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SURFACES CONTRACTING WITH SPEED $|A|^2$

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Abstract

We show that closed strictly convex surfaces contracting with normal velocity equal to $|A|^2$ shrink to a point in finite time. After appropriate rescaling, they converge to spheres. We describe our algorithm to find the main test function.

1. Introduction

We consider closed strictly convex surfaces M_t in $\mathbb{R}^3$ that contract with normal velocity equal to the square of the norm of the second fundamental form

$$\frac{d}{dt}X = -|A|^2\nu. \tag{1.1}$$

This is a parabolic flow equation. We obtain a solution on a maximal time interval $[0, T)$, $0 < T < \infty$. For $t \uparrow T$, the surfaces converge to a point. After appropriate rescaling, they converge to a round sphere. We say that the surfaces M_t converge to a "round point". The key step in the proof, Theorem 5.1, is to show that

$$\max_{M_t} \left(\frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \right) \tag{1.2}$$

is non-increasing in time.

Here, we used standard notation as explained in Section 3.

Our main theorem is

Theorem 1.1. *For any smooth closed strictly convex surface M in $\mathbb{R}^3$, there exists a smooth family of closed strictly convex surfaces M_t, $t \in [0,T)$, solving (1.1) with $M_0 = M$. For $t \uparrow T$, M_t converges to a point Q. The rescaled surfaces $(M_t - Q) \cdot (6(T-t))^{-1/3}$ converge smoothly to the unit sphere $\mathbb{S}^2$.*

We will also consider other normal velocities for which similar results hold. Therefore, we have to find quantities like (1.2) that are monotone during the flow and vanish precisely for spheres. In general, this is a complicated issue. In order to find these test quantities, we used

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$\lvert A\rvert^2$	$\dfrac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$
K [**7**]	$(\lambda_1-\lambda_2)^2$
H^2	$\dfrac{(\lambda_1+\lambda_2)^3(\lambda_1-\lambda_2)^2}{\left(\lambda_1^2+\lambda_2^2\right)\lambda_1\lambda_2}$
H^3	$\dfrac{\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^2}{\left(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2\right)\lambda_1\lambda_2}$
H^4	$\dfrac{\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)(\lambda_1+\lambda_2)^6(\lambda_1-\lambda_2)^2}{\lambda_1^2\lambda_2^2}$
$\lvert A\rvert^2+\beta H^2$, $0\le\beta\le 5$	$\dfrac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$
$\operatorname{tr} A^3$	$\dfrac{\left(3\lambda_1^2+2\lambda_1\lambda_2+3\lambda_2^2\right)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$
$\operatorname{tr} A^\alpha$, $\alpha=2,\,4,\,5,\,6$	$\dfrac{\left(\lambda_1^{\alpha-2}+\lambda_2^{\alpha-2}\right)(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$
$H\lvert A\rvert^2$	$\dfrac{(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$
$\lvert A\rvert^4$	$\dfrac{\left(\lambda_1^4+2\lambda_1^3\lambda_2+4\lambda_1^2\lambda_2^2+2\lambda_1\lambda_2^3+\lambda_2^4\right)(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)\lambda_1\lambda_2}$

Table 1. Monotone quantities.

an algorithm that checks, based on randomized tests, whether possible candidates fulfill certain inequalities. These inequalities guarantee especially that we can apply the maximum principle to prove monotonicity. We used that algorithm only to propose useful quantities. The presented proof does not depend on it. So far, all approved candidates turned out to be appropriate for proving convergence to a round point. In Table 1, we have collected some normal velocities F (1^{st} column) and quantities w (2^{nd} column) such that $\max_{M_t} w$ is non-increasing in time for surfaces contracting with normal velocity F. In each case, we obtain convergence to round points for smooth closed strictly convex initial surfaces M_0. The proofs for F different from $|A|^2$ will appear elsewhere.

We could not find similar monotone quantities for arbitrary strictly convex hypersurfaces in $\mathbb{R}^k$, $k \geq 4$, contracting with normal velocities of homogeneity larger than one.

There are many papers concerning convex hypersurfaces contracting to a point. Convergence to round points for convex hypersurfaces contracting with certain normal velocities homogeneous of degree one is proved in [**11**, **3**, **4**, **5**]. For homogeneities larger than one, appropriate initial pinching ensures also convergence to round points [**3**, **14**]. The Gauß curvature flow shrinks strictly convex hypersurfaces to points [**16**]. If the homogeneity is less than one, there are examples, where hypersurfaces do not become spherical [**8**, **1**].

In [**7**], Ben Andrews shows that convex surfaces moving by Gauß curvature converge to round points. This normal velocity is homogeneous of degree two in the principal curvatures. He does not require any pinching condition for the initial surface. Our paper extends this result to other flow equations. We consider also normal velocities of degree larger than one and do not have to impose any pinching condition on the initial surface. Any smooth strictly convex surface converges to a round point.

The rest of this paper is organized as follows. In Section 3, we explain our notation. We show in Section 4, that surfaces converge to a point in finite time. Section 5 concerns the key step, Theorem 5.1, the proof of the monotonicity of our test function during the flow. We state some consequences of this monotonicity in Section 6 and finish the proof of Theorem 1.1 in Section 7. Finally, we describe our algorithm to find test functions in Section 8.

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3. Notation

We use $X = X(x, t)$ to denote the embedding vector of a manifold M_t into $\mathbb{R}^3$ and $\frac{d}{dt}X = \dot{X}$ for its total time derivative. It is convenient to identify M_t and its embedding in $\mathbb{R}^3$. We choose ν to be the outer unit normal vector to M_t. The embedding induces a metric (g_{ij}) and a second fundamental form (h_{ij}). We use the Einstein summation convention. Indices are raised and lowered with respect to the metric or its inverse

$\left(g^{ij}\right)$. The principal curvatures λ_1, λ_2 are the eigenvalues of the second fundamental form with respect to the induced metric. A surface is called strictly convex, if all principal curvatures are strictly positive. We will assume this throughout the paper.

Symmetric functions of the principal curvatures are well-defined, we will use the mean curvature $H = \lambda_1 + \lambda_2$, the square of the norm of the second fundamental form $|A|^2 = \lambda_1^2 + \lambda_2^2$, $\operatorname{tr} A^k = \lambda_1^k + \lambda_2^k$, and the Gauß curvature $K = \lambda_1\lambda_2$. We write indices, preceded by semi-colons, e.g., $h_{ij;\,k}$, to indicate covariant differentiation with respect to the induced metric. It is often convenient to choose coordinate systems such that the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, and (h_{ij}) is diagonal, $(h_{ij}) = \operatorname{diag}(\lambda_1,\, \lambda_2)$, e.g.,

$$\sum \lambda_k h_{ij;\,k}^2 = \sum_{i,\,j,\,k=1}^{2} \lambda_k h_{ij;\,k}^2 = h^{kl} h^i_{j;\,k} h^j_{i;\,l} = h_{rs} h_{ij;\,k} h_{ab;\,l} g^{ia} g^{jb} g^{rk} g^{sl}.$$

Whenever we use this notation, we will also assume that we have fixed such a coordinate system. We will only use Euclidean coordinate systems for $\mathbb{R}^3$ so that $h_{ij;\,k}$ is symmetric according to the Codazzi equations.

A normal velocity F can be considered as a function of $(\lambda_1,\, \lambda_2)$ or $(h_{ij},\, g_{ij})$. We set $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, $F^{ij,\,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$. Note that in coordinate systems with diagonal h_{ij} and $g_{ij} = \delta_{ij}$ as mentioned above, F^{ij} is diagonal. For $F = |A|^2$, we have $F^{ij} = 2h^{ij} = 2\lambda_i g^{ij}$.

Recall, see e.g., [**11**, **13**, **12**], that for a hypersurface moving according to $\frac{d}{dt} X = -F\nu$, we have

$$\frac{d}{dt} g_{ij} = -2F h_{ij}, \tag{3.1}$$

$$\frac{d}{dt} h_{ij} = F_{;\,ij} - F h_i^k h_{kj}, \tag{3.2}$$

$$\frac{d}{dt} \nu^\alpha = g^{ij} F_{;\,i} X^\alpha_{;\,j}, \tag{3.3}$$

where Greek indices refer to components in the ambient space $\mathbb{R}^3$. In order to compute evolution equations, we use the Gauß equation and the Ricci identity for the second fundamental form

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \tag{3.4}$$

$$h_{ik;\,lj} = h_{ik;\,jl} + h_k^a R_{ailj} + h_i^a R_{aklj}. \tag{3.5}$$

We will also employ the Gauß formula and the Weingarten equation

$$X^\alpha_{;\,ij} = -h_{ij} \nu^\alpha \qquad \text{and} \qquad \nu^\alpha_{;\,i} = h_i^k X^\alpha_{;\,k}.$$

For tensors A and B, $A_{ij} \ge B_{ij}$ means that $(A_{ij} - B_{ij})$ is positive definite. Finally, we use c to denote universal, estimated constants.

4. Convergence to a Point

It is known, that (1.1) is a parabolic evolution equation for strictly convex initial data and that it has a solution on a maximal time interval.

We show that M_t stays uniformly strictly convex. The following lemma is similar to results in [**5**].

Lemma 4.1. *For a smooth closed strictly convex surface M in $\mathbb{R}^3$, flowing according to $\dot{X} = -|A|^2\nu$, the minimum of the principal curvatures is non-decreasing.*

Proof. Consider $M_{ij} = h_{ij} - \varepsilon g_{ij}$ with $\varepsilon > 0$ so small that M_{ij} is positive semi-definite for some time t_0. We wish to show that M_{ij} is positive semi-definite for $t > t_0$. Combine (3.2), (3.4), and (3.5) to obtain

$$\frac{d}{dt}h_{ij} - F^{kl}h_{ij;\,kl} = 2\,\mathrm{tr}\,A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2g^{kr}g^{ls}h_{kl;\,i}h_{rs;\,j}.$$

In the evolution equation for M_{ij}, we drop the positive definite terms involving derivatives of the second fundamental form

$$\frac{d}{dt}M_{ij} - F^{kl}M_{ij;\,kl} \ge 2\,\mathrm{tr}\,A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2\varepsilon|A|^2 h_{ij}.$$

Let ξ be a zero eigenvalue of M_{ij} with $|\xi| = 1$, $M_{ij}\xi^j = h_{ij}\xi^j - \varepsilon g_{ij}\xi^j = 0$. So, we obtain in a point with $M_{ij} \ge 0$

$$\begin{aligned}&\left(2\,\mathrm{tr}\,A^3 h_{ij} - 3|A|^2 h_i^k h_{kj} + 2\varepsilon|A|^2 h_{ij}\right)\xi^i\xi^j\\ &\quad = 2\varepsilon\,\mathrm{tr}\,A^3 - 3\varepsilon^2|A|^2 + 2\varepsilon^2|A|^2\\ &\quad \ge 2\varepsilon^2|A|^2 - \varepsilon^2|A|^2 > 0\end{aligned}$$

and the maximum principle for tensors [**9**, **10**] gives the result. q.e.d.

The next result shows that $|A|^2$ stays uniformly bounded as long as M_t encloses a ball of fixed positive radius. A similar estimate is used in [**16**].

Lemma 4.2. *For a strictly convex solution of* (1.1), $|A|^2$ *is uniformly bounded in terms of the radius R of an enclosed sphere $B_R(x_0)$, $\max_{M_0} \frac{|A|^2}{\langle X - x_0,\nu\rangle - \frac{1}{2}R}$, and $\max_{M_0}|X - x_0|$. More precisely, we have*

(4.1)
$$\sup_t \max_{M_t} |A|^2 \le \max\left\{\max_{M_0}|X - x_0| \cdot \max_{M_0} \frac{|A|^2}{\langle X - x_0,\,\nu\rangle - \frac{1}{2}R},\ \frac{18}{R^2}\right\}.$$

Proof. We may assume that $x_0 = 0$. Let $\alpha = \frac{1}{2}R$. Then α is a positive lower bound for $\langle X,\,\nu\rangle - \alpha$. Standard computations [**11**, **12**, **13**] yield the evolution equations

$$\frac{d}{dt}X^\beta - F^{ij}X^\beta_{;\,ij} = |A|^2\nu^\beta,$$

$$\frac{d}{dt}\nu^\beta - F^{ij}\nu^\beta_{;\,ij} = 2\operatorname{tr} A^3\nu^\beta,$$
$$\frac{d}{dt}\langle X,\,\nu\rangle - F^{ij}\langle X,\,\nu\rangle_{;\,ij} = -3|A|^2 + 2\operatorname{tr} A^3\langle X,\,\nu\rangle,$$
$$\frac{d}{dt}|A|^2 - F^{ij}\big(|A|^2\big)_{;\,ij} = 2|A|^2\operatorname{tr} A^3.$$

In a critical point of $\frac{|A|^2}{|X,\nu|-\alpha}$, we obtain

$$\frac{d}{dt}\log\frac{|A|^2}{\langle X,\,\nu\rangle-\alpha} - F^{ij}\left(\log\frac{|A|^2}{\langle X,\,\nu\rangle-\alpha}\right)_{;\,ij} = \frac{3|A|^2 - 2\operatorname{tr} A^3\alpha}{\langle X,\,\nu\rangle-\alpha}.$$

Note that $\langle X,\,\nu\rangle-\alpha \le \max_{M_0}|X|$ as a sphere of radius $\max_{M_0}|X|$, centered at the origin, will enclose any M_t. We only have to prove that we preserve the bound in Equation (4.1), when $\max_{M_t}\frac{|A|^2}{\langle X,\nu\rangle-\alpha}$ increases. Then we have $0 \le 3|A|^2 - 2\operatorname{tr} A^3\alpha$ at a point, where $\max_{M_t}\frac{|A|^2}{\langle X,\nu\rangle-\alpha}$ is attained. This inequality and elementary calculations for convex surfaces give

$$|A|^2 \le 2^{1/3}\cdot\big(\operatorname{tr} A^3\big)^{2/3} \le 2\frac{\big(\operatorname{tr} A^3\big)^2}{\big(|A|^2\big)^2} \le \frac{9}{2\alpha^2}$$

at such a maximum point and the Lemma follows. q.e.d.

We obtain that the second fundamental form of the surface stays bounded as long as M_t encloses some ball. The estimates of Krylov, Safonov, Evans (see also [**2**]), and Schauder imply that the solution stays smooth. Then, similarly as in [**16**], the positive lower bound on the minimum principal curvature implies that the surfaces converge to a point in finite time.

5. A Monotone Quantity

Theorem 5.1. *For a family of smooth closed strictly convex surfaces M_t in $\mathbb{R}^3$ flowing according to $\dot X = -|A|^2\nu$,*

$$\max_{M_t}\frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{2\lambda_1\lambda 2} = \max_{M_t}\frac{H\cdot\big(2|A|^2-H^2\big)}{H^2-|A|^2} \equiv \max_{M_t} w \tag{5.1}$$

is non-increasing in time.

An immediate consequence of this theorem is

Corollary 5.2. *The only homothetically shrinking smooth closed strictly convex surfaces M_t, solving the flow equation $\dot X = -|A|^2\nu$ in $\mathbb{R}^3$, are spheres.*

Proof. The quantity $\frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2}$ is positive homogeneous of degree one in the principal curvatures and non-negative. If M is homothetically shrinking, Theorem 5.1 implies that $(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2 = 0$ everywhere.

Thus, M_t is umbilic and [**15**, Lemma 7.1] implies that M_t is a sphere.
q.e.d.

Proof of Theorem 5.1. We combine (3.1), (3.2), (3.4), and (3.5) in order to get the following evolution equations

$$\frac{d}{dt}H - F^{ij}H_{;\,ij} = -\left(|A|^2\right)^2 + 2H\operatorname{tr}A^3 + 2\sum h^2_{ij;\,k} \tag{5.2}$$

and

$$\frac{d}{dt}|A|^2 - F^{ij}\left(|A|^2\right)_{;\,ij} = 2|A|^2\operatorname{tr}A^3. \tag{5.3}$$

For the rest of the proof, we consider a critical point of $w|_{M_t}$ for some $t > 0$, where $w > 0$. It suffices to show that $\tilde{w} := \log w$ is non-increasing in such a point. Then, our theorem follows.

We rewrite $\tilde{w}$

$$\begin{aligned}\tilde{w} &= \log H + \log\left(2|A|^2 - H^2\right) - \log(H^2 - |A|^2)\\ &\equiv \log A + \log B - \log C.\end{aligned}$$

In a critical point of $\tilde{w}$, we obtain

$$\begin{aligned}\frac{d}{dt}\tilde{w} - F^{ij}\tilde{w}_{;\,ij} = {}& \frac{1}{A}\left(\frac{d}{dt}A - F^{ij}A_{;\,ij}\right) + \frac{1}{B}\left(\frac{d}{dt}B - F^{ij}B_{;\,ij}\right)\\ &- \frac{1}{C}\left(\frac{d}{dt}C - F^{ij}C_{;\,ij}\right)\\ &- \frac{1}{AB}F^{ij}(A_{;\,i}B_{;\,j} + A_{;\,j}B_{;\,i})\end{aligned}$$

and

$$\begin{aligned}0 &= \frac{1}{H}H_{;\,k} + \frac{1}{2|A|^2 - H^2}\left(2|A|^2 - H^2\right)_{;\,k} - \frac{1}{H^2 - |A|^2}\left(H^2 - |A|^2\right)_{;\,k}\\ &= \frac{2\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2}{\lambda_1\left(\lambda_1^2 - \lambda_2^2\right)}h_{11;\,k} + \frac{2\lambda_2^2 + \lambda_1\lambda_2 + \lambda_1^2}{\lambda_2\left(\lambda_2^2 - \lambda_1^2\right)}h_{22;\,k}.\end{aligned}$$

So, we deduce that

$$h_{22;\,1} = \frac{\lambda_2}{\lambda_1}\frac{2\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2}{2\lambda_2^2 + \lambda_1\lambda_2 + \lambda_1^2}h_{11;\,1} \equiv a_1 h_{11;\,1}$$

and a similar formula holds for $h_{11;\,2}$, $h_{11;\,2} = a_2 \cdot h_{22;\,2}$. We now combine all these results and obtain in a straightforward calculation

$$\begin{aligned}\frac{d}{dt}\tilde{w} - F^{ij}\tilde{w}_{;\,ij} = {}& \left(\frac{1}{H} - \frac{2H}{2|A|^2 - H^2} - \frac{2H}{H^2 - |A|^2}\right)\cdot\left(\frac{d}{dt}H - F^{ij}H_{;\,ij}\right)\\ &+ \left(\frac{2}{2|A|^2 - H^2} + \frac{1}{H^2 - |A|^2}\right)\\ &\cdot\left(\frac{d}{dt}|A|^2 - F^{ij}\left(|A|^2\right)_{;\,ij}\right)\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{6}{2|A|^2-H^2}+\frac{2}{H^2-|A|^2}\right)F^{ij}H_{;\,i}H_{;\,j}\\
&-\frac{2}{H\cdot(2|A|^2-H^2)}F^{ij}\left(\left(|A|^2\right)_{;\,i}H_{;\,j}+\left(|A|^2\right)_{;\,j}H_{;\,i}\right)\\
=&-\frac{\lambda_1^4+\lambda_1^3\lambda_2+4\lambda_1^2\lambda_2^2+\lambda_1\lambda_2^3+\lambda_2^4}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\\
&\quad\cdot\left(-\left(|A|^2\right)^2+2H\operatorname{tr}A^3\right)\\
&+\frac{(\lambda_1+\lambda_2)^2}{2(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\cdot 2|A|^2\operatorname{tr}A^3\\
&-2\frac{\lambda_1^4+\lambda_1^3\lambda_2+4\lambda_1^2\lambda_2^2+\lambda_1\lambda_2^3+\lambda_2^4}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\sum h_{ij;\,k}^2\\
&+2\frac{\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2}{(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\sum\lambda_k h_{ii;\,k}h_{jj;\,k}\\
&-\frac{8}{(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)}\sum\lambda_k(\lambda_i+\lambda_j)h_{ii;\,k}h_{jj;\,k}\\
=&-\frac{\lambda_1^8+3\lambda_1^7\lambda_2+4\lambda_1^6\lambda_2^2+9\lambda_1^5\lambda_2^3-2\lambda_1^4\lambda_2^4}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\\
&-\frac{9\lambda_1^3\lambda_2^5+4\lambda_1^2\lambda_2^6+3\lambda_1\lambda_2^7+\lambda_2^8}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\\
&+\frac{\lambda_1^7+2\lambda_1^6\lambda_2+2\lambda_1^5\lambda_2^2+3\lambda_1^4\lambda_2^3}{(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\\
&+\frac{3\lambda_1^3\lambda_2^4+2\lambda_1^2\lambda_2^5+2\lambda_1\lambda_2^6+\lambda_2^7}{(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\\
&-2\frac{\lambda_1^4+\lambda_1^3\lambda_2+4\lambda_1^2\lambda_2^2+\lambda_1\lambda_2^3+\lambda_2^4}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\\
&\quad\cdot\left(\left(1+3a_1^2\right)\cdot h_{11;\,1}^2+\left(1+3a_2^2\right)\cdot h_{22;\,2}^2\right)\\
&+2\frac{\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2}{(\lambda_1-\lambda_2)^2\lambda_1\lambda_2}\\
&\quad\cdot\left(\lambda_1(1+a_1)^2\cdot h_{11;\,1}^2+\lambda_2(1+a_2)^2\cdot h_{22;\,2}^2\right)\\
&-\frac{16}{(\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2)}\cdot\left(\lambda_1(\lambda_1+\lambda_2a_1)(1+a_1)\cdot h_{11;\,1}^2\right.\\
&\qquad\left.+\lambda_2(\lambda_2+\lambda_1a_2)(1+a_2)\cdot h_{22;\,2}^2\right)\\
=&-4\frac{K^2}{H}\\
&-2\frac{\left(5\lambda_1^8-4\lambda_1^7\lambda_2+46\lambda_1^6\lambda_2^2\right)\lambda_2}{(\lambda_1+\lambda_2)\left(\lambda_1-\lambda_2\right)^2\left(\lambda_1^2+\lambda_1\lambda_2+2\lambda_2^2\right)^2\lambda_1^3}h_{11;\,1}^2
\end{aligned}
$$

$$- 2\frac{\left(48\lambda_1^5\lambda_2^3 + 72\lambda_1^4\lambda_2^4 + 44\lambda_1^3\lambda_2^5\right)\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2\left(\lambda_1^2+\lambda_1\lambda_2+2\lambda_2^2\right)^2\lambda_1^3}h_{11;\,1}^2$$
$$- 2\frac{\left(34\lambda_1^2\lambda_2^6 + 8\lambda_1\lambda_2^7 + 3\lambda_2^8\right)\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2\left(\lambda_1^2+\lambda_1\lambda_2+2\lambda_2^2\right)^2\lambda_1^3}h_{11;\,1}^2$$
$$- 2\frac{\left(5\lambda_2^8 - 4\lambda_2^7\lambda_1 + 46\lambda_2^6\lambda_1^2\right)\lambda_1}{(\lambda_2+\lambda_1)(\lambda_2-\lambda_1)^2\left(\lambda_2^2+\lambda_2\lambda_1+2\lambda_1^2\right)^2\lambda_2^3}h_{22;\,2}^2$$
$$- 2\frac{\left(48\lambda_2^5\lambda_1^3 + 72\lambda_2^4\lambda_1^4 + 44\lambda_2^3\lambda_1^5\right)\lambda_1}{(\lambda_2+\lambda_1)(\lambda_2-\lambda_1)^2\left(\lambda_2^2+\lambda_2\lambda_1+2\lambda_1^2\right)^2\lambda_2^3}h_{22;\,2}^2$$
$$- 2\frac{\left(34\lambda_2^2\lambda_1^6 + 8\lambda_2\lambda_1^7 + 3\lambda_1^8\right)\lambda_1}{(\lambda_2+\lambda_1)(\lambda_2-\lambda_1)^2\left(\lambda_2^2+\lambda_2\lambda_1+2\lambda_1^2\right)^2\lambda_2^3}h_{22;\,2}^2$$
$$\le 0.$$

We finally, apply the maximum principle and our theorem follows. q.e.d.

6. Direct Consequences

We obtain a pinching estimate

Corollary 6.1. *For a smooth closed strictly convex surface M_t in $\mathbb{R}^3$, flowing according to $\dot{X} = -|A|^2\nu$, there exists $c = c(M_0)$ such that $0 < \frac{1}{c} \le \frac{\lambda_1}{\lambda_2} \le c$.*

Proof. Choose $\varepsilon > 0$ such that $\lambda_1,\, \lambda_2 > \varepsilon$ at $t = 0$. Theorem 5.1 and Lemma 4.1 imply that

$$2\varepsilon\frac{\left(\frac{\lambda_1}{\lambda_2}-1\right)^2}{\frac{\lambda_1}{\lambda_2}} = 2\varepsilon\frac{(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2} \le \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2} \le c.$$

We obtain the upper bound on $\frac{\lambda_1}{\lambda_2}$ claimed above. Similarly, we obtain an upper bound on $\frac{\lambda_2}{\lambda_1}$. q.e.d.

Let ρ_+ be the minimal radius of enclosing spheres and ρ_- the maximal radius of enclosed spheres. The quotient of these radii can be estimated as follows

Corollary 6.2. *Under the assumptions of Corollary* 6.1*, ρ_+/ρ_- is bounded above by a constant depending only on the constant $c(M_0)$ in Corollary* 6.1*.*

Proof. Combine Corollary 6.1, [**5**, Theorem 5.1], and [**5**, Lemma 5.4]. q.e.d.

We also obtain a bound for $|\lambda_1 - \lambda_2|$

Corollary 6.3. *For a smooth closed strictly convex surface M_t in $\mathbb{R}^3$, flowing according to $\dot{X} = -|A|^2\nu$, there exists a constant $c = c(M_0)$ such that $|\lambda_1 - \lambda_2| \le c \cdot \left(|A|^2\right)^{1/4} \le c \cdot \sqrt{H}$.*

Proof. This is a direct consequence of Theorem 5.1 and Corollary 6.1. q.e.d.

As in [**7**], this estimate on $|\lambda_1 - \lambda_2|$ is "better" than scaling invariant. It is crucial for the rest of the proof of Theorem 1.1.

Let us recall a form of the maximum principle for evolving hypersurfaces.

Lemma 6.4. *Let M_t and $\tilde{M}_t$ be two smooth closed strictly convex solutions to* (1.1) *on some time interval* $[0, T^*)$*. If M_0 encloses $\tilde{M}_0$, then M_t encloses $\tilde{M}_t$ for any $t \in [0, T^*)$.*

Proof. This is a standard consequence of the maximum principle. q.e.d.

The next result describes the evolution of spheres.

Lemma 6.5. *Spheres $\partial B_{r(t)}(x_0)$ solve* (1.1) *for $t \in [0, T)$ with $r(t) = (6(T-t))^{1/3}$ and $T = \frac{1}{6}r^3(0)$.*

Proof. The evolution equation for the radius of a sphere is

$$\dot{r}(t) = -\frac{2}{r^2(t)}.$$

q.e.d.

As a consequence, we can estimate the life span of a solution in terms of inner and outer radii.

Lemma 6.6. *Let $\rho_+(t)$ and $\rho_-(t)$ be the inner and outer radii of M_t, respectively. Assume that M_t is a smooth closed strictly convex solution of* (1.1) *on a maximal time interval* $[0, T)$*. Then, we have for $t \in [0, T)$*

$$\frac{1}{6}\rho_-^3(t) \le T - t \le \frac{1}{6}\rho_+^3(t).$$

Proof. As M_t contracts to a point, we deduce from Lemma 6.4 that $T - t$ is bounded below by the life span of $\partial B_{\rho_-(t)}$ evolving according to (1.1). So the lower bound follows from Lemma 6.5. The upper bound is obtained similarly. q.e.d.

7. Convergence to a Round Point

We closely follow the corresponding part of [**7**].

Proposition 7.1. *Define* $q(t) := \frac{1}{4\pi} \int\limits_{M_t} KX$. *Then*

$$\left| \langle X - q, \nu \rangle - \frac{1}{8\pi} \int\limits_{M_t} H \right| \le \frac{1}{4\pi} \cdot \sup_{M_t} |\lambda_1 - \lambda_2| \cdot \mathcal{H}^2(M_t),$$

where $\mathcal{H}^2(M_t)$ *denotes the area of* M_t.

Proof. This is [**7**, Proposition 4]. q.e.d.

We define $r_+(t)$ to be the minimal radius of a sphere, centered at $q(t)$, that encloses M_t. Similarly, we define $r_-(t)$ to be the maximal radius of a sphere, centered at $q(t)$, that is enclosed by M_t.

Lemma 7.2. *Under the assumptions of Theorem* 1.1*, for* $T - t$ *sufficiently small,* r_+ *and* r_- *are estimated as follows*

$$r_+(t) \le (6(T-t))^{1/3} \cdot \left(1 + c \cdot (T-t)^{1/6}\right),$$
$$r_-(t) \ge (6(T-t))^{1/3} \cdot \left(1 - c \cdot (T-t)^{1/6}\right),$$

and

$$1 \le \frac{r_+}{r_-} \le 1 + c \cdot (T-t)^{1/6}.$$

Proof. Denote the bounded component of $\mathbb{R}^3 \setminus M_t$ by E_t. The transformation formula for integrals implies that

$$\frac{1}{4\pi} \int\limits_{M_t} KX = \frac{1}{4\pi} \int\limits_{\mathbb{S}^2} X\left(\nu^{-1}(\cdot)\right).$$

So, we see that $q(t) \in E_t$. We have

$$r_+ = \max_{M_t} \langle X - q(t), \nu \rangle, \qquad r_- = \min_{M_t} \langle X - q(t), \nu \rangle,$$
$$\rho_+ = \min_{p \in \mathbb{R}^3} \max_{M_t} \langle X - p, \nu \rangle, \quad \text{and} \quad \rho_- = \max_{p \in E_t} \min_{M_t} \langle X - p, \nu \rangle.$$

Recall the first variation formula for a vector field Y along M_t

$$\int\limits_{M_t} H \langle Y, \nu \rangle = \int\limits_{M_t} \operatorname{div}_{M_t} Y$$

and get for $p \in E_t$ such that $\rho_+ = \max_{M_t} \langle X - p, \nu \rangle$

$$\int\limits_{M_t} H \ge \frac{1}{\rho_+} \int\limits_{M_t} H \cdot \langle X - p, \nu \rangle = \frac{1}{\rho_+} \int\limits_{M_t} \operatorname{div}_{M_t} X = \frac{1}{\rho_+} \int\limits_{M_t} 2 = \frac{2}{\rho_+} \mathcal{H}^2(M_t).$$

We employ Proposition 7.1 and deduce that

$$\begin{aligned} r_- &\ge \frac{1}{8\pi}\int\limits_{M_t} H \cdot \left\{1 - 2\left(\int_{M_t} H\right)^{-1} \cdot \sup_{M_t} |\lambda_1 - \lambda_2| \cdot \mathcal{H}^2(M_t)\right\} \\ &\ge \frac{1}{8\pi}\int\limits_{M_t} H \cdot \left\{1 - \rho_+ \cdot \sup_{M_t} |\lambda_1 - \lambda_2|\right\}. \end{aligned}$$

We estimate as follows

$$\begin{aligned} \rho_+ \cdot \sup_{M_t} |\lambda_1 - \lambda_2| &\le c \cdot \rho_+ \cdot \left(|A|^2\right)^{1/4} && \text{by Corollary 6.3} \\ &\le c \cdot \rho_+ \cdot \left(c + \frac{c}{\rho_-^2}\right)^{1/4} && \text{by Lemma 4.2} \\ &\le c \cdot (T-t)^{1/6} \end{aligned}$$

by Corollary 6.2 and Lemma 6.6 for $(T-t)$ small. So, we obtain

$$r_-(t) \ge \frac{1}{8\pi}\int\limits_{M_t} H \cdot \left(1 - c \cdot (T-t)^{1/6}\right). \tag{7.1}$$

Similar calculations yield

$$r_+(t) \le \frac{1}{8\pi}\int\limits_{M_t} H \cdot \left(1 + c \cdot (T-t)^{1/6}\right). \tag{7.2}$$

We employ Lemma 6.6

$$r_- \le \rho_- \le (6(T-t))^{1/3} \le \rho_+ \le r_+$$

and obtain for $(T-t)$ small

$$\begin{aligned} &(6(T-t))^{1/3} \cdot \left(1 - c \cdot (T-t)^{1/6}\right) \\ &\quad \le \frac{1}{8\pi}\int\limits_{M_t} H \le (6(T-t))^{1/3} \cdot \left(1 + c \cdot (T-t)^{1/6}\right). \end{aligned}$$

Using (7.1) and (7.2) gives the claimed estimates on r_- and r_+, and r_+/r_- is bounded as stated above. q.e.d.

Corollary 7.3. *Under the assumptions of Theorem* 1.1, *we have the estimate*

$$|q(t) - Q| \le c \cdot (T-t)^{1/3+1/18},$$

where $Q = \lim_{t\uparrow T} q(t) = \lim_{t\uparrow T} M_t$. *Therefore, we obtain estimates of the same form as in Lemma* 7.2, *if we define* r_+ *and* r_- *using* Q *instead of* $q(t)$.

Proof. Fix $t_0 \in [0, T)$. A sphere of radius

$$r(t_0) = (6(T - t_0))^{1/3} \left(1 + c(T - t_0)^{1/6}\right)$$

at time t_0, centered at $q(t_0)$ as defined in Proposition 7.1, which evolves according to (1.1), will enclose M_t for all $t_0 \le t < T$. Thus, its radius at time $t = T$ is an upper bound for $|q(t_0) - Q|$. According to Lemma 6.5, the radius of that sphere evolves as follows

$$r(t) = \left(6\left(\frac{1}{6}r^3(t_0) - (t - t_0)\right)\right)^{1/3}.$$

Therefore, we get

$$|q(t_0) - Q| \le r(T) = (6(T - t_0))^{1/3} \cdot \left(\left(1 + c(T - t_0)^{1/6}\right)^3 - 1\right)^{1/3}$$

and the claimed estimate follows. q.e.d.

We wish to check, that we can apply a Harnack inequality [**6**, Theorem 5.17]. For $F = F(\lambda_i)$, $\lambda_i > 0$, we define

$$\Phi(\kappa_i) := -F\left(\kappa_i^{-1}\right).$$

We say that Φ is α-concave, if $\Phi = \operatorname{sgn}\alpha \cdot B^\alpha$ for some B, where B is positive and concave. The function Φ is called the dual function to F.

Lemma 7.4. *The dual function to $F = |A|^2 = \lambda_1^2 + \lambda_2^2$ is α-concave for $\alpha \le -2$.*

Proof. It is convenient to use [**6**, (5.4)]. We leave the details to the reader. q.e.d.

We are now able to improve our velocity bounds.

Lemma 7.5. *Under the assumptions of Theorem* 1.1, *we obtain*

$$\begin{aligned} &2(6(T - t))^{-2/3} \cdot \left(1 - c \cdot (T - t)^{1/12}\right) \\ &\quad \le |A|^2 \le 2(6(T - t))^{-2/3} \cdot \left(1 + c \cdot (T - t)^{1/12}\right) \end{aligned}$$

everywhere on M_t for $(T - t)$ sufficiently small.

Proof. We may assume that $T - t > 0$ is so small that we can use the results obtained before. Parameterize M_t by $\mathbb{S}^2$ such that the normal image of M_t at $X(z, t)$ equals $z \in \mathbb{S}^2$. Let us define the support function s of M_t as

$$s(z, t) := \langle X(z, t), z\rangle.$$

Its evolution equation, see e.g., [**6**], is

$$\frac{d}{dt}s(z, t) = -|A|^2(z, t). \tag{7.3}$$

The α-concavity proved in Lemma 7.4 allows us to use [**6**, Theorem 5.17]. We obtain for $0 < t_1 < t_2 < T$ and $z \in \mathbb{S}^2$, for two points $(z,\, t_1)$ and $(z,\, t_2)$ with the same normal,

$$\frac{|A|^2(z,\, t_2)}{|A|^2(z,\, t_1)} \geq \left(\frac{t_1}{t_2}\right)^{2/3}. \tag{7.4}$$

Let us assume that $q(t)$ is the origin for some fixed time t. As M_t lies between $\partial B_{r_+(t)}(0)$ and $\partial B_{r_-(t)}(0)$, $M_{t+\tau}$ lies outside $B_{\left(r_-^3(t)-6\tau\right)^{1/3}}(0)$ for any $0 < \tau < T - t$, so

$$r_-(t) \leq s(\cdot,\, t) \leq r_+(t) \quad \text{and} \quad \left(r_-^3 - 6\tau\right)^{1/3} \leq s(\cdot,\, t+\tau). \tag{7.5}$$

Set $\tau = r_-^{5/2}(t) \cdot (r_+(t) - r_-(t))^{1/2}$ and observe that $t + \tau < T$, if $(r_+ - r_-)^{1/2} \leq \frac{1}{6} r_-^{1/2}$ (by Lemma 6.6), or, if $T - t$ is sufficiently small (by Lemma 7.2). We estimate

$$\begin{aligned}
|A|^2(z,\, t) &\leq \inf_{0\leq\tilde\tau\leq\tau} \left\{ \left(\frac{t+\tilde\tau}{t}\right)^{2/3} \cdot |A|^2(z,\, t+\tilde\tau) \right\} && \text{by (7.4)}\\
&\leq \left(\frac{t+\tau}{t}\right)^{2/3} \cdot \frac{1}{\tau} \cdot \int\limits_t^{t+\tau} |A|^2(z,\, \tilde\tau) d\tilde\tau\\
&\leq \left(1 + \frac{\tau}{t}\right)^{2/3} \cdot \frac{1}{\tau} \cdot (s(z,\, t) - s(z,\, t+\tau)) && \text{by (7.3)}\\
&\leq \left(1 + \frac{\tau}{t}\right) \cdot \frac{1}{\tau} \cdot \left(r_+(t) - \left(r_-^3(t) - 6\tau\right)^{1/3}\right) && \text{by (7.5)}\\
&= \frac{1 + \frac{\tau}{t}}{r_-^2 \cdot \left(\frac{r_+}{r_-} - 1\right)^{1/2}}\\
&\qquad \cdot \left(\frac{r_+}{r_-} - \left(1 - 6 \cdot \left(\frac{r_+}{r_-} - 1\right)^{1/2}\right)^{1/3}\right).
\end{aligned}$$

The maximal existence time T is bounded below in terms of the radius of a sphere enclosed by M_0. So, we may also assume that t is bounded below by a positive constant. A very crude estimate gives

$$\tau \leq r_-^{5/2} \cdot r_+^{1/2} \leq r_+^3 \leq c \cdot (T - t),$$

so we obtain

$$\frac{1 + \frac{\tau}{t}}{r_-^2} \leq (6(T-t))^{-2/3} \cdot \left(1 + c \cdot (T-t)^{1/6}\right).$$

Observe that for $|x| \leq \frac{1}{2}$, we have

$$-(1-x)^{1/3} \leq -1 + \tfrac{1}{3}x + \tfrac{1}{3}x^2.$$

We conclude for small $(T-t)$

$$\begin{aligned}|A|^2(z,t) &\le (6(T-t))^{-2/3}\cdot\left(1+c\cdot(T-t)^{1/6}\right)\left(2+13\left(\frac{r_+}{r_-}-1\right)^{1/2}\right)\\ &\le 2\cdot(6(T-t))^{-2/3}\cdot\left(1+c\cdot(T-t)^{1/12}\right).\end{aligned}$$

For the lower bound on $|A|^2$, we proceed similarly and use $\tau = r_-^{5/2}(t)\cdot(r_+(t)-r_-(t))^{1/2}$,

$$\left(r_-^3+6\tau\right)^{1/3}\le s(z,t-\tau)\quad\text{and}\quad s(z,t)\le r_+(t).$$

q.e.d.

We have the following estimate for the principal curvatures

Lemma 7.6. *Under the assumptions of Theorem* 1.1*, we obtain*

$$\begin{aligned}&(6(T-t))^{-1/3}\cdot\left(1-c\cdot(T-t)^{1/12}\right)\\ &\qquad\le \lambda_1,\,\lambda_2\le(6(T-t))^{-1/3}\cdot\left(1+c\cdot(T-t)^{1/12}\right)\end{aligned}$$

on M_t for small $(T-t)$.

Proof. As $H^2 = 2|A|^2-(\lambda_1-\lambda_2)^2$, we obtain

$$\begin{aligned}\lambda_1 &= \tfrac12(\lambda_1+\lambda_2)+\tfrac12(\lambda_1-\lambda_2)\\ &= \tfrac12\sqrt{2|A|^2-(\lambda_1-\lambda_2)^2}+\tfrac12(\lambda_1-\lambda_2).\end{aligned}\tag{7.6}$$

Combining Lemmata 6.3 and 7.5, we get $|\lambda_1-\lambda_2|\le c\cdot(T-t)^{-1/6}$. We use Lemma 7.5 and (7.6). The claimed inequality follows. q.e.d.

Proof of Theorem 1.1. Lemma 7.6 implies, that, everywhere on M_t, the quotient λ_1/λ_2 tends to 1 as $t\uparrow T$. Then we can apply known results, see e.g., [**3**, Theorem 2], to conclude that the rescaled surfaces converge smoothly to the unit sphere $\mathbb{S}^2\subset\mathbb{R}^3$. q.e.d.

A standard way of rescaling [**5**] is to consider the embeddings $\tilde X(\cdot,t)$,

$$\tilde X(z,t) := (6(T-t))^{-1/3}(X(z,t)-Q)$$

with Q as in Theorem 1.1. Define the time function $\tau(t) := \frac16\log T-\frac16\log(T-t)$. Then we have, using suggestive notation, the following evolution equation

$$\frac{d}{d\tau}\tilde X = -\left|\tilde A\right|^2\tilde\nu+2\tilde X$$

and our a priori estimates imply, that, for $\tau\to\infty$, $\tilde M_t$ converges exponentially to $\mathbb{S}^2$.

8. Finding Monotone Quantities

8.1. The Algorithm. We use a sieve algorithm and start with symmetric rational functions of the principal curvatures as candidates for test functions, e.g.,

$$w = \frac{p_1(\lambda_1, \lambda_2)}{p_2(\lambda_1, \lambda_2)} = \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}.$$

Here, $p_1 \neq 0$ and $p_2 \neq 0$ are homogeneous polynomials.

In the end, we want to find functions w such that $W := \sup_{M_t} w$ is monotone and ensures convergence to round spheres.

We check, whether these test functions w fulfill the following conditions.

1) a) $p_1(\lambda_1, \lambda_2), p_2(\lambda_1, \lambda_2) \geq 0$ for $0 < \lambda_1, \lambda_2$,
   b) $p_1(\lambda_1, \lambda_2) = 0$ for $\lambda_1 = \lambda_2 > 0$.
2) $\deg p_1 > \deg p_2$.
3) $\frac{\partial w(1,\lambda_2)}{\partial \lambda_2} < 0$ for $0 < \lambda_2 < 1$ and $\frac{\partial w(1,\lambda_2)}{\partial \lambda_2} > 0$ for $\lambda_2 > 1$.
4) $\frac{d}{dt} w - F^{ij} w_{;ij} \leq 0$
   a) for terms without derivatives of (h_{ij}),
   b) for terms involving derivatives of (h_{ij}), if $w_{;i} = 0$ for $i = 1, 2$.

8.2. Motivation and Randomized Tests. For all flow equations considered, spheres contract to points and stay spherical. So, we can only find monotone quantities, if $\deg p_1 \leq \deg p_2$ or $p_1(\lambda, \lambda) = 0$.

If $\deg p_1 < \deg p_2$, we obtain that W is decreasing on any self-similarly shrinking surface. So, this does not imply convergence to a sphere. The counterexamples in [**3**] show for normal velocities of homogeneity larger than 1, that the pinching ratio $\sup_{M_t} \lambda_2/\lambda_1$ (for $\lambda_2 > \lambda_1$) will increase during the flow for appropriate initial surfaces. Therefore, we require in step (2), that $\deg p_1 > \deg p_2$.

Condition (3) ensures that the quantity decreases, if the eigenvalues approach each other.

In step (4a) and (4b), we check that we can apply the maximum principle. Here, we have to use various differentiation rules.

In all these steps, inequalities are tested by evaluating both sides at random numbers. After enough testing, all candidates for which the above inequalities, evaluated at random numbers, were not violated, could be used to prove convergence to a round point.

Alternatively, for surfaces, we can avoid using random numbers, compute evolution equations algebraically, and use Sturm's algorithm to test for non-negativity.

We expect that similar algorithms will be used to find (monotone) test functions for other (geometric) problems.

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