J. DIFFERENTIAL GEOMETRY
71 (2005) 307-313

LAGRANGIAN HOMOLOGY CLASSES WITHOUT REGULAR MINIMIZERS

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Abstract

We show that there is an integral homology class in a Kähler–Einstein surface that can be represented by a lagrangian two-sphere, but that a minimizer of area among lagrangian two-spheres representing this class has isolated singularities with non-flat tangent cones.

1. Introduction

In this note we show that there is an integral homology class in a Kähler–Einstein surface that can be represented by a lagrangian two-sphere, but that a minimizer of area among lagrangian two-spheres representing this class is singular. Note that we do not consider branch points as singularities. Therefore, in the unconstrained case, this result is false since a minimizer of area in a homology class is a branched immersion.

To put this result in context, recall the constrained variational theory developed in [**5**]. Consider a homology class in a Kähler surface that can be represented by a lagrangian map of a compact surface (a *lagrangian homology class*) and minimize area among such maps. Then in [**5**], it is shown that a lagrangian minimizer exists, that the map is Lipschitz and is an immersion except at a finite number of isolated points that are either (i) branch points, or (ii) singular points with non-flat tangent cone. The tangent cones can be described precisely and it can be shown that there is a Maslov index associated to each tangent cone (and hence to each singular point). If the map is a minimizer, this index is ± 1. The sum of these indices equals the pairing of the first Chern class of the Kähler surface with the homology class of the minimizer. Thus, when this pairing is non-zero, a lagrangian minimizer must have singular points. However, if the Kähler surface is Kähler–Einstein, then this pairing vanishes and it is possible that the minimizer is always regular. More precisely, one could speculate that on a minimizer, a pair of singularities with indices 1 and -1 could be shown to "cancel". This is

The author was partially supported by NSF grant DMS-0304587.
Received 07/23/2004.

formulated as a question in [**4**]. The examples of this paper show that this is not the case.

2. Preliminaries

In this section, we review basic results in Kähler geometry and the geometry of the K3 surfaces that will be used in the proof of our result. For proofs, see [**1**] and [**2**].

Let X be a K3 surface, that is, X is a compact, complex, simply connected surface with trivial canonical bundle. Let

$$L = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H,$$

define the intersection form on a vector space of real dimension 22. Set $L_{\mathbb{C}} = L \otimes \mathbb{C}$ with the intersection form extended complex linearly. For any $\Omega \in L_{\mathbb{C}}$, we denote $[\Omega] \in \mathcal{P}(L_{\mathbb{C}})$ the corresponding line. It is known that $H^2(X, \mathbb{Z})$ is free of rank 22 and the intersection form on $H^2(X, \mathbb{Z})$ is given by L. In particular, $b^2_+ = 3$ and $b^2_- = 19$. A marking of X is a choice of basis,

$$\{\alpha_1, \dots, \alpha_8, \beta_1, \dots, \beta_8, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3\}$$

of $H^2(X, \mathbb{Z})$ that induces the intersection form L. Equivalently, a marking of X is the choice of an isometry $\phi : H^2(X, \mathbb{Z}) \to L$. The period domain $\mathcal{D}$ of X is the projectivization of the set:

$$\{\Omega \in L_{\mathbb{C}} : \Omega \cdot \Omega = 0, \ \ \Omega \cdot \overline{\Omega} > 0\}.$$

The complex dimension of $\mathcal{D}$ equals 20. If Ω is a holomorphic $(2, 0)$-form on X, then the identities $\Omega \cdot \Omega = 0$ and $\Omega \cdot \overline{\Omega} > 0$ show that a marking of X determines a point $[\Omega] \in \mathcal{D}$, called the *period point of* X. The first main theorem we require is the weak Torelli theorem:

Theorem 2.1. *Two $K3$ surfaces are isomorphic* (*as complex surfaces*) *if and only if there are markings for them such that the corresponding period points are the same.*

The second main theorem we require is the surjectivity of the period map:

Theorem 2.2. *All points of the period domain $\mathcal{D}$ occur as period points of marked $K3$ surfaces.*

A class $\omega \in H^{1,1}(X, \mathbb{R})$ that can be represented by a Kähler form is called a *Kähler class*. Clearly, a Kähler class satisfies $\omega \cdot \omega > 0$ and $\omega \cdot \Omega = 0$. Note that the set $\{x \in H^{1,1}(X, \mathbb{R}) : x \cdot x > 0\}$ consists of two disjoint connected cones and that the Kähler classes, if they exist, all belong to one of these two cones. This cone is called the *positive cone*. Additional conditions on the Kähler classes arise from the Picard lattice. Let $j : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$ and define the Picard lattice $S_X = H^{1,1}(X, \mathbb{R}) \cap \mathrm{Im} j(H^2(X, \mathbb{Z}))$. An element $\sigma \in S_X$ is called *divisorial* if there exists a divisor D whose associated line bundle has Chern class σ. Then σ is called effective if, in addition, D can be chosen effective. The Kähler cone is defined to be the convex subcone of the positive cone consisting of those classes that have positive inner product with any effective class in S_X. The Kähler cone contains all Kähler classes. When X is a K3 surface, the characterization of the Kähler cone becomes particularly simple. A non-singular curve γ in X is called nodal if $\gamma \cdot \gamma = -2$.

Theorem 2.3. *For a $K3$ surface, the Kähler cone consists of the classes $\omega \in H^{1,1}(X, \mathbb{R})$ that satisfy:*

(i) $\omega \cdot \omega > 0$,
(ii) $\omega \cdot \Omega = 0$ *and*
(iii) $\omega \cdot \gamma > 0$, *for all nodal curves γ in X.*

It is a consequence of the surjectivity of the refined period map (see [**1**] for details) that every class in the Kähler cone is a Kähler class. Consequently, Yau's theorem on the existence of Ricci flat metrics on K3 surfaces can be stated as:

Theorem 2.4. *Let (X, ω) be a $K3$ surface where $\omega \in H^{1,1}(X, \mathbb{R})$ lies in the Kähler cone. Then there is a unique Ricci flat metric g on X whose Kähler form represents the class ω.*

The Ricci flat metric g is hyperkähler. That is, there is a two-sphere of complex structures on X, the *hyperkähler line of g*, such that each complex structure together with g determines a Kähler form. The Kähler forms are parameterized by the two-sphere determined by ω, Re Ω and Im Ω.

If X is a Kähler surface and Σ is a possibly singular holomorphic curve of genus g in X the *adjunction formula* is:

$$\Sigma \cdot \Sigma \geq c_1(X) \cdot \Sigma + 2g - 2,$$

with equality when Σ is non-singular. When X is a K3 surface, this becomes:

$$\Sigma \cdot \Sigma \geq 2g - 2 \geq -2.$$

If X is a K3 surface, we say a (singular) holomorphic curve Σ is a (-2)-curve if $\Sigma \cdot \Sigma = -2$ (equivalently, if its Poincare dual α satisfies $\alpha \cdot \alpha = -2$). From the adjunction formula, it follows that if Σ is a (-2)-curve then Σ is a non-singular rational curve.

We conclude this section with some results on lagrangian stationary submanifolds of Kähler–Einstein manifolds (see [**5**]). Let N be a Kähler–Einstein manifold and Σ be a lagrangian submanifold. We call Σ *lagrangian stationary* if the volume is stationary for arbitrary smooth variations through lagrangian submanifolds.

Theorem 2.5. *A closed immersed lagrangian submanifold in a Kähler–Einstein manifold is a classical minimal submanifold if and only if it is lagrangian stationary.*

Consequently,

Corollary 2.6. *A closed, immersed, lagrangian stationary submanifold in a Calabi–Yau manifold N is a special lagrangian submanifold. In particular, if N is a $K3$ surface with a hyperkähler metric g, then a closed, lagrangian stationary branched immersion Σ is a J-holomorphic curve with respect to a complex structure J on the hyperkähler line of g.*

Note that these results require regularity of the lagrangian stationary submanifold.

3. The Result

We begin with:

Lemma 3.1. *There is an Ω in the period domain (i.e., $\Omega \cdot \Omega = 0$ and $\Omega \cdot \bar{\Omega} > 0$) such that Ω is an irrational class and therefore, the complex structure J determined by Ω has no nodal curves.*

Proof. Let a marking of X be given by,

$$\{\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8, \xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3\}.$$

Recall that $\xi_i \cdot \xi_j = 0$, $\eta_i \cdot \eta_j = 0$, for all i, j, that $\xi_i \cdot \eta_j = 0$ for $i \neq j$ and that $\xi_i \cdot \eta_i = 1$ for all i. Therefore, $(\xi_i - \eta_i) \cdot (\xi_j + \eta_j) = 0$ for all i, j and $(\xi_i - \eta_i)^2 = -2$, $(\xi_i + \eta_i)^2 = 2$ for all i.

Define the period point Ω as follows:

$$\Omega \cdot \alpha_k = a_k, \quad \text{for} \;\; k = 1, \ldots, 8,$$
$$\Omega \cdot \beta_k = ib_k, \quad \text{for} \;\; k = 1, \ldots, 8,$$
$$\Omega \cdot (\xi_j - \eta_j) = s_j + it_j, \quad \text{for} \;\; j = 1, 2, 3,$$
$$\Omega \cdot (\xi_j + \eta_j) = \sigma_j + i\tau_j, \quad \text{for} \;\; j = 1, 2, 3,$$

where the $a_k, b_k, s_j, t_j, \sigma_j, \tau_j$ are non-zero real scalars. Assume that the vectors:

$$s = (s_1, s_2, s_3), \;\; t = (t_1, t_2, t_3),$$

and

$$\sigma = (\sigma_1, \sigma_2, \sigma_3), \;\; \tau = (\tau_1, \tau_2, \tau_3).$$

satisfy $s \cdot t = 0$ and $\sigma \cdot \tau = 0$. Then $\mathrm{Re}\, \Omega \cdot \mathrm{Im}\, \Omega = 0$. Choosing $|\sigma|$ and $|\tau|$ sufficiently large implies that $\mathrm{Re}\, \Omega \cdot \mathrm{Re}\, \Omega > 0$ and $\mathrm{Im}\, \Omega \cdot \mathrm{Im}\, \Omega > 0$. Therefore, $\Omega \cdot \bar{\Omega} > 0$. Scaling either $\mathrm{Re}\ \Omega$ or $\mathrm{Im}\ \Omega$, we can assume $\mathrm{Re}\, \Omega \cdot \mathrm{Re}\, \Omega = \mathrm{Im}\, \Omega \cdot \mathrm{Im}\, \Omega$ and therefore, $\Omega \cdot \Omega = 0$. Choose the $\{a_k\}$, $\{s_j\}$ and $\{\sigma_j\}$ so that no rational linear combination of them vanishes and similarly choose the $\{b_k\}$, $\{t_j\}$ and $\{\tau_j\}$ so that no rational linear combination of them vanishes. By the Torelli theorem, Ω defines a complex structure. Since the pairing of Ω with any integral homology class is non-zero, no integral homology class can be represented by a holomorphic curve. (Note that the vectors σ and τ have the property that no rational linear combination of their components vanishes. For technical use below, we require that, in addition, the vector $\rho = (\rho_1, \rho_2, \rho_3)$ satisfying $\sigma \cdot \rho = \tau \cdot \rho = 0$ and $|\rho| = 1$ also has this property.) q.e.d.

Note that the complex structure J determined by the period point Ω is non-algebraic. Fix two integral classes α_j and β_k with self-intersection -2 and zero intersection. For simplicity, we will denote them α_1 and β_1.

Lemma 3.2. *There is a Kähler class ω in the Kähler cone of Ω (i.e., $\omega \cdot \omega > 0$ and $\omega \cdot \Omega = 0$) such that $\omega \cdot \alpha_1 = 1$, $\omega \cdot \beta_1 = 1$ and such that on every integral class γ not in the integral lattice of $\{\alpha_1, \beta_1\}$, $\omega \cdot \gamma$ is irrational (and, in particular, $\omega \cdot \gamma \neq 0$).*

Proof. Define the Kähler class ω as follows:

$$\begin{aligned}
\omega \cdot \alpha_1 &= 1 \\
\omega \cdot \alpha_k &= A_k,\ k = 2, \dots, 8, \\
\omega \cdot \beta_1 &= 1 \\
\omega \cdot \beta_k &= B_k,\ k = 2, \dots, 8, \\
\omega \cdot (\xi_j - \eta_j) &= C_j,\ j = 1, 2, 3, \\
\omega \cdot (\xi_j + \eta_j) &= \lambda\sigma_j + \mu\tau_j + \nu\rho_j,\ j = 1, 2, 3,
\end{aligned}$$

where the vectors $\sigma = (\sigma_1, \sigma_2, \sigma_3), \tau = (\tau_1, \tau_2, \tau_3), \rho = (\rho_1, \rho_2, \rho_3)$ are given above. Choose A_k, B_k, C_j such that every class δ in the integral lattice of $\{\alpha_1, \dots, \alpha_8, \beta_1, \dots, \beta_8, \xi_1 - \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3\}$ not of the form $\{m\alpha_1 + n\beta_1; m, n \in \mathbb{Z}\}$ has the property that $\omega \cdot \delta$ is irrational. Choose λ so that Re $\Omega \cdot \omega = 0$ and μ so that Im $\Omega \cdot \omega = 0$. Then, $\omega \cdot \Omega = 0$. Choose ν sufficiently large so that $\omega \cdot \omega > 0$. Since there are no nodal curves, the condition that $\omega \cdot \gamma > 0$ for all nodal curves is vacuous. It follows that ω lies in the Kähler cone determined by Ω.

Using that no rational linear combination of the components of ρ vanishes and the choice of A_k, B_k, C_j above, it is possible to choose an appropriate irrational ν so that every integral class γ not in the integral lattice of $\{\alpha_1, \beta_1\}$ has the property that $\omega \cdot \gamma$ is irrational. q.e.d.

According to Yau's theorem, the complex structure J and the Kähler class ω determine a hyperkähler metric g on X whose Kähler form lies in the class ω. We will abuse notation and denote the Kähler form ω.

The class $\alpha_1 - \beta_1$ is a lagrangian class for the Kähler form ω (that is, $\omega(\alpha_1 - \beta_1) = 0$). Moreover, because of the irrationality condition on ω, the only integral lagrangian classes for ω are integral multiples of

$\alpha_1 - \beta_1$. Since X is simply connected, $\alpha_1 - \beta_1$ is a spherical class and there is a smooth immersion $f : S^2 \to X$ that represents $\alpha_1 - \beta_1$. Using that $\alpha_1 - \beta_1$ is a lagrangian class there is a smooth homotopy of f to a lagrangian immersion $\ell : S^2 \to X$. In particular, there is an (immersed) lagrangian two-sphere in X that represents $\alpha_1 - \beta_1$.

The following theorem is our main result.

Theorem 3.3. *There is some class in the integral lattice of* $\{\alpha_1 - \beta_1\}$ *that has an area minimizer among lagrangian two-spheres* (*for the hyperkähler metric* g) *that is not regular* (*i.e., has singular points with non-flat tangent cones*).

Proof. Suppose, to the contrary, that every area minimizer among lagrangian two-spheres is regular. (Recall that branch points are allowed.) Consider an area minimizing sequence of lagrangian two-spheres that represent $\alpha_1 - \beta_1$. First, suppose that it converges to a lagrangian area minimizing two-sphere C without bubbling. Then C represents $\alpha_1 - \beta_1$. By assumption, C is regular so it is both minimal and lagrangian [**5**] and therefore there is a a complex structure J' on the hyperkähler line for which C is J'-holomorphic. But,

$$C \cdot C = (\alpha_1 - \beta_1) \cdot (\alpha_1 - \beta_1) = -4,$$

and this contradicts the adjunction formula. Next, suppose that the sequence converges with bubbling. Each bubble B is a lagrangian two-sphere that minimizes area among lagrangian two-spheres in some lagrangian homology class. Therefore, by assumption, B is regular. Also $[B] = p(\alpha_1 - \beta_1)$, where $p \in \mathbb{Z}$ and $p \neq 0$, since these are the only non-trivial lagrangian homology classes. As above, since B is regular and a lagrangian area minimizer, there is a a complex structure J' on the hyperkähler line for which B is J'-holomorphic. But,

$$B \cdot B = p(\alpha_1 - \beta_1) \cdot p(\alpha_1 - \beta_1) = -4p^2 \leq -4,$$

and this again contradicts the adjunction formula. The result follows. q.e.d.

The author and Micallef adapted the technique used in their study of area minimizers in a K3 to prove a version of Theorem 3.3 [**3**]. However, the Kähler structure used in [**3**] is quite special. In particular, it lies near the boundary of the moduli space of Calabi–Yau metrics. The technique used here produces quite general Kähler structures that, in particular, lie away from the boundary of moduli space. It is not yet clear how to characterize the Kähler structures and lagrangian classes in a K3 that have singular minimizers though the construction given here suggests that they may be “generic” in a suitable sense.

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