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LOCAL RIGIDITY OF 3-DIMENSIONAL CONE-MANIFOLDS

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Abstract

We study the local deformation space of 3-dimensional cone-manifold structures of constant curvature $\kappa \in \{-1, 0, 1\}$ and cone-angles $\leq \pi$. Under this assumption on the cone-angles the singular locus will be a trivalent graph. In the hyperbolic and the spherical case our main result is a vanishing theorem for the first L^2-cohomology group of the smooth part of the cone-manifold with coefficients in the flat bundle of infinitesimal isometries. We conclude local rigidity from this. In the Euclidean case we prove that the first L^2-cohomology group of the smooth part with coefficients in the flat tangent bundle is represented by parallel forms.

1. Introduction

A 3-dimensional cone-manifold is a 3-manifold C equipped with a singular geometric structure. More precisely, C carries a length metric, which is in the complement of an embedded geodesic graph Σ induced by a smooth Riemannian metric of constant sectional curvature $\kappa \in \mathbb{R}$. Σ is called the singular locus and $M = C \setminus \Sigma$ the smooth part of C. Neighbourhoods of singular points are modelled on cones of curvature κ over 2-dimensional cone-manifolds diffeomorphic to S^2. One associates with each edge contained in Σ the so-called cone-angle, which is a positive real number. If all cone-angles are $\leq \pi$, then a connected component of Σ is either a (connected) trivalent graph or a circle.

3-dimensional cone-manifolds arise naturally in the geometrization of 3-dimensional orbifolds, cf. [**Thu**]. The concept of cone-manifold can be viewed as a generalization of the concept of geometric orbifold, where the cone-angles are no longer restricted to the set of orbifold-angles, which are rational multiples of π.

The deformation space of cone-manifold structures on a given cone-3-manifold C with fixed topological type (C, Σ) plays a significant role in the proof of the Orbifold Theorem, which has recently been completed by M. Boileau, B. Leeb and J. Porti, cf. [**BLP1**] and [**BLP2**]. The proof of the Orbifold Theorem in the general case requires the analysis

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of cone-manifold structures with cone-angles $\leq \pi$, where the singular locus is allowed to have trivalent vertices. The case, where the singular locus is a union of circle components, i.e., a link in C, has earlier been settled by M. Boileau and J. Porti, cf. [**BP**].

In this article we investigate local properties of the deformation space of cone-manifold structures with cone-angles $\leq \pi$. We consider the general case under this cone-angle restriction, where trivalent vertices are allowed. In particular we prove local rigidity in the spherical and in the hyperbolic case.

In the hyperbolic case there are some important results known. There is on the one hand Garland-Weil local rigidity (cf. [**Gar**]), which applies in any dimension ≥ 3 to the space of complete, finite-volume hyperbolic structures on a given hyperbolic manifold. On the other hand, C. Hodgson and S. Kerckhoff proved a local rigidity result for 3-dimensional hyperbolic cone-manifolds, cf. [**HK**]. Their proof applies to the case, where the singular locus Σ is a link in C, but where the cone-angles are allowed to be $\leq 2\pi$.

Our main technical result is a vanishing theorem for L^2-cohomology on the smooth part M of the cone-manifold C with coefficients in the flat vector-bundle of infinitesimal isometries. L^2-cohomology is by definition the cohomology of the subcomplex of the de-Rham complex, which consists of those forms ω such that ω and $d\omega$ are L^2-bounded.

Theorem 1.1. *Let C be a 3-dimensional cone-manifold of curvature $\kappa \in \{-1, 0, 1\}$ with cone-angles $\leq \pi$. Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be the vector-bundle of infinitesimal isometries of $M = C \setminus \Sigma$ with its natural flat connection. In the Euclidean case let $\mathcal{E}_{\mathrm{trans}} \subset \mathcal{E}$ be the parallel subbundle of infinitesimal translations. Then in the hyperbolic and the spherical case*

$$H^1_{L^2}(M, \mathcal{E}) = 0,$$

while in the Euclidean case

$$H^1_{L^2}(M, \mathcal{E}_{\mathrm{trans}}) \cong \{\omega \in \Omega^1(M, \mathcal{E}_{\mathrm{trans}}) \mid \nabla\omega = 0\}.$$

The proof of this theorem is analytic in nature. The main difficulty is caused by the non-completeness of the metric on M. On a complete Riemannian manifold the Hodge-Laplace operator on differential forms is known to be essentially selfadjoint, cf. [**BL1**] and the references therein. This is something we cannot expect to hold here.

On the other hand, the fact that the singularities of the metric are of iterated cone type allows us to apply separation of variables techniques. This has already been explored by J. Cheeger, cf. [**Ch1**].

One main ingredient is a Hodge-theorem for cone-manifolds, which allows us to identify L^2-cohomology spaces with the kernel of a certain selfadjoint extension of the Laplacian on forms. The second one is a Bochner-Weitzenböck formula for the Laplacian on 1-forms with values in the flat vector-bundle $\mathcal{E}$, resp. the parallel subbundle $\mathcal{E}_{\text{trans}} \subset \mathcal{E}$ in the Euclidean case.

The essence of the Bochner technique is that the Weitzenböck formula may be used to bound the Laplacian on compactly supported 1-forms from below: $\langle \Delta\omega, \omega \rangle_{L^2} \geq C \langle \omega, \omega \rangle_{L^2}$ for all $\omega \in \Omega^1_{cp}(M, \mathcal{E})$ and some $C > 0$. If we can show that this lower bound extends to hold for the selfadjoint extension given to us by the Hodge-theorem, we can conclude $H^1(M, \mathcal{E}) = 0$. In the Euclidean case, where one does not get a positive lower bound, one has to vary this argument a little.

In the complete, finite-volume case this settles everything in view of the essential selfadjointness of the Hodge-Laplacian (cf. [**Gar**]). In our case it requires a more detailed study of the selfadjoint extensions of the Hodge-Laplacian. Here we use techniques introduced by J. Brüning and R. Seeley, cf. [**BS**], along with some basic functional analytic properties of the de-Rham complex presented in a very convenient form in [**BL1**].

In the hyperbolic and in the spherical case we may conclude local rigidity from this; let us now briefly discuss the results:

If $\Sigma \subset C$ is the singular locus, for $\varepsilon > 0$ let $U_\varepsilon(\Sigma)$ be the smooth part of the ε-tube of Σ in C, i.e., $U_\varepsilon(\Sigma) = B_\varepsilon(\Sigma) \cap M$. Let $M_\varepsilon = M \setminus U_\varepsilon(\Sigma)$, which is topologically a manifold with boundary. Let μ_i be the meridian curve around the i-th edge of Σ.

In the hyperbolic case, the holonomy representation of the smooth but incomplete hyperbolic structure on M lifts to a representation

$$\mathrm{hol} : \pi_1 M \longrightarrow \widetilde{\mathrm{Isom}^+}\mathbf{H}^3 = \mathrm{SL}_2(\mathbb{C}).$$

Let $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ denote the set of representations of $\pi_1 M$ in $\mathrm{SL}_2(\mathbb{C})$ equipped with the compact-open topology. The set-theoretic quotient of the representation variety $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ by the conjugation action of $\mathrm{SL}_2(\mathbb{C})$ equipped with the quotient topology is denoted by $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$. For a representation $\rho \in R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ let $t_{\mu_i}(\rho) = \operatorname{tr} \rho(\mu_i) \in \mathbb{C}$. Clearly the functions t_{μ_i} are invariant under conjugation and descend to $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$.

The above defined spaces may be badly behaved in general, but near the holonomy representation of a hyperbolic cone-manifold structure we can establish smoothness and the following parametrization:

Theorem 1.2. *Let C be a hyperbolic cone-3-manifold with cone-angles $\leq \pi$. Let $\{\mu_1, \ldots, \mu_N\}$ be the family of meridians, where N is the number of edges contained in Σ. Then the map*

$$X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^N, \chi \mapsto (t_{\mu_1}(\chi), \ldots, t_{\mu_N}(\chi))$$

is locally biholomorphic near $\chi = [\mathrm{hol}]$.

The quotient space $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ may be considered, at least locally, as the deformation space of hyperbolic structures on M. Hyperbolic cone-manifold structures correspond to representations, where the meridians μ_i map to elliptic elements in $\mathrm{SL}_2(\mathbb{C})$. Therefore the previous theorem implies local rigidity in the following strong sense:

Corollary 1.3 (local rigidity)**.** *Let C be a hyperbolic cone-3-manifold with cone-angles $\leq \pi$. Then the set of cone-angles $\{\alpha_1, \dots, \alpha_N\}$, where N is the number of edges contained in Σ, provides a local parametrization of the space of hyperbolic cone-manifold structures near the given structure on M. In particular, there are no deformations leaving the cone-angles fixed.*

In the spherical case, the holonomy representation of the smooth, but incomplete spherical structure on M lifts to a product representation

$$\mathrm{hol} = (\mathrm{hol}_1, \mathrm{hol}_2) : \pi_1 M \longrightarrow \widetilde{\mathrm{Isom}^+\mathbf{S}^3} = \mathrm{SU}(2) \times \mathrm{SU}(2).$$

For a representation $\rho \in R(\pi_1 M, \mathrm{SU}(2))$ let $t_{\mu_i}(\rho) = \operatorname{tr} \rho(\mu_i) \in \mathbb{R}$. Again the functions t_{μ_i} are invariant under conjugation and descend to $X(\pi_1 M, \mathrm{SU}(2))$.

Following [**Por**] we will say that a cone-3-manifold C is *Seifert fibered* if C carries a Seifert fibration such that the components of Σ are leaves of the fibration. In particular Σ is a link and $M = C \setminus \Sigma$ is a Seifert fibered 3-manifold. In the statement of the following result we have to include the additional hypothesis "C not Seifert fibered" to ensure that the representations $\mathrm{hol}_i : \pi_1 M \to \mathrm{SU}(2)$ are non-abelian.

Theorem 1.4. *Let C be a spherical cone-3-manifold with cone-angles $\leq \pi$, which is not Seifert fibered. Let $\{\mu_i, \dots, \mu_N\}$ be the family of meridians, where N is the number of edges contained in Σ. Then the map*

$$X(\pi_1 M, \mathrm{SU}(2)) \to \mathbb{R}^N, \chi_i \mapsto (t_{\mu_1}(\chi_i), \dots, t_{\mu_N}(\chi_i))$$

is a local diffeomorphism near $\chi_i = [\mathrm{hol}_i]$ for $i \in \{1, 2\}$.

As in the hyperbolic case we conclude local rigidity from this:

Corollary 1.5 (local rigidity)**.** *Let C be a spherical cone-3-manifold with cone-angles $\leq \pi$, which is not Seifert fibered. Then the set of cone-angles $\{\alpha_1, \dots, \alpha_N\}$, where N is the number of edges contained in Σ, provides a local parametrization of the space of spherical cone-manifold structures near the given structure on M. In particular, there are no deformations leaving the cone-angles fixed.*

The geometric significance of the cohomological result in the Euclidean case is subject to further investigation.

The results of this article are contained in my doctoral thesis. I would like to thank Bernhard Leeb, my thesis advisor, for his support and encouragement. I am indebted to Joan Porti for answering many of my questions concerning representation varieties and related things. Furthermore, I would like to thank Daniel Grieser for explaining various

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2. Cone-manifolds

For $\kappa \in \mathbb{R}$ let sn_κ and cs_κ be the unique solutions of the ODE

$$f''(r) + \kappa f(r) = 0$$

subject to the initital conditions

$$\mathrm{sn}_\kappa(0) = 0 \quad \text{and} \quad \mathrm{sn}'_\kappa(0) = 1$$
$$\mathrm{cs}_\kappa(0) = 1 \quad \text{and} \quad \mathrm{cs}'_\kappa(0) = 0.$$

If (N, g^N) is a Riemannian manifold we define for $\kappa \in \mathbb{R}$ and $\varepsilon > 0$ (and $\varepsilon < \pi/\sqrt{\kappa}$ if $\kappa > 0$) the ε-truncated κ-*cone* over N to be the space

$$\mathrm{cone}_{\kappa,(0,\varepsilon)} N = (0, \varepsilon) \times N$$

equipped with the Riemannian metric

$$g = dr^2 + \mathrm{sn}^2_\kappa(r) g^N.$$

A *cone-surface* S of curvature $\kappa \in \mathbb{R}$ is a compact, oriented surface which carries a length metric with the property that there are a finite number of points $\{x_1, \dots, x_k\} \subset S$ (the *cone-points*) and numbers $\{\alpha_1, \dots, \alpha_k\} \subset \mathbb{R}^k_+$ (the *cone-angles*), such that $N = S \setminus \{x_1, \dots, x_k\}$ is a smooth Riemannian manifold of curvature κ and furthermore the smooth part of the ε-ball around each cone-point $U_\varepsilon(x_i) = B_\varepsilon(x_i) \cap N$ is isometric to the κ-cone over the circle of length α_i.

We will also use the notation $\operatorname{int} S = S \setminus \{x_1, \dots, x_k\}$ for the smooth part of a cone-surface S. For $\kappa \in \{-1, 0, 1\}$ we will call S respectively hyperbolic, Euclidean or spherical. Let us call the homeomorphism type of $(S, \{x_1, \dots, x_k\})$ the topological type of S.

Using a version of the Gauss-Bonnet theorem for cone-surfaces, it is easy to classify the spherical cone-surfaces S with cone-angles $\le \pi$. The underlying space has to be S^2 and there are two types:

$$S = \begin{cases} \mathbf{S}^2(\alpha, \beta, \gamma) & \text{or} \\ \mathbf{S}^2(\alpha, \alpha). \end{cases}$$

Here $\mathbf{S}^2(\alpha, \beta, \gamma)$ is the double of a spherical triangle with angles $\alpha/2, \beta/2, \gamma/2$ and $\mathbf{S}^2(\alpha, \alpha)$ is the double of a spherical bigon with angles $\alpha/2, \alpha/2$. Spherical cone-surfaces with cone-angles $\le \pi$ are rigid, i.e., they are determined up to isometry by the topological type and the set of cone-angles.

A *cone*-3-*manifold* C of curvature $\kappa \in \mathbb{R}$ is a compact, oriented 3-manifold which carries a length metric with the property that there is a distinguished subset $\Sigma \subset C$ (the *singular locus*) such that $M = C \setminus \Sigma$ is a smooth Riemannian manifold of curvature κ and furthermore the smooth part of the ε-ball around each singular point $U_\varepsilon(x) = B_\varepsilon(x) \cap M$ is isometric to the κ-cone over $\operatorname{int} S_x$ for a spherical cone-surface S_x.

We will also use the notation $\operatorname{int} C = C \setminus \Sigma$ for the smooth part of a cone-3-manifold C. For $\kappa \in \{-1, 0, 1\}$ we will call C respectively hyperbolic, Euclidean or spherical. Let us call the homeomorphism type of (C, Σ) the topological type of C.

If $x \in \Sigma$ is a singular point then we call S_x the link of x in C. The hypothesis that the underlying space C is a manifold implies that the links of singular points are cone-surfaces with underlying space S^2. If the cone-angles are $\leq \pi$ we in particular obtain that links of singular points are either $\mathbf{S}^2(\alpha, \beta, \gamma)$ or $\mathbf{S}^2(\alpha, \alpha)$. This implies that the singular locus Σ is a trivalent graph embedded geodesically into C.

Cone-manifolds with cone-angles $\leq 2\pi$ satisfy a lower curvature bound in the triangle comparison sense and may be studied from a synthetic point of view. This is pursued in [**BLP2**].

Basic material on the geometry of 2- and 3-dimensional cone-manifolds as well as an outline of the authors' approach to the Orbifold Theorem can be found in [**CHK**].

3. Analysis on cone-manifolds

By analysis on a cone-manifold C we mean analysis on $M = C \setminus \Sigma$, the smooth part of C. M is a smooth Riemannian manifold, but incomplete if Σ is nonempty. This causes the main difficulties here.

In this chapter we discuss some functional analytic properties of differential operators on noncompact manifolds. In contrast to the compact situation one has to distinguish more carefully between a differential operator acting on smooth, compactly supported sections of some vector-bundle and its closed realizations as an unbounded operator on the Hilbert space of L^2-sections.

3.1. Differential operators on noncompact manifolds. Let (M, g) be a Riemannian manifold (possibly noncompact and possibly incomplete) and let $(\mathcal{E}, h^{\mathcal{E}}), (\mathcal{F}, h^{\mathcal{F}})$ be hermitian vector-bundles over M. The naturally associated L^2-spaces $L^2(\mathcal{E})$, resp. $L^2(\mathcal{F})$, only depend on the quasi-isometry classes of the metrics g and $h^{\mathcal{E}}$, resp. $h^{\mathcal{F}}$.

We consider a differential operator P acting on sections of $\mathcal{E}$ as an unbounded, densely defined operator with domain the compactly supported sections:

$$P : L^2(\mathcal{E}) \supset \operatorname{dom} P = C^\infty_{\mathrm{cp}}(\mathcal{E}) \longrightarrow L^2(\mathcal{F}).$$

The *formal adjoint* of a differential operator P

$$P^t : L^2(\mathcal{F}) \supset \operatorname{dom} P^t = C^\infty_{\mathrm{cp}}(\mathcal{F}) \longrightarrow L^2(\mathcal{E})$$

is uniquely defined by the relation $\langle Ps, t \rangle = \langle s, P^t t \rangle$ to hold for all $s \in C^\infty_{\mathrm{cp}}(\mathcal{E})$ and $t \in C^\infty_{\mathrm{cp}}(\mathcal{F})$. P^t is again a differential operator, hence densely defined. P is said to be *symmetric* (or *formally selfadjoint*) if $\mathcal{E} = \mathcal{F}$ and $\langle Ps, t\rangle = \langle s, Pt\rangle$ for all $s, t \in C^\infty_{\mathrm{cp}}(\mathcal{E})$.

The formal adjoint is not to be confused with the *adjoint* P^* in the sense of unbounded operator theory. The domain of P^* is given as follows:

$$\operatorname{dom} P^* = \{s \in L^2(\mathcal{F}) | u \mapsto \langle Pu, s\rangle \text{ bounded for } u \in \operatorname{dom} P\}.$$

Since P is densely defined there is a unique $t \in L^2(\mathcal{E})$ such that $\langle Pu, s\rangle = \langle u, t\rangle$ holds for all $u \in \operatorname{dom} P$. Then let $P^* s = t$ by definition. P^* is a closed operator. Recall that a linear operator A is called (graph-)*closed* if $\operatorname{dom} A$ equipped with the graph norm $\|x\|_A = (\|x\|^2 + \|Ax\|^2)^{\frac{1}{2}}$ is complete.

P^* obviously extends P^t (which we as usual denote by $P^t \subset P^*$), in particular P^* is densely defined. Note that P is symmetric if and only if $P \subset P^*$. A natural question to ask is if P admits closed extensions, and this is in fact always the case. Define

$$P_{\max} = (P^t)^*$$

and

$$P_{\min} = P^{**}.$$

P^{**} is well-defined since P^* is densely defined. P^{**} then equals $\overline{P}$, the (graph-) closure of P, i.e., the domain of $P_{\min}$ can be characterized as follows:

$$\begin{aligned}\operatorname{dom} P_{\min} = \{s \in L^2(\mathcal{E}) | \exists (s_n)_{n\in\mathbb{N}} \subset \operatorname{dom} P \text{ such that } s_n \to s \text{ in } L^2(\mathcal{E})\\ \text{and } (Ps_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } L^2(\mathcal{F})\},\end{aligned}$$

and $P_{\min}(s) = \lim_{n\to\infty} Ps_n$.

We say that $Ps = t$ in the distributional sense if $\langle s, P^t u\rangle = \langle t, u\rangle$ holds for all $u \in C^\infty_{\mathrm{cp}}(\mathcal{F})$. The domain of $P_{\max}$ may then be written as:

$$\operatorname{dom} P_{\max} = \{s \in L^2(\mathcal{E}) | Ps \in L^2(\mathcal{F})\},$$

and $P_{\max}(s) = Ps$ in the distributional sense. Clearly $P_{\min} \subset P_{\max}$ and both are closed extensions of P. $P_{\max}$ is maximal with respect to having $C^\infty_{\mathrm{cp}}(\mathcal{F})$ in the domain of its adjoint, i.e., $P^*_{\max}$ still extends P^t.

If P is symmetric we ask for selfadjoint extensions. Recall that a closed symmetric operator A is called *selfadjoint* if $A = A^*$. P is called *essentially selfadjoint* if $P_{\min}$ is selfadjoint. Since for a symmetric operator one has $P_{\max} = P^*$, this is the case if and only if $P_{\min} = P_{\max}$. Selfadjoint extensions need not exist in general.

On the other hand, if we assume that our operator P is semibounded, there is alway a distinguished selfadjoint extension which preserves the lower bound. This feature will turn out to be particularly useful.

P *semibounded* means by definition that there exists $c \in \mathbb{R}$ such that $\langle s, Ps\rangle \geq c\langle s, s\rangle$ for all $s \in \operatorname{dom} P$. Recall that a semibounded quadratic form $q : \operatorname{dom} q \times \operatorname{dom} q \to L^2$ with lower bound c is *closed* if and only if $\operatorname{dom} q$ equipped with the norm $\|x\|_q = (q(x)+(1-c)\|x\|^2)^{1/2}$ is complete.

Theorem 3.1 (the Friedrichs extension, cf. [**RS**])**.** *Let P be a semibounded symmetric operator and let $q(s,t) = \langle s, Pt\rangle$ for $s,t \in \operatorname{dom} P$. Then q is a closable quadratic form and the closure $\overline{q}$ is the quadratic form of a unique selfadjoint operator P_F, the so-called Friedrichs extension of P. $\operatorname{dom} P_F$ is contained in $\operatorname{dom} \overline{q}$ and P_F is the only selfadjoint extension of P with this property. Furthermore, P_F satisfies the same lower bound as P.*

In the formulation of the following theorem as for the rest of the article we adopt the usual convention $\operatorname{dom} AB = \{x \in \operatorname{dom} B | Bx \in \operatorname{dom} A\}$.

Theorem 3.2 (von Neumann, cf. [**RS**])**.** *Let A be a closed densely defined operator. Then A^*A is selfadjoint.*

For a differential operator of the form $P = D^tD$ we obtain for its quadratic form $q(s) = \langle Ds, Ds\rangle \geq 0$ and therefore $\operatorname{dom} \overline{q} = \operatorname{dom} D_{\min}$. A consequence of von Neumann's theorem (Theorem 3.2) is (with $A = D_{\min}$) that $D^t_{\max} D_{\min}$ is a selfadjoint extension of P. On the other hand, $\operatorname{dom} D^t_{\max} D_{\min}$ is certainly contained in $\operatorname{dom} D_{\min} = \operatorname{dom} \overline{q}$. Therefore we get as an important corollary:

Corollary 3.3. *$D^t_{\max} D_{\min}$ is the Friedrichs extension of D^tD.*

3.2. The de-Rham complex. Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be a flat vector-bundle equipped with a hermitian metric $h^{\mathcal{E}}$. The metric $h^{\mathcal{E}}$ will not necessarily be assumed to be parallel with respect to $\nabla^{\mathcal{E}}$. We denote the exterior derivative coupled with the flat connection again by d. As an operator

$$d : \Omega^{\bullet}_{\mathrm{cp}}(M, \mathcal{E}) \to \Omega^{\bullet+1}_{\mathrm{cp}}(M, \mathcal{E}),$$

d is uniquely determined by the relation $d(\alpha \otimes s) = d\alpha \otimes s + (-1)^{|\alpha|} \alpha \otimes \nabla s$, where α is an ordinary form and s a section of $\mathcal{E}$.

Since $d^i_{\max}(\operatorname{dom} d^i_{\max}) \subset \operatorname{dom} d^{i+1}_{\max}$ and $d^{i+1}_{\max} \circ d^i_{\max} = 0$, we can consider the $d_{\max}$-complex

$$\ldots \longrightarrow \operatorname{dom} d^i_{\max} \xrightarrow{d^i_{\max}} \operatorname{dom} d^{i+1}_{\max} \longrightarrow \ldots$$

In fact, $d_{\max}$ is a particular choice of *ideal boundary condition*, cf. [**Ch1**], and the $d_{\max}$-complex is a particular instance of a so-called *Hilbert complex*, see [**BL1**] for the definition and a general discussion.

Recall that the *Hodge-Dirac* operator $D = d + d^t$ decomposes as a direct sum $D = D^{\mathrm{ev}} \oplus D^{\mathrm{odd}}$, where

$$D^{\mathrm{ev}} : \Omega^{\mathrm{ev}}_{\mathrm{cp}}(M, \mathcal{E}) \longrightarrow \Omega^{\mathrm{odd}}_{\mathrm{cp}}(M, \mathcal{E})$$

and

$$D^{\mathrm{odd}} = (D^{\mathrm{ev}})^t : \Omega^{\mathrm{odd}}_{\mathrm{cp}}(M, \mathcal{E}) \longrightarrow \Omega^{\mathrm{ev}}_{\mathrm{cp}}(M, \mathcal{E}).$$

We obtain closed extensions of D, D^{ev} and D^{odd} by setting

$$D(d_{\max}) = d_{\max} + d^t_{\min}$$

and

$$D(d_{\max})^{\text{ev/odd}} = (d_{\max} + d^t_{\min})^{\text{ev/odd}}.$$

Here we adopt the usual convention $\operatorname{dom} A + B = \operatorname{dom} A \cap \operatorname{dom} B$. Note in particular that $d^t_{\min} = d^*_{\max}$. Since $d_{\max}$ and $d^*_{\max}$ are closed operators and $(\ker d_{\max})^{\perp}$ and $(\ker d^*_{\max})^{\perp}$ are orthogonal, it follows that $D(d_{\max})^{\text{odd}} = (D(d_{\max})^{\text{ev}})^*$ and in particular that $D(d_{\max})$ is a selfadjoint extension of D.

Note that we do *not* claim that in general the extension $D(d_{\max})$ equals the maximal extension of D itself.

Recall that the *Hodge-Laplace* operator is the square of the Hodge-Dirac operator:

$$\Delta = D^2 = dd^t + d^t d.$$

Von Neumann's Theorem (Theorem 3.2) implies that

$$\Delta(d_{\max}) = D(d_{\max})^2 = d_{\max} d^t_{\min} + d^t_{\min} d_{\max}$$

is a selfadjoint extension of Δ. Note again that this extension need not be equal to the maximal extension of Δ.

Lemma 3.4. $\Delta_F = D_{\max} D_{\min}$

Proof. The assertion follows from Corollary 3.3. q.e.d.

We single out the following consequence since it is the basis for our main line of argument towards the adaptation of the classical Bochner technique in our singular context.

Corollary 3.5. *If D is essentially selfadjoint, then $\Delta_F = \Delta(d_{\max})$.*

Proof. If D is essentially selfadjoint, then since $D(d_{\max})$ is a selfadjoint extension of D, we obtain $D_{\min} = D(d_{\max}) = D_{\max}$. Now the assertion follows from the previous lemma. q.e.d.

Once essential selfadjointness of D is established, this result allows one to extend lower bounds obtained for Δ on compactly supported forms to $\Delta(d_{\max})$ on its respective domain. Our concern for this particular extension will become clear from the next section.

3.3. Hodge theory. To define L^2-*cohomology* we consider the following subcomplex of the de-Rham complex:

$$\begin{aligned}\Omega^i_{L^2}(M, \mathcal{E}) &= \{\omega \in \Omega^i(M, \mathcal{E}) | w \in L^2 \text{ and } dw \in L^2\} \\ &= \operatorname{dom} d^i_{\max} \cap \Omega^i(M, \mathcal{E}),\end{aligned}$$

which we will refer to as the smooth L^2-complex. L^2-cohomology is by definition the cohomology of the smooth L^2-complex, i.e.,

$$H^i_{L^2}(M, \mathcal{E}) = \ker d^i \cap \Omega^i_{L^2}(M, \mathcal{E}) / d^{i-1} \Omega^{i-1}_{L^2}(M, \mathcal{E}).$$

Let us denote the cohomology of the $d_{\max}$-complex by

$$H^i_{\max} = \ker d^i_{\max} / \operatorname{im} d^{i-1}_{\max}.$$

We define the $d_{\max}$-*harmonic* i-forms to be

$$\mathcal{H}^i_{\max} = \ker d^i_{\max} \cap \ker (d^{i-1})^t_{\min}.$$

The following theorem is due to Cheeger, cf. [**Ch1**], the corresponding statement in a slightly more general setting may be found in [**BL1**].

Theorem 3.6. *The inclusion* $\Omega^i_{L^2}(M, \mathcal{E}) \hookrightarrow \operatorname{dom} d^i_{\max}$ *induces an isomorphism on the level of cohomology:* $H^i_{L^2}(M, \mathcal{E}) \cong H^i_{\max}$.

There is a basic Hodge theorem for the $d_{\max}$-complex, which goes back to Kodaira, cf. [**Kod**], while [**BL1**] prove a similar statement in the context of Hilbert complexes.

Theorem 3.7 (weak Hodge-decomposition)**.** *For each* i *there is an orthogonal decomposition*

$$L^2(\Lambda^i T^* M \otimes \mathcal{E}) = \mathcal{H}^i_{\max} \oplus \overline{\operatorname{im} d^{i-1}_{\max}} \oplus \overline{\operatorname{im} (d^i)^t_{\min}}$$

and furthermore

$$\mathcal{H}^i_{\max} = \ker \Delta^i(d_{\max}) = \ker D(d_{\max}) \cap L^2(\Lambda^i T^* M \otimes \mathcal{E}).$$

We define a map

$$\begin{aligned} \iota : \mathcal{H}^i_{\max} &\longrightarrow H^i_{\max} \\ \alpha &\longmapsto \alpha + \operatorname{im} d^{i-1}_{\max}. \end{aligned}$$

Injectivity of ι is equivalent to $\operatorname{im} d^{i-1}_{\max} \cap \ker (d^{i-1})^t_{\min} = 0$, which is always the case, since

$$\overline{\operatorname{im} d^{i-1}_{\max}} = (\ker (d^{i-1}_{\max})^*)^{\perp} = (\ker (d^{i-1})^t_{\min})^{\perp}.$$

Surjectivity of ι is equivalent to

$$\operatorname{im} d^{i-1}_{\max} = \overline{\operatorname{im} d^{i-1}_{\max}}\,;$$

therefore we obtain the following enhancement of the Hodge decomposition, which is due to Cheeger (cf. [**Ch1**]) in the case of the $d_{\max}$-complex. Again a more general statement may be found in [**BL1**].

Theorem 3.8 (strong Hodge-decomposition)**.** *If* $\operatorname{im} d^{i-1}_{\max}$ *is closed for all* i*, then for each* i *there is an orthogonal decomposition*

$$L^2(\Lambda^i T^* M \otimes \mathcal{E}) = \mathcal{H}^i_{\max} \oplus \operatorname{im} d^{i-1}_{\max} \oplus \operatorname{im} (d^i)^t_{\min}$$

and furthermore $\iota : \mathcal{H}^i_{\max} \to H^i_{\max}$ *is an isomorphism.*

A sufficient condition for $d^{i-1}_{\max}$ to have closed range is finite dimensionality of $H^i_{\max}$ on the one hand, since $\ker d^i_{\max}/\operatorname{im} d^{i-1}_{\max}$ finite dimensional implies that $\operatorname{im} d^{i-1}_{\max}$ is closed in $\ker d^i_{\max}$, hence in $L^2(\Lambda^i T^*M \otimes \mathcal{E})$. Note that by the closed-range theorem, $(d^i_{\max})^*$ has closed range if and only if $d^i_{\max}$ has closed range.

On the other hand, if $D(d_{\max})^{\mathrm{ev}}$ has closed range, then $d^i_{\max}$ and $(d^{i+1}_{\max})^*$ will have closed range for all i even. Similarly, if $D(d_{\max})^{\mathrm{odd}}$ has closed range, then $d^i_{\max}$ and $(d^{i+1}_{\max})^*$ will have closed range for all i odd. Since $D(d_{\max})^{\mathrm{odd}} = (D(d_{\max})^{\mathrm{ev}})^*$, the closed-range theorem implies that $D(d_{\max})^{\mathrm{ev}}$ has closed range if and only $D(d_{\max})^{\mathrm{odd}}$ has closed range.

It is easy to show that $D(d_{\max})^{\mathrm{ev}}$ has closed range if $\operatorname{dom} D(d_{\max})^{\mathrm{ev}}$ equipped with the graph norm embeds into $L^2(\Lambda^{\mathrm{ev}} T^*M \otimes \mathcal{E})$ compactly. This latter condition is related to the question of discreteness of the spectra of the operators $D(d_{\max})$ and $\Delta(d_{\max})$. Recall that an operator is said to have discrete spectrum if its spectrum consists of a discrete set of eigenvalues with finite multiplicities.

4. Spectral properties of cone-manifolds

In this chapter we apply the techniques of Brüning and Seeley to analyze the closed extensions of the Hodge-Dirac operator on a 3-dimensional cone-manifold. The main reference for the first order case will be [**BS**]. The analysis relies heavily on the fact that the spaces we consider are *locally conical*, i.e., neighbourhoods of points are isometric to (κ-)cones over spaces of lower dimension. This allows us to apply separation of variables techniques.

To keep the exposition self-contained here, we describe these techniques in detail. Furthermore we adopt a more elementary viewpoint than in [**BS**], in particular giving a direct argument for discreteness of the relevant operators.

Let us further mention that [**BS**] deal with isolated conical singularities only, i.e., the links of singular points are compact smooth Riemannian manifolds, where in our case we have to allow the links of singular points to be again singular, namely the spherical cone-surfaces $\mathbf{S}^2(\alpha, \beta, \gamma)$ and $\mathbf{S}^2(\alpha, \alpha)$. This requires some extra arguments which we will provide as we expose the theory.

There has been a lot of work on Hodge-theory and L^2-cohomology of Riemannian manifolds with conical singularities, besides [**Ch1**] and [**Ch2**] see for example [**BL2**].

4.1. Separation of variables. Let (N, g^N) be a Riemannian manifold of dimension n and let us consider $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} N$ with the Riemannian metric $g = dr^2 + \operatorname{sn}^2_\kappa(r) g^N$. We may think of N as the (smooth part of the) link S_x of a singular point x in a cone-manifold, U_ε serving as a model for the (smooth part of the) ε-neighbourhood $U_\varepsilon(x)$ of a singular point x in M.

Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be a flat vector-bundle over U_ε. We will identify the fibers of $\mathcal{E}$ along radial geodesics via parallel translation using $\nabla^{\mathcal{E}}$. In particular we may canonically identify $\mathcal{E}|_{U_\varepsilon} = (0,\varepsilon)\times\mathcal{E}|_N$. Let us further assume that $\mathcal{E}$ is equipped with a metric $h^{\mathcal{E}}$, which is not necessarily parallel with respect to $\nabla^{\mathcal{E}}$. We will assume instead:

A1 The limit $h_0^{\mathcal{E}} := \lim_{r\to 0} h^{\mathcal{E}}(r)$ exists as a smooth metric on $\mathcal{E}|_N$ and is parallel with respect to $\nabla^{\mathcal{E}}$. (The limit is defined using the canonical identification $\mathcal{E}|_{U_\varepsilon} = (0,\varepsilon)\times\mathcal{E}|_N$ as above.)

Now $h_0^{\mathcal{E}}$ extends to a parallel metric on $\mathcal{E}|_{U_\varepsilon}$, which we continue to denote by $h_0^{\mathcal{E}}$. We may write

$$h^{\mathcal{E}}(\sigma,\tau) = h_0^{\mathcal{E}}(A\sigma,\tau)$$

for $\sigma,\tau \in \Gamma(U_\varepsilon,\mathcal{E})$, where $A \in \Gamma(U_\varepsilon, \operatorname{End}\mathcal{E})$ is symmetric with respect to $h_0^{\mathcal{E}}$. Let us continue to denote the flat connection on $\operatorname{End}\mathcal{E}$ by $\nabla^{\mathcal{E}}$. We will further assume:

A2 $A^{-1}(\nabla^{\mathcal{E}}A) \in \Omega^1(U_\varepsilon, \operatorname{End}\mathcal{E})$ is bounded with respect to g and $h^{\mathcal{E}}$.

Remark 4.1.

1) A2 implies that $h^{\mathcal{E}}$ and $h_0^{\mathcal{E}}$ are quasi-isometric on U_ε, since for $\sigma \in \Gamma(U_\varepsilon,\mathcal{E})$ satisfying $\nabla^{\mathcal{E}}_{\partial/\partial r}\sigma = 0$ we have

$$\left|\frac{d}{dr}\log\frac{h^{\mathcal{E}}(\sigma,\sigma)}{h_0^{\mathcal{E}}(\sigma,\sigma)}\right| = \frac{\left|h^{\mathcal{E}}(A^{-1}(\nabla^{\mathcal{E}}_{\partial/\partial r}A)\sigma,\sigma)\right|}{h^{\mathcal{E}}(\sigma,\sigma)} \le C$$

on the complement of the zero-set of σ, where C is the bound on $A^{-1}(\nabla^{\mathcal{E}}A)$ given by A2.

2) If the cross-section N is compact, then A2 is a direct consequence of A1, in the general case A2 is an additional assumption.
3) If $h^{\mathcal{E}}$ is already parallel with respect to $\nabla^{\mathcal{E}}$, then $h_0^{\mathcal{E}} = h^{\mathcal{E}}$ and A1 and A2 are trivially satisified.

Let d denote the exterior covariant derivative coupled with $\nabla^{\mathcal{E}}$ and let d^t denote the formal adjoint of d with respect to $h^{\mathcal{E}}$. Similarly let d_0^t denote the formal adjoint of d with respect to $h_0^{\mathcal{E}}$. If $\iota(\nabla^{\mathcal{E}}A)$ denotes interior multiplication with the $\operatorname{End}\mathcal{E}$-valued 1-form $\nabla^{\mathcal{E}}A$, then we have:

Lemma 4.2. *$d^t = d_0^t - A^{-1}\iota(\nabla^{\mathcal{E}}A)$.*

Proof. If L_0^2 denotes the L^2-space with respect to g and $h_0^{\mathcal{E}}$, we have

$$\begin{aligned}\langle A\, d^t\eta,\xi\rangle_{L_0^2} &= \langle A\eta, d\xi\rangle_{L_0^2} = \langle \eta, d(A\xi) - \nabla^{\mathcal{E}}A\wedge\xi\rangle_{L_0^2}\\ &= \langle A(d_0^t\eta) - \iota(\nabla^{\mathcal{E}}A)\eta,\xi\rangle_{L_0^2}\end{aligned}$$

for $\eta \in \Omega_{cp}^{p+1}(U_\varepsilon, \mathcal{E})$ and $\xi \in \Omega_{cp}^{p}(U_\varepsilon, \mathcal{E})$. In the last line we have used that $h_0^{\mathcal{E}}$ is parallel with respect to $\nabla^{\mathcal{E}}$, hence $\nabla^{\operatorname{End}\mathcal{E}} A$ has values in the symmetric (w.r.t. $h_0^{\mathcal{E}}$) endomorphisms of $\mathcal{E}$. q.e.d.

With $D = d + d^t$ and $D_0 = d + d_0^t$ we therefore have

$$D = D_0 - A^{-1}\iota(\nabla^{\mathcal{E}} A).$$

Following [**BS**], we identify p-forms on the model neighbourhood U_ε with pairs of r-dependent forms on N via

$$(\phi, \psi) \mapsto \operatorname{sn}_\kappa(r)^{(p-1)-\frac{n}{2}}\phi \wedge dr + \operatorname{sn}_\kappa(r)^{p-\frac{n}{2}}\psi,$$

where $\phi \in \Gamma(\pi_N^* \Lambda^{p-1} T^* N \otimes \mathcal{E})$ and $\psi \in \Gamma(\pi_N^* \Lambda^{p} T^* N \otimes \mathcal{E})$. This correspondence preserves L^2-norms, if we use the parallel metric $h_0^{\mathcal{E}}$:

$$\int_0^\varepsilon \int_0 |\phi|_0^2 \, dr dvol_N = \int_{U_\varepsilon} \operatorname{sn}_\kappa(r)^{2(p-1)-n} |\phi \wedge dr|_0^2 \, dvol_{U_\varepsilon}$$

and

$$\int_0^\varepsilon \int_N |\psi|_0^2 \, dr dvol_N = \int_{U_\varepsilon} \operatorname{sn}_\kappa(r)^{2p-n} |\psi|_0^2 \, dvol_{U_\varepsilon}.$$

With respect to these decompositions the exterior differential has the following matrix form on U_ε:

$$d^p = \begin{bmatrix} \operatorname{sn}_\kappa(r)^{-1} d_N^{p-1} & (-1)^p \left\{ \frac{\partial}{\partial r} + (p - \frac{n}{2}) \operatorname{ct}_\kappa(r) \right\} \\ 0 & \operatorname{sn}_\kappa(r)^{-1} d_N^{p} \end{bmatrix}.$$

By passing to the formal adjoints using $h_0^{\mathcal{E}}$ we obtain:

$$(d_0^t)_p = \begin{bmatrix} \operatorname{sn}_\kappa(r)^{-1} (d_N^t)_{p-1} & 0 \\ (-1)^p \left\{ \frac{\partial}{\partial r} + (\frac{n}{2} - p + 1) \operatorname{ct}_\kappa(r) \right\} & \operatorname{sn}_\kappa(r)^{-1} (d_N^t)_p \end{bmatrix}.$$

We may identify r-dependent forms on N of arbitrary degree with either even forms on U_ε via

$$(\phi^0, \dots, \phi^n) \mapsto \sum_i \operatorname{sn}_\kappa(r)^{2i+1-\frac{n}{2}} \phi^{2i+1} \wedge dr + \sum_i \operatorname{sn}_\kappa(r)^{2i-\frac{n}{2}} \phi^{2i},$$

or odd forms on U_ε via

$$(\phi^0, \dots, \phi^n) \mapsto \sum_i \operatorname{sn}_\kappa(r)^{2i-\frac{n}{2}} \phi^{2i} \wedge dr + \sum_i \operatorname{sn}_\kappa(r)^{2i+1-\frac{n}{2}} \phi^{2i+1}.$$

We obtain that the even part of the Hodge-Dirac operator associated with $h_0^{\mathcal{E}}$ may be written on U_ε as

$$D_0^{ev} = \frac{\partial}{\partial r} + \frac{1}{\operatorname{sn}_\kappa(r)} B_\kappa(r),$$

where

$$B_\kappa(r) = D_N + \begin{bmatrix} \operatorname{cs}_\kappa(r) c_0 & & \\ & \ddots & \\ & & \operatorname{cs}_\kappa(r) c_n \end{bmatrix}$$

with

$$c_p = (-1)^p(p - \tfrac{n}{2}).$$

Note that $\lim_{r\to 0} B_\kappa(r)$ is independent of $\kappa \in \mathbb{R}$, more precisely we have

$$\lim_{r\to 0} B_\kappa(r) = D_N + \begin{bmatrix} c_0 & & \\ & \ddots & \\ & & c_n \end{bmatrix}.$$

Definition 4.3 (model operator)**.** Let $B = \lim_{r\to 0} B_\kappa(r)$ and

$$P_B^\kappa = \frac{\partial}{\partial r} + \frac{1}{\mathrm{sn}_\kappa(r)} B.$$

If the assumptions A1 and A2 hold, the operator P_B^κ may be used as a model operator for D^{ev} on U_ε, since it captures its essential analytic features. This is made precise by the following lemma:

Lemma 4.4. *If* A1 *and* A2 *hold, then*

$$\mathrm{dom}(D^{\mathrm{ev}})_{\max/\min} = \mathrm{dom}(P_B^\kappa)_{\max/\min}$$

and the graph norms $\|\cdot\|_{D^{\mathrm{ev}}}$ *and* $\|\cdot\|_{P_B^\kappa}$ *are equivalent.*

Proof. Since

$$\frac{B_\kappa(r) - B}{\mathrm{sn}_\kappa(r)} = \frac{\mathrm{cs}_\kappa(r) - 1}{\mathrm{sn}_\kappa(r)} \begin{bmatrix} c_0 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$$

and

$$\lim_{r\to 0} \frac{\mathrm{cs}_\kappa(r) - 1}{\mathrm{sn}_\kappa(r)} = 0\,,$$

we see that D_0^{ev} differs from P_B^κ just by a bounded 0-th order term. If the assumptions A1 and A2 hold, then the L^2-norms defined by using $h^{\mathcal{E}}$, respectively $h_0^{\mathcal{E}}$, are equivalent and D_0^{ev} differs from D^{ev} again by a bounded 0-th order term. This implies the assertion. q.e.d.

4.2. The radial equation. The operator B is symmetric on $\Omega^\bullet_{cp}(N, \mathcal{E})$. Note also that B does not depend on the radial variable $r \in (0, \varepsilon)$ any more. If B is essentially selfadjoint and has discrete spectrum, we use the spectral decomposition of $L^2(\Lambda^\bullet T^*N, \mathcal{E})$ with respect to B to transform the model operator P_B^κ into a family of operators P_b^κ on the interval $(0, \varepsilon)$, where b ranges over the spectrum of B.

For $b \in \mathbb{R}$ let

$$P_b^\kappa = \frac{\partial}{\partial r} + \frac{b}{\mathrm{sn}_\kappa(r)}.$$

We will consider P_b^κ acting on $C^\infty_{\mathrm{cp}}(0,1)$. Furthermore let $P_b = P_b^0$, i.e.,

$$P_b = \frac{\partial}{\partial r} + \frac{b}{r}.$$

It is enough to study the operator P_b in view of the following lemma:

Lemma 4.5. *It is* $\mathrm{dom}(P_b^\kappa)_{\max/\min} = \mathrm{dom}(P_b)_{\max/\min}$ *and the graph norms* $\|\cdot\|_{P_b^\kappa}$ *and* $\|\cdot\|_{P_b}$ *are equivalent.*

Proof. Since $P_b^\kappa - P_b = \varphi(r)b$ with

$$\varphi(r) = \frac{1}{\mathrm{sn}_\kappa(r)} - \frac{1}{r}$$

and

$$\lim_{r\to 0}\varphi(r) = 0,$$

we see that P_b^κ differs from P_b just by a bounded 0-th order term. In the same way as before this implies the assertion. q.e.d.

It is useful to observe that

$$(P_b f)(r) = r^{-b}\tfrac{\partial}{\partial r}(r^b f),$$

therefore $P_b f = 0$ if and only if

$$f(r) = f(1)r^{-b}$$

and $P_b f = g$ if and only if

$$f(r) = f(1)r^{-b} + r^{-b}\int_1^r \varrho^b g(\varrho)d\varrho.$$

For any subinterval $(\delta, 1) \subset (0,1)$ the graph norm of P_b is equivalent to the ordinary H^1-norm, since $\frac{1}{r} \in L^\infty(\delta,1)$. H^1-functions - more generally: $W^{1,1}$-functions - on $(\delta,1)$ are absolutely continuous on $[\delta,1]$, hence differentiable almost everywhere. For absolutely continuous functions the fundamental theorem of calculus holds, i.e., $\varphi \in AC([\delta,1])$ if and only if $\varphi(r) = \varphi(1) + \int_1^r \varphi'(\varrho)d\varrho$ for $r \in [\delta,1]$. Therefore the above integral representation remains valid for $f \in \mathrm{dom}(P_b)_{\max}$ (take $\varphi(r) = r^b f(r)$). It follows in particular that $f \in \mathrm{dom}(P_b)_{\max}$ is continuous on $(0,1)$ and has a continuous boundary value at $r = 1$, i.e., $f \in C^0((0,1])$.

Following [**BS**] we define two integral operators acting on $L^2(0,1)$:

$$(T_{b,1}g)(r) = r^{-b}\int_1^r \varrho^b g(\varrho)d\varrho,$$

where b is arbitrary, and

$$(T_{b,0}g)(r) = r^{-b}\int_0^r \varrho^b g(\varrho)d\varrho,$$

for $b > -\frac{1}{2}$. Note that $b > -\frac{1}{2}$ implies that $r^b \in L^2(0,1)$ and therefore with the Cauchy-Schwarz inequality $\int_0^r \varrho^b g(\varrho)d\varrho < \infty$.

We start from the following estimates in [**BS**], which easily follow from the Cauchy-Schwarz inequality:

Lemma 4.6 (Lemma 2.1 in [**BS**])**.** *For $g \in L^2(0,1)$ and $r \in (0,1)$ we have*

$$|(T_{b,0}g)(r)| \leq r^{\frac{1}{2}}(2b+1)^{-\frac{1}{2}} \left(\int_0^r |g(\varrho)|^2 d\varrho \right)^{\frac{1}{2}}$$

for $b > -\frac{1}{2}$, and

$$|(T_{b,1}g)(r)| \leq \begin{cases} r^{\frac{1}{2}}|2b+1|^{-\frac{1}{2}}\|g\|_{L^2(0,1)}, & b < -\frac{1}{2} \\ r^{\frac{1}{2}}|\log r|^{\frac{1}{2}}\|g\|_{L^2(0,1)}, & b = -\frac{1}{2} \\ r^{-b}(2b+1)^{-\frac{1}{2}}\|g\|_{L^2(0,1)}, & b > -\frac{1}{2} \end{cases},$$

in particular $T_{b,1}g \in L^2(0,1)$ if $b < \frac{1}{2}$.

From this we may derive decay estimates for $f \in \mathrm{dom}(P_b)_{\max}$:

Lemma 4.7 (decay estimates)**.** *Let $f \in \mathrm{dom}(P_b)_{\max}$. Then for $r \in (0,1)$ and with $g = P_b f$ we have*

$$|f(r)| \leq \begin{cases} r^{\frac{1}{2}}(2b+1)^{-\frac{1}{2}} \left(\int_0^r |g(\varrho)|^2\right)^{\frac{1}{2}}, & b \geq \frac{1}{2} \\ r^{-b}|f(1)| + r^{-b}(2b+1)^{-\frac{1}{2}}\|g\|_{L^2(0,1)}, & b \in (-\frac{1}{2}, \frac{1}{2}) \\ r^{\frac{1}{2}}|f(1)| + r^{\frac{1}{2}}|\log r|^{\frac{1}{2}}\|g\|_{L^2(0,1)}, & b = -\frac{1}{2} \\ r^{-b}|f(1)| + r^{\frac{1}{2}}|2b+1|^{-\frac{1}{2}}\|g\|_{L^2(0,1)}, & b < -\frac{1}{2} \end{cases}.$$

Proof. The estimates for $b < \frac{1}{2}$ follow directly from the integral representation

$$f(r) = r^{-b}f(1) + (T_{b,1}g)(r)$$

and the corresponding estimates for $T_{b,1}g$ from the previous lemma. For the case $b \geq \frac{1}{2}$ we observe that for $b \geq \frac{1}{2}$ (in fact already for $b > -\frac{1}{2}$) $r^b \in L^2(0,1)$, hence $r^b g \in L^1(0,1)$ by the Cauchy-Schwarz inequality. This implies that $r^b f$ has its distributional derivative in $L^1(0,1)$ and is therefore absolutely continuous on $[0,1]$. We obtain

$$f(r) = r^{-b}C + (T_{b,0}g)(r)$$

with $C = \lim_{r\to 0} r^b f(r)$. Now $r^{-b} \notin L^2(0,1)$ for $b \geq \frac{1}{2}$, therefore $C = 0$, so the estimate for $T_{b,0}g$ gives the result. q.e.d.

Corollary 4.8. *Let $f \in \mathrm{dom}(P_b)_{\max}$ and $r \in (0,1)$. If $b \notin (-\frac{1}{2}, \frac{1}{2})$, then*

$$|f(r)| \leq C(b) r^{\frac{1}{2}}(1 + |\log r|^{\frac{1}{2}})\|f\|_{P_b},$$

in particular $f \in C^0([0,1])$ with $f(0) = 0$, while if $b \in (-\frac{1}{2}, \frac{1}{2})$, then

$$|f(r)| \leq C(b) r^{-b}\|f\|_{P_b}.$$

Proof. The case $b \geq \frac{1}{2}$ follows directly from the above estimates. For the other cases we again refer to the integral representation

$$f(r) = r^{-b} f(1) + (T_{b,1} g)(r)$$

and observe that $r^{-b} f(1) \in L^2(0,1)$ for $b < \frac{1}{2}$. Therefore the bound on $T_{b,1} g$ translates into a bound on $|f(1)|$ in terms of $\|f\|_{L^2(0,1)}$ and $\|g\|_{L^2(0,1)}$. This plugged into the decay estimates gives the result, which clearly implies that $f(r) = o(1)$ as $r \to 0$ in the first case. q.e.d.

The following statement is implicitly contained in the parametrix construction of Brüning and Seeley, cf. [**BS**]:

Proposition 4.9 (integration by parts)**.** *Let $\varphi \in C^\infty(0,1)$ be a cut-off function with $\varphi \equiv 1$ near 0 and $\varphi \equiv 0$ near 1. For $u \in \operatorname{dom}(P_b)_{\max}$ let $f = \varphi u \in \operatorname{dom}(P_b)_{\max}$, and let $g \in \operatorname{dom}(P_b^t)_{\max}$. Then for $b \notin (-\frac{1}{2}, \frac{1}{2})$ the following holds:*

$$\langle (P_b)_{\max} f, g \rangle_{L^2(0,1)} = \left\langle f, (P_b^t)_{\max} g \right\rangle_{L^2(0,1)}.$$

Proof. With $(P_b)^t = -P_{-b}$ we calculate

$$\begin{aligned}
\langle (P_b)_{\max} f, g \rangle_{L^2(0,1)} &= \int_0^1 \left(\frac{\partial f}{\partial r} + \frac{rf}{b} \right) g \\
&= \lim_{\delta \to 0} \left\{ \int_\delta^1 \left(\frac{\partial f}{\partial r} \right) g + \int_\delta^1 \left(\frac{rf}{b} \right) g \right\} \\
&= \lim_{\delta \to 0} \left\{ [fg]_\delta^1 - \int_\delta^1 f \left(\frac{\partial g}{\partial r} \right) + \int_\delta^1 f \left(\frac{rg}{b} \right) \right\} \\
&= \lim_{\delta \to 0} \{ f(1) g(1) - f(\delta) g(\delta) \} + \left\langle f, (P_b^t)_{\max} g \right\rangle_{L^2(0,1)}.
\end{aligned}$$

Now $f(1) = 0$ and $\lim_{\delta \to 0} f(\delta) g(\delta) = 0$ according to the decay estimates. Therefore

$$\lim_{\delta \to 0} \{ f(1) g(1) - f(\delta) g(\delta) \} = 0$$

and we obtain the result. q.e.d.

This statement becomes wrong, if we allow $b \in (-\frac{1}{2}, \frac{1}{2})$. To see this, let $f(r) = \varphi(r) r^{-b}$ with φ as above and $g(r) = r^b$. Note that $P_b(r \mapsto r^{-b}) = P_b^t(r \mapsto r^b) = 0$, so clearly $f \in \operatorname{dom}(P_b)_{\max}$ and $g \in \operatorname{dom}(P_b^t)_{\max}$. But on the other hand

$$\lim_{\delta \to 0} \{ f(1) g(1) - f(\delta) g(\delta) \} = 0 - \lim_{\delta \to 0} f(\delta) g(\delta) = -1,$$

so we have a boundary contribution.

The preceding result allows us to conclude that we do not have to impose boundary conditions for P_b at 0, if (and only if) $b \notin (-\frac{1}{2}, \frac{1}{2})$.

Corollary 4.10. *Let $\varphi \in C^\infty(0,1)$ be a cut-off function with $\varphi \equiv 1$ near 0 and $\varphi \equiv 0$ near 1. For $u \in \operatorname{dom}(P_b)_{\max}$ let $f = \varphi u \in \operatorname{dom}(P_b)_{\max}$. Then $f \in \operatorname{dom}(P_b)_{\min}$ for $b \notin (-\frac{1}{2}, \frac{1}{2})$.*

Proof. For all $g \in \operatorname{dom}(P_b^t)_{\max}$ we have

$$\langle (P_b)_{\max} f, g\rangle_{L^2(0,1)} = \left\langle f, (P_b^t)_{\max} g\right\rangle_{L^2(0,1)}.$$

This means that $f \in \operatorname{dom}(P_b^t)^*_{\max} = \operatorname{dom}(P_b)_{\min}$. q.e.d.

Let $P_B^\kappa = \frac{\partial}{\partial r} + \operatorname{sn}_\kappa(r)^{-1} B$ acting on $C^\infty_{\mathrm{cp}}((0,1)\times N)$. We will assume that B is essentially selfadjoint on $C^\infty_{\mathrm{cp}}(N)$, i.e., in equivalent terms $B_{\max} = B_{\min}$, since B is symmetric. We will furthermore assume that B has discrete spectrum. Let $\{\Psi_b\}_{b\in\operatorname{spec} B}$ be an orthonormal basis of $L^2(N)$ consisting of eigensections of B, where as usual each eigenvalue is repeated according to its multiplicity. By interior elliptic regularity, the Ψ_b are smooth. There are orthogonal decompositions

$$L^2(N) = \overline{\bigoplus_{b\in\operatorname{spec} B} \mathbb{R}\otimes\langle\Psi_b\rangle}$$

and

$$L^2((0,1)\times N) = \overline{\bigoplus_{b\in\operatorname{spec} B} L^2(0,1)\otimes\langle\Psi_b\rangle},$$

where the closure is taken with respect to the corresponding L^2-norm. For $f \in L^2((0,1)\times N)$ we have an L^2-convergent expansion

$$f = \sum_{b\in\operatorname{spec} B} f_b\otimes\Psi_b,$$

where

$$f_b(r) = \int_N (f(r,x), \Psi_b(x))\,dx.$$

Obviously we have

$$\|f\|^2_{L^2((0,1)\times N)} = \sum_{b\in\operatorname{spec} B} \|f_b\|^2_{L^2(0,1)}.$$

Lemma 4.11. *Let $f, g \in L^2((0,1)\times N)$. Then $P_B^\kappa f = g$ if and only if $P_b^\kappa f_b = g_b$ for all $b \in \operatorname{spec} B$. In particular $f \in \operatorname{dom}(P_B^\kappa)_{\max}$ if and only if $f_b \in \operatorname{dom}(P_b^\kappa)_{\max}$ for all $b \in \operatorname{spec} B$.*

Proof. Let us assume first that $P_B^\kappa f = g$ with $f, g \in L^2((0,1)\times N)$, i.e., by definition $\langle f, P_B^{\kappa,t}\phi\rangle_{L^2} = \langle g, \phi\rangle_{L^2}$ for all $\phi \in C^\infty_{\mathrm{cp}}((0,1)\times N)$.

If $\varphi \in C^\infty_{\mathrm{cp}}(0,1)$ is an arbitrary cut-off function, we claim that this relation remains valid for $\phi = \varphi\Psi_b$ and $b \in \operatorname{spec} B$. Since by assumption $B_{\max} = B_{\min}$ we may choose sequences $\Psi_{b,n} \in C^\infty_{\mathrm{cp}}(N)$, which approximate Ψ_b with respect to $\|\cdot\|_B$. Then it follows immediately that $\varphi\Psi_{b,n}$

approximate $\varphi\Psi_b$ with respect to $\|\cdot\|_{P_B^{\kappa,t}}$. Since $\varphi\Psi_{b,n} \in C^\infty_{\mathrm{cp}}((0,1)\times N)$ we have

$$\langle f, P_B^{\kappa,t}(\varphi\Psi_{b,n})\rangle_{L^2} = \langle g, \varphi\Psi_{b,n}\rangle_{L^2}$$

for all n. By continuity we obtain

$$\langle f, P_B^{\kappa,t}(\varphi\Psi_b)\rangle_{L^2} = \langle g, \varphi\Psi_b\rangle_{L^2},$$

which proves the subclaim. Now the left-hand side of this equation equals

$$\int_0^1 \int_N \Big(f, P_B^{\kappa,t}(\varphi\Psi_b)\Big) = \int_0^1 P_b^{\kappa,t}\varphi \int_N (f,\Psi_b) = \int_0^1 f_b P_b^{\kappa,t}\varphi,$$

whereas the right-hand side is given by

$$\int_0^1 \int_N (g, \varphi\Psi_b) = \int_0^1 \varphi \int_N (g,\Psi_b) = \int_0^1 g_b\varphi.$$

Since φ was arbitrary, this means that $P_b^\kappa f_b = g_b$ for all $b \in \operatorname{spec} B$.

Conversely, if $P_b^\kappa f_b = g_b$ holds for all $b \in \operatorname{spec} B$, we have to show that

$$\langle f, P_B^{\kappa,t}\phi\rangle_{L^2} = \langle g, \phi\rangle_{L^2}$$

is true for all $\phi \in C^\infty_{\mathrm{cp}}((0,1)\times N)$. Now

$$\langle f, P_B^{\kappa,t}\phi\rangle_{L^2} = \sum_{b\in\operatorname{spec} B} \langle f_b, (P_B^{\kappa,t}\phi)_b\rangle_{L^2(0,1)}$$

and

$$\langle g, \phi\rangle_{L^2} = \sum_{b\in\operatorname{spec} B} \langle g_b, \phi_b\rangle_{L^2(0,1)},$$

so we obtain the result, since $(P_B^{\kappa,t}\phi)_b = P_b^{\kappa,t}\phi_b$. q.e.d.

Lemma 4.12. *Let $f \in \operatorname{dom}(P_B^\kappa)_{\max}$. Then $f \in \operatorname{dom}(P_B^\kappa)_{\min}$ if and only if $f_b \in \operatorname{dom}(P_b^\kappa)_{\min}$ for all $b \in \operatorname{spec} B$.*

Proof. The proof essentially uses the observation that $f \in \operatorname{dom}(P_B^\kappa)_{\min}$ if and only if $\langle P_B^\kappa f, g\rangle_{L^2} = \langle f, P_B^{\kappa,t} g\rangle_{L^2}$ for all $g \in \operatorname{dom}(P_B^{\kappa,t})_{\max}$. Now the left-hand side of the equation in question equals

$$\sum_{b\in\operatorname{spec} B} \langle (P_B^\kappa f)_b, g_b\rangle_{L^2(0,1)} = \sum_{b\in\operatorname{spec} B} \langle P_b^\kappa f_b, g_b\rangle_{L^2(0,1)},$$

since $f_b \in \operatorname{dom}(P_b^\kappa)_{\max}$ and $g_b \in \operatorname{dom}(P_b^{\kappa,t})_{\max}$, while the right-hand side is given by

$$\sum_{b\in\operatorname{spec} B} \langle f_b, (P_B^{\kappa,t} g)_b\rangle_{L^2(0,1)} = \sum_{b\in\operatorname{spec} B} \langle f_b, P_b^{\kappa,t} g_b\rangle_{L^2(0,1)}.$$

We get that $f \in \operatorname{dom}(P_B^\kappa)_{\min}$ if and only if

$$\langle P_b^\kappa f_b, g_g\rangle_{L^2(0,1)} = \langle f_b, P_B^{\kappa,t} g_b\rangle_{L^2(0,1)}$$

for all $g_b \in \operatorname{dom}(P_b^{\kappa,t})_{\max}$, i.e., that $f_b \in \operatorname{dom}(P_b^{\kappa})_{\min}$ for all $b \in \operatorname{spec} B$.
q.e.d.

The following lemma will turn out to be decisive in the question of essential selfadjointness of D on cone-manifolds.

Lemma 4.13. *Let $\varphi \in C^\infty(0,1)$ be a cut-off function with $\varphi \equiv 1$ near 0 and $\varphi \equiv 0$ near 1. For $u \in \operatorname{dom}(P_B^{\kappa})_{\max}$ let $f = \varphi u \in \operatorname{dom}(P_B^{\kappa})_{\max}$. Then $f \in \operatorname{dom}(P_B^{\kappa})_{\min}$ if $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$.*

Proof. This follows from the above discussion together with Corollary 4.10 and Lemma 4.5. q.e.d.

In the following we derive certain compactness properties which will be relevant for the question of discreteness of $D(d_{\max})$ and $\Delta(d_{\max})$ on cone-manifolds.

Lemma 4.14. *The embedding $\operatorname{dom}(P_b)_{\max} \hookrightarrow L^2(0,1)$ is compact for all $b \in \mathbb{R}$.*

Proof. Given a sequence $f_n \in \operatorname{dom}(P_b)_{\max}$ with bound $\|f_n\|_{P_b} \leq C$ independent of n, we have to extract a subsequence convergent in $L^2(0,1)$. On any subinterval $(\delta,1) \subset (0,1)$ the graph norm of P_b is equivalent to the ordinary H^1-norm, since $\frac{1}{r} \in L^\infty(\delta,1)$. Recall that the embedding $H^1(\delta,1) \hookrightarrow C^0([\delta,1])$ is compact by Rellich's theorem. Therefore we obtain a locally uniformly convergent subsequence, which we again denote by f_n. As a consequence of the decay estimates (cf. Corollary 4.8) we have

$$|f_n(r)| \leq C(b) r^{\frac{1}{2}}(1 + |\log r|^{\frac{1}{2}}) \|f_n\|_{P_b} \leq C'(b) r^{\frac{1}{2}}(1 + |\log r|^{\frac{1}{2}})$$

if $b \notin (-\frac{1}{2}, \frac{1}{2})$, and

$$|f_n(r)| \leq C(b) r^{-b} \|f_n\|_{P_b} \leq C'(b) r^{-b}$$

if $b \in (-\frac{1}{2}, \frac{1}{2})$. The functions $r^{\frac{1}{2}}(1+|\log r|^{\frac{1}{2}})$ and r^{-b} with $b < \frac{1}{2}$ are certainly in $L^2(0,1)$. In any case we conclude with Lebesgue's dominated convergence theorem, that f_n is convergent in $L^2(0,1)$. q.e.d.

Corollary 4.15. *The embedding $\operatorname{dom}(P_b^{\kappa})_{\max} \hookrightarrow L^2(0,1)$ is compact for all $b \in \mathbb{R}$.*

Proof. This is a direct consequence of the previous lemma in view of Lemma 4.5. q.e.d.

For $b \in \mathbb{R}$ we define

$$\widetilde{P}_b^{\kappa} = \begin{cases} (P_b^{\kappa})_{\max}, & b \in (-\frac{1}{2}, \frac{1}{2}) \\ (P_b^{\kappa})_{\min}, & b \notin (-\frac{1}{2}, \frac{1}{2}) \end{cases} .$$

This determines a closed extension $\widetilde{P}^{\kappa}_{B}$ of P^{κ}_{B} such that

$$\operatorname{dom} \widetilde{P}^{\kappa}_{B} = \overline{\bigoplus_{b \in \operatorname{spec} B} \operatorname{dom} \widetilde{P}^{\kappa}_{b} \otimes \Psi_b},$$

where the closure is taken with respect to the graph norm $\|\cdot\|_{P^{\kappa}_{B}}$. Note in particular that $\widetilde{P}^{\kappa}_{B} = (P^{\kappa}_{B})_{\min}$ if $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$.

Lemma 4.16. *The embedding* $\operatorname{dom} \widetilde{P}^{\kappa}_{B} \hookrightarrow L^2((0,1) \times N)$ *is compact.*

Proof. The previous lemma implies that for all $b \in \operatorname{spec} B$ the embedding $(L^{\kappa}_{b})_{\max} : \operatorname{dom}(P^{\kappa}_{b})_{\max} \hookrightarrow L^2(0,1)$ is compact. We derive an upper bound for the operator norm of $(L^{\kappa}_{b})_{\min} : \operatorname{dom}(P^{\kappa}_{b})_{\min} \hookrightarrow L^2(0,1)$, where $\operatorname{dom}(P^{\kappa}_{b})_{\min}$ is equipped with the graph norm $\|\cdot\|_{P^{\kappa}_{b}}$. For $f \in C^{\infty}_{\mathrm{cp}}(0,1)$ we have

$$P^{\kappa,t}_{b} P^{\kappa}_{b} f = -\frac{\partial^2 f}{\partial r^2} + \frac{b(b + \mathrm{cs}_{\kappa}(r)) f}{\mathrm{sn}^2_{\kappa}(r)},$$

and therefore integration by parts applied twice yields

$$\begin{aligned} \|P^{\kappa}_{b} f\|^2_{L^2(0,1)} &= \left\langle P^{\kappa,t}_{b} P^{\kappa}_{b} f, f \right\rangle_{L^2(0,1)} \\ &= \int_0^1 \left|\frac{\partial f}{\partial r}\right|^2 + \int_0^1 \frac{b(b+\mathrm{cs}_{\kappa}(r)) f^2}{\mathrm{sn}^2_{\kappa}(r)} \\ &\geq C_{\kappa}(b)\, \|f\|^2_{L^2(0,1)}, \end{aligned}$$

where $C_{\kappa}(b) \nearrow \infty$ as $|b| \to \infty$. Since $C^{\infty}_{\mathrm{cp}}(0,1)$ is dense in $\operatorname{dom}(P^{\kappa}_{b})_{\min}$ we obtain

$$\begin{aligned} \|(L^{\kappa}_{b})_{\min}\|^2 &= \sup_{f \in C^{\infty}_{\mathrm{cp}}(0,1)\setminus\{0\}} \frac{\|f\|^2}{\|f\|^2 + \|P^{\kappa}_{b} f\|^2} \\ &\leq \frac{1}{1 + C_{\kappa}(b)}, \end{aligned}$$

i.e., for large eigenvalues of B the operator norm of $(L^{\kappa}_{b})_{\min}$ is uniformly small.

Let L denote the embedding $\operatorname{dom} \widetilde{P}^{\kappa}_{B} \hookrightarrow L^2((0,1) \times N)$. Furthermore for $a > 0$ let $\pi^{<a}$ denote the projection onto the eigenspaces corresponding to eigenvalues b with $|b| < a$. Since there are only finitely many such eigenvalues,

$$L^{<a} = \pi^{<a} \circ L$$

is a compact operator and by the above estimates

$$\|L - L^{<a}\|^2 = \sup_{|b| \geq a} \|(L^{\kappa}_{b})_{\min}\|^2 \leq \frac{1}{1 + C_{\kappa}(a)},$$

for a large enough. In particular, for $a \to \infty$ we obtain that L is a limit of compact operators with respect to the operator norm and is therefore itself compact. q.e.d.

4.3. Spectral properties of cone-surfaces. Let S now be a cone-surface and $(\mathcal{F}, \nabla^{\mathcal{F}})$ a flat vector-bundle over $N = \operatorname{int} S$ equipped with a metric $h^{\mathcal{F}}$. Particular attention will be paid to the spherical cone-surfaces $\mathbf{S}^2(\alpha, \beta, \gamma)$ and $\mathbf{S}^2(\alpha, \alpha)$, which appear as links of singular points in a 3-dimensional cone-manifold.

We wish to investigate spectral properties of the operators $D(d_{\max})$ and $\Delta(d_{\max})$ by separation of variables. In view of Lemma 4.4 and Lemma 4.13 the following requirements are natural:

Definition 4.17. Let S be a cone-surface and $(\mathcal{F}, \nabla^{\mathcal{F}})$ a flat vector-bundle over $N = \operatorname{int} S$ equipped with a metric $h^{\mathcal{F}}$. If $\{x_i\}$ are the cone-points, we call $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ *cone-admissible* if for all i:

1) Assumptions A1 and A2 hold for $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ restricted to $U_\varepsilon(x_i)$, hence the model operator $P^\kappa_{B_i}$ is defined.
2) $\operatorname{spec} B_i \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ holds.

Remark 4.18. Since the cross-section S^1_α is compact in this case, it would be enough to require A1 here, cf. Remark 4.1.

We will see in the following that Definition 4.17 implicitly contains restrictions on the cone-angles of S and the holonomy of the flat bundle $(\mathcal{F}, \nabla^{\mathcal{F}})$ around the cone-points:

Let $S^1_\alpha = \mathbb{R}/\alpha\mathbb{Z}$ be the circle of length α and let $\operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha$ be the ε-truncated κ-cone over S^1_α, i.e.,

$$\operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha = (0, \varepsilon) \times S^1_\alpha$$

with metric

$$dr^2 + \operatorname{sn}^2_\kappa(r) d\theta^2$$

where $r \in (0, \varepsilon)$ and $\theta \in \mathbb{R}/\alpha\mathbb{Z}$. Recall that if x is a cone-point, the smooth part of the ε-ball around x will be isometric to

$$U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha.$$

In this situation the model operator for the even part of the Hodge-Dirac operator on the cone is given by

$$P^\kappa_B = \frac{\partial}{\partial r} + \frac{1}{\operatorname{sn}_\kappa(r)} B$$

with

$$B = D_{S^1_\alpha} + \begin{bmatrix} -\frac{1}{2} & \\ & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & d^t_{S^1_\alpha} \\ d_{S^1_\alpha} & -\frac{1}{2} \end{bmatrix}.$$

We determine the spectrum of the operator B, but let us discuss the case with trivial coefficient bundle first. If we identify functions and 1-forms on S^1_α via

$$\begin{aligned} C^\infty(S^1_\alpha) &\longrightarrow \Omega^1(S^1_\alpha) \\ g &\longmapsto g \cdot d\theta, \end{aligned}$$

we may write

$$D_{S^1_\alpha} = \begin{bmatrix} 0 & -\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\theta} & 0 \end{bmatrix}.$$

It is easily verified that

$$\operatorname{spec} D_{S^1_\alpha} = \left\{ \frac{2\pi n}{\alpha}, n \in \mathbb{Z} \right\},$$

and therefore we obtain

$$\operatorname{spec} B = \left\{ -\frac{1}{2} + \frac{2\pi n}{\alpha}, n \in \mathbb{Z} \right\}.$$

We see that $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ if $\alpha \leq 2\pi$ in the case of trivial coefficients.

Let us now add a flat bundle to the situation. Let $\mathbb{C}(a)$ be the flat $U(1)$-bundle over S^1_α with holonomy e^{ia}, $a \in \mathbb{R}$. Without loss of generality we may assume that $a \in [0, 2\pi)$. Note that the bundles $\mathbb{C}(a)$ are topologically trivial. Any unitarily flat bundle on S^1_α decomposes as a direct sum of these. A flat connection is given by

$$\nabla^{\mathbb{C}(a)} = d - i\frac{a}{\alpha} d\theta.$$

The associated Hodge-Dirac operator may be written as

$$D_{S^1_\alpha, \mathbb{C}(a)} = \begin{bmatrix} 0 & -\frac{\partial}{\partial\theta} + i\frac{a}{\alpha} \\ \frac{\partial}{\partial\theta} - i\frac{a}{\alpha} & 0 \end{bmatrix}.$$

We obtain

$$\operatorname{spec} D_{S^1_\alpha, \mathbb{C}(a)} = \left\{ \pm \left| \frac{2\pi n - a}{\alpha} \right|, n \in \mathbb{Z} \right\},$$

and therefore

$$\operatorname{spec} B = \left\{ -\frac{1}{2} \pm \left| \frac{2\pi n - a}{\alpha} \right|, n \in \mathbb{Z} \right\}.$$

We see that $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ if either $a = 0$ and $\alpha \leq 2\pi$ or $\alpha \leq a \leq 2\pi - \alpha$. In the latter case we must in particular have that $\alpha \leq \pi$.

Remark 4.19. The previous discussion shows that if S has cone-angles $\leq \pi$ and $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is an orthogonally flat bundle which decomposes locally around the cone-points as a direct sum of trivial bundles $\mathbb{R}$ and bundles of type $\mathbb{C}(a)$ with $\alpha \leq a \leq 2\pi - \alpha$, then $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ will be cone-admissible in the sense of Definition 4.17.

4.3.1. Discreteness. In this section we investigate discreteness of the operators $D(d_{\max})$ and $\Delta(d_{\max})$ on a cone-surface. Recall that a selfadjoint operator A is called discrete if its spectrum is discrete, i.e., if $\operatorname{spec} A$ consists of a discrete set of eigenvalues with finite multiplicities. A necessary and sufficient condition for A to be discrete is the compactness of the embedding $\operatorname{dom} A \hookrightarrow L^2$, where $\operatorname{dom} A$ is equipped with the graph norm $\|\cdot\|_A$.

For simplicity we state the results concerning discreteness under the stronger hypothesis that $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible, though we do not need the assumption $\operatorname{spec} B_i \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ for $i \in \{1, \dots, k\}$ as far as discreteness is concerned.

Proposition 4.20. *The embedding* $\operatorname{dom} D^{\mathrm{ev}}_{\max} \hookrightarrow L^2(\Lambda^{\mathrm{ev}} T^* N \otimes \mathcal{F})$ *is compact if* $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ *is cone-admissible.*

Proof. We construct a partition of unity on S in the following way: Letting $\{x_1, \dots, x_k\}$ be the set of cone-points, we choose $\varepsilon > 0$ such that the $U_\varepsilon(x_i)$ are disjoint. We choose cut-off functions φ_i supported inside $U_\varepsilon(x_i)$ with $\varphi_i = \varphi_i(r)$ and $\varphi_i \equiv 1$ near the cone-point x_i. Then we define $\varphi_{\mathrm{int}} = 1 - \sum_{i=1}^k \varphi_i$. Let $u_n \in \operatorname{dom} D^{\mathrm{ev}}_{\max}$ be a sequence with $\|u_n\|_{D^{\mathrm{ev}}} \leq C$.

We claim that $\varphi_{\mathrm{int}} u_n$ has a subsequence which is convergent in L^2: Let $\Omega \subset N$ be a relatively compact domain with smooth boundary, such that $\operatorname{supp} \varphi_{\mathrm{int}} \subset \Omega$. Then by the usual elliptic regularity results, $\varphi_{\mathrm{int}} u_n \in H^1_0(\Omega)$. Furthermore by the standard elliptic estimate

$$\|\varphi u_n\|^2_{H^1(\Omega)} \leq C \left(\|\varphi u_n\|^2_{L^2(\Omega)} + \|D^{\mathrm{ev}} \varphi u_n\|^2_{L^2(\Omega)} \right) = C \, \|\varphi u_n\|^2_{D^{\mathrm{ev}}_\Omega}.$$

Now by Rellich's theorem $H^1_0(\Omega)$ embeds into $L^2(\Omega)$ compactly, which proves the subclaim.

Thus we are reduced to a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha$, i.e., given a sequence $f_n = \varphi u_n$ with $\|f_n\|_{P^\kappa_B} \leq C$, we have to extract a subsequence convergent in $L^2((0,1) \times S^1_\alpha)$. The operator B is essentially selfadjoint and discrete, since the cross-section of the cone is nonsingular in this case. Therefore the discussion from the last section applies. It is a consequence of Corollary 4.10 that $\varphi u_n \in \operatorname{dom} \widetilde{P}^\kappa_B$, therefore Lemma 4.16 yields the result. q.e.d.

As a consequence we obtain that strong Hodge-decomposition holds for the $d_{\max}$-complex on a cone-surface if $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible. Here we remind the reader of Theorem 3.8 and the remark thereafter.

We summarize our results concerning Hodge-decomposition on a cone-surface in the following statement:

Theorem 4.21 (Hodge-theorem for cone-surfaces)**.** *If S is a cone-surface and $(\mathcal{F}, \nabla^{\mathcal{F}})$ a flat vector-bundle over $N = \operatorname{int} S$ together with* *a metric $h^{\mathcal{F}}$ such that $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible, then there is an orthogonal decomposition*

$$L^2(\Lambda^i T^*N \otimes \mathcal{F}) = \mathcal{H}^i_{\max} \oplus \operatorname{im} d^{i-1}_{\max} \oplus \operatorname{im}(d^i)^t_{\min},$$

and the map $\iota : \mathcal{H}^i_{\max} \to H^i_{\max}$ is an isomorphism. Furthermore, the inclusion $\Omega^i_{L^2}(N, \mathcal{F}) \to \operatorname{dom} d^i_{\max}$ induces an isomorphism $H^i_{L^2}(N, \mathcal{F}) \cong H^i_{\max}$.

Since $D^{\mathrm{odd}} = (D^{\mathrm{ev}})^t$, we get by the same arguments that the embeddings $\operatorname{dom} D^{\mathrm{odd}}_{\max} \hookrightarrow L^2(\Lambda^{\mathrm{odd}} T^*N \otimes \mathcal{F})$ and $\operatorname{dom} D_{\max} \hookrightarrow L^2(\Lambda^{\bullet} T^*N \otimes \mathcal{F})$ are again compact.

Proposition 4.22. *The operators $D(d_{\max})$ and $\Delta(d_{\max})$ are discrete on a cone-surface if $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible.*

Proof. Since dom $D\,(d_{\max})$ and dom $\Delta\,(d_{\max})$ are continuously contained in $\operatorname{dom} D_{\max}$, this follows from compactness of the embedding $\operatorname{dom} D_{\max} \hookrightarrow L^2(\Lambda^{\bullet} T^*N \otimes \mathcal{F})$. q.e.d.

4.3.2. Selfadjointness. In this section we establish essential selfadjointness of the Hodge-Dirac operator D on a cone-surface if $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible. In contrast to the the previous section, we will now make strong use of the assumption $\operatorname{spec} B_i \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ for $i \in \{1, \dots, k\}$ in Definition 4.17 since we wish to apply Lemma 4.13.

Proposition 4.23. *$D^{\mathrm{ev}}_{\max} = D^{\mathrm{ev}}_{\min}$ on a cone-surface if $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible.*

Proof. Given $u \in \operatorname{dom} D^{\mathrm{ev}}_{\max}$ we have to show that $u \in \operatorname{dom} D^{\mathrm{ev}}_{\min}$. We choose a partition of unity on S as in the proof of Proposition 4.20.

We claim that $\varphi_{\mathrm{int}} u \in \operatorname{dom} D^{\mathrm{ev}}_{\min}$: As we have already observed in the proof of Proposition 4.20, if $\Omega \subset N$ is a relatively compact domain with smooth boundary such that $\operatorname{supp} \varphi_{\mathrm{int}} \subset \Omega$, then $\varphi_{\mathrm{int}} u \in H^1_0(\Omega)$. Now $C^\infty_{\mathrm{cp}}(\Omega)$ is dense in $H^1_0(\Omega)$, therefore we find a sequence $f_n \in C^\infty_{\mathrm{cp}}(\Omega)$ such that f_n approximates $f = \varphi_{\mathrm{int}} u$ with respect to the H^1-norm. But since D^{ev} maps $H^1(\Omega)$ continuously to $L^2(\Omega)$, f_n approximates f also with respect to the graph norm of D^{ev}, which proves the subclaim.

It remains to prove that $\varphi_i u \in \operatorname{dom} D^{\mathrm{ev}}_{\min}$ for $i \in \{1, \dots, k\}$. But here we are again in a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} S^1_\alpha$. It is therefore sufficient to show that $f = \varphi u \in \operatorname{dom}(P^\kappa_B)_{\min}$ for $u \in \operatorname{dom}(P^\kappa_B)_{\max}$ and φ a cut-off function of the above type. Now since $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible, $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ will be satisfied. Then Lemma 4.13 implies that $f \in \operatorname{dom}(P^\kappa_B)_{\min}$, hence in $\operatorname{dom} D^{\mathrm{ev}}_{\min}$. q.e.d.

Corollary 4.24. *The operator D is essentially selfadjoint on a cone-surface if $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible.*

Proof. We have

$$D = \begin{bmatrix} 0 & (D^{\mathrm{ev}})^t \\ D^{\mathrm{ev}} & 0 \end{bmatrix}$$

considered as an operator

$$\Omega^{\mathrm{ev}}_{\mathrm{cp}}(N, \mathcal{F}) \oplus \Omega^{\mathrm{odd}}_{\mathrm{cp}}(N, \mathcal{F}) \longrightarrow \Omega^{\mathrm{ev}}_{\mathrm{cp}}(N, \mathcal{F}) \oplus \Omega^{\mathrm{odd}}_{\mathrm{cp}}(N, \mathcal{F})$$

and therefore

$$D_{\min} = \begin{bmatrix} 0 & (D^{\mathrm{ev}})^t_{\min} \\ D^{\mathrm{ev}}_{\min} & 0 \end{bmatrix}$$

and

$$D_{\max} = \begin{bmatrix} 0 & (D^{\mathrm{ev}})^t_{\max} \\ D^{\mathrm{ev}}_{\max} & 0 \end{bmatrix}.$$

This shows that $D_{\max} = D_{\min}$, i.e., D is essentially selfadjoint. q.e.d.

Corollary 4.25. *$\Delta_F = \Delta(d_{\max})$ on a cone-surface if $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is cone-admissible.*

Proof. This follows from essential selfadjointness of D together with Corollary 3.5. q.e.d.

4.3.3. The first eigenvalue. Let λ_1 be the smallest positive eigenvalue of $\Delta^0(d_{\max})$ on the smooth part of $\mathbf{S}^2(\alpha, \beta, \gamma)$ (resp. $\mathbf{S}^2(\alpha, \alpha)$) with coefficients in a flat vector-bundle $(\mathcal{F}, \nabla^{\mathcal{F}})$. Here we will derive a lower bound on λ_1, which will be sufficient for later purposes. Comparison with the smooth case suggests that this bound might not be optimal.

Proposition 4.26. *Let S be either $\mathbf{S}^2(\alpha, \beta, \gamma)$ or $\mathbf{S}^2(\alpha, \alpha)$ and $(\mathcal{F}, \nabla^{\mathcal{F}})$ a flat vector-bundle over $N = \operatorname{int} S$ equipped with a metric $h^{\mathcal{F}}$. If $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is orthogonally flat and cone-admissible, then $\mathcal{H}^1_{\max} = 0$. Moreover, under the same hypothesis, if λ_1 denotes the smallest positive eigenvalue of $\Delta^0(d_{\max})$, then $\lambda_1 \geq 1$.*

Proof. Since $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is orthogonally flat, we may apply the standard Weitzenböck formula on $\mathcal{F}$-valued 1-forms

$$\Delta\omega = \nabla^t \nabla \omega + (\mathrm{Ric} \otimes \mathrm{id})\omega,$$

where the action of the Ricci tensor on a scalar-valued 1-form α is determined by the relation

$$g(\mathrm{Ric}(\alpha), \beta) = \mathrm{Ric}(\alpha, \beta)$$

for all $\beta \in \Omega^1(N, \mathbb{R})$. In two dimensions the Ricci tensor of a spherical metric (i.e., of constant curvature $\kappa = 1$) is given by

$$\mathrm{Ric}(\cdot\,, \cdot) = g(\cdot\,, \cdot),$$

so we end up with

$$\Delta\omega = \nabla^t \nabla \omega + \omega.$$

For $\omega \in \Omega^1_{\mathrm{cp}}(N,\mathcal{F})$ integration by parts yields

$$\begin{aligned}\int_N (\Delta\omega,\omega) &= \int_N (\nabla^t\nabla\omega,\omega) + \int_N |\omega|^2 \\ &= \int_N |\nabla\omega|^2 + \int_N |\omega|^2 \geq \int_N |\omega|^2.\end{aligned}$$

This means we have a lower bound for Δ on $\Omega^1_{\mathrm{cp}}(N,\mathcal{F})$:

$$\langle \Delta\omega,\omega\rangle_{L^2} \geq \|\omega\|^2_{L^2}.$$

Since $\Delta(d_{\max}) = \Delta_F$ if $(\mathcal{F},\nabla^{\mathcal{F}},h^{\mathcal{F}})$ is cone-admissible and the Friedrichs extension preserves lower bounds, we obtain

$$\langle \Delta(d_{\max})\omega,\omega\rangle_{L^2} \geq \|\omega\|^2_{L^2}$$

for all $\omega \in \operatorname{dom}\Delta^1(d_{\max})$. This proves the first part of the assertion. Now for $f \in E_{\lambda_1}$, the λ_1-eigenspace of $\Delta^0(d_{\max})$, $f \neq 0$, let $\omega = d_{\max} f$. Then $w \neq 0$ and $\Delta^1(d_{\max})\omega = d_{\max} d^t_{\min} d_{\max} f = \lambda_1 \omega$. This yields the estimate $\lambda_1 \geq 1$. q.e.d.

4.4. Spectral properties of cone-3-manifolds. Let in the following C be a cone-3-manifold and $(\mathcal{E},\nabla^{\mathcal{E}})$ a flat vector-bundle over $M = \operatorname{int} C$ equipped with a metric $h^{\mathcal{E}}$. Again we wish to investigate spectral properties of the operators $D(d_{\max})$ and $\Delta(d_{\max})$ by separation of variables. We require:

Definition 4.27. Let C be a 3-dimensional cone-manifold and $(\mathcal{E},\nabla^{\mathcal{E}})$ a flat vector-bundle over $M = \operatorname{int} C$ equipped with a metric $h^{\mathcal{E}}$. We call $(\mathcal{E},\nabla^{\mathcal{E}},h^{\mathcal{E}})$ *cone-admissible* if for all $x \in \Sigma$:

1) Assumptions A1 and A2 hold for $(\mathcal{E},\nabla^{\mathcal{E}},h^{\mathcal{E}})$ restricted to $U_\varepsilon(x)$, hence the model operator $P^{\kappa}_{B_x}$ is defined.
2) B_x is essentially selfadjoint and $\operatorname{spec} B_x \cap (-\frac{1}{2},\frac{1}{2}) = \varnothing$ holds.

Remark 4.28. If we compare this definition with the cone-surface case, we note that a new issue arises, namely that we have to include essential selfadjointness of the operator B on the cross-section of the model cone into the definition. This issue was not present in the cone-surface case, since there the cross-section of the model cone was compact.

Let $x \in \Sigma$ be a singular point. For the local analysis around x we consider two cases:

1) x is a vertex
2) x lies on a singular edge.

In the first case, the smooth part of the ε-ball around x will be isometric to

$$U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} \operatorname{int} \mathbf{S}^2(\alpha,\beta,\gamma),$$

and in the second case to

$$U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} \operatorname{int} \mathbf{S}^2(\alpha,\alpha).$$

We treat the two cases simultaneously. Let N denote either $\operatorname{int} \mathbf{S}^2(\alpha,\beta,\gamma)$ or $\operatorname{int} \mathbf{S}^2(\alpha,\alpha)$ in the following.

Suppose that $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ satisfies assumptions A1 and A2 on U_ε, in particular that $h_0^{\mathcal{E}} = \lim_{r\to 0} h^{\mathcal{E}}(r)$ exists and is parallel with respect to $\nabla^{\mathcal{E}}$. Recall that the model operator for the even part of the Hodge-Dirac operator on the κ-cone with two-dimensional cross-section N is given by

$$P_B^\kappa = \frac{\partial}{\partial r} + \frac{1}{\operatorname{sn}_\kappa(r)} B$$

with

$$B = D_N + \begin{bmatrix} -1 & & \\ & 0 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} -1 & d_N^t & \\ d_N & 0 & d_N^t \\ & d_N & 1 \end{bmatrix}.$$

Let us now assume that $(\mathcal{E}, \nabla^{\mathcal{E}}, h_0^E)$ restricted to the 2-dimensional cross-section N is cone-admissible. Then D_N and in particular the operator B will be essentially selfadjoint. The Hodge-$\star$-operator defines a linear isometry

$$\star : L^2(\Lambda^p T^* N \otimes \mathcal{E}) \longrightarrow L^2(\Lambda^{n-p} T^* N \otimes \mathcal{E}),$$

where in this case $n = 2$. Note furthermore that these two conditions together imply that $\mathcal{H}^1_{\max} = 0$ via Proposition 4.26.

We determine the spectrum of B in the following. For $\lambda \geq 0$ let E_λ be the λ-eigenspace of

$$\Delta(d_{\max}) = \Delta^0(d_{\max}) \oplus \Delta^1(d_{\max}) \oplus \Delta^2(d_{\max}).$$

Let $\lambda > 0$ be an eigenvalue and f_λ a corresponding eigensection of $\Delta^0(d_{\max})$ with $\|f_\lambda\|_{L^2}{=}1$. Then

$$\left\{ f_\lambda, \frac{1}{\sqrt{\lambda}} df_\lambda, \frac{1}{\sqrt{\lambda}} \star df_\lambda, \star f_\lambda \right\}$$

form an orthonormal basis of a D_N-invariant subspace $E_{f_\lambda} \subset E_\lambda$. It is a consequence of Theorem 4.21 that the E_{f_λ} provide an orthogonal decomposition of E_λ for f_λ pairwise orthogonal. With respect to the given basis of E_{f_λ} we have

$$D_N|_{E_{f_\lambda}} = \begin{bmatrix} 0 & \sqrt{\lambda} & & \\ \sqrt{\lambda} & 0 & & \\ & & 0 & -\sqrt{\lambda} \\ & & -\sqrt{\lambda} & 0 \end{bmatrix}$$

and correspondingly

$$B|_{E_{f_\lambda}} = \begin{bmatrix} -1 & \sqrt{\lambda} & & \\ \sqrt{\lambda} & 0 & & \\ & & 0 & -\sqrt{\lambda} \\ & & -\sqrt{\lambda} & 1 \end{bmatrix}.$$

For $\lambda = 0$ we observe that if there exists $f_0 \in \mathcal{H}^0_{\max}$ with $\|f_0\|_{L^2} = 1$, then $\{f_0, f_0 \otimes dvol\}$ form an orthonormal basis of $E_{f_0} \subset E_0 = \mathcal{H}^0_{\max} \oplus \mathcal{H}^2_{\max}$ and we obtain

$$B|_{E_{f_0}} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}.$$

Note that possibly $E_0 = 0$. Therefore we obtain for the spectrum of B

$$\operatorname{spec} B \subset \{-1, 1\} \cup \left\{ \pm\frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda} \,\middle|\, \lambda \in \operatorname{spec} \Delta^0(d_{\max}), \lambda > 0 \right\}.$$

We see that $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ if $\lambda_1 \geq \frac{3}{4}$, which we can guarantee under the given conditions by means of Proposition 4.26.

Remark 4.29. As a consequence of the previous discussion we observe that a sufficient condition for $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ to be cone-admissible in the sense of Definition 4.27 is that assumptions A1 and A2 hold and the restriction of $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}}_0)$ to the link S_x of a singular point x is cone-admissible in the sense of Definition 4.17 for all $x \in \Sigma$.

4.4.1. Discreteness. In this section we investigate discreteness of the operators $D(d_{\max})$ and $\Delta(d_{\max})$ on a 3-dimensional cone-manifold.

For simplicity we state the results concerning discreteness under the stronger hypothesis that $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible, though we do not need the assumption $\operatorname{spec} B_x \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ for $x \in \Sigma$ as far as discreteness is concerned.

Proposition 4.30. *The embedding* $\operatorname{dom} D^{\mathrm{ev}}_{\max} \hookrightarrow L^2(\Lambda^{\mathrm{ev}} T^*M \otimes \mathcal{E})$ *is compact if* $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ *is cone-admissible.*

Proof. Since Σ is compact we find finitely many $x_i \in \Sigma$ such that the $B_\varepsilon(x_i)$ cover Σ. Then $\{M, B_\varepsilon(x_i)\}$ is a finite open cover of C. We fix a partition of unity $\{\varphi_{\mathrm{int}}, \varphi_i\}$ subordinate to this cover. Let $U_\varepsilon(x_i) = B_\varepsilon(x_i) \cap M$.

Now let $u_n \in \operatorname{dom} D^{\mathrm{ev}}_{\max}$ be a sequence with $\|u_n\|_{D^{\mathrm{ev}}} \leq C$. Clearly $\varphi_{\mathrm{int}} u_n$ has a subsequence convergent in L^2: This follows in the same way as in the cone-surface case (cf. the proof of Proposition 4.20).

On the other hand $U_\varepsilon(x)$ will be isometric to $\operatorname{cone}_{\kappa,(0,\varepsilon)} \operatorname{int} \mathbf{S}^2(\alpha, \beta, \gamma)$ if x is a vertex or $\operatorname{cone}_{\kappa,(0,\varepsilon)} \operatorname{int} \mathbf{S}^2(\alpha, \alpha)$ if x is an edge point. Thus we are reduced to a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} N$. Without loss of generality we may assume that $\varphi = \varphi(r)$ if r is the radial variable and $\varphi(r) = 1$ for r small. If this is not the case we just replace φ by a second cut-off function $\widetilde{\varphi} \in C^\infty_{\mathrm{cp}}(U_\varepsilon(x))$ which satisfies these assumptions and in addition $\widetilde{\varphi} = 1$ near $\operatorname{supp} \varphi$, and we replace u_n by $\widetilde{u}_n = \varphi u_n$. Since $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible, the operator B will be essentially selfadjoint. B will have discrete spectrum as a consequence of Proposition 4.20. As in the cone-surface case we obtain that $\varphi u_n \in \operatorname{dom} \widetilde{P}^\kappa_B$. We may now use Lemma 4.16 to conclude the result. q.e.d.

We obtain that strong Hodge-decomposition holds for the $d_{\max}$-complex on a cone-3-manifold if $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible.

We summarize our results concerning Hodge-decomposition on a cone-3-manifold in the following statement:

Theorem 4.31 (Hodge-theorem for cone-3-manifolds)**.** *If C is a cone-3-manifold and $(\mathcal{E}, \nabla^{\mathcal{E}})$ a flat vector-bundle over $M = \operatorname{int} C$ together with a metric $h^{\mathcal{E}}$ such that $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible, then there is an orthogonal decomposition*

$$L^2(\Lambda^i T^* M \otimes \mathcal{E}) = \mathcal{H}^i_{\max} \oplus \operatorname{im} d^{i-1}_{\max} \oplus \operatorname{im}(d^i)^t_{\min},$$

and the map $\iota : \mathcal{H}^i_{\max} \to H^i_{\max}$ is an isomorphism. Furthermore, the inclusion $\Omega^i_{L^2}(N, \mathcal{F}) \to \operatorname{dom} d^i_{\max}$ induces an isomorphism $H^i_{L^2}(N, \mathcal{F}) \cong H^i_{\max}$.

Since $D^{\mathrm{odd}} = (D^{\mathrm{ev}})^t$, we get by the same arguments that the embeddings $\operatorname{dom} D^{\mathrm{odd}}_{\max} \hookrightarrow L^2(\Lambda^{\mathrm{odd}} T^* M \otimes \mathcal{E})$ and $\operatorname{dom} D_{\max} \hookrightarrow L^2(\Lambda^{\bullet} T^* M \otimes \mathcal{E})$ are again compact.

Proposition 4.32. *The operators $D(d_{\max})$ and $\Delta(d_{\max})$ are discrete on a cone-3-manifold if $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible.*

Proof. Since dom $D\,(d_{\max})$ and dom $\Delta\,(d_{\max})$ are continuously contained in $\operatorname{dom} D_{\max}$, this follows from compactness of the embedding $\operatorname{dom} D_{\max} \hookrightarrow L^2(\Lambda^{\bullet} T^* M \otimes \mathcal{E})$. q.e.d.

4.4.2. Selfadjointness. In this section we establish essential selfadjointness of the Hodge–Dirac operator D on a cone-3-manifold if $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible. Here the condition $\operatorname{spec} B_x \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ for all $x \in \Sigma$ is essential.

Proposition 4.33. $D^{\mathrm{ev}}_{\max} = D^{\mathrm{ev}}_{\min}$ *on a cone-3-manifold if $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible.*

Proof. Given $u \in \operatorname{dom} D^{\mathrm{ev}}_{\max}$ we have to show that $u \in \operatorname{dom} D^{\mathrm{ev}}_{\min}$. We choose a partition of unity on C as in the proof of Proposition 4.30.

Clearly $\varphi_{int} u \in \operatorname{dom} D^{\mathrm{ev}}_{\min}$: This follows in the same way as in the cone-surface case (cf. the proof of Proposition 4.23).

It remains to prove that $\varphi_i u \in \operatorname{dom} D^{\mathrm{ev}}_{\min}$. Again this brings us back to a situation on the cone $U_\varepsilon = \operatorname{cone}_{\kappa,(0,\varepsilon)} N$, where $N = \operatorname{int} \mathbf{S}^2(\alpha, \beta, \gamma)$ or $N = \operatorname{int} \mathbf{S}^2(\alpha, \alpha)$. It is therefore sufficient to show that $f = \varphi u \in \operatorname{dom}(P^\kappa_B)_{\min}$ for $u \in \operatorname{dom}(P^\kappa_B)_{\max}$ and φ a cut-off function of the above type. Since $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible, B is essentially selfadjoint and has discrete spectrum. Moreover, the condition $\operatorname{spec} B \cap (-\frac{1}{2}, \frac{1}{2}) = \varnothing$ will be satisfied. Then Lemma 4.13 implies that $f \in \operatorname{dom}(P^\kappa_B)_{\min}$, hence in $\operatorname{dom} D^{\mathrm{ev}}_{\min}$. q.e.d.

Corollary 4.34. *The operator D is essentially selfadjoint on a cone-3-manifold if $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible.*

Proof. This follows from $D^{\mathrm{ev}}_{\max} = D^{\mathrm{ev}}_{\min}$ in the same way as in the cone-surface case. q.e.d.

Corollary 4.35. $\Delta_F = \Delta(d_{\max})$ *on a cone-*3*-manifold if* $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ *is cone-admissible.*

Proof. This follows from essential selfadjointness of D together with Corollary 3.5. q.e.d.

5. The Bochner technique

5.1. Infinitesimal isometries. For simplicity consider $\mathbf{M}^3_\kappa$ for $\kappa \in \{-1, 0, 1\}$. Let $G = \mathrm{Isom}^+\, \mathbf{M}^3_\kappa$ and $\mathfrak{g}$ its Lie-algebra. $\mathfrak{g}$ may be identified with the Lie-algebra of Killing vectorfields. Note however, that the Lie-bracket in $\mathfrak{g}$ corresponds to the negative of the vectorfield commutator under this identification:

$$ad_{\mathfrak{g}}(X)Y = [X, Y]_{\mathfrak{g}} = -[X, Y] = -\mathcal{L}_X Y.$$

Fix a point $p \in \mathbf{M}^3_\kappa$ and let $K = \mathrm{Stab}_G(p)$. Note that $K \cong \mathrm{SO}(T_p\mathbf{M}^3_\kappa)$, since G acts simply transitively on frames in constant curvature. Then we have the usual decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie-algebra of K. Recall that

$$\mathfrak{k} = \{X \in \mathfrak{g} \,|\, X(p) = 0\}$$

and

$$\mathfrak{p} = \{X \in \mathfrak{g} \,|\, (\nabla X)(p) = 0\}.$$

There are isomorphisms

$$\mathfrak{p} \cong T_p\mathbf{M}^3_\kappa,\ X \mapsto X(p)$$

and (in our constant-curvature situation)

$$\mathfrak{k} \cong \mathfrak{so}(T_p\mathbf{M}^3_\kappa),\ X \mapsto A_X(p) := (\nabla X)(p).$$

We have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, since $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$ (resp. -1) eigenspace of the Cartan-involution on $\mathfrak{g}$ induced by the geodesic involution on $\mathbf{M}^3_\kappa$ about p.

Let X be a Killing vectorfield. Let γ be a geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = Y(p)$. Then X will be a Jacobi vectorfield along γ. We obtain

$$\begin{aligned} 0 &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}X + R(X, \dot\gamma)\dot\gamma \\ &= (\nabla_{\dot\gamma}A_X)\dot\gamma + R(X, \dot\gamma)\dot\gamma. \end{aligned}$$

Therefore we have

$$(\nabla_Y A_X)Y + R(X, Y)Y = 0.$$

The expression $(\nabla_Y A_X)Z + R(X, Y)Z$ is symmetric in Y and Z. Therefore we obtain by polarization

$$(\nabla_Y A_X)Z + R(X, Y)Z = 0 \qquad (\bigstar)$$

if X is a Killing vectorfield.

Lemma 5.1. *Under the identification* $\mathfrak{g} = \mathfrak{so}(T_p\mathbf{M}^3_\kappa) \oplus T_p\mathbf{M}^3_\kappa$ *the Lie-bracket corresponds to*

$$[(A,X),(B,Y)] = ([A,B] - R(X,Y), AY - BX),$$

where $[A,B]$ *is the commutator in* $\mathfrak{so}(T_p\mathbf{M}^3_\kappa)$ *and* R *the Riemannian curvature tensor.*

Proof. Let $X, Y \in \mathfrak{p}, Z \in \mathfrak{p}$. From equation ($\bigstar$) we obtain

$$\begin{aligned} A_{[X,Y]_\mathfrak{g}} Z(p) &= -\nabla_Z([X,Y])(p) = -\nabla_Z\nabla_X Y(p) + \nabla_Z\nabla_Y X(p) \\ &= -(\nabla_Z A_Y)X(p) + (\nabla_Z A_X)Y(p) \\ &= R(Y,Z)X(p) + R(Z,X)Y(p) = -R(X,Y)Z(p). \end{aligned}$$

Let $X, Y \in \mathfrak{k}$, $Z \in \mathfrak{p}$.

$$\begin{aligned} A_{[X,Y]_\mathfrak{g}} Z(p) &= -\nabla_Z([X,Y])(p) = -\nabla_{[X,Y]} Z(p) - [Z,[X,Y]](p) \\ &= [X,[Y,Z]](p) + [Y,[Z,X]](p) \\ &= [X, \nabla_Y Z - \nabla_Z Y](p) + [Y, \nabla_Z X - \nabla_X Z](p) \\ &= \nabla_{\nabla_Z Y} X(p) - \nabla_{\nabla_Z X} Y(p) = [A_X, A_Y]Z(p). \end{aligned}$$

Let $X \in \mathfrak{k}, Y \in \mathfrak{p}$.

$$\begin{aligned} [X,Y]_\mathfrak{g}(p) &= -\left(\nabla_X Y - \nabla_Y X\right)(p) \\ &= \nabla_Y X(p) = A_X Y(p). \end{aligned}$$

This is sufficient since $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. q.e.d.

Note that the usual formula for the curvature tensor of a symmetric space

$$R(X,Y)Z(p) = -[[X,Y],Z](p), \; X, Y, Z \in \mathfrak{p}$$

is contained in the statement.

Corollary 5.2. $Ad_G(g)(A,X) = (Ad_K(g)A, gX)$ *for* $g \in K$.

Let $\mathcal{E} = \mathfrak{so}(T\mathbf{M}^3_\kappa) \oplus T\mathbf{M}^3_\kappa$. $\mathcal{E}$ is a bundle of Lie-algebras with a flat connection $\nabla^\mathcal{E}$, such that a section $\sigma = (A,X)$ is parallel if and only if X is a Killing vectorfield and $A = A_X$.

Lemma 5.3. *The flat connection on* $\mathcal{E}$ *is given by*

$$\nabla^\mathcal{E}_Y(A,X) = (\nabla_Y A - R(Y,X), \nabla_Y X - AY),$$

where ∇ *denotes the Levi-Civita connection on* $T\mathbf{M}^3_\kappa$ *and on* $\mathfrak{so}(T\mathbf{M}^3_\kappa)$.

Proof. If ∇^0 and ∇^1 are connections on a vector-bundle $\mathcal{E}$, then the difference $\alpha = \nabla^0 - \nabla^1$ is a 1-form with values in $\operatorname{End}\mathcal{E}$. If $\nabla^0\sigma = 0$, then $-\nabla^1_Y\sigma = \alpha(Y)\sigma$ for all $Y \in T\mathbf{M}^3_\kappa$.

Let $\nabla^0 = \nabla^\mathcal{E}$ and $\nabla^1 = \nabla$. A Killing vectorfield X determines a parallel section $\sigma_X = (A_X, X)$. From equation ($\bigstar$) we have

$$(\nabla_Y A_X)Z = -R(X,Y)Z = R(Y,X)Z,$$

and from the very definition

$$\nabla_Y X = A_X Y,$$

hence $\alpha(Y)(A, X) = (-R(Y, X), -AY)$. q.e.d.

In fact $\mathcal{E} = \mathbf{M}^3_\kappa \times \mathfrak{g}$ and $\nabla^{\mathcal{E}}$ is just the trivial connection d written in terms of the subbundles $T\mathbf{M}^3_\kappa$ and $\mathfrak{so}(T\mathbf{M}^3_\kappa)$.

Corollary 5.4. *$\nabla^{\mathcal{E}}_Y \sigma = \nabla_Y \sigma + ad_{\mathfrak{g}}(Y)\sigma$ for $\sigma \in \Gamma(\mathcal{E}), Y \in T\mathbf{M}^3_\kappa$.*

Proof. Lemma 5.1 implies that $\alpha(Y)\sigma = ad(Y)\sigma$. q.e.d.

We have a natural metric on $\mathcal{E}$, namely

$$h^{\mathcal{E}} = (\,\cdot\,,\cdot\,)_{\mathfrak{so}(T\mathbf{M}^3_\kappa)} \oplus (\,\cdot\,,\cdot\,)_{T\mathbf{M}^3_\kappa},$$

where

$$(A, B)_{\mathfrak{so}(T\mathbf{M}^3_\kappa)} = -\frac{1}{2}\,\mathrm{tr}(AB).$$

Recall the definition of the Killing form

$$B_{\mathfrak{g}}(a, b) = \mathrm{tr}(ad_{\mathfrak{g}}(a) ad_{\mathfrak{g}}(b))$$

for $a, b \in \mathfrak{g}$. $B_{\mathfrak{g}}$ is a symmetric bilinear form, which is $Ad_G(g)$-invariant for all $g \in G$. This implies in particular that $ad_{\mathfrak{g}}(a)$ is antisymmetric with respect to $B_{\mathfrak{g}}$ for all $a \in \mathfrak{g}$.

We wish to express $B_{\mathfrak{g}}$ in terms of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. First of course the relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ imply that $\mathfrak{k}$ and $\mathfrak{p}$ are $B_{\mathfrak{g}}$-orthogonal. The following computation is left to the reader:

Lemma 5.5. *The restrictions of $B_{\mathfrak{g}}$ to $\mathfrak{k} = \mathfrak{so}(T_p\mathbf{M}^3_\kappa)$ and $\mathfrak{p} = T_p\mathbf{M}^3_\kappa$ are given as follows:*

$$\begin{aligned} B_{\mathfrak{g}}|_{\mathfrak{k}}(\,\cdot\,,\cdot\,) &= -4(\,\cdot\,,\cdot\,)_{\mathfrak{so}(T_p\mathbf{M}^3_\kappa)} \\ B_{\mathfrak{g}}|_{\mathfrak{p}}(\,\cdot\,,\cdot\,) &= -4\kappa(\,\cdot\,,\cdot\,)_{T_p\mathbf{M}^3_\kappa}. \end{aligned}$$

We obtain as an immediate consequence:

Corollary 5.6. *If $\kappa = 1$, then $ad(Y)$ is antisymmetric with respect to $h^{\mathcal{E}}$ for $Y \in T\mathbf{M}^3_\kappa$, in particular $\nabla^{\mathcal{E}} h^{\mathcal{E}} = 0$. If $\kappa = -1$, then $ad(Y)$ is symmetric with respect to $h^{\mathcal{E}}$ for $Y \in T\mathbf{M}^3_\kappa$.*

For $\kappa = -1$ we want to calculate the precise deviation of $h^{\mathcal{E}}$ from being parallel. With $\nabla^{\mathcal{E}} = \nabla + ad$ we get using the fact that $h^{\mathcal{E}}$ is parallel with respect to ∇:

$$\begin{aligned} (\nabla^{\mathcal{E}}_X h^{\mathcal{E}})(\sigma, \tau) &= X(h^{\mathcal{E}}(\sigma, \tau)) - h^{\mathcal{E}}(\nabla^{\mathcal{E}}_X \sigma, \tau) - h^{\mathcal{E}}(\sigma, \nabla^{\mathcal{E}}_X \tau) \\ &= -h^{\mathcal{E}}(ad(X)\sigma, \tau) - h^{\mathcal{E}}(\sigma, ad(X)\tau) \\ &= -2h^{\mathcal{E}}(ad(X)\sigma, \tau). \end{aligned}$$

Let $h^{\mathcal{E}}_0$ denote the metric on $\mathcal{E}$ obtained by parallel extension of $h^{\mathcal{E}}(p)$ with respect to $\nabla^{\mathcal{E}}$ for $p \in \mathbf{M}^3_\kappa$. If we write $h^{\mathcal{E}}(\sigma, \tau) = h^{\mathcal{E}}_0(A\sigma, \tau)$ with

$A \in \Gamma(\operatorname{End}\mathcal{E})$ symmetric, we obtain using the fact that $h_0^{\mathcal{E}}$ is parallel with respect to $\nabla^{\mathcal{E}}$:

$$h_0^{\mathcal{E}}((\nabla_X^{\mathcal{E}} A)\sigma,\tau) = (\nabla_X^{\mathcal{E}} h^{\mathcal{E}})(\sigma,\tau) = -2h^{\mathcal{E}}(ad(X)\sigma,\tau)$$
$$= -2h_0^{\mathcal{E}}(A\,ad(X)\sigma,\tau),$$

and in particular we have proved:

Lemma 5.7. *If $\kappa = -1$, then $A^{-1}(\nabla^{\mathcal{E}} A) = -2ad$ and is therefore bounded on $\mathbf{M}^3_\kappa$ with respect to $h^{\mathcal{E}}$.*

Let us now consider M, the nonsingular part of a cone-3-manifold C. The condition that M is locally modelled on $\mathbf{M}^3_\kappa$ is usually expressed in terms of the *developing map*

$$\operatorname{dev} : (\widetilde{M}, p_0) \longrightarrow (\mathbf{M}^3_\kappa, p)$$

and the *holonomy representation*

$$\operatorname{hol} : \pi_1(M, x_0) \longrightarrow G = \operatorname{Isom}^+ \mathbf{M}^3_\kappa,$$

where dev is a local isometry and $\pi_1(M)$-equivariant with respect to the deck-action on $\widetilde{M}$ and the action via hol on $\mathbf{M}^3_\kappa$. For details we ask the reader to consult Section 6.1.

We again denote by $\mathcal{E}$ the bundle $\mathfrak{so}(TM) \oplus TM$. Since being a Killing vectorfield is a local condition, we again have a flat connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ with the property that parallel sections correspond to Killing vectorfields. The formula for $\nabla^{\mathcal{E}}$ given in Lemma 5.3 applies as well. In contrast to the model-space situation, $\mathcal{E}$ will now have holonomy. It is easy to see that the holonomy of $\mathcal{E}$ along a loop $\gamma \in \pi_1(M, x_0)$ is given by $Ad \circ \operatorname{hol}(\gamma)$ if we identify $\mathcal{E}_{x_0}$ with $\mathfrak{g}$. Therefore we obtain an alternative description of $\mathcal{E}$:

$$\mathcal{E} = \widetilde{M} \times_{Ad \circ \operatorname{hol}} \mathfrak{g}$$

The Lie-algebra structure on $\mathcal{E}$ induced by this representation coincides with the one given in Lemma 5.1.

The same considerations apply to the two-dimensional situation as well if we replace $\mathbf{M}^3_\kappa$ and its isometry group with the corresponding two-dimensional objects. Here we restrict our attention to the spherical case. Let

$$S = \begin{cases} \mathbf{S}^2(\alpha,\beta,\gamma) & \text{or} \\ \mathbf{S}^2(\alpha,\alpha) & \end{cases}$$

in the following. Since $\operatorname{Isom}^+ \mathbf{S}^2 = \mathrm{SO}(3)$ we have a holonomy representation

$$\operatorname{hol} : \pi_1(\operatorname{int} S) \longrightarrow \operatorname{Isom}^+ \mathbf{S}^2 = \mathrm{SO}(3)$$

and developing map

$$\operatorname{dev} : \widetilde{\operatorname{int} S} \longrightarrow \mathbf{S}^2.$$

Let us denote the vector-bundle of infinitesimal isometries with its natural flat connection in this situation by $(\mathcal{F}, \nabla^{\mathcal{F}})$. We have

$$\mathcal{F} = \widetilde{\operatorname{int} S} \times_{Ad \circ \mathrm{hol}} \mathfrak{so}(3).$$

Since the adjoint representation of SO(3) on $\mathfrak{so}(3)$ is isomorphic to the standard representation of SO(3) on $\mathbb{R}^3$, we have alternatively

$$\mathcal{F} = \widetilde{\operatorname{int} S} \times_{\mathrm{hol}} \mathbb{R}^3.$$

Since hol preserves the standard scalar product on $\mathbb{R}^3$, we have a natural metric $h^{\mathcal{F}}$ on $\mathcal{F}$ which is parallel with respect to $\nabla^{\mathcal{F}}$.

Now if $x_i \in S$ is a cone-point with cone-angle α_i and $\gamma_i \in \pi_1(\operatorname{int} S)$ a loop around x_i, then $\mathrm{hol}(\gamma_i)$ is just rotation about the cone-angle α_i around some fixed axis in $\mathbb{R}^3$. Note that the axis of $\mathrm{hol}(\gamma_i)$ and the axis of $\mathrm{hol}(\gamma_j)$ need not coincide for $x_i \neq x_j$. This gives us a quite explicit description of $\mathcal{F}$. In particular we see that locally around the cone-points we have the following splitting

$$\mathcal{F}|_{S^1_{\alpha_i}} = \mathbb{C}(\alpha_i) \oplus \mathbb{R},$$

where $\mathbb{C}(\alpha_i)$ denotes the flat $U(1)$-bundle over $S^1_{\alpha_i}$ with holonomy $e^{i\alpha_i}$.

Next we describe the restriction of $\mathcal{E}$ to the links of singular points. Recall that if $x \in \Sigma$ is a singular point and S_x is its link, then

$$S_x = \mathbf{S}^2(\alpha, \beta, \gamma)$$

if x is a vertex, and

$$S_x = \mathbf{S}^2(\alpha, \alpha)$$

if x is an edge point.

Lemma 5.8. *Let S_x be the link of a singular point $x \in \Sigma$. Then the restriction of $\mathcal{E}$ to $\operatorname{int} S_x$ is given by:*

$$\mathcal{E}|_{\operatorname{int} S_x} = \mathcal{F} \oplus \mathcal{F},$$

where $\mathcal{F}$ is the flat vector-bundle of infinitesimal isometries on S_x.

Proof. The holonomy of $\pi_1(\operatorname{int} S_x)$ fixes a point $p \in \mathbf{M}^3_\kappa$ and is therefore contained in $K = \mathrm{Stab}_G(p) \cong \mathrm{SO}(T_p\mathbf{M}^3_\kappa)$. We have seen in Corollary 5.2 that $Ad_G(g) = (Ad_K(g), g)$ for $g \in K$ with respect to the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Again, since the adjoint representation and the standard representation of SO(3) are isomorphic, we obtain two copies of $\mathcal{F}$. q.e.d.

Proposition 5.9. *Let C be a spherical cone-3-manifold with cone-angles $\leq \pi$. Then $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$, the vector-bundle of infinitesimal isometries of $M = \operatorname{int} C$ with its natural flat connection and metric, is cone-admissible.*

Proof. Let $x \in \Sigma$ be a singular point. We have $\mathcal{E}|_{U_\varepsilon(x)} = \mathcal{F} \oplus \mathcal{F}$ via Lemma 5.8. Since $h^{\mathcal{E}}$ is parallel in the spherical case, we have $h_0^{\mathcal{E}} = h^{\mathcal{E}}$ and assumptions A1 and A2 are trivially satisfied on $U_\varepsilon(x)$, cf. Remark 4.1. Clearly $h^{\mathcal{E}} = h^{\mathcal{F}} \oplus h^{\mathcal{F}}$. $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is orthogonally flat and therefore via Remark 4.19 cone-admissible over $\operatorname{int} S_x$ if the cone-angles are $\leq \pi$. Then we may apply Remark 4.29 to conclude that $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible over M. q.e.d.

Proposition 5.10. *Let C be a hyperbolic cone-3-manifold with cone-angles $\leq \pi$. Then $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$, the vector-bundle of infinitesimal isometries of $M = \operatorname{int} C$ with its natural flat connection and metric, is cone-admissible.*

Proof. Letting $x \in \Sigma$ be a singular point, we have $\mathcal{E}|_{U_\varepsilon(x)} = \mathcal{F} \oplus \mathcal{F}$ via Lemma 5.8. $(\mathcal{F}, \nabla^{\mathcal{F}}, h^{\mathcal{F}})$ is orthogonally flat, so clearly $h_0^{\mathcal{E}} = \lim_{r \to 0} h^{\mathcal{E}}(r)$ exists and $h_0^{\mathcal{E}} = h^{\mathcal{F}} \oplus h^{\mathcal{F}}$, i.e., assumption A1 is satisfied. In view of Lemma 5.7, assumption A2 is also satisfied. The assertion follows now as in the spherical case. q.e.d.

In the Euclidean case for fixed $p \in \mathbf{E}^3$ we have a group homomorphism

$$\begin{aligned} \mathrm{rot} : \mathrm{Isom}^+ \mathbf{E}^3 &\longrightarrow \mathrm{Stab}_G(p) \cong \mathrm{SO}(T_p \mathbf{E}^3) \\ g &\longmapsto g + (p - g(p)). \end{aligned}$$

We may form the rotational part of the holonomy

$$\mathrm{rot} \circ \mathrm{hol} : \pi_1(M) \longrightarrow \mathrm{Stab}_G(p) \cong \mathrm{SO}(T_p \mathbf{E}^3).$$

On the other hand

$$\mathcal{E}_{\mathrm{trans}} := TM \subset \mathcal{E} = \mathfrak{so}(TM) \oplus TM$$

is via the explicit formula for $\nabla^{\mathcal{E}}$ in Lemma 5.3 easily seen to be a parallel subbundle of $\mathcal{E}$. Note that in contrast

$$\mathcal{E}_{rot} := \mathfrak{so}(TM) \subset \mathcal{E} = \mathfrak{so}(TM) \oplus TM$$

is *not* parallel.

Since the rotational part of the holonomy is nothing but the holonomy of the flat tangent bundle, we obtain

$$\mathcal{E}_{\mathrm{trans}} = \widetilde{M} \times_{\mathrm{rot} \circ \mathrm{hol}} \mathbb{R}^3.$$

In the same way as before one shows:

Lemma 5.11. *Let C be a Euclidean cone-3-manifold. The restriction of $\mathcal{E}_{\mathrm{trans}}$ to the link S_x of a singular point $x \in \Sigma$ is given as*

$$\mathcal{E}_{\mathrm{trans}}|_{\operatorname{int} S_x} = \mathcal{F},$$

where $\mathcal{F}$ is the flat vector-bundle of infinitesimal isometries on S_x. Furthermore $\mathcal{E}_{\mathrm{trans}}$ is cone-admissible if the cone-angles are $\leq \pi$.

5.2. Weitzenböck formulas.

5.2.1. The spherical and the Euclidean case. Let $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ be an orthogonally flat vector-bundle. Recall the standard Weitzenböck formula on $\mathcal{E}$-valued 1-forms:

$$\Delta\omega = \nabla^t \nabla \omega + (\mathrm{Ric} \otimes \mathrm{id})\omega.$$

For this formula to hold without extra terms we really need that the metric $h^{\mathcal{E}}$ is parallel with respect to $\nabla^{\mathcal{E}}$. Recall that the action of the Ricci tensor on a scalar-valued 1-form α is determined by the relation

$$g(\mathrm{Ric}(\alpha), \beta) = \mathrm{Ric}(\alpha, \beta)$$

for all $\beta \in \Omega^1(M, \mathbb{R})$. In three dimensions the Ricci tensor of a metric with constant curvature κ is given by

$$\mathrm{Ric}(\cdot\,,\cdot) = 2\kappa \cdot g(\cdot\,,\cdot),$$

so we end up with

$(\kappa = 1)$

$$\Delta\omega = \nabla^t \nabla \omega + 2 \cdot \omega$$

$(\kappa = 0)$

$$\Delta\omega = \nabla^t \nabla \omega$$

in the spherical and the Euclidean case.

5.2.2. The hyperbolic case. In the hyperbolic case we use a different type of Weitzenböck formula, due to Y. Matsushima and S. Murakami, cf. [**MM**]. We use the notation of [**HK**]. Let $\mathcal{E} = \mathfrak{so}(TM) \oplus TM$ be the vector-bundle of infinitesimal isometries and $\nabla^{\mathcal{E}}$ its natural flat connection. We continue to denote by $\nabla^{\mathcal{E}}$ the tensor-product connection on $\Lambda^\bullet T^*M \otimes \mathcal{E}$ induced by the Levi-Civita connection on M and the connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, whereas we denote by ∇ the Levi-Civita connection on $\Lambda^\bullet T^*M \otimes \mathcal{E}$.

Recall the relation $\nabla^{\mathcal{E}}_Y = \nabla_Y + ad(Y)$ for $Y \in TM$, where the endomorphism $ad(Y)$ is symmetric with respect to $h^{\mathcal{E}}$. Let in the following

$$\varepsilon : T^*M \otimes \Lambda^\bullet T^*M \to \Lambda^{\bullet+1} T^*M$$

denote exterior multiplication, and

$$\iota : TM \otimes \Lambda^\bullet T^*M \to \Lambda^{\bullet-1} T^*M$$

denote interior multiplication. Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame and $\{e^1, e^2, e^3\}$ the dual coframe. Then we have

$$d = \sum_{i=1}^{3} \varepsilon(e^i) \nabla^{\mathcal{E}}_{e_i} = \sum_{i=1}^{3} \varepsilon(e^i) \left(\nabla_{e_i} + ad(e_i)\right).$$

This implies

$$d^t = -\sum_{i=1}^{3} \iota(e_i)\left(\nabla_{e_i} - ad(e_i)\right).$$

Define

$$\mathcal{D} := \sum_{i=1}^{3} \varepsilon(e^i)\nabla_{e_i} \quad \text{and} \quad T := \sum_{i=1}^{3} \varepsilon(e^i) ad(e_i),$$

which implies

$$\mathcal{D}^t = -\sum_{i=1}^{3} \iota(e^i)\nabla_{e_i} \quad \text{and} \quad T^t = \sum_{i=1}^{3} \iota(e_i) ad(e_i).$$

We obviously have $d = \mathcal{D} + T$ and $d^t = \mathcal{D}^t + T^t$. Let $\Delta_{\mathcal{D}} = \mathcal{D}\mathcal{D}^t + \mathcal{D}^t\mathcal{D}$ and $H = TT^t + T^tT$. H is symmetric and non-negative. From the definitions we have

$$\begin{aligned} \Delta &= dd^t + d^t d \\ &= \Delta_{\mathcal{D}} + H + \mathcal{D}T^t + T\mathcal{D}^t + \mathcal{D}^t T + T^t\mathcal{D}. \end{aligned}$$

A computation in a local orthogonal frame shows that

$$\mathcal{D}T^t + T\mathcal{D}^t + \mathcal{D}^t T + T^t\mathcal{D} = 0$$

and

$$H = \sum_{i=1}^{3} ad(e_i)^2 + \sum_{i,j=1}^{3} \varepsilon(e^i)\iota(e_j) ad\left([e_i, e_j]\right).$$

This implies the following Weitzenböck formula, where a priori $\Delta_{\mathcal{D}}$ and H are non-negative.

Lemma 5.12 ([**MM**]). *$\Delta = \Delta_{\mathcal{D}} + H$.*

The following positivity property of H on 1-forms makes this formula particularly useful for us. The proof may again be obtained by a calculation in a local orthonormal frame.

Proposition 5.13 ([**MM**]). *There is a constant $C > 0$ such that*

$$(H\omega, \omega)_x \geq C(\omega, \omega)_x$$

for all $\omega \in \Omega^1(M, \mathcal{E})$ and $x \in M$.

5.3. A vanishing theorem. In this section we prove our main result about L^2-cohomology spaces of 3-dimensional cone-manifolds with coefficients in the flat vector-bundle of infinitesimal isometries. This completes the analytic part of our argument. For convenience we discuss the proof case by case.

5.3.1. The spherical case.

Theorem 5.14. *Let C be a spherical cone-3-manifold with cone-angles $\leq \pi$. Let $M = C \setminus \Sigma$ and $(\mathcal{E}, \nabla^{\mathcal{E}})$ be the vector-bundle of infinitesimal isometries of M with its natural flat connection. Then*

$$H^1_{L^2}(M, \mathcal{E}) = 0.$$

Proof. We recall the Weitzenböck formula for the Hodge-Laplace operator on $\mathcal{E}$-valued 1-forms, which in the spherical case (i.e., $\kappa = 1$) amounts to

$$\Delta\omega = \nabla^t \nabla \omega + 2\omega$$

for $\omega \in \Omega^1(M, \mathcal{E})$. For $\omega \in \Omega^1_{\mathrm{cp}}(M, \mathcal{E})$ integration by parts yields

$$\begin{aligned}\int_M (\Delta\omega, \omega) &= \int_M (\nabla^t \nabla\omega, \omega) + 2\int_M |\omega|^2 \\ &= \int_M |\nabla\omega|^2 + 2\int_M |\omega|^2 \\ &\geq 2\int_M |\omega|^2.\end{aligned}$$

This means we have a positive lower bound for Δ on $\Omega^1_{\mathrm{cp}}(M, \mathcal{E})$:

$$\langle \Delta\omega, \omega \rangle_{L^2} \geq C \langle \omega, \omega \rangle_{L^2}$$

with $C = 2$. Since $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible according to Proposition 5.9, we obtain

$$\Delta_F = \Delta(d_{\max})$$

via Corollary 4.35. Since the Friedrichs extension preserves lower bounds, we conclude

$$\mathcal{H}^1_{\max} = \ker \Delta^1(d_{\max}) = 0.$$

Finally Theorem 4.31 identifies L^2-cohomology with the $d_{\max}$-harmonic forms. This implies $H^1_{L^2}(M, \mathcal{E}) = 0$ and proves the theorem. q.e.d.

5.3.2. The Euclidean case.

Theorem 5.15. *Let C be a Euclidean cone-3-manifold with cone-angles $\leq \pi$. Let $\mathcal{E}_{\mathrm{trans}} \subset \mathcal{E}$ be the parallel subbundle of infinitesimal translations of $M = C \setminus \Sigma$. Then*

$$H^1_{L^2}(M, \mathcal{E}_{\mathrm{trans}}) \cong \{\omega \in \Omega^1(M, \mathcal{E}_{\mathrm{trans}}) \,|\, \nabla\omega = 0\}.$$

Proof. The Weitzenböck formula for the Hodge-Laplace operator on $\mathcal{E}_{\mathrm{trans}}$-valued 1-forms in the Euclidean case (i.e., $\kappa = 0$) amounts to

$$\Delta\omega = \nabla^t \nabla\omega$$

for $\omega \in \Omega^1(M, \mathcal{E}_{\mathrm{trans}})$. This implies with Corollary 3.3 that

$$\Delta^1_F = \nabla^t_{\max} \nabla_{\min}.$$

Since $\mathcal{E}_{\mathrm{trans}} \subset \mathcal{E}$ is cone-admissible according to Lemma 5.11, we obtain

$$\Delta^1_F = \Delta^1(d_{\max})$$

via Corollary 4.35. This implies that

$$\Delta^1(d_{\max}) = \nabla^t_{\max}\nabla_{\min}.$$

For $\omega \in \ker \Delta^1(d_{\max})$ we have

$$0 = \langle \Delta(d_{\max})\omega, \omega\rangle_{L^2} = \langle \nabla^t_{\max}\nabla_{\min}\omega, \omega\rangle_{L^2} = \|\nabla_{\min}\omega\|^2_{L^2}.$$

We conclude that $\omega \in \ker \nabla_{\min}$. On the other hand, if $\omega \in \ker \nabla_{\max}$, then clearly $\omega \in kerD_{\max}$. Since D is essentially selfadjoint according to Corollary 4.34, $\ker D_{\max} = \ker D_{\min} = \mathcal{H}_{\max}$. We obtain

$$\ker \nabla_{\max} \subset \mathcal{H}^1_{\max} \subset \ker \nabla_{\min},$$

which proves the theorem via Theorem 4.31, since $\mathcal{H}_{\max}$ consists of smooth forms. Note also that a parallel form ω will automatically be L^2-bounded, since ∇ is compatible with the metric on $\mathcal{E}_{\mathrm{trans}}$. q.e.d.

5.3.3. The hyperbolic case.

Theorem 5.16. *Let C be a hyperbolic cone-3-manifold with cone-angles $\leq \pi$. Let $M = C \setminus \Sigma$ and $(\mathcal{E}, \nabla^{\mathcal{E}})$ be the vector-bundle of infinitesimal isometries of M with its natural flat connection. Then*

$$H^1_{L^2}(M, \mathcal{E}) = 0.$$

Proof. The proof follows the same scheme as in the spherical case. For convenience of the reader we also give full details in this case.

We recall that in the hyperbolic case we have a Weitzenböck formula for the Hodge-Laplace operator for $\mathcal{E}$-valued 1-forms of the type

$$\Delta\omega = \mathcal{D}^t\mathcal{D}\omega + \mathcal{D}\mathcal{D}^t\omega + H\omega,$$

where

$$\langle H\omega, \omega\rangle_{L^2} \geq C\,\langle \omega, \omega\rangle_{L^2}$$

for $C > 0$ independent of $\omega \in \Omega^1(M, \mathcal{E})$. For $\omega \in \Omega^1_{\mathrm{cp}}(M, \mathcal{E})$ integration by parts yields

$$\begin{aligned}\int_M (\Delta\omega, \omega) &= \int_M (\mathcal{D}^t\mathcal{D}\omega, \omega) + \int_M (\mathcal{D}\mathcal{D}^t\omega, \omega) + \int_M (H\omega, \omega)\\ &= \int_M |\mathcal{D}\omega|^2 + \int_M |\mathcal{D}^t\omega|^2 + \int_M (H\omega, \omega)\\ &\geq C\int_M |\omega|^2.\end{aligned}$$

This means we have a positive lower bound for Δ on $\Omega^1_{\mathrm{cp}}(M, \mathcal{E})$:

$$\langle \Delta\omega, \omega\rangle_{L^2} \geq C\,\langle \omega, \omega\rangle_{L^2}$$

for $C > 0$. Since $(\mathcal{E}, \nabla^{\mathcal{E}}, h^{\mathcal{E}})$ is cone-admissible according to Proposition 5.10, we obtain

$$\Delta_F = \Delta(d_{\max})$$

via Corollary 4.35. Since the Friedrichs extension preserves lower bounds, we conclude

$$\mathcal{H}^1_{\max} = \ker \Delta^1(d_{\max}) = 0.$$

Finally Theorem 4.31 identifies L^2-cohomology with the $d_{\max}$-harmonic forms. This implies $H^1_{L^2}(M, \mathcal{E}) = 0$ and proves the theorem. q.e.d.

6. Deformation theory

In this chapter we study the deformation space of cone-manifold structures on a 3-dimensional cone-manifold of given topological type (C, Σ). It is convenient to use the more general framework of (X, G)-structures and deformations thereof, in particular since there is a quite general theorem of [**Gol**], which relates the local structure of the deformation space of (X, G)-structures to the local structure of $X(\pi_1 M, G)$. By $X(\pi_1 M, G)$ we denote the quotient of $R(\pi_1 M, G)$, the space of representations of $\pi_1 M$ in G, by the conjugation action of G.

The (X, G)-structures relevant for our situation will be $X = \mathbf{M}^3_\kappa$ and $G = \mathrm{Isom}^+ \mathbf{M}^3_\kappa$. In fact, by a theorem of [**Cul**], the holonomy representation of a 3-dimensional cone-manifold structure may always be lifted to the universal covering group of $\mathrm{Isom}^+ \mathbf{M}^3_\kappa$, which in the hyperbolic case is $\mathrm{SL}_2(\mathbb{C})$ and in the spherical case $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

We will use the L^2-vanishing theorem to analyze local properties of $\mathrm{SL}_2(\mathbb{C})$- and $\mathrm{SU}(2)$-representation spaces. From this we will be able to conclude local rigidity in the hyperbolic and in the spherical case.

6.1. (X, G)**-structures.** Let (X, g^X) be a Riemannian manifold upon which a Lie group G acts transitively by isometries. Let M be manifold of the same dimension as X. Then we say that M carries an (X, G)-structure if M is locally modelled on X, i.e., there is a covering of M by charts $\{\varphi_i : U_i \to X\}_{i \in I}$ such that for each connected component of C of $U_i \cap U_j$ there exists $g_{C,i,j} \in G$ such that $g_{C,i,j} \circ \varphi_i = \varphi_j$ on C. The collection of charts $\{\varphi_i : U_i \to X\}_{i \in I}$ is called an (X, G)-atlas and an (X, G)-structure on M is a maximal (X, G)-atlas. A detailed discussion of this kind of structure may be found in [**Gol**], which we will use as the main reference for this section.

Let us fix basepoints $x_0 \in M$ and $p_0 \in \pi^{-1}(x_0)$, where $\pi : \widetilde{M} \to M$ is the universal covering of M. Then an (X, G)-structure on M together with the germ of an (X, G)-chart $\varphi : U \to X$ around x_0 determines by analytic continuation of φ a local diffeomorphism

$$\mathrm{dev} : \widetilde{M} \longrightarrow X,$$

the developing map, and a representation

$$\mathrm{hol} : \pi_1(M, x_0) \longrightarrow G,$$

the holonomy representation, such that dev is equivariant with respect to hol, i.e.,

$$\mathrm{dev} \circ \gamma = \mathrm{hol}(\gamma) \circ \mathrm{dev}$$

for all $\gamma \in \pi_1(M, x_0)$. Conversely, a local diffeomorphism $\mathrm{dev} : \widetilde{M} \to X$ equivariant with respect to some representation $\mathrm{hol} : \pi_1(M, x_0) \to G$ as above, defines an (X, G)-structure on M together with the germ of an (X, G)-chart at x_0. Note that hol is uniquely determined by dev and the equivariance condition.

Let $\mathcal{D}'_{(X,G)}(M)$ be the space of developing maps with the topology of C^∞-convergence on compact sets. As usual we equip $R(\pi_1(M, x_0), G)$, the set of representations of $\pi_1(M, x_0)$ in G, with the compact-open topology. Associating its holonomy representation with a developing map yields a continuous map

$$\begin{aligned} \mathcal{D}'_{(X,G)}(M) &\longrightarrow R(\pi_1(M, x_0), G) \\ \mathrm{dev} &\longmapsto \mathrm{hol}\,. \end{aligned}$$

For simplicity we assume that M is diffeomorphic to the interior of a compact manifold with boundary $M \cup \partial M$, which is certainly the case for the object of our main concern, namely the smooth part of a 3-dimensional cone-manifold.

Following [**CHK**] we introduce the equivalence relation $\sim$ on the space of developing maps, which is generated by *isotopy* and *thickening*. Clearly $\mathrm{Diff}_0(M)$, the group of diffeomorphisms of M isotopic to the identity, acts on the space of developing maps. Two structures equivalent under this action will be called *isotopic*. On the other hand, if an (X, G) structure on M extends to $M \cup \partial M \times [0, \varepsilon)$ for some $\varepsilon > 0$, this gives rise to an (X, G)-structure on M, which we will call a *thickening* of the original structure. Let

$$\mathcal{D}_{(X,G)}(M) = \mathcal{D}'_{(X,G)}(M)/\sim\,.$$

We obtain a G-equivariant map

$$\begin{aligned} \mathcal{D}_{(X,G)}(M) &\longrightarrow R(\pi_1(M, x_0), G) \\ [\mathrm{dev}] &\longmapsto \mathrm{hol}\,. \end{aligned}$$

We define the *deformation space* of (X, G)-structures to be the quotient

$$\mathcal{T}_{(X,G)}(M) := \mathcal{D}_{(X,G)}(M)/G.$$

Let $X(\pi_1(M, x_0), G)$ denote the G-quotient of $R(\pi_1(M, x_0), G)$ by conjugation. Properties of this quotient in our particular context will be discussed in greater detail in subsequent sections.

Assuming that the action of G on $R(\pi_1(M,x_0),G)$ by conjugation is proper, this implies in particular by the G-equivariance of the above map, that the action of G on $\mathcal{D}_{(X,G)}(M)$ is also proper. In this situation the arguments of [**Gol**] (cf. also the discussion in [**CHK**]) yield the following theorem about the local structure of the deformation space of (X,G)-structures:

Theorem 6.1 (deformation theorem, cf. [**Gol**]). *If the action of G by conjugation on $R(\pi_1(M,x_0),G)$ is proper, then the map*

$$\begin{aligned} \mathcal{T}_{(X,G)}(M) &\longrightarrow X(\pi_1(M,x_0),G) \\ [\mathrm{dev}] &\longmapsto [\mathrm{hol}] \end{aligned}$$

is a local homeomorphism.

This theorem explains the meaning of representation varieties in the study of deformations of (X,G)-structures: Local properties of the deformation space of (X,G)-structures on M translate into local properties of $X(\pi_1(M,x_0),G)$ and vice versa.

By a theorem of M. Culler (cf. [**Cul**]) the holonomy representation of a cone-3-manifold may be lifted to the universal covering group of $\mathrm{Isom}^+ \mathbf{M}^3_\kappa$:

$$\widetilde{\mathrm{hol}} : \pi_1 M \longrightarrow \widetilde{\mathrm{Isom}^+}\mathbf{M}^3_\kappa.$$

In the hyperbolic case $\widetilde{\mathrm{Isom}^+}\mathbf{H}^3 = \mathrm{SL}_2(\mathbb{C})$. We obtain that the flat vector-bundle of infinitesimal isometries may be written as

$$\mathcal{E} = \widetilde{M} \times_{Ad\circ\widetilde{\mathrm{hol}}} \mathfrak{sl}_2(\mathbb{C}).$$

As a consequence $\mathcal{E}$ has a parallel complex structure, such that in particular all the cohomology spaces $H^i(M,\mathcal{E})$ are complex vector spaces.

In the spherical case $\widetilde{\mathrm{Isom}^+}\mathbf{S}^3 = \mathrm{SU}(2)\times\mathrm{SU}(2)$. Therefore the lift of the holonomy splits as a product representation

$$\widetilde{\mathrm{hol}} = (\mathrm{hol}_1, \mathrm{hol}_2) : \pi_1 M \longrightarrow \mathrm{SU}(2)\times\mathrm{SU}(2),$$

and in particular the flat vector-bundle of infinitesimal isometries splits as a direct sum of parallel subbundles:

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2,$$

where

$$\mathcal{E}_i = \widetilde{M} \times_{Ad\circ\mathrm{hol}_i} \mathfrak{su}(2).$$

Consequently $H^i(M,\mathcal{E}) = H^i(M,\mathcal{E}_1)\oplus H^i(M,\mathcal{E}_2)$ for all i.

For notational convenience we will drop the distinction between hol and $\widetilde{\mathrm{hol}}$ from here.

6.2. The representation variety. In the following let Γ be a finitely generated discrete group. Once and for all we fix a presentation $\langle\gamma_1,\ldots,\gamma_n|(r_i)_{i\in I}\rangle$ of Γ. The cardinality of the indexset I may a priori be infinite, but most of the groups we deal with will turn out to be finitely presented. Let $G=\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SU}(2)$. The *representation variety* $R(\Gamma,G)$ is defined to be the set of group homomorphisms $\rho:\Gamma\to G$. $R(\Gamma,G)$ endowed with the compact-open topology is a Hausdorff space, compact in the case of $\mathrm{SU}(2)$.

The relations r_i define functions $f_i:G^n\to G$ such that $R(\Gamma,G)$ may be identified with the set $\{(A_1,\ldots,A_n)\in G^n|\,f_i(A_1,\ldots,A_n)=1\}$. Since $\mathrm{SL}_2(\mathbb{C})$ is a $\mathbb{C}$-algebraic (resp. $\mathrm{SU}(2)$ a $\mathbb{R}$-algebraic) group and the f_i are polynomial maps, $R(\Gamma,G)$ acquires the structure of a $\mathbb{C}$-algebraic (resp. $\mathbb{R}$-algebraic) set. Note that $R(\Gamma,G)$ won't be a smooth space in general.

The action of G on G^n by simultaneous conjugation leaves the set $R(\Gamma,G)\subset G^n$ invariant. Therefore the quotient $X(\Gamma,G)=R(\Gamma,G)/G$ is well defined. We endow $X(\Gamma,G)$ with the quotient topology. $X(\Gamma,G)$ will in general be neither smooth nor even Hausdorff. $X(\Gamma,G)$ as we have defined it should not be confused with a quotient constructed in the algebraic category. This usually requires arguments from geometric invariant theory, which we can avoid using here.

A smooth family of representations $\rho_t:\Gamma\to G$ with $\rho_0=\rho$ defines a group 1-cocycle $z:\Gamma\to\mathfrak{g}$, where

$$z(\gamma)=\tfrac{d}{dt}\Big|_{t=0}\rho_t(\gamma)\rho(\gamma)^{-1}$$

for $\gamma\in\Gamma$. Recall that $Z^1(\Gamma,\mathfrak{g})$, the space of 1-cocycles of Γ with coefficients in the representation $Ad\circ\rho:\Gamma\to\mathrm{GL}(\mathfrak{g})$, is the the space of maps $z:\Gamma\to\mathfrak{g}$ such that

$$z(ab)=z(a)+(Ad\circ\rho(a))\,z(b)$$

for all $a,b\in\Gamma$. A cocycle z is a coboundary if there exists some $v\in\mathfrak{g}$ such that

$$z(a)=v-(Ad\circ\rho(a))\,v$$

for all $a\in\Gamma$. Let $B^1(\Gamma,\mathfrak{g})$ be the space of 1-coboundaries. Now by definition

$$H^1(\Gamma,\mathfrak{g})=Z^1(\Gamma,\mathfrak{g})/B^1(\Gamma,\mathfrak{g})$$

is the first *group cohomology* group of Γ with coefficients in the representation $Ad\circ\rho:\Gamma\to\mathrm{GL}(\mathfrak{g})$. $H^1(\Gamma,\mathfrak{g})$ is a real vector space. Recall further that

$$H^0(\Gamma,\mathfrak{g})=Z^0(\Gamma,\mathfrak{g})=\{v\in\mathfrak{g}|(Ad\circ\rho(\gamma))v=v\,\forall\gamma\in\Gamma\}.$$

For more details on group cohomology, cf. [**Bro**] for instance.

We refer to $Z^1(\Gamma,\mathfrak{g})$ as the space of infinitesimal deformations of the representation ρ. We call a 1-cocycle z integrable, if there exists a (local) deformation ρ_t which is tangent to z in the above sense.

It is easy to see that $z \in B^1(\Gamma, \mathfrak{g})$ if and only if z is tangent to the orbit of G through ρ, i.e., there exists a smooth curve g_t in G with $g_0 = 1$ such that

$$z(\gamma) = \tfrac{d}{dt}\Big|_{t=0} g_t \rho(\gamma) g_t^{-1} \rho(\gamma)^{-1}$$

for $\gamma \in \Gamma$. A deformation $\rho_t(\gamma) = g_t \rho(\gamma) g_t^{-1}$ will be considered trivial.

We use the following observation due to A. Weil (cf. [**Wei**]): A map $z : \Gamma \to \mathfrak{g}$ defines a group 1-cocycle if and only if the map

$$\begin{aligned} (Ad \circ \rho, z) : \Gamma &\longrightarrow \mathrm{GL}(\mathfrak{g}) \ltimes \mathfrak{g} \\ \gamma &\longmapsto (Ad \circ \rho(\gamma), z(\gamma)) \end{aligned}$$

is a group homomorphism. $\mathrm{GL}(\mathfrak{g}) \ltimes \mathfrak{g}$ is the affine group of the vector-space $\mathfrak{g}$. Using the fixed presentation of Γ, this identifies $Z^1(\Gamma, \mathfrak{g})$ with a linear subspace of $\mathfrak{g}^n$. More precisely, the relations r_i determine linear functions $g_i : \mathfrak{g}^n \to \mathfrak{g}$, such that

$$Z^1(\Gamma, \mathfrak{g}) = \{(a_1, \ldots, a_n) \in \mathfrak{g}^n | g_i(a_1, \ldots, a_n) = 0 \, \forall i \in I\}.$$

On the other hand, $\ker df_i$ may be identified with a subspace of $\mathfrak{g}^n$ via

$$(\dot{A}_1, \ldots, \dot{A}_n) \mapsto (\dot{A}_1 A_1^{-1}, \ldots, \dot{A}_n A_n^{-1}).$$

With these identifications we have the following lemma:

Lemma 6.2. $Z^1(\Gamma, \mathfrak{g}) = \cap_{i \in I} \ker df_i$.

Proof. A straightforward calculation shows that $df_i(\dot{A}_1, \ldots, \dot{A}_n) = 0$ for $\dot{A}_i \in T_{A_i} G$ if and only if $g_i(a_1, \ldots, a_n) = 0$, where $a_i = \dot{A}_i A_i^{-1}$.
q.e.d.

If the equations $(f_i)_{i \in I}$ cut out $R(\Gamma, G)$ transversely near ρ, then the previous lemma identifies $Z^1(\Gamma, \mathfrak{g})$ with the tangent space of $R(\Gamma, G)$ at the point ρ. In particular ρ will be a smooth point. If furthermore the G-action on $R(\Gamma, G)$ by conjugation is free and proper, then $X(\Gamma, G)$ will be smooth near $\chi = [\rho]$ and the tangent space at χ may be identified with $H^1(\Gamma, \mathfrak{g})$.

6.3. Integration and group cohomology. We wish to represent group cocycles of $\pi_1 M$ with coefficients in the representation $Ad \circ \mathrm{hol} : \pi_1 M \to \mathfrak{g} = \mathfrak{isom}^+ \mathbf{M}^3_\kappa$ by differential forms on M with values in $\mathcal{E}$. This will be achieved by means of integration.

Let x_0 be a base point in M. Then for $\gamma \in \pi_1(M, x_0)$ and a closed 1-form $\omega \in \Omega^1(M, \mathcal{E})$ we define

$$\int_\gamma \omega = \int_0^1 \tau_{\gamma(t)}^{-1} \omega(\dot{\gamma}(t)) dt \in \mathcal{E}_{x_0},$$

where $\tau_{\gamma(t)}$ denotes the parallel transport along γ from $x_0 = \gamma(0)$ to $\gamma(t)$. Since ω is closed, the integral depends only on the homotopy class of γ. If we identify $\mathcal{E}_{x_0}$ with $\mathfrak{g}$, then we may set

$$z_\omega(\gamma) = \int_\gamma \omega \in \mathfrak{g}.$$

Alternatively, we may proceed as follows: The flat bundle $\mathcal{E}$ may be described as an associated bundle $\mathcal{E} = \widetilde{M} \times_{Ad \circ \mathrm{hol}} \mathfrak{g}$. 1-forms $\omega \in \Omega^1(M, \mathcal{E})$ correspond to 1-forms $\widetilde{\omega} \in \Omega^1(\widetilde{M}, \mathfrak{g})$ satisfying the following equivariance condition:

$$\gamma^* \widetilde{\omega} = (Ad \circ \mathrm{hol}(\gamma))\, \widetilde{\omega}$$

for all $\gamma \in \pi_1(M, x_0)$. For $\omega \in \Omega^1(M, \mathcal{E})$ closed consider $\widetilde{\omega} \in \Omega^1(\widetilde{M}, \mathfrak{g})$, which will again be closed. Let $p_0 \in \pi^{-1}(x_0)$ be a base point in $\widetilde{M}$. Now since $\widetilde{M}$ is simply connected, there exists a primitive $F \in C^\infty(M, \mathfrak{g})$ such that $dF = \widetilde{\omega}$. For $\gamma \in \pi_1(M, x_0)$ we define

$$z_\omega(\gamma) = \int_\gamma \omega = F(\gamma p_0) - F(p_0) \in \mathfrak{g}.$$

Since F is determined up to an additive constant, this is well defined.

Both definitions of the map $z_\omega : \pi_1 M \to \mathfrak{g}$ associated with the closed form $\omega \in \Omega^1(M, \mathcal{E})$ clearly agree. The proof of the following lemma is straightforward and left to the reader:

Lemma 6.3. *If $\omega \in \Omega^1(M, \mathcal{E})$ is closed, then z_ω defines a group cocycle, i.e., $z_\omega \in Z^1(\pi_1 M, \mathfrak{g})$. ω is exact if and only if $z_\omega \in B^1(\pi_1 M, \mathfrak{g})$.*

As a consequence of the preceding lemma, we obtain that the period map

$$\begin{aligned} P : H^1(M, \mathcal{E}) &\longrightarrow H^1(\pi_1 M, \mathfrak{g}) \\ [\omega] &\longmapsto [\gamma \mapsto \textstyle\int_\gamma \omega] \end{aligned}$$

is well defined and injective. Since we know from more general considerations (cf. [**Bro**, Theorem 5.2] for example) that $H^i(M, \mathcal{E}) \cong H^i(\pi_1 M, \mathfrak{g})$ for $i \in \{0, 1\}$, we find that the period map provides an explicit isomorphism between $H^1(M, \mathcal{E})$ and $H^1(\pi_1 M, \mathfrak{g})$.

6.4. Isometries.

6.4.1. Isometries of $\mathbf{H}^3$. The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbf{H}^3$ by Poincaré extension identifies $\mathrm{SL}_2(\mathbb{C})$ with the universal cover of $\mathrm{Isom}^+ \mathbf{H}^3 = \mathrm{PSL}_2(\mathbb{C})$. Here we use the upper half space model. Let $\phi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Isom}^+ \mathbf{H}^3$ denote the covering projection.

Semisimple elements in $\mathrm{SL}_2(\mathbb{C})$ project to semisimple isometries. A semisimple isometry ϕ has an invariant axis; this is the unique geodesic, where δ_ϕ, the displacement function of ϕ, assumes its minimum. If this minimum is positive, we call ϕ hyperbolic, otherwise elliptic. Parabolic elements in $\mathrm{SL}_2(\mathbb{C})$ project to parabolic isometries. Parabolic isometries have a unique fixed point at infinity. The following is well-known:

Lemma 6.4. $A, B \in \mathrm{SL}_2(\mathbb{C})$ *commute if and only if* $\phi(A), \phi(B)$ *are either semisimple isometries and preserve the same axis* γ *or* $\phi(A), \phi(B)$ *are parabolic isometries with the same fixed point at infinity.*

The stabilizer of an oriented geodesic γ is isomorphic to $\mathbb{C}^*$, more precisely, if we work in the upper half space model $\mathbf{H}^3 = \mathbb{C} \times \mathbb{R}_+$, then for $\gamma = \{0\} \times \mathbb{R}_+$ we obtain

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{C})}(\gamma) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}.$$

$S^1 \subset \mathbb{C}^*$ corresponds to pure rotations around γ, while $\mathbb{R} \subset \mathbb{C}^*$ corresponds to pure translations along γ. Recall that for a Killing vectorfield X on $\mathbf{H}^3$ we denote by $\sigma_X = (\nabla X, X) \in \mathfrak{sl}_2(\mathbb{C})$ the corresponding parallel section. In particular, if we choose cylindrical coordinates (r, θ, z) around γ, we see that

$$\sigma_{\partial/\partial\theta} = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

and

$$\sigma_{\partial/\partial z} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}).$$

Note in particular that $\sigma_{\partial/\partial\theta} = i\sigma_{\partial/\partial z}$. The factor $1/2$ comes from the fact that $\mathrm{SL}_2(\mathbb{C})$ is a twofold cover of $\mathrm{Isom}^+ \mathbf{H}^3$.

Let $A \in \mathrm{SL}_2(\mathbb{C})$ be semisimple and $\phi = \phi(A) \in \mathrm{Isom}^+ \mathbf{H}^3$. Then A is conjugate to $\mathrm{diag}(\lambda, \lambda^{-1})$ in $\mathrm{SL}_2(\mathbb{C})$ for $\lambda \in \mathbb{C}^*$. Now let $z \in \mathbb{C}/2\pi i\mathbb{Z}$ such that $\lambda = \exp(z)$. We define $\mathcal{L}(A) = 2z \in \mathbb{C}/2\pi i\mathbb{Z}$. Then $\mathcal{L}(A)$ is determined by A up to sign. $\mathcal{L}(A)$ is called the *complex length* of A.

For $A \neq \pm\,\mathrm{id}$ we can orient the axis γ of ϕ and remove the sign ambiguity of $\mathcal{L}$ consistently in a neighbourhood of A in $\mathrm{SL}_2(\mathbb{C})$. The real part of $\mathcal{L}(A)$ equals the (signed) translation length of ϕ along γ, while the imaginary part equals the angle of rotation around γ. We obtain

$$\mathrm{tr}\, A = 2\cosh(z) = \pm 2\cosh(\mathcal{L}(A)/2)$$

and by the inverse function theorem:

Lemma 6.5. *Let* $A \neq \pm\,\mathrm{id} \in \mathrm{SL}_2(\mathbb{C})$ *be semisimple. There exist neighbourhoods* U *of* A *in* $\mathrm{SL}_2(\mathbb{C})$ *and* V *of* $\mathrm{tr}\, A$ *in* $\mathbb{C}$ *and a biholomorphic map* $\phi : V \to \mathcal{L}(V) \subset \mathbb{C}$ *such that* $\mathrm{tr}(U) \subset V$ *and* $\phi \circ \mathrm{tr} = \mathcal{L}$ *on* U.

6.4.2. Isometries of $\mathbf{S}^3$. We identify $\mathbf{S}^3$ with the unit quaternions, i.e., $\mathbf{S}^3 = \{x \in \mathbb{H} : |x| = 1\}$. If we view the quaternions as a subalgebra of $\mathbb{C}^{2\times 2}$ via

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$\mathbf{S}^3$ gets identified with the group SU(2) via

$$\mathbf{S}^3 \longrightarrow \mathrm{SU}(2)$$
$$a + bj \longmapsto \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. The map

$$\phi : \mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SO}(4)$$
$$(A, B) \longmapsto (x \mapsto AxB^{-1})$$

exhibits $\mathrm{SU}(2) \times \mathrm{SU}(2)$ as the universal cover of $\mathrm{Isom}^+ \mathbf{S}^3 = \mathrm{SO}(4)$. Note that the diagonal matrices

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} : \lambda \in S^1 \right\} \subset \mathrm{SU}(2)$$

correspond to the geodesic $\gamma = \mathbb{C} \cap \mathbf{S}^3$, where as usual $\mathbb{C}$ is identified with $\mathbb{R} \oplus \mathbb{R}i \subset \mathbb{H}$. For any geodesic $\gamma \subset \mathbf{S}^3$ let us denote by $\gamma^\perp$ the geodesic which lies in the plane orthogonal to γ. In the above case $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3 = (\mathbb{R}j \oplus \mathbb{R}k) \cap \mathbf{S}^3$, which corresponds to the set of matrices

$$\left\{ \begin{pmatrix} 0 & \lambda \\ -\overline{\lambda} & 0 \end{pmatrix} : \lambda \in S^1 \right\} \subset \mathrm{SU}(2).$$

A spherical isometry may be put in a standard form: namely, if an isometry is represented as $\phi = \phi(A, B)$ with $A, B \in \mathrm{SU}(2)$, then by conjugation we may achieve that $A = \mathrm{diag}(\lambda, \overline{\lambda})$ and $B = \mathrm{diag}(\mu, \overline{\mu})$ with $\lambda, \mu \in S^1$. The matrix A corresponds to $\lambda \in \mathbb{C} \cap \mathbf{S}^3$ and B to $\mu \in \mathbb{C} \cap \mathbf{S}^3$ if we identify SU(2) with $\mathbf{S}^3$ as above. Then for $x \in \mathbf{S}^3$ we have $\phi(x) = \lambda x \overline{\mu}$, such that ϕ preserves the Hopf-fibrations, which are associated with the complex structures $x \mapsto ix$ and $x \mapsto xi$ on $\mathbb{H}$.

In particular, ϕ preserves $\gamma = \mathbb{C} \cap \mathbf{S}^3$ and $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3$. More precisely we have $\phi(\eta) = \lambda\overline{\mu}\eta$ for $\eta \in S^1 = \mathbb{C} \cap \mathbf{S}^3$, and $\phi(\eta j) = (\lambda\mu\eta)j$ for $\eta j \in \mathbb{C}j \cap \mathbf{S}^3$. Note that γ and $\gamma^\perp$ are the common fibers of the two fibrations, which are transverse everywhere else.

If $\mu = 1$, then ϕ translates along the fibers of the Hopf-fibration obtained by left-multiplication with S^1. In particular the displacement of ϕ is constant on $\mathbf{S}^3$. Similarly, if $\lambda = 1$, then ϕ translates along the fibres of the Hopf-fibration obtained by right-multiplication with S^1. Again the displacement of ϕ will be constant on $\mathbf{S}^3$.

If $\lambda = \mu$, then ϕ is a pure rotation around γ, or equivalently, a pure translation along $\gamma^\perp$. Similarly, if $\lambda = \overline{\mu}$, then ϕ is a pure rotation around $\gamma^\perp$, or equivalently, a pure translation along γ.

Recall that for a Killing vectorfield X on $\mathbf{S}^3$ we denote by $\sigma_X = (\nabla X, X) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ the corresponding parallel section. In particular, if we choose cylindrical coordinates (r, θ, z) around γ, we see that

$$\sigma_{\partial/\partial\theta} = \left(\frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

and

$$\sigma_{\partial/\partial z} = \left(\frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

The factors $1/2$ arise from the fact that $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is a twofold cover of $\mathrm{Isom}^+ \mathbf{S}^3$. The following is immediate from the above discussion:

Lemma 6.6. $\phi_1, \phi_2 \in \mathrm{Isom}^+ \mathbf{S}^3$ *commute if and only they preserve the same pair of orthogonal axes* $\{\gamma, \gamma^\perp\}$.

Since $\phi(A, B)1 = 1$ if and only if $A = B \in \mathrm{SU}(2)$ we obtain:

Lemma 6.7. $\phi = \phi(A, B) \in \mathrm{Isom}^+ \mathbf{S}^3$ *has a fixed point if and only if A is conjugate to B within* $\mathrm{SU}(2)$.

We want to define an analogue of the complex length in the spherical case. If $\phi = \phi(A, B)$ with A conjugate to $\mathrm{diag}(\lambda, \overline{\lambda})$ and B conjugate to $\mathrm{diag}(\mu, \overline{\mu})$, then let $x \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\lambda = \exp(ix)$ and $y \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\mu = \exp(iy)$. We define $\mathcal{L}_1(A, B) = x - y$ and $\mathcal{L}_2(A, B) = x + y$. Then $\mathcal{L}(A, B) = (\mathcal{L}_1(A, B), \mathcal{L}_2(A, B)) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$ is determined by A and B up to an overall sign and up to switching components.

Let in the following $A \neq \pm\,\mathrm{id}$ and $B \neq \pm\,\mathrm{id}$. If ϕ preserves a pair of orthogonal axes $\{\gamma, \gamma^\perp\}$, these ambiguities can be removed in a neighbourhood of (A, B) by orienting γ. Let us again call $\mathcal{L}(A, B)$ the "complex" length of $(A, B) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$. $\mathcal{L}_1(A, B)$ equals the (signed) translation length along γ, while $\mathcal{L}_2(A, B)$ equals the (signed) translation length along $\gamma^\perp$. We obtain

$$\mathrm{tr}\, A = 2\cos x = \pm 2 \cos\big((\mathcal{L}_1(A, B) + \mathcal{L}_2(A, B))/2 \big)$$

and

$$\mathrm{tr}\, B = 2\cos y = \pm 2 \cos\big((-\mathcal{L}_1(A, B) + \mathcal{L}_2(A, B))/2 \big)\,.$$

We set $\mathrm{Tr}_1(A, B) = \mathrm{tr}\, A$, $\mathrm{Tr}_2(A, B) = \mathrm{tr}\, B$ and $\mathrm{Tr} = (\mathrm{Tr}_1, \mathrm{Tr}_2)$. By the inverse function theorem we obtain:

Lemma 6.8. *Let* $(A, B) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$ *with* $A \neq \pm\,\mathrm{id}$ *and* $B \neq \pm\,\mathrm{id}$. *There exist neighbourhoods U of (A, B) in* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *and V of* $\mathrm{Tr}(A, B)$ *in $\mathbb{R}^2$ and a diffeomorphism $\phi : V \to \mathcal{L}(V) \subset \mathbb{R}^2$ such that* $\mathrm{Tr}(U) \subset V$ *and $\phi \circ \mathrm{Tr} = \mathcal{L}$ on U.*

6.5. Cohomology computations. Let C be a 3-dimensional cone-manifold with cone-angles $\leq \pi$. Under this cone-angle bound, a connected component of the singular locus Σ will either be a circle or a (connected) trivalent graph, cf. Chapter 2, see also [**CHK**] and [**BLP2**]. Let $M_\varepsilon = M \setminus B_\varepsilon(\Sigma)$, where $B_\varepsilon(\Sigma)$ is the open ε-tube around Σ. Let

$U_\varepsilon(\Sigma) = B_\varepsilon(\Sigma)\setminus\Sigma$. Then M_ε is topologically a manifold with boundary, which is a deformation retract of M. ∂M_ε consists of tori and surfaces of higher genus. $\partial M_\varepsilon = \partial U_\varepsilon(\Sigma)$ is a deformation retract of $U_\varepsilon(\Sigma)$.

Without loss of generality we may assume in the following that Σ is connected.

6.5.1. The torus case. Let $\Sigma = S^1$. Then $U_\varepsilon(\Sigma)$ is given as $(0,\varepsilon)\times T^2$, where $T^2 = \mathbb{R}^2/\Lambda$ and Λ is the lattice generated by $(\theta, z) \mapsto (\theta+\alpha, z)$ and $(\theta, z) \mapsto (\theta - t, z+l)$. The metric is given as $g = dr^2 + \mathrm{sn}_\kappa^2(r)d\theta^2 + \mathrm{cs}_\kappa^2(r)dz^2$. Here α, t and l are the parameters, which determine the geometry of $U_\varepsilon(\Sigma)$, namely the cone-angle, the twist and the length of the singular tube. Note that a function f in the coordinates (r,θ,z) descends to a function on $U_\varepsilon(\Sigma)$ if and only if $f(r,\theta,z) = f(r,\theta+\alpha,z)$ and $f(r,\theta,z+l) = f(r,\theta+t,z)$. Note also that $H^i(U_\varepsilon(\Sigma),\cdot) = H^i(T^2,\cdot)$ for any local coefficient system.

The forms $d\theta$ and dz are invariant under Λ and descend to forms on T^2, which generate the de-Rham cohomology of the torus in degree 1, i.e., $H^1(T^2,\mathbb{R}) = \mathbb{R}\cdot[d\theta]\oplus\mathbb{R}\cdot[dz]$.

Similarly, $\partial/\partial\theta$ and $\partial/\partial z$ descend to Killing-vectorfields on $U_\varepsilon(\Sigma)$. To be more specific, $\partial/\partial\theta$ is an infinitesimal rotation around the singular axis and $\partial/\partial z$ an infinitesimal translation along the same axis. Consequently, $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ make up parallel sections of the bundle $\mathcal{E}$, i.e., $\sigma_{\partial/\partial\theta}, \sigma_{\partial/\partial z} \in H^0(T^2,\mathcal{E})$.

Lemma 6.9. *If the cone-angles are $\leq \pi$, then in the hyperbolic and the spherical case*

$$H^0(T^2,\mathcal{E}) = \mathbb{R}\cdot\sigma_{\partial/\partial\theta}\oplus\mathbb{R}\cdot\sigma_{\partial/\partial z}.$$

Proof. Let λ be the longitudinal and μ be the meridian loop. Clearly

$$\begin{aligned} H^0(T^2,\mathcal{E}) &\cong Z^0(\pi_1 T^2, \mathfrak{g}) \\ &= \{v \in \mathfrak{g} : (Ad\circ \mathrm{hol}(\gamma))v = v\ \forall \gamma\in\pi_1T^2\}, \end{aligned}$$

which we view as the infinitesimal centralizer of the holonomy representation restricted to the torus. We compute the centralizer $Z(\mathrm{hol}(\pi_1T^2))$ in each case.

In the hyperbolic case, let $A = \mathrm{hol}(\lambda) \in \mathrm{SL}_2(\mathbb{C})$ and $B = \mathrm{hol}(\mu) \in \mathrm{SL}_2(\mathbb{C})$. Since hol is the holonomy of a hyperbolic cone-manifold structure with cone-angles $\leq \pi$, we may assume that $A = \mathrm{diag}(\eta,\eta^{-1})$ and $B = \mathrm{diag}(\xi,\xi^{-1})$ with $\eta,\xi \neq \pm 1$. Then it is easy to see that $Z(\mathrm{hol}(\pi_1T^2)) = \{\mathrm{diag}(\zeta,\zeta^{-1}), \zeta\in\mathbb{C}^*\}$, hence $Z^0(\pi_1T^2,\mathfrak{sl}_2(\mathbb{C}))\cong\mathbb{R}^2$. Since $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ are closed and linearly independent, the result follows.

In the spherical case, $\mathrm{hol} : \pi_1T^2 \to \mathrm{SU}(2)\times\mathrm{SU}(2)$ splits as a product representation $\mathrm{hol} = (\mathrm{hol}_1,\mathrm{hol}_2)$ with $\mathrm{hol}_i : \pi_1T^2\to\mathrm{SU}(2)$ for $i\in\{1,2\}$. We then have $Z(\mathrm{hol}(\pi_1T^2)) = Z(\mathrm{hol}_1(\pi_1T^2))\times Z(\mathrm{hol}_2(\pi_1T^2))$. Let $A_i = \mathrm{hol}_i(\lambda)\in\mathrm{SU}(2)$ and $B_i = \mathrm{hol}_i(\mu)\in\mathrm{SU}(2)$. Without loss of generality we assume that $A_i = \mathrm{diag}(\eta_i, \overline{\eta}_i)$ and $B_i = \mathrm{diag}(\xi_i, \overline{\xi}_i)$ with $\eta_i, \xi_i \in S^1$. Since hol is the holonomy of a spherical cone-manifold structure with cone-angles $\leq \pi$, $\mathrm{hol}(\mu)$ must be a nontrivial rotation. This implies that $\{\xi_1, \overline{\xi}_1\} = \{\xi_2, \overline{\xi}_2\} \neq \{\pm 1\}$. Then it follows that $Z(\mathrm{hol}_i(\pi_1 T^2)) = \{\mathrm{diag}(\zeta, \overline{\zeta}), \zeta \in S^1\}$ implying that $Z^0(\pi_1 T^2, \mathfrak{su}(2)) \cong \mathbb{R}$. As above, $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ provide a basis for $H^0(T^2, \mathcal{E})$. q.e.d.

We define forms

$$\begin{aligned} \omega_{ang} &= d\theta \otimes \sigma_{\partial/\partial\theta} \\ \omega_{shr} &= d\theta \otimes \sigma_{\partial/\partial z} \\ \omega_{tws} &= dz \otimes \sigma_{\partial/\partial\theta} \\ \omega_{len} &= dz \otimes \sigma_{\partial/\partial z}. \end{aligned}$$

Since $\sigma_{\partial/\partial\theta}$ and $\sigma_{\partial/\partial z}$ are parallel, these forms are closed. They will be tangent to the corresponding geometric deformations of the singular tube, i.e., ω_{ang} is supposed to change the cone-angle α, similarly for t and l. ω_{shr} will be tangent to a deformation, which leads out of the class of cone-metrics (which may be called a "shearing"-deformation). This will be made precise.

Lemma 6.10. *The forms ω_{ang} and ω_{shr} are not L^2 on $U_\varepsilon(\Sigma)$, whereas the forms ω_{tws} and ω_{len} are bounded on $U_\varepsilon(\Sigma)$ and hence L^2.*

Proof. The metric on $U_\varepsilon(\Sigma)$ is given by $g = dr^2 + \mathrm{sn}_\kappa^2(r) d\theta^2 + \mathrm{cs}_\kappa^2(r) dz^2$. Hence $dvol = \mathrm{sn}_\kappa(r)\,\mathrm{cs}_\kappa(r) dr \wedge d\theta \wedge dz$. For a 1-form $\omega = \alpha \otimes \sigma_X$ with $\alpha \in \Omega^1(U_\varepsilon(\Sigma))$ and $X \in \Gamma(TU_\varepsilon(\Sigma))$ we have $|\omega|^2 = |\alpha|^2 \left(|\nabla X|^2 + |X|^2\right)$. Clearly

$$|d\theta|^2 = \frac{1}{\mathrm{sn}_\kappa^2(r)},\ |dz|^2 = \frac{1}{\mathrm{cs}_\kappa^2(r)},\ \left|\tfrac{\partial}{\partial\theta}\right|^2 = \mathrm{sn}_\kappa^2(r),\ \left|\tfrac{\partial}{\partial z}\right|^2 = \mathrm{cs}_\kappa^2(r).$$

Let

$$\left\{e_1 = \tfrac{\partial}{\partial r},\ e_2 = \mathrm{sn}_\kappa(r)^{-1}\tfrac{\partial}{\partial\theta},\ e_3 = \mathrm{cs}_\kappa(r)^{-1}\tfrac{\partial}{\partial z}\right\}$$

be an orthonormal frame for $TU_\varepsilon(\Sigma)$. A straightforward calculation shows that with respect to this frame

$$\nabla\tfrac{\partial}{\partial\theta} = \begin{pmatrix} 0 & -\mathrm{cs}_\kappa(r) & 0 \\ \mathrm{cs}_\kappa(r) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \Gamma(\mathfrak{so}(TU_\varepsilon(\Sigma))$$

and

$$\nabla\tfrac{\partial}{\partial z} = \begin{pmatrix} 0 & 0 & \kappa\,\mathrm{sn}_\kappa(r) \\ 0 & 0 & 0 \\ -\kappa\,\mathrm{sn}_\kappa(r) & 0 & 0 \end{pmatrix} \in \Gamma(\mathfrak{so}(TU_\varepsilon(\Sigma)),$$

such that

$$\left|\nabla\tfrac{\partial}{\partial\theta}\right|^2 = \mathrm{cs}_\kappa^2(r),\ \left|\nabla\tfrac{\partial}{\partial z}\right|^2 = \kappa^2\,\mathrm{sn}_\kappa^2(r).$$

We obtain

$$|\omega_{ang}|^2 = \frac{\mathrm{sn}_\kappa^2(r) + \mathrm{cs}_\kappa^2(r)}{\mathrm{sn}_\kappa^2(r)},\ |\omega_{shr}|^2 = \frac{\mathrm{cs}_\kappa^2(r) + \kappa^2\,\mathrm{sn}_\kappa^2(r)}{\mathrm{sn}_\kappa^2(r)}$$

and

$$|\omega_{tws}|^2 = \frac{\mathrm{sn}_\kappa^2(r) + \mathrm{cs}_\kappa^2(r)}{\mathrm{cs}_\kappa^2(r)},\ |\omega_{len}|^2 = \frac{\mathrm{cs}_\kappa^2(r) + \kappa^2\,\mathrm{sn}_\kappa^2(r)}{\mathrm{cs}_\kappa^2(r)}.$$

In the first case we observe that $|\omega_{ang}|^2 dvol \sim |\omega_{shr}|^2 dvol \sim \mathrm{sn}_\kappa(r)^{-1}$, which is not integrable for $r \in (0, \varepsilon)$. In the second case we find ω_{tws} and ω_{len} bounded and therefore L^2-integrable. q.e.d.

Lemma 6.11. *If the cone-angles are $\leq \pi$, then in the hyperbolic and the spherical case*

$$H^1(T^2, \mathcal{E}) = \mathbb{R} \cdot [\omega_{ang}] \oplus \mathbb{R} \cdot [\omega_{shr}] \oplus \mathbb{R} \cdot [\omega_{tws}] \oplus \mathbb{R} \cdot [\omega_{len}].$$

Proof. Since $H^0(T^2, \mathcal{E}) = \mathbb{R} \cdot \sigma_{\partial/\partial\theta} \oplus \mathbb{R} \cdot \sigma_{\partial/\partial z}$, we have a short exact sequence of flat vector-bundles

$$0 \to \underline{\mathbb{R}}^2 \to \mathcal{E} \to \mathcal{E}/\underline{\mathbb{R}}^2 \to 0.$$

Here we denote by $\underline{\mathbb{R}}^k$ the trivial vector-bundle of real rank k together with the trivial flat connection.

We claim that the natural map $H^1(T^2, \mathbb{R}^2) \to H^1(T^2, \mathcal{E})$ is an isomorphism. In the spherical case we can use the parallel metric on $\mathcal{E}$ to split the short exact coefficient sequence. Then clearly $H^0(\mathcal{E}/\underline{\mathbb{R}}^2) = 0$ and we may use Poincaré duality to conclude that $\mathcal{E}/\underline{\mathbb{R}}^2$ is acyclic. Now the result follows from the long exact cohomology sequence.

In the hyperbolic case we can use the parallel Killing form B to split the coefficient sequence, if B restricted to $\underline{\mathbb{R}}^2$ is nondegenerate. We use the local formula for the Killing form in Lemma 5.5 with $\kappa = -1$:

$$B(\sigma_X, \sigma_Y) = -4(\nabla X, \nabla Y) + 4(X, Y).$$

From the calculations in the previous lemma we obtain

$$\begin{aligned} B(\sigma_{\partial/\partial\theta}, \sigma_{\partial/\partial\theta}) &= -4\cosh^2(r) + 4\sinh^2(r) = -4 \\ B(\sigma_{\partial/\partial z}, \sigma_{\partial/\partial z}) &= -4\sinh^2(r) + 4\cosh^2(r) = 4 \\ B(\sigma_{\partial/\partial\theta}, \sigma_{\partial/\partial z}) &= 0, \end{aligned}$$

which shows that $B|_{\underline{\mathbb{R}}^2}$ is nondegenerate. Then the result follows as above. q.e.d.

We wish to calculate the periods of the differential forms $\omega_{ang}, \omega_{shr}$, ω_{tws} and ω_{len}. Let $x_0 = (0,0)$ be the basepoint of T^2. For $\gamma \in \pi_1 T^2$ and $\omega \in \Omega^1(T^2, \mathcal{E})$ closed, we have a well-defined integral

$$\int_\gamma \omega = \int_0^1 \tau_{\gamma(t)}^{-1} \omega(\dot{\gamma}(t)) dt$$

where $\tau_{\gamma(t)}$ denotes the parallel transport along γ from $x_0 = \gamma(0)$ to $\gamma(t)$. Recall that the map $\gamma \mapsto z_\omega(\gamma) = \int_\gamma \omega$ defines a group cocycle, if we identify $\mathcal{E}_{x_0}$ with $\mathfrak{g}$.

Note that if ω is of the form $\omega = \alpha \otimes \sigma$ with $\nabla\sigma = 0$, then $\int_\gamma \omega$ is very easy to compute:

$$\int_\gamma \omega = \int_\gamma \alpha \cdot \sigma_{x_0}.$$

This remark applies in particular to $\omega_{ang}, \omega_{shr}, \omega_{tws}$ and ω_{len}. We concentrate on the values of the corresponding group cocycles $z_{ang}, z_{shr}, z_{tws}$ and z_{len} on the meridian $\mu \in \pi_1 T^2$, $\mu(0) = x_0$. We obtain

$$z_{ang}(\mu) = \int_\mu \omega_{ang} = \alpha \cdot (\sigma_{\partial/\partial\theta})_{x_0}$$

$$z_{shr}(\mu) = \int_\mu \omega_{shr} = \alpha \cdot (\sigma_{\partial/\partial z})_{x_0}$$

and

$$z_{tws}(\mu) = z_{len}(\mu) = 0.$$

6.5.2. The higher genus case. Let Σ be a connected graph with trivalent vertices. Then $F_g = \partial U_\varepsilon(\Sigma)$ is a surface of genus $g = (N+3)/3$, where N is the number of edges contained in Σ. $U_\varepsilon(v)$, the smooth part of the ε-ball around a vertex $v \in \Sigma$, is homotopy equivalent to a pair of pants P.

Lemma 6.12. *If the cone-angles are $\leq \pi$, then $H^0(F_g, \mathcal{E}) = 0$.*

Proof. If we restrict the holonomy of M to $U_\varepsilon(v)$, the smooth part of the ε-ball around a vertex $v \in \Sigma$, then $\mathrm{hol}(\pi_1(U_\varepsilon(v))$ fixes a point $p \in \mathbf{M}^3_\kappa$. $U_\varepsilon(v)$ deformation-retracts to $P \subset \partial U_\varepsilon(\Sigma) = F_g$. Using the presentation $\pi_1(P) = \langle \mu_1, \mu_2, \mu_3 | \mu_1\mu_2\mu_3 = 1\rangle$, we obtain that the $\mathrm{hol}(\mu_i)$, $i \in \{1,2,3\}$, project to nontrivial rotations with mutually distinct axes. This implies that $Z(\mathrm{hol}(\pi_1 F_g)) = \{\pm 1\}$. q.e.d.

Corollary 6.13. *If the cone-angles are $\leq \pi$, then in the hyperbolic case $H^1(F_g, \mathcal{E}) \cong \mathbb{C}^{6g-6}$ and in the spherical case $H^1(F_g, \mathcal{E}_i) \cong \mathbb{R}^{6g-6}$.*

Proof. Using the parallel Killing form B on $\mathcal{E}$ in the hyperbolic case, resp. the parallel metric on $\mathcal{E}_i$ in the spherical case, we conclude that $H^2(F_g, \mathcal{E}) = H^2(F_g, \mathcal{E}_i) = 0$ using Poincaré duality. Now for any flat bundle $\mathcal{F}$ over F_g one has $\chi(F_g, \mathcal{F}) = \dim \mathcal{F} \cdot \chi(F_g) = \dim \mathcal{F} \cdot (2-2g)$, this implies in particular that $\dim H^1(F_g, \mathcal{F}) = -\dim \mathcal{F} \cdot (2-2g)$ if $H^0(F_g, \mathcal{F}) = H^2(F_g, \mathcal{F}) = 0$. q.e.d.

Away from the vertices, the singular locus $U_\varepsilon(\Sigma)$ can be given coordinates (r, θ_i, z_i) with $r \in (0, \varepsilon)$, $\theta_i \in \mathbb{R}/\alpha_i\mathbb{Z}$ and $z_i \in (\delta, l_i - \delta)$ for some $\delta > 0$. Here α_i is the cone-angle around the i-th edge and l_i its length. Then the metric is given by $g = dr^2 + \mathrm{sn}^2_\kappa(r) d\theta_i^2 + \mathrm{cs}^2_\kappa(r) dz_i^2$.

We choose a function $\varphi_i = \varphi_i(z_i)$ such that $\varphi_i(\delta) = 0$, $\varphi_i(l_i-\delta) = l_i-\delta$ and $d\varphi_i|_{(\delta,2\delta)} = d\varphi_i|_{(l_i-2\delta,l_i-\delta)} = 0$. Then $d\varphi_i \in \Omega^1(U_\varepsilon(\Sigma))$ is well-defined and so are

$$\omega^i_{tws} = d\varphi_i \otimes \sigma_{\partial/\partial\theta_i}$$
$$\omega^i_{len} = d\varphi_i \otimes \sigma_{\partial/\partial z_i}.$$

Note that these forms are supported away from the vertices of the singularity.

Lemma 6.14. *The differential forms ω^i_{tws} and ω^i_{len} are bounded on $U_\varepsilon(\Sigma)$, hence in particular L^2.*

Proof. This essentially amounts to the same computation as in the torus case. q.e.d.

Lemma 6.15. *The cohomology classes of the closed differential forms*

$$\{\omega^1_{tws}, \omega^1_{len}, \ldots, \omega^N_{tws}, \omega^N_{len}\}$$

are linearly independent in $H^1(F_g, \mathcal{E})$.

Proof. Suppose we have a nontrivial linear relation between the above classes in $H^1(F_g, \mathcal{E})$, say

$$t_1\omega^1_{tws} + l_1\omega^1_{len} + \cdots + t_N\omega^N_{tws} + l_N\omega^N_{len} = d\sigma$$

for some $\sigma \in \Gamma(F_g, \mathcal{E})$. Since the forms ω^i_{tws} and ω^i_{len} are supported away from the vertices, we obtain $d\sigma = 0$ in a neighbourhood of each vertex v_i. A neighbourhood $U_\varepsilon(v_i)$ of a vertex is homotopy equivalent to the thrice-punctured sphere P. Since $H^0(P, \mathcal{E}) = 0$, we have $\sigma|_{U_\varepsilon(v_i)} = 0$ for each vertex. Therefore we obtain a nontrivial linear relation on at least one of the tori $T^2_i = \mathbb{R}^2/\alpha_i\mathbb{Z} + l_i\mathbb{Z}$, where σ_i denotes the restriction of σ to a neighbourhood of the i-th edge:

$$t_i\omega^i_{tws} + l_i\omega^i_{len} = d\sigma_i,$$

which is a contradiction in view of Lemma 6.11, since $d\varphi_i$ is cohomologous to dz_i on T^2_i. q.e.d.

6.6. Local structure of the representation variety.

6.6.1. The torus case. Let $\iota : T^2 \to M$ be the inclusion of a torus boundary component. The map ι induces a group homomorphism $\iota_* : \pi_1 T^2 \to \pi_1 M$ and hence a map $\iota^* : R(\pi_1 M, G) \to R(\pi_1 T^2, G)$ for $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SU}(2)$ respectively.

Lemma 6.16. *Let $\rho = \iota^*_{T^2}\,\mathrm{hol} : \pi_1 T^2 \to \mathrm{SL}_2(\mathbb{C})$. Then ρ is a smooth point of $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$. The local $\mathbb{C}$-dimension of $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ around ρ equals 4. Furthermore, the tangent space $T_\rho R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$.*

Proof. We identify $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ with the (affine algebraic) set $\{(A,B) \in \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) | [A,B] = 1\}$. The kernel of the differential of the commutator map $\ker d_{(A,B)}[\,\cdot\,,\cdot\,]$ may be identified with the space of 1-cocycles $Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$. We have $\dim_{\mathbb{C}} Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C})) = 4$ from the cohomology computations. Note that this implies that $d_{(A,B)}[\,\cdot\,,\cdot\,]$ is *not* surjective at $(A,B) = \rho$. W.l.o.g. we may assume that $\rho = (\mathrm{diag}(\lambda, \lambda^{-1}), \mathrm{diag}(\mu, \mu^{-1}))$ with $\lambda, \mu \in \mathbb{C}^*$. We define a map

$$\begin{aligned} F : \mathbb{C}^* \times \mathbb{C}^* \times \mathrm{SL}_2(\mathbb{C}) &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \\ (\lambda, \mu, A) &\longmapsto \left(A\, \mathrm{diag}(\lambda, \lambda^{-1}) A^{-1}, A\, \mathrm{diag}(\mu, \mu^{-1}) A^{-1}\right). \end{aligned}$$

We claim that $\mathrm{rank}_{\mathbb{C}} F = 4$ at $(\lambda, \mu, 1)$. The image of F is certainly contained in $R(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$, such that an easy application of the implicit function theorem (cf. [**Wei**], [**Rag**, Lemma 6.8]) yields the result. Consider the standard $\mathbb{C}$-basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$\left\{ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Clearly $\mathbb{C} \cdot h$ exponentiates to $Z(\rho(\pi_1 T^2)) = \{\mathrm{diag}(\eta, \eta^{-1}) | \eta \in \mathbb{C}^*\}$, the stabilizer of ρ under the conjugation action of $\mathrm{SL}_2(\mathbb{C})$. Now it is easily verified that

$$\{dF(1,0,0), dF(0,1,0), dF(0,0,x), dF(0,0,y)\}_{(\lambda,\mu,1)}$$

are linearly independent if $\lambda \neq \pm 1$ or $\mu \neq \pm 1$. This implies that $\mathrm{rank}_{\mathbb{C}} F$ at $(\lambda, \mu, 1)$ is at least 4, but since $\mathrm{im}\, d_{(\lambda,\mu,1)} F \subset Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$, it has to equal 4. q.e.d.

Corollary 6.17. $\chi = [\iota_{T^2}^* \mathrm{hol}]$ *is a smooth point of* $X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$. *The local* $\mathbb{C}$*-dimension of* $X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ *around* χ *equals* 2. *Furthermore, the tangent space* $T_\chi X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$ *may be identified with* $H^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$.

Proof. The restriction of F to $\mathbb{C}^* \times \mathbb{C}^* \times \{1\}$ provides a local slice to the action through ρ, upon which the stabilizer of ρ acts trivially. The tangent space to the orbit through ρ may be identified with $B^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$. We know that $\dim_{\mathbb{C}} H^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C})) = 2$ from the cohomology computations. q.e.d.

For $\gamma \in \Gamma$ we define a function $t_\gamma : R(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}$ by $t_\gamma(\rho) = \mathrm{tr}\, \rho(\gamma)$. If ρ is a smooth point of $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then t_γ is smooth near ρ. Since tr is invariant under conjugation, t_γ descends to a map on the quotient $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, which we again refer to as t_γ. If $\chi = [\rho]$ is a smooth point of $X(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then t_γ is smooth near χ.

Let $\rho = \iota_{T^2}^* \mathrm{hol}$ and let $z \in Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$ be given. If we have a deformation of ρ, i.e., a family of representations $\rho_t : \pi_1 T^2 \to \mathrm{SL}_2(\mathbb{C})$ with $\rho_0 = \rho$, which is tangent to z, i.e., $z(\gamma) = \frac{d}{dt}\big|_{t=0} \rho_t(\gamma) \rho(\gamma)^{-1}$ for all $\gamma \in \pi_1 T^2$, we have that the infinitesimal change of the trace of $\rho(\gamma)$ is given as

$$dt_\gamma(z) = \tfrac{d}{dt}\big|_{t=0} \operatorname{tr} \rho_t(\gamma) = \operatorname{tr}\left(z(\gamma)\rho(\gamma)\right).$$

We wish to apply this to $z_{ang}, z_{shr}, z_{tws}$ and z_{len}. Let $\mu \in \pi_1 T^2$ be the meridian and $\lambda \in \pi_1 T^2$ the longitude. We assume that

$$\rho(\lambda) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

and

$$\rho(\mu) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

with $\eta, \xi \neq \pm 1$. Then ρ preserves the axis $\gamma = \{0\} \times \mathbb{R}_+ \subset \mathbf{H}^3$, if we work in the upper half-space model $\mathbf{H}^3 = \mathbb{C} \times \mathbb{R}_+$. If we use cylindrical coordinates (r, θ, z) around γ, then we have already observed that

$$\sigma_{\partial/\partial\theta} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

and

$$\sigma_{\partial/\partial z} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}).$$

Let us concentrate on the value of the cocycles $z_{ang}, z_{shr}, z_{tws}$ and z_{len} on the meridian $\mu \in \pi_1 T^2$. We obtain

$$z_{ang}(\mu) = \frac{\alpha}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$
$$z_{shr}(\mu) = \frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}),$$

while

$$z_{tws}(\mu) = z_{len}(\mu) = 0.$$

As a consequence we obtain for the infinitesimal change of trace

$$dt_\mu(z_{ang}) = (i\alpha/2)(\xi - \xi^{-1}) \in \mathbb{C}$$
$$dt_\mu(z_{shr}) = (\alpha/2)(\xi - \xi^{-1}) \in \mathbb{C},$$

while

$$dt_\mu(z_{tws}) = dt_\mu(z_{len}) = 0.$$

Note that $\xi - \xi^{-1} \neq 0$ since $\xi \neq \pm 1$. Since the cohomology classes of the cocycles $\{z_{ang}, z_{shr}, z_{tws}, z_{len}\}$ provide a $\mathbb{R}$-basis of $H^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$, we obtain as a consequence of the above calculations:

Lemma 6.18. *The function t_μ has $\mathbb{C}$-rank 1 at $\chi = [\iota_{T^2}^* \operatorname{hol}]$. In particular, the level-set $V = \{t_\mu \equiv t_\mu(\chi)\}$ is locally around χ a smooth, half-dimensional submanifold of $X(\pi_1 T^2, \mathrm{SL}_2(\mathbb{C}))$. Furthermore, the cohomology class of the cocycle z_{len} provides a $\mathbb{C}$-basis for $T_\chi V$. The cohomology classes of the cocycles $\{z_{tws}, z_{len}\}$ provide a $\mathbb{R}$-basis of $T_\chi V$.*

We now turn to the spherical case.

Lemma 6.19. *Let $\rho_i = \iota_{T^2}^* \operatorname{hol}_i : \pi_1 T^2 \to \mathrm{SU}(2)$. Then ρ_i is a smooth point of $R(\pi_1 T^2, \mathrm{SU}(2))$. The local $\mathbb{R}$-dimension of $R(\pi_1 T^2, \mathrm{SU}(2))$ around ρ_i equals 4. Furthermore, the tangent space $T_{\rho_i} R(\pi_1 T^2, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 T^2, \mathfrak{su}(2))$.*

Proof. As above we define a map

$$\begin{aligned} F : S^1 \times S^1 \times \mathrm{SU}(2) &\longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2) \\ (\lambda, \mu, A) &\longmapsto \left(A \operatorname{diag}(\lambda, \lambda^{-1}) A^{-1}, A \operatorname{diag}(\mu, \mu^{-1}) A^{-1} \right). \end{aligned}$$

We consider the standard $\mathbb{R}$-basis of $\mathfrak{su}(2)$:

$$\left\{ i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}.$$

Now $\mathbb{R} \cdot i$ exponentiates to $Z(\rho(\pi_1 T^2)) = \{\operatorname{diag}(\eta, \eta^{-1}) | \eta \in S^1\}$, the stabilizer of ρ under the conjugation action of $\mathrm{SU}(2)$. It is easily verified that

$$\{dF(1,0,0), dF(0,1,0), dF(0,0,j), dF(0,0,k)\}_{(\lambda,\mu,1)}$$

are linearly independent if $\lambda \neq \pm 1$ or $\mu \neq \pm 1$. The result follows as above. q.e.d.

Corollary 6.20. *$\chi_i = [\iota_{T^2}^* \operatorname{hol}_i]$ is a smooth point of $X(\pi_1 T^2, \mathrm{SU}(2))$. The local $\mathbb{R}$-dimension of $X(\pi_1 T^2, \mathrm{SU}(2))$ around χ_i equals 2. Furthermore, the tangent space $T_{\chi_i} X(\pi_1 T^2, \mathrm{SU}(2))$ may be identified with $H^1(\pi_1 T^2, \mathfrak{su}(2))$.*

Proof. The restriction of F to $S^1 \times S^1 \times \{1\}$ provides a local slice to the action through ρ_i, upon which the stabilizer of ρ acts trivially. The tangent space to the orbit through ρ_i may be identified with $B^1(\pi_1 T^2, \mathfrak{su}(2))$. From the cohomology computations we have $\dim_{\mathbb{R}} H^1(\pi_1 T^2, \mathfrak{su}(2)) = 2$. q.e.d.

For $\gamma \in \Gamma$ we define a function $t_\gamma : R(\Gamma, \mathrm{SU}(2)) \to \mathbb{R}$ by $t_\gamma(\rho) = \operatorname{tr} \rho(\gamma)$. If ρ is a smooth point of $R(\Gamma, \mathrm{SU}(2))$, then t_γ is smooth near ρ. Since tr is invariant under conjugation, t_γ descends to a map on the quotient $X(\Gamma, \mathrm{SU}(2))$, which we again refer to as t_γ. If $\chi = [\rho]$ is a smooth point of $X(\Gamma, \mathrm{SU}(2))$, then t_γ is smooth in a neighbourhood of χ.

For a representation $\rho = (\rho_1, \rho_2) : \Gamma \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\gamma \in \Gamma$ let $T_\gamma^i(\rho) = t_\gamma(\rho_i)$. This defines an $\mathbb{R}^2$-valued function $T_\gamma = (T_\gamma^1, T_\gamma^2)$ on $R(\Gamma, \mathrm{SU}(2) \times \mathrm{SU}(2))$, which we view as a “complex” trace function.

Let $\rho = \iota_{T^2}^* \operatorname{hol}$ and let $z = (z_1, z_2) \in Z^1(\pi_1 T^2, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$ be given. The infinitesimal change of the trace of $\rho(\gamma)$ is given as

$$dT_\gamma(z) = (dt_\gamma(z_1), dt_\gamma(z_2)).$$

We wish to apply this to $z_{ang}, z_{shr}, z_{tws}$ and z_{len}. Let $\lambda \in \pi_1 T^2$ be the meridian and $\mu \in \pi_1 T^2$ the longitude. We assume that

$$\rho(\lambda) = \left(\begin{pmatrix} \eta_1 & 0 \\ 0 & \overline{\eta}_1 \end{pmatrix}, \begin{pmatrix} \eta_2 & 0 \\ 0 & \overline{\eta}_2 \end{pmatrix} \right) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$$

and

$$\rho(\mu) = \left(\begin{pmatrix} \xi_1 & 0 \\ 0 & \overline{\xi}_1 \end{pmatrix}, \begin{pmatrix} \xi_2 & 0 \\ 0 & \overline{\xi}_2 \end{pmatrix} \right) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$$

with $\xi_1 = \xi_2 =: \xi$ and $\xi \neq \pm 1$, since $\rho(\mu)$ is a nontrivial rotation. Then ρ preserves the pair of axes $\{\gamma, \gamma^\perp\}$, where $\gamma = \mathbb{C} \cap \mathbf{S}^3$ and $\gamma^\perp = \mathbb{C}j \cap \mathbf{S}^3$. If we use cylindrical coordinates (r, θ, z) around γ, then we have already observed that

$$\sigma_{\partial/\partial\theta} = \left(\frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

and

$$\sigma_{\partial/\partial z} = \left(\frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

In particular, this implies that $\sigma_{\partial/\partial\theta} + \sigma_{\partial/\partial z} \in \Gamma(U_\varepsilon(\Sigma), \mathcal{E}_1)$, and on the other hand $\sigma_{\partial/\partial\theta} - \sigma_{\partial/\partial z} \in \Gamma(U_\varepsilon(\Sigma), \mathcal{E}_2)$. Therefore we have

$$\omega_{tws} + \omega_{len} \in \Omega^1(U_\varepsilon(\Sigma), \mathcal{E}_1)$$

and

$$\omega_{tws} - \omega_{len} \in \Omega^1(U_\varepsilon(\Sigma), \mathcal{E}_2).$$

Again, we concentrate on the value of the cocycles $z_{ang}, z_{shr}, z_{tws}$ and z_{len} on the meridian $\mu \in \pi_1 T^2$. We obtain

$$z_{ang}(\mu) = \left(\frac{\alpha}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\alpha}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

$$z_{shr}(\mu) = \left(\frac{\alpha}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\alpha}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2),$$

while

$$z_{tws}(\mu) = z_{len}(\mu) = 0.$$

As a consequence we obtain for the infinitesimal change of trace

$$dT_\mu(z_{ang}) = \alpha(-\operatorname{Im}\xi, -\operatorname{Im}\xi) \in \mathbb{R}^2$$

$$dT_\mu(z_{shr}) = \alpha(-\operatorname{Im}\xi, +\operatorname{Im}\xi) \in \mathbb{R}^2,$$

while

$$dT_\mu(z_{tws}) = dT_\mu(z_{len}) = 0.$$

Note that $\operatorname{Im}\xi = \frac{1}{2i}(\xi - \overline{\xi}) \neq 0$ since $\xi \neq \pm 1$. Since the cohomology classes of the cocycles $\{z_{ang}, z_{shr}, z_{tws}, z_{len}\}$ provide a $\mathbb{R}$-basis of $H^1(\pi_1 T^2, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$, we obtain as a consequence of the above calculations:

Lemma 6.21. *The function t_μ has $\mathbb{R}$-rank 1 at $\chi_i = [\iota_{T^2}^* \operatorname{hol}_i]$. In particular, the level-set $V_i = \{t_\mu \equiv t_\mu(\chi_i)\}$ is locally around χ_i a smooth, half-dimensional submanifold of $X(\pi_1 T^2, \mathrm{SU}(2))$. Furthermore the cohomology class of the cocycle $z_{tws} + z_{len}$ provides a $\mathbb{R}$-basis of $T_{\chi_1} V_1$, and the cohomology class of the cocycle $z_{tws} - z_{len}$ provides a $\mathbb{R}$-basis of $T_{\chi_2} V_2$.*

6.6.2. The higher genus case. Let $\iota : F_g \to M$ be the inclusion of a boundary component of higher genus $g \geq 2$. ι induces a group homomorphism $\iota_* : \pi_1 F_g \to \pi_1 M$ and a map $\iota^* : R(\pi_1 M, G) \to R(\pi_1 F_g, G)$ for $G = \mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SU}(2)$ respectively.

Lemma 6.22. *Let $\rho : \pi_1 F_g \to \mathrm{SL}_2(\mathbb{C})$ be irreducible. Then ρ is a smooth point of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$. The local $\mathbb{C}$-dimension of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ around ρ equals $6g-3$. $T_\rho R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$.*

Proof. We identify $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ with the (affine algebraic) set

$$\left\{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{SL}_2(\mathbb{C})^{2g} | f(A_1, B_1, \ldots, A_g, B_g) = 1\right\},$$

where $f(A_1, B_1, \ldots, A_g, B_g) = [A_1, B_1] \cdot \ldots \cdot [A_g, B_g]$. $\ker d_\rho f$ may be identified with the space of 1-cocycles $Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$. From the cohomology computations we know that $\dim_{\mathbb{C}} H^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C})) = 6g-6$. Since ρ is irreducible, we have $Z^0(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C})) = 0$, which implies $\dim_{\mathbb{C}} Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C})) = 6g-3$. Hence $\operatorname{rank}_{\mathbb{C}} d_\rho f = 3$, i.e., $d_\rho f$ is surjective. Now the implicit function theorem implies that $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ is smooth at ρ with $T_\rho R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C})) = Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$. q.e.d.

Corollary 6.23. *Let $\rho = \iota_{F_g}^* \operatorname{hol} : \pi_1 F_g \to \mathrm{SL}_2(\mathbb{C})$. Then ρ is a smooth point of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$. The local $\mathbb{C}$-dimension of $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ around ρ equals $6g-3$. Furthermore $T_\rho R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$.*

Proof. Clearly ρ is irreducible: If $v \in \Sigma$ is a singular vertex and we restrict hol further to $U_\varepsilon(v)$, which deformation-retracts to a pair of pants $P \subset F_g$, then $\iota_P^* \operatorname{hol}$ preserves a point $p \in \mathbf{H}^3$. Now if ρ was reducible, then $\iota_P^* \operatorname{hol}$ would preserve a geodesic, which is a contradiction. q.e.d.

Although the following is a well-known fact about the action of $\mathrm{SL}_2(\mathbb{C})$ on the irreducible part of $R(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, for convenience of the reader we give a proof:

Lemma 6.24. *The action of $\mathrm{SL}_2(\mathbb{C})$ on $R_{irr}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ is proper.*

Proof. Let X be a G-space. If we have a continuous G-equivariant map from X to a proper G-space Y, then X itself will be a proper G-space. We construct a continuous, $\mathrm{SL}_2(\mathbb{C})$-equivariant map

$$\begin{aligned} R_{irr}(\Gamma, \mathrm{SL}_2(\mathbb{C})) &\longrightarrow \mathbf{H}^3 \\ \rho &\longmapsto \operatorname{center}(\rho), \end{aligned}$$

where the "center" of a representation will be the point in $\mathbf{H}^3$, which is displaced the least in average by the generators of the group. More precisely, let us fix a presentation $\langle \gamma_1, \dots, \gamma_n | (r_i)_{i \in I} \rangle$ of Γ. Note that the (modified) displacement function of $A \in \mathrm{SL}_2(\mathbb{C})$

$$\begin{aligned} \delta_A : \mathbf{H}^3 &\longrightarrow \mathbb{R} \\ x &\longmapsto \cosh d(x, Ax) - 1 \end{aligned}$$

is a convex function in general. It is strictly convex if A is parabolic. If A is semisimple, it is strictly convex along any geodesic different from the axis of A. We define

$$f_\rho(x) = \frac{1}{n} \sum_{i=1}^{n} \delta_{\rho(\gamma_i)}.$$

If we have a sequence $x_n \in \mathbf{H}^3$ which converges to $x_\infty \in \partial_\infty \mathbf{H}^3$, then since ρ is irreducible, there has to be at least one $\rho(\gamma_i)$ that does not fix x_∞. Then it follows that $\delta_{\rho(\gamma_i)}(x_n) \to \infty$. Therefore f_ρ is proper. If we take any geodesic γ, again since ρ is irreducible, there has to be at least one $\rho(\gamma_i)$ such that $\delta_{\rho(\gamma_i)}$ is strictly convex along γ. Therefore f_ρ is strictly convex.

As a proper and strictly convex function, f_ρ assumes its minimum at a unique point in $\mathbf{H}^3$, which we define to be the center of ρ.

If we have a sequence of representations ρ_n converging to ρ with respect to the compact-open topology on $R_{irr}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$, then f_{ρ_n} converges to f_ρ uniformly on compact sets. Therefore the map center is continuous. Since $\delta_{BAB^{-1}}(x) = \delta_A(B^{-1}x)$ we obtain that the map center is $\mathrm{SL}_2(\mathbb{C})$-equivariant.

This, together with the fact that the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbf{H}^3$ is proper, proves the lemma. q.e.d.

Corollary 6.25. $\chi = [\iota_{F_g}^* \mathrm{hol}]$ *is a smooth point of* $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$. *The local* $\mathbb{C}$*-dimension of* $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ *around* χ *equals* $6g - 6$. $T_\chi X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ *may be identified with* $H^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C}))$.

Proof. Since the action of $\mathrm{SL}_2(\mathbb{C})$ is proper, we have a local slice to the action. We recall that the stabilizer of ρ, $Z(\rho(\pi_1 F_g))$, equals $\{\pm 1\}$. Therefore $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ is locally around χ the quotient of a free $\mathrm{PSL}_2(\mathbb{C})$ action and therefore smooth. q.e.d.

The meridian curves around the singularity give rise to a pair-of-pants decomposition of F_g. Let $\{\mu_1, \dots, \mu_N\}$ be the family of meridians, where $N = 3g - 3$. This may be used to give an alternative construction of $R(\pi_1(F_g), \mathrm{SL}_2(\mathbb{C}))$, which is better suited for certain purposes.

Let P denote the thrice-punctured sphere, i.e., a pair of pants. The fundamental group of P is the free group on 2 generators. We will use the following slightly redundant presentation:

$$\pi_1 P = \langle \mu_1, \mu_2, \mu_3 | \mu_1 \mu_2 \mu_3 = 1 \rangle.$$

It follows that

$$R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) = \{(A_1, A_2, A_3) \in \mathrm{SL}_2(\mathbb{C})^3 | A_1 A_2 A_3 = 1\}.$$

Clearly the map $f : \mathrm{SL}_2(\mathbb{C})^3 \to \mathrm{SL}_2(\mathbb{C}), (A_1, A_2, A_3) \mapsto A_1 A_2 A_3$ is a submersion, such that $R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) = f^{-1}(1)$ is a smooth submanifold of $\mathbb{C}$-dimension 6.

Let $\iota_i : S^1 \to P$ be the inclusion of the i-th boundary circle. Then the induced map $\iota_i^* : R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C}))$ corresponds to the projection $pr_i : R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) \to \mathrm{SL}_2(\mathbb{C}), (A_1, A_2, A_3) \mapsto A_i$, which is also a submersion.

The verification of the following statement is elementary and left to the reader:

Lemma 6.26. *Let $\rho = \iota_P^* \mathrm{hol}$ be the restriction of the holonomy of a hyperbolic cone-manifold structure to a pair of pants P. Then the differentials $\{dt_{\mu_1}, dt_{\mu_2}, dt_{\mu_3}\}$ are $\mathbb{C}$-linearly independent in $T_\rho^* R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$.*

Since $\rho = \iota_P^* \mathrm{hol}$ is irreducible, we can use Lemma 6.24 to conclude that $\chi = [\rho]$ is a smooth point in $X(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$. The local $\mathbb{C}$-dimension of $X(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$ around χ is 3. The functions $\{t_{\mu_1}, t_{\mu_2}, t_{\mu_3}\}$ are local holomorphic coordinates on $X(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$ near χ.

We build up $R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ from $R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$ using two basic operations:

1) glue a pair of pants P to a connected surface with boundary S along a boundary circle, call the resulting connected surface S'
2) glue a connected surface S along two different boundary circles, call the resulting connected surface S'.

In the first case $\pi_1 S' = \pi_1 S \amalg_{\pi_1 S^1} \pi_1 P$ by van Kampen's theorem and we have

$$R(\pi_1 S', \mathrm{SL}_2(\mathbb{C})) = R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \times_{R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C}))} R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$$

via the maps

$$\iota_{S^1 \hookrightarrow S}^* : R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C}))$$

and

$$\iota_{S^1 \hookrightarrow P}^* : R(\pi_1 P, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S^1, \mathrm{SL}_2(\mathbb{C})),$$

which will be transversal since the latter one is a submersion. Therefore $\rho = \iota_{S'}^* \mathrm{hol}$ is a smooth point in $R(\pi_1 S', \mathrm{SL}_2(\mathbb{C}))$ since $\rho_S = \iota_S^* \mathrm{hol}$ is a smooth point in $R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))$ and $\rho_P = \iota_P^* \mathrm{hol}$ is a smooth point in $R(\pi_1 P, \mathrm{SL}_2(\mathbb{C}))$.

In the second case $\pi_1 S'$ splits as an HNN-extension of $\pi_1 S$. More precisely, if $\mu_1, \mu_2 \in \pi_1 S$ are the loops around the boundary circles, which will be identified, then $\pi_1 S' = \langle \pi_1 S, \lambda | \lambda \mu_1 \lambda^{-1} = \mu_2 \rangle$. In this case we have

$$\begin{aligned} R(\pi_1 S', \mathrm{SL}_2(\mathbb{C})) &= \{(\rho_S, B) | B\rho_S(\mu_1)B^{-1} = \rho_S(\mu_2)\} \\ &\subset R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \times \mathrm{SL}_2(\mathbb{C}) \end{aligned}$$

as a consequence. We show that the map

$$\begin{aligned} f : R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})) \times \mathrm{SL}_2(\mathbb{C}) &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ (\rho_S, B) &\longmapsto B\rho_S(\mu_1)B^{-1}\rho_S(\mu_2)^{-1} \end{aligned}$$

is a submersion near $\rho = \iota_{S'}^* \mathrm{hol}$. This implies that $\rho = (\rho_S, B)$ is a smooth point in $R(\pi_1 S', \mathrm{SL}_2(\mathbb{C}))$.

Surjectivity of df at ρ can be established as follows: Let $A_1 = \rho_S(\mu_1)$ and $A_2 = \rho_S(\mu_2)$. Clearly the map $B \mapsto BA_1B^{-1}A_2^{-1}$ has $\mathbb{C}$-rank 2. Since $\{dt_{\mu_1}, dt_{\mu_2}\}$ are linearly independent, we can construct a deformation $t \mapsto (\rho_S)_t$ with $(\rho_S)_t(\mu_2) = A_2$ and $dt_{\mu_1}(\dot{\rho}_S) \neq 0$. This deformation will be transverse to $\mathrm{im}(B \mapsto BA_1B^{-1}A_2^{-1})$.

From the construction given above the following is immediate:

Lemma 6.27. *The differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ with $N = 3g - 3$ are linearly independent over $\mathbb{C}$ in $T_\rho^* R(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ for $\rho = \iota_{F_g}^* \mathrm{hol}$.*

Clearly

$$z_{tws}^i(\mu_j) = \int_{\mu_j} \omega_{tws}^i = 0$$

and

$$z_{len}^i(\mu_j) = \int_{\mu_j} \omega_{len}^i = 0.$$

Therefore

$$dt_{\mu_j}(z_{tws}^i) = 0$$

and

$$dt_{\mu_j}(z_{len}^i) = 0.$$

As a consequence of this we obtain the following lemma.

Lemma 6.28. *The level-set $V = \{t_{\mu_1} \equiv t_{\mu_1}(\chi), \ldots, t_{\mu_N} \equiv t_{\mu_N}(\chi)\}$ is locally a smooth, half-dimensional submanifold of $X(\pi_1 F_g, \mathrm{SL}_2(\mathbb{C}))$ around $\chi = [\iota_{F_g}^* \mathrm{hol}]$. Furthermore, the cohomology classes of the cocycles $\{z_{len}^1, \ldots, z_{len}^N\}$ provide a $\mathbb{C}$-basis of $T_\chi V$. Similarly, the cohomology classes of the cocycles $\{z_{tws}^1, z_{len}^1, \ldots, z_{tws}^N, z_{len}^N\}$ provide a $\mathbb{R}$-basis for $T_\chi V$.*

We now turn to the spherical case.

Lemma 6.29. *Let $\rho : \pi_1 F_g \to \mathrm{SU}(2)$ be irreducible. Then ρ is a smooth point of $R(\pi_1 F_g, \mathrm{SU}(2))$. The local $\mathbb{R}$-dimension of $R(\pi_1 F_g, \mathrm{SU}(2))$ around ρ equals $6g-3$. $T_\rho R(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{su}(2))$.*

Proof. This follows as in the case of $\mathrm{SL}_2(\mathbb{C})$ from the cohomology computations and the implicit function theorem. q.e.d.

Corollary 6.30. *Let $\rho_i = \iota^*_{F_g}\,\mathrm{hol}_i : \pi_1\,F_g \to \mathrm{SU}(2)$. Then ρ_i is a smooth point of $R(\pi_1 F_g, \mathrm{SU}(2))$. The local $\mathbb{R}$-dimension of $R(\pi_1 F_g, \mathrm{SU}(2))$ around ρ_i equals $6g-3$. Furthermore $T_{\rho_i}R(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $Z^1(\pi_1 F_g, \mathfrak{su}(2))$.*

Proof. Clearly the ρ_i are both irreducible: If $v \in \Sigma$ is a singular vertex and we restrict $\mathrm{hol} = (\mathrm{hol}_1, \mathrm{hol}_2)$ further to $U_\varepsilon(v)$, which deformation-retracts to a pair of pants $P \subset F_g$, then $\iota^*_P\,\mathrm{hol}$ preserves a point $p \in \mathbf{S}^3$. Without loss of generality we may assume that $p = 1 \in \mathbf{S}^3 \subset \mathbb{H}$. Then since

$$\mathrm{Stab}_{\mathrm{SU}(2)\times\mathrm{SU}(2)}(1) = \{(A, A) : A \in \mathrm{SU}(2)\},$$

we obtain that $\iota^*_P\,\mathrm{hol}_1 = \iota^*_P\,\mathrm{hol}_2$. Now if ρ_1 or ρ_2 were reducible, then $\iota^*_P\,\mathrm{hol}$ would preserve a geodesic, which is a contradiction. q.e.d.

Corollary 6.31. *$\chi_i = [\iota^*_{F_g}\,\mathrm{hol}_i]$ is a smooth point of $X(\pi_1 F_g, \mathrm{SU}(2))$. The local $\mathbb{R}$-dimension of $X\,(\pi_1\,F_g,\,\mathrm{SU}\,(2))$ around χ_i equals $6g-6$. $T_{\chi_i}X(\pi_1 F_g, \mathrm{SU}(2))$ may be identified with $H^1(\pi_1 F_g, \mathfrak{su}(2))$.*

Proof. Since the group $\mathrm{SU}(2)$ is compact, the properness of the action is granted. We recall that the stabilizer of ρ_i, $Z(\rho_i(\pi_1 F_g))$, equals $\{\pm 1\}$. Therefore $X(\pi_1 F_g, \mathrm{SU}(2))$ is near χ_i a quotient of a free $\mathrm{PSU}(2)$ action and therefore smooth. q.e.d.

Lemma 6.32. *The differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ with $N = 3g-3$ are linearly independent in $T^*_{\rho_i}R(\pi_1 F_g, \mathrm{SU}(2))$ for $\rho_i = \iota^*_{F_g}\,\mathrm{hol}_i$.*

Proof. The arguments in the hyperbolic case apply without essential change. q.e.d.

We obtain finally:

Lemma 6.33. *The level-set $V_i = \{t_{\mu_1} \equiv t_{\mu_1}(\chi_i), \ldots, t_{\mu_N} \equiv t_{\mu_N}(\chi_i)\}$ is locally a smooth, half-dimensional submanifold of $X(\pi_1 F_g, \mathrm{SU}(2))$ around $\chi_i = [\iota^*_{F_g}\,\mathrm{hol}_i]$. The cohomology classes of the cocycles $\{z^1_{tws} + z^1_{len}, \ldots, z^N_{tws} + z^N_{len}\}$ provide a $\mathbb{R}$-basis for $T_{\chi_1}V_1$, and similarly the cohomology classes of the cocycles $\{z^1_{tws} - z^1_{len}, \ldots, z^N_{tws} - z^N_{len}\}$ provide a $\mathbb{R}$-basis for $T_{\chi_2}V_2$.*

6.7. Local rigidity.

Lemma 6.34. *Let C be a hyperbolic or a spherical cone-3-manifold with cone-angles $\leq \pi$. Then:*

1) *The natural map $H^1(M, \mathcal{E}) \to H^1(\partial M_\varepsilon, \mathcal{E})$ is injective.*
2) $\dim H^1(M, \mathcal{E}) = \frac{1}{2}\dim H^1(\partial M_\varepsilon, \mathcal{E})$.

In the spherical case, the assertions hold for the parallel subbundles $\mathcal{E}_i \subset \mathcal{E}$ individually.

Proof. Let us look at a part of the long exact cohomology sequence of the pair $(M_\varepsilon, \partial M_\varepsilon)$ with coefficients in $\mathcal{E}$. The natural map $q : H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) \to H^1(M_\varepsilon, \mathcal{E})$ factors through L^2-cohomology, since $H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) = H^1_{cp}(M, \mathcal{E})$:

$$\begin{tikzcd} {} \arrow[r] & H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) \arrow[r, "q"] \arrow[d, equal] & H^1(M_\varepsilon, \mathcal{E}) \arrow[r, "r"] & H^1(\partial M_\varepsilon, \mathcal{E}) \arrow[r] & {} \\ & H^1_{cp}(M, \mathcal{E}) \arrow[r] & H^1_{L^2}(M, \mathcal{E}) \arrow[u] & & \end{tikzcd}$$

Since by our vanishing theorem $H^1_{L^2}(M, \mathcal{E}) = 0$, we have that q is the zero map and $r : H^1(M_\varepsilon, \mathcal{E}) \to H^1(\partial M_\varepsilon, \mathcal{E})$ is injective.

Since the Killing form B on $\mathcal{E}$ (resp. the parallel metric $h^{\mathcal{E}}$ in the spherical case) provides a non-degenerate coefficient pairing, we can apply Poincaré duality to conclude that $H^2(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) \cong H^1(M_\varepsilon, \mathcal{E})^*$ and $H^2(M_\varepsilon, \mathcal{E}) \cong H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E})^*$. The Poincaré duality isomorphisms are natural, such that we obtain the following commutative diagram:

$$\begin{tikzcd} & H^1(M_\varepsilon, \mathcal{E})^* \arrow[r, "q^*"] & H^1(M_\varepsilon, \partial M_\varepsilon, \mathcal{E})^* & \\ {} \arrow[r] & H^2(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) \arrow[r] \arrow[u, "\cong"', "\text{P.D.}"] & H^2(M_\varepsilon, \mathcal{E}) \arrow[r] \arrow[u, "\cong"', "\text{P.D.}"] & {} \end{tikzcd}$$

Since $q^* = 0$, we obtain the following short exact sequence:

$$\begin{tikzcd} & & & H^1(M_\varepsilon, \mathcal{E})^* & \\ 0 \arrow[r] & H^1(M_\varepsilon, \mathcal{E}) \arrow[r] & H^1(\partial M_\varepsilon, \mathcal{E}) \arrow[r] & H^2(M_\varepsilon, \partial M_\varepsilon, \mathcal{E}) \arrow[r] \arrow[u, "\cong"', "\text{P.D.}"] & 0 \end{tikzcd}$$

This implies that $\dim H^1(M_\varepsilon, \mathcal{E}) = \frac{1}{2} \dim H^1(\partial M_\varepsilon, \mathcal{E})$. In the spherical case these arguments apply to the parallel subbundles $\mathcal{E}_i \subset \mathcal{E}$. q.e.d.

6.7.1. The hyperbolic case. The following is a well-known fact about the holonomy representation of a hyperbolic cone-manifold structure, and for convenience of the reader we give a proof:

Lemma 6.35. *The holonomy of a hyperbolic cone-manifold structure* $\mathrm{hol} : \pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ *is irreducible.*

Proof. Let us assume that the holonomy representation is reducible. Then there is a point $x_\infty \in \partial_\infty \mathbf{H}^3$ fixed by the holonomy. The volume decreasing flow, which moves each point x with unit speed towards x_∞ along the unique geodesic connecting x and x_∞, may then be pulled back via the developing map to a volume decreasing flow on M. This is a contradiction since M has finite volume. q.e.d.

Lemma 6.36. *Let* hol : $\pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ *be the holonomy of a hyperbolic cone-manifold structure with cone-angles* $\leq \pi$. *Then* hol *is a smooth point of* $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$. *The* $\mathbb{C}$*-dimension of* $R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ *around* hol *equals* $\tau + 3 - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components contained in* ∂M_ε. $T_{\mathrm{hol}} R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ *may be identified with* $Z^1(\pi_1 M, \mathfrak{sl}_2(\mathbb{C}))$.

Proof. We follow the discussion in M. Kapovich's book (cf. [**Kap**]), which essentially amounts to a transversality argument. The key to the proof is the following splitting of M_ε:

Lemma 6.37 ([**Kap**, Lm. 8.46])**.** *There is a system of disjoint* 1*-handles* $\{H_1, \ldots, H_t\}$ *in* M_ε *attached to* ∂M_ε *such that* $M_1 := M_\varepsilon \setminus \mathrm{int}(\cup_i H_i)$ *is a handlebody.*

As a consequence M_ε may be written as a union

$$M_\varepsilon = M_1 \cup_S M_2,$$

where S is a surface of genus $g = 1 + t - \chi(\partial M_\varepsilon)/2$. M_2 is homotopy equivalent to the wedge product of the components of ∂M_ε and $t-b+1$ circles, where b is the number of components of ∂M_ε. Therefore we obtain by van Kampen's theorem

$$\pi_1 M_\varepsilon = \pi_1 M_1 \amalg_{\pi_1 S} \pi_1 M_2,$$

where $\pi_1 M_1$ is the free group on g generators, and $\pi_1 M_2$ splits as a free product of the fundamental groups of the components of ∂M_ε and $t-b+1$ $\mathbb{Z}$-factors. Consequently we obtain for the representation varieties

$$R(\pi_1 M_\varepsilon, \mathrm{SL}_2(\mathbb{C})) = R(\pi_1 M_1, \mathrm{SL}_2(\mathbb{C})) \times_{R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))} R(\pi_1 M_2, \mathrm{SL}_2(\mathbb{C}))$$

via the maps

$$\mathrm{res}_1 : R(\pi_1 M_1, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))$$

and

$$\mathrm{res}_2 : R(\pi_1 M_2, \mathrm{SL}_2(\mathbb{C})) \to R(\pi_1 S, \mathrm{SL}_2(\mathbb{C})).$$

$R(\pi_1 M_1, \mathrm{SL}_2(\mathbb{C}))$ and $R(\pi_1 M_2, \mathrm{SL}_2(\mathbb{C}))$ are smooth near the restriction of the holonomy of a hyperbolic cone-manifold structure. Note that $\pi_1 S$ surjects onto $\pi_1 M_\varepsilon$. Since hol is irreducible, this will also be the case for ι_S^* hol, which is therefore seen to be a smooth point of $R(\pi_1 S, \mathrm{SL}_2(\mathbb{C}))$.

Therefore it is sufficient to show that res_1 and res_2 meet transversally at ι_S^* hol. This will follow from the equation

$$\begin{aligned} &\dim_{\mathbb{C}} Z^1(\pi_1 M_1, \mathfrak{sl}_2(\mathbb{C})) + \dim_{\mathbb{C}} Z^1(\pi_1 M_2, \mathfrak{sl}_2(\mathbb{C})) \\ &= \dim_{\mathbb{C}} Z^1(\pi_1 S, \mathfrak{sl}_2(\mathbb{C}) + \dim_{\mathbb{C}} Z^1(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C})), \end{aligned}$$

if we use the identification

$$\begin{aligned} Z^1(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C})) &= \{(z_1, z_2) \mid d\,\mathrm{res}_1(z_1) = d\,\mathrm{res}_2(z_2)\} \\ &\subset Z^1(\pi_1 M_1, \mathfrak{sl}_2(\mathbb{C})) \oplus Z^1(\pi_1 M_2, \mathfrak{sl}_2(\mathbb{C})). \end{aligned}$$

To obtain the desired equation, we have to calculate the dimensions of the cocycle spaces; note that $Z^1(\Gamma \amalg \Gamma', \mathfrak{g}) = Z^1(\Gamma, \mathfrak{g}) \oplus Z^1(\Gamma', \mathfrak{g})$:

1) $\dim_{\mathbb{C}} Z^1(\pi_1 M_1, \mathfrak{sl}_2(\mathbb{C})) = 3 + 3t - \frac{3}{2}\chi(\partial M_\varepsilon)$, since $\pi_1 M_1$ is the free group on $g = 1 + t - \chi(\partial M_\varepsilon)/2$ generators.
2) $\dim_{\mathbb{C}} Z^1(\pi_1 M_2, \mathfrak{sl}_2(\mathbb{C})) = \tau - 3\chi(\partial M_\varepsilon) + 3t + 3$, since $\dim_{\mathbb{C}} Z^1(\pi_1 T^2, \mathfrak{sl}_2(\mathbb{C}))$ equals 4 at $\iota_{T^2}^*$ hol, $\dim_{\mathbb{C}} Z^1(\pi_1 F_g, \mathfrak{sl}_2(\mathbb{C})) = -3\chi(F_g) + 3$ at $\iota_{F_g}^*$ hol and the fundamental group of a wedge of $t - b + 1$ circles is the free group on that number of generators.
3) $\dim_{\mathbb{C}} Z^1(\pi_1 S, \mathfrak{sl}_2(\mathbb{C})) = 6t - 3\chi(\partial M_\varepsilon) + 3$, since ι_S^* hol is irreducible.
4) $\dim_{\mathbb{C}} Z^1(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C})) = \tau - \frac{3}{2}\chi(\partial M_\varepsilon) + 3$, since $\dim_{\mathbb{C}} H^1(M_\varepsilon, \mathcal{E})$ equals $\frac{1}{2}\dim_{\mathbb{C}} H^1(\partial M_\varepsilon, \mathcal{E})$ by Lemma 6.34; furthermore hol is irreducible, therefore $Z^0(\pi_1 M_\varepsilon, \mathfrak{sl}_2(\mathbb{C})) = 0$.

This finishes the proof. q.e.d.

Corollary 6.38. $\chi = [\mathrm{hol}]$ *is a smooth point of* $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})$. *The* $\mathbb{C}$-*dimension of* $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})$ *around* χ *equals* $\tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components in* ∂M_ε. $T_\chi X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ *may be identified with* $H^1(\pi_1 M, \mathfrak{sl}_2(\mathbb{C}))$.

Proof. $Z(\mathrm{hol}(\pi_1 M)) = \{\pm 1\}$ since hol is irreducible. Using Lemma 6.24 we proceed in the same way as in the surface case. q.e.d.

We are now ready to state and prove the main result in the hyperbolic case:

Theorem 6.39. *Let* C *be a hyperbolic cone-3-manifold with cone-angles* $\leq \pi$. *Let* $\{\mu_1, \ldots, \mu_N\}$ *be the family of meridians, where* N *is the number of edges contained in* Σ. *Then the map*

$$X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{C}^N, \chi \mapsto (t_{\mu_1}(\chi), \ldots, t_{\mu_N}(\chi))$$

is locally biholomorphic near $\chi = [\mathrm{hol}]$.

Proof. Without loss of generality we may assume that Σ is connected. Then we have to consider two cases:

1) Σ is a circle, i.e., $\partial M_\varepsilon = T^2$
2) Σ is a connected, trivalent graph, i.e., $\partial M_\varepsilon = F_g$.

Let us recall what we have already established: In each of the above cases, the level-set of the trace functions

$$V = \{t_{\mu_1} \equiv t_{\mu_1}(\chi), \ldots, t_{\mu_N} \equiv t_{\mu_N}(\chi)\}$$

is a smooth, half-dimensional submanifold of $X(\pi_1 \partial M_\varepsilon, \mathrm{SL}_2(\mathbb{C}))$, since the differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ are $\mathbb{C}$-linearly independent in $H^1(\pi_1 \partial M_\varepsilon, \mathrm{SL}_2(\mathbb{C}))^*$ at χ. If we work in the de-Rham realization of $H^1(\pi_1 \partial M_\varepsilon, \mathrm{SL}_2(\mathbb{C}))$, the classes of the differential forms

$$\{\omega_{len}^1, \ldots, \omega_{len}^N\}$$

provide a $\mathbb{C}$-basis of $T_\chi V$. Furthermore, these forms are L^2-bounded on $U_\varepsilon(\Sigma)$.

On the other hand, the restriction map $H^1(M,\mathcal{E}) \to H^1(\partial M_\varepsilon, \mathcal{E})$ is injective with half-dimensional image. This means that $X(\pi_1 M, SL_2(\mathbb{C}))$ is immersed into $X(\pi_1 \partial M_\varepsilon, SL_2(\mathbb{C}))$ as a half-dimensional submanifold.

We claim that the submanifolds V and $X(\pi_1 M, SL_2(\mathbb{C}))$ are transversal in $X(\pi_1 \partial M_\varepsilon, SL_2(\mathbb{C}))$ at χ. It is sufficient to show that $T_\chi V$ and $\mathrm{im}(H^1(M,\mathcal{E}) \to H^1(\partial M_\varepsilon,\mathcal{E}))$ intersect trivially in $H^1(\partial M_\varepsilon, \mathcal{E})$.

Let $\omega \in \Omega^1(M,\mathcal{E})$ be a closed form such that $[\omega]|_{\partial M_\varepsilon} \in T_\chi V$. In particular, since the forms ω^i_{len} are L^2-bounded on $U_\varepsilon(\Sigma)$, $\omega + d\sigma$ will be L^2-bounded on $U_\varepsilon(\Sigma)$ for some $\sigma \in \Gamma(U_\varepsilon(\Sigma), \mathcal{E})$. We choose a cut-off function φ, which is 1 in a neighbourhood of Σ and which is supported in $U_\varepsilon(\Sigma)$. Then $\varphi\sigma$ extends to a section on M, such that $\omega + d(\varphi\sigma)$ is L^2-bounded on M. Since $H^1_{L^2}(M,\mathcal{E}) = 0$, this implies that $[\omega] = 0$ in $H^1(M,\mathcal{E})$ and therefore $[\omega]|_{\partial M_\varepsilon} = 0$.

It follows that the differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ are $\mathbb{C}$-linearly independent already in $H^1(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))^*$. q.e.d.

The complex length of the i-th meridian is related to its trace via

$$t_{\mu_i}(\rho) = \pm 2\cosh(\mathcal{L}(\rho(\mu_i))/2).$$

Locally the set of representations $\rho : \pi_1 M \to \mathrm{SL}_2(\mathbb{C}))$ such that $\mathcal{L}(\rho(\mu_i))$ is purely imaginary for all $i \in \{1, \ldots, N\}$ corresponds to hyperbolic cone-manifold structures on M. The cone-angle α_i is just the imaginary part of $\mathcal{L}(\rho(\mu_i))$, therefore we obtain using Lemma 6.5:

Corollary 6.40 (local rigidity). *Let C be a hyperbolic cone-3-manifold with cone-angles $\leq \pi$. Then the set of cone-angles $\{\alpha_1, \ldots, \alpha_N\}$, where N is the number of edges contained in Σ, provides a local parametrization of the space of hyperbolic cone-manifold structures near the given structure on M. In particular, there are no deformations leaving the cone-angles fixed.*

6.7.2. The spherical case.

Lemma 6.41. *Let* $\mathrm{hol} : \pi_1 M \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ *be the holonomy of a spherical cone-manifold structure. Then* hol_1 *and* hol_2 *are both non-abelian, unless Σ is a link and M is Seifert fibered.*

Proof. Let us assume that hol_1 is abelian. Then we may assume that the holonomy is contained in $S^1 \times \mathrm{SU}(2)$. This means that the Hopf-fibration on $\mathbf{S}^3 \subset \mathbb{H}$ obtained by left-multiplication with $S^1 \subset \mathbb{H}$ is preserved by the holonomy and may be pulled back via the developing map to a Seifert fibration on M. If hol_2 is abelian, then the Hopf-fibration obtained by right-multiplication with $S^1 \subset \mathbb{H}$ will be invariant under the holonomy, and the same argument applies. In both cases the singular locus Σ has to be a link, since in the presence of vertices hol_1 and hol_2 are clearly irreducible. q.e.d.

Lemma 6.42. *Let* $\mathrm{hol}_i : \pi_1 M \to \mathrm{SU}(2)$ *be a component of the holonomy of a spherical cone-manifold structure with cone-angles* $\leq \pi$. *If* M *is not Seifert fibered, then* hol_i *is a smooth point of* $R(\pi_1 M, \mathrm{SU}(2))$. *The* $\mathbb{R}$*-dimension of* $R(\pi_1 M, \mathrm{SU}(2))$ *around* hol_i *equals* $\tau + 3 - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components in* ∂M_ε. $T_{\mathrm{hol}_i} R(\pi_1 M, \mathrm{SU}(2))$ *may be identified with* $Z^1(\pi_1 M, \mathfrak{su}(2))$.

Proof. The arguments in the hyperbolic case apply directly, the $\mathbb{R}$-dimensions of the $\mathfrak{su}(2)$-cocycle spaces are equal to the $\mathbb{C}$-dimensions of the corresponding $\mathfrak{sl}_2(\mathbb{C})$-cocycle spaces. q.e.d.

Corollary 6.43. $\chi_i = [\mathrm{hol}_i]$ *is a smooth point of* $X(\pi_1 M, \mathrm{SU}(2))$. *The* $\mathbb{R}$*-dimension of* $X(\pi_1 M, \mathrm{SU}(2))$ *around* χ_i *equals* $\tau - \frac{3}{2}\chi(\partial M_\varepsilon)$, *where* τ *is the number of torus components in* ∂M_ε. $T_{\chi_i} X(\pi_1 M, \mathrm{SU}(2))$ *may be identified with* $H^1(\pi_1 M, \mathfrak{su}(2))$.

Proof. The action of $\mathrm{SU}(2)$ on $R(\pi_1 M, \mathrm{SU}(2))$ is proper since $SU(2)$ is a compact group. Since hol_i is non-abelian by Lemma 6.41, we have that $Z(\mathrm{hol}_i(\pi_1 M)) = \{\pm 1\}$. Now the result follows as in the surface case. q.e.d.

The main result in the spherical case is the following theorem:

Theorem 6.44. *Let* C *be a spherical cone-3-manifold with cone-angles* $\leq \pi$, *which is not Seifert fibered. Let* $\{\mu_i, \ldots, \mu_N\}$ *be the family of meridians, where* N *is the number of edges contained in* Σ. *Then the map*

$$X(\pi_1 M, \mathrm{SU}(2)) \to \mathbb{R}^N, \chi_i \mapsto (t_{\mu_1}(\chi_i), \ldots, t_{\mu_N}(\chi_i))$$

is a local diffeomorphism near $\chi_i = [\mathrm{hol}_i]$ *for* $i \in \{1, 2\}$.

Proof. The proof proceeds exactly along the same lines as in the hyperbolic case. The level-sets of the trace-functions

$$V_i = \{t_{\mu_1} \equiv t_{\mu_1}(\chi_i), \ldots, t_{\mu_N} \equiv t_{\mu_N}(\chi_i)\}$$

are smooth, half-dimensional submanifolds of $X(\pi_1 M_\varepsilon, \mathrm{SU}(2))$ near χ_i for $i \in \{1, 2\}$. The classes of the differential forms

$$\{\omega_{tws}^1 + \omega_{len}^1, \ldots, \omega_{tws}^N + \omega_{len}^N\}$$

provide a basis for $T_{\chi_1} V_1$, while the classes of the forms

$$\{\omega_{tws}^1 - \omega_{len}^1, \ldots, \omega_{tws}^N - \omega_{len}^N\}$$

provide a basis for $T_{\chi_1} V_1$. These forms are L^2-bounded on $U_\varepsilon(\Sigma)$. The same argument as in the hyperbolic case shows, that $T_{\chi_i} V_i$ and $\mathrm{im}(H^1(M, \mathcal{E}_i) \to H^1(\partial M_\varepsilon, \mathcal{E}_i))$ are transversal for $i \in \{1, 2\}$. It follows that the differentials $\{dt_{\mu_1}, \ldots, dt_{\mu_N}\}$ are $\mathbb{R}$-linearly independent already in $H^1(\pi_1 M, \mathrm{SU}(2))^*$ at χ_i for $i \in \{1, 2\}$. q.e.d.

Locally around hol the set of representations $\rho = (\rho_1, \rho_2)$ such that $t_{\mu_i}(\rho_1) = t_{\mu_i}(\rho_2)$, equivalently $\mathcal{L}_1(\rho(\mu_i)) = 0$, for all $i \in \{1, \ldots, N\}$ corresponds to spherical cone-manifold structures on M. The cone-angle $\alpha_i = \mathcal{L}_2(\rho(\mu_i))$ is related to the trace of the meridian via

$$t_{\mu_i}(\rho_1) = t_{\mu_i}(\rho_2) = \pm 2\cos(\alpha_i/2).$$

Therefore we obtain using Lemma 6.8:

Corollary 6.45 (local rigidity). *Let C be a spherical cone-3-manifold with cone-angles $\leq \pi$, which is not Seifert fibered. Then the set of cone-angles $\{\alpha_1, \ldots, \alpha_N\}$, where N is the number of edges contained in Σ, provides a local parametrization of the space of spherical cone-manifold structures near the given structure on M. In particular, there are no deformations leaving the cone-angles fixed.*

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