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MORSE INTERPOLATION FOR HAMILTONIAN GKM SPACES

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Abstract

Let M be a compact Hamiltonian $T-$space, with finite fixed point set M^T. An equivariant class is determined by its restriction to M^T, and to each fixed point $p \in M^T$ and generic component of the moment map, there corresponds a canonical class τ_p. For a special class of Hamiltonian $T-$spaces, the value $\tau_{p,q}$ of τ_p at a fixed point q can be determined through an iterated interpolation procedure, and we obtained a formula for $\tau_{p,q}$ as a sum over ascending chains from p to q. In general the number of such chains is huge, and the main result of this paper is a procedure to reduce the number of relevant chains, through a systematic degeneration of the interpolation direction. The resulting formula for $\tau_{p,q}$ resembles, via the localization formula, an integral over a space of chains, and we prove that, for complex Grassmannians, $\tau_{p,q}$ can indeed be expressed as the integral of an equivariant form over a smooth Schubert variety.

1. Hamiltonian GKM Spaces

1.1. Equivariant cohomology of Hamiltonian GKM spaces. Let T be a torus and let (M, ω) be a connected, compact, Hamiltonian $T-$space, with finite fixed point set M^T, and moment map $\phi : M \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ is the dual of the Lie algebra of T. Let $H_T^*(M) = H_T^*(M; \mathbb{R})$ be the $T-$equivariant cohomology of M; then $H_T^*(M)$ is a free module over $H_T^*(pt) = \mathbb{S}(\mathfrak{t}^*)$, the symmetric algebra of $\mathfrak{t}^*$. The main purpose of this paper is to give an explicit combinatorial construction of a basis of $H_T^*(M)$ as a module, for a special class of Hamiltonian $T-$spaces.

Hamiltonian $T-$spaces are equivariantly formal. The inclusion map $i : M^T \hookrightarrow M$ induces an injective map $i^* : H_T^*(M) \to H_T^*(M^T)$, and

$$H_T^*(M^T) = \bigoplus_{p \in M^T} H_T^*(p) = \bigoplus_{p \in M^T} \mathbb{S}(\mathfrak{t}^*) = \mathrm{Maps}(M^T, \mathbb{S}(\mathfrak{t}^*)).$$

Hence one can regard a class $f \in H_T^*(M)$ as a map that attaches a polynomial $f_p \in \mathbb{S}(\mathfrak{t}^*)$ to each fixed point $p \in M^T$, and for this reason we will refer to equivariant cohomology classes just by specifying their

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values at the fixed points. Not all such maps represent cohomology classes; a map $f\colon M^T \to \mathbb{S}(\mathfrak{t}^*)$ represents a cohomology class only if it satisfies certain compatibility conditions.

Example 1. If $M = \mathbb{C}P^1$, with the action $T \times M \to M$,

$$e^{it} \cdot [z_0 : z_1] = [z_0 : e^{i\alpha(t)} z_1],$$

for some nonzero weight $\alpha \in \Lambda_T \subset \mathfrak{t}^*$, then $M^T = \{[1:0],[0:1]\}$, and a map $f : M^T \to \mathbb{S}(\mathfrak{t}^*)$ represents a cohomology class if and only if

$$f([1:0]) \equiv f([0:1]) \pmod{\alpha} \quad \text{in} \quad \mathbb{S}(\mathfrak{t}^*). \tag{1}$$

Let $C = S^1$ be a generic circle in T, such that $M^C = M^T$. If $\xi \in \mathfrak{t}$ is an infinitesimal generator of C, then the moment map for the Hamiltonian $C-$action,

$$\phi^\xi : M \to \mathbb{R} \quad , \quad \phi^\xi(q) = \langle \phi(q), \xi \rangle,$$

is a perfect Morse function, whose critical points are precisely the fixed points, and each critical point has even index. Fix a $C-$invariant Riemannian metric on M. For every fixed point $p \in M^T$, the unstable manifold of ϕ^ξ at p is $T-$ invariant, and supports a class $\tau_p \in H_T^*(M)$; moreover, $\{\tau_p\}_{p \in M^T}$ is a basis of $H_T^*(M)$ as a module over $H_T^*(pt)$. The main goal of this paper is to provide a combinatorial construction of the classes τ_p, as maps $\tau_p\colon M^T \to \mathbb{S}(\mathfrak{t}^*)$, for a special class of Hamiltonian $T-$spaces, for which all compatibility conditions are of the type (1).

A *Hamiltonian GKM space* is a compact Hamiltonian $T-$space M such that the fixed point set M^T is finite and, for every fixed point $p \in M^T$, the weights of the isotropy (complex) representation of T on the tangent space T_pM (with a compatible almost complex structure) are non-collinear. A consequence of this second condition is that the connected components of the set of points fixed by a codimension one subtorus are either points or copies of $\mathbb{C}P^1$. Therefore, by a theorem of Chang and Skjelbred ([**CS**]) the compatibility conditions that a map $f : M^T \to \mathbb{S}(\mathfrak{t}^*)$ has to satisfy in order to represent a cohomology class are all of the form (1). These conditions are nicely encoded into the associated GKM graph. This is a regular graph $\Gamma = (V, E)$, with oriented edges labeled by weights of T. The vertices of this graph correspond to fixed points, M^T, and the edges are constructed as follows: if $p \in M^T$ is a fixed point and $\alpha_{p,i} \in \Lambda_T \in \mathfrak{t}^*$ is a weight of the isotropy representation of T on T_pM, then $H_{p,i} = \exp(\ker \alpha_{p,i}) \subset T$ is a codimension one subtorus of T. The connected component of $M^{H_{p,i}}$ containing p is a $\mathbb{C}P^1$, and there is exactly one more fixed point, $q \in M^T$, in this connected component. Then the vertices p and q are joined by an edge of Γ, and the oriented edge $e = [p, q]$ is labeled by $\alpha_e := \alpha_{p,i}$. In this case, the oriented edge $\bar{e} = [q, p]$ is labeled by $\alpha_{\bar{e}} := -\alpha_{p,i}$. (For more details on this construction, see [**GZ1**].)

Theorem 1 ([**GKM**], [**TW**]). *The maps $f\colon M^T \to \mathbb{S}(\mathfrak{t}^*)$ which represent equivariant cohomology classes are the ones that satisfy the following compatibility conditions: For every edge $e = [p,q]$ of the GKM graph Γ,*

$$f(q) \equiv f(p) \pmod{\alpha_e} \quad \textit{in} \quad \mathbb{S}(\mathfrak{t}^*).$$

Moreover, the GKM graph (V, E, α) contains all the information needed to compute integrals of equivariant forms: by the localization theorem, if $f \in H_T^*(M)$, then

$$\int_M f = \sum_{p \in M^T} \frac{f(p)}{\prod \alpha_{p,i}}, \tag{2}$$

where, for each term of the sum, the product in the denominator is over the weights of the isotropy representation at that fixed point.

1.2. Examples. An important class of Hamiltonian GKM spaces is given by homogeneous spaces of the form G/K, where G is a compact, connected, semisimple Lie group, and K is a closed subgroup of the same rank. The general situation is described in [**GHZ**]; here we present some particular cases.

1.2.1. Flag varieties. Let

$$\widetilde{T} = \{\operatorname{diag}(e^{it_1}, \dots, e^{it_n}) \;:\; t_1, \dots, t_n \in \mathbb{R}\}$$

be the diagonal torus in $U(n)$, the group of unitary matrices of order n. The Lie algebra of $\widetilde{T}$ is

$$\tilde{\mathfrak{t}} = \{\operatorname{diag}(it_1, \dots, it_n) \;:\; t_1, \dots, t_n \in \mathbb{R}\} \simeq \mathbb{R}^n$$

and a basis of the dual of $\tilde{\mathfrak{t}}$ is given by $\{x_1, \dots, x_n\}$, with

$$x_k(\operatorname{diag}(it_1, \dots, it_n)) = t_k.$$

The torus $\widetilde{T}$ acts linearly on $\mathbb{C}^n$ by

$$\operatorname{diag}(e^{it_1}, \dots, e^{it_n}) \cdot (z_1, \dots, z_n) = (e^{it_1} z_1, \dots, e^{it_n} z_n),$$

and this action induces an action of $\widetilde{T}$ on $\mathcal{F}l_n = \mathcal{F}l_n(\mathbb{C})$, the variety of full flags in $\mathbb{C}^n$. The diagonal circle in $\widetilde{T}$ acts trivially on $\mathcal{F}l_n(\mathbb{C})$, hence only the subtorus

$$T = \{\operatorname{diag}(e^{it_1}, \dots, e^{it_n}) \;:\; t_1, \dots, t_n \in \mathbb{R}, t_1 + \cdots + t_n = 0\}$$

acts effectively on $\mathcal{F}l_n(\mathbb{C})$. Then $\mathcal{F}l_n(\mathbb{C})$ is a Hamiltonian GKM space for the $T-$action, and the corresponding GKM graph is the permutahedron: The vertices of Γ correspond to elements of the permutation group S_n, and two vertices are joined by an edge if and only if the corresponding permutations differ by a transposition. If $e = (w, \tau_{ij} w)$ (with $i < j$) is an edge of Γ, then

$$\alpha_e = \begin{cases} x_j - x_i = \alpha_i + \cdots + \alpha_{j-1}, & \text{if } w^{-1}(i) < w^{-1}(j), \\ x_i - x_j = -(\alpha_i + \cdots + \alpha_{j-1}), & \text{if } w^{-1}(i) > w^{-1}(j), \end{cases}$$

where $\alpha_i = x_{i+1} - x_i$, for all $i = 1, \dots, n-1$. (The conditions $i < j$ and $w^{-1}(i) > w^{-1}(j)$ simply state that the *values* i, j are inverted in w.) The equivariant cohomology ring is

$$H_T^*(\mathcal{F}l_n(\mathbb{C})) \simeq \{f \colon S_n \to \mathbb{R}[\alpha_1, \dots, \alpha_{n-1}] \ : \\ f(w) = f(\tau_{ij} w) \text{ on } x_i = x_j \text{ for all } 1 \leqslant i < j \leqslant n\}.$$

1.2.2. Complex Grassmannians. The $T-$action on $\mathbb{C}^n$ described in Section 1.2.1 also induces an effective action on $Gr(k, n)$, the Grassmannian variety of $k-$dimensional complex subspaces in $\mathbb{C}^n$, and this Grassmannian is also a Hamiltonian GKM space. The fixed points (vertices of the GKM graph) correspond bijectively to $S_n/(S_k \times S_{n-k})$. This coset space is identified to both the set of permutations with exactly one descent, at $(k, k+1)$, and with the set of $k-$element subsets of the set $[n] = \{1, \dots, n\}$.

The edges of the GKM graph are of the form $e = ([w], [\tau_{ij} w])$, where $[w]$ is the class in $S_n/(S_k \times S_{n-k})$ of a permutation $w \in S_n$. The label of the edge $e = ([w], [\tau_{ij} w])$ is, again,

$$\alpha_e = \begin{cases} x_j - x_i = \alpha_i + \dots + \alpha_{j-1}, & \text{if } w^{-1}(i) < w^{-1}(j); \\ x_i - x_j = -(\alpha_i + \dots + \alpha_{j-1}), & \text{if } w^{-1}(i) > w^{-1}(j). \end{cases}$$

(It is not hard to see that the definition of α_e does not depend on the chosen representative for the class $[w]$.)

The GKM graph of the Grassmannian has an alternative description, as the Johnson graph $J(n, k)$: the vertices of the graph are the $k-$element subsets of $[n]$, and two subsets S_1 and S_2 are joined by an edge if and only if $S_1 \cap S_2$ has $k-1$ elements. If $S_2 = S_1 \cup \{j\} \setminus \{i\}$, then the edge $e = S_1 S_2$ is labeled by $\alpha_e = x_j - x_i$.

1.2.3. Subvarieties of Flag Varieties. For a subset $q = \{q_1, \dots, q_k\}$ of the set $[n]$, we define a map $i_q \colon \mathbb{C}^k \to \mathbb{C}^n$ by $i_q(e_i) = e_{q_i}$, where $\{e_1, \dots, e_k\}$ is the canonical basis of $\mathbb{C}^k$ and $\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{C}^n$. Using this map we construct a map $i_q \colon \mathcal{F}l_k \to \mathcal{F}l_n$ as follows.

For $j \in [n]$, let $j_0 = \max\{s : q_s \leqslant j\}$, with the convention that, if the set is empty, then $j_0 = 0$. If

$$V_\bullet : \ V_1 \subset V_2 \subset \ldots \subset V_k$$

is a flag in $\mathcal{F}l_k$, then $i_q(V_\bullet) = V'_\bullet \in \mathcal{F}l_n$, where

$$V'_j = i_q(V_{j_0}) \oplus \Big[\bigoplus_{\substack{s \leqslant j \\ s \notin q}} \mathbb{C} e_s\Big],$$

where $V_0 = 0$. Let

$$\mathcal{F}l_k(q) = i_q(\mathcal{F}l_k) \subset \mathcal{F}l_n$$

Example 2. If $q = \{2, 4, 5\} \subset [5]$, and $V_\bullet = \{V_1 \subset V_2 \subset V_3\} \in \mathcal{F}l_3$, then $i_q(V_\bullet) = \{V'_1 \subset V'_2 \subset V'_3 \subset V'_4 \subset V'_5\} \in \mathcal{F}l_5$, where

j	j_0	V'_j
1	0	$\mathbb{C}e_1$
2	1	$i_q(V_1) \oplus \mathbb{C}e_1$
3	1	$i_q(V_1) \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_3$
4	2	$i_q(V_2) \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_3$
5	3	$i_q(V_3) \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_3$

A subset $q = \{q_1, \ldots, q_k\}$ of the set $[n]$ also induces $i_q \colon S_k \to S_n$ by $i_q(w) = w'$, where

$$w'(j) = \begin{cases} j, & \text{if } j \notin q, \\ q_{w(i)}, & \text{if } j = q_i. \end{cases}$$

Let $S_k(q) = i_q(S_k) \subset S_n$.

The torus T induces a GKM action on $\mathcal{F}l_k(q)$, and the corresponding GKM graph is the subgraph of S_n whose set of vertices is $S_k(q)$. The edge $e = (w, \tau_{q_i q_j} w)$ of $S_k(q)$ is labeled by $\alpha_e = \pm(x_{q_j} - x_{q_i})$, where the sign is plus if (q_i, q_j) is not an inversion (as values) in w, and minus otherwise.

1.2.4. Schubert Varieties. The final example we give in this section is that of smooth Schubert varieties in $\mathcal{F}l_k(q)$. This example will play a key role in Section 3.2. Let $w \in S_k$ be a 3412− and 4231−avoiding permutation, and let $X(w)$ be the Schubert variety in $\mathcal{F}l_k$ associated to the T−fixed point corresponding to w. Then $X(w)$ is a smooth Schubert variety ([**LS**], [**C**]), and its image,

$$X_q(w) = i_q(X(w)) \subset i_q(\mathcal{F}l_k) = \mathcal{F}l_k(q) \subset \mathcal{F}l_n,$$

is a T−invariant smooth subvariety of $\mathcal{F}l_k(q)$. The fixed point set is $(X_q(w))^T = i_q((X(w))^T) \subset S_k(q)$, and $(X(w))^T = X^w$, the set of permutations in S_k that are below w in the Bruhat order on S_k. Let $X_q^w = i_q(X^w) \subset S_k(q)$.

1.3. Geometric construction of generators. For every fixed point $p \in M^T$, τ_p is a homogeneous class of degree $2\eta(p)$, where $2\eta(p)$ is the index of the Morse function ϕ^ξ at p. The Morse function ϕ^ξ induces a natural partial order on the set of fixed points: we start by setting $p \preccurlyeq q$ if q is in the closure of the unstable manifold at p, and then extend this order by transitivity. We call this order the *Morse order* on M^T generated by ξ. Then τ_p is supported on the union of the closures of unstable manifolds at fixed points $q \succcurlyeq p$. Moreover, at p, the class τ_p coincides with the equivariant Euler class of the normal bundle of the unstable manifold at p.

If the index function is strictly compatible with the Morse order (i.e., $\eta(a) < \eta(b)$ whenever $a \prec b$), then τ_p is uniquely defined by the three conditions above. When the index condition is not satisfied, τ_p is not uniquely determined by these three conditions. For every fixed point $p \in M^T$, Guillemin and Kogan constructed the *local index map at* p, as a map $\mathcal{I}_p \colon H_T^*(M) \to \mathbb{S}(\mathfrak{t}^*)$ (see [**GK**] for details of their geometric construction), and used these indices to construct a canonical class τ_p: the unique class that satisfies the following conditions

1) $\tau_p \in H_T^{2\eta(p)}$;
2) If $p \not\preccurlyeq q$, then $\tau_{p,q} = 0$, where $\tau_{p,q} := \tau_p(q)$ is the value of τ_p at q;
3) $\mathcal{I}_p(\tau_p) = 1$;
4) $\mathcal{I}_q(\tau_p) = 0$ if $q \neq p$.

(If the second condition is satisfied, then the third one is just a restatement of the fact that the restriction of τ_p to p is the equivariant Euler class.)

The main result of this paper is a combinatorial formula for $\tau_{p,q}$, for general Hamiltonian GKM spaces.

2. Multivariable Interpolation

The definitions and constructions of this section are motivated by the geometric constructions of Section 1; we will also see that some of the constructions that first appear in the combinatorial setting have nice geometric interpretations.

2.1. Abstract 1-skeleta. Let Γ be a regular graph, V_Γ the set of vertices of Γ, and E_Γ the set of oriented edges of Γ. For $p \in V_\Gamma$, we denote by E_p the set of oriented edges with initial vertex p.

Definition 1. A *connection* on Γ is a collection $\theta = (\theta_e)_{e \in E_\Gamma}$ of bijective maps

$$\theta_e : E_p \to E_q,\ e = (p,q) \in E_\Gamma,$$

indexed by the set of oriented edges of Γ, such that for every oriented edge $e = (p,q)$, $\theta_e(e) = \bar{e}$ and $\theta_{\bar{e}} = \theta_e^{-1}$, where $\bar{e} = (q,p)$.

Let $\mathfrak{t}$ be an n-dimensional real vector space (which will be thought of as the Lie algebra of a torus T), $\mathfrak{t}^*$ the dual of $\mathfrak{t}$, and $\mathbb{S}(\mathfrak{t}^*)$ the symmetric algebra of $\mathfrak{t}^*$, identified with the algebra of polynomial functions on $\mathfrak{t}$.

Definition 2. An *abstract one-skeleton* is a pair (Γ, α) consisting of a regular graph Γ and a function $\alpha : E_\Gamma \to \mathfrak{t}^*$ (called an *axial function*), such that:

1) For every vertex $p \in V_\Gamma$, the vectors
$$\{\alpha_e \ : \ e \in E_p\}$$
are pairwise linearly independent;
2) For every edge $e = (p,q) \in E_\Gamma$,
$$\alpha_{\bar{e}} = -\alpha_e;$$

3) There is a connection θ on Γ such that for every edge $e = (p, q)$ and every edge $e' \in E_p - \{e\}$,

$$\alpha_{\theta_e(e')} - \alpha_{e'} = c_{e,e'}\alpha_e, \quad \text{with } c_{e,e'} \in \mathbb{R}.$$

Definition 3. A map $f : V_\Gamma \to \mathbb{S}(\mathfrak{t}^*)$ is a *cohomology class* on (Γ, α) if for every edge $e = (p, q)$ of Γ,

$$f(q) \equiv f(p) \pmod{\alpha_e} \text{ in } \mathbb{S}(\mathfrak{t}^*).$$

The *cohomology ring* of (Γ, α) is the subring $H^*_\alpha(\Gamma)$ of $\mathrm{Maps}(V_\Gamma, \mathbb{S}(\mathfrak{t}^*))$ consisting of cohomology classes.

Constant maps are cohomology classes, hence $\mathbb{S}(\mathfrak{t}^*) \hookrightarrow H^*_\alpha(\Gamma)$ and $H^*_\alpha(\Gamma)$ is an $\mathbb{S}(\mathfrak{t}^*)$-module. In [**Za1**] we determined a general formula for constructing generators of $H^*_\alpha(\Gamma)$ as an $\mathbb{S}(\mathfrak{t}^*)-$module. Those generators are the combinatorial analogues of the classes τ_p discussed in Section 1, and here we present an improved version of the formula from [**Za1**].

Definition 4. A *polarizing vector* is a vector $\xi \in \mathfrak{t}$ such that

$$\alpha_e(\xi) \neq 0$$

for all edges $e \in E_\Gamma$.

A polarizing vector ξ defines a pre-order on V_Γ: for an edge $e = (p, q)$, define $p \prec q$ if $\alpha_e(\xi) > 0$, and extend this relation by transitivity. We will assume that this relation is an *order* on V (that is, there is no vertex p of Γ such that $p \prec p$), and we call this order the *Morse order* defined by ξ. It is not hard to see that this assumption is equivalent to the existence of a function $\phi \colon V \to \mathbb{R}$ such that, for every edge $e = (p, q)$ of Γ, $\phi(p) < \phi(q) \Leftrightarrow p \prec q$. We say that an edge $e = (p, q)$ *points upward* if $p \prec q$ and *points downward* if $q \prec p$. For a vertex $p \in V_\Gamma$, let

$$E_p^- = \{e = (p, q) \ : \ \alpha_e(\xi) < 0\}$$

be the set of downward-pointing edges originating at p, and

$$V_p^- = \{q \ : \ (p, q) \in E_p^-\}$$

the set of down-neighbors of p. We define the *index of* p, *ind*(p), to be the number of elements of E_p^-. The *flow-up* of p, F_p, is the set of vertices that can be reached from p along *ascending chains*, i.e., chains with no downward pointing edges. Similarly, the *flow-down* of p, F_p^-, is the set of vertices that can be reached from p along *descending chains*, i.e., chains with no upward pointing edges. Note that p belongs to both F_p and F_p^-.

2.2. Abstract local indices. For a class $f \in H_\alpha^*(\Gamma)$, let $f_s = f(s)$ be the value of f at $s \in V_\Gamma$.

Let $q \in V_\Gamma$ be a vertex of Γ. If V_q^- is the set of down-neighbors of q, then $\theta = f_q$ is a solution of the system of congruences

$$\theta \equiv f_s \pmod{\alpha_{sq}}, \quad s \in V_q^-. \tag{3}$$

Solving this system is essentially a multivariable Lagrange interpolation problem, and we reduce it to one variable interpolation using the polarizing vector ξ. Let $\mathfrak{t}_\xi^* \subset \mathfrak{t}^*$ be the annihilator of ξ, let $y_1, \dots, y_{n-1}$ be a basis of $\mathfrak{t}_\xi^*$, and let $x \in \mathfrak{t}^*$ such that $x(\xi) = 1$. Then $\{x, y_1, \dots, y_{n-1}\}$ is a basis of $\mathfrak{t}^*$, and every vector $u \in \mathfrak{t}^*$ such that $u(\xi) \neq 0$ can be written uniquely as $u(x,y) = m(x - L(y))$, where $m = u(\xi) \in \mathbb{R}$ and $L(y) \in \mathfrak{t}_\xi^*$.

For $u \in \mathfrak{t}^*$ such that $u(\xi) \neq 0$, let $\rho_u \colon \mathfrak{t}^* \to \mathfrak{t}^*$,

$$\rho_u(\beta) = \beta - \frac{\beta(\xi)}{u(\xi)} u.$$

If $u = m(x - L(y))$, then $\rho_u(\beta) = \beta(L(y), y) \in \mathfrak{t}_\xi^*$. Let $\rho_u \colon \mathbb{S}(\mathfrak{t}^*) \to \mathbb{S}(\mathfrak{t}^*)$ be the algebra morphism that extends the linear map $\rho_u \colon \mathfrak{t}^* \to \mathfrak{t}^*$, and for $s \in V_q^-$, let $\rho_{sq} = \rho_{\alpha_{sq}}$. With this notation, a particular solution of the system (3) is

$$\theta_q^0 = \sum_{s \in V_q^-} \left(\prod_{t \in V_q^- \setminus \{s\}} \frac{\alpha_{tq}}{\rho_{sq}(\alpha_{tq})} \right) \rho_{sq}(f_s),$$

hence

$$f_q = \theta_q^0 + \psi \prod_{s \in V_q^-} \alpha_{sq}, \tag{4}$$

for some unique $\psi \in \mathbb{S}(\mathfrak{t}^*)$.

Definition 5. Let $f \in H_\alpha^*(\Gamma)$ be a cohomology class. The *local index* (with respect to ξ) of f at $q \in V$ is the unique $\psi \in \mathbb{S}(\mathfrak{t}^*)$ such that (4) holds. The *local index map at* q is the map $\mathcal{I}_q = \mathcal{I}_q^\xi \colon H_\alpha^*(\Gamma) \to \mathbb{S}(\mathfrak{t}^*)$ that attaches to each cohomology class f its local index at q.

The local index map $\mathcal{I}_q$ is a morphism of $\mathbb{S}(\mathfrak{t}_\xi^*)-$modules, but not of $\mathbb{S}(\mathfrak{t}^*)-$modules. For $f \in H_\alpha^*(\Gamma)$ and $q \in V$, we have

$$\mathcal{I}_q(f) = (-1)^{\operatorname{ind}(q)} \left(\frac{f_q}{\prod_{s \in V_q^-} \alpha_{qs}} + \sum_{s \in V_q^-} \frac{\rho_{sq}(f_s)}{\alpha_{sq} \prod_{t \in V_q^- \setminus \{s\}} \rho_{sq}(\alpha_{qt})} \right),$$

and, modulo a sign convention, this is the combinatorial version of the local index of Guillemin and Kogan [**GK**, Formula 7.2].

Fix a polarizing $\xi \in \mathfrak{t}$.

Theorem 2. *For every vertex $p \in V$, there exists at most one cohomology class $\tau_p \in H^*_\alpha(\Gamma)$ such that*

$$\mathcal{I}_q(\tau_p) = \begin{cases} 1, & \text{if } q = p; \\ 0, & \text{if } q \neq p. \end{cases}$$

When such a class τ_p exists, it is called *the Thom class* of p (with respect to ξ). Necessary and sufficient conditions for the existence of families of Thom classes have been given in [**GZ2**].

Let $f \in H^*_\alpha(\Gamma)$. It is not hard to see that if $\mathcal{I}_q(f) = 0$ for all $q \preccurlyeq q'$, for some $q' \in V$, then $f(q) = 0$ for all $q \preccurlyeq q'$. This observation immediately implies that the Thom class τ_p is supported on the flow-up from p. Moreover, $\mathcal{I}_p(\tau_p) = 1$ implies that τ_p is normalized by the condition

$$\tau_{p,p} = \tau_p(p) = \prod_{s \in V_p^-} \alpha_{sp}.$$

2.3. Iterated interpolations. We now assume that the Thom class τ_p exists and we determine a formula for the computation of its values at vertices situated in the flow-up from p. Let q be such a vertex. Then

$$\tau_{p,q} = \mathcal{I}_q(\tau_p)\left(\prod_{s \in V_q^-} \alpha_{sq}\right) + \sum_{s \in V_q^-}\left(\prod_{t \in V_q^- \setminus \{s\}} \frac{\alpha_{tq}}{\rho_{sq}(\alpha_{tq})}\right)\rho_{sq}(\tau_{p,s}).$$

If $q = p$, then the second term is zero, and if $p \prec q$ then the first term is zero. If the shortest descending chain from q to p has at least two edges, then iterating the interpolation one more time we get

$$\tau_{p,q} = \sum_{s \in V_q^-}\sum_{r \in V_s^-}\left(\prod_{t \in V_q^- \setminus \{s\}} \frac{\alpha_{tq}}{\rho_{sq}(\alpha_{tq})}\right)\left(\prod_{t \in V_s^- \setminus \{r\}} \frac{\rho_{sq}(\alpha_{ts})}{\rho_{rs}(\alpha_{ts})}\right)\rho_{rs}(\tau_{p,r}),$$

since $\rho_{sq} \circ \rho_{rs} = \rho_{rs}$. Continuing the iteration we get a formula for $\tau_{p,q}$ as a sum of contributions of descending chains from q. But τ_p is supported on the flow-up from p, and hence the only nonzero contributions will be those corresponding to ascending chains

$$\gamma\colon \quad p = p_0 \to p_1 \to \ldots \to p_{m-1} \to p_m = q.$$

The contribution of such a chain is

$$E(\gamma) = \left(\prod_{\substack{t \in V_{p_m}^- \\ t \neq p_{m-1}}} \frac{\alpha_{tp_m}}{\rho_{p_{m-1}p_m}(\alpha_{tp_m})}\right)\left(\prod_{\substack{t \in V_{p_{m-1}}^- \\ t \neq p_{m-2}}} \frac{\rho_{p_{m-1}p_m}(\alpha_{tp_{m-1}})}{\rho_{p_{m-2}p_{m-1}}(\alpha_{tp_{m-1}})}\right)\cdots$$

$$\cdots\left(\prod_{\substack{t \in V_{p_2}^- \\ t \neq p_1}} \frac{\rho_{p_2p_3}(\alpha_{tp_2})}{\rho_{p_1p_2}(\alpha_{tp_2})}\right)\left(\prod_{\substack{t \in V_{p_1}^- \\ t \neq p}} \frac{\rho_{p_1p_2}(\alpha_{tp_1})}{\rho_{pp_1}(\alpha_{tp_1})}\right)\rho_{pp_1}(\tau_{p,p})$$

and, after regrouping the terms, it can be written as (see also [**GZ3**]):

$$E(\gamma) = \left(\prod_{\substack{s \in V_q^- \\ s \neq p_{m-1}}} \alpha_{sq} \right) \left(\prod_{k=1}^{m} \Theta_{p_{k-1}p_k} \right) \left(\prod_{k=1}^{m-1} \rho_{p_k p_{k+1}}(\alpha_{p_{k-1}p_k}) \right)^{-1},$$

where, for an ascending edge $r \to s$,

$$\Theta_{rs} = \frac{\prod_{t \in V_r^-} \rho_{rs}(\alpha_{tr})}{\prod_{t \in V_s^- \setminus \{r\}} \rho_{rs}(\alpha_{ts})}.$$

Example 3. In S_3 (see Section 1.2.1),

$$\Theta_{123,321} = \frac{1}{\rho_{\alpha_1+\alpha_2}(\alpha_1)\rho_{\alpha_1+\alpha_2}(\alpha_2)},$$

and $\Theta_{pq} = 1$ for all other edges.

There are two ascending chains from $p = (213)$ to $q = (321)$,

$$\gamma_1 \colon\ (213) \to (231) \to (321) \quad \text{and} \quad \gamma_2 \colon\ (213) \to (312) \to (321),$$

and their contributions are

$$E(\gamma_1) = \frac{\alpha_1(\alpha_1+\alpha_2)}{\rho_{\alpha_2}(\alpha_1+\alpha_2)} = \frac{\alpha_1(\alpha_1+\alpha_2)}{\rho_{\alpha_2}(\alpha_1)} \quad \text{and} \quad E(\gamma_2) = \frac{\alpha_2(\alpha_1+\alpha_2)}{\rho_{\alpha_1}(\alpha_2)}.$$

Therefore

$$\tau_{(213),(321)} = E(\gamma_1) + E(\gamma_2) = \alpha_1 + \alpha_2.$$

Note that although both $E(\gamma_1)$ and $E(\gamma_2)$ depend on the polarizing direction ξ, their sum doesn't. That suggests that we could try to consistently eliminate ξ from each $E(\gamma)$. Unfortunately, we can't simply take the limit of $E(\gamma)$ as ξ goes to 0, since in general this limit *doesn't exist*. But we could try to send ξ to 0 one coordinate at a time, and we describe this operation in a later section.

2.4. Special bases. We will need a basis of $\mathfrak{t}^*$ of a special type, and we devote this section to constructing such bases.

Let U be a real vector space and $\mathcal{B} = \{x_1, \dots, x_n\}$ a basis of U.

Definition 6. A vector $v = a_1x_1 + \cdots + a_nx_n \in U$ is called $\mathcal{B}$*−positive* if $a_k \geqslant 0$ for all $k = 1, \dots, n$, and is called $\mathcal{B}$*−negative* if $-v$ is $\mathcal{B}$−positive.

We will denote by $U_{\mathcal{B}}^+$ the set of $\mathcal{B}$−positive vectors (that is, the positive cone in U generated by $\mathcal{B}$) and by $U_{\mathcal{B}}^-$ the set of $\mathcal{B}$−negative vectors in U.

Lemma 1. *Let U be an n-dimensional real vector space, let U^* be its dual, let $S \subset U^*$ be a finite subset and let $\xi \in U$ such that $\alpha(\xi) \neq 0$ for all $\alpha \in S$. Then there exists a basis $\mathcal{B}$ of U^* such that*

$$\{\alpha \in S;\ \alpha(\xi) > 0\} = S \cap U_{\mathcal{B}}^+ \quad \textit{and} \quad \{\alpha \in S;\ \alpha(\xi) < 0\} = S \cap U_{\mathcal{B}}^-.$$

Proof. Let $(.,.)$ be a fixed scalar product on U^*, and $\alpha_0 \in U^*$ such that $\alpha(\xi) = (\alpha, \alpha_0)$, for all $\alpha \in U^*$. Let $\{y_1, \ldots, y_n\}$ be an orthonormal basis of U^*, such that

$$\alpha_0 = \frac{|\alpha_0|}{\sqrt{n}}(y_1 + \cdots + y_n),$$

and, for $\epsilon > 0$, let

$$y_k^\epsilon = y_k + \left(\epsilon - \frac{1}{|\alpha_0|\sqrt{n}}\right)\alpha_0.$$

Then $\mathcal{B}^\epsilon = \{y_1^\epsilon, \ldots, y_n^\epsilon\}$ is a basis of U^*. Since

$$\lim_{\epsilon \to 0} y_k^\epsilon \in \alpha_0^\perp,$$

there exists $\epsilon > 0$ such that $\mathcal{B} = \mathcal{B}^\epsilon$ has the required properties. (Intuitively, the process is analogous to "opening an umbrella.") q.e.d.

If (Γ, α) is an abstract one-skeleton, let $S = \alpha(E_\Gamma) \subset \mathfrak{t}^*$ and let $\xi \in \mathfrak{t}$ be a generic polarizing vector. By Lemma 1, there exists a basis $\mathcal{B} = \{x_1, .., x_n\}$ of $\mathfrak{t}^*$ such that, for every edge $e \in E_\Gamma$, $\alpha_e(\xi) > 0$ if and only if α_e is $\mathcal{B}-$positive and $\alpha_e(\xi) < 0$ if and only if α_e is $\mathcal{B}-$negative.

We choose and fix such a basis $\mathcal{B} = \{x_1, \ldots, x_n\}$ of $\mathfrak{t}^*$ and identify $\mathbb{S}(\mathfrak{t}^*)$ with $\mathbb{R}[x_1, \ldots, x_n]$. A vector $\beta \in \mathfrak{t}^*$ is *positive* ($\beta \succ 0$) if it is $\mathcal{B}-$positive; similarly, β is *negative* ($\beta \prec 0$) if it is $\mathcal{B}-$negative.

Let $\mathcal{B}^* = \{b_1, \ldots, b_n\}$ be the basis of $\mathfrak{t}$ dual to the basis $\mathcal{B}$ of $\mathfrak{t}^*$, and let $(\xi_1, \ldots, \xi_n)$ be the coordinates of the polarizing vector ξ in this basis; that is, $\xi = \xi_1 b_1 + \cdots + \xi_n b_n$.

Example 4. In the examples discussed in Sections 1.2.1 and 1.2.2, $\mathcal{B}^* = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a basis of $\mathfrak{t}^*$. Let $\{\varepsilon_1, \ldots, \varepsilon_{n-1}\}$ be the basis of $\mathfrak{t}$ dual to the basis $\mathcal{B}^*$ of $\mathfrak{t}^*$, and ξ be a vector with strictly positive coordinates in this basis, $\xi = \xi_1\varepsilon_1 + \cdots + \xi_{n-1}\varepsilon_{n-1} \in \mathfrak{t}$. Then ξ is polarizing and $\mathcal{B}^*$ is a special basis of $\mathfrak{t}^*$ compatible with ξ.

2.5. Relevant chains. Chain contributions $E(\gamma)$ are rational expressions in variables $x_1, \ldots, x_n$, $\xi_1, \ldots, \xi_n$, hence

$$E(\gamma) \in \mathbb{R}(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n),$$

the field of fractions of $\mathbb{R}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. Let $\pi \in S_n$ be a permutation of the set $[n]$.

Definition 7. For $E \in \mathbb{R}(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, we define

$$E_\pi = \lim_{\xi_{\pi(n)} \to 0}\left(\lim_{\xi_{\pi(n-1)} \to 0}\left(..\left(\lim_{\xi_{\pi(1)} \to 0} E\right)..\right)\right) \in \mathbb{R}(x_1, \ldots, x_n),$$

if all the limits exist.

Example 5. Consider the two chains described in Example 3. With respect to the polarizing vector ξ and the special basis given in Example 4, we have

$$E(\gamma_1) = \frac{\xi_2\alpha_1(\alpha_1+\alpha_2)}{\xi_2\alpha_1 - \xi_1\alpha_2} \quad \text{and} \quad E(\gamma_2) = \frac{\xi_1\alpha_2(\alpha_1+\alpha_2)}{\xi_1\alpha_2 - \xi_2\alpha_1}.$$

Let $w = (12)$ be the identity permutation of $\{1,2\}$. Then

$$(E(\gamma_1))_w = \lim_{\xi_2\to 0}\left(\lim_{\xi_1\to 0} E(\gamma_1)\right) = \lim_{\xi_2\to 0}(\alpha_1+\alpha_2) = \alpha_1+\alpha_2$$
$$(E(\gamma_2))_w = \lim_{\xi_2\to 0}\left(\lim_{\xi_1\to 0} E(\gamma_2)\right) = \lim_{\xi_2\to 0} 0 = 0.$$

We will show that $(E(\gamma))_\pi$ *does* exist for all permutations π and all ascending chains γ.

Definition 8. Let $\beta \in \mathfrak{t}^*$ and let $(\beta^1, \dots, \beta^n)$ the coordinates of β in the basis $\mathcal{B} = \{x_1, \dots, x_n\}$; hence $\beta = \beta^1 x_1 + \cdots + \beta^n x_n$. Let $\pi \in S_n$.

1) The *support* of β is the set $\mathrm{supp}(\beta) = \{i;\ \beta^i \neq 0\}$.
2) The $\pi-$*altitude* of β is $\mathrm{alt}_\pi(\beta) = \max\{i;\ \beta^{\pi(i)} \neq 0\}$.

Lemma 2. *Let $p \to q \to r$ be a chain. Then*

(5)

$$\left(\frac{1}{\rho_{qr}(\alpha_{pq})}\right)_\pi = \begin{cases} 0, & \text{if } alt_\pi(\alpha_{pq}) > alt_\pi(\alpha_{qr}) \\ \frac{1}{\alpha_{pq}}, & \text{if } alt_\pi(\alpha_{pq}) < alt_\pi(\alpha_{qr}) \\ \frac{\alpha_{qr}^{\pi(m)}}{\alpha_{qr}^{\pi(m)}\alpha_{pq} - \alpha_{pq}^{\pi(m)}\alpha_{qr}}, & \text{if } alt_\pi(\alpha_{pq}) = alt_\pi(\alpha_{qr}) = m, \end{cases}$$

hence it is defined, and it is non-zero if and only if $alt_\pi(\alpha_{pq}) \leqslant alt_\pi(\alpha_{qr})$.

Proof. Let $m = alt_\pi(\alpha_{qr})$ and $k = alt_\pi(\alpha_{pq})$. Then

$$\frac{1}{\rho_{qr}(\alpha_{pq})}$$
$$= \frac{\alpha_{qr}^{\pi(1)}\xi_{\pi(1)} + \cdots + \alpha_{qr}^{\pi(m)}\xi_{\pi(m)}}{(\alpha_{qr}^{\pi(1)}\xi_{\pi(1)} + \ldots + \alpha_{qr}^{\pi(m)}\xi_{\pi(m)})\alpha_{pq} - (\alpha_{pq}^{\pi(1)}\xi_{\pi(1)} + \ldots + \alpha_{pq}^{\pi(k)}\xi_{\pi(k)})\alpha_{qr}}.$$

If $k < m$, then

$$\lim_{\xi_{\pi(k)}\to 0}\left(\cdots\left(\lim_{\xi_{\pi(1)}\to 0}\frac{1}{\rho_{qr}(\alpha_{pq})}\right)\cdots\right)$$
$$= \frac{\alpha_{qr}^{\pi(k+1)}\xi_{\pi(k+1)} + \cdots + \alpha_{qr}^{\pi(m)}\xi_{\pi(m)}}{(\alpha_{qr}^{\pi(k+1)}\xi_{\pi(k+1)} + \cdots + \alpha_{qr}^{\pi(m)}\xi_{\pi(m)})\alpha_{pq}} = \frac{1}{\alpha_{pq}},$$

and it remains unchanged when we take the rest of the limits, since there are no more variables ξ_i.

If $alt_\pi(\alpha_{pq}) = k \geqslant m$, then

$$\lim_{\xi_{\pi(m-1)}\to 0}\left(\dots\left(\lim_{\xi_{\pi(1)}\to 0}\frac{1}{\rho_{qr}(\alpha_{pq})}\right)\dots\right)$$
$$= \frac{\alpha_{qr}^{\pi(m)}\xi_{\pi(m)}}{\alpha_{qr}^{\pi(m)}\xi_{\pi(m)}\alpha_{pq} - (\alpha_{pq}^{\pi(m)}\xi_{\pi(m)} + \dots + \alpha_{pq}^{\pi(k)}\xi_{\pi(k)})\alpha_{qr}}.$$

If $k > m$, then taking one more limit we get 0 in the enumerator and a nonzero quantity in the denominator; hence the limit, and all remaining limits, are zero.

If $k = m$, then $\xi_{\pi(m)}$ cancels out and

$$\lim_{\xi_{\pi(m-1)}\to 0}\left(\dots\left(\lim_{\xi_{\pi(1)}\to 0}\frac{1}{\rho_{qr}(\alpha_{pq})}\right)\dots\right) = \frac{\alpha_{qr}^{\pi(m)}}{\alpha_{qr}^{\pi(m)}\alpha_{pq} - \alpha_{pq}^{\pi(m)}\alpha_{qr}};$$

no more ξ_i's are present, hence that is the value of the final limit. q.e.d.

Lemma 3. *Let e be an ascending edge and $\pi \in S_n$. Then $(\Theta_e)_\pi$ is defined and nonzero.*

Proof. Recall that for an ascending edge $p \to q$,

$$\Theta_{pq} = \frac{\prod_{r\in V_p^-}\rho_{pq}(\alpha_{rp})}{\prod_{s\in V_q^-\setminus\{p\}}\rho_{pq}(\alpha_{sq})}.$$

Let θ be a connection on Γ compatible with the axial function α (see Definition). If $r \in V_p^-$ such that $\theta_{pq}(pr) = qs$, with $s \in V_q^- \setminus \{p\}$, then

$$\alpha_{qs} \equiv \alpha_{pr} \pmod{\alpha_{pq}},$$

and then $\rho_{pq}(\alpha_{pr}) = \rho_{pq}(\alpha_{qs})$. Therefore the corresponding terms in Θ_{pq} will cancel each other out, and the only terms that remain in Θ_{pq} after these cancellations are the terms corresponding to

1) vertices $r \in V_p^-$ such that $\theta_{pq}(pr) = qs$, with $s \notin V_q^-$, and
2) vertices $s \in V_q^- \setminus \{p\}$ such that $\theta_{qp}(qs) = pr$, with $r \notin V_p^-$.

In the first case, $\alpha_{pr} \prec 0$ and $\alpha_{qs} \succ 0$, and hence all the coordinates of α_{pr} in the basis $\mathcal{B}$ are non-positive and all the coordinates of α_{qs} are non-negative. Since $\alpha_{qs} = \alpha_{pr} + c\alpha_{pq}$ for some $c \in \mathbb{R}$, it follows that $\operatorname{supp}(\alpha_{pr}) \subseteq \operatorname{supp}(\alpha_{pq})$, and therefore $alt_\pi(\alpha_{pr}) \leqslant alt_\pi(\alpha_{pq})$. Therefore $(\rho_{pq}(\alpha_{rp}))_\pi$ is defined and

$$(\rho_{pq}(\alpha_{rp}))_\pi = \begin{cases} \alpha_{rp}, & \text{if } alt_\pi(\alpha_{pr}) < alt_\pi(\alpha_{pq}) \\ \alpha_{rp} - \frac{\alpha_{rp}^{\pi(h)}}{\alpha_{pq}^{\pi(h)}}\alpha_{pq}, & \text{if } alt_\pi(\alpha_{pq}) = alt_\pi(\alpha_{rp}) = h. \end{cases}$$

A completely similar argument shows that if s is a vertex of the second type, then $alt_\pi(\alpha_{qs}) \leqslant alt_\pi(\alpha_{pq})$

$$(\rho_{pq}(\alpha_{sq}))_\pi = \begin{cases} \alpha_{sq}, & \text{if } alt_\pi(\alpha_{sq}) < alt_\pi(\alpha_{pq}) \\ \alpha_{sq} - \frac{\alpha_{sq}^{\pi(h)}}{\alpha_{pq}^{\pi(h)}}\alpha_{pq}, & \text{if } alt_\pi(\alpha_{pq}) = alt_\pi(\alpha_{sq}) = h. \end{cases}$$

Therefore $(\Theta_e)_\pi$ is defined and nonzero for all ascending edges. q.e.d.

We say that an ascending chain γ is $\pi-$*relevant* if $(E(\gamma))_\pi \neq 0$. Using the previous two lemmas we have the following criterion for identifying relevant chain.

Theorem 3. *If* $\gamma\colon\ p_0 \to p_1 \to \cdots \to p_{m-1} \to p_m$ *is an ascending chain and* $\pi \in S_n$*, then* $(E(\gamma))_\pi$ *is defined, and* $(E(\gamma))_\pi \neq 0$ *if and only if*

$$alt_\pi(\alpha_{p_0p_1}) \leqslant alt_\pi(\alpha_{p_1p_2}) \leqslant \cdots \leqslant alt_\pi(\alpha_{p_{m-1}p_m}).$$

The criterion to eliminate unnecessary chains is most effective when the $\pi-$altitudes are as different as possible, and this can be achieved by choosing the basis $\mathcal{B}$ to contain as many vectors from $\alpha(E_\Gamma)$ as possible. For one-skeletons corresponding to flag varieties we can choose $\mathcal{B}$ to consist entirely of vectors in $\alpha(E_\Gamma)$, and the number of relevant chains drops dramatically. For example, in the case of S_5 (corresponding to $\mathcal{F}l_5$), there are 44062 ascending chains from (12435) to (54321), but only 18 of them are (4321)-relevant.

3. Application: Grassmannians

3.1. The Johnson graph. We return now to the example presented in Section 1.2.2, where the abstract one-skeleton is based on the Johnson graph $J(n,k)$. Recall that the vertices are the $k-$element subsets of $[n] = \{1,\dots,n\}$, and two vertices p and q are joined by an edge if and only if $\#(p \cap q) = k-1$; that is, if q is obtained by replacing an element $i \in p$ by an element $j \notin p$. We use the notation

$$p \xrightarrow{(i,j)} q$$

for such an edge (p,q). The axial function $\alpha\colon E_\Gamma \to \mathfrak{t}^*$ attaches to the oriented edge $p \to q$ the vector $\alpha_{p,q} = x_j - x_i$, denoted by α_{ij}.

Let $\mathcal{B} = \{\alpha_1,\dots,\alpha_{n-1}\}$ be the special basis described in Example 4, and $\xi = \xi_1\varepsilon_1 + \cdots + \xi_{n-1}\varepsilon_{n-1} \in \mathfrak{t}$ be a vector with strictly positive coordinates. Then ξ is a polarizing vector and $\mathcal{B}$ is a special basis of $\mathfrak{t}^*$ compatible with ξ, as in Section 2.4. If $i < j$, then $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$.

The Morse order on the vertices of $J(n,k)$ induced by ξ is the Bruhat order: for two subsets $p = \{p_1 < \cdots < p_k\}$ and $q = \{q_1 < \cdots < q_k\}$ of $[n]$, $p \preccurlyeq q$ if and only if $p_j \leqslant q_j$ for every $j = 1,\dots,k$.

3.2. Spaces of relevant chains. Let $p \preccurlyeq q$. A chain

$$\gamma \; : \; p = v_0 \xrightarrow{(i_1,j_1)} v_1 \to \cdots \to v_{m-1} \xrightarrow{(i_m,j_m)} v_m = q \tag{6}$$

is ascending if and only if $i_h < j_h$ for every $h = 1, \dots, m$. Let w_0 be the reverse order permutation of $[n-1]$, $w_0 = (n-1 \dots 21)$. If $i < j$, then the w_0−altitude of the weight

$$\alpha_{ij} = \alpha_{i+1} + \cdots \alpha_j = \alpha_{w_0(n-i-1)} + \cdots + \alpha_{w_0(n-j)}$$

is $alt_{w_0}(\alpha_{ij}) = n - i - 1$. Hence the ascending chain (6) is w_0−relevant (from now on, just *relevant*) if and only if $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_m$. But we can't have $i_h = i_{h+1}$, since i_h is not in v_h. Therefore, the relevant chains are the chains (6) that satisfy the following conditions:

1) $i_h < j_h$ for every $h = 1, \dots, m$;
2) $i_1 > i_2 > \cdots > i_m$.

These conditions imply that an element that has been added can't be replaced; therefore the elements $j_1, \dots, j_m$ are all distinct and in q. This remark allows us to associate to γ a permutation $w = w(\gamma) \in S_k$ as follows:

Definition 9. Let $p = \{p_1 < \cdots < p_k\}$, $q = \{q_1 < \cdots < q_k\}$, and let γ be the chain (6). We associate to γ a permutation $w = w(\gamma) \in S_k$, as follows:

1) If $p_i = i_r$, then $w(i)$ is defined by $j_r = q_{w(i)}$;
2) If $p_i \notin \{i_1, \dots, i_m\}$, then $w(i)$ is defined by $p_i = q_{w(i)}$.

To make the relation between a relevant chain γ and its associated permutation $w(\gamma)$ more suggestive, we represent the chain as

$$\gamma \; : \quad p \xrightarrow{(p_k,q_{w(k)})} \cdots \xrightarrow{(p_1,q_{w(1)})} q; \tag{7}$$

if $p_i = q_{w(i)}$, then the corresponding "edge" is a loop that starts and ends at the same vertex, and we delete this loop from our chain. If $w' = i_q(w) \in S_k(q) \subset S_n$, then (7) can be written as

$$\gamma \; : \quad p \xrightarrow{(p_k,w'(q_k))} \cdots \xrightarrow{(p_1,w'(q_1))} q.$$

Example 6. In $J(n, 3)$, with $n \geqslant 5$, the relevant chains from the vertex $p = \{1, 2, 4\}$ to the vertex $q = \{2, 4, 5\}$ are

$$\gamma_1 : \{1,2,4\} \xrightarrow{(4,5)} \{1,2,5\} \xrightarrow{(2,4)} \{1,4,5\} \xrightarrow{(1,2)} \{2,4,5\}, \quad w(\gamma_1) = (123),$$

$$\gamma_2 : \{1,2,4\} \xrightarrow{(2,5)} \{1,5,4\} \xrightarrow{(1,2)} \{2,5,4\}, \qquad w(\gamma_2) = (132),$$

$$\gamma_3 : \{1,2,4\} \xrightarrow{(4,5)} \{1,2,5\} \xrightarrow{(1,4)} \{4,2,5\}, \qquad w(\gamma_3) = (213),$$

$$\gamma_4 : \{1,2,4\} \xrightarrow{(1,5)} \{5,2,4\}, \qquad w(\gamma_4) = (312).$$

(We did not rearrange the 3-element sets after each exchange to make it clearer how the permutation is associated to the chain.)

Let $\Omega_{p,q}^{rlv}$ be the space of relevant chains from p to q, and $\Phi\colon \Omega_{p,q}^{rlv} \to S_k$ be the map that sends each relevant chain to its associated permutation. Then Φ is injective, hence $\Omega_{p,q}^{rlv}$ is parametrized by $W_{p,q} = \Phi(\Omega_{p,q}^{rlv}) \subset S_k$, and, as we proved in [**Za2**], $W_{p,q}$ has a very nice description.

Let $w_{p,q} \in S_k$ be a permutation defined inductively, from k down, by

$$w_{p,q}(i) = \min\{j \ : \ j \neq w_{p,q}(i+1), \dots, w_{p,q}(k) \text{ and } p_i \leqslant q_j\}.$$

For example, if $p = \{1,2,4\}$ and $q = \{2,4,5\}$, then

$$p_1 < p_2 = q_1 < p_3 = q_2 < q_3$$

and therefore $w_{p,q} = (312)$. Note that

$$W_{p,q} = \{(123),(213),(132),(312)\}$$

is the set of permutations in S_3 below (312) in the Bruhat order on S_3.

In Section 1.2.1, the fixed points of the $T-$action on $\mathcal{F}l_k$ correspond bijectively to permutations in S_k. In [**Za2**, Theorem 1.4]) we proved the following theorem.

Theorem 4. *For every pair $p \preccurlyeq q$ we have*

$$W_{p,q} = (X(w_{p,q}))^T = X^{w_{p,q}} \simeq X_q^{w_{p,q}}.$$

We also proved that $w_{p,q}$ avoids the pattern (231) ([**Za2**, Theorem 1.1]), and therefore it avoids the patterns (3412) and (4231). Hence $X(w_{p,q})$ is a *smooth* Schubert variety in $\mathcal{F}l_k$ ([**LS**],[**C**]), and $X_q(w_{p,q})$ is a smooth subvariety in $\mathcal{F}l_k(q)$ (see Section 1.2.4). We prove in Section 3.4 that, via the localization theorem, $\tau_{p,q}$ can be expressed as an integral over this space.

3.3. The contribution of a relevant chain. In this section we compute

$$(E(\gamma))_{w_0} = \lim_{\xi_1 \to 0}(\lim_{\xi_2 \to 0}(\dots(\lim_{\xi_{n-1} \to 0} E(\gamma))\dots)),$$

the contribution of a relevant chain γ, after taking the w_0-limit.

Since $alt_{w_0}(\alpha_{v_{h-1}v_h}) < alt_{w_0}(\alpha_{v_h v_{h+1}})$, it follows from (5) that

$$\left[\frac{1}{\rho_{v_h v_{h+1}}(\alpha_{v_{h-1}v_h})}\right]_{w_0} = \frac{1}{\alpha_{v_{h-1}v_h}}.$$

Let e be an ascending edge $r \xrightarrow{(i,j)} s$. Most of the terms in Θ_{rs} will cancel each other out. The only remaining terms correspond to the vertices t such that $r \to t$ and $t \to s$ are both ascending edges. There are two possible types:

$$r \xrightarrow{(h,j)} t \xrightarrow{(i,h)} s, \qquad \text{with } i < h < j \text{ and } h \in r \cap s$$

and

$$r \xrightarrow{(i,h)} t \xrightarrow{(h,j)} s, \qquad \text{with } i < h < j \text{ and } h \notin r \cap s.$$

Then

$$\Theta_{rs} = \left[\prod_{\substack{i<h<j \\ h \in r\cap s}} \frac{1}{\rho_{ij}(\alpha_{ih})} \right] \left[\prod_{\substack{i<h<j \\ h \notin r\cap s}} \frac{1}{\rho_{ij}(\alpha_{hj})} \right]$$

and, after a direct computation,

$$[\Theta_{rs}]_{w_0} = \left[\prod_{\substack{i<h<j \\ h \in r\cap s}} \frac{1}{-\alpha_{hj}} \right] \left[\prod_{\substack{i<h<j \\ h \notin r\cap s}} \frac{1}{\alpha_{hj}} \right] = (-1)^{n_{rs}} \prod_{i<h<j} \frac{1}{\alpha_{hj}},$$

where $n_{rs} = \#\{h : i < h < j \text{ and } h \in r \cap s\}$.

Putting everything together, we have shown that, if

$$\gamma : \qquad p = v_0 \xrightarrow{(i_1,j_1)} v_1 \to \cdots \to v_{m-1} \xrightarrow{(i_m,j_m)} v_m = q$$

is a relevant chain from p to q, then

$$\begin{aligned}(E(\gamma))_{w_0} =& \tau_{q,q} \prod_{s=1}^{m} \left[\left[\prod_{\substack{i_s<h<j_s \\ h \in v_s\cap v_{s-1}}} \frac{1}{-\alpha_{hj_s}} \right] \left[\prod_{\substack{i_s<h<j_s \\ h \notin v_s\cap v_{s-1}}} \frac{1}{\alpha_{hj_s}} \right] \right] = \\ =& \tau_{q,q} \prod_{s=1}^{m} \left[(-1)^{n_{p_{s-1}p_s}} \prod_{i_s \leqslant h<j_s} \frac{1}{\alpha_{hj_s}} \right].\end{aligned}$$

For the fixed path γ, the elements of q are divided into two sets: the first set, $\{j_1, \ldots, j_m\}$, consists of elements that have been added along γ, and the second set, denoted by $q - \gamma$, is the complement of the first in q. Then

$$\tau_{q,q} = \left[\prod_{s=1}^{m} \prod_{\substack{h<j_s \\ h \notin q}} \alpha_{hj_s} \right] \left[\prod_{j \in q-\gamma} \prod_{\substack{h<j \\ h \notin q}} \alpha_{hj} \right],$$

hence

$$(E(\gamma))_{w_0} = \frac{P(\gamma)}{Q(\gamma)},$$

where

$$P(\gamma) = \tau_{q,q} \prod_{s=1}^{m} \left[\prod_{\substack{i_s \leqslant h<j_s \\ h \notin q}} \frac{1}{\alpha_{hj_s}} \right] = \left[\prod_{s=1}^{m} \prod_{\substack{h=1 \\ h \notin q}}^{i_s-1} \alpha_{hj_s} \right] \left[\prod_{j \in q-\gamma} \prod_{\substack{h=1 \\ h \notin q}}^{j-1} \alpha_{hj} \right]$$

and

$$Q(\gamma) = \prod_{s=1}^{m} \left[(-1)^{n_{p_{s-1}p_s}} \prod_{\substack{i_s \leqslant h < j_s \\ h \in q}} \alpha_{hj_s} \right].$$

3.4. Chain integrals. By the localization formula (2), for an equivariant class $F \in H_T^*(X_q(w_{p,q}))$,

$$\int_{X_q(w_{p,q})} F = \sum_{w \in (X_q(w_{p,q}))^T} \frac{F(w)}{e(w)} = \sum_{w \in X_q^{w_{p,q}}} \frac{F(w)}{e(w)}, \tag{8}$$

with $e(w) = \prod \alpha_{w,w'}$, where the product is over all neighbors of w in $X_q^{w_{p,q}}$. On the other hand,

$$(\tau_{p,q})_{w_0} = \sum_{\gamma \in \Omega_{p,q}^{rlv}} (E(\gamma))_{w_0} = \sum_{w \in X_q^{w_{p,q}}} \frac{P(\gamma_w)}{Q(\gamma_w)}, \tag{9}$$

where

$$\gamma_w \,:\, p \xrightarrow{(p_k, w(q_k))} \cdots \xrightarrow{(p_1, w(q_1))} q$$

is the relevant chain parametrized by $w \in X_q^{w_{p,q}} \subset S_k(q) \subset S_n$.

Let $F_{p,q} \colon X_q^{w_{p,q}} \to \mathbb{R}[\alpha_1, \ldots, \alpha_{n-1}]$ be defined by

$$\begin{aligned} F_{p,q}(w) = P(\gamma_w) &= \left[\prod_{\substack{s=1 \\ p_s < w(q_s)}}^{k} \prod_{\substack{h=1 \\ h \notin q}}^{p_s - 1} \alpha_{hw(q_s)} \right] \left[\prod_{\substack{s=1 \\ p_s = w(q_s)}}^{k} \prod_{\substack{h=1 \\ h \notin q}}^{w(q_s)-1} \alpha_{hw(q_s)} \right] \\ &= \prod_{s=1}^{k} \prod_{\substack{h=1 \\ h \notin q}}^{p_s - 1} \alpha_{hw(q_s)}. \end{aligned}$$

We will show that $Q(\gamma_w) = e(w)$, and that $F_{p,q}$ is the restriction of a class $F_{p,q} \in H_T^*(X_q(w_{p,q}))$ to $(X_q(w_{p,q}))^T = X_q^{w_{p,q}}$. Since $\tau_{p,q}$ does not depend on ξ, we have that $(\tau_{p,q})_{w_0} = \tau_{p,q}$. To summarize, we will prove the following theorem.

Theorem 5. *If $p \preccurlyeq q$, then*

$$\tau_{p,q} = \int_{X_q(w_{p,q})} F_{p,q}.$$

Proof. Let $w \in X_q^{w_{p,q}}$, and

$$\gamma_w \colon p = V_k \xrightarrow{(p_k, w(q_k))} V_{k-1} \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{(p_1, w(q_1))} V_0 = q$$

the corresponding relevant chain. We first prove that $Q(\gamma_w) = e(w)$.

For $s = 1, \ldots, k$, the vertex V_s is given by

$$V_s = \{p_1, p_2, \ldots, p_s, w(q_{s+1}), \ldots, w(q_k)\},$$

and

$$\begin{aligned} n_{V_sV_{s-1}} = &\quad \#\{h\colon p_s \leqslant h < w(q_s) \text{ and } h \in V_s \cap V_{s-1}\} = \\ = &\quad \#\{r\colon r > s \text{ and } p_s < p_r \leqslant w(q_r) < w(q_s)\}. \end{aligned}$$

Therefore

$$(-1)^{n_{V_sV_{s-1}}} \prod_{\substack{p_s \leqslant h < w(q_s) \\ h \in q}} \alpha_{hw(q_s)} = (-1)^{n_{V_sV_{s-1}}} \prod_{\substack{p_s \leqslant w(q_r) < w(q_s) \\ r=1}}^{k} \alpha_{w(q_r)w(q_s)}$$

$$= \left[\prod_{\substack{r=1 \\ p_r < p_s \leqslant w(q_r) < w(q_s)}}^{s-1} \alpha_{w(q_r)w(q_s)} \right] \left[\prod_{\substack{r=s+1 \\ p_s < p_r \leqslant w(q_r) < w(q_s)}}^{k} (-\alpha_{w(q_r)w(q_s)}) \right];$$

hence

$$Q(\gamma_w) = \prod_{\substack{w' = \tau_{w(q_r)w(q_s)} w \\ p_s \leqslant w(q_r) < w(q_s)}} \alpha_{w,w'}. \tag{10}$$

Let $w(q_r) < w(q_s)$ and $w' = \tau_{w(q_r)w(q_s)} w$. Then $p_r \leqslant w(q_r) < w(q_s)$ and

$$w' \in X_q^{w_{p,q}} \iff w' \text{ parametrizes a relevant chain } \iff \left\{ \begin{array}{c} p_r \leqslant w'(q_r) \\ \text{and} \\ p_s \leqslant w'(q_s) \end{array} \right\}$$

$$\iff \left\{ \begin{array}{c} p_r \leqslant w(q_s) \\ \text{and} \\ p_s \leqslant w(q_r) \end{array} \right\} \iff p_s \leqslant w(q_r) < w(q_s),$$

which, together with (10), proves that $Q(\gamma_w) = e(w)$.

Next we show that $F_{p,q} \in H_T^*(X_q(w_{p,q}))$. Let $1 \leqslant w(q_i) < w(q_j) \leqslant k$, such that $w' = \tau_{w(q_i)w(q_j)} w \in X_q^{w_{p,q}}$. Then

$$F_{p,q}(w') = \left[\prod_{\substack{s=1 \\ s \neq i,j}}^{k} \prod_{\substack{h=1 \\ h \notin q}}^{p_s - 1} \alpha_{hw(q_s)} \right] \prod_{\substack{h=1 \\ h \notin q}}^{p_i - 1} \alpha_{hw(q_j)} \prod_{\substack{h=1 \\ h \notin q}}^{p_j - 1} \alpha_{hw(q_i)}$$

and

$$F_{p,q}(w) = \left[\prod_{\substack{s=1 \\ s \neq i,j}}^{k} \prod_{\substack{h=1 \\ h \notin q}}^{p_s - 1} \alpha_{hw(q_s)} \right] \prod_{\substack{h=1 \\ h \notin q}}^{p_i - 1} \alpha_{hw(q_i)} \prod_{\substack{h=1 \\ h \notin q}}^{p_j - 1} \alpha_{hw(q_j)}.$$

But $\alpha_{hw(q_j)} - \alpha_{hw(q_i)} = \alpha_{w(q_i)w(q_j)}$ for all h, so $F_{p,q}(w') - F_{p,q}(w)$ is a multiple of $\alpha_{w(q_i)w(q_j)} = \pm \alpha_{w,w'}$, and hence $F_{p,q}$ does define a cohomology class in $H_T^*(X_q(w_{p,q}))$. Moreover, (8) and (9) imply that

$$\tau_{p,q} = (\tau_{p,q})_{w_0} = \sum_{\gamma_w \in \Omega_{p,q}^{rlv}} (E(\gamma_w))_{w_0} = \sum_{w \in X_q^{w_{p,q}}} \frac{P(\gamma_w)}{Q(\gamma_w)} = \sum_{w \in X_q^{w_{p,q}}} \frac{F(w)}{e(w)}$$
$$= \sum_{w \in (X_q(w_{p,q}))^T} \frac{F(w)}{e(w)} = \int_{X_q(w_{p,q})} F.$$

q.e.d.

For flag varieties, $\tau_{p,q}$ can also be computed using divided difference operators, and that method leads naturally to expressing $\tau_{p,q}$ as a sum over subwords of a fixed word. For complete flags, those subwords correspond bijectively to relevant chains, while for Grassmannians, in general there are more subwords than relevant chains. We discuss the relationship between the two approaches (divided differences and Morse interpolation) in [**Za3**].

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