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ALGEBRAIC GROUPS OVER A 2-DIMENSIONAL LOCAL FIELD: IRREDUCIBILITY OF CERTAIN INDUCED REPRESENTATIONS

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To Raoul Bott, with admiration

Abstract

Let G be a split reductive group over a local field $\mathbf{K}$, and let $G((t))$ be the corresponding loop group. In [1], we have introduced the notion of a representation of (the group of $\mathbf{K}$-points) of $G((t))$ on a pro-vector space. In addition, we have defined an induction procedure, which produced $G((t))$-representations from usual smooth representations of G. We have conjectured that the induction of a cuspidal irreducible representation of G is irreducible. In this paper, we prove this conjecture for $G = SL_2$.

1. The result

1.1. Notation. The notation in this paper follows closely that of [1]. Let us remind the main characters. We denote by Set_0 the category of finite sets, and $\mathbf{S}et := \mathrm{Ind}(\mathrm{Pro}(Set_0))$, $\mathbb{S}et = \mathrm{Ind}(\mathrm{Pro}(\mathbf{S}et))$. By $Vect_0$, we denote the category of finite-dimensional vector spaces over $\mathbb{C}$, $Vect = \mathrm{Ind}(Vect_0)$ is the category of all vector spaces, and $\mathbb{V}ect$ is the category $\mathrm{Pro}(Vect)$ of pro-vector spaces.

Let G be a split reductive group over $\mathbf{K}$; $\mathbf{G}$ the corresponding group-object of $\mathbf{S}et$. We have the pro-algebraic group of arcs $G[[t]]$ and for $n \in \mathbb{N}$, we denote by $G^n \subset G[[t]]$ the corresponding congruence subgroup. By $\mathbf{G}[[t]]$ (resp., $\mathbf{G}^n \subset \mathbf{G}[[t]]$), we denote the corresponding group-objects of $\mathrm{Pro}(\mathbf{S}et)$.

Finally, $\mathbb{G} = \mathbf{G}((t))$ is the group-object of $\mathbb{S}et$, which is our main object of study. We denote by $\mathrm{Rep}(\mathbb{G})$ the category of representations of $\mathbb{G}$ on $\mathbb{V}ect$, cf. [1], Sect. 2.

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1.2. Let us recall the formulation of Conjecture 4.7 of [1]. Recall that we have an exact functor $r^{\mathbb{G}}_{\mathbf{G}} : \mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}(\mathbf{G}, \mathbb{V}ect)$, and its right adjoint, denoted $i^{\mathbb{G}}_{\mathbf{G}}$ and called the induction functor.

The functors $r^{\mathbb{G}}_{\mathbf{G}}$ and $i^{\mathbb{G}}_{\mathbf{G}}$ are direct loop-group analogs of the Jacquet and induction functors for usual reductive groups over $\mathbf{K}$.

Let π be an irreducible cuspidal representation of $\mathbf{G}$, and set $\Pi := i^{\mathbb{G}}_{\mathbf{G}}(\pi)$. In [1], Sect. 4.5, it was shown that the cuspidality assumption on π implies that the natural map

$$r^{\mathbb{G}}_{\mathbf{G}}(\Pi) = r^{\mathbb{G}}_{\mathbf{G}} \circ i^{\mathbb{G}}_{\mathbf{G}}(\pi) \to \pi \tag{1}$$

is an isomorphism. In particular, this implies that

$$\mathrm{End}_{\mathrm{Rep}(\mathbb{G})}(\Pi) \simeq \mathrm{Hom}_{\mathrm{Rep}(\mathbf{G},\mathbb{V}ect)}(r^{\mathbb{G}}_{\mathbf{G}}(\Pi), \pi) \simeq \mathrm{Hom}_{\mathrm{Rep}(\mathbf{G},\mathbb{V}ect)}(\pi, \pi) \simeq \mathbb{C}.$$

We have formulated:

Conjecture 1.3. The object $\Pi \in \mathrm{Rep}(\mathbb{G})$ is irreducible.

In this paper, we will prove:

Theorem 1.4. *Conjecture 1.3 holds for $G = SL_2$.*

Note that in [1], Conjecture 1.3 was stated slightly more generally, when we allow representations of a central extension $\widehat{\mathbb{G}}$ with a given central charge. The proof of Theorem 1.4 generalizes to this set-up in a straightforward way.

It should be remarked that from the definition of the category of representations of $G((t))$, it is not at all clear that $G((t))$ admits any non-trivial irreducible representations. Therefore, the fact that the above-mentioned irreducibility conjecture holds is somewhat surprising.

1.5. We will now consider a functor $\mathrm{Rep}(\mathbf{G}, \mathbb{V}ect) \to \mathrm{Rep}(\mathbb{G})$, which will be the *left* adjoint of the functor $r^{\mathbb{G}}_{\mathbf{G}}$.

First, recall from [2], Proposition 2.7, that the functor $\mathrm{Coinv}_{\mathbf{G}^1} : \mathrm{Rep}(\mathbf{G}^1, \mathbb{V}ect) \to \mathbb{V}ect$ does admit a left adjoint, denoted $\mathrm{Inf}^{\mathbf{G}^1}$.

Proposition 1.6. *The functor*

$$\mathrm{Coinv}_{\mathbf{G}^1} : \mathrm{Rep}(\mathbf{G}[[t]], \mathbb{V}ect) \to \mathrm{Rep}(\mathbf{G}, \mathbb{V}ect)$$

admits a left adjoint.

Proof.
For $\pi = (\mathbb{V}, \rho) \in \mathrm{Rep}(\mathbf{G}, \mathbb{V}ect)$, consider the functor $\mathrm{Rep}(\mathbf{G}[[t]], \mathbb{V}ect) \to Vect$ given by

$$\Pi \mapsto \mathrm{Hom}_{\mathrm{Rep}(\mathbf{G},\mathbb{V}ect)}(\pi, \mathrm{Coinv}_{\mathbf{G}^1}(\Pi)).$$

We claim that it is enough to show that this functor is pro-representable. Indeed, this follows by combining Lemma 1.2, Proposition 2.5 and Lemma 2.7 of [1].

Consider the object $\mathrm{Inf}^{\mathbf{G}^1}(\mathbb{V}) \in \mathrm{Rep}(\mathbf{G}^1, \mathbb{V}ect)$, where $\mathbb{V}$ is regarded just as a pro-vector space, and

$$\mathrm{Coind}_{\mathbf{G}^1}^{\mathbf{G}[[t]]}(\mathrm{Inf}^{\mathbf{G}^1}(\mathbb{V})) \in \mathrm{Rep}(\mathbf{G}[[t]], \mathbb{V}ect),$$

where $\mathrm{Coind}_{\mathbf{G}^1}^{\mathbf{G}[[t]]}$ is as in [2], Corollary 2.34.

Evidently,

$$\mathrm{Hom}_{\mathrm{Rep}(\mathbf{G},\mathbb{V}ect)}(\pi, \mathrm{Coinv}_{\mathbf{G}^1}(\Pi)) \hookrightarrow \mathrm{Hom}_{\mathbb{V}ect}(\mathbb{V}, \mathrm{Coinv}_{\mathbf{G}^1}(\Pi)),$$

and the latter, in turn, identifies with

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Rep}(\mathbf{G}^1,\mathbb{V}ect)}\left(\mathrm{Inf}^{\mathbf{G}^1}(\mathbb{V}), \Pi\right)\\ &\quad\simeq \mathrm{Hom}_{\mathrm{Rep}(\mathbf{G}[[t]],\mathbb{V}ect)}\left(\mathrm{Coind}_{\mathbf{G}^1}^{\mathbf{G}[[t]]}(\mathrm{Inf}^{\mathbf{G}^1}(\mathbb{V})), \Pi\right).\end{aligned}$$

Hence, the pro-representability follows from Proposition 1.4 of [1].

q.e.d.

We will denote the resulting functor by $\mathrm{Inf}_{\mathbf{G}}^{\mathbf{G}[[t]]}$. Note that by construction, for a representation π of $\mathbf{G}$, we have a surjection

$$\mathrm{Coind}_{\mathbf{G}^1}^{\mathbf{G}[[t]]}(\mathrm{Inf}^{\mathbf{G}^1}(\pi)) \twoheadrightarrow \mathrm{Inf}_{\mathbf{G}}^{\mathbf{G}[[t]]}(\pi).$$

By composing $\mathrm{Inf}_{\mathbf{G}}^{\mathbf{G}[[t]]}$ with the functor $\mathrm{Coind}_{\mathbf{G}[[t]]}^{\mathbb{G}} : \mathrm{Rep}(\mathbf{G}[[t]], \mathbb{V}ect) \to \mathrm{Rep}(\mathbb{G})$, we obtain a functor, left adjoint to $r_{\mathbf{G}}^{\mathbb{G}}$.

We will now formulate the main step in the proof of Theorem 1.4. Note that if π is a cuspidal representation of $\mathbf{G}$, isomorphism (1) implies that we have a canonical map

$$\mathrm{Coind}_{\mathbf{G}[[t]]}^{\mathbb{G}}(\mathrm{Inf}_{\mathbf{G}}^{\mathbf{G}[[t]]}(\pi)) \to \Pi. \tag{2}$$

We will deduce Theorem 1.4 from the following one:

Theorem 1.7. *If $G = SL_2$, the map of (2) is surjective.*

Of course, we conjecture that the map (2) is surjective for any G, but we are unable to prove that at the moment.

1.8. Let us show how Theorem 1.7 implies Theorem 1.4. Suppose that Π' is a non-zero sub-object of Π and let $\Pi'' := \Pi/\Pi'$ be the quotient. By definition of the induction functor, we have a map in $\mathrm{Rep}(\mathbf{G}, \mathbb{V}ect)$.

$$r_{\mathbf{G}}^{\mathbb{G}}(\Pi') \to \pi.$$

Using Proposition 2.4. of [**1**], we obtain that $r^{\mathbb{G}}_{\mathbf{G}}(\Pi')$ must surject onto π, since the latter was assumed irreducible. Since the functor $r^{\mathbb{G}}_{\mathbf{G}}$ is exact (cf. Lemma 2.6. of *loc.cit.*), and because of isomorphism (1), this implies that $r^{\mathbb{G}}_{\mathbf{G}}(\Pi'') = 0$.

However,

$$\mathrm{Hom}_{\mathrm{Rep}(\mathbb{G})}(\mathrm{Coind}^{\mathbb{G}}_{\mathbf{G}[[t]]}(\mathrm{Inf}^{\mathbf{G}[[t]]}_{\mathbf{G}}(\pi), \Pi'') \simeq \mathrm{Hom}_{\mathrm{Rep}(\mathbf{G},\mathbb{V}ect)}(\pi, r^{\mathbb{G}}_{\mathbf{G}}(\Pi'').$$

By Theorem 1.7, this implies that $\Pi'' = 0$.

2. The key lemma

2.1. The rest of the paper is devoted to the proof of Theorem 1.7. We will slightly abuse the notation, and for a scheme Y over $\mathbf{K}$, we will make no distinction between the corresponding object $\mathbf{Y} \in \mathbf{S}et$ and $Y(\mathbf{K})$, regarded as a topological space.

Recall the affine Grassmannian $\mathrm{Gr}_G = G((t))/G[[t]]$ of G, and the corresponding object $\mathbf{Gr}_G \in \mathrm{Ind}(\mathbf{S}et)$. Let us represent Gr_G as the direct limit of closures of $G[[t]]$-orbits, $\overline{\mathrm{Gr}}^{\lambda}_G$, with respect to the natural partial ordering on the set of dominant coweights.

Let us also denote by $\widetilde{\mathrm{Gr}}_G$ the ind-scheme $G((t))/G^1$, which is a principal G-bundle over Gr_G. Let $\widetilde{\mathrm{Gr}}^{\lambda}_G$ and $\widetilde{\overline{\mathrm{Gr}}}^{\lambda}_G$ denote the preimages in $\widetilde{\mathrm{Gr}}_G$ of the $G[[t]]$-orbit Gr^{λ}_G and its closure, respectively. Let $\widetilde{\mathbf{Gr}}^{\lambda}_G$ and $\widetilde{\overline{\mathbf{Gr}}}^{\lambda}_G$ denote the corresponding objects of $\mathbf{S}et$.

By construction (cf. [**1**], Sect. 3.9), as a $\mathbf{G}[[t]]$-representation, Π is the inverse limit of $\overline{\Pi}^{\lambda}$, where each $\overline{\Pi}^{\lambda}$ is the vector space consisting of locally constant $\mathbf{G}$-equivariant functions on $\widetilde{\overline{\mathbf{Gr}}}^{\lambda}_G$ with values in π.

Set Π^{λ} be the kernel of $\overline{\Pi}^{\lambda} \to \underset{\lambda'<\lambda}{\oplus} \overline{\Pi}^{\lambda'}$. Let ev denote the natural evaluation map $\Pi \to \Pi^0 \simeq \pi$, which sends a function $f \in \mathrm{Funct}^{lc}(\widetilde{\overline{\mathbf{Gr}}}^{\lambda}_G, \pi)$ to $f(1)$. More generally, for $\widetilde{g} \in \widetilde{\overline{\mathbf{Gr}}}^{\lambda}_G$, we will denote by $ev_{\widetilde{g}}$ the map $\Pi \to \pi$, corresponding to evaluation at $\widetilde{g}$.

To prove Theorem 1.7, we must show that the composition

$$\mathrm{Coind}^{\mathbb{G}}_{\mathbf{G}[[t]]}\left(\mathrm{Inf}^{\mathbf{G}[[t]]}_{\mathbf{G}}(\pi)\right) \to \Pi \to \overline{\Pi}^{\lambda} \tag{3}$$

is surjective for every λ. We will argue by induction. Therefore, let us first check that the map of (3) is indeed surjective for $\lambda = 0$.

We have a natural map

$$\mathrm{Inf}^{\mathbf{G}^1}(\pi) \to \mathrm{Inf}^{\mathbf{G}[[t]]}_{\mathbf{G}}(\pi) \to \mathrm{Coind}^{\mathbb{G}}_{\mathbf{G}[[t]]}\left(\mathrm{Inf}^{\mathbf{G}[[t]]}_{\mathbf{G}}(\pi)\right), \tag{4}$$

and its composition with

$$\mathrm{Coind}_{\mathbf{G}[[t]]}^{\mathbb{G}}\left(\mathrm{Inf}_{\mathbf{G}}^{\mathbf{G}[[t]]}(\pi)\right) \to \Pi \overset{\mathrm{ev}}{\to} \pi$$

is the natural surjection $\mathrm{Inf}^{\mathbf{G}^1}(\pi) \to \pi$.

Thus, we have to carry out the induction step. We will suppose that the composition

$$\mathrm{Coind}_{\mathbf{G}[[t]]}^{\mathbb{G}}\left(\mathrm{Inf}_{\mathbf{G}}^{\mathbf{G}[[t]]}(\pi)\right) \to \Pi \to \overline{\Pi}^{\lambda'}$$

is surjective for $\lambda' < \lambda$, and we must show that

$$\mathrm{Coind}_{\mathbf{G}[[t]]}^{\mathbb{G}}\left(\mathrm{Inf}_{\mathbf{G}}^{\mathbf{G}[[t]]}(\pi)\right) \underset{\overline{\Pi}^{\lambda}}{\times} \Pi^{\lambda} \to \Pi^{\lambda} \tag{5}$$

is surjective as well.

2.2. For λ as above, let t^λ be the corresponding point in $\mathbf{G}((t))$. By a slight abuse of notation, we will denote by the same symbol its image in Gr_G and $\widetilde{\mathrm{Gr}}_G$.

Consider the action of $G^1 \subset G((t))$ on Gr_G given by

$$g \times x = \mathrm{Ad}_{t^\lambda}(g) \cdot x.$$

Let $Y \subset \mathrm{Gr}_G$ be the closure of $\mathrm{Ad}_{t^\lambda}(G^1) \cdot \overline{\mathrm{Gr}}_G^\lambda$. Let G_λ be a finite-dimensional quotient of G^1, through which it acts on Y.

We will denote by $\mathbf{Y}$ and $\mathbf{G}_\lambda$, respectively, the corresponding objects of $\mathbf{S}et$. Let Π_Y denote the quotient of Π, equal to the space of $\mathbf{G}$-equivariant locally constant π-valued functions on the set of $\mathbf{K}$-points of the preimage of Y in $\widetilde{\mathrm{Gr}}_G$.

Let $N \subset G$ be the maximal unipotent subgroup. Since λ is dominant, $\mathrm{Ad}_{t^\lambda}(N[[t]])$ is a subgroup of $N[[t]]$. Let $N^\lambda \subset N[[t]]$ be any normal subgroup of finite codimension, contained in $\mathrm{Ad}_{t^\lambda}(N[[t]])$. (Later, we will specify to the case when $G = SL_2$; then $N \simeq G_a$ and is abelian, and we will take $N^\lambda = \mathrm{Ad}_{t^\lambda}(N[[t]])$.) Let N_λ denote the quotient $N[[t]]/N^\lambda$, and let $\mathbf{N}_\lambda$ be the corresponding group-object in $\mathbf{S}et$.

Let now $K_{\mathbf{N}}$ be an open compact subgroup in $\mathbf{N}_\lambda$, and $K_{\mathbf{G}_\lambda}$ an open compact subgroup in $\mathbf{G}_\lambda$.

Now, we are ready to state our main technical claim, Main Lemma 2.4. However, before doing that, let us explain the idea behind this lemma:

From the isomorphism (1), we will obtain that for any $f \in \Pi_Y$ and a *large* compact subgroup $K_{\mathbf{G}_\lambda}$ as above, the integral $f' := \int_{k \in K_{\mathbf{G}_\lambda}} f^k$ "localizes" near t^λ, i.e., f' will be 0 outside a "small" ball around t^λ. We will then average f' with respect to a fixed open subgroup $K_{\mathbf{N}}$ of $\mathbf{N}_\lambda$, and obtain a new element, denoted by $f'' \in \overline{\Pi}^\lambda$.

The Main Lemma 2.4 will insure that the compact subgroup $K_{\mathbf{G}_\lambda}$ can be chosen so that f'' will still be localized near t^λ, and such the resulting elements f'' for various subgroups $K_{\mathbf{N}}$, and their translations by elements of $\mathbf{G}((t))$, span Π^λ.

2.3. In precise terms, we proceed as follows. Consider the operator $A_{K_{\mathbf{N}},K_{\mathbf{G}_\lambda}} : \Pi \to \pi$ given by

$$f \mapsto \int\limits_{n\in K_{\mathbf{N}}} \int\limits_{k\in K_{\mathbf{G}_\lambda}} ev_{t^\lambda}(f^{n\cdot k}),$$

where the integral is taken with respect to the Haar measures on both groups. (In the above formula, $f \mapsto f^x$ denotes the action of $x \in \mathbf{G}((t))$ on Π.) By the definition of Π_Y, the above map factors through $\Pi \twoheadrightarrow \Pi_Y \to \pi$.

For a point $\widetilde{g} \in \widetilde{\overline{\mathbf{Gr}}}{}^\lambda_G$, we have a map $A_{\widetilde{g},K_{\mathbf{G}_\lambda}} : \Pi \to \pi$ given by

$$f \mapsto \int\limits_{k\in K_{\mathbf{G}_\lambda}} ev_{\widetilde{g}}(f^k).$$

This map also factors through Π_Y.

Our main technical claim, which we prove for $G = SL_2$ is the following. (We do not know whether an analogous statement holds for groups G of higher rank.)

Main Lemma 2.4. *For $v \in \pi$, an open compact subgroup $K_{\mathbf{N}} \subset \mathbf{N}_\lambda$ and an open compact subset $\mathbf{X} \subset \mathbf{Gr}^\lambda_G$ containing t^λ, there exists a finite-dimensional subspace $\mathsf{F}(v) \subset \Pi_Y$ and an increasing exhausting family of compact subgroups $K^\alpha_{\mathbf{G}_\lambda}(v) \subset \mathbf{G}_\lambda$ such that:*

(1) *For all sufficiently large indices α, the vector v would belong to the image of $A_{K_{\mathbf{N}},K^\alpha_{\mathbf{G}_\lambda}(v)}(\mathsf{F}(v))$.*

(2) *For every $f \in \mathsf{F}(v)$ and for all sufficiently large indices α, the vector $A_{\widetilde{g},K^\alpha_{\mathbf{G}_\lambda}(v)}(f)$ will vanish, unless the image of $\widetilde{g}$ under $\widetilde{\overline{\mathbf{Gr}}}{}^\lambda_G \to \overline{\mathbf{Gr}}{}^\lambda_G$ belongs to $\mathbf{X}$.*

2.5. Let us show how Lemma 2.4 implies the induction step in the proof of Theorem 1.7.

Recall that the orbit of the point t^λ under the action of $N[[t]]$ is open in Gr^λ_G. For an open compact subgroup $K_{\mathbf{N}} \subset \mathbf{N}_\lambda$, let $\mathbf{X} \subset \mathbf{Gr}^\lambda_G$ be its orbit under $K_{\mathbf{N}}$. Let $(\overline{\Pi}^\lambda)^{K_{\mathbf{N}}} \subset \overline{\Pi}^\lambda$ be the subspace of $K_{\mathbf{N}}$-invariants. We have a direct sum decomposition

$$(\overline{\Pi}^\lambda)^{K_{\mathbf{N}}} = \mathbf{V}_1 \oplus \mathbf{V}_2,$$

where the first direct summand consists of functions that vanish on the preimage of $\mathbf{X}$, and the second one functions of sections that vanish outside the preimage of $\mathbf{X}$. We have $\mathbf{V}_2 \subset \Pi^\lambda$ and the map ev_{t^λ} identifies $\mathbf{V}_2$ with π.

We claim that it suffices to show that the image of the map

$$\text{(6)} \qquad \mathrm{Coind}^{\mathbb{G}}_{\mathbf{G}[[t]]}\left(\mathrm{Inf}^{\mathbf{G}[[t]]}_{\mathbf{G}}(\pi)\right) \to \Pi \to \overline{\Pi}^\lambda \to (\overline{\Pi}^\lambda)^{K_{\mathbf{N}}},$$

where the last arrow is given by averaging with respect to $K_{\mathbf{N}}$, contains $\mathbf{V}_2$.

Indeed, let $G[[t]]_\lambda$ be a finite-dimensional quotient through which $G[[t]]$ acts on Gr^λ_G, and let $\mathbf{G}[[t]]_\lambda$ be the corresponding group-object of $\mathbf{S}et$. The vector space Π^λ is spanned by elements of the following form. Each is invariant under some (small) open compact subgroup $K_{\mathbf{G}[[t]]_\lambda} \subset \mathbf{G}[[t]]_\lambda$, and is supported on a preimage in $\widetilde{\mathbf{Gr}}^\lambda_G$ of a single $K_{\mathbf{G}[[t]]_\lambda}$-orbit on $\mathbf{Gr}^\lambda_G$. By $\mathbf{G}[[t]]$-invariance, we can assume that the orbit in question is that of the element $t^\lambda \in \mathbf{Gr}^\lambda_G$.

By setting $K_{\mathbf{N}} := \mathbf{N}_\lambda \cap K_{\mathbf{G}[[t]]_\lambda}$, we obtain that any element of the form specified above is contained in the corresponding $\mathbf{V}_2$.

We will show that Main Lemma 2.4 implies that $\mathbf{V}_2$ belongs to the image of the map

$$\mathrm{Inf}^{\mathbf{G}^1}(\pi) \to \mathrm{Coind}^{\mathbb{G}}_{\mathbf{G}[[t]]}\left(\mathrm{Inf}^{\mathbf{G}[[t]]}_{\mathbf{G}}(\pi)\right) \to (\overline{\Pi}^\lambda)^{K_{\mathbf{N}}},$$

where first the arrow is the composition of the map of (4), followed by the action of t^λ.

For that, let us write down in explicit terms the composition

$$\text{(7)} \qquad \mathrm{Inf}^{\mathbf{G}^1}(\pi) \to \mathrm{Coind}^{\mathbb{G}}_{\mathbf{G}[[t]]}\left(\mathrm{Inf}^{\mathbf{G}[[t]]}_{\mathbf{G}}(\pi)\right) \to \Pi \to \Pi_Y.$$

First, let us observe that the resulting map factors through the surjection $\mathrm{Inf}^{\mathbf{G}^1}(\pi) \twoheadrightarrow \mathrm{Inf}^{\mathbf{G}_\lambda}(\pi)$. Secondly, let us recall (cf. [2], Sect. 2.8) that $\mathrm{Inf}^{\mathbf{G}_\lambda}(\pi)$ is the inductive limit, taken in $\mathbb{V}ect$, over finite-dimensional subspaces $\mathsf{F}' \subset \pi$ of

$$\text{“}\underset{\underset{\alpha}{\longleftarrow}}{\lim}\text{”}\ \mathrm{Distr}_c(\mathbf{G}_\lambda / K^\alpha_{\mathbf{G}_\lambda}) \otimes \mathsf{F}',$$

where $K^\alpha_{\mathbf{G}_\lambda}$ runs through any exhausting family of open compact subgroups of $\mathbf{G}_\lambda$.

By (1), the map $\Pi_Y \overset{ev_{t^\lambda}}{\to} \pi$ induces an isomorphism $\mathrm{Coinv}_{\mathbf{G}_\lambda}(\Pi_Y) \simeq \pi$. For a given finite-dimensional subspace F', let us choose a finite-dimensional subspace $\mathsf{F} \subset \Pi_Y$ which projects surjectively onto F', and for every index α, consider the map

$$\mathrm{Distr}_c(\mathbf{G}_\lambda/K^\alpha_{\mathbf{G}_\lambda})\otimes \mathsf{F}\to \Pi_Y$$

given by

$$\mu\otimes f\mapsto \mu\star f,$$

where $f\in\mathsf{F}$ and $\mu\in\mathrm{Distr}_c(\mathbf{G}_\lambda/K^\alpha_{\mathbf{G}_\lambda})$ is regarded as an element of the Hecke algebra.

The resulting system of maps (eventually in α) factors through $\mathrm{Distr}_c(\mathbf{G}_\lambda/K^\alpha_{\mathbf{G}_\lambda})\otimes\mathsf{F}\twoheadrightarrow\mathrm{Distr}_c(\mathbf{G}_\lambda/K^\alpha_{\mathbf{G}_\lambda})\otimes\mathsf{F}'$, and defines the map in (7).

Let us now recall that if $\mathbb{W}=\varprojlim \mathbf{W}_\alpha$ is a pro-vector space mapping to a vector space $\mathbf{V}$, the surjectivity of this map means that the eventually defined maps $\mathbf{W}_\alpha\to\mathbf{V}$ are all surjective, or, which is the same, that $\forall v\in\mathbf{V}$, $v\in\mathrm{Im}(\mathbf{W}_\alpha)$ for those indices α, for which the map $\mathbb{W}\to\mathbf{V}$ factors through $\mathbf{W}_\alpha\to\mathbf{V}$.

For a vector $v\in\pi$, let $\mathsf{F}(v)$ be the finite-dimensional subspace of Π_Y, given by Lemma 2.4, and let $K^\alpha_{\mathbf{G}_\lambda}(v)$ be the corresponding system of subgroups. Let $\mathsf{F}'(v)$ denote the image of $\mathsf{F}(v)$ in π.

Consider the composition:

$$\mathrm{Distr}_c(\mathbf{G}_\lambda/K^\alpha_{\mathbf{G}_\lambda}(v))\otimes\mathsf{F}(v)\to\Pi_Y\to\overline{\Pi}^\lambda\to(\overline{\Pi}^\lambda)^{K_\mathbf{N}}.$$

Let us take the unit element in $\mathrm{Distr}_c(\mathbf{G}_\lambda/K^\alpha_{\mathbf{G}_\lambda}(v))$, corresponding to the Haar measure on $K^\alpha_{\mathbf{G}_\lambda}(v)$. We obtain a map $\mathsf{F}(v)\to(\overline{\Pi}^\lambda)^{K_\mathbf{N}}$.

By Lemma 2.4(2), the image of this map is contained in $\mathbf{V}_2$. When we further compose it with the evaluation map $\mathbf{V}_2\hookrightarrow\Pi_Y\to\pi$, we obtain a map $\mathsf{F}(v)\to\pi$ equal to $A_{K_\mathbf{N},K^\alpha_{\mathbf{G}_\lambda}(v)}$, whose image contains v, by Lemma 2.4(1).

This establishes the required surjectivity.

3. Proof of Main Lemma 2.4

3.1. For a given subgroup $K_\mathbf{N}\subset\mathbf{N}_\lambda$, a subset $\mathbf{X}\subset\mathbf{Gr}^\lambda_G$ and an arbitrary finite-dimensional subspace $\mathsf{F}\subset\Pi_Y$, we will construct a family of open compact subgroups $K_{\mathbf{G}_\lambda}\subset\mathbf{G}_\lambda$, such that the expressions

$$A_{K_\mathbf{N},K_{\mathbf{G}_\lambda}}(f)\text{ and }A_{\widetilde{g},K_{\mathbf{G}_\lambda}}(f)$$

for $f\in\mathsf{F}$ can be evaluated explicitly.

From now on, we will fix $G=SL_2$. We will change the notation slightly, and identify the set of dominant coweights with $\mathbb{N}$; in which case, we will replace λ by l and $t^\lambda \in \mathbf{G}((t))$ becomes the matrix

$$\begin{pmatrix} t^l & 0 \\ 0 & t^{-l} \end{pmatrix}.$$

Let us translate our initial subscheme Y by $t^{-\lambda}$, in which case, the point t^λ itself will go over to the unit point $1_{\mathrm{Gr}_G} \in \mathrm{Gr}_G$, and $t^{-\lambda} \cdot Y$ will be contained in $\overline{\mathrm{Gr}}^{2l}_G$. (We denote by Gr^r_G the $G[[t]]$-orbit of the point $\begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$ in Gr_G, and by $\overline{\mathrm{Gr}}^r_G$ its closure.) For the purposes of Lemma 2.4, we can replace $t^{-\lambda} \cdot Y$ by the entire $\overline{\mathrm{Gr}}^{2l}_G$, with the standard action of the congruence subgroup G^1.

Note that the action of G^1 on $\overline{\mathrm{Gr}}^{2l}_G$ (resp., $\widetilde{\overline{\mathrm{Gr}}}^{2l}_G$) factors through G^1/G^{4l} (resp., G^1/G^{4l+1}).

For an integer r, let us denote by G_r the quotient G^1/G^{2r+1}, and by N_r the quotient $t^{-r} \cdot N[[t]]/N[[t]]$. We will write elements of $\mathbf{N}_r$ as $\sum\limits_{1 \le i \le r} t^{-i} \cdot n_i$ with $n_i \in \mathbf{K}$, and thus think of it as an r-dimensional vector space over $\mathbf{K}$.

Similarly, we will identify $\mathbf{G}_r := \mathbf{G}^1/\mathbf{G}^{2r+1}$ with a $6r$-dimensional vector space over $\mathbf{K}$, by writing its elements as matrices:

$$\mathrm{Id} + \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

and $k_{lm} = \sum\limits_{1 \le i \le 2r} t^i \cdot (k_{lm})_i$. In particular, we can speak of $O_{\mathbf{K}}$-lattices in $\mathbf{G}_r$, where $O_{\mathbf{K}} \subset \mathbf{K}$ is the ring of integers.

3.2. In what follows, for a point $g \in \mathbf{G}((t))$, we will denote by $\widetilde{g}$ (resp., $\overline{g}$) its image in $\widetilde{\mathbf{Gr}}_G$ (resp., $\mathbf{Gr}_G$).

Thus, we are interested in computing the integral

$$\int\limits_{k \in K_{\mathbf{G}_r}} ev(f^{g \cdot k}),$$

when g is such that either $g \in \mathbf{N}_r$, or the corresponding point $\overline{g} \in \mathbf{Gr}_G$ lies in $\overline{\mathbf{Gr}}^r_G - \mathbf{X}$, where $\mathbf{X}$ is a fixed open compact subset of $\overline{\mathbf{Gr}}^r_G$ containing 1_{Gr_G}, and $f \in \mathsf{F}$, where F is a fixed finite-dimensional subspace of $\overline{\Pi}^r$.

Let $\mathfrak{p}$ denote the projection $\widetilde{\mathrm{Gr}}_G \to \mathrm{Gr}_G$. Let $\mathfrak{s}$ be a continuous section $\overline{\mathbf{Gr}}^r_G \to \widetilde{\overline{\mathbf{Gr}}}^r_G$, such that $\mathfrak{s}(1_{\mathrm{Gr}_G}) = 1_{\widetilde{\mathrm{Gr}}_G}$. A choice of such section defines an isomorphism

$$\widetilde{\overline{\mathbf{Gr}}}^r_G \simeq \overline{\mathbf{Gr}}^r_G \times \mathbf{G}.$$

We will denote by $\mathfrak{q}$ the resulting map $\widetilde{\overline{\mathbf{Gr}}}^r_G \to \mathbf{G}$.

Let us fix an open neighbourhood $\mathbf{Z}$ of 1_{Gr_G} in $\overline{\mathbf{Gr}}^r_G$ small enough so that

$$f(\mathfrak{s}(x)) = ev(f)$$

for $x \in \mathbf{Z}$ and $f \in \mathsf{F}$. Let $K_{\mathbf{G}}(\mathsf{F})$ be an open compact subgroup of $\mathbf{G}$, such that $ev(f) \in \pi$ is $K_{\mathbf{G}}(\mathsf{F})$-invariant for $f \in \mathsf{F}$.

Let $K_{\mathbf{N}_r}$ be an open compact subgroup of $\mathbf{N}_r$.

Proposition 3.3. *There exists an $O_{\mathbf{K}}$-lattice $K_{\mathbf{G}_r} \subset \mathbf{G}_r$, which contains any given open subgroup of $\mathbf{G}_r$, such that the following is satisfied:*

(1) *There exists an open compact subgroup $K^{sm}_{\mathbf{N}_r} \subset K_{\mathbf{N}_r}$ such that:*

(1a) *For $g = k \cdot n \in \mathbf{G}((t))$ with $k \in K_{\mathbf{G}_r}$ and $n \in K^{sm}_{\mathbf{N}_r}$, the corresponding point $\overline{g} \in \overline{\mathbf{Gr}}^l_G$ belongs to $\mathbf{Z}$.*

(1b) *For g as above, the left coset of $\mathfrak{q}(\widetilde{g}) \in \mathbf{G}$ with respect to $K_{\mathbf{G}}(\mathsf{F}) \subset \mathbf{G}$ equals that of*

$$\begin{pmatrix} 1 & -\sum\limits_{1\leq i\leq r} (k_{12})_{2i} \cdot n_i^2 \\ 0 & 1. \end{pmatrix}$$

(1c) *The integral $\int_{k\in K_{\mathbf{G}_r}} ev(f^{n\cdot k})$ vanishes if $n \in K_{\mathbf{N}_r} - K^{sm}_{\mathbf{N}_r}$ and $f \in \mathsf{F}$.*

(2) *If $g \in \mathbf{G}((t))$, such that $\overline{g} \in \overline{\mathbf{Gr}}^r_G - \mathbf{X}$, the integral $\int_{k\in K_{\mathbf{G}_r}} ev(f^{g\cdot k})$ vanishes.*

3.4. Let us deduce Main Lemma 2.4 from Proposition 3.3. Given a vector $v \in \pi$, let us first define the subspace $\mathsf{F}(v) \in \overline{\Pi}^r$.

Recall that $\mathbf{N} \simeq \mathbf{K}$ is the maximal unipotent subgroup of $\mathbf{G} = SL_2(\mathbf{K})$, and let $\mathbf{N}^*$ denote the Pontriagin dual group. Since $\mathbf{N}^*$ is also (non-canonically) isomorphic to $\mathbf{K}$, we have a valuation map $\nu : \mathbf{N}^* \to \mathbb{Z}$, defined up to a shift. In particular, we can consider the subalgebra $\mathrm{Funct}_{val}(\mathbf{N}^*) :\simeq \mathrm{Funct}(\mathbb{Z})$ inside the algebra $\mathrm{Funct}_{lc}(\mathbf{N}^*)$ of all locally constant functions on $\mathbf{N}^*$.

Any smooth representation of $\mathbf{N}$, and in particular π, can be thought of as a module over the algebra of $\mathrm{Funct}_{lc}(\mathbf{N}^*)$, such that every element of this module has compact support. If a representation is cuspidal, this means that the support of every section is disjoint from $0 \in \mathbf{N}^*$.

Therefore, if v is a vector in a cuspidal representation π, the vector space $\mathrm{Funct}_{val}(\mathbf{N}^*) \cdot v \subset \pi$ is finite-dimensional. We denote this vector subspace by $\mathsf{F}'(v)$ and let $\mathsf{F}(v) \subset \overline{\Pi}^r$ to be any subspace surjecting onto $\mathsf{F}'(v)$ by means of ev. We claim that $\mathsf{F}(v)$ satisfies the requirements of Main Lemma 2.4.

Property (2) in the lemma is satisfied due to Proposition 3.3(2). To check property (1), we will rewrite $A_{K_{\mathbf{N}},K_{\mathbf{G}_\lambda}}(f)$ more explicitly in terms of the action of $\mathbf{G}$ on π.

Note that by Proposition 3.3(1c), the integral

$$\int_{n\in K_{\mathbf{N}_r}}\int_{k\in K_{\mathbf{G}_r}} ev(f^{n\cdot k})$$

equals the integral over a smaller domain, namely,

$$\int_{n\in K^{sm}_{\mathbf{N}_r}}\int_{k\in K_{\mathbf{G}_r}} ev(f^{n\cdot k}).$$

By Proposition 3.3(1a) and (1b), the latter can be rewritten as

$$\int_{n\in K^{sm}_{\mathbf{N}_r}}\int_{k\in K_{\mathbf{G}_r}} \begin{pmatrix} 1 & \sum\limits_{1\le i\le l} (k_{12})_{2i}\cdot n_i^2 \\ 0 & 1 \end{pmatrix} \cdot ev(f). \tag{8}$$

For $n = \Sigma\, t^{-i}\cdot n_i \in \mathbf{N}_r$, consider the map $\phi_n : \mathbf{G}_r \to \mathbf{N}$, given by

$$k = \begin{pmatrix} 1+\sum\limits_i t^i\cdot (k_{11})_i & \sum\limits_i t^i\cdot (k_{12})_i \\ \sum\limits_i t^i\cdot (k_{21})_i & 1+\sum\limits_i t^i\cdot (k_{22})_i \end{pmatrix} \mapsto \begin{pmatrix} 1 & \sum\limits_{1\le i\le r} (k_{12})_{2i}\cdot n_i^2 \\ 0 & 1. \end{pmatrix}$$

Thus, the expression in (8) can be rewritten as

$$\int_{n\in K^{sm}_{\mathbf{N}_r}} (\phi_n)_*\,(\mu(K_{\mathbf{G}_r}))\cdot ev(f), \tag{9}$$

where $\mu(K_{\mathbf{G}_r})$ denotes the Haar measure of $K_{\mathbf{G}_r}$, and $(\phi_n)_*\,(\mu(K_{\mathbf{G}_r}))$ is its push-forward under ϕ_n, regarded as a distribution on $\mathbf{N}$.

Note, however, that when we identify $\mathbf{G}_r \simeq \mathbf{G}^1/\mathbf{G}^{2r+1}$ with a linear space over $\mathbf{K}$, the Haar measure on this group goes over to a linear Haar measure. From this, we obtain that for each $n\in\mathbf{N}_r$, the distribution $(\phi_n)_*\,(\mu(K_{\mathbf{G}_r}))$, thought of as a function on $\mathbf{N}^*$, is the characteristic function of some $O_{\mathbf{K}}$-lattice in $\mathbf{N}^*$. Moreover, this lattice grows as $n\to 0$.

In particular, $(\phi_n)_*\,(\mu(K_{\mathbf{G}_r}))$, as a function on $\mathbf{N}^*$, belongs to $\mathrm{Funct}_{val}(\mathbf{N}^*)$, and the integral $\int_{n\in K^{sm}_{\mathbf{N}_r}}(\phi_n)_*\,(\mu(K_{\mathbf{G}_r}))$, being *positive* at every point of $\mathbf{N}^*$, defines an invertible element of $\mathrm{Funct}_{val}(\mathbf{N}^*)$.

Hence,

$$\begin{aligned} v &\in (\phi_n)_*\,(\mu(K_{\mathbf{G}_r}))\cdot(\mathrm{Funct}_{val}(\mathbf{N}^*)\cdot v) \\ &= \mathrm{Im}\left(\int_{n\in K^{sm}_{\mathbf{N}_r}} (\phi_n)_*\,(\mu(K_{\mathbf{G}_r}))\cdot ev(\mathsf{F}(v))\right), \end{aligned}$$

which is what we had to show.

4. Proof of Proposition 3.3

4.1. We will construct the subgroup $K_{\mathbf{G}_r}$ by induction with respect to the parameter r. (For property (2), we take $\mathbf{X} \cap \overline{\mathbf{Gr}}_G^{r-1}$ as the corresponding open compact subset of $\overline{\mathbf{Gr}}_G^{r-1}$.)

When $r = 0$, all the subgroups in question are trivial. So, we can assume having constructed the subgroups $K_{\mathbf{G}_{r-1}}$ and $K^{sm}_{\mathbf{N}_{r-1}}$, and let us perform the induction step. The key observation is provided by the following lemma:

Lemma 4.2. *Let $\mathbf{X}$ be a compact subset of $\mathbf{G}r^r_G$ and $f \in \overline{\Pi}^r$. Then, the integral $\int_{k \in K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}} ev_{\widetilde{g}}(f^k) = 0$ if $K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}$ is a sufficiently large compact subgroup of $\mathbf{G}^{2r}/\mathbf{G}^{2r+1}$, and $\mathfrak{p}(\widetilde{g}) \in \mathbf{X}$.*

Proof. Since Gr^r_G is a $G[[t]]$-orbit of $t^\lambda := \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$ and since G^{2r} is normalized by $G[[t]]$ and acts trivially on $\overline{\mathrm{Gr}}^r_G$, by the compactness of $\mathbf{X}$, the assertion of the lemma reduces to the fact that

$$\int\limits_{k \in K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}} ev_{t^\lambda}(f^k) = 0$$

for every sufficiently large subgroup $K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}$.

Note that for $k \in \mathbf{G}^{2r}$ written as

$$\begin{pmatrix} 1 + t^{2r} \cdot k_{11} & t^{2r} \cdot k_{12} \\ t^{2r} \cdot k_{21} & 1 + t^{2r} \cdot k_{22} \end{pmatrix},$$

$\mathrm{Ad}_{t^\lambda}(k) \in \mathbf{G}[[t]]$ projects to the element

$$\begin{pmatrix} 1 & k_{12} \\ 0 & 1 \end{pmatrix} \in \mathbf{G} = \mathbf{G}[[t]]/\mathbf{G}^1.$$

We have

$$ev_{t^\lambda}(f^k) = \mathrm{Ad}_{t^\lambda}(k) \cdot ev_{t^\lambda}(f).$$

Therefore, the integral in question equals the averaging of the vector $ev_{t^\lambda}(f) \in \pi$ over a compact subgroup of the maximal unipotent subgroup of G. Moreover, this subgroup grows together with $K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}$. Hence, our assertion follows from the cuspidality of π. q.e.d.

4.3. To carry out the induction step, we first choose $K'_{\mathbf{G}_r} \subset \mathbf{G}_r$ to be any $O_{\mathbf{K}}$-lattice, which projects onto $K_{\mathbf{G}_{r-1}} \subset \mathbf{G}_{r-1}$.

By continuity and the compactness of $K'_{\mathbf{G}_r}$, there exists an $O_{\mathbf{K}}$-lattice $\mathbf{L} \subset \mathbf{K}$, such that for $K^{sm}_{\mathbf{N}_r} = K^{sm}_{\mathbf{N}_{r-1}} + t^{-r}\mathbf{L}$ the following holds:

(a$'$) For $g = k' \cdot n \in \mathbf{G}((t))/\mathbf{G}^1$ with $k \in K'_{\mathbf{G}_r}$, $n \in K^{sm}_{\mathbf{N}_r}$, the point $\overline{g} \in \overline{\mathbf{Gr}}^r_G$ belongs to $\mathbf{Z}$.

(b$'$) For g as above, the left coset of $\mathfrak{q}(\widetilde{g}) \in \mathbf{G}$ with respect to $K_{\mathbf{G}}(\mathsf{F}) \subset \mathbf{G}$ equals that of

$$\begin{pmatrix} 1 & -\sum\limits_{1 \le i \le r} (k_{12})_{2i} \cdot n_i^2 \\ 0 & 1. \end{pmatrix}$$

(c$'$) For $f \in \mathsf{F}$, $f(k' \cdot (n' + t^{-r} n_r)) = f(k' \cdot n')$ for $n' \in K^{sm}_{\mathbf{N}_{r-1}}$, $k' \in K'_{\mathbf{G}_r}$, and $n_r \in \mathbf{L}$.

Note that for any $n \in \mathbf{N}_r$, $k' \in \mathbf{G}_r$ and

$$k = \begin{pmatrix} 1 + t^{2r} \cdot k_{11} & t^{2r} \cdot k_{12} \\ t^{2r} \cdot k_{21} & 1 + t^{2r} \cdot k_{22} \end{pmatrix} \in \mathbf{G}^{2r},$$

we have:

$$k \cdot k' \cdot n = k' \cdot n \cdot \begin{pmatrix} 1 & -k_{12} \cdot n_r^2 \\ 0 & 1 \end{pmatrix} \bmod \mathbf{G}^1. \tag{10}$$

The group $K_{\mathbf{G}_r}$ will be obtained from $K'_{\mathbf{G}_r}$ by adding to it an (arbitrarily large) lattice in $\mathbf{G}^{2r}/\mathbf{G}^{2r+1}$.

Note that since G^{2r} acts trivially on $\overline{\mathrm{Gr}}^r_G$, any such subgroup would satisfy condition (1a) of Proposition 3.3, because $K_{\mathbf{G}_r}$ satisfies (a$'$) above. It will also automatically satisfy (1b) in view of (10) and (b$'$) above. Thus, we have to arrange so that $K_{\mathbf{G}_r}$ satisfies conditions (1c) and (2) of Proposition 3.3.

4.4. By Lemma 4.2, we can find an open compact subgroup $K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}} \subset \mathbf{G}^{2r}/\mathbf{G}^{2r+1}$, such that the integrals

$$\int\limits_{k \in K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}} ev(f^{n \cdot k' \cdot k})$$

would vanish for $f \in \mathsf{F}$, $k' \in K'_{\mathbf{G}_r}$ and $n \in K_{\mathbf{N}_r}$ is such that $n_r \notin \mathbf{L}$.

Let us enlarge the initial $K'_{\mathbf{G}_r}$ by adding to it any $O_{\mathbf{K}}$-lattice in $\mathbf{G}^{2r}/\mathbf{G}^{2r+1}$ containing the above $K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}$. We claim that the resulting subgroup satisfies condition (1c) of Proposition 3.3.

Indeed, let $n = n' + t^{-r} n_r$, $n' \in \mathbf{N}_{r-1}, n_r \in \mathbf{K}$ be an element in $K_{\mathbf{N}_r} - K^{sm}_{\mathbf{N}_r}$. If $n_r \notin \mathbf{L}$, the integral vanishes by the choice of $K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}$. Thus, we can assume that $n_r \in \mathbf{L}$, but $n' \notin K^{sm}_{\mathbf{N}_{r-1}}$. But then, the integral vanishes by (c$'$) and the induction hypothesis.

Now, let us deal with condition (2) of Proposition 3.3. By the induction hypothesis, the integrals

$$\int\limits_{k\in K'_{\mathbf{G}_r}} ev(f^{g\cdot k}) \tag{11}$$

vanish when $\overline{g}\in\overline{\mathbf{Gr}}^{r-1}_G-(\overline{\mathbf{Gr}}^{r-1}_G\cap\mathbf{X})$.

Hence, by continuity and since $K'_{\mathbf{G}_r}$ is compact, there exists a neighbourhood $\mathbf{X}_1$ of $\overline{\mathbf{Gr}}^{r-1}_G-(\overline{\mathbf{Gr}}^{r-1}_G\cap\mathbf{X})$ in $\overline{\mathbf{Gr}}^{r}_G-\mathbf{X}$, such that the integral (11) will vanish for the same subgroup $K'_{\mathbf{G}_r}$ and all g for which $\overline{g}\in\mathbf{X}_1$.

The sought-for subgroup $K_{\mathbf{G}_r}$ will be again obtained from the initial $K'_{\mathbf{G}_r}$ by adding to it an arbitrarily large open compact subgroup of $\mathbf{G}^{2r}/\mathbf{G}^{2r+1}$. We claim that for any such $K_{\mathbf{G}_r}$ the integral

$$\int\limits_{k\in K_{\mathbf{G}_r}} ev(f^{g\cdot k}) \tag{12}$$

will still vanish for $\overline{g}\in\mathbf{X}_1$.

This follows from the fact that the G^{2r}-action on Gr^r_G is trivial, and hence, for $k\in\mathbf{G}^{2r}$, $k'\in\mathbf{G}_r$ and $g\in\mathbf{G}((t))$ projecting to $\overline{g}\in\overline{\mathbf{Gr}}^{r}_G$,

$$f(k\cdot k'\cdot g)=k_1\cdot f(k'\cdot g)$$

for some $k_1\in\mathbf{G}$.

We choose the suitable subgroup in $\mathbf{G}^{2r}/\mathbf{G}^{2r+1}$ as follows. Set $\mathbf{X}_2=(\overline{\mathbf{Gr}}^{r}_G-\mathbf{X})-\mathbf{X}_1$. This is a compact subset of $\mathbf{Gr}^r_G$, and let us apply Lemma 4.2 to the compact set $K'_{\mathbf{G}_r}\cdot\mathbf{X}_2\subset\mathbf{Gr}^r_G$.

We obtain that there exists an open compact subgroup $K_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}\subset\mathbf{G}^{2r}/\mathbf{G}^{2r+1}$, such that

$$\int\limits_{k\in K'_{\mathbf{G}^{2r}/\mathbf{G}^{2r+1}}} ev(f^{g\cdot k'\cdot k})=0$$

for $\overline{g}\in\mathbf{X}_2$, $k'\in K'_{\mathbf{G}_r}$.

Let $K_{\mathbf{G}_r}$ be the resulting subgroup of $\mathbf{G}_r$. We claim that it does satisfy condition (2) of Proposition 3.3. Indeed, consider again the integral (12) for $\overline{g}\in\overline{\mathbf{Gr}}^{r}_G-\mathbf{X}=\mathbf{X}_1\cup\mathbf{X}_2$.

We already know that it vanishes for $\overline{g}\in\mathbf{X}_1$. And if $\overline{g}\in\mathbf{X}_2$, it vanishes by the choice of $K_{\mathbf{G}_r}$.

This completes the induction step in the proof of Proposition 3.3.

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