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ON STABILITY AND THE CONVERGENCE OF THE KÄHLER-RICCI FLOW

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Abstract

Assuming uniform bounds for the curvature, the exponential convergence of the Kähler-Ricci flow is established under two conditions which are a form of stability: the Mabuchi energy is bounded from below, and the dimension of the space of holomorphic vector fields in an orbit of the diffeomorphism group cannot jump up in the limit.

1. Introduction

The normalized Kähler-Ricci flow exists for all times, and converges when the first Chern class is negative or zero [**4**, **35**]. However, when the first Chern class is positive, there are very few known cases of convergence. In one complex dimension, Hamilton [**19**] used entropy estimates to show convergence under the assumption of an initial metric of everywhere positive scalar curvature. This last assumption was removed later by Chow [**12**]. In higher dimensions, convergence was established only in the case of positive biholomorphic sectional curvature, first by X.X. Chen and Tian [**9**, **10**] using Liouville energy functionals, and then by Cao, B.L. Chen, and Zhu [**5**] using the recent injectivity radius bound of Perelman [**25**].

The convergence of the Kähler-Ricci flow for a Kähler manifold X of positive Chern class can be expected to be a difficult issue, since the limit would give a Kähler-Einstein metric, and not all Fano manifolds admit such metrics. According to a well-known conjecture of S.T. Yau [**36**], the existence of a Kähler-Einstein metric should be equivalent to the stability of X in the sense of geometric invariant theory. It is then an important problem to relate stability to the convergence of the Kähler-Ricci flow.

In this paper, we take a first step in this direction. More specifically, we consider the convergence of the Kähler-Ricci flow on a compact

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Kähler manifold (X, ω_0) under two assumptions which are both a form of stability:

(A) The Mabuchi energy $\nu_{\omega_0}(\phi)$ is bounded from below.

(B) Let J be the complex structure of X, viewed as a tensor. Then the C^∞ closure of the orbit of J under the diffeomorphism group of X does not contain any complex structure J_∞ with the property that the space of holomorphic vector fields with respect to J_∞ has dimension strictly higher than the dimension of the space of holomorphic vector fields with respect to J.

It is well-known by the work of Bando and Mabuchi [**3**] that the existence of a Kähler-Einstein metric implies condition (A). This condition is also indirectly related to the notion of K-stability [**14**, **33**]. Indeed, if X is imbedded into $\mathbf{CP}^N$ by sections of the anti-pluricanonical bundle, then K-stability can be viewed as a condition on the asymptotic behavior of the Mabuchi energy functional $\nu_{\omega_0}(\phi)$ along the orbits of $GL(N+1)$. In particular, it has been shown by Donaldson [**14**] that K-stability implies the lower boundedness of the Mabuchi energy for toric varieties of dimension $n = 2$.

The condition (B) is arguably an even more direct manifestation of stability. The stability of a geometric structure should be a condition insuring that the moduli space of such structures be Hausdorff. All the tensors J in the same orbit of the diffeomorphism group of X define the same holomorphic structure. If, as ruled out by condition (B), the dimension of the space of holomorphic vector fields jumps up in the limit, then the limit would be different, and the moduli space of holomorphic structures would not be Hausdorff.

Theorem 1. *Let (X, J) be a compact complex manifold of dimension n. Let $\dot{g}_{\bar{k}j} = -R_{\bar{k}j} + \mu g_{\bar{k}j}$ be the normalized Kähler-Ricci flow, with initial metric $g_{\bar{k}j}(0)$, and $\mu g_{\bar{k}j}(0)$ a Kähler metric in the first Chern class of X. Here μn denotes the total scalar curvature. Assume that the Riemann curvature tensor is uniformly bounded along the flow.*

1. *If condition* (A) *holds, then we have for any $s \geq 0$*

$$\lim_{t\to\infty} ||R_{\bar{k}j}(t) - \mu g_{\bar{k}j}(t)||_{(s)} = 0, \tag{1.1}$$

   where $||\cdot||_{(s)}$ denotes the Sobolev norm of order s with respect to the metric $g_{\bar{k}j}(t)$.
2. *If both conditions* (A) *and* (B) *hold, and if the diameter of X is uniformly bounded along the flow, then the Kähler-Ricci flow converges exponentially fast in C^∞ to a Kähler-Einstein metric.*

A recent but as yet unpublished work of Perelman shows that the scalar curvature and the diameter of the manifold always remain bounded under the Kähler-Ricci flow. Thus the diameter and curvature assumptions in Theorem 1 may be significantly less restrictive than they appear at first sight. The following theorem provides a setting where the curvature assumptions in Theorem 1 can be obtained by combining Perelman's result with a relatively mild curvature positivity condition on the initial metric:

Theorem 2. *Let X be a compact Kähler manifold of dimension $n = 2$, which satisfies the conditions* (A) *and* (B)*. Assume that the scalar curvature and the diameter of X are bounded from above along the Kähler-Ricci flow with a Kähler initial metric in the first Chern class of X. Assume further that*

(C) *The initial metric $g_{\bar{k}j}(0)$ has non-negative Ricci curvature and its traceless curvature operator is* 2*-nonnegative.*

Then the Kähler-Ricci flow converges exponentially fast in C^∞ to a Kähler-Einstein metric.

The notion of 2-nonnegativity for the Riemann curvature operator was introduced by H. Chen [**8**]. The condition (C) was introduced in [**28**], where it was shown in 2 dimensions to be preserved by the Kähler-Ricci flow. It should perhaps be mentioned that the case of an initial metric with positive bisectional curvature in arbitrary dimension is another case where the uniform boundedness curvature conditions of Theorem 1 would be satisfied by the work of Cao, B.L. Chen and Zhu [**5**], or if we invoke Perelman's unpublished result. Indeed, by earlier work of Mok [**22**] and Bando [**2**], the positivity of the bisectional curvature is preserved. Thus the boundedness of the scalar curvature implies the boundedness of the bisectional curvature, and hence of the sectional curvature.

Without stability assumptions, the Kähler-Ricci flow is expected to produce either singularities or solitons in the infinite time limit. For such results, under similar assumptions of uniform bounds on the curvature, see the recent works of N. Sesum [**30**, **31**].

2. Part 1 of Theorem 1: Sobolev estimates

In this section, we prove the first statement in Theorem 1.

• We start with some preliminary considerations about the Kähler-Ricci flow and the Mabuchi functional. Since the initial metric $g_{\bar{k}j}(0)$ is in $c_1(X)$, and since the Kähler-Ricci flow manifestly preserves the first Chern class, we can write at all times

$$R_{\bar{k}j} - \mu\, g_{\bar{k}j} = \partial_j \partial_{\bar{k}} h \tag{2.1}$$

for some smooth real scalar function $h = h(t)$ defined up to a (time-dependent) additive constant. We fix an arbitrary smooth choice of such constants. Since all our estimates ultimately depend only on ∇h, such choices are immaterial. Now h flows according to

$$\dot{h} = \Delta h + \mu h + c \tag{2.2}$$

where c is a time-dependent constant, and $\Delta = \nabla^{\bar{k}}\nabla_{\bar{k}}$. To see this, it suffices to differentiate the defining equation for h with respect to time. Since $\dot{R}_{\bar{k}j} = -\partial_j\partial_{\bar{k}}(g^{l\bar{p}}\dot{g}_{\bar{p}l}) = \partial_j\partial_{\bar{k}}R$, we obtain $\partial_j\partial_{\bar{k}}R + \mu\partial_j\partial_{\bar{k}}h = \partial_j\partial_{\bar{k}}\dot{h}$, and hence $R + \mu h + c' = \dot{h}$, where c' is a constant. But (2.1) also implies that $R - \mu n = \Delta h$, so that the desired identity follows with $c = c' + \mu n$.

• Next, the Mabuchi energy functional $\nu_{\omega_0}(\phi)$ is the functional on the space of Kähler potentials $\{\phi : \; g_{\bar{k}j} = g_{\bar{k}j}(0) + \partial_j\partial_{\bar{k}}\phi > 0\}$ defined by its variation

$$\dot{\nu}_{\omega_0}(\phi) = -\frac{1}{V}\int_X \dot{\phi}(R - \mu n)\omega^n, \tag{2.3}$$

where the Kähler form ω is defined by $\omega = \frac{\sqrt{-1}}{2\pi}g_{\bar{k}j}dz^j \wedge d\bar{z}^k$ and $V = \int_X \omega^n$. To determine $\dot{\phi}$ in the case of the Kähler-Ricci flow, we rewrite the defining equation $\dot{g}_{\bar{k}j} = -R_{\bar{k}j} + \mu g_{\bar{k}j} = -\partial_j\partial_{\bar{k}}h$ in terms of ϕ. This gives $\dot{\phi} = -h + c''$, with c'' another time-dependent constant. Substituting this in the preceding equation, we get

$$\dot{\nu}_{\omega_0}(\phi) = -\frac{1}{V}\int_X |\nabla h|^2\omega^n. \tag{2.4}$$

• We return now to the proof of part 1 of Theorem 2 proper. Our first step is to show that the lower bound for the Mabuchi energy functional together with the uniform boundedness of the scalar curvature implies that

$$\int_X |\nabla h|^2\omega^n \to 0, \quad t \to \infty. \tag{2.5}$$

Now integrating $\dot{\nu}_{\omega_0}(\phi)$ and using the lower bound for $\nu_{\omega_0}(\phi)$ gives

$$\frac{1}{V}\int_0^T dt \int_X |\nabla h|^2\omega^n = \nu_{\omega_0}(\phi_0) - \nu_{\omega_0}(\phi_T) \le C. \tag{2.6}$$

for all $T > 0$, which implies that $\int_X |\nabla h|^2\omega^n$ converges to 0 along some sequence of times tending to ∞. To get full convergence, we consider the flow of $|\nabla h|^2$. We have the following Bochner-Kodaira formula

$$\Delta|\partial_j h|^2 = g^{j\bar{k}}\Delta(\partial_j h)\,\overline{\partial_k h} + g^{j\bar{k}}\partial_j h\,\overline{\Delta\partial_k h} + R^{j\bar{k}}\partial_j h\,\overline{\partial_k h} + |\bar{\nabla}\nabla h|^2 + |\nabla\nabla h|^2. \tag{2.7}$$

Comparing this with the time variation

$$(|\partial_j h|^2)^{\dot{}} = g^{j\bar{k}}(\partial_j h)^{\dot{}}\,\overline{\partial_k h} + g^{j\bar{k}}\partial_j h\,\overline{(\partial_k h)^{\dot{}}} + R^{j\bar{k}}\partial_j h\,\overline{\partial_k h} - \mu|\nabla h|^2 \tag{2.8}$$

and the flow $(\partial_j h)^{\cdot} = \Delta(\partial_j h) + \mu \partial_j h$ for $\partial_j h$, we find

$$(|\nabla h|^2)^{\cdot} - \Delta |\nabla h|^2 = -|\bar\nabla \nabla h|^2 - |\nabla\nabla h|^2 + \mu |\nabla h|^2. \tag{2.9}$$

Let $Y(t) = \int_X |\nabla h|^2 \omega^n$. Since $(\omega^n)^{\cdot} = (-R + \mu n)\omega^n$, we obtain

$$\dot Y = \mu(n+1)Y - \int_X |\nabla h|^2 R\, \omega^n - \int_X |\bar\nabla\nabla h|^2 \omega^n - \int_X |\nabla\nabla h|^2 \omega^n. \tag{2.10}$$

Say $|R| \le C$. Then $\dot Y \le (\mu(n+1) + C)Y$, and hence

$$Y(t) \le Y(s) e^{(\mu(n+1)+C)(t-s)}$$

for all $t \ge s$. Since the bound (2.6) implies that $\sum_{m=0}^{\infty} \int_m^{m+1} dt Y(t) < \infty$, there must be a sequence $t_m \in [m, m+1)$ with $Y(t_m) \to 0$. The previous bound implies $Y(t) \le Y(t_m) e^{\mu(n+1)+C}$ for all $t \in [m, m+1)$, and hence $Y(t) \to 0$ as $t \to \infty$.

• The next step is to extend this convergence to the higher derivatives of h. Since $Y(t)$ is now known to tend to 0, the integration of (2.10) gives

$$\begin{aligned} &\int_0^\infty dt \int_X |\bar\nabla\nabla h|^2 \omega^n + \int_0^\infty dt \int_X |\nabla\nabla h|^2 \omega^n \\ &\quad = Y(0) + \mu(n+1) \int_0^\infty dt \int_X |\nabla h|^2 \omega^n - \int_0^\infty dt \int_X |\nabla h|^2 R\, \omega^n. \end{aligned} \tag{2.11}$$

This implies that the L^2 norms of $\bar\nabla\nabla h$ and $\nabla\nabla h$ tend to 0 along some subsequence of times going to infinity. To establish convergence, we need as previously the flows for $\bar\nabla\nabla h$ and $\nabla\nabla h$. It is convenient to set up a systematic induction argument as follows. Set

$$\begin{aligned} h_{\bar K J} &= \nabla_{j_s} \cdots \nabla_{j_1} \nabla_{\bar k_r} \cdots \nabla_{\bar k_1} h, \\ h_{\bar K J} \cdot \overline{h'_{\bar K J}} &= g^{L\bar K} g^{J\bar M} h_{\bar K J} \overline{h'_{\bar L M}}, \\ |\nabla^s \bar\nabla^r h|^2 &= h_{\bar K J} \cdot \overline{h_{\bar K J}}, \\ g^{L\bar K} g^{J\bar M} &= g^{j_1 \bar m_1} \cdots g^{j_s \bar m_s} g^{l_1 \bar k_1} \cdots g^{l_r \bar k_r} \end{aligned} \tag{2.12}$$

for $K = (k_1 \cdots k_r)$, $J = (j_1 \cdots j_s)$, $L = (l_1 \cdots l_r)$, $M = (m_1 \cdots m_s)$. Instead of Bochner-Kodaira formulas for the complex Laplacian $\nabla^{\bar p}\nabla_{\bar p}$, it is simpler to use the real Laplacian $\Delta_{\mathbf{R}} = \Delta + \bar\Delta$, which gives at once

$$\begin{aligned} &\frac{1}{2}\Delta_{\mathbf{R}} |\nabla^s \bar\nabla^r h|^2 \\ &\quad = \frac{1}{2}\Delta_{\mathbf{R}} h_{\bar K J} \cdot \overline{h_{\bar K J}} + h_{\bar K J} \cdot \overline{\frac{1}{2}\Delta_{\mathbf{R}} h_{\bar K J}} + |\nabla h_{\bar K J}|^2 + |\bar\nabla h_{\bar K J}|^2. \end{aligned} \tag{2.13}$$

The time evolution of $|\nabla^s\bar\nabla^r h|^2$ is given by

$$(|\nabla^s\bar\nabla^r h|^2)^{\dot{}} = \dot h_{\bar K J}\cdot\overline{h_{\bar K J}} + h_{\bar K J}\cdot\overline{\dot h_{\bar K J}} - \mu(r+s)|\nabla^s\bar\nabla^r h|^2 + \sum_{\alpha=1}^{r} R^{l_\alpha\bar k_\alpha} h_{\bar k_1\cdots\bar k_\alpha\cdots\bar k_r\, J}\bar h^{J\bar k_r\cdots}{}_{l_\alpha}{}^{\cdots\bar k_1} + \sum_{\beta=1}^{s} R^{j_\beta\bar m_\beta} h_{\bar K\, j_1\cdots j_\beta\cdots j_s}\bar h^{j_1\cdots}{}_{\bar m_\beta}{}^{\cdots j_s\bar K}. \tag{2.14}$$

Now we need the flow for the tensor $h_{\bar K J}$. By induction, we find

$$\begin{aligned}\dot h_{\bar K J} = {} & \frac{1}{2}\Delta_{\mathbf{R}} h_{\bar K J} + \mu h_{\bar K J} - \frac{1}{2}\sum_{\alpha=1}^{r} R_{\bar k_\alpha}{}^{\bar m_\alpha} h_{\bar k_1\cdots\bar m_\alpha\cdots\bar k_r\, J} \\ & - \frac{1}{2}\sum_{\beta=1}^{s} R^{m_\beta}{}_{j_\beta} h_{\bar K j_1\cdots m_\beta\cdots j_s} \\ & - \sum_{1\le\alpha<\beta\le s} R^{m_\alpha}{}_{j_\alpha}{}^{m_\beta}{}_{j_\beta} h_{\bar K j_1\cdots m_\alpha\cdots m_\beta\cdots j_s} \\ & - \sum_{1\le\alpha<\beta\le r} R_{\bar k_\alpha}{}^{\bar m_\alpha}{}_{\bar k_\beta}{}^{\bar m_\beta} h_{\bar k_1\cdots\bar m_\alpha\cdots\bar m_\beta\cdots\bar k_r\, J} \\ & + \sum_{\alpha=1}^{r}\sum_{\beta=1}^{s} R_{\bar k_\alpha j_\beta}{}^{\bar m_\alpha n_\beta} h_{\bar k_1\cdots\bar m_\alpha\cdots\bar k_r j_1\cdots n_\beta\cdots j_s} \\ & + \sum_{u=1}^{r+s-1} D^u Riem \star D^{r+s-u} h.\end{aligned} \tag{2.15}$$

Here D denotes covariant differentiation in either j or $\bar j$ indices, and $D^u h$ and $D^u Riem$ denote all tensors obtained by u covariant differentiations of h and of the Riemann curvature tensor respectively. The $\star$ symbol indicates general pairings of these tensors. The last term in the above equation is a lower order term which is actually absent when $r = s = 1$. Assembling the equations (2.13), (2.14), and (2.15), we obtain

$$\begin{aligned}(|\nabla^s\bar\nabla^r h|^2)^{\dot{}} = {} & \frac{1}{2}\Delta_{\mathbf{R}}|\nabla^s\bar\nabla^r h|^2 - |\nabla^{s+1}\bar\nabla^r h|^2 - |\bar\nabla\nabla^s\bar\nabla^r h|^2 \\ & + \mu(2-r-s)|\nabla^s\bar\nabla^r h|^2 \\ & + 2\sum_{\alpha=1}^{r}\sum_{\beta=1}^{s} R_{\bar k_\alpha j_\beta}{}^{\bar m_\alpha n_\beta} h_{\bar k_1\cdots\bar m_\alpha\cdots\bar k_r j_1\cdots n_\beta\cdots j_s}\bar h^{\bar K J} \\ & - 2\sum_{1\le\alpha<\beta\le s} R^{m_\alpha}{}_{j_\alpha}{}^{m_\beta}{}_{j_\beta} h_{\bar K j_1\cdots m_\alpha\cdots m_\beta\cdots j_s}\bar h^{\bar K J}\end{aligned} \tag{2.16}$$

$$- 2 \sum_{1\leq\alpha<\beta\leq r} R_{\bar{k}_\alpha}{}^{\bar{m}_\alpha}{}_{\bar{k}_\beta}{}^{\bar{m}_\beta} h_{\bar{k}_1\cdots\bar{m}_\alpha\cdots\bar{m}_\beta\cdots k_r\, J}\bar{h}^{\bar{K}J}$$
$$+ 2 \sum_{u=1}^{r+s-1} D^u Riem \star D^{r+s-u}h \star \overline{\nabla^s\bar{\nabla}^r h}.$$

Set $Y_{r,s}(t) = \int_X |\nabla^s\bar{\nabla}^r h|^2\omega^n$. According to [**20**, **32**], the uniform boundedness of the Riemann curvature tensor in the Ricci flow implies the uniform boundedness of the covariant derivatives of the Riemann curvature tensor of any fixed order. The argument applies verbatim to the normalized Kähler-Ricci flow. Thus the previous identity implies that

$$\dot{Y}_{r,s}(t) \leq C_1\, Y_{r,s}(t) + C_2 \left(\int_X |D^{r+s-u}h|^2\omega^n\right)^{1/2} Y_{r,s}^{1/2}(t)$$
$$- \int_X |\nabla^{s+1}\bar{\nabla}^r h|^2\omega^n - \int_X |\bar{\nabla}\nabla^s\bar{\nabla}^r h|^2\omega^n, \tag{2.17}$$

where a summation over $1 \leq u \leq r+s-1$ is understood, we have bounded all curvature terms by constants, and applied the Cauchy-Schwarz inequality to the lower order terms. This implies for any $a \geq b$

$$\begin{aligned} &Y_{r,s}(a) - Y_{r,s}(b) \\ &\leq C_1 \int_b^a dt\, Y_{r,s}(t) + C_2 \int_b^a dt \left(\int_X |D^{r+s-u}h|^2\omega^n\right)^{1/2} Y_{r,s}^{1/2}(t) \\ &\leq C_1 \int_b^\infty Y_{r,s}(t) \\ &\quad + C_2 \left(\int_b^\infty dt \int_X |D^{r+s-u}h|^2\omega^n\right)^{1/2} \left(\int_b^\infty dt\, Y_{r,s}(t)\right)^{1/2}. \end{aligned} \tag{2.18}$$

We argue now by induction. Assume that

$$\begin{aligned} \int_0^\infty dt \int_X |D^v h|^2\omega^n &< \infty, \qquad \text{for } v \leq r+s \\ \int_X |D^v h|^2\omega^n &\to 0, \qquad \text{for } v < r+s. \end{aligned} \tag{2.19}$$

Then $Y_{r,s}(t)$ is in particular integrable on $[0,\infty)$, and arguing as before, we can choose $b_m \in [m, m+1)$ with $Y_{r,s}(b_m) \to 0$. The estimate (2.18) applied with $b = b_m$ and m large enough shows that $Y_{r,s}(t) \to 0$. Since any covariant derivative of h of order $r+s$ differs from covariant derivatives of the form $\nabla^s\bar{\nabla}^r h$ and $\nabla^s\bar{\nabla}^r h$ by $D^u Riem \star D^{r+s-u}h$ with $u \geq 1$, and since all derivatives of the Riemann curvature tensor are bounded, it follows from the second induction hypothesis and the fact that $Y_{r,s}(t) \to 0$ that $\int_X |D^{r+s}h|^2\omega^n \to 0$.

To establish the first induction hypothesis at order $r+s+1$, we return to the equation (2.17). Integrating from 0 to ∞, and applying again the

Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}(2.20)\qquad &\int_0^\infty dt \int_X |\nabla^{s+1}\bar\nabla^r h|^2\omega^n + \int_0^\infty \int_X |\bar\nabla\nabla^s\bar\nabla^r h|^2\omega^n \\ &\leq Y_{r,s}(0) + C_1\int_0^\infty dt\, Y_{r,s}(t) \\ &\quad + C_2\left(\int_0^\infty dt\int_X |D^{r+s-u}h|^2\omega^n\right)^{1/2}\left(\int_0^\infty dt\, Y_{r,s}(t)\right)^{1/2}.\end{aligned}$$

This implies that the L^2 norms of $\nabla^{s+1}\bar\nabla^r h$ and $\bar\nabla\nabla^s\bar\nabla^r h$ are integrable with respect to time on $[0,\infty)$. Using the first induction hypothesis and again the uniform boundedness of the Riemann curvature tensor and its derivatives, we can deduce that $\int_X |D^{r+s+1}h|^2\omega^n$ is integrable with respect to time on $[0,\infty)$. This completes the induction argument. Since $R_{\bar k j} - \mu g_{\bar k i}$ is given by $\partial_{\bar k}\partial_j h$, the L^2 convergence to 0 of all covariant derivatives of h implies the convergence to 0 of all Sobolev norms of $R_{\bar k j} - \mu g_{\bar k j}$. The proof of the first part of Theorem 1 is complete.

3. Part 2 of Theorem 1: convergence of the metrics

Assuming now condition (B) and the uniform boundedness of the diameter of X, we wish to establish the convergence of the metrics $g_{\bar k j}(t)$ as $t\to\infty$. The uniform boundedness of the curvature together with the uniform boundedness of the diameter imply the uniform boundedness from below of the injectivity radius [**7**]. Since the volume is fixed under the normalized flow, the uniform control of the diameter and the injectivity radius imply the uniform control of the Sobolev constant. Thus the equation (1.1) now implies

$$(3.1)\qquad \sup_X |D^p\dot g_{\bar k j}(t)|_t = \sup_X |D^p(R_{\bar k j}(t) - \mu g_{\bar k j}(t))|_t \to 0,$$

for any fixed integer p, where we have introduced the lower index t to stress that the norms are taken with respect to the metric $g_{\bar k j}(t)$. However, the convergence to 0 of $|\dot g_{\bar k j}(t)|_t$ does not guarantee the convergence of the metrics themselves. In fact, it does not even guarantee that they are uniformly equivalent.

We also note that the uniform boundedness of the volume, diameter, injectivity radius, and curvature implies that the metrics $g_{\bar k j}(t)$ have uniform bounded geometry, in the sense of Gromov [**17**], in fact uniform bounded C^∞ geometry, since all covariant derivatives of the curvature are also bounded uniformly. Thus, by passing to a subsequence and applying the C^∞ version of Gromov compactness due to Hamilton [**21**], we can find diffeomorphisms F_{t_j} so that the pull-backs $(F_{t_j})_*(g(t_j))$ converge in C^∞. But we have no control over the diffeomorphisms F_{t_j} and cannot deduce from this the convergence of the metrics $g_{\bar k j}(t_j)$ themselves. As we saw earlier, this issue of diffeomorphisms underlies the notion of stability, and it appears now central to the problem of convergence of the flow. We shall see later, however, that Gromov compactness can be put to good use in the proper context.

• To overcome these difficulties, we shall establish the exponential decay of $|\dot{g}_{\bar{k}j}(t)|_t$. Let $Y = \int_X |\nabla h|^2 \omega^n$ as before. Then we have

(3.2)
$$\dot{Y} = -\int_X |\nabla h|^2 (R - \mu n)\omega^n - \int_X \nabla^j h \nabla^{\bar{k}} h (R_{\bar{k}j} - \mu g_{\bar{k}j})\omega^n - 2\int_X |\bar{\nabla}\bar{\nabla} h|^2 \omega^n.$$

This follows from either the equation (2.9) or (2.10), the fact that $(\omega^n)^{\dot{}} = (-R + \mu n)\omega^n$, and the Bochner-Kodaira formula for vector fields $V = (V^j)$,

$$||\nabla V||^2 = ||\bar{\nabla} V||^2 + \int_X R_{\bar{k}j} V^j \bar{V}^{\bar{k}} \, \omega^n \tag{3.3}$$

applied to $V^j = g^{j\bar{k}} \partial_{\bar{k}} h = \nabla^j h$ (note that $\nabla^j h = (\bar{\nabla} h)^j$). The identity (3.2) shows that the only chance of getting exponential decay is by obtaining a strictly positive lower bound for $\int_X |\bar{\nabla}\bar{\nabla} h|^2 \omega^n = \int_X |\bar{\partial} V|^2 \omega^n$.

• Let λ_t be the lowest strictly positive eigenvalue of the complex Laplacian $-\Delta_t = -g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} = -\bar{\partial}^\dagger \bar{\partial}$ on $T^{1,0}(X)$ vector fields. By the elliptic theory, we have

$$\lambda_t \, ||V - \pi_t V||^2 \leq \int_X |\bar{\nabla} V|^2 \omega^n, \tag{3.4}$$

where π_t is the orthogonal projection with respect to $g_{\bar{k}j}(t)$ on the space $H^0(X, T^{1,0})$ of holomorphic vector fields. When V is the gradient of the function h as in our present case, it turns out that $||\pi_t V||^2$ is exactly the Futaki invariant of the manifold X, applied to $\pi_t V$. Recall that the Futaki invariant Fut is the character defined on the space of holomorphic vector fields by [**15**]

$$Fut(W) = \int_X (Wh)\omega^n, \qquad W \in H^0(X, T^{1,0}). \tag{3.5}$$

It is independent of the choice of metric within the Kähler class. The main observation for our purposes is

$$||\pi_t V||^2 = Fut(\pi_t V) \tag{3.6}$$

when $V^j = \nabla^j h$. To see this, we note $\langle \pi_t V, \pi_t V \rangle = \langle V, \pi_t^2 V \rangle = \langle V, \pi_t V \rangle = \langle \bar{\nabla} h, \pi_t V \rangle = \int_X \overline{(\pi_t V)(h)} \omega^n = \overline{Fut(\pi_t V)} = Fut(\pi_t V)$. Thus the inequality (3.2) implies

$$\begin{aligned} \dot{Y} &\leq -2\,\lambda_t \, Y + 2\,\lambda_t \, Fut(\pi_t(\nabla^j h)) \\ &\quad - \int_X |\nabla h|^2 (R - \mu n)\omega^n - \int_X \nabla^j h \, \nabla^{\bar{k}} h (R_{\bar{k}j} - \mu g_{\bar{k}j})\omega^n. \end{aligned} \tag{3.7}$$

This key inequality holds in all generality for the normalized Kähler-Ricci flow.

• Assume now all the conditions stated for part 2 of Theorem 1. Then the lower boundedness of the Mabuchi energy functional implies that the Futaki invariant vanishes identically (see [**34**]). Furthermore, $|\nabla^j h\,\nabla^{\bar k} h(R_{\bar k j}-\mu g_{\bar k j})|\leq |\nabla h|^2|R_{\bar k j}-\mu g_{\bar k j}|_t$, so that the convergence of $|R_{\bar k j}-\mu g_{\bar k j}|_t$ to 0 established in (1.1) implies that for any $\epsilon>0$, we have

$$\dot Y \leq (-2\,\lambda_t+\epsilon)\,Y, \qquad t\in[T_\epsilon,\infty), \tag{3.8}$$

for T_ϵ large enough. Thus establishing exponential convergence reduces to showing that λ_t is uniformly bounded from below by a positive constant.

• The bounds from below for λ_t do not appear to be accessible by Bochner-Kodaira techniques, as these apply to a negative bundle instead of a positive one, as is here the case. Rather, they should reflect the bounded geometry, just as in the case of the lowest eigenvalue for the scalar Laplacian. In that case, we can either construct explicitly the Green's function as in [**1**] or make use of estimates for the lowest eigenvalue such as Cheeger's [**6**] in terms of the isoperimetric and Sobolev constants [**11**]. However, no simple characterization of λ_t seems available in the case of vector fields. Instead, we shall establish the following estimate by using a complex version of the Gromov compactness theorem:

Theorem 3. *Let X be a compact, complex manifold of dimension n. Assume that its complex structure J is stable in the sense that it satisfies the condition* (B) *stated in the Introduction. Fix $V>0$, $D>0$, $\delta>0$, and constants C_k. Then there exists an integer N and a constant $C=C(V,D,\delta,C_k,n,N)>0$ such that*

$$C\,||W||^2\leq ||\bar\partial W||^2, \qquad W\perp H^0(X,T^{1,0}), \tag{3.9}$$

for all Kähler metrics g on X whose volumes and diameters are bounded above by V and D respectively, whose injectivity radius is bounded from below by $\delta>0$, and the k-th derivatives of whose curvature tensors are uniformly bounded by C_k, for all $k\leq N$.

Assuming Theorem 3 for the moment, we deduce that there exists a positive constant c so that $\lambda_t>2c$ for all t. Thus for t large enough, $Y(t)$ satisfies the differential inequality

$$\dot Y(t)\leq -c\,Y(t) \tag{3.10}$$

from which it follows that $Y(t)$ decreases exponentially fast

$$\int_X |\nabla h|^2\omega^n\leq C\,e^{-c\,t}. \tag{3.11}$$

• Once the exponential decay of the L^2 norm of ∇h has been established, it is not difficult to deduce the exponential decay of the L^2 norms $Y_{r,s}$ of $\bar\nabla^r \nabla^s h$, where all norms are taken with respect to the metric $g_{\bar k j}(t)$. For example, the inequality (2.17) implies that

(3.12)
$$\begin{aligned}
&\dot Y_{r,s}(t)\\
&\leq -2c\,Y_{r,s}(t) + \left(C_1 + \frac{1}{2} + 2c\right) Y_{r,s}(t)\\
&\quad + C_2 \int_X |D^{r+s-u} h|^2 \omega^n - \int_X |\nabla^{s+1}\bar\nabla^r h|^2 \omega^n - \int_X |\bar\nabla \nabla^s \bar\nabla^r h|^2 \omega^n,
\end{aligned}$$

and on the right hand side, a summation over all indices u in the range $1 \leq u \leq r+s-1$ is understood. For any $\epsilon > 0$, there exists a constant $c(\epsilon)$ independent of t so that

$$\begin{aligned}
Y_{r,s}(t) \leq \epsilon &\left(\int_X |\nabla^{s+1}\bar\nabla^r h|^2 \omega^n + \int_X |\bar\nabla \nabla^s \bar\nabla^r h|^2 \omega^n\right)\\
&+ c(\epsilon) \int |D^{r+s-1} h|^2 \omega^n.
\end{aligned} \tag{3.13}$$

By choosing ϵ small enough, we can deduce that

$$\begin{aligned}
\dot Y_{r,s}(t) &\leq -2c\,Y_{r,s}(t) + C_3 \sum_{1\leq u\leq r+s-1} \int_X |D^{r+s-u} h|^2 \omega^n\\
&\leq -2c\,Y_{r,s}(t) + C_4\, e^{-ct},
\end{aligned} \tag{3.14}$$

where in the last inequality, we have assumed by induction that all L^2 norms of $D^{r+s-u}h$ decay exponentially. Integrating between t and 0, we see that $Y_{r,s}(t)$ decays exponentially also.

• From the exponential decay of the L^2 norms of $\nabla^p h$ for all $p \geq 1$, we deduce from the Sobolev imbedding theorem with uniform constants that we have exponential decay of the C^k norms

$$\sup_X |\nabla^k h|_t^2 \leq C_k\, e^{-ct}. \tag{3.15}$$

• The next step is to show that all metrics $g_{\bar k j}(t)$ are uniformly equivalent, that is, bounded by one another up to a constant independent of t. According to a lemma of Hamilton ([**18**], Lemma 14.2), it suffices to show that

$$\int_T^\infty \sup_X |\dot g_{\bar k j}|_t \, dt < \infty. \tag{3.16}$$

Since $\dot g_{\bar k j}(t) = \partial_{\bar k}\partial_j h$, the preceding result implies that $|\dot g_{\bar k j}|_t$ decays exponentially as $t \to \infty$. This implies the desired inequality, and hence all metrics $g_{\bar k j}(t)$ are uniformly equivalent.

• Since all metrics $g_{\bar{k}j}(t)$ are now known to be equivalent, we can now write for some strictly positive constant c

$$
\begin{aligned}
(3.17) \qquad & |g_{\bar{k}j}(T)W^j\bar{W}^{\bar{k}} - g_{\bar{k}j}(S)W^j\bar{W}^{\bar{k}}| \\
&\leq \int_S^T \sup_X |\dot{g}_{\bar{k}j}|_t |W|_t^2 \, dt \\
&\leq C\, |W|_{t=0}^2 \int_S^T \sup_X |\dot{g}_{\bar{k}j}|_t \, dt, \\
&\leq C'\, |W|_{t=0}^2 \, (e^{-cT/2} - e^{-cS/2}).
\end{aligned}
$$

This tends to 0 exponentially as $S, T \to \infty$. Thus the metrics $g_{\bar{k}j}(t)$ converge exponentially fast as $t \to \infty$ to some metric $g_{\bar{k}j}(\infty)$, which is also equivalent to all the $g_{\bar{k}j}(t)$'s. Iterating the arguments shows that the convergence is in C^∞. Since $\partial_{\bar{k}}\partial_j h$ tends to 0, the metric $g_{\bar{k}j}(\infty)$ is clearly Kähler-Einstein. The proof of Theorem 1 is complete. q.e.d.

4. Lower bounds for the $\bar{\partial}$ operator on vector fields and Gromov compactness

It remains to prove Theorem 3. This theorem is essentially a consequence of the following Kähler version of the Gromov compactness theorem, combined with some elementary perturbation theory for Laplacians:

Theorem 4. *Let X be a compact smooth manifold. Let $(g(t), J(t))$ be any sequence of metrics $g(t)$ and complex structures $J(t)$ on X such that $g(t)$ is Kähler with respect to $J(t)$. Assume that the $g(t)$'s have bounded geometry, in the sense that their volumes, diameters, curvatures, and covariant derivatives of their curvature tensor are all bounded from above, and their injectivity radii are all bounded from below. Then there exists a subsequence t_j, and a sequence of diffeomorphisms $F_{t_j} : X \to X$ such that the pull-back metrics $\tilde{g}(t_j) = F_{t_j}^* g(t_j)$ converge in C^∞ to a smooth metric $\tilde{g}(\infty)$, and the pull-back complex structure tensors $\tilde{J}(t_j) = F_{t_j}^* J(t_j)$ converge in C^∞ to an integrable complex structure tensor $\tilde{J}(\infty)$. Furthermore, the metric $\tilde{g}(\infty)$ is Kähler with respect to the complex structure $\tilde{J}(\infty)$.*

Proof of Theorem 4. The C^∞ part of the theorem, without reference to complex structures and Kähler forms, is actually the version of the Gromov compactness theorem established by Hamilton [**20**], where C^k uniform bounds on the curvature are assumed, instead of just C^0 bounds as in the original version of Gromov, Peters [**25**], Greene and Wu [**16**]. (Hamilton's version is even more difficult, because no diameter bound is assumed. In that case, the diffeomorphisms F_{t_j} map a sequence of exhausting compact subsets of X into a sequence of exhausting compact subsets of a limiting manifold $\tilde{X}$, which may not be compact. One also needs to choose reference points P_t and reference frames at these points.) Thus, passing to a subsequence, we assume that there exist diffeomorphisms so that $\tilde{g}(t)$ converges, and concentrate on finding a subsequence t_j so that the complex structures $\tilde{J}(t_j)$ converge also.

First we recall the definitions: To say that J is a complex structure is to say that $J^2 = -I$ and its Nijenhuis tensor vanishes. To say that a metric g is compatible with a complex structure J is to say that $g(u, v) = g(Ju, Jv)$ for all $u, v \in TX$. In local coordinates, $g_{ij}u^i v^j = g_{kl}J_i^k u^i J_j^l v^i$, in other words

$$g_{ij} = J_i^k g_{kl} J_j^l. \tag{4.1}$$

Let $g = \tilde{g}(\infty)$ and, whenever convenient, write also $g(t) = g_t$, $J(t) = J_t$ for all values of t including ∞. Let ∇ be the Riemannian connection associated to g. It suffices to show that there are constants C_α such that

$$|\nabla^\alpha J_t|_g \leq C_\alpha \quad \text{for all } t. \tag{4.2}$$

We do this first in the case $\alpha = 0$: Since $g_t \to g$, it suffices to prove

$$|J_t|_{g_t} \leq C_\alpha \quad \text{for all } t. \tag{4.3}$$

Working in normal coordinates for g_t, the equation (4.1) implies, for each i, that

$$1 = \sum_k (J_t)_i^k (J_t)_i^k.$$

Thus $|J_t|_{g_t} = n$, and this proves (4.3).

Now we prove (4.2) by induction: It is true when $\alpha = 0$. Since $\nabla_t J_t = 0$ (this is the definition of "Kähler") we have

$$\nabla^\alpha J_t = \nabla^{\alpha-1}(\nabla - \nabla_t)J_t. \tag{4.4}$$

Let $H_t = \nabla - \nabla_t$. Then $(H_t)g_t = \nabla g_t$. In other words,

$$(H_t)_{ki}^p (g_t)_{pj} + (H_t)_{kj}^p (g_t)_{pi} = \nabla_k (g_t)_{ij}. \tag{4.5}$$

Permuting the indices gives

$$(H_t)_{ij}^p (g_t)_{pk} + (H_t)_{ik}^p (g_t)_{pj} = \nabla_i (g_t)_{jk} \tag{4.6}$$
$$(H_t)_{jk}^p (g_t)_{pi} + (H_t)_{ji}^p (g_t)_{pk} = \nabla_j (g_t)_{ki}.$$

Thus

$$2(H_t)_{ij}^p = (g_t)^{pk}[\nabla_j (g_t)_{ki} + \nabla_i (g_t)_{jk} - \nabla_k (g_t)_{ij}]. \tag{4.7}$$

This shows that H_t and all its derivatives are uniformly bounded: indeed, H_t converges in C^∞ to $H_\infty = 0$. It follows from (4.4) that J_t and its derivatives are bounded. Thus a subsequence converges, and the limit $\tilde{J}_\infty$ is clearly a complex structure. Since $H_t J_t = \nabla J_t$, we get

$H_\infty \tilde{J}_\infty = \nabla \tilde{J}_\infty$, but $H_\infty = 0$. Thus $\tilde{J}_\infty = \tilde{J}(\infty)$ is Kähler, and the proof of Theorem 4 is complete. q.e.d.

Proof of Theorem 3. The main step in the proof is to show that, if $(g(t), J(t))$ are Kähler metrics which converge in C^∞ to $(g(\infty), J(\infty))$, where $g(\infty)$ is a Kähler metric with respect to the complex structure $J(\infty)$, and if the dimension of the space of holomorphic vector fields is the same for all $N \leq t \leq \infty$, then

$$\lim_{t\to\infty}\lambda_t = \lambda_\infty, \tag{4.8}$$

where λ_t is the lowest strictly positive eigenvalue of the Laplacian $-\Delta_t = -\nabla_t^{\bar{j}}\nabla_{t\bar{j}}$ on the space $T_t^{1,0}(X)$ of complex tangent vectors with respect to $(g(t), J(t))$. Assuming this for the moment, Theorem 3 can be proven by contradiction: if it does not hold, then there exists a sequence of metrics $g(t)$ with $\lambda_t \to 0$. Passing to a subsequence, we can apply Theorem 4, and obtain diffeomorphisms F_t so that the metrics $\tilde{g}(t) = (F_t)_*(g(t))$ and complex structures $\tilde{J}(t) = (F_t)_*(J(t))$ converge in C^∞ to a metric $\tilde{g}(\infty)$ and complex structure $\tilde{J}(\infty)$. By the preceding inequality, the lowest eigenvalues of $\tilde{g}(t)$ tend then to a strictly positive limit. But F_t is a biholomorphic isometry between the space X equipped with the Kähler structure $(g(t), J(t))$ and the Kähler structure $(\tilde{g}(t), \tilde{J}(t))$. Thus the eigenvalues of $(\tilde{g}(t), \tilde{J}(t))$ are the same as the eigenvalues of $(g(t), J(t))$. This contradiction proves Theorem 3.

• We turn now to the proof of the eigenvalue limit (4.8). Let $||\cdot||_{H_t^{(s)}}$ be the Sobolev norm of order s on $T(X)$, taken with respect to the metric $g(t)$. Since the metrics $g(t)$ converge, we have the following inequalities

$$\begin{aligned} &|\langle U,V\rangle_t - \langle U,V\rangle_\infty| \leq c_t\,||U||_\infty ||V||_\infty, \quad U,V \in C^\infty(X,T(X)),\\ &C_t^{-1}||V||_{H_t^{(s)}} \leq ||V||_{H_\infty^{(s)}} \leq C_t||V||_{H_t^{(s)}}, \quad V \in C^\infty(X,T(X)) \end{aligned} \tag{4.9}$$

with constants $c_t \to 0$, $C_t \to 1$ as $t \to \infty$. Here $\langle\cdot,\cdot\rangle_t$ denotes the inner product with respect to g_t. Furthermore, there exist constants C independent of t so that the elliptic a priori estimate for Δ_t holds uniformly in t:

$$||V||_{H_t^{(1)}} \leq C(\langle \Delta_t V, V\rangle_t + ||\cdot||^2_{H_t^{(0)}}), \qquad V \in C^\infty(X, T^{1,0}(X)). \tag{4.10}$$

Let $\{\phi_t^{(\alpha)}\}_{1\leq\alpha\leq N}$ be an orthonormal set of eigenvectors for Δ_t

$$\Delta_t\phi_t^{(\alpha)} = 0, \qquad \langle\phi_t^{(\alpha)}, \phi_t^{(\beta)}\rangle_t = \delta^{\alpha\beta}. \tag{4.11}$$

For each α, the uniform a priori estimate implies that $\phi_t^{(\alpha)}$ is uniformly bounded in $H_t^{(1)}$. But the uniform equivalence of all Sobolev norms $||\cdot||_t^{(1)}$ to $||\cdot||_\infty^{(1)}$ implies that $\phi_t^{(\alpha)}$ is uniformly bounded in $H_\infty^{(1)}$. By

Rellich's lemma, it follows that there is a subsequence $\phi_{t_j}^{(\alpha)}$ which converges in $L^2_\infty(X)$. This implies $\langle\phi_\infty^{(\alpha)}, \phi_\infty^{(\beta)}\rangle = \lim_{j\to\infty}\langle\phi_t^{(\alpha)}, \phi_t^{(\beta)}\rangle_t = \delta^{\alpha\beta}$. Furthermore, $\Delta_\infty\phi_\infty^{(\alpha)} = \lim_{j\to\infty}\Delta_t\phi_t^{(\alpha)} = 0$ in the sense of distributions, which implies by elliptic regularity that $\phi_\infty^{(\alpha)}$ is actually smooth, and the equation holds in the standard sense. Thus we have shown that there is a *subsequence* $t_j \to \infty$ so that for each α, the sequence $\phi_{t_j}^\alpha$ converges in L^2 to an orthonormal set $\phi_\infty^{(\alpha)} \in H^0_\infty(X, T^{1,0}(X))$.

• We make use now of the assumption that the dimensions of $K_t = \ker(\Delta_t)$ and $K_\infty = \ker(\Delta_\infty)$ are the same. Thus, by passing to a subsequence, we can assume the existence of orthonormal bases $\{\phi_t^{(\alpha)}\}$ for $K_t = \ker(\Delta_t)$ for $1 \leq t \leq \infty$, with $\phi_t^{(\alpha)}$ converging to $\phi_\infty^{(\alpha)}$ as $t \to \infty$. We now show that

$$\mathrm{liminf}_{t\to\infty}\lambda_t \geq \lambda_\infty. \tag{4.12}$$

Let $K_t^\perp$ be the orthogonal complement of K_t in $L^2(X, T_t^{1,0}(X))$, with respect to the metric $g(t)$. Let $\psi_t \in K_t^\perp$ be a lowest eigenfunction of Δ_t,

$$\Delta_t\psi_t = \lambda_t\psi_t, \qquad \psi_t \in K_t^\perp, \quad ||\psi_t||_t^2 = 1. \tag{4.13}$$

Fix any $\epsilon > 0$. Assume that there exists a sequence $t_j \to \infty$ so that

$$\lambda_{t_j} \leq (1-\epsilon)\lambda_\infty \tag{4.14}$$

for all t_j. We abbreviate t_j by t for the sake of notational simplicity. Then $||\Delta_t\psi_t||_{L_t^2} = \lambda_t$ is bounded, and the uniform elliptic a priori estimate implies that $||\psi_t||_{H_t^{(2)}}$ is uniformly bounded. In fact, the same argument applied to Δ_t^2 and its corresponding uniform elliptic a priori estimate implies that $||\psi_t||_{H_t^{(4)}}$ is uniformly bounded. Thus we may assume that ψ_t converges in $H_\infty^{(2)}$. Clearly the limit is in $K_\infty^\perp$. Now let Π denote the orthogonal projection from $T(X)$ to $K_\infty^\perp$. It follows immediately from the convergence of $(g(t), J(t))$ to $(g(\infty), J(\infty))$, and the convergence of the orthonormal basis $\{\phi_t^{(\alpha)}\}$ of K_t to the orthonormal basis $\{\phi_\infty^{(\alpha)}\}$ of K_∞ that

$$||\Pi\psi_t - \psi_t||_{H_\infty^{(2)}} \to 0, \tag{4.15}$$

for any $\psi_t \in K_t^\perp$ converging to a vector field in $K_\infty^\perp$. Since $\Delta_t - \Delta_\infty$ is a differential operator of second order whose coefficients tend to 0 in C^∞, we have $||\Delta_t - \Delta_\infty||_{\mathrm{Hom}\,(H_\infty^{(2)}, H_\infty^{(0)})} \to 0$, where the norm denotes the operator norm from $H_\infty^{(2)}$ to $H_\infty^{(0)}$. Thus

$$\langle\Delta_t\psi_t, \psi_t\rangle_t = \langle\Delta_\infty(\Pi\psi_t), \Pi\psi_t\rangle_\infty - o(1) \geq \lambda_\infty\,||\Pi\psi_t||_\infty^2 - o(1), \tag{4.16}$$

where $o(1)$ denotes terms tending to 0 as $t \to \infty$. But $||\Pi\psi_t||_\infty^2 \to 1$ as $t \to \infty$. This contradicts (4.14) for t large enough. Our statement about the liminf λ_t follows at once.

• Similarly, we can show that $\limsup \lambda_t \leq \lambda_\infty$, even without the assumption about the ranks of K_t not jumping up in the limit. Indeed, for any fixed $\epsilon > 0$, choose $\psi \in Ker(\Delta_\infty)^\perp$ with $||\psi||_\infty = 1$ and $\lambda_\infty \leq \langle \Delta_\infty\psi, \psi\rangle_\infty \leq \lambda_\infty + \epsilon$. If Π_t denotes the orthogonal projection from $T(X)$ to $K_t^\perp$, then it is easy to see that $\Pi_t\psi \in K_t^\perp$ satisfies $\langle \Delta_t(\Pi_t\psi), \Pi_t\psi\rangle_t \to \langle \Delta_\infty\psi, \psi\rangle_\infty$. Since $\lambda_t \leq \langle \Delta_t\psi_t, \psi_t\rangle_t$, this gives the desired estimate. The proof of Theorem 3 is complete. q.e.d.

5. Proof of Theorem 2

When the dimension of X is 2, under the assumption of non-negativity of the Ricci curvature and 2-nonnegativity of the traceless curvature operator for the initial metric, the non-negativity of the Ricci curvature is preserved for all times [**28**]. (The definition of the traceless curvature operator is given below. Its 2-nonnegativity means that the sum of its two lowest eigenvalues is non-negative.) Thus the boundedness of the scalar curvature implies the boundedness of the eigenvalues and hence of the Ricci curvature.

Next, we consider the curvature tensor $R_{\bar a b \bar c d}$. On a Kähler manifold, the Riemann curvature operator can be viewed as an operator on the space of real $(1,1)$-forms. The traceless curvature operator $Op(S)$ is the projection of the Riemann curvature operator on the subspace of traceless real (1,1)-forms (see [**28**], eq.(2.8)). Now, under the assumption (C), the eigenvalues $m_1 \leq m_2 \leq m_3$ of the traceless curvature operator $Op(S)$ are bounded: indeed, if they are all non-negative, then this follows immediately from the boundedness of the scalar curvature since $m_1 + m_2 + m_3 = R/2$. Otherwise, the 2-nonnegativity implies that at most one eigenvalue m_1 is negative, and that $0 \leq m_1 + m_2$. But then

$$\frac{1}{2}R = (m_1 + m_2) + m_3 \geq m_3 \geq 0, \tag{5.1}$$

and hence m_3 is uniformly bounded. Since $0 \leq m_2 \leq m_3$, so is m_2. Finally,

$$0 \leq \frac{1}{2}R = m_1 + m_2 + m_3 \Rightarrow |m_1| \leq m_2 + m_3 \tag{5.2}$$

and thus $|m_1|$ is also uniformly bounded. The boundedness of the eigenvalues of $Op(S)$ implies that its entries are also uniformly bounded. This is because the matrix for $Op(S)$ is symmetric, and thus diagonalizable by unitary matrices. Next, the curvature operator $R_{\bar a b \bar c d} = Op(R)$ can be written as

$$Op(R) = \begin{pmatrix} R/2 & S \\ S^t & Op(S) \end{pmatrix} \tag{5.3}$$

where S is the traceless part of the Ricci curvature $S_{\bar{a}b} = R_{\bar{a}b} - R\delta_{\bar{a}b}/n$. We deduce that the entries of $Op(R)$ are all bounded. Hence its eigenvalues are also bounded, establishing the uniform boundedness of the Riemann curvature tensor $R_{\bar{a}b\bar{c}d}$. Theorem 2 follows now from Theorem 1. q.e.d.

6. Remarks

• It is clear from the proof of Theorem 1 that, for any $\epsilon > 0$, the rate of exponential convergence can be taken to be $\lambda_\infty - \epsilon$, where λ_∞ is the lowest strictly positive eigenvalue of the complex Laplacian on the space $T^{1,0}(X)$ of vector fields with respect to the Kähler-Einstein metric $g_{\bar{k}j}(\infty)$.

• Several specific notions of stability have been by now proposed in the literature [**13**, **14**, **24**, **27**, **29**, **33**]. At the present time, the relations between these various notions are still obscure. Nor has any precise relation between any of them and the convergence of the Kähler-Ricci flow been as yet proved. This is clearly an important direction for further investigation, and it can be hoped that the methods of the present paper would be useful.

• From the proof of part 1 of Theorem 1, it is clear that the assumption that $R \geq 0$ (but not necessarily uniformly bounded from above), combined with lower bounds for the Mabuchi energy also suffices to show that $\int_X |Dh|^2\omega^n \to 0$ and $\int_0^\infty dt \int_X |D^2h|^2\omega^n < \infty$.

• The preceding inequality implies that $\int_0^\infty \int_X (R - \mu n)^2\omega^n < \infty$. This inequality was instrumental in [**9**], where it was established under the stronger assumption of positive biholomorphic sectional curvature, using a Moser-Trudinger type inequality and suitable generalizations of Liouville energy functionals.

• The flow for $|D^r h|^2$ for general r requires bounds for the full Riemann curvature tensor and its derivatives. However, certain lower order derivatives can still be bounded under the assumption of lower bounds for the Mabuchi functional and weaker curvature assumptions. For example, under the weaker assumption that $|R|$ remains bounded, we still have

$$\int_X |\nabla\bar{\nabla}h|^2\omega^n \to 0. \tag{6.1}$$

Indeed, an integration by parts shows that the L^2 norm of $\nabla\bar\nabla h$ is the same as the L^2 norm of Δh. The flow of Δh is easily derived from that of $h_{\bar k j}$, which is given in (2.15), with no lower order terms as we had noted earlier. We find $(\Delta h)^{\dot{}} = \Delta(\Delta h) + R^{j\bar k} h_{\bar k j}$, and hence

$$(|\Delta h|^2)^{\dot{}} = \Delta(|\Delta h|^2) - 2|\nabla \Delta h|^2 + 2|\nabla\bar\nabla h|^2 \Delta h + 2\mu(\Delta h)^2. \tag{6.2}$$

If we let $Y = \int_X |\Delta h|^2 \omega^n = \int_X |\nabla\bar\nabla h|^2 \omega^n$, it follows that

$$\begin{aligned}\dot Y = &-2\int_X |\nabla \Delta h|^2 \omega^n + 2\int_X |\nabla\bar\nabla h|^2 \Delta h \omega^n \\ &+ \int_X (\Delta h)^2 (-R + \mu(n+2))\omega^n \leq C\,Y\end{aligned} \tag{6.3}$$

since $\Delta h = R - \mu n$ is bounded in absolute value by assumption. As we saw earlier, this differential inequality together with the integrability of $Y(t)$ over $[0,\infty)$ implies that $Y(t) \to 0$ as $t \to \infty$. Substituting this back in the above equation, we see also that $\int_0^\infty \int_X |\nabla(\Delta h)|^2 \omega^n < \infty$.

• It would be interesting to find additional conditions, such as the k-nonnegativity of the traceless curvature operator, which would preserve the non-negativity of the Ricci curvature under the Kähler-Ricci flow in higher dimensions. It is already known that the positivity of the Ricci curvature is not preserved by the Ricci flow on complete manifolds of dimensions 4 or higher, thanks to the counterexamples constructed by L. Ni [**23**].

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