs are bounded below. Suppose $|t| > D$. From the new parametrization F we obtain

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(t+t_c) = 1, \\ \frac{dy}{dt} &= \frac{\log\lambda}{\cosh(2t\log\lambda)}, \qquad \text{and} \\ \frac{dz}{dt} &= \log\lambda\tanh(2t\log\lambda).\end{aligned}$$

Since $|t| > D$ and $\cosh(x)$ is monotonic, we know $\cosh(2t\log\lambda) > \cosh(2D\log\lambda)$. Further, by our choice of D, we know $\cosh(2D\log\lambda) = 2$. Therefore $\tanh^2(2t\log\lambda) = 1 - \frac{1}{\cosh^2(2t\log\lambda)} > 1 - \frac{1}{4} = \frac{3}{4}$, so

$$\left|\frac{dz}{dx}\right| = |\log\lambda\tanh(2t\log\lambda)| > \frac{\sqrt{3}\log\lambda}{2}.$$

Also, $\sinh^2(2t\log\lambda) = \cosh^2(2t\log\lambda) - 1 > 3$, so

$$\left|\frac{dz}{dy}\right| = |\sinh(2t\log\lambda)| > \sqrt{3}.$$

Let $A = \min\left(\frac{\sqrt{3}\log\lambda}{2}, \sqrt{3}\right)$. Then both $\left|\frac{dz}{dx}\right|, \left|\frac{dz}{dy}\right| > A$, and by an argument similar to the one used in the trapped flow line case, the hyperbolic length of a leg is at most a uniform constant multiple K_1 of the length of the vertical segment from the boundary of the horoball to the top of the leg.

We have just shown that the legs of ℓ are uniform quasigeodesics. Since the crown of ℓ has bounded length, in order to prove that all of ℓ is a uniform quasigeodesic, one just needs to compare the whole length of ℓ with the distance between its endpoints. If z_0 is the z value at the top of a leg and $|\ |_{\mathbb{H}^3}$ denotes the length of a path in the hyperbolic metric,

$$\begin{aligned}|\ell|_{\mathbb{H}^3} &= 2|\mathrm{leg}|_{\mathbb{H}^3} + |\mathrm{crown}|_{\mathbb{H}^3} \\ &\leq 2K_1\log(z_0) + B(\lambda).\end{aligned}$$

Let $\Delta x, \Delta z$ be the change in x and z coordinates along one leg of ℓ. Since $|\tanh(x)| < 1$, we know $|\frac{dz}{dx}| < \log\lambda$. There exists a constant L_1 so that if the length of the flow line ℓ is greater than L_1, then $\Delta x = \Delta t > \max(1, \frac{1}{\log\lambda})$. Then, if the length of ℓ is greater than L_1,

$$\begin{aligned}z_0 &= \Delta z + 1 \\ &\leq \Delta x\log\lambda + 1 \\ &< 2\Delta x\log\lambda,\end{aligned}$$

since $\Delta x > \frac{1}{\log\lambda}$. So, if ℓ is longer than L_1

$$\begin{aligned}
|\ell|_{\mathbb{H}^3} &\leq 2K_1\log(z_0) + B(\lambda) \\
&\leq 2K_1\log(2\Delta x\log\lambda) + B(\lambda) \\
&\leq 2K_1\log(\Delta x) + C(\lambda) \quad \text{for some constant } C(\lambda) \\
&\leq 2K_1\log E + C(\lambda) \quad (\text{note } \log E > 0 \text{ since } E \geq \Delta x > 1) \\
&\leq K_1(2\log E) + C(\lambda) \\
&\leq K_1(\cosh^{-1}(1 + \frac{E^2}{2}) + \log 2) + C(\lambda) \quad \text{since } E > 2\sqrt{2} - 2 \\
&= K_1(d(a,b) + \log 2) + C(\lambda) \\
&\leq K_1 d(a,b) + D(\lambda),
\end{aligned}$$

where $D(\lambda)$ depends only on λ. Taking $L_{\mathrm{r}} = \max(L_1, D(\lambda))$ and $K_{\mathrm{r}} = K_1$ gives the result. q.e.d.

Proof of Lemma 1. If $K_{\mathrm{c}} = \max\{K_{\mathrm{t}}, K_{\mathrm{r}}\}$ and $L_{\mathrm{c}} = L_{\mathrm{r}}$, then every segment of flow line inside a horoball is a $(K_{\mathrm{c}}, L_{\mathrm{c}})$-quasigeodesic. q.e.d.

1.7. Proof of Lemma 2. In this section we prove Lemma 2, which says that every flow line visits a cylinder T in $\widehat{M}$ (and hence a horoball in $\widetilde{M}$) at most once.

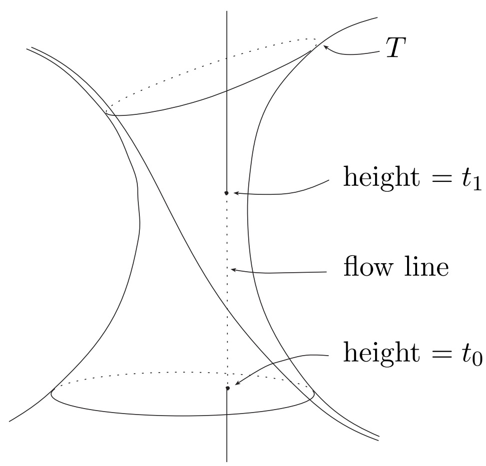

Figure 10. A typical flow line intersection with T.

Proof of Lemma 2. For a fixed cusp neighborhood T, we may choose local coordinates r, θ, t. In these coordinates, each flow line which intersects T is determined by some fixed choice of r and θ, and is parameterized by the coordinate t. As flow lines are straight lines in these coordinates, a typical flow line which intersects T will enter the cylinder at some height t_0 and will remain inside the cylinder until some height

$t_1 > t_0$, where it will exit. It is clear from the shape of a cusp that a flow line may enter any given sector of T at most one time. Every returning flow line remains in the same sector of T as t grows, and every trapped flow line stays on the same line radiating from the origin. Therefore a flow line may visit a covering cylinder T in $\widehat{M}$ at most once. Figure 10 illustrates this observation in the case that T is a 2-pronged cusp. q.e.d.

2. Flow Lines Are Near Geodesics

In this section we will find a uniform constant R_Φ such that every flow line segment lies in a neighborhood of radius R_Φ of the geodesic connecting its endpoints. We will establish this result for flow line segments, but we need only pass to longer and longer segments of flow line to have the same result for entire flow lines.

2.1. Pushed Flow Lines Lie Near Strings of Beads. Identify F with a fixed fiber F_x in M, where $F_x = p^{-1}(x)$ for some $x \in \mathbb{S}^1$. Let $F_N = F \cap N$, and let $N_\infty \to N$ be the infinite cyclic cover of the neutered space dual to F_N. We may identify N_∞ with $F_N \times \mathbb{R}$ in such a way that the action of $n \in \mathbb{Z}$ by covering transformations on N_∞ corresponds to $\tau^n : F_N \times \mathbb{R} \to F_N \times \mathbb{R}$ where $\tau(p,t) = (\Psi^{-n}(p), t+n)$. Note that $N_\infty \subseteq M_\infty$, where $M_\infty \to M$ is the corresponding infinite cyclic cover of M.

2.1.1. Pushed Flow Segments are Quasigeodesic in N.

Proposition 2.1. *There exist constants $K_p > 1, L_p > 0$ such that every pushed flow line segment is a (K_p, L_p)-quasigeodesic in the induced hyperbolic metric on N_∞, and hence a (K_p, L_p)-quasigeodesic in N.*

Proof. Any two metrics on the compact set N are quasi-isometric. We will prove that there exist $K_\mathrm{p}, L_\mathrm{p}$ satisfying the proposition using the singular Solv metric on N; from this the result will follow for the hyperbolic metric on N.

We want to compare the length in the modified metric along a pushed flow line between two points with the length in the modified metric of a geodesic segment between the two points. To do this, we show an even stronger result: that the maximum length of a pushed flow line between successive lifts of F is no more than a bounded multiple of the minimum (neutered space) distance between the lifts.

Let $\pi_2 : F_N \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor. From the structure of the flow and the pushing map operation, it follows that π_2 restricted to any pushed flow line is monotone, so once any pushed flow line passes through one copy of the fiber $F_N \times \{t\}$ for $t \in \mathbb{R}$, it never returns. Thus, the length of any given pushed flow line segment is no more than the maximum length of a pushed flow line segment between $F_N \times \{0\}$ and $F_N \times \{1\}$ multiplied by the number n of copies of $F_N \times [0,1]$ the pushed flow line intersects.

Every segment of flow line in $F \times [0,1]$ has singular Solv length 1. By the choice made on page 216, the hyperbolic distance between horoballs is at least 1000. Recall that a cylinder is covered by a horoball. The hyperbolic metric is $(K,0)$-quasi-isometric (for some $K > 0$) to the singular Solv metric, so the singular Solv distance between cylinders is at least $1000/K$. Therefore each segment meets at most $1 + K/1000$ cylinders.

Let T be a cylinder in $F \times \mathbb{R}$. Then $A = (\partial T) \cap (F \times [0,1])$ is a smooth, compact annulus. Recall from the discussion on page 227 that the pushing map produces a foliation on the boundary of the neutered space N. Therefore pushing all flow line segments inside $T \cap (F \times [0,1])$ out to the boundary annulus A gives a foliation on A. This foliation is transverse to the fibers $F \times t$ for the following reason: The fibers $F \times t$ are horizontal in $F \times \mathbb{R}$. Under the map f defined in Equation (2), a constant value for t gives a constant value for the x coordinate in the upper half space model, and from Figure 5 we see that the foliation on the boundary of the neutered space is transverse to every line of constant x value. The leaves of this foliation vary continuously as you travel around A, therefore there is a uniform upper bound to the singular Solv length of any pushed flow line segment on A. This gives a uniform upper bound $B > 1$ to the singular Solv length of any pushed flow line segment in $F_N \times [0,1]$.

Let ϵ be the minimum distance between the compact surfaces $F_N \times \{0\}$ and $F_N \times \{1\}$ in N_∞. Choose $K_{\mathrm{p}} > \max(\frac{2B}{\epsilon}(1+\frac{K}{1000}),1)$, and let $L_{\mathrm{p}} = 1$. Let $\overline{\ell}$ be a finite segment of pushed flow line of length at least L_{p} with endpoints $a \in F_N \times 0$ and $b \in F_N \times \{t\}$, where $t \in [n, n+1)$ for some integer $n \geq L_{\mathrm{p}} = 1$. Then

$$\begin{aligned} d_{\overline{\ell}}(a,b) &\leq B\left(1+\frac{K}{1000}\right)\cdot(n+1) \\ &< 2B\left(1+\frac{K}{1000}\right)\cdot n \quad \text{(since } B>1\text{)} \\ &< K_{\mathrm{p}}\epsilon \cdot n \\ &\leq K_{\mathrm{p}} \cdot d_{N_\infty}(a,b). \end{aligned}$$

Therefore every pushed flow line is a $(K_{\mathrm{p}}, L_{\mathrm{p}})$-quasigeodesic. q.e.d.

2.1.2. Quasigeodesic Segments Lie Near Strings of Beads. Recall that a string of beads γ_+ is the geodesic γ together with all the beads which γ intersects. Let $CH(\gamma_+)$ denote the convex hull of γ_+, given by the union of all the tetrahedra with all four vertices in γ_+, and let $\pi_{\mathrm{CH}} : \widetilde{N} \to CH(\gamma_+)$ be nearest point retraction onto this convex hull. The following 3 lemmas will be used to show that any quasigeodesic segment in the path metric on $\widetilde{N}$ must lie close to the string of

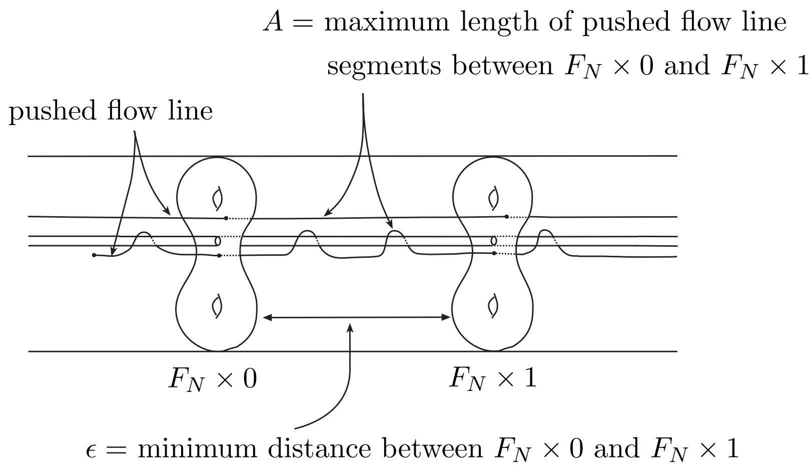

Figure 11. Distance comparison in the infinite cyclic cover of N.

beads connecting its endpoints. Lemma 5 is a standard fact from hyperbolic geometry, while Lemmas 6 and 7 are straightforward applications of the thin triangles property of hyperbolic space (Lemma 7 follows from Lemma 6).

Lemma 5. *If c is a path segment which is a hyperbolic distance at least r from $CH(\gamma_+)$, then $|c| \geq \sinh r \cdot |\pi_{CH}(c)|$.*

Lemma 6. *Every point in a geodesic tetrahedra in hyperbolic space lies within distance 2δ of an edge, where δ is the thin triangles constant $\log(1+\sqrt{2})$.*

Lemma 7. *Every point in $CH(\gamma_+)$ lies within distance 4δ of γ_+.*

We will use Lemma 5 to show that a pushed flow line segment cannot stray very far away from the string of beads connecting its endpoints, for if it does, there exists a path in $\widetilde{N}$ with the same endpoints which is much shorter than the pushed flow line segment, contradicting that the pushed flow line is quasigeodesic in $\widetilde{N}$.

Proposition 2.2. *Let $\widetilde{N} \subseteq \mathbb{H}^3$ be the neutered space described above. Given $K > 1$, $L > 0$, there exists $R > 0$ such that if q is any (K, L)-quasigeodesic segment in $\widetilde{N}$ and γ is the hyperbolic geodesic segment in $\mathbb{H}^3$ connecting the endpoints of q, then q lies within hyperbolic distance R of the hyperbolic convex hull of the string of beads along γ.*

Proof. We will show that if a neutered space quasigeodesic gets too far (in the hyperbolic metric) from the hyperbolic convex hull of the string of beads connecting its endpoints, then projecting the path onto the convex hull (followed by pushing the projected path out of the horoballs)

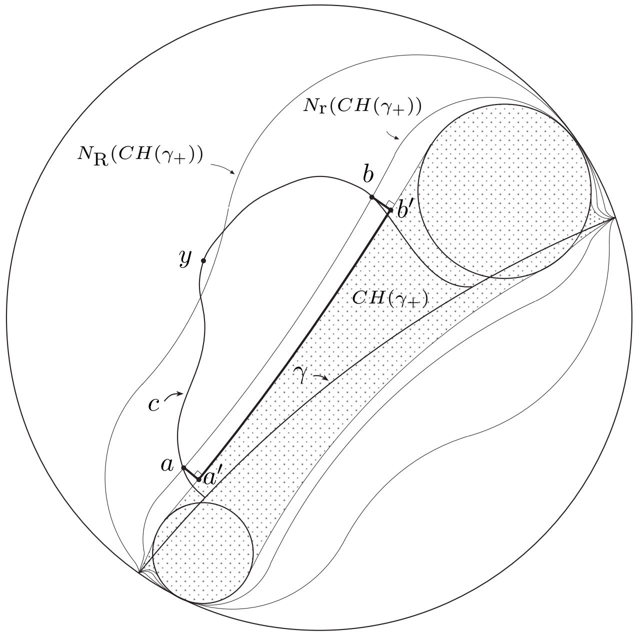

Figure 12. Quasigeodesic segments lie near convex hulls of strings of beads.

produces a much shorter path in the neutered space (in the path metric on $\widetilde{N}$), contradicting that the original path was quasigeodesic.

Suppose q is any (K, L)-quasigeodesic segment in $\widetilde{N}$ and γ is the hyperbolic geodesic segment in $\mathbb{H}^3$ connecting the endpoints of q. Choose $r > \max(L, \sinh^{-1}(2Ke^{4\delta}))$ and let $R = 2Kre^{r+4\delta} + r$, where δ is the thin triangles constant. Then q lies completely inside the hyperbolic R-neighborhood of the convex hull of the string of beads along γ, for suppose a point y in q falls outside this neighborhood. Then there is a subsegment c of q (containing y), which is at least hyperbolic distance r from the convex hull of the string of beads, of hyperbolic length at least

$$|c| \geq 2(R - r) = 4Kre^{r+4\delta} \tag{4}$$

(see Figure 12). Note that, by our choice of r, we also have $|c| > L$. Let a and b be the endpoints of this subsegment c, and let s be the path $A \cup \pi_{\mathrm{CH}}(c) \cup B$, where A and B are paths of hyperbolic length r connecting a to $a' = \pi_{\mathrm{CH}}(a)$ and b to $b' = \pi_{\mathrm{CH}}(b)$, respectively. Paths A, B, and $\pi_{\mathrm{CH}}(c)$ need not be in $\widetilde{N}$, but may pass through many horoballs not on the string of beads. Let $\overline{A}$, $\overline{B}$, $\overline{\pi_{\mathrm{CH}}(c)}$, and $\overline{s}$ be the results of pushing A, B, $\pi_{\mathrm{CH}}(c)$, and s out of the horoballs into $\widetilde{N}$; then $\overline{s} = \overline{A} \cup \overline{\pi_{\mathrm{CH}}(c)} \cup \overline{B}$. We must find upper bounds for the hyperbolic lengths of these subpaths of $\overline{s}$.

The path $\pi_{\mathrm{CH}}(c)$ lies along the boundary of the convex hull of the string of beads γ_+. To understand by how much $\pi_{\mathrm{CH}}(c)$ lengthens when pushed out of all horoballs into the neutered space, we must know how deeply $\pi_{\mathrm{CH}}(c)$ can enter any given horoball. Let H be a horoball that $\pi_{\mathrm{CH}}(c)$ enters. Note that H cannot be a bead along γ_+ since H contains a nontrivial subsegment of $\pi_{\mathrm{CH}}(c)$. By Lemma 7, every point in $H \cap CH(\gamma_+)$ lies within hyperbolic distance 4δ of γ_+. But by our initial choice of horoballs, H lies a distance at least 1000 from all the other horoballs. Thus $H \cap CH(\gamma_+)$ lies with hyperbolic distance 4δ of γ. So no point of $CH(\gamma_+)$ inside H can be further than 4δ from the boundary of H. Therefore, pushing $\pi_{\mathrm{CH}}(c)$ from the interior of H onto the boundary of H can increase its length by no more than the factor $e^{4\delta}$, and indeed

$$|\overline{\pi_{\mathrm{CH}}(c)}| \quad \leq \quad e^{4\delta} \cdot |\pi_{\mathrm{CH}}(c)|. \tag{5}$$

Similarly, since every point of A and B is within $r+4\delta$ of γ, no point of A or B inside a horoball H can be further than $r+4\delta$ from the boundary of H; therefore each of $\overline{A}$ and $\overline{B}$ will be no longer than $e^{r+4\delta} \cdot r$. Therefore

$$\begin{aligned} K \cdot |\overline{s}| &= K \cdot |\overline{A} \cup \overline{\pi_{\mathrm{CH}}(c)} \cup \overline{B}| \\ &\leq K \cdot (e^{r+4\delta} \cdot r + e^{4\delta} \cdot |\pi_{\mathrm{CH}}(c)| + e^{r+4\delta} \cdot r) && \text{by Equation 5} \\ &= 2Kre^{r+4\delta} + Ke^{4\delta} \cdot |\pi_{\mathrm{CH}}(c)| \\ &\leq \frac{|c|}{2} + Ke^{4\delta} \cdot |\pi_{\mathrm{CH}}(c)| && \text{by Equation 4} \\ &< \frac{|c|}{2} + \frac{Ke^{4\delta}}{\sinh r}|c| && \text{by Lemma 5} \\ &< \frac{|c|}{2} + \frac{Ke^{4\delta}}{2Ke^{4\delta}}|c| \\ &\leq \frac{|c|}{2} + \frac{|c|}{2} \\ &= |c|. \end{aligned}$$

Notice that since c is a subsegment of the (K, L)-quasigeodesic segment with $|c| > L$, $|c| < K \cdot d_{\widetilde{N}}(a, b)$. Hence $K \cdot |\overline{s}| < |c| < K \cdot d_{\widetilde{N}}(a, b)$, so $|\overline{s}| < d_{\widetilde{N}}(a, b)$, a contradiction to the quasigeodesic behavior of c. q.e.d.

Proposition 2.3. *There exists $R_s > 0$ such that every pushed flow line segment lies within hyperbolic distance R_s of the string of beads connecting its endpoints.*

Proof. Since pushed flow lines are $(K_{\mathrm{p}}, L_{\mathrm{p}})$-quasigeodesics in $\widetilde{N}$, by the previous proposition there exists a constant $R_{\mathrm{p}} > 0$ so that every pushed flow line segment lies within a hyperbolic neighborhood of radius R of the *convex hull* of the string of beads connecting its endpoints. Thus, in light of Lemma 7, every pushed flow line segment lies within $R_{\mathrm{s}} = R_{\mathrm{p}} + 4\delta$ of the string of beads connecting its endpoints. q.e.d.

2.2. Flow Segments Lie Near Strings of Beads. We now know that a pushed flow line segment lies in a neighborhood of the string of beads along the geodesic connecting its endpoints. To prove that flow line segments lie near the string of beads connecting their endpoints, we only need to prove that the segments of flow line which lie inside a horoball lie in a uniform neighborhood of the string of beads. This does not follow automatically from what we have already done, for given a large neighborhood of a string of beads, most horoballs intersecting this neighborhood are not beads on this string. A segment of flow line which lies inside one of the beads trivially lies inside a neighborhood of the string of beads, but the non-bead horoballs need special consideration.

Proposition 2.4. *Every flow line segment ℓ can be extended to a flow line segment ℓ^+ which lies inside neighborhood of radius R_f of the string of beads along the geodesic connecting the endpoints of ℓ^+.*

Proof. Let ℓ be a flow line segment. This flow line segment is either part of a trapped flow line (one that spends infinite backwards or forwards time in a cusp) or it isn't.

Case 1: Suppose ℓ is not a subsegment of a trapped flow line. Extend ℓ a bit further if necessary to get ℓ^+ so that the endpoints of ℓ^+ lie outside the cusps; then ℓ^+ and its associated pushed flow line $\overline{\ell^+}$ have the same endpoints. Let γ be the hyperbolic geodesic connecting these endpoints; then $\overline{\ell^+}$ lies inside the R_{s}-neighborhood of γ_+, the string of beads along γ by Proposition 2.3. The flow line ℓ^+ agrees with $\overline{\ell^+}$ outside the cusps, thus $\ell^+ \cap \overline{\ell^+}$ lies inside the R_{s}-neighborhood of γ_+ as well.

Consider a subsegment of ℓ^+ which lies inside a cusp, with endpoints a and b on the boundary of the cusp (see Figure 13). Since the a and b are points on the pushed flow line, they lie within a distance R_{s} of the string of beads along the geodesic. Therefore, a and b lie within R_{s} of $CH(\gamma_+)$, the convex hull of the string of beads γ_+. Let γ' be the geodesic segment connecting a and b. Since a geodesic segment is furthest from a convex set at its endpoints, we know that γ' lies within R_{s} of $CH(\gamma_+)$. Therefore, by Lemma 7, γ' lies within distance $R_{\mathrm{s}} + 4\delta$ of γ_+. But a flow line is quasigeodesic inside the cusps, and therefore ℓ^+ lies within a neighborhood of some radius R_{c} of γ'. Hence, the portions of ℓ^+ inside the cusp (and thus the entire flow line segment ℓ^+) lie within a neighborhood of radius $R_{\mathrm{f}} = R_{\mathrm{c}} + R_{\mathrm{s}} + 4\delta$ of the string of beads along γ.

Case 2: Suppose now that ℓ is a subsegment of a trapped flow line. Extend this flow line segment all the way to its end, where the trapping horoball meets the sphere at infinity. Let $\overline{\ell}$ be the associated pushed flow line. The portion of ℓ inside the trapping horoball has θ coordinate a multiple of $\frac{\pi}{2}$, and therefore is mapped under f to a Euclidean straight

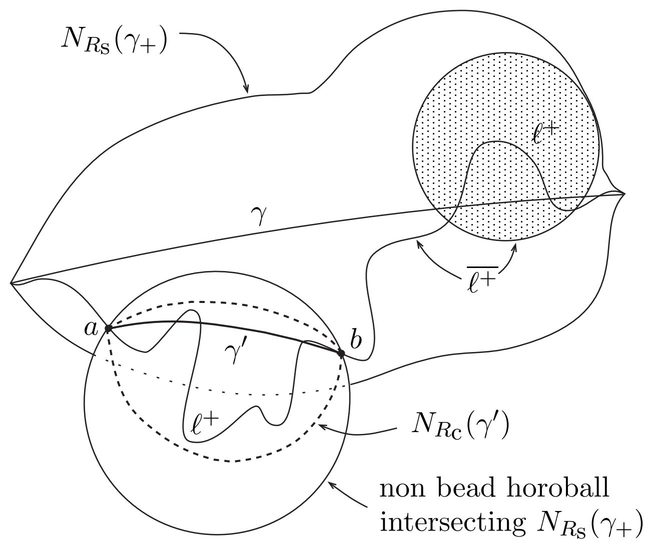

Figure 13. Flow line segments lie inside $R_f = R_c + R_s + 4\delta$ neighborhoods of strings of beads.

line of slope $\log \lambda$ in the x, z plane in the standard cusp in the upper half space model of $\mathbb{H}^3$. Hence ℓ projects to the boundary of the horoball as a horocycle (rather than just as a single point). Thus $\overline{\ell}$ has the same endpoints as ℓ, with at least one of these endpoints lying on the sphere at infinity. Let γ be the geodesic connecting these endpoints. Then the trapping horoball(s) are beads on the the string of beads along γ, so it is immediate that the portion of ℓ inside the trapping horoball lies within the string of beads, and hence any neighborhood of the string of beads, along γ. The argument for the rest of ℓ follows exactly as above. Therefore, every flow line segment lies within a neighborhood $R_f = R_c + R_s + 4\delta$ of the string of beads connecting its endpoints. q.e.d.

2.3. Flow Line Segments Lie Close to Geodesics. We now know that every flow line segment is close to a string of beads. The goal of this section is to show that every flow line segment is close to the *string* in the string of beads.

The proof proceeds as follows: Consider a flow line segment ℓ which lies in an R_f neighborhood of a bead. We will show that the subsegments of ℓ which lie near the bead but do not enter the bead have a bounded length M_f depending only on R_f, and therefore these subsegments are close to the string. On the other hand, since flow lines are quasigeodesic inside cusps, the subsegments of ℓ which lie inside the bead lie within R_c of some geodesic segment in the bead. We will show this geodesic lies within an $R_f + M_f$ neighborhood of the string, and therefore the subsegment of ℓ inside the bead lies within $R_f + M_f + R_c$ of the string.

Lemma 8 (Limited Time Lemma). *For any $R > 0$, there is an $M_R > 0$ such that every segment of flow line in $\mathbb{H}^3$ which lies within hyperbolic distance R of a horoball but does not enter the horoball has hyperbolic length at most M_R.*

To prove the Limited Time Lemma, we will first find a bound for the length of ℓ in the singular Solv metric on M. Then we will use a compactness argument to produce a bound for the hyperbolic length of ℓ. Let $d_{\widetilde{F^+}}(p,q)$ be the distance between points p and q in the singular Euclidean metric on $\widetilde{F^+}$, and let $d_{\widetilde{M^+}}(p,q)$ be the distance between p and q in the singular Solv metric on $\widetilde{M^+}$. We will need to make use of the lemma below:

Lemma 9. *Given $K > 0$ there exists $L > 0$ such that if $p, q \in \widetilde{F^+} \times 0$ and $d_{\widetilde{F^+}}(p,q) > L$ then $d_{\widetilde{M^+}}(p,q) > K$.*

Proof of Lemma 9. Suppose not. Then for each $n \in \mathbb{N}$ there exist $p_n, q_n \in \widetilde{F^+} \times 0$ such that $d_{\widetilde{F^+}}(p_n, q_n) > n$ and $d_{\widetilde{M^+}}(p_n, q_n) \leq K$. Apply covering transformations to the pairs p_n, q_n such that the p_n all lie in a given (compact) fundamental domain D. Then, up to subsequences, we may assume p_n converges to $p \in D$. Let B be the ball in $\widetilde{M^+}$ of (singular Solv) radius $K+1$ centered at p. Then for n sufficiently large, $p_n, q_n \in B$. Now B is compact; therefore $\widetilde{F^+} \cap B$ is compact. But $d_{\widetilde{F^+}}(p_n, q_n) > n \to \infty$ as $n \to \infty$, which contradicts compactness of $\widetilde{F^+} \cap B$. q.e.d.

Proof of Limited Time Lemma. Let C be the compact neighborhood of a cusp line c in M^+ such that $C - c$ is a cusp of M. Let H be a horoball in $\mathbb{H}^3$ covering $C - c$, and let $Z = N_R(H)$ be the neighborhood of H of hyperbolic radius R. Suppose $\widetilde{\ell}$ is a segment of flow line in $\mathbb{H}^3$ contained in $Z - H$.

Let $\widehat{\ell}, \widehat{H}$, and $\widehat{Z}$ be the projections of $\widetilde{\ell}, H$, and Z to $\widehat{M}$, and let $\widehat{c} \subseteq \widetilde{M^+}$ be the cusp line in the closure of $\widehat{H}$ covering c in M^+. Then $\widehat{\ell} \in \widehat{Z} - \widehat{H}$. We will first show that there is a universal bound to the singular Solv length of $\widehat{\ell}$ (and hence $\widetilde{\ell}$) by bounding the distance between $\widehat{\ell}$ and $\widehat{c}$ in terms of the orbit of a point on $\widehat{\ell}$ in F^+ under the monodromy Ψ. Then we will use a compactness argument to produce an upper bound on the hyperbolic length of $\widetilde{\ell}$.

The cusp line c is a closed flow line in M^+; without loss of generality (by passing to a finite cover) we may assume that c has period 1. Choose a fixed fiber $\widetilde{F^+}$ in $\widetilde{M^+} \simeq \widetilde{F^+} \times \mathbb{R}$ and identify $\widetilde{F^+}$ with $\widetilde{F^+} \times 0$. Define $p_0 = \widehat{c} \cap \widetilde{F^+}$. Let $\widehat{\Psi^+} : \widetilde{F^+} \to \widetilde{F^+}$ be the lift of Ψ^+ such that $\widehat{\Psi^+}(p_0) = p_0$, and let $\tau : \widetilde{F^+} \times \mathbb{R} \to \widetilde{F^+} \times \mathbb{R}$ be the generator for the covering translations such that $\tau(p,t) = ((\widehat{\Psi^+})^{-1} p, t+1)$. Let S be the set of integers n for which $\widehat{\ell} \cap \tau^n(\widetilde{F^+}) \neq \emptyset$. For each $n \in S$, define $q_n = \tau^{-n}(\widehat{\ell} \cap \tau^n(\widetilde{F^+}))$. Then $\widehat{\ell} \subset q_0 \times \mathbb{R}$. Therefore

$$\begin{aligned} q_n &= \tau^{-n}(\widehat{\ell} \cap \tau^n(\widetilde{F^+})) \\ &= \tau^{-n}((q_0 \times \mathbb{R}) \cap \tau^n(\widetilde{F^+})) \\ &= \tau^{-n}(q_0, n) \\ &= \widehat{\Psi^+}^n(q_0, 0). \end{aligned}$$

As we have identified $\widetilde{F^+}$ with $\widetilde{F^+} \times 0$, we can write $q_n = \widehat{\Psi^+}^n(q_0)$. So for all $n \in S$,

$$d_{\widetilde{M^+}}(p_0, q_n) = d_{\widetilde{M^+}}(p_0, \widehat{\Psi^+}^n(q_0)).$$

We will show that if n is large, then $d_{\widetilde{F^+}}(p_0, \widehat{\Psi^+}^n(q_0))$ is large. Then Lemma 9 will imply that $d_{\widetilde{M^+}}(p_0, \widehat{\Psi^+}^n(q_0))$ is large, hence p_0 and $\widehat{\Psi^+}^n(q_0)$ are far apart in in the singular Solv metric on $\widetilde{M^+}$.

Since $\widehat{H}$ is a neighborhood of p_0 which is invariant under τ, and since $\widehat{\ell}$ stays outside $\widehat{H}$, we know that $\widehat{\Psi^+}^n(q_0)$ is bounded away from p_0 for all $n \in \mathbb{Z} \cap [n_0, n_1]$. Thus there exists a uniform $\epsilon > 0$ (depending only on C, independent of our choice of ℓ) such that for all $n \in \mathbb{Z} \cap [n_0, n_1]$,

$$d_{\widetilde{F^+}}(p_0, \widehat{\Psi^+}^n(q_0)) > \epsilon.$$

The stable and unstable foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ on $\widetilde{F^+}$ give pseudo-metrics d_+ and d_- on $\widetilde{F^+}$ using the transverse measures of the foliations. Since Ψ^+ is pseudo-Anosov, for any points p and q on F^+,

$$\begin{aligned} d_+(\Psi p, \Psi q) &= \lambda \cdot d_+(p, q), \qquad \text{and} \\ d_-(\Psi p, \Psi q) &= \lambda^{-1} \cdot d_-(p, q). \end{aligned}$$

Away from the singularities in $\widetilde{F^+}$, $d_+ + d_-$ is locally the taxicab metric. The following comparison holds at the infinitesimal level between $d_+ + d_-$ and the singular Solv metric on $\widetilde{F^+}$:

$$\frac{1}{2} \leq \frac{\sqrt{dx^2 + dy^2}}{|dx| + |dy|} \leq 1.$$

Integrating this along smooth curves allows us to approximate distance $d_{\widetilde{F^+}}(p, q)$ between any points $p, q \in \widetilde{F^+}$ using $d_+(p, q) + d_-(p, q)$, as follows:

$$\frac{1}{2} \leq \frac{d_{\widetilde{F^+}}(p, q)}{d_+(p, q) + d_-(p, q)} \leq 1.$$

Let $a = d_+(p_0, q_0)$, and $b = d_-(p_0, q_0)$. Then

$$\begin{aligned} d_+(p_0, \widehat{\Psi^+}^n(q_0)) &= d_+(\widehat{\Psi^+}^n(p_0), \widehat{\Psi^+}^n(q_0)) = \lambda^n a, \qquad \text{and} \\ d_-(p_0, \widehat{\Psi^+}^n(q_0)) &= \lambda^{-n} b. \end{aligned}$$

Therefore, since $\widehat{\ell}$ stays outside $\widehat{H}$, we know that for all $n \in \mathbb{Z} \cap [n_0, n_1]$,

$$\begin{aligned} \lambda^n a + \lambda^{-n} b &= d_+(p_0, \widehat{\Psi^+}^n(q_0)) + d_-(p_0, \widehat{\Psi^+}^n(q_0)) \\ &\geq d_{\widetilde{F^+}}(p_0, \widehat{\Psi^+}^n(q_0)) \\ &> \epsilon. \end{aligned}$$

Thus the values of a and b are bounded below.

In M^+, the intersection of every copy of the fiber surface F with Z is a disk. Therefore, $\widehat{Z} \cap (\widetilde{F^+} \times 0)$ is a disk as well, and is therefore compact. Thus there exists $K > 0$ (depending only on R) such that for any $q \in \widetilde{F^+} \times 0$, if $d_{\widetilde{M^+}}(p_0, q) > K$, then $q \notin \widehat{Z}$. By Lemma 9, for this K there is an $L > 0$ (depending only on K) such that for any $q \in \widetilde{F^+} \times 0$, if $d_{\widetilde{F^+}}(p_0, q) > L$, then $q \notin \widehat{Z}$. Now, for this L, there exists a $B > 0$ depending only on L (and hence on R), λ, and ϵ, such that for all $|n| > B$, $\lambda^n a + \lambda^{-n} b > 2L$. Then, for $|n| > B$,

$$d_{\widetilde{F^+}}(p_0, \widehat{\Psi^+}^n(q_0)) \geq \frac{1}{2}(\lambda^n a + \lambda^{-n} b) > L$$

Therefore, for $|n| > B$, $\widehat{\Psi^+}^n(q_0) \notin \widehat{Z}$.

Since $\widehat{\ell} \subseteq \widehat{Z} - \widehat{H}$, we know $\widehat{\ell}$ crosses at most $2B$ translates of $\widetilde{F^+} \times 0$ in $\widehat{M}$, and therefore $\widehat{\ell}$ has t coordinate that changes by at most $2B + 2$. Therefore, $\widehat{\ell}$ has singular Solv length at most $2B + 2$. The projection of $\widehat{\ell}$ to M is contained in the compact set $Z - C$, and all metrics on a compact set are quasi-isometric; it follows that there is a universal upper bound M_R (depending only on R) to the hyperbolic length of ℓ, and hence the hyperbolic length of $\widetilde{\ell}$. q.e.d.

2.3.1. Flow Line Segments Lie Inside Neighborhoods.

Theorem 2.5. *There is a constant $R_\Phi > 0$ such that every segment of flow line ℓ of the suspension flow on M can be extended to a flow line segment ℓ^+ which lies within the neighborhood of radius R_Φ of the geodesic connecting the endpoints of ℓ^+.*

The proof will make use of the following lemma:

Lemma 10. *Let $R > 0$. Suppose H_1 and H_2 are horoballs on a string of beads γ_+ such that $N_R(H_1) \cap N_R(H_2) \neq \emptyset$. Then there is a $D > 0$ depending only on R such that every point in $N_R(H_1) \cap N_R(H_2)$ lies within $D + 2\delta$ of the string γ.*

Proof. Let α be the geodesic segment perpendicular to both H_1 and H_2, and let d be the distance between H_1 and H_2. Then $N_R(H_1) \cap N_R(H_2)$ is a compact set whose diameter depends only on d and R. As d gets larger, the diameter of $N_R(H_1) \cap N_R(H_2)$ gets smaller; therefore since R is fixed and d is bounded below by 1000 (page 216), there is a maximum diameter D of $N_R(H_1) \cap N_R(H_2)$ depending only on R. Let

x be midpoint of α; then $x \in N_R(H_1) \cap N_R(H_2)$ and thus every point in $N_R(H_1) \cap N_R(H_2)$ lies within D of x.

Let $g_1 \in H_1$ and $g_2 \in H_2$ be the endpoints of the subsegment of γ which lies between H_1 and H_2, and let $a_1 \in H_1$ and $a_2 \in H_2$ be the endpoints of α. Create a hyperbolic quadrilateral by connecting a_i to g_i by a geodesic segment β_i for $i = 1, 2$. Then β_1 and β_2 are distance at least $1000 > 4\delta$ apart. By thin triangles, we know x lies within 2δ of $\beta_1 \cup \gamma \cup \beta_2$. Since x is the midpoint of α, we know x is not within 2δ of β_1 or β_2; therefore x lies within a distance at most 2δ of some point on γ. Therefore, every point in $N_R(H_1) \cap N_R(H_2)$ lies within $D + 2\delta$ of γ.
q.e.d.

Proof of Theorem 2.5. Let ℓ be a flow line segment, γ be the geodesic segment with the same endpoints, and γ_+ be the string of beads along γ. Then by Proposition 2.4, ℓ can be extended to a flow line segment ℓ^+, with endpoints outside or on the boundary of the beads, which lies within a neighborhood of radius R_{f} of γ_+. Away from the beads, ℓ^+ lies within R_{f} of the string, so we only need to consider the portions of ℓ^+ which lie within R_{f} of a bead. Let a and b be the points where ℓ^+ last enters and first exits the R_{f} neighborhood of a bead H.

We will now show a and b must lie close to γ. Let H_1 be the bead adjacent to H along γ that is nearest to a. If $N_{R_{\mathrm{f}}}(H)$ and $N_{R_{\mathrm{f}}}(H_1)$ do not intersect, then it is clear that a must lie within R_{f} of γ. If $N_{R_{\mathrm{f}}}(H) \cap N_{R_{\mathrm{f}}}(H_1) \neq \emptyset$, then by Lemma 10 every point in $\ell \cap N_{R_{\mathrm{f}}}(H) \cap N_{R_{\mathrm{f}}}(H_1)$ lies within $D + 2\delta$ of γ. Hence a (and by a similar argument, b) lies within $R_{\mathrm{D}} = \max(R_{\mathrm{f}}, D + 2\delta)$ of γ.

Let M_{f} be the constant that comes from the Limited Time Lemma (Lemma 8) for $R = R_{\mathrm{f}}$. This segment of ℓ^+ between a and b is either shorter than or longer than M_{f}. If the segment is shorter than M_{f}, then the entire segment is within $R_{\mathrm{D}} + M_{\mathrm{f}}$ of the geodesic, by measuring along the segment back to endpoint a (at most M_{f}), then joining up to the string with a path of length at most R_{D}.

So assume that the segment is longer than M_{f}. Then by the Limited Time Lemma, the flow line segment must enter the cusp (see Figure 14) The flow line segment does not enter the cusp more than once, by Lemma 2. Therefore this flow line segment consists of a segment from a to the point a' where the flow line meets the cusp (of length at most M_{f}), connected to a segment from a' to b' which lies entirely within the cusp, connected to a segment from b' to b of length at most M_{f}. As above in the case of the short flow line segment, the subsegments between a and a' and between b and b' are within $R_{\mathrm{D}} + M_{\mathrm{f}}$ of the geodesic, by measuring along the subsegments. It remains only to consider the portion of the flow line inside the cusp.

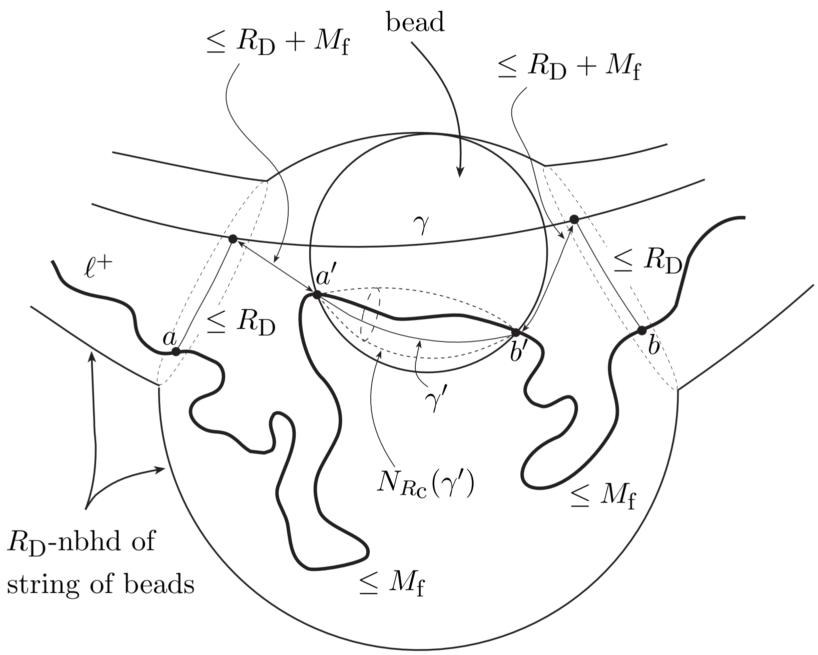

Figure 14. The flow line segment ℓ^+ lies within $R_\Phi = R_{\rm D} + M_{\rm f} + R_{\rm c}$ of γ.

The flow lines are quasigeodesic in the cusp, therefore the portion of ℓ^+ inside H lies within a neighborhood of radius $R_{\rm c}$ of the geodesic connecting its endpoints a' and b'; call it γ'. The maximum distance between two geodesic segments occurs at one of the endpoints of the segments; thus the entire geodesic segment γ' lies within a $R_{\rm D} + M_{\rm f}$ neighborhood of γ. Therefore, every point on ℓ^+ between a' and b' lies within a distance of $R_{\rm c} + R_{\rm D} + M_{\rm f}$ of γ. Taking $R_\Phi = R_{\rm c} + R_{\rm D} + M_{\rm f}$ gives the result. q.e.d.

3. Flow Line Progress

In this section, we complete the proof by showing that flow lines satisfy part 2 of the definition of tracking: that inside the R_Φ neighborhood of the geodesic connecting its endpoints, a flow line makes progress from one end of the tube to the other.

Theorem 3.1. *There exists $Q > 0$ such that the following is true: Let ℓ be a flow line and let γ be the geodesic connecting its endpoints. Let $\pi_\gamma : \mathbb{H}^3 \to \gamma$ denote nearest point retraction onto γ. Then if p, q are any two points on ℓ such that the length of the segment of ℓ between p and q is at least Q, $d(\pi_\gamma(p), \pi_\gamma(q)) \geq 1$.*

Proof. If not, then for every $n \in \mathbb{N}$ there exists p_n, q_n on a flow line ℓ_n such that $d(\pi_n(p_n), \pi_n(q_n)) < 1$, but the length along ℓ_n between p_n

and q_n is greater than n (where π_n denotes nearest point retraction onto γ_n and γ_n is the geodesic connecting the endpoints of ℓ_n). Let $\overline{p_n q_n}$ be the subsegment of ℓ_n between p_n and q_n. Note that by the results in Section 2, ℓ_n and hence $\overline{p_n q_n}$ is contained in $N_{R_\Phi}(\gamma_n)$.

Suppose there is a subsequence of $\{p_n\}$, which after relabelling we may assume is the original sequence, such that each $\pi_n(p_n)$ lies a distance at least $R_\Phi + 1$ inside a horoball H_n. Then since $p_n \in \ell_n$, we know p_n lies within R_Φ of γ_n; therefore p_n lies inside H_n. Similarly, since $d(\pi_n(p_n), \pi_n(q_n)) < 1$, we know that each q_n lies inside H_n as well. We proved earlier that a flow line in $\widetilde{M}$ enters a horoball at most once; therefore $\overline{p_n q_n}$ lies entirely inside H_n. So we have a sequence of flow line segments $\overline{p_n q_n}$ inside horoballs H_n whose lengths go to infinity, but that do not make progress along their corresponding geodesics γ_n. This contradicts the fact that all segments of flow line inside cusps are uniform quasigeodesics. Therefore, this case cannot happen.

Therefore, all p_n (and all q_n) project under π_n to points in a compact subset of M. We may assume up to covering translations and subsequences that p_n converges to p and q_n converges to q. Since Φ is a suspension flow, the orbit space $\mathcal{O}$ of $\widetilde{\Phi}$ can be identified to $\widetilde{F}$. Thus $\mathcal{O}$ is homeomorphic to $\mathbb{R}^2$ and is therefore Hausdorff. For every n, the points p_n and q_n project to the same point in the $\mathcal{O}$. Since $\mathcal{O}$ is Hausdorff, it follows that the limits of the projections of p_n and q_n into $\mathcal{O}$ are the same, and hence p and q lie on the same flow line. The flow line segment $\overline{pq}$ has a flow product neighborhood; therefore the flow line segments $\overline{p_n q_n}$ converge to $\overline{pq}$. Thus the length of $\overline{p_n q_n}$ must be bounded above, contradicting our original choice of p_n and q_n. Therefore, flow lines must make uniform progress along the geodesics connecting their endpoints. This completes the proof. q.e.d.

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