

## COARSE ALEXANDER DUALITY AND DUALITY GROUPS

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### Abstract

We study discrete group actions on coarse Poincaré duality spaces, e.g., acyclic simplicial complexes which admit free cocompact group actions by Poincaré duality groups. When  $G$  is an  $(n - 1)$  dimensional duality group and  $X$  is a coarse Poincaré duality space of formal dimension  $n$ , then a free simplicial action  $G \curvearrowright X$  determines a collection of “peripheral” subgroups  $H_1, \dots, H_k \subset G$  so that the group pair  $(G, \{H_1, \dots, H_k\})$  is an  $n$ -dimensional Poincaré duality pair. In particular, if  $G$  is a 2-dimensional 1-ended group of type  $FP_2$ , and  $G \curvearrowright X$  is a free simplicial action on a coarse  $PD(3)$  space  $X$ , then  $G$  contains surface subgroups; if in addition  $X$  is simply connected, then we obtain a partial generalization of the Scott/Shalen compact core theorem to the setting of coarse  $PD(3)$  spaces. In the process, we develop coarse topological language and a formulation of coarse Alexander duality which is suitable for applications involving quasi-isometries and geometric group theory.

### 1. Introduction

In this paper, we study metric complexes (e.g., metric simplicial complexes) which behave homologically in the large-scale like  $\mathbb{R}^n$ , and discrete group actions on them. One of our main objectives is a partial generalization of the Scott/Shalen compact core theorem for 3-manifolds ([37], see also [26]) to the setting of coarse Poincaré duality spaces and Poincaré duality groups of arbitrary dimension. In the one ended case, the compact core theorem says that if  $X$  is a contractible 3-manifold and  $G$  is a finitely generated one-ended group acting discretely and freely on  $X$ , then the quotient  $X/G$  contains a compact core — a compact submanifold  $Q$  with (aspherical) incompressible boundary so that the inclusion  $Q \rightarrow X/G$  is a homotopy equivalence. The proof of the compact core theorem relies on standard tools in 3-manifold theory like

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transversality, which has no appropriate analog in the 3-dimensional coarse Poincaré duality space setting, and the Loop Theorem, which has no analog even for manifolds when the dimension is at least 4.

We now formulate our analog of the core theorem. For our purpose, the appropriate substitute for a finitely generated, one-ended, 2-dimensional group  $G$  will be a *duality group* of dimension  $n - 1$ . (By the dimension of a group, we will always mean the cohomological dimension over  $\mathbb{Z}$ .) We recall [6] that a group  $G$  is a  $k$ -dimensional duality group if  $G$  is of type  $FP$ ,  $H^i(G; \mathbb{Z}G) = 0$  for  $i \neq k$ , and  $H^k(G; \mathbb{Z}G)$  is torsion-free. (We never make use of the last assumption about  $H^k(G; \mathbb{Z}G)$  in our paper.) Examples of duality groups include:

- A. Freely indecomposable 2-dimensional groups of type  $FP_2$ ; for instance, torsion free one-ended 1-relator groups.
- B. The fundamental groups of compact aspherical manifolds with incompressible aspherical boundary [6].
- C. The product of two duality groups.
- D. Torsion free  $S$ -arithmetic groups [9].

Instead of 3-dimensional contractible manifolds, we work with a class of metric complexes which we call “coarse  $PD(n)$  spaces”. We defer the definition to the main body of the paper (see Section 6 and Appendix 11), but we note that important examples include universal covers of closed aspherical  $n$ -dimensional PL-manifolds, acyclic complexes  $X$  with  $H_c^*(X) \simeq H_c^*(\mathbb{R}^n)$  which admit free cocompact simplicial group actions, and uniformly acyclic  $n$ -dimensional PL-manifolds with bounded geometry. We recall that an  $n$ -dimensional Poincaré duality group ( $PD(n)$  group) is a duality group  $G$  with  $H^n(G; \mathbb{Z}G) \simeq \mathbb{Z}$ . Our group-theoretic analog for the compact core will be an  $n$ -dimensional Poincaré duality pair ( $PD(n)$  pair), i.e., a group pair  $(G, \{H_1, \dots, H_k\})$  whose double with respect to the  $H_i$ 's is an  $n$ -dimensional Poincaré duality group, [14]. In this case, the “peripheral” subgroups  $H_i$  are  $PD(n - 1)$  groups. See Section 3 for more details.

**Theorem 1.1.** *Let  $X$  be a coarse  $PD(n)$  space, and let  $G$  be an  $(n - 1)$ -dimensional duality group acting freely and discretely on  $X$ . Then:*

1.  $G$  contains subgroups  $H_1, \dots, H_k$  (which are canonically defined up to conjugacy by the action  $G \curvearrowright X$ ) so that  $(G, \{H_i\})$  is a  $PD(n)$  pair.
2. There is a connected  $G$ -invariant subcomplex  $K \subset X$  so that  $K/G$  is compact, the stabilizer of each component of  $X - K$  is conjugate to one of the  $F_i$ 's, and each component of  $\overline{X - K}/G$  is one-ended.

Thus, the duality groups  $G$  which appear in the above theorem behave homologically like the groups in example B. As far as we know, Theorem 1.1 is new even in the case that  $X \simeq \mathbb{R}^n$ , when  $n \geq 4$ . Theorem 1.1 and Lemma 11.6 imply

**Corollary 1.2.** *Let  $\Gamma$  be a  $n$ -dimensional Poincaré duality group. Then, any  $(n - 1)$ -dimensional duality subgroup  $G \subset \Gamma$  contains a finite collection  $H_1, \dots, H_k$  of  $PD(n - 1)$  subgroups so that the group pair  $(G, \{H_i\})$  is a  $PD(n)$  pair; moreover, the subgroups  $H_1, \dots, H_k$  are canonically determined by the embedding  $G \rightarrow \Gamma$ .*

**Corollary 1.3.** *Suppose that  $G$  is a group of type  $FP_2$ ,  $\dim(G) \leq 2$ , and  $G$  acts freely and simplicially on a coarse  $PD(3)$  space. Then:*

1. *Each 1-ended factor of  $G$  admits the structure of a  $PD(3)$  pair.*
2. *Either  $G$  contains a surface group, or  $G$  is free. In particular, an infinite index  $FP_2$  subgroup of a 3-dimensional Poincaré duality group contains a surface subgroup or is free.*

*Proof.* Let  $G = F * (*_i G_i)$  be a free product decomposition where  $F$  is a finitely generated free group, and each  $G_i$  is finitely generated, freely indecomposable, and non-cyclic. Then, by Stallings' theorem on ends of groups, each  $G_i$  is one-ended, and hence is a 2-dimensional duality group. Since  $\dim(G) \leq 2$ , this group is not a  $PD(3)$ -group. By Theorem 1.1, each  $G_i$  has structure of a  $PD(3)$ -pair  $(G, \{H_1, \dots, H_k\})$ . Each  $H_i$  is a  $PD(2)$  subgroups, and therefore these subgroups are surface groups. q.e.d.

**Remark 1.4.** Each  $PD(2)$  group over a commutative ring  $\mathcal{R}$  with a unit is the fundamental group of a 2-dimensional orbifold, see [16, 17] for  $\mathcal{R} = \mathbb{Z}$ , [10] in case when  $\mathcal{R}$  is a field and [31, 29] in the general case.

We believe that Corollary 1.3 still holds if one relaxes the  $FP_2$  assumption to finite generation, and we conjecture that any finitely generated group which acts freely, simplicially but not cocompactly, on a coarse  $PD(3)$  space is finitely presented. We note that Bestvina and Brady [2] construct 2-dimensional groups which are  $FP_2$ , but not finitely presented.

In Theorem 1.1 and Corollary 1.2, one can ask to what extent the peripheral structure – the subgroups  $H_1, \dots, H_k$  – are uniquely determined by the duality group  $G$ . We prove an analog of the uniqueness theorem for peripheral structure [27] for fundamental groups of acylindrical 3-manifolds with aspherical incompressible boundary:

**Theorem 1.5.** *Let  $(G, \{H_i\}_{i \in I})$  be a  $PD(n)$  pair, where  $G$  is not a  $PD(n-1)$  group, and  $H_i$  does not coarsely separate  $G$  for any  $i$ . If  $(G, \{F_j\}_{j \in J})$  is a  $PD(n)$  pair, then there is a bijection  $\beta : I \rightarrow J$  such that  $H_i$  is conjugate to  $F_{\beta(i)}$  for all  $i \in I$ .*

**Remark 1.6.** In a recent paper [38], Scott and Swarup give a group-theoretic proof of Johannson's theorem, see also [39].

**Remark 1.7.** The results and methods of this paper, in particular Theorem 1.1, Corollaries 1.2, 1.3, and Theorem 1.5, remain valid (with minor modifications) if one replaces the coefficient ring  $\mathbb{Z}$  with an arbitrary commutative ring with unit. In Corollary 1.3, the conclusion in the second case is that  $G$  either contains a surface group, or is virtually free.

We were led to Theorem 1.1 and Corollary 1.3 by our earlier work on hyperbolic groups with one-dimensional boundary [28]; in that paper, we conjectured that every torsion-free hyperbolic group  $G$  whose boundary is homeomorphic to the Sierpinski carpet is the fundamental group of a compact hyperbolic 3-manifold with totally geodesic boundary. In the same paper, we showed that such a group  $G$  is part of a canonically defined  $PD(3)$  pair and that our conjecture would follow if one knew that  $G$  were a 3-manifold group. One approach to proving this is to produce an algebraic counterpart to the Haken hierarchy for Haken 3-manifolds in the context of  $PD(3)$  pairs. We say that a  $PD(3)$  pair  $(G, \{H_1, \dots, H_k\})$  is *Haken* if it admits a non-trivial splitting. (If  $k > 0$  then such a splitting always exists.) One would like to show that Haken  $PD(3)$  pairs always admit non-trivial splittings over  $PD(2)$  pairs whose peripheral structure is compatible with that of  $G$ . Given this, one can create a hierarchical decomposition of the group  $G$ , and try to show that the terminal groups correspond to fundamental groups of 3-manifolds with boundary. The corresponding 3-manifolds might then be glued together along boundary surfaces to yield a 3-manifold with fundamental group  $G$ . At the moment, the biggest obstacle in this hierarchy program appears to be the first step; and the two theorems above provide a step toward overcoming it.

**Remark 1.8.** It is a difficult open problem due to Wall whether each  $PD(n)$  group  $G$  (that admits a compact  $K(G, 1)$ ) is isomorphic to the fundamental group of a compact aspherical  $n$ -manifold (here,  $n \geq 3$ ), see [30]. The case of  $n = 1$  is quite easy, for  $n = 2$ , the positive solution is due to Eckmann, Linnell and Müller [16, 17]. Partial results for  $n = 3$  were obtained by Kropholler [32] and Thomas [42]. If the assumption that  $G$  has finite  $K(G, 1)$  is omitted, then there is a counter-example due to Davis [13]; he constructed  $PD(n)$  groups (for each  $n \geq 4$ ) which

do not admit finite Eilenberg–MacLane spaces. For  $n \geq 5$ , the positive answer would follow from Borel Conjecture [30].

As an application of Theorems 1.1 and Corollary 1.3 and the techniques used in their proof, we give examples of  $(n - 1)$ -dimensional groups which cannot act freely on coarse  $PD(n)$  spaces (in particular, they cannot be subgroups of  $PD(n)$  groups), see Section 9 for details:

1. A 2-dimensional one-ended group of type  $FP_2$  with positive Euler characteristic cannot act on a coarse  $PD(3)$  space. The semi-direct product of two finitely generated free groups is such an example.
2. For  $i = 1, \dots, \ell$ , let  $G_i$  be a duality group of dimension  $n_i$  and assume that for  $i = 1, 2$  the group  $G_i$  is not a  $PD(n_i)$  group. Then, the product  $G_1 \times \dots \times G_\ell$  cannot act on a coarse  $PD(n)$  space where  $n - 1 = n_1 + \dots + n_\ell$ . The case when  $n = 3$  is due to Kropholler, [32].
3. If  $G_1$  is a  $k$ -dimensional duality group and  $G_2$  is the the Baumslag–Solitar group  $BS(p, q)$  (where  $p \neq \pm q$ ), then the direct product  $G_1 \times G_2$  cannot act on a coarse  $PD(3 + k)$  space. In particular,  $BS(p, q)$  cannot act on a coarse  $PD(3)$  space (unless  $|p| = |q| = 1$ ).

**Remark 1.9.** Peter Kropholler had proven that a Baumslag–Solitar group as above cannot be embedded in a  $PD(3)$  group  $G$ , under an assumption on centralizers of elements of  $G$ .

4. An  $(n - 1)$ -dimensional group  $G$  of type  $FP_{n-1}$  which contains infinitely many conjugacy classes of coarsely non-separating maximal  $PD(n - 1)$  subgroups cannot act freely on a coarse  $PD(n)$  space.

Our theme is related to the problem of finding an  $n$ -thickening of an aspherical polyhedron  $P$  up to homotopy, i.e., finding a homotopy equivalence  $P \rightarrow M$  where  $M$  is a compact manifold with boundary and  $\dim(M) = n$ . If  $k = \dim(P)$ , then we may immerse  $P$  in  $\mathbb{R}^{2k}$  by general position, and obtain a  $2k$ -manifold thickening  $M$  by “pulling back” a regular neighborhood. Given an  $n$ -thickening  $P \rightarrow M$ , we may construct a free simplicial action of  $G = \pi_1(P)$  on a coarse  $PD(n)$  space by modifying the geometry of  $\text{Int}(M)$  and passing to the universal cover. In particular, if  $G$  cannot act on a coarse  $PD(n)$  space, then no such  $n$ -thickening can exist. In the paper with Bestvina [3], we give examples of finite  $k$ -dimensional aspherical polyhedra  $P$  whose fundamental groups cannot act freely simplicially on any coarse  $PD(n)$  space for  $n < 2k$ , and hence the polyhedra  $P$  do not admit  $n$ -thickening for  $n < 2k$ .

We conclude the discussion of our results with a couple of questions:

**Question 1.10.** Is there a uniform embedding of a Baumslag–Solitar group  $B(p, q)$  (with  $|p| = |q| \neq 1$ ) into the fundamental group of a compact 3-manifold?

Note that one can easily construct a uniform embedding of  $B(p, q)$  into a uniformly contractible 3-manifold  $M$  of bounded geometry, however it seems difficult to find an  $M$  which is the universal cover of a compact 3-manifold.

**Question 1.11.** Is it true that  $PD(3)$  groups  $\Gamma$  are *coherent*, i.e., every finitely generated subgroup of  $\Gamma$  is also finitely presented (or even  $FP_2$ )? It seems unclear even if finitely generated *normal* subgroups in  $\Gamma$  are finitely presented.

More generally,

**Question 1.12.**

1. Suppose that  $G$  is a finitely generated group acting freely and simplicially on a coarse  $PD(3)$  space. Is it true that  $G$  is of type  $FP_2$ ?
2. Suppose that a finitely generated group  $G$  admits a uniform embedding into a coarse  $PD(3)$  space (e.g., a uniformly contractible 3-manifold). Is it true that  $G$  is of type  $FP_2$ ?

Below is a heuristic explanation of why Theorem 1.1 is true. Suppose that the space  $X$  in question is the hyperbolic space  $\mathbb{H}^n$ . Suppose in addition that  $G \subset \text{Isom}(X)$  is a *convex-cocompact* discrete group of isometries, i.e., there exists a closed convex  $G$ -invariant subset  $C \subset X$  with compact quotient  $C/G$ . The hypothesis that  $G$  is an  $(n - 1)$ -dimensional duality group means that its boundary (i.e., the limit set  $\Lambda(G) \subset S^{n-1}$ ) has the same homology as a wedge of  $(n - 2)$ -spheres. Then, Alexander duality implies that each component of the complement of the discontinuity domain  $\Omega(G) = S^{n-1} \setminus \Lambda(G)$  is acyclic. Moreover, since  $G$  is convex-cocompact, there are only finitely many  $G$ -orbits of such components and the stabilizer  $H_i$  of such a component acts on it cocompactly. Therefore, each  $H_i$  is a  $PD(n - 1)$ -group. Thus, we obtain a collection of peripheral subgroups  $\{H_1, \dots, H_k\}$  and it follows that  $(G, \{H_1, \dots, H_k\})$  is a  $PD(n)$  pair.

To give an idea of the actual proof of Theorem 1.1, consider the case when the coarse  $PD(n)$ -space  $X$  happens to be  $\mathbb{R}^n$  with a uniformly acyclic bounded geometry triangulation. We take combinatorial tubular neighborhoods  $N_R(K)$  of a  $G$ -orbit  $K$  in  $X$  and analyze the structure of connected components of  $X - N_R(K)$ . Following Schwartz, we call a connected component  $C$  of  $X - N_R(K)$  *deep* if  $C$  is not contained in

any tubular neighborhood of  $K$ . When  $G$  is a group of type  $FP_n$ , using Alexander duality one shows that deep components of  $X - N_R(K)$  stabilize: there exists  $R_0$  so that no deep component of  $X - N_{R_0}(K)$  breaks up into multiple deep components as  $R$  increases beyond  $R_0$ . If  $G$  is an  $(n - 1)$ -dimensional duality group, then the idea is to show that the stabilizers of deep components of  $X - N_{R_0}(K)$  are  $PD(n - 1)$ -groups, which is the heart of the proof. These groups define the peripheral subgroups  $H_1, \dots, H_k$  of the  $PD(n)$  pair structure  $(G, \{H_1, \dots, H_k\})$  for  $G$ .

When  $X$  is a coarse  $PD(n)$ -space rather than  $\mathbb{R}^n$ , one does not have Alexander duality since Poincaré duality need not hold locally. However, there is a coarse version of Poincaré duality which we use to derive an appropriate coarse analogue of Alexander duality; this extends Richard Schwartz’s coarse Alexander duality from the manifold context to the coarse  $PD(n)$  spaces. Roughly speaking this goes as follows. If  $K \subset \mathbb{R}^n$  is a subcomplex, then Poincaré duality gives an isomorphism

$$H_c^*(K) \rightarrow H_{n-*}(\mathbb{R}^n, \mathbb{R}^n - K).$$

This fails when we replace  $\mathbb{R}^n$  by a general coarse  $PD(n)$  space  $X$ . We prove, however, that for a certain constant  $D$ , there are homomorphisms defined on tubular neighborhoods of  $K$ :

$$P_{R+D} : H_c^k(N_{D+R}(K)) \rightarrow H_{n-k}(X, Y_R), \text{ where } Y_R := \overline{X - N_R(K)},$$

which determine an *approximate isomorphism*. This means that for every  $R$ , there is an  $R'$  (one may take  $R' = R + 2D$ ) so that the homomorphisms  $a$  and  $b$  in the following commutative diagram are zero:

$$\begin{array}{ccccccc} \ker(P_{R'}) & \rightarrow & H_c^k(N_{R'}(K)) & \xrightarrow{P_{R'}} & H_{n-k}(X, Y_{R'-D}) & \rightarrow & \text{coker}(P_{R'}) \\ a \downarrow & & \downarrow & & \downarrow & & b \downarrow \\ \ker(P_R) & \rightarrow & H_c^k(N_R(K)) & \xrightarrow{P_R} & H_{n-k}(X, Y_{R-D}) & \rightarrow & \text{coker}(P_R) \end{array}$$

This coarse version of Poincaré duality leads to coarse Alexander duality, which suffices for our purposes.

In this paper, we develop and use ideas in coarse topology which originated in earlier work by a number of authors: [8, 20, 22, 24, 34, 35, 36]. Other recent papers involving similar ideas include [10, 43, 18, 19]. We would like to stress, however, the difference between our framework and versions of coarse topology in the literature. In [34, 24, 25], coarse topological invariants appear as direct/inverse limits of anti-Čech systems. By passing to the limit (or even working with pro-categories à la Grothendieck), one inevitably loses quantitative information which is essential in many applications of coarse topology to quasi-isometries and geometric group theory. The notion of approximate isomorphism mentioned above (see Section 4) retains this information.

In the main body of the paper, we deal with a special class of metric complexes, namely metric simplicial complexes. This makes the exposition more geometric, and, we believe, more transparent. Also, this special case suffices for many of the applications to quasi-isometries and geometric group theory. In Appendix (Section 11), we explain how the definitions, theorems, and proofs can be modified to handle general metric complexes.

**Organization of the paper.** In Section 2, we introduce metric simplicial complexes and recall notions from coarse topology. Section 3 reviews some facts and definitions from cohomological group theory, duality groups, and group pairs. In Section 4, we define approximate isomorphisms between inverse and direct systems of abelian groups, and compare these with Grothendieck's pro-morphisms. Section 5 provides finiteness criteria for groups, and establishes approximate isomorphisms between group cohomology and cohomologies of nested families of simplicial complexes. In Section 6, we define coarse  $PD(n)$  spaces, give examples, and prove coarse Poincaré duality for coarse  $PD(n)$  spaces. In Section 7, we prove coarse Alexander duality and apply it to coarse separation. In Section 8, we prove Theorems 1.1, Proposition 8.10, and variants of Theorem 1.1. In Section 9, we apply coarse Alexander duality and Theorem 1.1 to show that certain groups cannot act freely on coarse  $PD(n)$  spaces. In Section 10, we give a brief account of coarse Alexander duality for uniformly acyclic triangulated manifolds of bounded geometry. The reader interested in manifolds and not in Poincaré complexes can use this as a replacement of Theorem 7.5.

**Suggestions to the reader.** Readers familiar with Grothendieck's pro-morphisms may wish to read the second part of section 4, which will allow them to translate statements about approximate isomorphisms into pro-language. Readers who are not already familiar with pro-morphisms may simply skip this. Those who are interested in finiteness properties of groups may find Section 5, especially Theorems 5.11 and Corollary 5.14, of independent interest.

## 2. Geometric preliminaries

**Metric simplicial complexes.** Let  $X$  be the geometric realization of a connected locally finite simplicial complex. Henceforth, we will conflate simplicial complexes with their geometric realizations. We will metrize the 1-skeleton  $X^1$  of  $X$  by declaring each edge to have unit length and taking the corresponding path-metric. Such an  $X$  with the metric on  $X^1$  will be called a *metric simplicial complex*. The complex  $X$  is said to



have *bounded geometry* if all links have a uniformly bounded number of simplices; this is equivalent to saying that the metric space  $X^1$  is locally compact and every  $R$ -ball in  $X^1$  can be covered by at most  $C = C(R, r)$   $r$ -balls for any  $r > 0$ . In particular,  $\dim(X) < \infty$ . If  $K \subset X$  is a subcomplex and  $r$  is a positive integer, then we define (combinatorial)  $r$ -tubular neighborhood  $N_r(K)$  of  $K$  to be  $r$ -fold iterated closed star of  $K$ ,  $St^r(K)$ ; we declare  $N_0(K)$  to be  $K$  itself. Note that for  $r > 0$ ,  $N_r(K)$  is the closure of its interior. The diameter of  $K$  is defined to be the diameter of its zero-skeleton, and  $\partial K$  denotes the frontier of  $K$ , which is a subcomplex. For each vertex  $x \in X$  and  $R \in \mathbb{Z}_+$ , we let  $B(x, R)$  denote  $N_R(\{x\})$ , the “ $R$ -ball centered at  $x$ ”.

**Remark 2.1.** The definition of metric complexes, which generalize metric simplicial complexes, appears in Appendix 11.

**Coarse Lipschitz and uniform embeddings.** We recall that a map  $f : X \rightarrow Y$  between metric spaces is called  $(L, A)$ -Lipschitz if

$$d(f(x), f(x')) \leq Ld(x, x') + A$$

for any  $x, x' \in X$ . A map is *coarse Lipschitz* if it is  $(L, A)$ -Lipschitz for some  $L, A$ . A coarse Lipschitz map  $f : X \rightarrow Y$  is called a *uniform embedding* if there is a proper function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  (a *distortion function*) such that

$$d(f(x), f(x')) \geq \phi(d(x, x'))$$

for all  $x, x' \in X$ .

Throughout the paper, we will use simplicial (co)chain complexes and integer coefficients. If  $C_*(X)$  is the simplicial chain complex and  $A \subset C_*(X)$ , then the *support of  $A$* , denoted  $\text{Support}(A)$ , is the smallest subcomplex  $K \subset X$  so that  $A \subset C_*(K)$ . Throughout the paper, we will assume that morphisms between simplicial chain complexes preserve the usual augmentation.

If  $X, Y$  are metric simplicial complexes as above, then a homomorphism

$$h : C_*(X) \rightarrow C_*(Y)$$

is said to be *coarse Lipschitz* if for each simplex  $\sigma \subset X$ ,  $\text{Support}(h(C_*(\sigma)))$  has uniformly bounded diameter. The *Lipschitz constant of  $h$*  is

$$\max_{\sigma} \text{diam}(\text{Support}(h(C_*(\sigma)))).$$

A homomorphism  $h$  is said to be a *uniform embedding* if it is coarse Lipschitz and there exists a proper function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  (a *distortion function*) such that for each subcomplex  $K \subset X$  of diameter  $\geq r$ ,

Support  $(h(C_*(K)))$  has diameter  $\geq \phi(r)$ . We will apply this definition only to chain mappings and chain homotopies. (Recall that there is a standard way to triangulate the product  $\Delta^k \times [0, 1]$ ; we can use this to triangulate  $X \times [0, 1]$  and hence view it as a metric simplicial complex.) We say that a homomorphism  $h : C_*(X) \rightarrow C_*(X)$  has displacement  $\leq D$  if for every simplex  $\sigma \subset X$ ,  $\text{Support}(h(C_*(\sigma))) \subset N_D(\sigma)$ .

We may adapt all of the definitions from the previous paragraph to mappings between other (co)chain complexes associated with metric simplicial complexes, such as the compactly supported cochain complex  $C_c^*(X)$ .

**Coarse topology.** An  $n$ -dimensional metric simplicial complex  $X$  is said to be *uniformly acyclic* if for every  $R_1$ , there is an  $R_2$  such that for each subcomplex  $K \subset X$  of diameter  $\leq R_1$  the inclusion  $K \rightarrow N_{R_2}(K)$  induces zero on reduced homology groups. Such a function  $R_2 = R_2(R_1)$  will be called an *acyclicity function* for  $C_*(X)$ . Let  $C_c^*(X)$  denote the complex of compactly supported simplicial cochains, and suppose  $\alpha : C_c^n(X) \rightarrow \mathbb{Z}$  is an augmentation for  $C_c^*(X)$ , i.e., a homomorphism which is zero on all coboundaries. Then, the pair  $(C_c^*(X), \alpha)$  is called *uniformly acyclic* if there is an  $R_0 > 0$  and a function  $R_2 = R_2(R_1)$  so that for all  $x \in X^0$  and all  $R_1 \geq R_0$ ,

$$\text{Im}(H_c^*(X, \overline{X - B(x, R_1)}) \rightarrow H_c^*(X, \overline{X - B(x, R_2)}))$$

maps isomorphically onto  $H_c^*(X)$  under  $H_c^*(X, \overline{X - B(x, R_2)}) \rightarrow H_c^*(X)$ , and  $\alpha$  induces an isomorphism  $\bar{\alpha} : H_c^n(X) \rightarrow \mathbb{Z}$ .

Let  $K \subset X$  be a subcomplex of a metric simplicial complex  $X$ . For every  $R \geq 0$ , we say that an element  $c \in H_k(X - N_R(K))$  is *deep* if it lies in

$$\text{Im}(H_k(X - N_{R'}(K)) \rightarrow H_k(X - N_R(K)))$$

for every  $R' \geq R$ ; equivalently,  $c$  is deep if it belongs to the image of

$$\varprojlim_r H_k(X - N_r(K)) \longrightarrow H_k(X - N_R(K)).$$

We let  $H_k^{\text{Deep}}(X - N_R(K))$  denote the subgroup of deep homology classes of  $X - N_R(K)$ . Hence, we obtain an inverse system  $\{H_k^{\text{Deep}}(X - N_R(K))\}$ . We say that the deep homology *stabilizes* at  $R_0$  if the projection homomorphism

$$\varprojlim_R H_k^{\text{Deep}}(X - N_R(K)) \rightarrow H_k^{\text{Deep}}(X - N_{R_0}(K))$$

is injective.

Specializing the above definition to the case  $k = 0$ , we arrive at the definition of deep complementary components. If  $R \geq 0$ , a component

$C$  of  $X - N_R(K)$  is called *deep* if it is not contained within a finite neighborhood of  $K$ . A subcomplex  $K$  *coarsely separates*  $X$  if there is an  $R$  so that  $X - N_R(K)$  has at least two deep components. A deep component  $C$  of  $X - N_R(K)$  is said to be *stable* if for each  $R' \geq R$  the component  $C$  meets exactly one deep component of  $X - N_{R'}(K)$ .  $K$  is said to *coarsely separate*  $X$  into (exactly)  $m$  components if there is an  $R$  so that  $X - N_R(K)$  consists of exactly  $m$  stable deep components.

Note that  $H_0^{\text{Deep}}(X - N_R(K))$  is freely generated by elements corresponding to deep components of  $X - N_R(K)$ . The deep homology  $H_0^{\text{Deep}}(X - N_R(K))$  stabilizes at  $R_0$  if and only if all deep components of  $X - N_{R_0}(K)$  are stable.

If  $G \curvearrowright X$  is a simplicial action of a group on a metric simplicial complex, then one orbit  $G(x)$  coarsely separates  $X$  if and only if every  $G$ -orbit coarsely separates  $X$ ; hence, we may simply say that  $G$  coarsely separates  $X$ . If  $H$  is a subgroup of a finitely generated group  $G$ , then we say that  $H$  coarsely separates  $G$  if  $H$  coarsely separates some (and hence any) Cayley graph of  $G$ .

Let  $Y, K$  be subcomplexes of a metric simplicial complex  $X$ . We say that  $Y$  coarsely separates  $K$  in  $X$  if there is  $R > 0$  and two distinct components  $C_1, C_2 \subset X - N_R(Y)$  so that the distance function  $d_Y(\cdot) := d(\cdot, Y)$  is unbounded on both  $K \cap C_1$  and  $K \cap C_2$ . The subcomplex  $Y$  will coarsely separate  $X$  in this case.

### 3. Group theoretic preliminaries

**Resolutions, cohomology and relative cohomology.** Let  $G$  be group and  $K$  be an Eilenberg–MacLane space for  $G$ . If  $\mathcal{M}$  is a system of local coefficients on  $K$ , then we have homology and cohomology groups of  $K$  with coefficients in  $\mathcal{M}$ :  $H_*(K; \mathcal{M})$  and  $H^*(K; \mathcal{M})$ . Now, let  $A$  be a  $\mathbb{Z}G$ -module. We recall that a *resolution* of  $A$  is an exact sequence of  $\mathbb{Z}G$ -modules:

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0.$$

Every  $\mathbb{Z}G$ -module has a unique projective resolution up to chain homotopy equivalence. If  $M$  is a  $\mathbb{Z}G$ -module, then the *cohomology of  $G$  with coefficients in  $M$* ,  $H^*(G; M)$ , is defined as the homology of chain complex  $\text{Hom}_{\mathbb{Z}G}(P_*, M)$  where  $P_*$  is a projective resolution of the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$ ; the *homology of  $G$  with coefficients in  $M$* ,  $H_*(G; M)$ , is the homology of the chain complex  $P_* \otimes_{\mathbb{Z}G} M$ . Using the 1–1 correspondence between  $\mathbb{Z}G$ -modules,  $M$  and local coefficient

systems  $\mathcal{M}$  on an Eilenberg–MacLane space  $K$ , we get natural isomorphisms  $H_*(K; \mathcal{M}) \simeq H_*(G; M)$  and  $H^*(K; \mathcal{M}) \simeq H^*(G; M)$ . Henceforth, we will use the same notation to denote  $\mathbb{Z}G$ -modules and the corresponding local systems on  $K(G, 1)$ 's.

**Group pairs.** We now discuss relative (co)homology following [7]. Let  $G$  be a group, and  $\mathcal{H} := \{H_i\}_{i \in I}$  an indexed collection of (not necessarily distinct) subgroups. We refer to  $(G, \mathcal{H})$  as a *group pair*. Let  $\Pi_i K(H_i, 1) \xrightarrow{f} K(G, 1)$  be the map induced by the inclusions  $H_i \rightarrow G$ , and let  $K$  be the mapping cylinder of  $f$ . We, therefore, have a pair of spaces  $(K, \Pi_i K(H_i, 1))$  since the domain of a map naturally embeds in the mapping cylinder. Given any  $\mathbb{Z}G$ -module  $M$ , we define the relative cohomology  $H^*(G, \mathcal{H}; M)$  (respectively homology  $H_*(G, \mathcal{H}; M)$ ) to be the cohomology (resp. homology) of the pair  $(K, \Pi_i K(H_i, 1))$  with coefficients in the local system  $M$ . As in the absolute case, one can compute relative (co)homology groups using projective resolutions, see [7]. For each  $i \in I$ , let

$$\cdots \rightarrow Q_n(i) \rightarrow \cdots \rightarrow Q_0(i) \rightarrow \mathbb{Z} \rightarrow 0$$

be a resolution of  $\mathbb{Z}$  by projective  $\mathbb{Z}H_i$ -modules, and let

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

be a resolution of  $\mathbb{Z}$  by projective  $\mathbb{Z}G$ -modules. The inclusions  $H_i \rightarrow G$  induce  $\mathbb{Z}H_i$ -chain mappings  $f_i : Q_*(i) \rightarrow P_*$ , unique up to chain homotopy. We define a  $\mathbb{Z}G$ -chain complex  $Q_*$  to be  $\bigoplus_i (\mathbb{Z}G \otimes_{\mathbb{Z}H_i} Q_*(i))$  with an augmentation

$$Q_0 \rightarrow \bigoplus_i (\mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z})$$

induced by the augmentations  $Q_0(i) \rightarrow \mathbb{Z}$ ; the chain mappings  $f_i$  yield a  $\mathbb{Z}G$ -chain mapping  $f : Q_* \rightarrow P_*$ . We let  $C_*$  be the algebraic mapping cylinder of  $f$ : this is the chain complex with  $C_i := P_i \oplus Q_{i-1} \oplus Q_i$  with the boundary homomorphism given by

$$(3.1) \quad \partial(p_i, q_{i-1}, q_i) = (\partial p_i + f(q_{i-1}), -\partial q_{i-1}, \partial q_i + q_{i-1}).$$

We note that each  $C_i$  is clearly projective, a copy  $D_*$  of  $Q_*$  naturally sits in  $C_*$  as the third summand, and the quotient  $C_*/D_*$  is a chain complex of projective  $\mathbb{Z}G$ -modules. Proposition 1.2 of [7] implies that the relative homology (resp. cohomology) of the group pair  $(G, \mathcal{H})$  with coefficients in a  $\mathbb{Z}G$ -module  $M$  (defined as above using local systems on Eilenberg–MacLane spaces) is canonically isomorphic to homology of the chain complex  $(C_*/D_*) \otimes_{\mathbb{Z}G} M$  (resp.  $\text{Hom}_{\mathbb{Z}G}((C_*/D_*), M)$ ).

**Finiteness properties of groups.** The (cohomological) dimension  $\dim(G)$  of a group  $G$  is  $n$  if  $n$  is the minimal integer such that there exists a resolution of  $\mathbb{Z}$  by projective  $\mathbb{Z}G$ -modules:

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0.$$

Recall that  $G$  has cohomological dimension  $n$  if and only if  $n$  is the minimal integer so that  $H^k(G, M) = 0$  for all  $k > n$  and all  $\mathbb{Z}G$ -modules  $M$ . Moreover, if  $\dim(G) < \infty$ , then

$$\dim(G) = \sup\{n \mid H^n(G; F) \neq 0 \text{ for some free } \mathbb{Z}G\text{-module } F\},$$

see [12, Ch. VIII, Proposition 2.3]. If

$$1 \rightarrow G_1 \rightarrow G \rightarrow G_2 \rightarrow 1$$

is a short exact sequence then  $\dim(G) \leq \dim(G_1) + \dim(G_2)$ , [12, Ch. VIII, Proposition 2.4]. If  $G' \subset G$  is a subgroup then  $\dim(G') \leq \dim(G)$ .

A *partial resolution* of a  $\mathbb{Z}G$ -module  $A$  is an exact sequence  $\mathbb{Z}G$ -modules:

$$P_n \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0.$$

If  $A_*$ :

$$\cdots \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_0 \rightarrow A \rightarrow 0$$

is a chain complex, then we let  $[A_*]_n$  denote the  $n$ -truncation of  $A_*$ , i.e.,

$$A_n \rightarrow \cdots \rightarrow A_0 \rightarrow A \rightarrow 0.$$

A group  $G$  is of type  $FP_n$  if there exists a partial resolution of  $\mathbb{Z}$  by finitely generated projective  $\mathbb{Z}G$ -modules:

$$P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0.$$

The group  $G$  is of type  $FP$  (resp.  $FL$ ) if there exists a finite resolution of  $\mathbb{Z}$  by finitely generated projective (resp. free)  $\mathbb{Z}G$ -modules. A group pair  $(G, \{H_1, \dots, H_m\})$  (where  $H_i$ 's are subgroups of  $G$ ) is said to be of type  $FP$  if  $G$  and all  $H_i$ 's are of type  $FP$ .

**Lemma 3.2.**

1. If  $G$  is of type  $FP$ , then  $\dim(G) = n$  if and only if

$$n = \max\{i : H^i(G; \mathbb{Z}G) \neq 0\}.$$

2. If  $\dim(G) = n$  and  $G$  is of type  $FP_n$ , then there exists a resolution of  $\mathbb{Z}$  by finitely generated projective  $\mathbb{Z}G$ -modules:

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0.$$

In particular,  $G$  is of type  $FP$ .

*Proof.* The first assertion follows from [12, Ch. VIII, Proposition 5.2]. We prove 2. Start with a partial resolution

$$P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

where each  $P_i$  is finitely generated projective. By [12, Ch. VIII, Lemma 2.1], the kernel  $Q_n := \ker [P_{n-1} \rightarrow P_{n-2}]$  is projective. However,  $P_n$  maps onto  $Q_n$ , hence  $Q_n$  is also finitely generated. Thus, replacing  $P_n$  with  $Q_n$ , we get the required resolution. q.e.d.

Examples of groups of type  $FP$  and  $FL$  are given by fundamental groups of finite Eilenberg–MacLane complexes, or more generally, groups acting freely cocompactly on acyclic complexes. According to the theorem of Eilenberg–Ganea and Wall, if  $G$  is a finitely presentable group of type  $FL$ , then  $G$  admits a finite  $K(G, 1)$  of dimension  $\max(\dim(G), 3)$ .

Let  $G$  be a group, let  $\mathcal{H} := \{H_i\}_{i \in I}$  be an indexed collection of subgroups, and let

$$\epsilon : \bigoplus_i (\mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z}) \rightarrow \mathbb{Z}$$

be induced by the usual augmentation  $\mathbb{Z}G \rightarrow \mathbb{Z}$ . Then, the group pair  $(G, \mathcal{H})$  has finite type if the  $\mathbb{Z}G$ -module  $\text{Ker}(\epsilon)$  admits a finite length resolution by finitely generated projective  $\mathbb{Z}G$ -modules. If the index set  $I$  is finite and the groups  $G$  and  $H_i$  are of type  $FP$ , then one obtains the desired resolution of  $\text{Ker}(\epsilon)$  using the quotient  $C_*/D_*$  where  $(C_*, D_*)$  is the pair given by the algebraic mapping cylinder construction (3.1).

For the next three topics, the reader may consult [5, 6, 7, 12, 14].

**Duality groups.** Let  $G$  be a group of type  $FP$ . Then,  $G$  is an  $n$ -dimensional duality group if  $H^i(G; \mathbb{Z}G) = \{0\}$  when  $i \neq n = \dim(G)$ , and  $H^n(G; \mathbb{Z}G)$  is torsion-free, [6]. There is an alternate definition of duality groups involving isomorphisms  $H^i(G; M) \simeq H_{n-i}(G; D \otimes M)$  for a suitable dualizing module  $D$  and arbitrary  $\mathbb{Z}G$ -modules  $M$ , see [6, 12]. Examples of duality groups include:

1. The fundamental groups of compact aspherical manifolds with aspherical boundary, where the inclusion of each boundary component induces a monomorphism of fundamental groups.
2. Torsion-free S-arithmetic groups, [6, 9].
3. 2-dimensional one-ended groups of type  $FP_2$  [5, Proposition 9.17]; for instance torsion-free, one-ended, one-relator groups.
4. Any group which can act freely, cocompactly, and simplicially on an acyclic simplicial complex  $X$ , where  $H_c^i(X)$  vanishes except in dimension  $n$ , and  $H_c^n(X)$  is torsion-free.

**Poincaré duality groups.** These form a special class of duality groups. If  $G$  is an  $n$ -dimensional duality group and  $H^n(G; \mathbb{Z}G) = \mathbb{Z}$ , then  $G$  is an  $n$ -dimensional Poincaré duality group ( $PD(n)$  group). As in the case of duality groups, there is an alternate definition involving isomorphisms  $H^i(G; M) \simeq H_{n-i}(G; D \otimes M)$  where  $M$  is an arbitrary  $\mathbb{Z}G$ -module and the orientation  $\mathbb{Z}G$ -module  $D$  is isomorphic to  $\mathbb{Z}$  as an abelian group. Examples include:

1. Fundamental groups of closed aspherical manifolds.
2. Fundamental groups of aspherical finite Poincaré complexes. Recall that an (orientable) *Poincaré complex of formal dimension  $n$*  is a finitely dominated complex  $K$  together with a fundamental class  $[K] \in H_n(K; \mathbb{Z})$  so that the cap product operation  $[K] \cap : H^k(K; M) \rightarrow H_{n-k}(K; M)$  is an isomorphism for every local system  $M$  on  $K$  and for  $k = 0, \dots, n$ .
3. Any group which can act freely, cocompactly, and simplicially on an acyclic simplicial complex  $X$ , where  $X$  has the same compactly supported cohomology as  $\mathbb{R}^n$ .
4. Each torsion-free Gromov-hyperbolic group  $G$  whose boundary is a homology manifold with the homology of sphere (over  $\mathbb{Z}$ ), see [4]. Note that every such group is the fundamental group of a finite aspherical Poincaré complex, namely the  $G$ -quotient of a Rips complex of  $G$ .

Below are several useful facts about Poincaré duality groups (see [12]):

- (a) If  $G$  is a  $PD(n)$  group and  $G' \subset G$  is a subgroup, then  $G'$  is a  $PD(n)$  group if and only if the index  $[G : G']$  is finite.
- (b) If  $G$  is a  $PD(n)$  group which is contained in a torsion-free group  $G'$  as a finite index subgroup, then  $G'$  is a  $PD(n)$  group.
- (c) If  $G \times H$  is a  $PD(m)$  group, then  $G$  and  $H$  are  $PD(n)$  and  $PD(k)$  groups, where  $m = n + k$ .
- (d) If  $G \rtimes H$  is a semi-direct product where  $G$  is a  $PD(n)$ -group and  $H$  is a  $PD(k)$ -group, then  $G \rtimes H$  is a  $PD(n + k)$ -group. See [6, Theorem 3.5].

There are several questions about  $PD(n)$  groups and their relation with fundamental groups of aspherical manifolds. It was an open question going back to Wall [44] whether every  $PD(n)$  group is the fundamental group of a closed aspherical manifold. The answer to this is yes in dimensions 1 and 2, [40, 16, 17]. Recently, Davis in [13] gave examples for  $n \geq 4$  of  $PD(n)$  groups which do not admit a finite presentation, and these groups are clearly not fundamental groups of compact manifolds. This leaves open several questions:

1. Is every finitely presented  $PD(n)$  group the fundamental group of a compact aspherical manifold?
2. A weaker version of 1: Is every finitely presented  $PD(n)$  group the fundamental group of a finite aspherical complex? Equivalently, by Eilenberg–Ganea, one may ask if every such group is of type  $FL$ .
3. Does every  $PD(n)$  group act freely and cocompactly on an acyclic complex? We believe this question is open for groups of type  $FP$ . One can also ask if every  $PD(n)$  group acts freely and cocompactly on an acyclic  $n$ -manifold.

**Poincaré duality pairs.** Let  $G$  be an  $(n - 1)$ -dimensional group of type  $FP$ , and let  $H_1, \dots, H_k \subset G$  be  $PD(n - 1)$  subgroups of  $G$ . Then, the group pair  $(G, \{H_1, \dots, H_k\})$  is an  $n$ -dimensional Poincaré duality pair, or  $PD(n)$  pair, if the double of  $G$  over the  $H_i$ 's is a  $PD(n)$  group. We recall that the double of  $G$  over the  $H_i$ 's is the fundamental group of the graph of groups  $\mathcal{G}$ , where  $\mathcal{G}$  has two vertices labeled by  $G$ ,  $k$  edges with the  $i^{\text{th}}$  edge labeled by  $H_i$ , and edge monomorphisms are the inclusions  $H_i \rightarrow G$ . An alternate homological definition of  $PD(n)$  pairs is the following: A group pair  $(G, \{H_i\}_{i \in I})$  is a  $PD(n)$  pair if it has type  $FP$ , and  $H^*(G, \{H_i\}; \mathbb{Z}G) \simeq H_c^*(\mathbb{R}^n)$ . For a discussion of these and other equivalent definitions, see [7, 14]. We will sometimes refer to the system of subgroups  $\{H_i\}$  as the *peripheral structure* of the  $PD(n)$  pair, and the  $H_i$ 's as peripheral subgroups. The first class of examples of duality groups mentioned above have natural peripheral structure which makes them  $PD(n)$  pairs. In [28], we proved that if  $G$  is a torsion-free Gromov-hyperbolic group whose boundary is homeomorphic to the Sierpinski carpet  $S$ , then  $(G, \{H_1, \dots, H_k\})$  is a  $PD(3)$  group pair, where  $H_i$ 's are representatives of conjugacy classes of stabilizers of the peripheral circles of  $S$  in  $\partial_\infty G$ . If  $(G, \{H_1, \dots, H_k\})$  is a  $PD(n)$  pair, where  $G$  and each  $H_i$  admit a finite Eilenberg–MacLane space  $X$  and  $Y_i$  respectively, then the inclusions  $H_i \rightarrow G$  induce a map  $\sqcup_i Y_i \rightarrow X$  (well-defined up to homotopy) whose mapping cylinder  $C$  gives a Poincaré pair  $(C, \sqcup_i Y_i)$ , i.e., a pair which satisfies Poincaré duality for manifolds with boundary with local coefficients (where  $\sqcup_i Y_i$  serves as the boundary of  $C$ ). Conversely, if  $(X, Y)$  is a Poincaré pair where  $X$  is aspherical and  $Y$  is a union of aspherical components  $Y_i$ , then  $(\pi_1(X), \{\pi_1(Y_1), \dots, \pi_1(Y_k)\})$  is a  $PD(n)$  pair.

**Lemma 3.3.** *Let  $(G, \{H_i\})$  be a  $PD(n)$  pair, where  $G$  is not a  $PD(n - 1)$  group. Then, the subgroups  $H_i$  are pairwise non-conjugate maximal  $PD(n - 1)$  subgroups.*



*Proof.* If  $H_i$  is conjugate to  $H_j$  for some  $i \neq j$ , then the double  $\hat{G}$  of  $G$  over the peripheral subgroups would contain an infinite index subgroup isomorphic to the  $PD(n)$  group  $H_i \times \mathbb{Z}$ . The group  $\hat{G}$  is a  $PD(n)$  group, which contradicts property (a) of Poincaré duality groups listed above.

We now prove that each  $H_i$  is maximal. Suppose that  $H_i \subset H \subset G$ , where  $H \neq H_i$  is a  $PD(n - 1)$  group. Then,  $[H : H_i] < \infty$ . Pick  $h \in H - H_i$ . Then, there exists a finite index subgroup  $F_i \subset H_i$  which is normalized by  $h$ . Consider the double  $\hat{G}$  of  $G$  along the collection of subgroups  $\{H_i\}$ , and let  $\hat{G} \curvearrowright T$  be the associated action on the Bass-Serre tree. Since  $G$  is not a  $PD(n - 1)$  group,  $H_i \neq G$  for each  $i$ , and so there is a unique vertex  $v \in T$  fixed by  $G$ . The involution of the graph of groups defining  $\hat{G}$  induces an involution of  $\hat{G}$  which is unique up to an inner automorphism; let  $\tau : \hat{G} \rightarrow \hat{G}$  be an induced involution which fixes  $H_i$  element-wise. Then,  $G' := \tau(G)$  fixes a vertex  $v'$  adjacent to  $v$ , where the edge  $\overline{vv'}$  is fixed by  $H_i$ . So  $h' := \tau(h)$  belongs to  $\tau(G) = G'$ , but  $h'$  does not fix  $\overline{vv'}$ . Therefore, the fixed point sets of  $h$  and  $h'$  are disjoint, which implies that  $g := hh'$  acts on  $T$  as a hyperbolic automorphism. Since  $h' \in \text{Normalizer}(\tau(F_i)) = \text{Normalizer}(F_i)$ , we get  $g \in \text{Normalizer}(F_i)$ . Hence, the subgroup  $F$  generated by  $F_i$  and  $g$  is a semi-direct product  $F = F_i \rtimes \langle g \rangle$ , and  $\langle g \rangle \simeq \mathbb{Z}$  since  $g$  is hyperbolic. The group  $F$  is a  $PD(n)$  group (by property (d)) sitting as an infinite index subgroup of the  $PD(n)$  group  $\hat{G}$ , which contradicts property (a).  
 q.e.d.

#### 4. Algebraic preliminaries

In this Section, we introduce a notion of “morphism” between inverse systems. Approximate isomorphisms, which figure prominently in the remainder of the paper, are maps between inverse (or direct) systems which fail to be isomorphisms in a controlled way, and for many purposes are as easy to work with as isomorphisms.

##### Approximate morphisms between inverse and direct systems.

Recall that a partially ordered set  $I$  is *directed* if for each  $i, j \in I$  there exists  $k \in I$  such that  $k \geq i, j$ . An inverse system of (abelian) groups indexed by a directed set  $I$  is a collection of abelian groups  $\{A_i\}_{i \in I}$  and homomorphisms (*projections*)  $p_i^j : A_i \rightarrow A_j, i \geq j$  so that

$$p_i^i = id \text{ and } p_j^k \circ p_i^j = p_i^k$$

for any  $i \leq j \leq k$ . (One may weaken these assumptions, but they will suffice for our purposes.) We will often denote the inverse system by

$(A_\bullet, p_\bullet)$  or  $\{A_i\}_{i \in I}$ . Recall that a subset  $I' \subset I$  of a partially ordered set is *cofinal* if for every  $i \in I$  there is an  $i' \in I'$  so that  $i' \geq i$ .

Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be two inverse systems of (abelian) groups indexed by  $I$  and  $J$ , with the projection maps  $p_i^{i'} : A_i \rightarrow A_{i'}$  and  $q_j^{j'} : B_j \rightarrow B_{j'}$ . The directed sets appearing later in the paper will be order isomorphic to  $\mathbb{Z}_+$  with the usual order.

**Definition 4.1.** Let  $\alpha$  be an order preserving, partially defined, map from  $I$  to  $J$ . Then,  $\alpha$  is *cofinal* if it is defined on a subset of the form  $\{i \in I \mid i \geq i_0\}$  for some  $i_0 \in I$ , and the image of every cofinal subset  $I' \subset I$  is a cofinal subset  $\alpha(I') \subset J$ .

**Definition 4.2.** Let  $\alpha : I \rightarrow J$  be a cofinal map. Suppose that  $(\{A_i\}_{i \in I}, p_\bullet)$  and  $(\{B_j\}_{j \in J}, q_\bullet)$  are inverse systems. Then, a family of homomorphisms  $f_i : A_i \rightarrow B_{\alpha(i)}$ ,  $i \in I$ , is an  $\alpha$ -morphism from  $\{A_i\}_{i \in I}$  to  $\{B_j\}_{j \in J}$  if

$$(4.3) \quad q_{\alpha(i)}^{\alpha(i')} \circ f_i = f_i \circ p_i^{i'}$$

whenever  $i, i' \in I$  and  $i \geq i'$ . The *saturation*  $\hat{f}_\bullet$  of the  $\alpha$ -morphism  $f_\bullet$  is the collection of maps  $\hat{f}_i^j : A_i \rightarrow B_j$  of the form

$$q_{\alpha(k)}^j \circ f_k \circ p_i^k.$$

In view of (4.3), this definition is consistent, and  $\hat{f}_\bullet$  is compatible with the projection maps of  $A_\bullet$  and  $B_\bullet$ .

Suppose that  $\{A_i\}_{i \in I}$ ,  $\{B_j\}_{j \in J}$ ,  $\{C_k\}_{k \in K}$  are inverse systems,  $\alpha : I \rightarrow J$ ,  $\beta : J \rightarrow K$  are cofinal maps. Then, the composition of  $\alpha$ - and  $\beta$ -morphisms

$$f_\bullet : A_\bullet \rightarrow B_\bullet, \quad g_\bullet : B_\bullet \rightarrow C_\bullet$$

is a  $\gamma$ -morphism for the cofinal map  $\gamma = \beta \circ \alpha : I \rightarrow K$ . (The composition  $\beta \circ \alpha$  is defined on the subset  $\text{Domain}(\alpha) \cap \alpha^{-1}(\text{Domain}(\beta))$  which contains  $\{i : i \geq i_1\}$  where  $i_1$  is an upper bound for non-cofinal subset  $\alpha^{-1}(J - \text{Domain}(\beta))$  in  $I$ .)

**Definition 4.4.** Let  $A_\bullet \xrightarrow{f_\bullet} B_\bullet$  be an  $\alpha$ -morphism of inverse systems  $(A_\bullet, p_\bullet), (B_\bullet, q_\bullet)$ .

1. When  $I$  is totally ordered, we define  $\text{Im}(\hat{f}_\bullet^j)$ , the *image* of  $f_\bullet$  in  $B_j$ , to be  $\cup\{\text{Im}(\hat{f}_i^j : A_i \rightarrow B_j) \mid \alpha(i) \geq j\}$ .
2. Let  $\omega : I \rightarrow I$  be a function with  $\omega(i) \geq i$  for all  $i \in I$ . Then,  $f_\bullet$  is an  $\omega$ -approximate monomorphism if for every  $i \in I$ , we have

$$\text{Ker} \left( A_{\omega(i)} \xrightarrow{f_{\omega(i)}} B_{\alpha(\omega(i))} \right) \subset \text{Ker} \left( A_{\omega(i)} \xrightarrow{p_\bullet} A_i \right).$$

3. Suppose  $I$  is totally ordered. If  $\bar{\omega} : J \rightarrow J$  is a function with  $\bar{\omega}(j) \geq j$  for all  $j \in J$ , then  $f_\bullet$  is an  $\bar{\omega}$ -approximate epimorphism if for every  $j \in J$ , we have:

$$\text{Im}(B_{\bar{\omega}(j)} \xrightarrow{q_\bullet} B_j) \subset \text{Im}(\hat{f}_j^\bullet).$$

4. Suppose  $I$  is totally ordered. If  $\omega : I \rightarrow I$  and  $\bar{\omega} : J \rightarrow J$  are functions, then  $f$  is an  $(\omega, \bar{\omega})$ -approximate isomorphism if both 2 and 3 hold.

We will frequently suppress the functions  $\alpha, \omega, \bar{\omega}$  when speaking of morphisms, approximate monomorphisms (epimorphisms, isomorphisms).

Note that an  $\alpha$ -morphism induces a homomorphism between inverse limits, since for each cofinal subset  $J' \subset J$ , we have:

$$\lim_{\leftarrow j \in J} B_j \cong \lim_{\leftarrow j \in J'} B_j.$$

Similarly, an approximate monomorphism, resp. isomorphism, of inverse systems induces a monomorphism, resp. isomorphism, of their inverse limits. (Eric Swenson observed that similar assertion is false for approximate epimorphisms.) However, the converse is not true. For instance, let  $A_i := \mathbb{Z}$  for each  $i \in \mathbb{N}$ , where  $\mathbb{N}$  has the usual order. Let

$$p_i^{i-n} : A_i \rightarrow A_{i-n} \text{ be the index } n \text{ inclusion.}$$

It is clear that the inverse limit of this system is zero. We leave it to the reader to verify that the system  $(A_\bullet, p_\bullet)$  is not approximately isomorphic to zero-inverse system.

We have similar definitions for homomorphisms of direct systems. A direct system of (abelian) groups indexed by a directed set  $I$  is a collection of abelian groups  $\{A_i\}_{i \in I}$  and homomorphisms (*projections*)  $p_i^j : A_i \rightarrow A_j, i \leq j$  so that

$$p_i^i = id, \quad p_j^k \circ p_i^j = p_i^k$$

for any  $i \leq j \leq k$ . We often denote the direct system by  $(A_\bullet, p_\bullet)$ . Let  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  be two direct systems of (abelian) groups indexed by directed sets  $I$  and  $J$ , with projection maps  $p_i^{i'} : A_i \rightarrow A_{i'}$  and  $q_j^{j'} : B_j \rightarrow B_{j'}$ .

**Definition 4.5.** Let  $\alpha : I \rightarrow J$  be a cofinal map. Then, a family of homomorphisms  $f_i : A_i \rightarrow B_{\alpha(i)}, i \in I$ , is a  $\alpha$ -morphism of the direct systems  $\{A_i\}_{i \in I}$  and  $\{B_j\}_{j \in J}$  if

$$q_{\alpha(i)}^{\alpha(i')} \circ f_i = f_{i'} \circ p_i^{i'}$$

whenever,  $i \leq i'$ . We define the saturation  $\widehat{f}_\bullet$  the same way as for morphisms of inverse systems.

**Definition 4.6.** Let  $f_\bullet : A_\bullet \rightarrow B_\bullet$  be an  $\alpha$ -morphism of direct systems:

$$f_\bullet = \{f_i : A_i \rightarrow B_{\alpha(i)}, i \in I\}.$$

1. When  $I$  is totally ordered, we define  $Im(\widehat{f}_\bullet^j)$ , the *image* of  $f_\bullet$  in  $B_j$ , to be  $\cup\{Im(\widehat{f}_i^j) \mid \alpha(i) \leq j\}$ .
2. Let  $\omega : I \rightarrow I$  be a function with  $\omega(i) \geq i$  for all  $i \in I$ . Then,  $f_\bullet$  is an  $\omega$ -*approximate monomorphism* if for every  $i \in I$ , we have

$$Ker \left( A_i \xrightarrow{f_i} B_{\alpha(i)} \right) \subset Ker \left( A_i \xrightarrow{p_\bullet} A_{\omega(i)} \right).$$

3. Suppose  $I$  is totally ordered, and  $\bar{\omega} : J \rightarrow J$  is a function with  $\bar{\omega}(j) \geq j$  for all  $j \in J$ .  $f_\bullet$  is an  $\bar{\omega}$ -*approximate epimorphism* if for every  $j \in J$ , we have:

$$Im \left( B_j \xrightarrow{q_\bullet} B_{\bar{\omega}(j)} \right) \subset Im \left( \widehat{f}_\bullet^{\bar{\omega}(j)} \right).$$

4. Suppose  $I$  is totally ordered and  $\omega : I \rightarrow I$  and  $\bar{\omega} : J \rightarrow J$  are functions. Then,  $f$  is an  $(\omega, \bar{\omega})$ -*approximate isomorphism* if both 2 and 3 hold.

An inverse (direct) system  $A_\bullet$  is said to be constant if  $A_i = A_j$  and  $p_j^i = id$  for each  $i, j$ . An inverse (direct) system  $A_\bullet$  is *approximately constant* if there is an approximate isomorphism between it and a constant system (in either direction). Likewise, an inverse or direct system is *approximately zero* if it is approximately isomorphic to a zero system. The reader will notice that approximately zero systems are the same as pro-zero systems [1, Appendix 3], i.e., systems  $A_\bullet$  such that for each  $i \in I$ , there exists  $j \geq i$  such that  $p_j^i : A_j \rightarrow A_i$  (resp.  $p_i^j : A_i \rightarrow A_j$ ) is zero (see below).

The proof of the following lemma is straightforward and is left to the reader.

**Lemma 4.7.** *The composition of two approximate monomorphisms (epimorphisms, isomorphisms) is an approximate monomorphism (epimorphism, isomorphism).*

**Category-theoretic behavior of approximate morphisms and Grothendieck’s pro-categories.** The remaining material in this section relates to the category theoretic behavior of approximate morphisms and a comparison with pro-morphisms, and it will not be used elsewhere in the paper.

In what follows,  $(A_\bullet, p_\bullet)$  and  $(B_\bullet, q_\bullet)$  will once again denote inverse systems indexed by  $I$  and  $J$  respectively. However, for simplicity, we will assume that  $I$  and  $J$  are both totally ordered.

**Definition 4.8.** Let  $f_\bullet : A_\bullet \rightarrow B_\bullet$  be an  $\alpha$ -morphism with saturation  $\hat{f}_\bullet$ . The *kernel* of  $f_\bullet$  is the inverse system  $\{K_i\}_{i \in I}$  where  $K_i := \text{Ker}(f_i : A_i \rightarrow B_{\alpha(i)})$  with the projection maps obtained from the projections of  $A_\bullet$  by restriction. We define the *image* of  $f_\bullet$  to be the inverse system  $\{D_j\}_{j \in J}$  where  $D_j := \text{Im}(\hat{f}_\bullet^j)$ , with the projections coming from the projections of  $B_\bullet$ . Note that  $D_j$  is a subgroup of  $B_j$ ,  $j \in J$ . We also define the *cokernel*  $\text{coKer}(f_\bullet)$  of  $f_\bullet$ , as the inverse system  $\{C_j\}_{j \in J}$  where  $C_j := B_j/D_j$ .

An inverse (respectively direct) system of abelian groups  $A_\bullet$  is *pro-zero* if for every  $i \in I$ , there exists  $j \geq i$  such that  $p_j^i : A_j \rightarrow A_i$  (resp.  $p_i^j : A_i \rightarrow A_j$ ) is zero (see [1, Appendix 3]). Using this language, we may reformulate the definitions of approximate monomorphisms:

**Lemma 4.9.** *Let  $f_\bullet : A_\bullet \rightarrow B_\bullet$  be a morphism of inverse systems of abelian groups. Then:*

1.  $f_\bullet$  is an approximate monomorphism iff its kernel  $K_\bullet := \text{Ker}(f_\bullet)$  is pro-zero.
2.  $f_\bullet$  is an approximate epimorphism iff its cokernel is a pro-zero inverse system.
3.  $f_\bullet$  is an approximate isomorphism iff both  $\text{Ker}(f_\bullet)$  and  $\text{coKer}(f_\bullet)$  are pro-zero systems.

*Proof.* This is immediate from the definitions. q.e.d.

For a fixed cofinal map  $\alpha : I \rightarrow J$ , the collection of  $\alpha$ -morphisms from  $A_\bullet$  to  $B_\bullet$  forms an abelian group the obvious way. In order to compare morphisms  $A_\bullet \rightarrow B_\bullet$  with different index maps  $I \rightarrow J$ , we introduce an equivalence relation:

**Definition 4.10.** Let  $f : A_\bullet \rightarrow B_\bullet$  and  $g : A_\bullet \rightarrow B_\bullet$  be morphisms with saturations  $\hat{f}_\bullet$  and  $\hat{g}_\bullet$ . Then,  $f_\bullet$  is *equivalent*  $g_\bullet$  if there is a cofinal function  $\rho : J \rightarrow I$  so that for all  $j \in J$ , both  $\hat{f}_{\rho(j)}^j$  and  $\hat{g}_{\rho(j)}^j$  are defined, and they coincide.

This equivalence relation is compatible with composition of approximate morphisms. Hence, we obtain a category *Approx* where the objects are inverse systems of abelian groups and the morphisms are equivalence classes of approximate morphisms. An *approximate inverse* for an approximate morphism  $f_\bullet$  is an approximate morphism  $g_\bullet$  which inverts  $f_\bullet$  in *Approx*.

**Lemma 4.11.** *Suppose  $I, J \cong \mathbb{Z}_+$ ,  $D_\bullet$  is a sub inverse system of  $A_\bullet$  (i.e.,  $D_i \subset A_i$ ,  $i \in I$ ), and let  $Q_\bullet$  be the quotient system:  $Q_i := A_i/D_i$ . Then*

1. *The morphism  $A_\bullet \rightarrow Q_\bullet$  induced by the canonical epimorphisms  $A_i \rightarrow Q_i$  has an approximate inverse iff  $D_\bullet$  is a pro-zero system.*
2. *The morphism  $D_\bullet \rightarrow A_\bullet$  defined by the inclusion homomorphisms  $D_i \rightarrow A_i$  has an approximate inverse iff  $Q_\bullet$  is a pro-zero system.*
3. *If  $f_\bullet : A_\bullet \rightarrow B_\bullet$  is a morphism,  $\text{Ker}(f_\bullet)$  is zero (i.e.,  $\text{Ker}(f_\bullet)_i = \{0\}$  for all  $i \in I$ ), and  $\text{Im}(f_\bullet) = B_\bullet$ , then  $f_\bullet$  has an approximate inverse.*

*Proof.* We leave the “only if” parts of 1 and 2 to the reader.

When  $D_\bullet$  is pro-zero, the map  $\beta : I \rightarrow I$  defined by

$$\beta(i) := \max\{i' \mid D_i \subset \text{Ker}(A_i \rightarrow A_{i'})\}$$

is cofinal. Let  $g_\bullet : Q_\bullet \rightarrow A_\bullet$  be the  $\beta$ -morphism where  $g_i : A_i/D_i = Q_i \rightarrow A_{\beta(i)}$  is induced by the projection  $A_i \rightarrow A_{\beta(i)}$ . One checks that  $g_\bullet$  is an approximate inverse for  $A_\bullet \rightarrow Q_\bullet$ .

Suppose  $Q_\bullet$  is pro-zero. Define a cofinal map  $\beta : I \rightarrow I$  by

$$\beta(i) := \max\{i' \mid \text{Im}(A_i \rightarrow A_{i'}) \subset D_{i'}\},$$

and let  $g_\bullet : A_\bullet \rightarrow D_\bullet$  be the  $\beta$ -morphism where  $g_i : A_i \rightarrow D_{\beta(i)}$  is induced by the projection  $A_i \rightarrow A_{\beta(i)}$ . Then,  $g_\bullet$  is an approximate inverse for the inclusion  $D_\bullet \rightarrow A_\bullet$ .

Now, suppose  $f_\bullet : A_\bullet \rightarrow B_\bullet$  is an  $\alpha$ -morphism with zero kernel and cokernel. Let  $J' := \alpha(I) \subset J$ , and define  $\beta' : J' \rightarrow I$  by  $\beta'(j) = \min \alpha^{-1}(j)$ . Define a cofinal map  $\sigma : J \rightarrow J'$  by  $\sigma(j) := \max\{j' \in J' \mid j' \leq j\}$ ; let  $\beta : J \rightarrow I$  be the composition  $\beta' \circ \sigma$ , and define a  $\beta$ -morphism  $g_\bullet$  by  $g_j := f_{\beta(j)}^{-1} \circ q_j^{\sigma(j)}$ . Then,  $g_\bullet$  is the desired approximate inverse for  $f_\bullet$ . q.e.d.

**Lemma 4.12.** *Let  $f_\bullet : A_\bullet \rightarrow B_\bullet$  be a morphism.*

1. *If  $f_\bullet$  has an approximate inverse, then it is an approximate isomorphism.*
2. *If  $f_\bullet$  is an approximate isomorphism and  $I, J \cong \mathbb{Z}_+$ , then  $f_\bullet$  has an approximate inverse.*

*Proof.* Let  $f_\bullet : A_\bullet \rightarrow B_\bullet$  and  $g_\bullet : B_\bullet \rightarrow A_\bullet$  be  $\alpha$  and  $\beta$  morphisms respectively, and let  $g_\bullet$  be an approximate inverse for  $f_\bullet$ . Since  $h_\bullet := g_\bullet \circ f_\bullet$  is equivalent to  $id_{A_\bullet}$ , then for all  $i$  there is an  $i' \geq i$  so that  $\hat{h}_{i'}^i$  is defined and  $\hat{h}_{i'}^i = p_{i'}^i$ . Letting  $\gamma := \beta \circ \alpha$ , we have, by the definition of the saturation,  $\hat{h}_\bullet^i, p_{i'}^i = \hat{h}_{i'}^i = p_{\gamma(i)}^i \circ h_{i'}$ . So,  $\text{Ker}(h_{i'}) \subset \text{Ker}(p_{i'}^i)$ .

Thus,  $f_\bullet$  is an approximate monomorphism. The proof that  $f_\bullet$  is an approximate epimorphism is similar.

We now prove part 2. Let  $\{K_i\}_{i \in I}$  be the kernel of  $f_\bullet$ , let  $\{Q_i\}_{i \in I} = \{A_i/K_i\}_{i \in I}$  be the quotient system, and let  $\{D_j\}_{j \in J}$  be the image of  $f_\bullet$ . Then,  $f_\bullet$  may be factored as  $f_\bullet = t_\bullet \circ s_\bullet \circ r_\bullet$  where  $r_\bullet : A_\bullet \rightarrow Q_\bullet$  is induced by the epimorphisms  $A_i \rightarrow A_i/K_i$ ,  $s_\bullet : Q_\bullet \rightarrow D_\bullet$  is induced by the homomorphisms of quotients, and  $t_\bullet : D_\bullet \rightarrow B_\bullet$  is the inclusion. By Lemma 4.11,  $s_\bullet$  has an approximate inverse. When the kernel and cokernel of  $f_\bullet$  are pro-zero, then  $r_\bullet$  and  $t_\bullet$  also admit approximate inverses by Lemma 4.11. Hence,  $f_\bullet$  has an approximate inverse in this case. q.e.d.

Below, we relate the notions of  $\alpha$ -morphisms, approximate monomorphisms (epimorphisms, isomorphisms) with Grothendieck’s pro-morphisms. Strictly speaking, this is unnecessary for the purposes of this paper. However, it puts our definitions into perspective. Also, readers who prefer the language of pro-categories may use Lemma 4.14 and Corollary 4.15 to translate the theorems of Sections 6 and 7 into pro-theorems.

**Definition 4.13.** Let  $\{A_i\}_{i \in I}, \{B_j\}_{j \in J}$  be inverse systems. The group of pro-morphisms  $\text{proHom}(A_\bullet, B_\bullet)$  is defined as

$$\lim_{\substack{\longleftarrow \\ j \in J}} \lim_{\substack{\longleftarrow \\ i \in I}} \text{Hom}(A_i, B_j)$$

(see [23], [1, Appendix 2], [15, Ch II, Section 1]). The *identity pro-morphism* is the element of  $\text{proHom}(A_\bullet, A_\bullet)$  determined by

$$(id_{A_j})_{j \in I} \in \prod_j \lim_{\substack{\longleftarrow \\ i \in I}} \text{Hom}(A_i, A_j).$$

This yields a category *Pro-Abelian* where the objects are inverse systems of abelian groups and the morphisms are the pro-morphisms. A *pro-isomorphism* is an isomorphism in this category.

**Remark.** By relaxing the definition of inverse systems, this category becomes an abelian category, [1, Appendix 4]. However, we will not discuss this further.

By the definitions of direct and inverse limits, an element of  $\text{proHom}(A_\bullet, B_\bullet)$  can be represented by an admissible “sequence”

$$\left( \left[ h_{\rho(j)}^j : A_{\rho(j)} \rightarrow B_j \right] \right)_{j \in J}$$

of equivalence classes of homomorphisms  $h_{\rho(j)}^j : A_{\rho(j)} \rightarrow B_j$ ; here, two homomorphisms  $h_i^j : A_i \rightarrow B_j, h_k^j : A_k \rightarrow B_j$  are equivalent if there

exists  $\ell \geq i, k$  such that

$$h_i^j \circ p_\ell^i = h_k^j \circ p_\ell^k;$$

and the “sequence” is *admissible* if for each  $j \geq j'$ , there is an  $i \geq \max\{\rho(j), \rho(j')\}$  so that

$$q_j^{j'} \circ h_{\rho(j)}^j \circ p_i^{\rho(j)} = h_{\rho(j')}^{j'} \circ p_i^{\rho(j')}.$$

Given a cofinal map  $\alpha : I \rightarrow J$  between directed sets, we may construct a function  $\rho : J \rightarrow I$  so that  $\alpha(\rho(j)) \geq j$  for all  $j$ ; then any  $\alpha$ -morphism  $f_\bullet : A_\bullet \rightarrow B_\bullet$  induces an admissible sequence  $([f_{\rho(j)}^j : A_{\rho(j)} \rightarrow B_j]_{j \in J})$ . The corresponding element  $\text{pro}(f_\bullet) \in \text{proHom}(A_\bullet, B_\bullet)$  is independent of the choice of  $\rho$  by condition (4.3) of Definition 4.2.

**Remark.** Here we construct  $\rho$  using the axiom of choice, where we pick  $\rho(j) \in \alpha^{-1}(j)$ .

**Lemma 4.14.**

1. If  $f : A_\bullet \rightarrow B_\bullet$  and  $g : A_\bullet \rightarrow B_\bullet$  are morphisms, then  $\text{pro}(f) = \text{pro}(g)$  iff  $f_\bullet$  is equivalent to  $g_\bullet$ . In other words, *pro* descends to a faithful functor from *Approx* to *Pro-Abelian*.
2. When  $I, J \cong \mathbb{Z}_+$ , then every pro-morphism from  $A_\bullet$  to  $B_\bullet$  arises as  $\text{pro}(f_\bullet)$  for some approximate morphism  $f_\bullet : A_\bullet \rightarrow B_\bullet$ . Thus, *pro* descends to a fully faithful functor from *Approx* to *Pro-Abelian* in this case.

*Proof.* The first assertion follows readily from the definition of  $\text{proHom}(A_\bullet, B_\bullet)$  and Definition 4.10.

Suppose  $I, J \cong \mathbb{Z}_+$  and  $\phi \in \text{proHom}(A_\bullet, B_\bullet)$  is represented by an admissible sequence

$$\left( [h_{\rho_0(j)}^j : A_{\rho_0(j)} \rightarrow B_j]_{j \in J} \right).$$

We define  $\rho : J \rightarrow I$  and another admissible sequence  $(\bar{h}_{\rho(j)}^j : A_{\rho(j)} \rightarrow B_j)_{j \in J}$  representing  $\phi$  by setting  $\rho(0) = \rho_0(0)$ ,  $\bar{h}_{\rho(0)}^0 := h_{\rho_0(0)}^0$ , and inductively choosing  $\rho(j)$ ,  $\bar{h}_{\rho(j)}^j$  so that  $\rho(j) > \rho(j-1)$ ,  $\bar{h}_{\rho(j)}^j := h_{\rho_0(j)}^j \circ p_{\rho(j)}^{\rho_0(j)}$  and  $q_j^{j-1} \circ \bar{h}_{\rho(j)}^j = \bar{h}_{\rho(j-1)}^{j-1} \circ p_{\rho(j)}^{\rho(j-1)}$ . Note that the mapping  $\rho$  is strictly increasing and hence cofinal. Now, define a cofinal map  $\alpha : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  by setting  $\alpha(i) := \max\{j \mid \rho(j) \leq i\}$  for  $i \geq \rho(0) = \rho_0(0)$ . We then get an  $\alpha$ -morphism  $f_\bullet : A_\bullet \rightarrow B_\bullet$  where  $f_i := \bar{h}_{\rho(\alpha(i))}^{\alpha(i)} \circ p_i^{\rho(\alpha(i))}$ . Clearly,  $\text{pro}(f_\bullet) = (\bar{h}_{\rho(j)}^j)_{j \in J}$ . q.e.d.



**Corollary 4.15.** *Suppose  $I, J \cong \mathbb{Z}_+$  and  $f_\bullet : A_\bullet \rightarrow B_\bullet$  is a morphism. Then,  $f_\bullet$  is an approximate isomorphism iff  $\text{pro}(f_\bullet)$  is a pro-isomorphism.*

*Proof.* By Lemma 4.12,  $f_\bullet$  is an approximate isomorphism iff it represents an invertible element of *Approx*, and by Lemma 4.14, this is equivalent to saying that  $\text{pro}(f_\bullet)$  is invertible in *Pro-Abelian*. q.e.d.

### 5. Recognizing groups of type $FP_n$

The main result in this section is Theorem 5.11, which gives a characterization of groups  $G$  of type  $FP_n$  in terms of nested families of  $G$ -chain complexes, and Lemma 5.1 which relates the cohomology of  $G$  with the corresponding cohomology of the  $G$ -chain complexes. A related characterization of groups of type  $FP_n$  appears in [11]. We will apply Theorem 5.11 and Lemma 5.1 in Section 8 to show that peripheral subgroups  $H_i$  are of type FP.

Suppose for  $i = 0, \dots, N$ , we have an augmented chain complex  $A_*(i)$  of projective  $\mathbb{Z}G$ -modules, and for  $i = 1, \dots, N$ , we have an augmentation preserving  $G$ -equivariant chain map  $a_i : A_*(i - 1) \rightarrow A_*(i)$  which induces zero on reduced homology in dimensions  $< n$ . Let  $G$  be a group of type  $FP_k$ , and let

$$0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow \dots \leftarrow P_k$$

be a chain complex of finitely generated projective  $\mathbb{Z}G$ -modules. We assume that  $k \leq n \leq N$ .

**Lemma 5.1.** *Under the above conditions, we have:*

1. *There is an augmentation preserving  $G$ -equivariant chain mapping  $P_* \rightarrow A_*(n)$ .*
2. *If  $k < n$  and  $j_i : P_* \rightarrow A_*(0)$  are augmentation preserving  $G$ -equivariant chain mappings for  $i = 1, 2$ , then the compositions  $P_* \xrightarrow{j_i} A_*(0) \rightarrow A_*(k)$  are  $G$ -equivariantly chain homotopic.*

*Proof of 1.* We start with the diagram

$$\begin{array}{ccc} P_0 & & \\ \downarrow & & \\ \mathbb{Z} & \leftarrow & A_0(0). \end{array}$$

Then, projectivity of  $P_0$  implies that we can complete this to a commutative diagram by a  $\mathbb{Z}G$ -morphism  $f_0 : P_0 \rightarrow A_0(0)$ . Assume inductively that we have constructed a  $G$ -equivariant augmentation preserving chain mapping  $f_j : [P_*]_j \rightarrow A_*(j)$ . Then, the image of the

composition  $P_{j+1} \xrightarrow{\partial} P_j \xrightarrow{f_j} A_j(j) \rightarrow A_j(j+1)$  is contained in the image of  $A_{j+1}(j+1) \xrightarrow{\partial} A_j(j+1)$  since  $a_{j+1}$  induces zero on reduced homology. So, projectivity of  $P_{j+1}$  allows us to extend  $f_j$  to a  $G$ -equivariant chain mapping  $f_{j+1} : [P_*]_{j+1} \rightarrow A_*(j+1)$ . q.e.d.

*Proof of 2.* Similar to the proof of 1: Use induction and projectivity of the  $P_\ell$ 's. q.e.d.

We now assume in addition that  $P_*$  is a partial resolution of  $\mathbb{Z}$ . Then,

**Lemma 5.2.** *Suppose  $k < n$  and  $f : P_* \rightarrow A_*(0)$  is an augmentation preserving  $G$ -equivariant chain mapping. Then, for any  $\mathbb{Z}G$ -module  $M$ , the map*

$$H^i(f) : H^i(A_*(0); M) \rightarrow H^i(P_*; M)$$

*carries the image  $Im(H^i(A_*(n); M) \rightarrow H^i(A_*(0); M))$  isomorphically onto  $H^i(P_*; M)$  for  $i = 0, \dots, k - 1$ . The map*

$$H_i(f) : H_i(P_*; M) \rightarrow H_i(A_*(n); M)$$

*is an isomorphism onto the image of  $H_i(A_*(0); M) \rightarrow H_i(A_*(n); M)$  for  $i = 0, \dots, k - 1$ . The map*

$$H_k(f) : H_k(P_*; M) \rightarrow H_k(A_*(n); M)$$

*is onto the image of  $H_k(A_*(0); M) \rightarrow H_k(A_*(n); M)$ .*

*Proof.* Let  $\rho_* : [A_*(n)]_k \rightarrow P_*$  be a  $G$ -equivariant chain mapping constructed using the fact that  $H_i(P_*) = \{0\}$  for  $i < k$ . Consider the compositions

$$\alpha_{k-1} : [P_*]_{k-1} \xrightarrow{f_*} [A_*(0)]_{k-1} \rightarrow [A_*(n)]_{k-1} \xrightarrow{\rho_*} P_*$$

and

$$\beta_k : [A_*(0)]_k \rightarrow [A_*(n)]_k \xrightarrow{\rho_*} [P_*]_k \xrightarrow{f_*} [A_*(0)]_k \rightarrow A_*(n).$$

Both are ( $G$ -equivariantly) chain homotopic to the inclusions; the first one since  $P_*$  is a partial resolution, and the second by applying assertion 2 of Lemma 5.1 to the chain mapping  $[A_*(0)]_k \rightarrow A_*(0)$ . Assertion follows immediately from this. q.e.d.

We note that the above lemmas did not require any finiteness assumptions on the  $\mathbb{Z}G$ -modules  $A_i(j)$ . Suppose now that the group  $G$  satisfies assumptions in Lemma 5.2 and let  $G \curvearrowright X$  be a free simplicial action on a uniformly  $(n - 1)$ -acyclic locally finite metric simplicial complex  $X$ ,  $k \leq n - 1$ . Then, by part 1 of Lemma 5.1, we have a  $G$ -equivariant augmentation-preserving chain mapping  $f : P_* \rightarrow C_*(X)$ . Let  $K \subset X$

be the support of the image of  $f$ . It is clear that  $K$  is  $G$ -invariant and  $K/G$  is compact. As a corollary of the proof of the previous lemma, we get:

**Corollary 5.3.** *Under the above assumptions the direct system of reduced homology groups  $\{\tilde{H}_i(N_R(K))\}_{R \geq 0}$  is approximately zero for each  $i < k$ .*

*Proof.* Given  $R > 0$ , we consider the system of chain complexes  $A_*(0) := C_*(N_R(K))$ ,  $A_*(1) = A_*(2) = \dots = A_*(N) = C_*(X)$ . The mapping  $[A_*(0)]_k \xrightarrow{\beta_k} A_*(N) = C_*(X)$  from the proof of Lemma 5.1 is chain homotopic to the inclusion via a  $G$ -equivariant homotopy  $h_R$ . On the other hand, this map factors through  $P_*$ , hence it induces zero mapping of the reduced homology groups

$$\tilde{H}_i(N_R(K)) \xrightarrow{0} \tilde{H}_i(\text{Support}(\text{Im}(\beta_k))), \quad i < k.$$

The support of  $\text{Im}(h_R)$  is contained in  $N_{R'}(K)$  for some  $R' < \infty$ , since  $h_R$  is  $G$ -equivariant. Hence, the inclusion  $N_R(K) \rightarrow N_{R'}(K)$  induces zero map of  $\tilde{H}_i(\cdot)$  for  $i < k$ . q.e.d.

Before stating the next corollary, we recall the following fact:

**Lemma 5.4** (See [12]). *Let  $G \curvearrowright X$  be a discrete, free, cocompact action of a group on a simplicial complex. Then, the complex of compactly supported simplicial cochains  $C_c^*(X)$  is canonically isomorphic to the complex  $\text{Hom}_{\mathbb{Z}G}(C_*(X); \mathbb{Z}G)$ ; in particular, the compactly supported cohomology of  $X$  is canonically isomorphic to  $H^*(X/G; \mathbb{Z}G)$ .*

In the next corollary, we assume that  $G, P_*, X, f, K$  are as above, in particular,  $X$  is a uniformly  $(n-1)$ -acyclic locally finite metric simplicial complex, and for some  $k \leq n-1$ ,

$$P_k \rightarrow \dots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

is a resolution by finitely generated projective  $\mathbb{Z}G$  modules.

**Corollary 5.5.**

1. *For any local coefficient system ( $\mathbb{Z}G$ -module)  $M$ , the family of maps*

$$H^i(N_R(K)/G; M) \xrightarrow{f_R^i} H^i(P_*; M)$$

*defines a morphism between the inverse system*

$$\{H^i(N_R(K)/G; M)\}_{R \geq 0}$$

*and the constant inverse system  $\{H^i(P_*; M)\}_{R \geq 0}$  which is an approximate isomorphism when  $0 \leq i < k$ .*

2. *The map*

$$H_c^i(N_R(K)) \simeq H^i(N_R(K)/G; \mathbb{Z}G) \xrightarrow{f_R^i} H^i(P_*; \mathbb{Z}G)$$

*is an approximate isomorphism when  $0 \leq i < k$ .*

3. *The  $\mathbb{Z}G$ -chain map*

$$f_{R,*} : P_* \rightarrow C_*(N_R(K))$$

*induces a homomorphism of homology groups*

$$f_{R,i} : \tilde{H}_i(P_*; \mathbb{Z}G) \rightarrow \tilde{H}_i(N_R(K))$$

*which is an approximate isomorphism for  $0 \leq i < k$ .*

*Proof.*

1. According to Corollary 5.3 the direct system of reduced homology groups  $\{\tilde{H}_i(N_R(K))\}$  is approximately zero for each  $i < k$ . Thus, for  $N > k$ , we have a sequence of integers  $R_0 = 0 < R_1 < R_2 < \dots < R_N$  so that the maps

$$\tilde{H}_i(N_{R_j}(K)) \rightarrow \tilde{H}_i(N_{R_{j+1}}(K))$$

are zero for each  $j < N, i < k$ . We now apply Lemma 5.1 where  $A_*(j) := C_*(N_{R_j}(K))$ .

2. This follows from part 1 and Lemma 5.4.

3. Note that  $\tilde{H}_i(P_*; \mathbb{Z}G) \simeq \{0\}$  for  $i < k$ ; this follows directly from the definition of a group of type  $FP_k$ . Thus, the assertion follows from Corollary 5.3. q.e.d.

There is also an analog of Corollary 5.5 which does not require a group action:

**Lemma 5.6.** *Let  $X$  and  $Y$  be bounded geometry metric simplicial complexes, where  $Y$  is uniformly  $(k - 1)$ -acyclic and  $X$  is uniformly  $k$ -acyclic. Suppose  $C_*(Y) \xrightarrow{f} C_*(X)$  is a chain mapping which is a uniform embedding, and  $K := \text{Support}(\text{Im}(f)) \subset X$ . Then*

1. *The induced map on cohomology*

$$H_c^i(f) : H_c^i(N_R(K)) \rightarrow H_c^i(Y)$$

*defines a morphism between the inverse system  $\{H_c^i(N_R(K))\}_{R \geq 0}$  and the constant inverse system  $\{H_c^i(Y)\}_{R \geq 0}$  which is an approximate isomorphism for  $0 \leq i < k$ , and an approximate monomorphism for  $i = k$ .*

2. *The approximate isomorphism approximately respects support in the following sense. There is a function  $\zeta : \mathbb{N} \rightarrow \mathbb{N}$  so that if  $i < k$ ,  $S \subset Y$  is a subcomplex,*

$$T := \text{Support}(f_*(C_*(S))) \subset X$$

*is the corresponding subcomplex of  $X$ , and  $\alpha \in \text{Im}(H_c^i(Y, \overline{Y-S}) \rightarrow H_c^i(Y))$ , then  $\alpha$  belongs to the image of the composition*

$$H_c^i(N_R(K), \overline{N_R(K) - N_{\zeta(R)}(T)}) \rightarrow H_c^i(N_R(K)) \xrightarrow{H_c^i(f)} H_c^i(Y).$$

3. *The induced map*

$$\tilde{H}_i(f) : \{0\} \simeq \tilde{H}_i(Y) \rightarrow \tilde{H}_i(N_R(K))$$

*is an approximate isomorphism for  $0 \leq i < k$ .*

4. *All functions  $\omega, \bar{\omega}$  associated with the above approximate isomorphisms and the function  $\zeta$  can be chosen to depend only on the geometry of  $X, Y$  and  $f$ .*

*Proof.* Since  $f$  is a uniform embedding, using the uniform  $(k - 1)$ -acyclicity of  $Y$  and uniform  $k$ -acyclicity of  $X$ , we can construct a direct system  $\{\rho_R\}$  of uniform embeddings of the truncated chain complexes

$$[0 \leftarrow C_0(N_R(K)) \leftarrow \dots \leftarrow C_k(N_R(K))] \xrightarrow{\rho_R} [0 \leftarrow C_0(Y) \leftarrow \dots \leftarrow C_k(Y)]$$

so that the compositions  $f \circ \rho_R$  are chain homotopic to the inclusions

$$\begin{aligned} & [0 \leftarrow C_0(N_R(K)) \leftarrow \dots \leftarrow C_k(N_R(K))] \\ & \rightarrow [0 \leftarrow C_0(N_{R'}(K)) \leftarrow \dots \leftarrow C_k(N_{R'}(K))] \end{aligned}$$

(for  $R' = \omega(R)$ ) via chain homotopies of bounded support. Moreover, the restriction of the composition  $\rho_R \circ f$  to the  $(k - 1)$ -truncated chain complexes is chain homotopic to the identity via a chain homotopy with bounded support.

We first prove that the morphism of inverse systems defined by

$$H_c^i(f) : H_c^i(N_R(K)) \rightarrow H_c^i(Y)$$

is an approximate monomorphism. Suppose

$$\alpha \in \text{Ker}(H_c^i(f) : H_c^i(N_{R'}(K)) \rightarrow H_c^i(Y))$$

where  $R' = \omega(R)$ . Then,  $H^i(f \circ \rho_{R'}) (\alpha) = 0$ . But the restriction of  $H^i(f \circ \rho_{R'}) (\alpha)$  to  $N_R(K)$  is cohomologous to the restriction of  $\alpha$  to  $N_R(K)$ .

Since the restriction of the composition  $\rho_R \circ f$  to the  $(k - 1)$ -truncated chain complex  $[C_*(Y)]_{k-1}$  is chain homotopic to the identity, it follows that

$$H_c^i(f) : H_c^i(N_R(K)) \rightarrow H_c^i(Y)$$

is an epimorphism for  $R \geq 0$  and  $i < k$ .

Part 2 of the lemma follows immediately from the fact that  $\rho_R$  is a uniform embedding and the coarse Lipschitz property of the chain homotopies constructed above.

We omit the proof of part 3 as it is similar to that of part 2. q.e.d.

**Lemma 5.7.** *Let  $(X, d)$  and  $(X', d')$  be bounded geometry uniformly acyclic metric simplicial complexes,  $Z \subset X$  a subcomplex; suppose  $f : (Z, d|_Z) \rightarrow (X', d')$  is a uniform embedding, and set  $K := f(Z)$ . Then,  $f$  “induces” approximate isomorphisms of the direct and inverse systems*

$$\begin{aligned} \{H_*(N_R(Z))\}_{R \geq 0} &\rightarrow \{H_*(N_R(K))\}_{R \geq 0}, \\ \{H_c^*(N_R(Z))\}_{R \geq 0} &\rightarrow \{H_c^*(N_R(K))\}_{R \geq 0}. \end{aligned}$$

As in part 2 of Lemma 5.6, these approximate isomorphisms respect support, and as in part 4 of that lemma, the functions  $\omega, \bar{\omega}$  can be chosen to depend only on the geometry of  $X, X'$ , and  $f$ .

*Proof.* We argue as in the previous lemma. Since  $f$  is a uniform embedding, using the uniform acyclicity of  $X$  and  $X'$ , we construct direct systems  $\{\rho_R\}, \{\phi_r\}$  of uniform embeddings of chain complexes

$$C_*(N_R(Z)) \xrightarrow{\rho_R} C_*(N_{\alpha(R)}(K))$$

(extending  $f_* : C_*(Z) \rightarrow C_*(K)$ ) and

$$C_*(N_r(K)) \xrightarrow{\phi_r} C_*(N_{\beta(r)}(Z)),$$

so that the compositions  $\phi_{\alpha(R)} \circ \rho_R, \rho_{\beta(r)} \circ \phi_r$ , regarded as maps

$$C_*(N_R(Z)) \rightarrow C_*(N_{\omega(R)}(Z)), C_*(N_r(K)) \rightarrow C_*(N_{\bar{\omega}(r)}(K))$$

for certain  $\omega(R) \geq \alpha(R), \bar{\omega}(r) \geq \beta(r)$ , are chain homotopic to the inclusions

$$C_*(N_R(Z)) \rightarrow C_*(N_{\omega(R)}(Z)), \quad C_*(N_r(K)) \rightarrow C_*(N_{\bar{\omega}(r)}(K))$$

via chain homotopies of bounded support. Thus, the induced maps of homology (and compactly supported cohomology) groups are approximate inverses of each other. q.e.d.

Note that in the above discussion, we used finiteness assumptions on the group  $G$  to make conclusions about (co)homology of families of  $G$ -invariant chain complexes. Our next goal is to use existence of a family of chain complexes  $A_*(i)$  of *finitely generated* projective  $\mathbb{Z}G$  modules as in Lemma 5.1 to establish finiteness properties of the group  $G$  (Theorem 5.11). We begin with a homotopy-theoretic analog of Theorem 5.11.

**Proposition 5.8.** *Let  $G$  be a group, and let  $X(0) \xrightarrow{a_1} X(1) \xrightarrow{a_2} \dots \xrightarrow{a_{n+1}} X(n+1)$  be a diagram of free, simplicial  $G$ -complexes where  $X(i)/G$  is compact for  $i = 0, \dots, n+1$ . If the maps  $a_i$  are  $n$ -connected for each  $i$ , then there is an  $(n+1)$ -dimensional free, simplicial  $G$ -complex  $Y$  where  $Y/G$  is compact and  $Y$  is  $n$ -connected.*

*Proof.* We build  $Y$  inductively as follows. Start with  $Y_0 = G$  where  $G$  acts on  $Y_0$  by left translation, and let  $j_0 : Y_0 \rightarrow X(0)$  be any  $G$ -equivariant simplicial map. Inductively apply Lemma 5.9 below to the composition  $Y_i \xrightarrow{j_i} X(i) \rightarrow X(i+1)$  to obtain  $Y_{i+1}$  and a simplicial  $G$ -map  $j_{i+1} : Y_{i+1} \rightarrow X(i+1)$ . Set  $Y := Y_{n+1}$ . q.e.d.

Recall that a simplicial cell is a simplicial complex PL-homeomorphic to a single simplex.

**Lemma 5.9.** *Let  $Z$  and  $A$  be locally finite simplicial complexes with free cocompact simplicial  $G$ -actions, where  $\dim(Z) = k$ , and  $Z$  is  $(k-1)$ -connected. Let  $j : Z \rightarrow A$ , be a null-homotopic  $G$ -equivariant simplicial map. Then, we may construct a  $k$ -connected simplicial  $G$ -complex  $Z'$  by attaching (equivariantly) finitely many  $G$ -orbits of simplicial  $(k+1)$ -cells to  $Z$ , and a  $G$ -map  $j' : Z' \rightarrow A$  extending  $j$ .*

*Proof.* By replacing  $A$  with the mapping cylinder of  $j$ , we may assume that  $Z$  is a subcomplex of  $A$  and  $j$  is the inclusion map. Let  $A_k$  denote the  $k$ -skeleton of  $A$ . Since  $Z$  is  $(k-1)$ -connected, after subdividing  $A_k$  if necessary, we may construct a  $G$ -equivariant simplicial retraction  $r : A_k \rightarrow Z$ . For every  $(k+1)$ -simplex  $c$  in  $A$ , we attach a simplicial  $(k+1)$ -cell  $c'$  to  $Z$  using the composition of the attaching map of  $c$  with the retraction  $r$ . It is clear that we may do this  $G$ -equivariantly, and there will be only finitely many  $G$ -orbits of  $(k+1)$ -cells attached. We denote the resulting simplicial complex by  $Z'$ , and note that the inclusion  $j : Z \rightarrow A$  clearly extends (after subdivision of  $Z'$ ) to an equivariant simplicial map  $j' : Z' \rightarrow A$ .

We now claim that  $Z'$  is  $k$ -connected. Since we built  $Z'$  from  $Z$  by attaching  $(k+1)$ -cells, it suffices to show that  $\pi_k(Z) \rightarrow \pi_k(Z')$  is trivial. If  $\sigma : S^k \rightarrow Z$  is a simplicial map for some triangulation of  $S^k$ , we get a simplicial null-homotopy  $\tau : D^{k+1} \rightarrow A$  extending  $\sigma$ . Let  $D_k^{k+1}$  denote the  $k$ -skeleton of  $D^{k+1}$ . The composition  $D_k^{k+1} \xrightarrow{\tau} A \xrightarrow{r} Z \rightarrow Z'$  extends over each  $(k+1)$ -simplex  $\Delta$  of  $D^{k+1}$ , since  $\tau|_{\Delta} : \Delta \rightarrow A$  is either an embedding, in which case  $r \circ \tau|_{\partial\Delta} : \partial\Delta \rightarrow Z'$  is null homotopic by the construction of  $Z'$ , or  $\tau|_{\Delta} : \Delta \rightarrow A$  has image contained in a  $k$ -simplex

of  $A$ , and the composition

$$\partial\Delta \xrightarrow{\tau} A \xrightarrow{r} Z$$

is already null-homotopic. Hence, the composition  $S^k \xrightarrow{\sigma} Z \hookrightarrow Z'$  is null-homotopic. q.e.d.

The next lemma is a homological analog of Lemma 5.9 which provides the inductive step in the proof of Theorem 5.11.

In the following lemma we declare that  $H_k(P_*) := Z_k(P_*)$ .

**Lemma 5.10.** *Let  $G$  be a group. Suppose  $0 \leftarrow \mathbb{Z} \xleftarrow{\epsilon} P_0 \leftarrow \cdots \leftarrow P_k$  is a partial resolution by finitely generated projective  $\mathbb{Z}G$ -modules, and  $\mathbb{Z} \xleftarrow{\epsilon} A_0 \leftarrow \cdots \leftarrow A_{k+1}$  is an augmented chain complex of finitely generated projective  $\mathbb{Z}G$ -modules. Let  $j : P_* \rightarrow A_*$  be an augmentation preserving chain mapping which induces zero on homology groups. Then, we may extend  $P_*$  to a partial resolution  $P'_*$ :*

$$0 \leftarrow \mathbb{Z} \xleftarrow{\epsilon} P_0 \leftarrow \cdots \leftarrow P_k \leftarrow P_{k+1}$$

where  $P_{k+1}$  is finitely generated free, and  $j$  extends to a chain mapping  $j' : P'_* \rightarrow A_*$ .

*Proof.* By replacing  $A_*$  with the algebraic mapping cylinder of  $j$ , we may assume that  $P_*$  is embedded as a subcomplex of  $A_*$ ,  $j$  is the inclusion, and for  $i = 0, \dots, k$ , the chain group  $A_k$  splits as a direct sum of  $\mathbb{Z}G$ -modules  $A_i = P_i \oplus Q_i$  where  $Q_i$  is finitely generated and projective. Applying the projectivity of  $Q_i$ , we construct a chain retraction from the  $k$ -truncation  $[A_*]_k$  of  $A_*$  to  $P_*$ . Choose a finite set of generators  $a_1, \dots, a_\ell$  for the  $\mathbb{Z}G$ -module  $A_{k+1}$ . We let  $P_{k+1}$  be the free module of rank  $\ell$ , with basis  $a'_1, \dots, a'_\ell$ , and define the boundary operator  $\partial : P_{k+1} \rightarrow P_k$  by the formula  $\partial(a'_i) = r(\partial(a_i))$ . To see that  $H_k(P'_*) = 0$ , pick a  $k$ -cycle  $\sigma \in Z_k(P_*)$ . We have  $\sigma = \partial\tau$  for some  $\tau = \sum c_i a_i \in A_{k+1}$ . Then,  $\sigma = r(\partial\tau) = \sum c_i r(\partial a_i) = \sum c_i \partial a'_i$ ; so  $\sigma$  is null-homologous in  $P'_*$ . The extension mapping  $j' : P'_* \rightarrow A_*$  is defined by  $a'_i \mapsto a_i, 1 \leq i \leq \ell$ . q.e.d.

**Theorem 5.11.** *Suppose for  $i = 0, \dots, N$ , we have an augmented chain complex  $A_*(i)$  of finitely generated projective  $\mathbb{Z}G$ -modules, and for  $i = 1, \dots, N$ , we have an augmentation preserving  $G$ -equivariant chain map  $a_i : A_*(i-1) \rightarrow A_*(i)$  which induces zero on reduced homology in dimensions  $\leq n \leq N$ .*

*Then, there is a partial resolution*

$$0 \leftarrow \mathbb{Z} \leftarrow F_0 \leftarrow \cdots \leftarrow F_n$$



of finitely generated free  $\mathbb{Z}G$ -modules, and a  $G$ -equivariant chain mapping  $f : F_* \rightarrow A(n)$ . In particular,  $G$  is a group of type  $FP_n$ .

*Proof.* Define  $F_0$  to be the group ring  $\mathbb{Z}G$ , with the usual augmentation  $\mathbb{Z} \leftarrow \mathbb{Z}G$ . Then, construct  $F_i$  and a chain map  $F_i \rightarrow A_i(i)$  by applying the previous lemma inductively. q.e.d.

**Corollary 5.12.** *Suppose that  $G \curvearrowright X$  is a free simplicial action of a group  $G$  on a metric simplicial complex  $X$ . Suppose that we have a system of (non-empty)  $G$ -invariant simplicial subcomplexes  $X(0) \subset X(1) \subset \dots \subset X(N)$  so that:*

- (a)  $X(i)/G$  is compact for each  $i$ ,
- (b) The induced mappings  $\tilde{H}_i(X(k)) \rightarrow \tilde{H}_i(X(k+1))$  are zero for each  $i \leq n \leq N$  and  $0 \leq k < N$ .

Then, the group  $G$  is of type  $FP_n$ .

*Proof.* Apply Theorem 5.11 to  $A_*(i) := C_*(X(i))$ . q.e.d.

Note that the above corollary is the converse to Corollary 5.3. Thus,

**Corollary 5.13.** *Suppose that  $G \curvearrowright X$  is a free simplicial group action on a uniformly acyclic bounded geometry metric simplicial complex,  $K := G(\star)$ , where  $\star \in X$ . Then,  $G$  is of type  $FP$  if and only if the direct system of reduced homology groups  $\{\tilde{H}_*(N_R(K))\}$  is approximately zero.*

Combining Theorem 5.11 and Lemma 5.1, we get:

**Corollary 5.14.** *Suppose for  $i = 0, \dots, 2n+1$ , we have an augmented chain complex  $A_*(i)$  of finitely generated projective  $\mathbb{Z}G$ -modules, and for  $i = 1, \dots, 2n+1$ , we have augmentation preserving  $G$ -equivariant chain maps  $a_i : A_*(i-1) \rightarrow A_*(i)$  which induce zero on reduced homology in dimensions  $\leq n$ . Then:*

1. There is a partial resolution  $F_*$ :

$$0 \leftarrow \mathbb{Z} \leftarrow F_0 \leftarrow \dots \leftarrow F_n$$

by finitely generated free  $\mathbb{Z}G$ -modules and a  $G$ -equivariant chain mapping  $f_* : F_* \rightarrow A_*(n)$ . In particular,  $G$  is of type  $FP_n$ .

2. For any  $\mathbb{Z}G$ -module  $M$ , the map  $H^i(f) : H^i(A_*(n); M) \rightarrow H^i(F_*; M)$  carries the image  $\text{Im}(H^i(A(2n); M) \rightarrow H^i(A(n); M))$  isomorphically onto  $H^i(F_*; M)$  for  $i = 0, \dots, n-1$ .
3. The map  $H_i(f) : H_i(P_*; M) \rightarrow H_i(A_*(2n); M)$  is an isomorphism onto the image of  $H_i(A_*(n); M) \rightarrow H_i(A_*(2n); M)$ .

We now discuss a relative version of Corollaries 5.5 and 5.14. Let  $X$  be a uniformly acyclic bounded geometry metric simplicial complex, and  $G$  a group acting freely and simplicially on  $X$ ; thus,  $G$  has finite cohomological dimensions since  $X$  is acyclic and  $\dim(X) < \infty$ . Let  $K \subset X$  be a  $G$ -invariant subcomplex so that  $K/G$  is compact; and let  $\{C_\alpha\}_{\alpha \in I}$  be the deep components of  $X - K$ . Define  $Y_R := \overline{X - N_R(K)}$ ,  $Y_{\alpha,R} := C_\alpha \cap Y_R$ . We will assume that the system

$$\{\tilde{H}_j(Y_{\alpha,R})\}_{R \geq 0}$$

is approximately zero for each  $j, \alpha$ . In particular,  $\{\tilde{H}_0(Y_{\alpha,R})\}_{R \geq 0}$  is approximately zero, which implies that each  $C_\alpha$  is stable. Let  $H_\alpha$  denote the stabilizer of  $C_\alpha$  in  $G$ . Choose a set of representatives  $C_{\alpha_1}, \dots, C_{\alpha_k}$  from the  $G$ -orbits in the collection  $\{C_\alpha\}$ . For notational simplicity, we relabel  $\alpha_1, \dots, \alpha_k$  as  $1, \dots, k$ . Let  $H_i = H_{\alpha_i}$  be the stabilizer of  $C_i = C_{\alpha_i}$ . This defines a group pair  $(G, \{H_1, \dots, H_k\})$ . Let  $P_*$  be a finite length projective resolution of  $\mathbb{Z}$  by  $\mathbb{Z}G$ -modules, and for each  $i = 1, \dots, k$ , we choose a finite length projective resolution of  $\mathbb{Z}$  by  $\mathbb{Z}H_i$ -modules  $Q_*(i)$ . Using the construction described in Section 3 (see the discussion of the group pairs), we convert this data to a pair  $(C_*, D_*)$  of finite length projective resolutions (consisting of  $\mathbb{Z}G$ -modules). We recall that  $D_*$  decomposes in a natural way as a direct sum  $\bigoplus_\alpha D_*(\alpha)$  where each  $D(\alpha)$  is a resolution of  $\mathbb{Z}$  by projective  $\mathbb{Z}H_\alpha$ -modules. Now, construct a  $\mathbb{Z}H_i$ -chain mapping  $C_*(Y_{\alpha_i,0}) \rightarrow D_*(\alpha_i)$  using the acyclicity of  $D_*(\alpha_i)$ . We then extend this  $G$ -equivariantly to a mapping  $C_*(Y_0) \rightarrow D_*$ , and then to a  $\mathbb{Z}G$ -chain mapping  $\rho_0 : (C_*(X), C_*(Y_0)) \rightarrow (C_*, D_*)$ . By restriction, this defines a morphism of inverse systems  $\rho_R : (C_*(X), C_*(Y_R)) \rightarrow (C_*, D_*)$ .

**Lemma 5.15.** *The mapping  $\rho_\bullet$  induces approximate isomorphisms between relative (co)homology with local coefficients:*

$$H^*(G, \{H_i\}; M) \rightarrow H^*(C_*(X), C_*(Y_R); M) \simeq H^*(X/G, Y_R/G; M)$$

$$H_*(X/G, Y_R/G; M) \simeq H_*(C_*(X), C_*(Y_R); M) \rightarrow H_*(G, \{H_i\}; M)$$

for any  $\mathbb{Z}G$ -module  $M$ .

*Proof.* We will prove the lemma by showing that the maps  $\rho_R$  form an ‘‘approximate chain homotopy equivalence’’ in an appropriate sense.

For each  $i$ , we construct a  $\mathbb{Z}H_i$ -chain mapping  $D_*(i) \rightarrow C_*(Y_{i,R})$  using part 1 of Lemma 5.1 and the fact that

$$\{\tilde{H}_j(Y_{\alpha,R})\}_{R \geq 0}$$

is an approximately zero system. We then extend these to  $\mathbb{Z}G$ -chain mappings

$$f_R : (C_*, D_*) \rightarrow (C_*(X), C_*(Y_R)).$$

Using part 2 of Lemma 5.1, we can actually choose the mappings  $f_R$  so that they form a compatible system chain mappings up to chain-homotopy. The composition

$$\rho_R \circ f_R : (C_*, D_*) \rightarrow (C_*, D_*)$$

is  $\mathbb{Z}G$ -chain mapping, hence it is chain-homotopic to the identity. The composition

$$f_R \circ \rho_R : C_*(X, Y_R) \rightarrow C_*(X, Y_R)$$

need not be chain homotopic to the identity, but it becomes chain homotopic to the projection map when precomposed with the restriction  $C_*(X, Y_{R'}) \rightarrow C_*(X, Y_R)$  where  $R' \geq R$  is suitably chosen (by again using part 2 of Lemma 5.1 and the fact that

$$\{\tilde{H}_j(Y_{\alpha,R})\}_{R \geq 0}$$

is an approximately zero system). This clearly implies the induced homomorphisms on (co)homology are approximate isomorphisms. q.e.d.

### 6. Coarse Poincaré duality

We now introduce a class of metric simplicial complexes which satisfy coarse versions of Poincaré and Alexander duality, see Theorems 6.7, 7.5, 7.7.

From now on, we will adopt the convention of extending each (co)chain complex indexed by the non-negative integers to a complex indexed by the integers by setting the remaining groups equal to zero. So, for each (co)chain complex  $\{C_i, i \geq 0\}$ , we get the (co)homology groups  $H_i(C_*)$ ,  $H^i(C_*)$  defined for  $i < 0$ .

**Definition 6.1** (Coarse Poincaré duality spaces). A *coarse Poincaré duality space of formal dimension  $n$*  is a bounded geometry metric simplicial complex  $X$  so that  $C_*(X)$  is uniformly acyclic, and there is a constant  $D_0$  and chain mappings

$$C_*(X) \xrightarrow{\bar{P}} C_c^{n-*}(X) \xrightarrow{P} C_*(X)$$

so that

1.  $P$  and  $\bar{P}$  have displacement  $\leq D_0$  (see Section 2 for the definition of displacement).
2.  $\bar{P} \circ P$  and  $P \circ \bar{P}$  are chain homotopic to the identity by  $D_0$ -Lipschitz (see Section 2) chain homotopies  $\Phi : C_*(X) \rightarrow C_{*+1}(X)$ ,  $\bar{\Phi} : C_c^*(X) \rightarrow C_c^{*-1}(X)$ .

We will often refer to coarse Poincaré duality spaces of formal dimension  $n$  as *coarse PD( $n$ ) spaces*. Throughout the paper, we will reserve the letter  $D_0$  for the constant which appears in the definition of a coarse PD( $n$ ) space; we let  $D := D_0 + 1$ .

Note that for each coarse PD( $n$ ) space  $X$ , we have

$$H_c^*(X) \simeq H_{n-*}(X) \simeq H_{n-*}(\mathbb{R}^n) \simeq H_c^*(\mathbb{R}^n).$$

We will not need the bounded geometry and uniform acyclicity conditions until Theorem 7.7. Later in the paper, we will consider simplicial actions on coarse PD( $n$ ) spaces, and we will assume implicitly that the actions commute with the operators  $\bar{P}$  and  $P$ , and the chain homotopies  $\bar{\Phi}$  and  $\bar{\Psi}$ .

The next lemma gives important examples of coarse PD( $n$ ) spaces:

**Lemma 6.2.** *The following are coarse PD( $n$ ) spaces:*

1. *An acyclic metric simplicial complex  $X$  which admits a free, simplicial, cocompact action by a PD( $n$ ) group.*
2. *An  $n$ -dimensional, bounded geometry metric simplicial complex  $X$ , with an augmentation  $\alpha : C_c^n(X) \rightarrow \mathbb{Z}$  for the compactly supported simplicial cochain complex, so that  $(C_c^*(X), \alpha)$  is uniformly acyclic (see Section 2 for definitions).*
3. *A uniformly acyclic, bounded geometry metric simplicial complex  $X$  which is a topological  $n$ -manifold.*

*Proof.*

1. Let  $0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow \cdots \leftarrow P_n \leftarrow 0$  be a resolution of  $\mathbb{Z}$  by finitely generated projective  $\mathbb{Z}G$ -modules.  $X$  is acyclic, so we have  $\mathbb{Z}G$ -chain homotopy equivalences  $P_* \xrightarrow{\alpha} C_*(X)$  and  $\text{Hom}(P_*, \mathbb{Z}G) \simeq C_c^*(X)$  where  $\alpha$  is augmentation preserving. Hence, to construct the two chain equivalences needed in Definition 6.1, it suffices to construct a  $\mathbb{Z}G$ -chain homotopy equivalence  $p : P_* \rightarrow \text{Hom}(P_{n-*}, \mathbb{Z}G)$  of  $\mathbb{Z}G$ -modules (since the operators are  $G$ -equivariant conditions 1 and 2 of Definition 6.1 will be satisfied automatically). For this, see [12, p. 221].

2. We construct a chain mapping  $P : C_*(X) \rightarrow C_c^{n-*}(X)$  as follows. We first map each vertex  $v$  of  $X$  to an  $n$ -cocycle  $\beta \in C_c^n(X, \overline{X - B(v, R_0)})$  which maps to 1 under the augmentation  $\alpha$ , (such a  $\beta$  exists by the uniform acyclicity of  $(C_c^*(X), \alpha)$ ), and extend this to a homomorphism  $C_0(X) \rightarrow C_c^n(X)$ . By the uniform acyclicity of  $(C_c^*(X), \alpha)$ , we can extend this to a chain mapping  $P$ . By similar reasoning, we obtain a chain homotopy inverse  $\bar{P}$ , and construct chain homotopies  $\bar{P} \circ P \sim id$  and  $P \circ \bar{P} \sim id$ .

3.  $X$  is acyclic, and therefore orientable. An orientation of  $X$  determines an augmentation  $\alpha : C_c^n(X) \rightarrow \mathbb{Z}$ . The uniform acyclicity of  $X$  together with ordinary Poincaré duality implies that  $(C_c^*(X), \alpha)$  is uniformly acyclic. So, 3 follows from 2.

We remark that if  $G \curvearrowright X$  is a free simplicial action, then these constructions can be made  $G$ -invariant. q.e.d.

When  $K \subset X$  is a (non-empty) subcomplex we will consider the direct system of tubular neighborhoods  $\{N_R(K)\}_{R \geq 0}$  of  $K$  and the inverse system of the closures of their complements

$$\{Y_R := \overline{X - N_R(K)}\}_{R \geq 0}.$$

We get four inverse and four direct systems of (co)homology groups:

$$\begin{aligned} &\{H_c^k(N_R(K))\}, \{H_j(X, Y_R)\}, \{H_c^k(X, N_R(K))\}, \{H_j(Y_R)\} \\ &\{H_c^k(Y_R)\}, \{H_j(X, N_R(K))\}, \{H_c^k(X, Y_R)\}, \{H_j(N_R(K))\} \end{aligned}$$

with the usual restriction and projection homomorphisms. Note that by excision, we have isomorphisms

$$H_j(X, Y_R) \simeq H_j(N_R(K), \partial N_R(K)), \text{ etc.}$$

Extension by zero defines a group homomorphism  $C_c^k(N_{R+D}(K)) \xrightarrow{ext} C_c^k(X)$ . When we compose this with

$$C_c^k(X) \xrightarrow{P} C_{n-k}(X) \xrightarrow{proj} C_{n-k}(X, Y_R),$$

we get a well-defined induced homomorphism

$$P_{R+D} : H_c^k(N_{R+D}(K)) \rightarrow H_{n-k}(X, Y_R)$$

where  $D$  is as in Definition 6.1. We get, in a similar fashion, homomorphisms

$$(6.3) \quad H_c^k(N_{R+D}(K)) \xrightarrow{P_{R+D}} H_{n-k}(X, Y_R) \xrightarrow{\bar{P}_R} H_c^k(N_{R-D}(K))$$

$$(6.4) \quad H_c^k(Y_R) \xrightarrow{P_R} H_{n-k}(X, N_{R+D}(K)) \xrightarrow{\bar{P}_{R+D}} H_c^k(Y_{R+2D})$$

$$(6.5) \quad H_c^k(X, N_{R+D}(K)) \xrightarrow{P_{R+D}} H_{n-k}(Y_R) \xrightarrow{\bar{P}_R} H_c^k(X, N_{R-D}(K))$$

$$(6.6) \quad H_c^k(X, Y_R) \xrightarrow{P_R} H_{n-k}(N_{R+D}(K)) \xrightarrow{\bar{P}_{R+D}} H_c^k(X, Y_{R+2D})$$

Note that the homomorphisms in (6.3), (6.5) determine  $\alpha$ -morphisms between inverse systems and the homomorphisms in (6.4), (6.6) determine  $\beta$ -morphisms between direct systems, where  $\alpha(R) = R - D$ ,  $\beta(R) = R + D$  (see Section 4 for definitions). These operators inherit

the bounded displacement property of  $P$  and  $\bar{P}$ , see condition 1 of Definition 6.1. We let  $\omega(R) := R + 2D$ , where  $D$  is the constant from Definition 6.1.

**Theorem 6.7** (Coarse Poincaré duality). *Let  $X$  be a coarse  $PD(n)$  space,  $K \subset X$  be a subcomplex as above. Then, the morphisms  $P_\bullet, \bar{P}_\bullet$  in (6.3), (6.5) are  $(\omega, \omega)$ -approximate isomorphisms of inverse systems and the morphisms  $P_\bullet, \bar{P}_\bullet$  in (6.4), (6.6) are  $(\omega, \omega)$ -approximate isomorphisms of direct systems (see Section 4). In particular, if  $X \neq N_{R_0}(K)$  for any  $R_0$ , then the inverse systems  $\{H_c^n(N_R(K))\}_{R \geq 0}$  and  $\{H_n(Y_R)\}_{R \geq 0}$  are approximately zero.*

*Proof.* We will verify the assertion for the homomorphism  $P_\bullet$  in (6.3) and leave the rest to the reader. We first check that  $P_\bullet$  is an  $\omega$ -approximate monomorphism. Let

$$\xi \in Z_c^*(N_{R+2D}(K))$$

be a cocycle representing an element  $[\xi] \in \text{Ker}(P_{R+2D})$ , and let  $\xi_1 \in C_c^*(X)$  be the extension of  $\xi$  by zero. Then, we have

$$P(\xi_1) = \partial\eta + \zeta$$

where  $\eta \in C_{n-*}(X)$  and  $\zeta \in C_{n-*}(\overline{X - N_{R+D}(K)})$ . Applying  $\bar{P}$  and the chain homotopy  $\Phi$ , we get

$$\delta\Phi(\xi_1) + \Phi\delta(\xi_1) = \bar{P} \circ P(\xi_1) - \xi_1 = \bar{P}(\partial\eta + \zeta) - \xi_1$$

so

$$\xi_1 = \delta\bar{P}(\eta) + \bar{P}(\zeta) - \delta\Phi(\xi_1) - \Phi\delta(\xi_1).$$

The second and fourth terms on the right-hand side vanish upon projection to  $H_c^*(N_R(K))$ , so  $[\xi] \in \text{Ker}(H_c^*(N_{R+2D}(K)) \rightarrow H_c^*(N_R(K)))$ .

We now check that  $P_\bullet$  is an  $\omega$ -approximate epimorphism. Let

$$[\sigma] \in \text{Im}(H_{n-*}(X, \overline{X - N_{R+2D}(K)}) \rightarrow H_{n-*}(X, \overline{X - N_R(K)})),$$

then  $\sigma$  lifts to a chain  $\tau \in C_{n-*}(X)$  so that  $\partial\tau \in C_{n-*}(\overline{X - N_{R+2D}(K)})$ . Let  $[\tau] \in H_{n-*}(X, Y_{R+2D})$  be the corresponding relative homology class. Applying  $P$  and the chain homotopy  $\Phi$ , we get

$$P(\bar{P}(\tau)) - \tau = \partial\bar{\Phi}(\tau) + \bar{\Phi}(\partial\tau).$$

Since  $\bar{\Phi}(\partial\tau)$  vanishes in  $C_{n-*}(X, \overline{X - N_R(K)})$ , we get that

$$[\sigma] = P_{R+D}(\bar{P}_{R+2D}([\tau])).$$

The proof of the last assertion about  $\{H_c^n(N_R(K))\}_{R \geq 0}$  and  $\{H_n(Y_R)\}_{R \geq 0}$  follows since they are approximately isomorphic to zero systems  $H_0(X, Y_R)$  and  $H^0(X, N_R(K))$ . q.e.d.

**Corollary 6.8.** *Suppose  $W$  be a bounded geometry uniformly acyclic metric simplicial complex (with metric  $d_W$ ),  $Z \subset W$  and  $f : (Z, d_W|_Z) \rightarrow (X, d_X)$  be a uniform embedding to a coarse  $PD(n)$  space  $X$ .*

1.  $N_R(f(Z)) = X$  for some  $R$  iff  $\{H_c^n(N_R(Z))\}_{R \geq 0}$  is approximately isomorphic to the constant system  $\mathbb{Z}$ .
2. If  $W$  is a coarse  $PD(k)$ -space for  $k < n$ , then  $N_R(f(Z)) \neq X$  for any  $R$ .
3. If  $W = N_r(Z)$  for some  $r$  and  $W$  is a coarse  $PD(n)$ -space, then  $N_R(f(Z)) = X$  for some  $R$ . The thickness  $R$  depends only on  $r$ , and the geometry of  $W$ ,  $X$ , and  $f$ .

*Proof.* 1. Let  $K = f(Z)$ . The mapping  $f$  induces an approximate isomorphism between the inverse systems  $\{H_c^n(N_R(Z))\}_{R \geq 0}$  and  $\{H_c^n(N_R(K))\}_{R \geq 0}$  (see Lemma 5.7), and the latter is approximately isomorphic to  $\{H_0(X, \overline{X - N_R(K)})\}_{R \geq 0}$  by coarse Poincaré duality. Note that  $H_0(X, \overline{X - N_R(K)}) = 0$  unless  $N_R(K) = X$ , in which case  $H_0(X, \overline{X - N_R(K)}) = \mathbb{Z}$ . In the latter case,  $\{H_c^n(N_R(Z))\}_{R \geq 0}$  is approximately isomorphic to  $\mathbb{Z}$ . In the former case,  $\{H_c^n(N_R(Z))\}_{R \geq 0}$  is approximately zero.

2. If  $W$  is a coarse  $PD(k)$ -space, then by applying Theorem 6.7 to  $Z \subset W$ , we get that  $\{H_c^n(N_R(Z))\}_{R \geq 0}$  is approximately zero (recall our convention that both homology and cohomology groups are defined to be zero in negative dimensions). Thus, 2 follows from 1.

3. This follows by applying part 1 twice. q.e.d.

### 7. Coarse Alexander duality and coarse Jordan separation

In this section, as in the previous one, we extend complexes indexed by the non-negative integers to complexes indexed by  $\mathbb{Z}$ , by setting the remaining groups equal to zero.

Let  $X, K, D, Y_R$ , and  $\omega$  be as in the preceding section. Composing the morphisms  $P_\bullet$  and  $\bar{P}_\bullet$  with the boundary operators for long exact sequences of pairs, we obtain the compositions  $A_{R+D}$

$$(7.1) \quad H_c^*(N_{R+D}(K)) \xrightarrow{P_{R+D}} H_{n-*}(X, Y_R) \xrightarrow{\partial} \tilde{H}_{n-*}(Y_R)$$

and  $\bar{A}_{R+D}$

$$(7.2) \quad \tilde{H}_{n-*}(Y_{R+D}) \xrightarrow{\partial^{-1}} H_{n-*}(X, Y_{R+D}) \xrightarrow{\bar{P}_{R+D}} H_c^*(N_R(K)).$$

Similarly, composing the maps from (6.3)–(6.4) with boundary operators and their inverses, we get:

$$(7.3) \quad H_c^*(Y_R) \xrightarrow{A_R} \tilde{H}_{n-* -1}(N_{R+D}(K))$$

and

$$(7.4) \quad \tilde{H}_{n-* -1}(N_R(K)) \xrightarrow{\bar{A}_R} H_c^*(Y_{R+D}).$$

**Theorem 7.5** (Coarse Alexander duality).

1. The morphisms  $A_\bullet$  and  $\bar{A}_\bullet$  in (7.1)–(7.4) are  $(\omega, \omega)$ -approximate isomorphisms.
2. The maps  $A_\bullet$  in (7.1) and (7.3) have displacement at most  $D$ . The map  $\bar{A}_\bullet$  in (7.2) (respectively (7.4)) has displacement at most  $D$  in the sense that if  $\sigma \in Z_{n-* -1}(Y_{R+D})$  ( $\sigma \in Z_{n-* -1}(N_R(K))$ ), and  $\sigma = \partial\tau$  for  $\tau \in C_{n-*}(X)$ , then the support of  $\bar{A}_{R+D}([\sigma])$  (respectively  $\bar{A}_R([\sigma])$ ) is contained in  $N_D(\text{Support}(\tau))$ .

Like ordinary Alexander duality, this theorem follows directly from Theorem 6.7, and the long exact sequence for pairs.

Combining Theorem 7.5 with Corollary 5.5, we obtain:

**Theorem 7.6** (Coarse Alexander duality for  $FP_k$  groups). *Let  $X$  be a coarse  $PD(n)$  space, and let  $G, P_*, G \curvearrowright X, f$ , and  $K$  be as in the statement of Corollary 5.5. Then,*

1. The family of compositions

$$\tilde{H}_{n-i-1}(Y_{R+D}) \xrightarrow{\bar{A}} H_c^i(N_R(K)) \xrightarrow{f_R^i} H^i(P_*; \mathbb{Z}G)$$

defines an approximate isomorphism when  $i < k$ , and an approximate monomorphism when  $i = k$ . Recall that for  $i < k$ , we have a natural isomorphism  $H^i(P_*, \mathbb{Z}G) \simeq H^i(G, \mathbb{Z}G)$ .

2. The family of compositions

$$\tilde{H}_i(P_*; \mathbb{Z}G) \rightarrow \tilde{H}_i(N_R(K)) \xrightarrow{\bar{A}_R} H_c^{n-i-1}(Y_{R+D})$$

is an approximate isomorphism when  $i < k$ , and an approximate epimorphism when  $i = k$ . Recall that  $\tilde{H}_i(P_*; \mathbb{Z}G) = \{0\}$  for  $i < k$  since  $G$  is of type  $FP_k$ .

**Theorem 7.7** (Coarse Alexander duality for maps). *Suppose  $X$  is a coarse  $PD(n)$  space,  $X'$  is a bounded geometry uniformly  $(k-1)$ -acyclic metric simplicial complex, and  $f : C_*(X') \rightarrow C_*(X)$  is a chain map which is a uniform embedding. Let  $K := \text{Support}(f(C_*(X')))$ ,  $Y_R := \overline{X - N_R(K)}$ . Then:*



1. *The family of compositions*

$$\tilde{H}_{n-i-1}(Y_{R+D}) \xrightarrow{\bar{A}} H_c^i(N_R(K)) \xrightarrow{H_c^i(f_R)} H_c^i(X')$$

defines an approximate isomorphism when  $i < k$ , and an approximate monomorphism when  $i = k$ .

2. *The family of compositions*

$$\tilde{H}_i(X') \rightarrow \tilde{H}_i(N_R(K)) \xrightarrow{\bar{A}_R} H_c^{n-i-1}(Y_{R+D})$$

is an approximate isomorphism when  $i < k$ , and an approximate epimorphism when  $i = k$ . (The function  $\omega$  for the above approximate isomorphisms depends only on the distortion of  $f$ , the acyclicity functions for  $X$  and  $X'$ , and the bounds on the geometry of  $X$  and  $X'$ .)

3. Furthermore, these approximate isomorphisms approximately respect support in the following sense. There is a function  $\zeta : \mathbb{N} \rightarrow \mathbb{N}$  so that if  $i < k$ ,  $S \subset X'$  is subcomplex,  $T := \text{Support}(f_*(C_*(S))) \subset X$  is the corresponding subcomplex of  $X$ , and  $\alpha \in \text{Im}(H_c^i(X', \overline{X' - S}) \rightarrow H_c^i(X'))$ , then  $\alpha$  belongs to the image of the composition

$$\tilde{H}_{n-i-1}(Y_R \cap N_{\zeta(R)}(T)) \rightarrow \tilde{H}_{n-i-1}(Y_R) \xrightarrow{H_c^i(f) \circ \bar{A}} H_c^i(X').$$

4. If  $k = n + 1$ , then  $H_c^n(X') = \{0\}$  unless  $N_R(K) = X$  for some  $R$ .

*Proof.* Parts 1, 2 and 3 of Theorem follow from Lemma 5.6 and Theorem 7.5. Part 4 follows since for  $i = n$ ,  $\{\tilde{H}_{n-i-1}(Y_{R+D})\} = \{0\}$  is approximately isomorphic to the constant system  $\{H_c^n(X')\}$ . q.e.d.

We now give a number of corollaries of coarse Alexander duality.

**Corollary 7.8** (Coarse Jordan separation for maps). *Let  $X$  and  $X'$  be  $n$ -dimensional and  $(n-1)$ -dimensional coarse Poincaré duality spaces respectively, and let  $g : X' \rightarrow X$  be a uniform embedding. Then:*

1.  $g(X')$  coarsely separates  $X$  into (exactly) two components.
2. For every  $R$ , each point of  $N_R(g(X'))$  lies within uniform distance from each of the deep components of  $Y_R := \overline{X - N_R(g(X'))}$ .
3. If  $Z \subset X'$ ,  $X' \not\subset N_R(Z)$  for any  $R$  and  $h : Z \rightarrow X$  is a uniform embedding, then  $h(Z)$  does not coarsely separate  $X$ . Moreover, for any  $R_0$ , there is an  $R_1 > 0$  depending only on  $R_0$  and the geometry of  $X, X'$ , and  $h$  such that precisely one component of  $X - N_{R_0}(h(Z))$  contains a ball of radius  $R_1$ .

*Proof.* We have the following diagram:

$$\begin{array}{ccc} \tilde{H}_0(Y_R) & \xrightarrow{H_c^{n-1}(g) \circ \bar{A}} & H_c^{n-1}(X') = \mathbb{Z} \\ \uparrow & & \\ \varprojlim_R \tilde{H}_0^{\text{Deep}}(Y_R) & & \end{array}$$

where the family of morphisms  $H_c^{n-1}(g) \circ \bar{A}$  gives rise to an approximate isomorphism. Thus,

$$\varprojlim_R \tilde{H}_0^{\text{Deep}}(Y_R) = \mathbb{Z}$$

which implies 1. Let  $x \in N_R(K)$ . Then there exists a representative  $\alpha$  of a generator of  $H_c^{n-1}(X')$  such that  $H_c^{n-1}(g)(\alpha) \in C_c^{n-1}(X)$  is supported uniformly close to  $x$ . We apply Part 3 of Theorem 7.7 to the class  $[H_c^{n-1}(g)(\alpha)]$  to prove 2.

To prove part 3, we first note that by Corollary 6.8, we have  $X - N_R(h(Z)) \neq \emptyset$  for all  $R$ . By Lemma 5.7 and coarse Alexander duality (Theorem 7.5), the inverse system  $\{\tilde{H}_0(X - N_R(h(Z)))\}_{R \geq 0}$  is approximately zero. But this means that there is precisely one deep component of  $X - N_R(f(Z))$  for every  $R$ ; it also implies the second half of part 3. q.e.d.

As a special case of the above corollary, we have:

**Corollary 7.9** (Coarse Jordan separation for submanifolds). *Let  $X$  and  $X'$  be  $n$ -dimensional and  $(n-1)$ -dimensional uniformly acyclic PL-manifolds respectively, and let  $g : X' \rightarrow X$  be a uniform embedding. Then, the assertions 1, 2 and 3 from the preceding theorem hold.*

Similarly, to the Corollary 7.8, we get:

**Corollary 7.10** (Coarse Jordan separation for groups). *Let  $X$  be a coarse PD( $n$ )-space and  $G$  be a PD( $n-1$ )-group acting freely simplicially on  $X$ . Let  $K \subset X$  be a  $G$ -invariant subcomplex with  $K/G$  compact. Then:*

1.  $G$  coarsely separates  $X$  into (exactly) two components.
2. For every  $R$ , each point of  $N_R(K)$  lies within uniform distance from each of the deep components of  $\overline{X - N_R(K)}$ .

**Lemma 7.11.** *Let  $W$  be a bounded geometry metric simplicial complex which is homeomorphic to a union of  $W = \cup_{i \in I} W_i$  of  $k$  half-spaces  $W_i \simeq \mathbb{R}_+^{n-1}$  along their boundaries. Assume that for  $i \neq j$ , the union  $W_i \cup W_j$  is uniformly acyclic and is uniformly embedded in  $W$ . Let  $g : W \rightarrow X$  be a uniform embedding of  $W$  into a coarse PD( $n$ ) space  $X$ . Then,  $g(W)$  coarsely separates  $X$  into  $k$  components. Moreover,*

there is a unique cyclic ordering on the index set  $I$  so that for  $R$  sufficiently large, the frontier of each deep component  $C$  of  $X - N_R(g(W))$  is at finite Hausdorff distance from  $g(W_i) \cup g(W_j)$  where  $i$  and  $j$  are adjacent with respect to the cyclic ordering.

*Proof.* We have  $H_c^{n-1}(W) \simeq \mathbb{Z}^{k-1}$ , so, arguing analogously to Corollary 7.8, we see that  $g(W)$  coarsely separates  $X$  into  $k$  components. Applying coarse Jordan separation and the fact that no  $W_i$  coarsely separates  $W_j$  in  $W$ , we can define the desired cyclic ordering by declaring that  $i$  and  $j$  are consecutive iff  $g(W_i) \cup g(W_j)$  coarsely separates  $X$  into two deep components (Corollary 7.8), one of which is a deep component of  $X - g(W)$ . We leave the details to the reader. q.e.d.

**Lemma 7.12.** *Suppose  $G$  is a group of type  $FP_{n-1}$  of cohomological dimension  $\leq n - 1$ , and let  $P_*$ ,  $f$ ,  $G \curvearrowright X$ ,  $K \subset X$  and  $Y_R$  be as in Theorem 7.6. Then, every deep component of  $Y_R$  is stable for  $R \geq D$ ; in particular, there are only finitely many deep components of  $Y_R$  modulo  $G$ . If  $\dim(G) < n - 1$ , then there is only one deep component.*

*Proof.* The composition

$$(7.13) \quad \lim_{\leftarrow R} \tilde{H}_0^{\text{Deep}}(Y_R) \rightarrow \tilde{H}_0^{\text{Deep}}(Y_D) \xrightarrow{f_D^i \circ \bar{A}_D} H^{n-1}(P_*; \mathbb{Z}G)$$

is an isomorphism by Theorem 7.6. Therefore,

$$\tilde{H}_0^{\text{Deep}}(Y_R) \rightarrow \tilde{H}_0^{\text{Deep}}(Y_D)$$

is a monomorphism for any  $R \geq D$ , and hence every deep component of  $Y_D$  is stable. If  $\dim(G) < n - 1$ , then  $H^{n-1}(P_*, \mathbb{Z}G) = \{0\}$ , and by (7.13), we conclude that  $Y_D$  contains only one deep component. q.e.d.

Another consequence of coarse Jordan separation is:

**Corollary 7.14.** *Let  $G \curvearrowright X$  be a free simplicial action of a group  $G$  of type  $FP$  on a coarse  $PD(n)$  space  $X$ , and let  $K \subset X$  be a  $G$ -invariant subcomplex on which  $G$  acts cocompactly. By Lemma 7.12, there is an  $R_0$  so that all deep components of  $X - N_{R_0}(K)$  are stable; hence, we have a well-defined collection of deep complementary components  $\{C_\alpha\}$  and their stabilizers  $\{H_\alpha\}$ . If  $H \subset G$  is a  $PD(n - 1)$  subgroup, then one of the following holds:*

1.  $H$  coarsely separates  $G$ .
2.  $H$  has finite index in  $G$ , and so  $G$  is a  $PD(n - 1)$  group.
3.  $H$  has finite index in  $H_\alpha$  for some  $\alpha$ .

*In particular,  $G$  contains only finitely many conjugacy classes of maximal, coarsely non-separating  $PD(n-1)$  subgroups.*

*Proof.* We assume that  $H$  does not coarsely separate  $G$ . Pick a base-point  $\star \in K$ , and let  $W := H(\star)$  be the  $H$ -orbit of  $\star$ . Then, by Corollary 7.10, there is an  $R_1$  so that  $X - N_{R_1}(W)$  has two deep components  $C_+$ ,  $C_-$  and both are stable. Since  $H$  does not coarsely separate  $G$ , we may assume that  $K \subset N_{R_2}(C_-)$  for some  $R_2$ . Therefore,  $C_+$  has finite Hausdorff distance from some deep component  $C_\alpha$  of  $X - N_{R_0}(K)$ , and clearly, the Hausdorff distance between the frontiers  $\partial C_+$  and  $\partial C_\alpha$  is finite. Either  $H$  preserves  $C_+$  and  $C_-$ , or it contains an element  $h$  which exchanges the two. In the latter case,  $h(C_\alpha)$  is within finite Hausdorff distance from  $C_-$ ; so, in this case,  $K$  is contained in  $N_r(W)$  for some  $r$ , and this implies 2. When  $H$  preserves  $C_+$ , then we have  $H \subset H_\alpha$ , and since  $H$  acts cocompactly on  $\partial C_+$ , it also acts cocompactly on  $\partial C_\alpha$  and hence,  $[H_\alpha : H] < \infty$ . q.e.d.

## 8. The proof of Theorem 1.1

**Sketch of the proof of Theorem 1.1.** Consider an action  $G \curvearrowright X$  as in the statement of Theorem 1.1. Let  $K \subset X$  be a  $G$ -invariant subcomplex with  $K/G$  compact. By Lemma 7.12, the deep components of  $X - N_R(K)$  stabilize at some  $R_0$ , and hence, we have a collection of deep components  $C_\alpha$  and their stabilizers  $H_\alpha$ . Naively, one might hope that for some  $R \geq R_0$ , the tubular neighborhood  $N_R(K)$  is acyclic, and the frontier of  $N_R(K)$  breaks up into connected components which are in one-to-one correspondence with the  $C_\alpha$ 's, each of which is acyclic and has the same compactly supported cohomology as  $\mathbb{R}^{n-1}$ . Of course, this is too much to hope for, but there is a coarse analog which does hold. To explain this, we first note that the systems  $\tilde{H}_*(N_R(K))$  and  $H_c^*(N_R(K))$  are approximately zero and approximately constant respectively by Corollary 5.5. Applying coarse Alexander duality, we find that the systems  $\underline{H}_c^*(Y_R)$  and  $\tilde{H}_*(Y_R)$  corresponding to the complements  $Y_R := \overline{X - N_R(K)}$  are approximately zero and approximately constant, respectively. Instead of looking at the frontiers of the neighborhoods  $N_R(K)$ , we look at metric annuli  $A(r, R) := \overline{N_R(K) - N_r(K)}$  for  $r \leq R$ . One can try to compute the (co)homology of these annuli using a Mayer–Vietoris sequence for the covering  $X = N_R(K) \cup Y_r$ ; however, the input to this calculation is only approximate, and the system of annuli does not form a direct or inverse system in any useful way. Nonetheless, there are finite direct systems of nested annuli of arbitrary depth for which one can understand the (co)homology, and this allows

us to apply results from Section 5 to see that the  $H_\alpha$ 's are Poincaré duality groups.

**Remark.** There is an extra complication in calculating  $H_c^{n-1}$  for the annuli which we have omitted from this sketch.

**The proof of Theorem 1.1.** We now assume that  $G$  is a group of type  $FP$  acting freely and simplicially on a coarse  $PD(n)$  space  $X$ . This implies that  $\dim(G) \leq n$ , so, by Lemma 3.2, there is a resolution  $0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$  of  $\mathbb{Z}$  by finitely generated projective  $\mathbb{Z}G$ -modules. We may construct  $G$ -equivariant (augmentation preserving) chain mappings  $\rho : C_*(X) \rightarrow P_*$  and  $f : P_* \rightarrow C_*(X)$  using the acyclicity of  $C_*(X)$  and  $P_*$ ; the composition  $\rho \circ f : P_* \rightarrow P_*$  is  $\mathbb{Z}G$ -chain homotopic to the identity. If  $L \subset X$  is a  $G$ -invariant subcomplex for which  $L/G$  is compact, then we get an induced homomorphism

$$H^*(G; \mathbb{Z}G) \xrightarrow{H^*(\rho)} H^*(X/G; \mathbb{Z}G) \rightarrow H^*(L/G; \mathbb{Z}G) \simeq H_c^*(L);$$

abusing notation, we will denote this composition by  $H^*(\rho)$ .

Let  $K \subset X$  be a connected,  $G$ -invariant subcomplex so that  $K/G$  is compact and the image of  $f$  is supported in  $K$ . For  $R \geq 0$  set  $Y_R := \overline{X - N_R(K)}$ . Corollary 5.5 tells us that the families of maps

$$(8.1) \quad \{0\} \rightarrow \{\tilde{H}_*(P_*; \mathbb{Z}G)\} \rightarrow \{\tilde{H}_*(N_R(K))\}$$

$$(8.2) \quad H_c^*(f) : H_c^*(N_R(K)) \rightarrow H^*(G; \mathbb{Z}G) \simeq H^*(P; \mathbb{Z}G).$$

define approximate isomorphisms. Applying Theorems 7.6, we get approximate isomorphisms

$$(8.3) \quad \{0\} \rightarrow H_c^k(Y_R) \text{ for all } k$$

and

$$(8.4) \quad \phi_{k,R} : \tilde{H}_k(Y_R) \rightarrow H^{n-k-1}(P_*; \mathbb{Z}G) \simeq H^{n-k-1}(G; \mathbb{Z}G) \text{ for all } k.$$

We denote  $\phi_{*,D}$  by  $\tilde{\phi}_*$ .

We now apply Lemma 7.12 to see that every deep component of  $X - N_D(K)$  is stable. Let  $\{C_\alpha\}$  denote the collection of deep components of  $X - N_D(K)$ , and set  $Y_{R,\alpha} := Y_R \cap C_\alpha$  and  $Z_{R,\alpha} := \overline{X - Y_{R,\alpha}}$ . Note that for every  $\alpha$ , and  $D < r < R$ , we have  $Z_{R,\alpha} \cap Y_{r,\alpha} = \overline{N_R(K) - N_r(K)} \cap C_\alpha$ .

**Lemma 8.5.**

1. There is an  $R_0$  so that if  $R \geq R_0$ , then  $Y_{R,\alpha} = \overline{X - Z_{R,\alpha}}$  and  $Z_{R,\alpha} = N_{R-R_0}(Z_{R_0,\alpha})$ .
2. The systems  $\{\tilde{H}_k(Y_{R,\alpha})\}$ ,  $\{\tilde{H}_k(Z_{R,\alpha})\}$ ,  $\{H_c^k(Y_{R,\alpha})\}$ ,  $\{H_c^k(Z_{R,\alpha})\}$  are approximately zero for all  $k$ .

*Proof.* Pick  $R_0$  large enough that all shallow components of  $X - N_D(K)$  are contained in  $N_{R_0-1}(K)$ . Then, for all  $R \geq R_0$ ,  $\partial C_\alpha \cap Y_R = \emptyset$  and hence  $Y_{R,\alpha}$ , like  $Y_R$  itself, is the closure of its interior; this implies that  $Y_{R,\alpha} = \overline{X - \overline{X - Y_{R,\alpha}}} = \overline{X - Z_{R,\alpha}}$ . We also have  $Z_{R,\alpha} = N_R(K) \sqcup (\sqcup_{\beta \neq \alpha} C_\beta)$  for all  $R \geq R_0$ . Since  $\sqcup_{\beta \neq \alpha} N_R(C_\beta) \subset N_{R_0+R}(K) \cup (\sqcup_{\beta \neq \alpha} C_\beta)$ , we get

$$\begin{aligned} N_R(Z_{R_0,\alpha}) &= N_{R_0+R}(K) \cup (\sqcup_{\beta \neq \alpha} N_R(C_\beta)) \\ &= N_{R_0+R}(K) \cup (\sqcup_{\beta \neq \alpha} C_\beta) \\ &= Z_{R_0+R,\alpha}. \end{aligned}$$

Thus, we have proven 1.

To prove 2, we first note that  $\{\tilde{H}_0(Y_{R,\alpha})\}$  is approximately zero by the stability of the deep components  $C_\alpha$ . When  $R \geq R_0$ , then  $Z_{R,\alpha}$  is connected (since  $N_R(K)$  and each  $C_\beta$  are connected), and this says that  $\{\tilde{H}_0(Z_{R,\alpha})\}$  is approximately zero. When  $R \geq R_0$ , then  $Y_R$  is the disjoint union  $\sqcup_\alpha Y_{R,\alpha}$ , so we have direct sum decompositions  $H_k(Y_R) = \oplus_\alpha H_k(Y_{R,\alpha})$  and  $H_c^k(Y_R) = \oplus_\alpha H_c^k(Y_{R,\alpha})$  which are compatible projection homomorphisms. This together with (8.3) and (8.4) implies that  $\{\tilde{H}_k(Y_{R,\alpha})\}$  and  $\{H_c^k(Y_{R,\alpha})\}$  are approximately zero for all  $k$ . By part 1 and Theorem 7.5, we get that  $\{H_c^k(Z_{R,\alpha})\}$  and  $\{\tilde{H}_k(Z_{R,\alpha})\}$  are approximately zero for all  $k$ . q.e.d.

**Lemma 8.6.** *There is an  $R_{min} > D$  so that for any  $R \geq R_{min}$  and any integer  $M$ , there is a sequence  $R \leq R_1 \leq R_2 \leq \dots \leq R_M$  with the following property. Let  $A(i, j) := \overline{N_{R_j}(K) - N_{R_i}(K)} \subset Y_{R_i}$ , and  $A_\alpha(i, j) := A(i, j) \cap C_\alpha$ . Then, for each  $1 < i < j < M$ ,*

1. *The image of  $\tilde{H}_k(A(i, j)) \rightarrow \tilde{H}_k(A(i-1, j+1))$  maps isomorphically onto  $H^{n-k-1}(G; \mathbb{Z}G)$  under the composition*

$$\tilde{H}_k(A(i-1, j+1)) \rightarrow \tilde{H}_k(Y_D) \xrightarrow{\phi^k} H^{n-k-1}(G; \mathbb{Z}G)$$

*for  $0 \leq k \leq n-1$ . The homomorphism*

$$\tilde{H}_n(A(i, j)) \rightarrow \tilde{H}_n(A(i-1, j+1))$$

*is zero.*

2.  *$H^k(\rho) : H^k(G; \mathbb{Z}G) \rightarrow H_c^k(A(i, j))$  maps  $H^k(G; \mathbb{Z}G)$  isomorphically onto the image of  $H_c^k(A(i-1, j+1)) \rightarrow H_c^k(A(i, j))$  for  $0 \leq k < n-1$ .*
3. *There is a system of homomorphisms  $H_c^{n-1}(A_\alpha(i, j)) \xrightarrow{\theta_{i,j}^\alpha} \mathbb{Z}$  (compatible with the inclusions  $A_\alpha(i, j) \rightarrow A_\alpha(i-1, j+1)$ ) so that the*

image of  $H_c^{n-1}(A_\alpha(i-1, j+1)) \rightarrow H_c^{n-1}(A_\alpha(i, j))$  maps isomorphically to  $\mathbb{Z}$  under  $\theta_{i,j}^\alpha$ .

4. For each  $\alpha$ ,  $\tilde{H}_0(A_\alpha(i, j)) \xrightarrow{0} \tilde{H}_0(A_\alpha(i-1, j+1))$ .

*Proof.* We choose  $R_{\min}$  large enough so that for any  $R \geq R_{\min}$ , the following inductive construction is valid. Let  $R_1 := R$ . Using the approximate isomorphisms (8.1), (8.2), (8.3), (8.4), and Lemma 8.5, we inductively choose  $R_{i+1}$  so that:

- A.  $\tilde{H}_k(N_{R_i}(K)) \xrightarrow{0} \tilde{H}_k(N_{R_{i+1}}(K))$  for  $0 \leq k \leq n$ .
- B.  $\text{Im}(\tilde{H}_k(Y_{R_{i+1}}) \rightarrow \tilde{H}_k(Y_{R_i}))$  maps isomorphically to  $H^{n-k-1}(G; \mathbb{Z}G)$  under  $\phi_{k,R_i}$  for  $0 \leq k < n$ , and  $\text{Im}(\tilde{H}_k(Y_{R_{i+1}}) \rightarrow \tilde{H}_k(Y_{R_i}))$  is zero when  $k = n$ .
- C.  $\text{Im}(H_c^*(N_{R_{i+1}}(K)) \rightarrow H_c^*(N_{R_i}(K)))$  maps isomorphically onto  $H^*(G; \mathbb{Z}G)$  under  $H_c^*(f)$ .
- D.  $H_c^*(Y_{R_i}) \xrightarrow{0} H_c^*(Y_{R_{i+1}})$ .
- E. For each  $\alpha$ ,  $H_c^{n-1}(Y_{R_i,\alpha}) \xrightarrow{0} H_c^{n-1}(Y_{R_{i+1},\alpha})$ , and  $H_c^{n-1}(Z_{R_{i+1},\alpha}) \xrightarrow{0} H_c^{n-1}(Z_{R_i,\alpha})$ .
- F. For each  $\alpha$ ,  $\tilde{H}_0(Y_{R_{i+1},\alpha}) \xrightarrow{0} \tilde{H}_0(Y_{R_i,\alpha})$  and  $\tilde{H}_0(Z_{R_i,\alpha}) \xrightarrow{0} \tilde{H}_0(Z_{R_{i+1},\alpha})$ .

Now, take  $1 < i < j < M$ , and consider the map of Mayer–Vietoris sequences for the decompositions  $X = N_{R_j}(K) \cup Y_{R_i}$  and  $X = N_{R_{j+1}}(K) \cup Y_{R_{i-1}}$ :

$$\begin{array}{ccccccc}
 \tilde{H}_{k+1}(X) & \rightarrow & \tilde{H}_k(A(i, j)) & \rightarrow & \tilde{H}_k(N_{R_j}(K)) \oplus \tilde{H}_k(Y_{R_i}) & \rightarrow & \tilde{H}_k(X) \\
 \downarrow & & \downarrow & & \begin{array}{c} 0 \downarrow \downarrow \\ \downarrow \downarrow \end{array} & & \downarrow \\
 \tilde{H}_{k+1}(X) & \rightarrow & \tilde{H}_k(A(i-1, j+1)) & \rightarrow & \tilde{H}_k(N_{R_{j+1}}(K)) \oplus \tilde{H}_k(Y_{R_{i-1}}) & \rightarrow & \tilde{H}_k(X) \\
 & & \downarrow \phi_k \Big|_{A(i-1, j+1)} & & \downarrow \phi_k & & \\
 & & H^{n-k-1}(G, \mathbb{Z}G) & \rightarrow & H^{n-k-1}(G, \mathbb{Z}G) & & 
 \end{array}$$

Since  $\tilde{H}_*(X) = \{0\}$ , conditions A and B and the diagram imply the first part of assertion 1. The same Mayer–Vietoris diagram for  $k = n$  implies the second part.

Let  $0 \leq k < n - 1$ . Consider the commutative diagram of Mayer–Vietoris sequences:

$$\begin{array}{ccccccc}
 & & H^k(G, \mathbb{Z}G) & \rightarrow & H^k(G, \mathbb{Z}G) & & \\
 & & \downarrow H^k(\rho) & & \downarrow H^k(\rho) & & \\
 H_c^k(X) & \rightarrow & H_c^k(N_{R_{j+1}}(K)) \oplus H_c^k(Y_{R_{i-1}}) & \rightarrow & H_c^k(A(i-1, j+1)) & \rightarrow & H_c^{k+1}(X) \\
 \downarrow & & \downarrow 0 & & \downarrow & & \downarrow \\
 H_c^k(X) & \rightarrow & H_c^k(N_{R_j}(K)) \oplus H_c^k(Y_{R_i}) & \rightarrow & H_c^k(A(i, j)) & \rightarrow & H_c^{k+1}(X)
 \end{array}$$

Assertion 2 now follows from the fact that  $H_c^k(X) \cong H_c^{k+1}(X) = 0$ , conditions C and D, and the diagram.

Assertion 3 follows from condition E, the fact that  $H_c^n(X) \simeq \mathbb{Z}$ , and the following commutative diagram of Mayer–Vietoris sequences ( $\theta_{i,j}^\alpha$  is the coboundary operator in the sequence):

$$\begin{array}{ccccccc} H_c^{n-1}(Z_{R_{j+1},\alpha}) \oplus H_c^{n-1}(Y_{R_{i-1},\alpha}) & \rightarrow & H_c^{n-1}(A_\alpha(i-1, j+1)) & \xrightarrow{\theta_{i-1,j+1}^\alpha} & H_c^n(X) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 \downarrow & 0 \downarrow & & & \\ H_c^{n-1}(Z_{R_j,\alpha}) \oplus H_c^{n-1}(Y_{R_i,\alpha}) & \rightarrow & H_c^{n-1}(A_\alpha(i, j)) & \xrightarrow{\theta_{i,j}^\alpha} & H_c^n(X) & \rightarrow & 0 \end{array}$$

Assertion 4 follows from condition F and the following commutative diagram:

$$\begin{array}{ccccccc} \tilde{H}_1(X) & \rightarrow & \tilde{H}_0(A_\alpha(i, j)) & \rightarrow & \tilde{H}_0(Z_{R_j,\alpha}) \oplus \tilde{H}_0(Y_{R_i,\alpha}) & \rightarrow & \tilde{H}_0(X) \\ & & \downarrow & & 0 \downarrow & 0 \downarrow & \downarrow \\ \tilde{H}_1(X) & \rightarrow & \tilde{H}_0(A_\alpha(i-1, j+1)) & \rightarrow & \tilde{H}_0(Z_{R_{j+1},\alpha}) \oplus \tilde{H}_0(Y_{R_{i-1},\alpha}) & \rightarrow & \tilde{H}_0(X) \end{array}$$

q.e.d.

**Corollary 8.7.** *If  $G$  is an  $(n - 1)$ -dimensional duality group, then each deep component stabilizer is a  $PD(n - 1)$  group.*

*Proof.* Fix a deep component  $C_\alpha$  of  $X - N_D(K)$ , and let  $H_\alpha$  be its stabilizer in  $G$ . Let  $R = D$ ,  $M = 4k + 2$ , and apply the construction of Lemma 8.6 to get  $D \leq R_1 \leq R_2 \leq \dots \leq R_{4k+2}$  satisfying the conditions of Lemma 8.6.

Pick  $1 < i < j < M$ . The mappings  $\tilde{H}_\ell(A(i, j)) \rightarrow \tilde{H}_\ell(A(i - 1, j + 1))$  are zero for each  $\ell = 1, \dots, n$  by part 1 of Lemma 8.6, since  $H^k(G, \mathbb{Z}G) = 0$  for  $k < n - 1$ . Because  $A(p, q)$  is the disjoint union  $\amalg_\alpha A_\alpha(p, q)$  for all  $0 < p < q < M$ , we actually have  $\tilde{H}_\ell(A_\alpha(i, j)) \xrightarrow{0} \tilde{H}_\ell(A_\alpha(i - 1, j + 1))$  for  $1 \leq \ell \leq n$ . By part 4 of Lemma 8.6, the same assertion holds for  $\ell = 0$ . Applying Theorem 5.11 to the chain complexes  $C_*(A_\alpha(i, j))$ , we see that when  $k > 2n + 5$ ,  $H_\alpha$  is a group of type  $FP_n$ . Since  $\dim(H_\alpha) \leq \dim(G) = n - 1$ , it follows that  $H_\alpha$  is of type  $FP$  (see Section 3).

The mappings  $H_c^\ell(A_\alpha(i - 1, j + 1)) \rightarrow H_c^\ell(A_\alpha(i, j))$  are zero for  $0 \leq \ell < n - 1$  by part 2 of Lemma 8.6 and the fact that  $A(p, q) = \amalg_\alpha A_\alpha(p, q)$ . By parts 1 and 2 of Lemma 5.1, we have  $H^k(H_\alpha, \mathbb{Z}H_\alpha) = \{0\}$  for  $0 \leq k < n - 1$ , and  $H^{n-1}(H_\alpha, \mathbb{Z}H_\alpha) \simeq \mathbb{Z}$  by part 3 of Lemma 8.6. Hence,  $H_\alpha$  is a  $PD(n - 1)$  group. q.e.d.

**Remark.** For the remainder of the proof, we really only need to know that each deep component stabilizer is of type  $FP$ .



*Proof of Theorem 1.1 concluded.* Let  $C_1, \dots, C_k$  be a set of representatives for the  $G$ -orbits of deep components of  $X - N_R(K)$ , and let  $H_1, \dots, H_k \subset G$  denote their stabilizers. Recall that both  $G$  and each  $H_i$  are assumed to be of type  $FP$ , see Section 3. By Lemma 5.15, we have

$$H^*(G, \{H_i\}; \mathbb{Z}G) \simeq \varinjlim_R H_c^*(X, Y_R),$$

while  $\lim_R H_c^*(X, Y_R) \simeq \lim_R H_{n-*}(N_R(K))$  by Coarse Poincaré duality, and

$$\varinjlim_R H_*(N_R(K)) \simeq H_*(X) \simeq H_*(pt)$$

since homology commutes with direct limits. Therefore, the group pair  $(G, \{H_i\})$  satisfies one of the criteria for  $PD(n)$  pairs (see Section 3), and we have proven Theorem 1.1. q.e.d.

We record a variant of Theorem 1.1 which describes the geometry of the action  $G \curvearrowright X$  more explicitly:

**Theorem 8.8.** *Let  $G \curvearrowright X$  be as in Theorem 1.1, and let  $K \subset X$  be a  $G$ -invariant subcomplex with  $K/G$  compact. Then, there are  $R_0, R_1, R_2$  so that*

1. *The deep components  $\{C_\alpha\}_{\alpha \in I}$  of  $X - N_{R_0}(K)$  are all stable, there are only finitely many of them modulo  $G$ , and their stabilizers  $\{H_\alpha\}_{\alpha \in I}$  are  $PD(n - 1)$  groups.*
2. *For all  $\alpha \in I$ , the frontier  $\partial C_\alpha$  is connected, and  $N_{R_1}(\partial C_\alpha)$  has precisely two deep complementary components,  $E_\alpha$  and  $F_\alpha$ , where  $E_\alpha$  has Hausdorff distance at most  $R_2$  from  $C_\alpha$ . Unless  $G$  is a  $PD(n - 1)$  group, the distance function  $d(\partial C_\alpha, \cdot)$  is unbounded on  $K \cap F_\alpha$ .*
3. *The Hausdorff distance between  $X - \coprod_\alpha E_\alpha$  and  $K$  is at most  $R_2$ .*

*Proof.* This is clear from the discussion above. q.e.d.

We remark that there are  $\alpha_1 \neq \alpha_2 \in I$  so that the Hausdorff distance

$$d_H(\partial C_{\alpha_1}, \partial C_{\alpha_2}) < \infty$$

iff  $G$  is a  $PD(n - 1)$  group.

In Proposition 8.10, we generalize the following uniqueness theorem of the peripheral structure from 3-dimensional manifolds to  $PD(n)$  pairs:

**Theorem 8.9** (Johannson [27], see also [41]). *Let  $M$  be a compact connected acylindrical 3-manifold with aspherical incompressible boundary components  $S_1, \dots, S_m$ . Let  $N$  be a compact 3-manifold homotopy-equivalent to  $M$ , with incompressible boundary components  $Q_1, \dots, Q_n$ ,*

and  $\varphi : \pi_1(M) \rightarrow \pi_1(N)$  be an isomorphism. Then,  $\varphi$  preserves the peripheral structures of  $\pi_1(M)$  and  $\pi_1(N)$  in the following sense. There is a bijection  $\beta$  between the set of boundary components of  $M$  and the set of boundary components on  $N$  so that after relabeling via  $\beta$ , we have:  $\varphi(\pi_1(S_i))$  is conjugate to  $\pi_1(Q_i)$  in  $\pi_1(N)$ .

**Proposition 8.10.** *Let  $(G, \{H_i\}_{i \in I})$  be a  $PD(n)$  pair, where  $G$  is not a  $PD(n-1)$  group, and  $H_i$  does not coarsely separate  $G$  for any  $i$ . Now, let  $G \curvearrowright X$  be a free simplicial action on a coarse  $PD(n)$  space, and let  $(G, \{L_j\}_{j \in J})$  be the group pair obtained by applying Theorem 1.1 to this action. Then, there is a bijection  $\beta : I \rightarrow J$  so that  $H_i$  is conjugate to  $L_{\beta(i)}$  for all  $i \in I$ .*

*Proof.* Under the assumptions above, each  $H_i$  and  $L_j$  is a maximal  $PD(n-1)$  subgroup (see Lemma 3.3). By Corollary 7.14, each  $H_i$  is conjugate to some  $L_j$ , and by Lemma 3.3, this defines an injection  $\beta : I \rightarrow J$ . Consider the double  $\hat{G}$  of  $G$  over the  $L_j$ 's. Then, the double of  $G$  over the  $H_i$ 's sits in  $\hat{G}$ , and the index will be infinite unless  $\beta$  is a bijection. q.e.d.

We now establish a relation between the acylindricity assumption in Theorem 8.9 and coarse non-separation assumption in Proposition 8.10. We first note that if  $M$  is a compact 3-manifold with incompressible aspherical boundary components  $S_1, \dots, S_m$ , then  $M$  is acylindrical iff  $\pi_1(S_i) \cap g(\pi_1(S_j))g^{-1} = \{e\}$  whenever  $i \neq j$  or  $i = j$  but  $g \notin \pi_1(S_i)$ .

**Lemma 8.11.** *Suppose  $G$  is a duality group and  $G \curvearrowright X$  is a free simplicial action on a coarse  $PD(n)$  space, and let  $(G, \{H_j\}_{j \in J})$  be the group pair obtained by applying Theorem 1.1 to this action. Assume that  $H_i \cap (gH_jg^{-1}) = \{e\}$  whenever,  $i \neq j$  or  $i = j$  but  $g \notin H_i$ . Then, no  $H_i$  coarsely separates  $G$ .*

*Proof.* Let  $K_0 \subset X$  be a connected  $G$ -invariant subcomplex so that  $K_0/G$  is compact and all deep components of  $X - K_0$  are stable. Now, enlarge  $K_0$  to a subcomplex  $K \subset X$  by throwing in the shallow (i.e., non-deep) components of  $X - K_0$ ; then  $K$  is connected,  $G$ -invariant,  $K/G$  is compact, and all components of  $X - K$  are deep and stable. Let  $\{C_\alpha\}$  denote the components of  $X - K$ , and let  $C_i$  be a component stabilized by  $H_i$ . We will show that  $\partial C_i$  does not coarsely separate  $K$  in  $X$ . Since  $K \hookrightarrow X$  is a uniform embedding,  $G \curvearrowright K$  is cocompact, and  $H_i \curvearrowright \partial C_i$  is cocompact, this will imply the lemma.

For all components  $C_\alpha$  and all  $R$ , the intersection  $H_i \cap H_\alpha$  acts cocompactly on  $N_R(\partial C_i) \cap \bar{C}_\alpha$ , where  $H_\alpha$  is the stabilizer of  $C_\alpha$ ; when  $\alpha \neq i$  the group  $H_i \cap H_\alpha$  is trivial, so in this case  $\text{Diam}(N_R(\partial C_i) \cap \bar{C}_\alpha) < \infty$ .

For each  $R$ , there are only finitely many  $\alpha$  – modulo  $H_i$  – for which  $N_R(\partial C_i) \cap C_\alpha \neq \emptyset$ , so there is a constant  $D_1 = D_1(R)$  so that if  $\alpha \neq i$ , then  $\text{Diam}(N_R(\partial C_i) \cap C_\alpha) < D_1$ . Each  $\partial C_\alpha$  is connected and 1-ended, so we have an  $R_1 = R_1(R)$  so that if  $\alpha \neq i$ , and  $x, y \in \partial C_\alpha - N_{R_1}(\partial C_i)$ , then  $x$  may be joined to  $y$  by a path in  $\partial C_\alpha - N_{R_1}(\partial C_i)$ .

By Corollary 7.10, there is a function  $R_2 = R_2(R)$  so that if  $x, y \in K - N_{R_2}(\partial C_i)$ , then  $x$  may be joined to  $y$  by a path in  $X - N_{R_2}(\partial C_i)$ .

Pick  $R$ , and let  $R' = R_2(R_1(R))$ . If  $x, y \in K - N_{R'}(\partial C_i)$ , then they are joined by a path  $\alpha_{xy}$  in  $X - N_{R_1(R)}(\partial C_i)$ . For each  $\alpha \neq i$ , the portion of  $\alpha_{xy}$  which enters  $C_\alpha$  may be replaced by a path in  $\partial C_\alpha - N_{R_1}(\partial C_i)$ . So  $x$  may be joined to  $y$  in  $K - N_{R'}(\partial C_i)$ . Thus,  $\partial C_i$  does not coarsely separate  $K$  in  $X$ . q.e.d.

**Lemma 8.12.** *Let  $M$  be a compact 3-manifold with  $\partial M \neq \emptyset$ , with aspherical incompressible non-empty boundary components  $S_1, \dots, S_m$ . Then,  $M$  is acylindrical if and only if  $\pi_1(M)$  is not a surface group and no  $H_i = \pi_1(S_i) \subset \pi_1(M) = G$  coarsely separates  $G$ .*

*Proof.* The implication  $\Rightarrow$  follows from Lemma 8.11. To establish  $\Leftarrow$  assume that  $M$  is not acylindrical. This implies that there exists a non-trivial decomposition of  $\pi_1(M)$  as a graph of groups with a single edge group  $C$  which is a cyclic subgroup of some  $H_i$ . Thus,  $C$  coarsely separates  $G$ . Since  $[G : H_i] = \infty$ , it follows that  $H_i$  coarsely separates  $G$  as well. q.e.d.

**Corollary 8.13.** *Suppose  $G$  is not a  $PD(n - 1)$  group, both  $(G, \{H_i\}_{i \in I})$  and  $(G, \{L_j\}_{j \in J})$  are  $PD(n)$  pairs, no  $H_i$  coarsely separates  $G$ , and each  $L_j$  admits a finite Eilenberg–MacLane space. Then, there is a bijection  $\beta : I \rightarrow J$  so that  $H_i$  is conjugate to  $L_{\beta(i)}$  for all  $i \in I$ . Thus, the peripheral structure of  $G$  in this case is unique.*

*Proof.* Under the above assumptions, the double  $\hat{G}$  of  $G$  with respect to the collection of subgroups  $\{L_j\}_{j \in J}$  admits a finite Eilenberg–MacLane space  $K(\hat{G}, 1)$ . Thus, we can take as a coarse  $PD(n)$ -space  $X$  the universal cover of  $K(\hat{G}, 1)$ . Now, apply Proposition 8.10. q.e.d.

### 9. Applications

In this section, we discuss examples of  $(n - 1)$ -dimensional groups which cannot act on coarse  $PD(n)$  spaces.

**2-dimensional groups with positive Euler characteristic.** Let  $G$  be a group of type  $FP_2$  with cohomological dimension 2. If the  $\chi(G) > 0$ , then  $G$  cannot act freely simplicially on a coarse  $PD(3)$  space. To see this, note that by Mayer–Vietoris some one-ended free factor  $G'$  of  $G$  must have  $\chi(G') > 0$ . If  $G'$  acts on a coarse  $PD(3)$  space, then  $G'$  contains a collection  $\mathcal{H}$  of surface subgroups so that  $(G', \mathcal{H})$  is a  $PD(3)$  pair. Since the double of a  $PD(3)$  pair is a  $PD(3)$  group (which has zero Euler characteristic) by Mayer–Vietoris, we have  $\chi(G') \leq 0$ , which is a contradiction.

We are grateful to the referee for the following remark:

**Remark 9.1.** A generalization of the Chern–Hopf Conjecture asserts that if  $H$  is a  $2n$ -dimensional Poincaré duality group, then  $(-1)^n \chi(H) \geq 0$ . So, if this conjecture is true, then Theorem 1.1 implies that if  $G$  is a  $2n$ -dimensional duality group with  $(-1)^n \chi(G) < 0$ , then  $G$  cannot act freely and simplicially on a coarse  $PD(2n + 1)$  space.

**Bad products.** Suppose  $G = \prod_{i=1}^k G_i$  where each  $G_i$  is a duality group of dimension  $n_i$ , and  $G_1, G_2$  are not Poincaré duality groups. Then,  $G$  cannot act freely simplicially on a coarse  $PD(n)$  space, where  $n - 1 = \sum_{i=1}^k n_i$ .

*Proof.* Let  $G \curvearrowright X$  be a free simplicial action on a coarse  $PD(n)$  space.

*Step 1.*  $G$  contains a  $PD(n - 1)$  subgroup. This follows by applying Theorem 1.1 to  $G \curvearrowright X$ , since otherwise,  $G \curvearrowright X$  is cocompact and Lemma 5.4 would give  $H^n(G; \mathbb{Z}G) \simeq \mathbb{Z}$ , contradicting  $\dim(G) = n - 1$ .

We apply Theorem 1.1 to see that  $G \curvearrowright X$  defines deep complementary component stabilizers  $H_\alpha \subset G$  which are  $PD(n - 1)$  groups.

*Step 2.* Any  $PD(n - 1)$  subgroup  $V \subset G$  virtually splits as a product  $\prod_{i=1}^k V_i$  where  $V_i \subset G_i$  is a  $PD(n_i)$  subgroup. Consequently each  $G_i$  contains a  $PD(n_i)$  subgroup.

**Lemma 9.2.** A  $PD(m)$  subgroup  $V$  of a  $m$ -dimensional product group  $W := \prod_{i=1}^k W_i$  contains a finite index subgroup  $V'$  which splits as a product  $V' = \prod_{i=1}^k V_i$  where  $V_i \subset W_i$  is a Poincaré duality group of dimension  $\dim(W_i)$ .

*Proof.* Look at the kernels of the projections

$$\hat{p}_j : W \rightarrow \prod_{i \neq j} W_i$$

restricted to  $V$ . The dimension of the middle group in a short exact sequence has dimension at most the sum of the dimensions of the other two groups. Applying this to the exact sequence

$$1 \rightarrow W_j \cap V \rightarrow V \rightarrow \hat{p}_j(V) \rightarrow 1,$$

we get that  $W_j \cap V$  has the same dimension as  $W_j$ . Hence,  $\prod_j (W_j \cap V)$  has the same dimension as  $V$ , so it has finite index in  $V$  (see Section 3). Therefore,  $\prod_j (W_j \cap V)$  is a  $PD(n)$  group and so the factor groups  $(W_j \cap V)$  are  $PD(\dim(W_j))$  groups. q.e.d.

*Step 3.* No  $PD(n-1)$  subgroup  $V \subset G$  can coarsely separate  $G$ . This follows immediately from step 2 and:

**Lemma 9.3.** *For  $i = 1, 2$  let  $A_i \subset B_i$  be finitely generated groups, with  $[B_i : A_i] = \infty$ . Then,  $A_1 \times A_2$  does not coarsely separate  $B_1 \times B_2$ .*

*Proof.* Suppose that  $x = (x_1, x_2), y = (y_1, y_2)$  are points in the Cayley graphs of  $B_1, B_2$  which are at distance at least  $R$  from  $A := A_1 \times A_2$ . Without loss of generality, we may assume that  $d(x_1, A_1) \geq R/2$ . We then pick a point  $x'_2 \in B_2$  with distance at least  $R/2$  from  $A_2$  and connect  $x_2$  to  $x'_2$  by a path  $x_2(t)$  in the Cayley graph of  $B_2$ . The path  $(x_1, x_2(t))$  does not intersect  $N_{\frac{R}{2}}(A)$ . Applying similar argument to  $y$ , we reduce the proof to the case where  $d(x_i, A_i) \geq R/2$  and  $d(y_i, A_i) \geq R/2, i = 1, 2$ . Now, connect  $x_1$  to  $y_1$  by a path  $x_1(t)$ , and  $y_2$  to  $x_2$  by a path  $y_2(t)$ ; it is clear that the paths  $(x_1(t), x_2), (y_1, y_2(t))$  do not intersect  $N_{\frac{R}{4}}(A)$ . On the other hand, these paths connect  $x$  to  $(y_1, x_2)$  and  $y$  to  $(y_1, x_2)$ . q.e.d.

*Step 4.* By steps 1 and 2, we know that each  $G_i$  contains a  $PD(n_i)$  subgroup. Let  $L_i \subset G_i$  be a  $PD(n_i)$  subgroup for  $i > 1$ . Set  $L := G_1 \times (\prod_{i=2}^k L_i)$ . Observe that  $L$  is not a  $PD(n-1)$  group since  $G_1$  is not a  $PD(n_1)$  group. Therefore, no finite index subgroup of  $L$  can be a  $PD(n-1)$  subgroup, see Section 3.

*Step 5.* Choose a base-point  $\star \in X$ . We now apply Theorem 8.8 to the action  $L \curvearrowright X$  with  $K := L(\star)$ , and we let  $R_i, C_\alpha, H_\alpha, E_\alpha$ , and  $F_\alpha$  be as in the Theorem 8.8. Since  $L$  has infinite index in  $G$ , the distance function  $d(\partial C_\alpha, \cdot)$  is unbounded on  $G(\star) \cap E_\alpha$  for some  $\alpha \in I$ , while part 2 of Theorem 8.8 implies that  $d(\partial C_\alpha, \cdot)$  is unbounded on  $K \cap F_\alpha$ . Hence,  $H_\alpha$  coarsely separates  $G$ , which contradicts step 3. q.e.d.

**Baumslag–Solitar groups.** Pick  $p \neq \pm q$ , and let  $G := BS(p, q)$  denote the Baumslag–Solitar group with the presentation

$$(9.4) \quad \langle a, b \mid ba^p b^{-1} = a^q \rangle.$$

If  $G_1$  is a  $k$ -dimensional duality group, then the direct product  $G_1 \times G$  does not act freely simplicially on a coarse  $PD(3+k)$  space.

We will prove this when  $G_1 = \{e\}$ . The general case can be proved using straightforward generalization of the argument given below, once one applies the “Bad products” example above to see that  $G_1$  must be a  $PD(k)$  group if  $G_1 \times G$  acts on a coarse  $PD(3+k)$  space. Assume that  $G \curvearrowright X$  is a free simplicial action on a coarse  $PD(3)$  space. Choosing a base-point  $\star \in X$ , we obtain a uniform embedding  $G \rightarrow X$ .

We recall that the presentation (9.4) defines a graph of groups decomposition of  $G$  with one vertex labeled with  $\mathbb{Z}$ , one oriented edge labeled with  $\mathbb{Z}$ , and where the initial and final edge monomorphisms embed the edge group as subgroups of index  $p$  and  $q$  respectively. The Bass–Serre tree  $T$  corresponding to this graph of groups has the following structure. The action  $G \curvearrowright T$  has one vertex orbit and one edge orbit. For each vertex  $v \in T$ , the vertex stabilizer  $G_v$  is isomorphic to  $\mathbb{Z}$ . The vertex  $v$  has  $p$  incoming edges and  $q$  outgoing edges; the incoming (respectively outgoing) edges are cyclically permuted by  $G_v$  with ineffective kernel the subgroup of index  $p$  (respectively  $q$ ).

Let  $\bar{\Sigma}$  be the presentation complex corresponding to the presentation (9.4), and let  $\Sigma$  denote its universal cover. Then,  $\Sigma$  admits a natural  $G$ -equivariant fibration  $\pi : \Sigma \rightarrow T$ , with fibers homeomorphic to  $\mathbb{R}$ . For each vertex  $v \in T$ , the inverse image  $\pi^{-1}(v)$  has a cell structure isomorphic to the usual cell structure on  $\mathbb{R}$ , and  $G_v$  acts freely transitively on the vertices. For each edge  $e \subset T$ , the inverse image  $\pi^{-1}(e) \subset \Sigma$  is homeomorphic to a strip. The cell structure on the strip may be obtained as follows. Take the unit square in  $\mathbb{R}^2$  with the left edge subdivided into  $p$  segments and the right edge subdivided into  $q$  segments; then glue the top edge to the bottom edge by translation and take the induced cell structure on the universal cover. The edge stabilizer  $G_e$  acts simply transitively on the 2-cells of  $\pi^{-1}(e)$ .

We may view  $\Sigma$  as a bounded geometry metric simplicial complex by taking a  $G$ -invariant triangulation of  $\Sigma$ . Given  $k$  distinct ideal boundary points  $\xi_1, \dots, \xi_k \in \partial_\infty T$  and a base-point  $\star \in T$ , we consider the geodesic rays  $\overline{\star\xi_i} \subset T$ , take the disjoint union of their inverse images  $Y_i := \pi^{-1}(\overline{\star\xi_i}) \subset \Sigma$  and glue them together along the copies of  $\pi^{-1}(\star) \subset \pi^{-1}(\overline{\star\xi_i})$ . The resulting complex  $Y$  inherits bounded geometry metric simplicial complex structure from  $\Sigma$ . The reader will verify the following assertions:

1.  $Y$  is uniformly contractible.
2. For  $i \neq j$ , the union  $Y_i \cup Y_j \subset Y$  is uniformly contractible and the inclusion  $Y_i \cup Y_j \rightarrow Y$  is a uniform embedding.
3. The natural map  $Y \rightarrow \Sigma$  is a uniform embedding.
4. The cyclic ordering induced on the  $Y_i$ 's by the uniform embedding which is the composition  $C_*(Y) \rightarrow C_*(\Sigma) \rightarrow C_*(X)$  (see Lemma 7.11) defines a continuous  $G$ -invariant cyclic ordering on  $\partial_\infty T$ .

Let  $a$  be the generator of  $G_v$  for some  $v \in T$ . Setting  $e_k := (pq)^k$ , the sequence  $g_k := a^{e_k}$  – viewed as elements in  $\text{Isom}(T)$  – converges to the identity as  $k \rightarrow \infty$ . So, the sequence of induced homeomorphisms of the ideal boundary of  $T$  converges to the identity. The invariance of the cyclic ordering clearly implies that  $g_k$  acts trivially on the ideal boundary of  $T$  for large  $k$ . This implies that  $g_k$  acts trivially on  $T$  for large  $k$ . Since this is absurd,  $G$  cannot act discretely and simplicially on a coarse  $PD(3)$  space.

**Remark 9.5.** The complex  $\Sigma$  – and hence  $BS(p, q)$  – can be uniformly embedded in a coarse  $PD(3)$  space homeomorphic to  $\mathbb{R}^3$ . To see this, we proceed as follows. First take a proper PL embedding  $T \rightarrow \mathbb{R}^2$  of the Bass–Serre tree into  $\mathbb{R}^2$ . For each co-oriented edge  $\vec{e}$  of  $T \subset \mathbb{R}^2$ , we take product cell structure on the half-slab  $P(\vec{e}) := \pi^{-1}(e) \times \mathbb{R}_+$  where  $\mathbb{R}_+$  is given the usual cell structure. We now perform two types of gluings. First, for each co-oriented edge  $\vec{e}$ , we glue the half-slab  $P(\vec{e})$  to  $\Sigma$  by identifying  $\pi^{-1}(e) \times 0$  with  $\pi^{-1}(e) \subset \Sigma$ . Now, for each pair  $\vec{e}_1, \vec{e}_2$  of adjacent co-oriented edges, we glue  $P(\vec{e}_1)$  to  $P(\vec{e}_2)$  along  $\pi^{-1}(v) \times \mathbb{R}_+$  where  $v = e_1 \cap e_2$ . It is easy to see that after suitable subdivision, the resulting complex  $X$  becomes a bounded geometry, uniformly acyclic 3-dimensional PL manifold homeomorphic to  $\mathbb{R}^3$ .

**Higher genus Baumslag–Solitar groups.** Note that  $BS(p, q)$  is the fundamental group of the following complex  $K = K_1(p, q)$ . Take the annulus  $A$  with the boundary circles  $C_1, C_2$ . Let  $B$  be another annulus with the boundary circles  $C'_1, C'_2$ . Map  $C'_1, C'_2$  to  $C_1, C_2$  by mappings  $f_1, f_2$  of degrees  $p$  and  $q$  respectively. Then,  $K$  is obtained by gluing  $A$  and  $B$  by  $f_1 \sqcup f_2$ . Below, we describe a “higher genus” generalization of this construction. Instead of the annulus  $A$ , take a surface  $S$  of genus  $g \geq 1$  with two boundary circles  $C_1, C_2$ . Then, repeat the above construction of  $K$  by gluing the annulus  $B$  to  $S$  via the mappings  $C'_1 \rightarrow C_1, C'_2 \rightarrow C_2$  of the degrees  $p, q$  respectively. The fundamental group  $G = G_g(p, q)$  of the resulting complex  $K_g(p, q)$  has the presentation

$$\langle a_1, b_1, \dots, a_g, b_g, c_1, c_2, t : [a_1, b_1] \cdots [a_g, b_g] c_1 c_2 = 1, t c_2^q t^{-1} = c_1^p \rangle.$$

One can show that the group  $G_g(p, q)$  is torsion-free and Gromov-hyperbolic [28]. Note that the universal cover  $\tilde{K}$  of the complex  $K_g(p, q)$  does not fiber over the Bass–Serre tree  $T$  of the HNN-decomposition of  $G$ . Nevertheless there is a properly embedded  $c_1$ -invariant subcomplex in  $\tilde{K}$  which ( $c_1$ -invariantly) fibers over  $T$  with the fiber homeomorphic to  $\mathbb{R}$ . This allows one to repeat the arguments given above for the group  $BS(p, q)$  and show that the group  $G_g(p, q)$  cannot act simplicially freely on a coarse  $PD(3)$  space (unless  $p = \pm q$ ). However, in [28], we show that  $G_g(p, q)$  contains a finite index subgroup isomorphic to the fundamental group of a compact 3-manifold with boundary.

**Groups with too many coarsely non-separating Poincaré duality subgroups.** By Corollary 7.14, if  $G$  is of type  $FP$ , and  $G \curvearrowright X$  is a free simplicial action on a coarse  $PD(n)$  space, then there are only finitely many conjugacy classes of coarsely non-separating maximal  $PD(n - 1)$  subgroups in  $G$ .

We now construct an example of a 2-dimensional group of type  $FP$  which has infinitely many conjugacy classes of coarsely non-separating maximal surface subgroups; this example does not fit into any of the classes described above. Let  $S$  be a 2-torus with one hole, and let  $\{a, b\} \subset H_1(S)$  be a set of generators. Consider a sequence of embedded loops  $\gamma_k \subset S$  which represent  $a + kb \in H_1(S)$ , for  $k = 0, 1, \dots$ . Let  $\Sigma$  be a 2-torus with two holes. Glue the boundary torus of  $S \times S^1$  homeomorphically to one of the boundary tori of  $\Sigma \times S^1$  so that the resulting manifold  $M$  is not Seifert fibered. Consider the sequence  $T_k \subset M$  of embedded incompressible tori corresponding to  $\gamma_k \times S^1 \subset S \times S^1 \subset M$ . Let  $L \subset \pi_1(M)$  be the infinite cyclic subgroup generated by the homotopy class of  $\gamma_0$ . Finally, we let  $G$  be the double of  $\pi_1(M)$  over the cyclic subgroup  $L$ , i.e.,  $G := \pi_1(M) *_L \pi_1(M)$ . Then, the reader may verify the following:

1. Let  $H_i \subset \pi_1(M) \subset G$  be the image of the fundamental group of the torus  $T_i$  for  $i > 0$  (which is well-defined up to conjugacy). Then, each  $H_i$  is maximal in  $G$ , and the  $H_i$ 's are pairwise non-conjugate in  $G$ .
2. Each  $H_i \subset \pi_1(M)$  coarsely separates  $\pi_1(M)$  into precisely two deep components.
3. For each  $i > 0$ , the subgroup  $H_i \subset \pi_1(M)$  coarsely separates some conjugate of  $L$  in  $\pi_1(M)$ .
4. It follows from 3 that  $H_i$  is coarsely non-separating in  $G$  for  $i > 0$ .
5.  $G$  is of type  $FP$  and has dimension 2.

Therefore,  $G$  cannot act freely simplicially on a coarse  $PD(3)$  space.



**10. Appendix: Coarse Alexander duality in brief**

We will use terminology and notation from Section 2.

**Theorem 10.1.** *Let  $X$  and  $Y$  be bounded geometry uniformly acyclic metric simplicial complexes, where  $X$  is an  $n$ -dimensional PL manifold. Let  $f : C_*(Y) \rightarrow C_*(X)$  be a chain map which is a uniform embedding, and let  $K \subset X$  be the support of  $f(C_*(Y)) \subset C_*(X)$ . For every  $R$ , we may compose the Alexander duality isomorphism A.D. with the induced map on compactly supported cohomology:*

$$(10.2) \quad \tilde{H}_{n-k-1}(X \setminus N_R(K)) \xrightarrow{A.D.} H_c^k(N_R(K)) \xrightarrow{H_c^k(f)} H_c^k(Y);$$

we call this composition  $A_R$ . Then,

1. For every  $R$ , there is an  $R'$  so that
- $$(10.3) \quad Ker(A_{R'}) \subset Ker(\tilde{H}_{n-k-1}(X - N_{R'}(K)) \rightarrow \tilde{H}_{n-k-1}(X \setminus N_R(K))).$$
2.  $A_R$  is an epimorphism for all  $R \geq 0$ .
  3. All deep components of  $X \setminus K$  are stable; their number is  $1 + rank(H_c^{n-1}(Y))$ .
  4. If  $Y$  is an  $(n - 1)$ -dimensional manifold, then for all  $R$ , there is a  $D$  so that any point in  $N_R(K)$  lies within distance  $D$  of both the deep components of  $X - N_R(K)$ .

The functions  $R' = R'(R)$  and  $D = D(R)$  depend only on the geometry of  $X$  and  $Y$  (via their dimensions and acyclicity functions), and on the coarse Lipschitz constant and distortion of  $f$ .

*Proof.*

*Step 1.* We construct a coarse Lipschitz chain map  $g : C_*(X) \rightarrow C_*(Y)$  as follows. For each vertex  $x \in X, y \in Y$ , we let  $[x], [y]$  denote the corresponding element of  $C_0(X), C_0(Y)$ . To define  $g_0 : C_0(X) \rightarrow C_0(Y)$ , we map  $[x]$  for each vertex  $x \in X \subset C_0(X)$  to  $[y]$ , where we choose a vertex  $y \in Y \subset C_0(Y)$  for which the distance  $d(x, \text{Support}(f(y)))$  is minimal, and extend this homomorphism  $\mathbb{Z}$ -linearly to a map  $C_0(X) \rightarrow C_0(Y)$ . Now, assume inductively that  $g_j : C_j(X) \rightarrow C_j(Y)$  has been defined by  $j < i$ . For each  $i$ -simplex  $\sigma \in C_i(X)$ , we define  $g_i(\sigma)$  to be a chain bounded by  $g_{i-1}(\partial\sigma)$  (where  $\text{Support}(g_i(\sigma))$  lies inside the ball supplied by the acyclicity function of  $Y$ ). Using a similar inductive procedure to construct chain homotopies, one verifies:

- a) For every  $R$ , there is an  $R'$  so that the composition

$$(10.4) \quad C_*(N_R(K)) \xrightarrow{g_*} C_*(Y) \rightarrow C_*(K) \rightarrow C_*(N_{R'}(K))$$

is chain homotopic to the inclusion by an  $R'$ -Lipschitz chain homotopy with displacement  $< R'$ .

b) There is a  $D$  so that

$$C_*(Y) \xrightarrow{f} C_*(K) \xrightarrow{g} C_*(Y)$$

is a chain map with displacement at most  $D$  and  $g \circ f$  is chain homotopic to  $id_{C_*(Y)}$  by a  $D$ -Lipschitz chain map with displacement  $< D$ .

*Step 2.* Pick  $R$ , and let  $R'$  be as in (a) above. If

$$\alpha \in \text{Ker}(H_c^k(N_{R'}(K)) \xrightarrow{H_c^k(f)} H_c^k(Y)),$$

then,  $\alpha$  is in the kernel of the composition

$$H_c^k(N_{R'}(K)) \xrightarrow{H_c^k(f)} H_c^k(Y) \xrightarrow{H_c^k(g)} H_c^k(N_R(K))$$

which coincides with the restriction  $H_c^k(N_{R'}(K)) \rightarrow H_c^k(N_R(K))$  by (a) above. Similarly, the composition

$$H_c^k(Y) \xrightarrow{H_c^k(g)} H_c^k(N_R(K)) \xrightarrow{H_c^k(f)} H_c^k(Y)$$

is the identity, so  $H_c^k(f)$  is an epimorphism. Applying the Alexander duality isomorphism to these two assertions, we get parts 1 and 2.

*Step 3.* Let  $C$  be a deep component of  $X - K$ . Suppose  $C_1, C_2$  are deep components of  $X - N_R(K)$  with  $C_i \subset C$ . Picking points  $x_i \in C_i$ , the difference  $[x_1] - [x_2]$  determines an element of  $\tilde{H}_0(X - N_R(K))$  lying in  $\text{Ker}(\tilde{H}_0(X - N_R(K)) \rightarrow \tilde{H}_0(X - K))$ . Hence,

$$A_R([x_1] - [x_2]) = A_0(p_R([x_1] - [x_2])) = A_0(0) = 0$$

where  $p_R : \tilde{H}_0(X - N_R(K)) \rightarrow \tilde{H}_0(X - K)$  is the projection. Since  $C_1$  and  $C_2$  are deep, for any  $R' \geq R$ , there is a  $c \in \tilde{H}_0(X - N_{R'}(K))$  which projects to  $[x_1] - [x_2] \in \tilde{H}_0(X - N_R(K))$ . But then  $A_{R'}(c) = 0$  and part 1 forces  $[x_1] - [x_2] = 0$ . This proves that  $C_1 = C_2$ , and hence that all deep components of  $X - K$  are stable. The number of deep components of  $X - K$  is

$$1 + \text{rank} \left( \varprojlim_R \tilde{H}_0(X - N_R(K)) \right),$$

and by part 1, this clearly coincides with  $1 + \text{rank}(H_c^{n-1}(Y))$ . Thus, we have proved 2.

*Step 4.* To prove part 4, we let  $C_1, C_2$  be the two deep components of  $X - K$  guaranteed to exist by part 3. Pick  $x \in N_R(K)$ , and let  $R'$  be as in part 1. Since  $f$  is coarse Lipschitz chain map, there is a  $y \in Y$  with  $d(x, \text{Support}(f([y]))) < D_1$  where  $D_1$  is independent of  $x$  (but does depend on  $R$ ). Choose a cocycle  $\alpha \in C_c^{n-1}(Y)$  representing the generator of  $H_c^{n-1}(Y)$  which is supported in an  $(n - 1)$ -simplex containing

$y$ . Then, the image  $\alpha'$  of  $\alpha$  under  $C_c^{n-1}(Y) \xrightarrow{C_c^{n-1}(g)} C_c^{n-1}(N_{R'}(K))$  is a cocycle supported in  $B(x, D_2) \cap N_{R'}(K)$  where  $D_2$  depends on  $R'$ , but is independent of  $x$ . Applying the Alexander duality isomorphism to  $[\alpha'] \in H_c^{n-1}(N_{R'}(K))$ , we get an element  $c \in \tilde{C}_0(X - N_{R'}(K))$  which is supported in  $B(x, D_2 + 1) \cap (X - N_{R'}(K))$ , and which maps under  $A_{R'}$  to  $[\alpha] \in H_c^{n-1}(Y)$ .

**Remark.** The above Alexander duality isomorphism is ultimately induced by taking the cap product with the fundamental class of  $H_n^{lf}(X)$ , the locally finite homology group of  $X$ .

Picking  $x_i \in C_i$  far from  $K$ , we have  $[x_1] - [x_2] \in \tilde{H}_0(X - N_{R'}(K))$  and  $A_{R'}([x_1] - [x_2]) = \pm[\alpha]$ . By part 1, it follows that the images of  $c$  and  $[x_1] - [x_2]$  under the map  $\tilde{H}_0(X - N_{R'}(K)) \rightarrow \tilde{H}_0(X - N_R(K))$  coincide up to sign. In other words,  $\text{support}(c) \cap C_i \neq \emptyset$ , so we have shown that  $d(x, C_i) < D_2$  for each  $i = 1, 2$ . q.e.d.

### 11. Appendix: Metric complexes

In this section, we discuss the definition of metric complexes, and explain how one can modify statements and proofs from the rest of the paper so that they work with metric complexes rather than metric simplicial complexes.

We have several reasons for working with objects more general than metric simplicial complexes. First of all, Poincaré duality groups are not known to act freely cocompactly on acyclic simplicial complexes (or even on simplicial complexes that are acyclic through dimension  $n + 1$ ). Second, many maps arising in our arguments (e.g., retraction maps and chain maps associated with a uniform embedding) are chain mappings which are not realizable using  $PL$  maps. Also, one would like to have natural constructions like mapping cylinders for chain mappings of geometric origin.

#### 11.1. Metric complexes.

**Definition 11.1.** A metric space  $X$  has bounded geometry if there is a constant  $a > 0$  such that for every  $x, x' \in X$ , we have  $d(x, x') > a$ , and for every  $R \geq 0$ , every  $R$ -ball contains at most  $N = N(R)$  points.

We observe that this definition relates to the usual notion of a *Riemannian manifold* of bounded geometry as follows. Recall that a complete Riemannian manifold is said to have *bounded geometry* if its injectivity radius is bounded away from zero and the sectional curvature

is bounded both from above and from below. For  $0 < r < \infty$ , pick a maximal  $r$ -net  $X \subset M$  in such a manifold and consider  $X$  as a metric space with the metric induced from  $M$ . Then, the metric space  $X$  has bounded geometry in the sense of the above definition.

In the remainder of this section,  $X$  and  $X'$  will denote bounded geometry metric spaces.

A *free module over  $X$*  is a triple  $(M, \Sigma, p)$  where  $M$  is the free  $\mathbb{Z}$ -module with basis  $\Sigma$ , and  $\Sigma \xrightarrow{p} X$  is a map. (This definition can be generalized to the category of projective modules  $M$  over  $X$  by considering the pair  $(M, \text{supp})$  where  $\text{supp} : M \rightarrow (\text{bounded subsets of } X)$  is the *support* map for the elements  $m \in P$ .) We will refer to the space  $X$  as the *control space*, and  $p$  as the *projection map*. A free module over  $X$  has *finite type* if  $\#p^{-1}(x)$  is uniformly bounded independent of  $x \in X$ . We will often suppress the basis  $\Sigma$  and the projection  $p$  in our notation for free modules over  $X$ . A  *$D$ -morphism* from a free module  $(M, \Sigma, p)$  over  $X$  to a free module  $(M', \Sigma', p')$  over  $X'$  is a pair  $(f, \hat{f})$  where  $f : X \rightarrow X'$  is a map,  $\hat{f} : M \rightarrow M'$  is module homomorphism such that for all  $\sigma \in \Sigma$ ,  $\hat{f}(\sigma) \in \text{span}((p')^{-1}(B(f(p(\sigma)), D)))$ . A morphism  $(f, \hat{f})$  is *coarse Lipschitz* (resp. a *uniform embedding*) if the map of control spaces  $f$  is coarse Lipschitz (resp. a uniform embedding). When  $X = X'$ , we say that  $(f, \hat{f})$  has displacement (at most)  $D$  if  $f = \text{id}_X$  and  $(f, \hat{f})$  defines a  $D$ -morphism.

A *chain complex over  $X$*  is a chain complex  $C_*$  where each  $C_i$  is a free module over  $X$ , and the boundary operators  $\partial_i : C_i \rightarrow C_{i-1}$  have bounded displacement (depending on  $i$ ). A chain map (resp. chain homotopy) between a chain complex  $C_*$  over  $X$  and a chain complex  $C'_*$  over  $X'$  is a chain map (resp. chain homotopy)  $C_* \rightarrow C'_*$  which induces bounded displacement morphisms  $C_i \rightarrow C'_i$  (resp.  $C_i \rightarrow C'_{(i+1)'}$ ) for each  $i$ . Note that any chain complex over  $X$  has a natural augmentation  $\epsilon : C_0 \rightarrow \mathbb{Z}$  which maps each element of  $\Sigma_0$  to  $1 \in \mathbb{Z}$ . A *metric complex* is a pair  $(X, C_*)$  where

1.  $X$  is a bounded geometry metric space and  $C_*$  is a chain complex over  $X$ .
2. Each  $(C_i, \Sigma_i, p_i)$  is a free module over  $X$  of finite type.
3. The projection map  $p_0$  is onto.

The space  $X$  is called the *control space* of the metric complex  $(X, C_*)$ .

**Example 11.2.** If  $Y$  is a metric simplicial complex, we may define two closely related metric complexes:

1. Let  $X$  be the zero-skeleton of  $Y$ , equipped with the induced metric. We orient each simplex in  $Y$ , and let  $C_*$  be the simplicial chain

complex, where the basis  $\Sigma_i$  is just the collection of oriented  $i$ -simplices. We then define the projection  $p_i : \Sigma_i \rightarrow X$  by setting  $p_i(\sigma)$  equal to some vertex of  $\sigma$ , for each  $\sigma \in \Sigma_i$ .

2. Let  $X'$  be the zero-skeleton of the first barycentric subdivision  $Sd(Y)$ , equipped with the induced metric. We consider the subcomplex of the singular chain complex of  $Y$  generated by the singular simplices of the form  $\sigma : \Delta_k \rightarrow Y$  where  $\sigma$  is an affine isomorphism from the standard  $k$ -simplex to a  $k$ -simplex in  $Y$ ; these maps form the basis  $\Sigma'_k$  for  $C'_k$ , and we define  $p' : \Sigma_* \rightarrow X$  by projecting each  $\sigma \in \Sigma_*$  to its barycenter.

If  $C_*$  is a chain complex over  $X$ , and  $W \subset C_*$ , then the *support of  $W$* ,  $\text{supp}(W)$ , is the image under  $p$  of the smallest subset of  $\Sigma_*$  whose span contains  $W$ .

If  $K \subset X$ , we define the *(sub)complex over  $K$* , denoted  $C[K]$ , to be the metric subcomplex  $(K, C'_*)$  where the basis  $\Sigma'_*$  for the chain complex  $C'_*$  is the largest subset of  $\Sigma_*$  such that  $p(\Sigma'_*) \subset K$  and  $\text{span}(\Sigma'_*)$  is a subcomplex of the chain complex  $C_*$ . In other words, the triple  $(C'_i, \Sigma'_i, p'_i)$  can be described inductively as follows. Start with  $\Sigma'_0 = p_0^{-1}(K)$ , and inductively, let

$$\Sigma'_i := \{\sigma \in \Sigma_i \mid p_i(\sigma) \in K \text{ and } \partial_i(\sigma) \in C'_{i-1}\}.$$

By abusing notation, we shall refer to the homology groups  $H_*(C_*[K])$  (resp. compactly supported cohomology groups) as the *homology* (resp. compactly supported cohomology) of  $K$ .

If  $L \subset X$ , then  $[C_*(L)]_k$ , the “ $k$ -skeleton of  $C_*$  over  $L$ ”, is defined as the  $k$ -truncation of  $C_*[L]$ :

$$C_0[L] \leftarrow C_1[L] \leftarrow \dots \leftarrow C_k[L].$$

If  $(X, C_*)$  is a metric complex,  $K \subset X$ , then we have a chain complex  $C_*[X, K]$  (and hence homology groups  $H_*[X, K]$ ) for the pair  $[X, K]$  defined by the formula  $C_*[X, K] := C_*[X]/C_*[K]$ . Likewise, we may define the cochain complexes

$$C^*[X, K] := \text{Hom}(C_*[X, K], \mathbb{Z})$$

and cohomology of pairs  $H^*[X, K]$ . The *compactly supported cochain complex*  $C^*_c[X, L]$  of  $[X, L]$  is the direct limit  $\lim H^*[X, X - K]$  where  $K \subset X$  ranges over compact subsets disjoint from  $L$ . The compactly supported cochain complex is clearly isomorphic to the subcomplex of  $C^*[X, L]$  consisting of cochains  $\alpha$  with  $\alpha(\sigma) = 0$  for all, but finitely many  $\sigma \in \Sigma_*$ . The *support* of  $\alpha \in C^*[X]$  is  $\{p_*(\sigma) \mid \sigma \in \Sigma_*, \alpha(\sigma) \neq 0\}$ . Note that there is a constant  $D$  depending on  $k$  such that for all  $\alpha \in C^k[X, L]$ , we have  $\text{Supp}(\alpha) \subset N_D(X - L)$ .

If  $K \subset X$ , we define an equivalence relation on  $p_0^{-1}(K) \subset \Sigma_0$  by saying that  $\sigma \sim \sigma'$  if  $\sigma - \sigma'$  is homologous to zero in  $C_*[K]$ . We call the equivalence classes of the relation the *components* of  $K$ . By abusing notation, we will also refer to the projection of such component to  $X$  is called a “*component*” of  $K$ . Note that uniform 0-acyclicity of  $(X, C_*)$  implies that there exists  $r_0 > 0$  so that for each “component”  $L \subset K$ , there exists a component of  $C_0[N_{r_0}(L)]$  which contains  $C_0[L]$ .

With this in mind, deep components of  $X - K$ , stable deep components and coarse separation in  $X$  are defined as in Section 2. For instance, a component  $L \subset \Sigma_0$  of  $X - K$  is deep if  $p_0(L)$  is not contained in  $N_R(K)$  for any  $R$ .

The *deep homology* classes and *stabilization of the deep homology* of the complement  $X - K$  are defined similarly to the case of metric simplicial complexes.

The relation between the deep components and the deep 0-homology classes is the same as in the case of metric simplicial complexes.

If  $[\sigma] \in H_0^{\text{Deep}}(C_*[X - K])$  and  $\sigma \in \Sigma_0$ , then  $\sigma$  belongs to a deep component of  $X - K$  and this component does not depend on the choice of  $\sigma$  representing  $[\sigma]$ . Vice-versa, if  $L \subset \Sigma_0$  is a deep component of  $X - K$ , then each  $\xi \in \text{Span}(L)$  determines an element of  $H_0^{\text{Deep}}(C_*[X - K])$ .

The deep homology  $H_0^{\text{Deep}}(C_*[X - N_R(K)])$  stabilizes at  $R_0$  iff all deep components of  $X - N_{R_0}(K)$  are stable.

Note also that for each  $k \in \mathbb{Z}_+$ , there exists  $r > 0$  so that the following holds for each  $K \subset X$ :

Suppose that  $L_\alpha \subset X$ ,  $\alpha \in A$ , is a collection of “components” of  $X - K$  so that  $d(L_\alpha, L_\beta) \geq r$  for all  $\alpha \neq \beta$ . Then,

$$[C_*(\cup_{\alpha \in A} L_\alpha)]_k = \oplus_{\alpha \in A} [C_*(L_\alpha)]_k.$$

An action of a group  $G$  on a metric complex  $(X, C_*)$  is a pair  $(\rho, \hat{\rho})$  where  $G \curvearrowright X$  and  $G \curvearrowright \Sigma_*$  are actions,  $\hat{\rho}$  induces an action  $G \curvearrowright C_*$  by chain isomorphisms, and  $p_* : \Sigma_* \rightarrow X$  is  $G$ -equivariant with respect to  $\rho$  and  $\hat{\rho}$ . For many of our results, a more general notion of action (or quasi-action) would suffice here. An action  $G \curvearrowright (X, C_*)$  is *free* (resp. *discrete*, *cocompact*) provided the action  $G \curvearrowright X$  is free (resp. discrete, cocompact). We can identify  $C_c^*[X]$  with  $\text{Hom}_{\mathbb{Z}G}(C_*, \mathbb{Z}G)$  whenever  $G$  acts freely cocompactly on a metric complex  $(X, C_*)$ , [12, Lemma 7.4].

We say that a metric complex  $(X, C_*)$  is *uniformly  $k$ -acyclic* if for each  $R$ , there is an  $R' = R'(R)$  such that for all  $x \in X$ , the inclusion

$$C_*[B(x, R)] \rightarrow C_*[B(x, R')]$$

induces zero in reduced homology  $\tilde{H}_j$  for all  $j = 0 \dots k$ . We say that  $(X, C_*)$  is *uniformly acyclic* if it is uniformly  $k$ -acyclic for every  $k$ . Observe that a group  $G$  acts freely cocompactly on a uniformly  $(k - 1)$ -acyclic metric complex iff it is a group of type  $FP_k$ , and it acts freely cocompactly on a uniformly acyclic metric complex iff it is a group of type  $FP_\infty$ .

The next lemma implies that for uniformly 0-acyclic metric complexes  $(X, C_*)$ , the metric space  $X$  is “uniformly properly equivalent” to a path-metric space.

**Lemma 11.3.** *Suppose  $(X, C_*)$  is a uniformly 0-acyclic metric complex. For any subset  $Y \subset X$  and any  $r > 0$ , let  $G_r(Y)$  be the graph with vertex set  $Y$ , with  $y, y' \in Y$  joined by an edge iff  $d(y, y') < r$ . Let  $d_{G_r} : Y \times Y \rightarrow \mathbb{Z} \cup \{\infty\}$  be the combinatorial distance in  $G_r$  (the distance between points in the distinct components of  $G_r$  is infinite). Then, the following hold:*

1. *Let  $r_0$  be the displacement of  $\partial_1 : (C_1, \Sigma_1, p_1) \rightarrow (C_0, \Sigma_0, p_0)$ . If  $r \geq r_0$ , then  $(X, d_{G_r}) \xrightarrow{id_X} (X, d)$  is a uniform embedding (here,  $G_r = G_r(X)$ ). In particular,  $d_{G_r}(x, x') < \infty$  for all  $x, x' \in X$ .*
2. *For all  $R$ , there exists  $R' = R'(R)$  such that if  $\sigma, \sigma' \in \Sigma_0$ ,  $d(p_0(\sigma), p_0(\sigma')) \leq R$  and  $K \subset X$ , then either  $\sigma$  and  $\sigma'$  belong to the same component of  $X - K$ , or  $d(p_0(\sigma), K) < R'$  and  $d(p_0(\sigma'), K) < R'$ .*

*Proof.* Pick  $r \geq r_0$ . To prove 1, it suffices to show that for all  $R$ , there is an  $N$  such that if  $d(x, x') < R$ , then  $d_{G_r}(x, x') < N$ .

Pick  $R$  and  $x, x' \in X$  with  $d(x, x') < R$ . Choose  $\sigma \in p_0^{-1}(x)$  and  $\sigma' \in p_0^{-1}(x')$ . By the uniform 0-acyclicity of  $X$ , there is an  $R' = R'(R)$  such that  $\sigma - \sigma'$  represents zero in  $H_0[B(x, R')]$ . So

$$\sigma - \sigma' = \sum a_i \tau_i$$

where  $\tau_i \in p_1^{-1}(B(x, R'))$  and  $\partial \tau_i \in C_0[B(x, R')]$  for all  $i$ . Let  $Z \subset X$  be the set of vertices lying in the same component of  $G_r(B(x, R'))$  as  $x$ . Then,

$$\sum_{\tau_i \in p_1^{-1}(Z)} a_i \partial_1 \tau_i$$

has augmentation zero, forcing  $\sigma' \in p_0^{-1}(Z)$ . It follows that  $d_{G_r}(x, x') \leq \#B(x, R') \leq N = N(R)$ .

Part 2 follows immediately from the uniform 0-acyclicity of  $X$ . q.e.d.

Recall that if  $X$  is a metric space and  $d \in [0, \infty)$ , the *Rips complex*  $\text{Rips}_D(X)$  is defined as follows: The vertices of  $\text{Rips}_D(X)$  are points in

$X$ . Distinct points  $x_0, x_1, \dots, x_n \in X$  span an  $n$ -simplex in  $\text{Rips}_D(X)$  if

$$d(x_i, x_j) \leq D, \quad \forall 0 \leq i, j \leq n.$$

Note that  $\text{Rips}_0(X) = X$ . Then, for  $r \leq R$ , we have a natural embeddings

$$\text{Rips}_r(X) \rightarrow \text{Rips}_R(X).$$

We metrize each connected component of  $\text{Rips}_D(X)$  by using the path metric so that each simplex is isometric to the regular Euclidean simplex with edges of the unit length.

Suppose that  $X$  is a bounded geometry metric space, consider the sequence of Rips complexes

$$X \rightarrow \text{Rips}_1(X) \rightarrow \text{Rips}_2(X) \rightarrow \text{Rips}_3(X) \rightarrow \dots$$

of  $X$ . Then arguing analogously to the proof of Lemma 5.10, one proves

**Proposition 11.4.**  *$X$  is the control space of a uniformly acyclic complex  $C_*$  iff the sequence of Rips complexes  $\text{Rips}_j(X)$  is uniformly pro-acyclic.*

Using the above definitions, one can translate the results from Sections 2 and 5 into the language of metric complexes by

1. Replacing metric simplicial complexes  $X$  with metric complexes  $(X, C_*)$ .
2. Replacing simplicial subcomplexes  $K \subseteq X$  with subsets of the control space  $X$ .
3. Replacing tubular neighborhoods  $N_R(K)$  of simplicial subcomplexes of metric simplicial complexes with metric  $R$ -neighborhoods  $N_R(K)$  of subsets  $K$  of the control space  $X$ .
4. Replacing the simplicial chain complex  $C_*(K)$  (resp.  $C_c^*(K)$ ) with  $C_*[K]$  (resp.  $C_c^*[K]$ ), and likewise for homology and compactly supported cohomology.
5. Replacing coarse Lipschitz and uniform embeddings (resp. chain maps, chain homotopies) with coarse Lipschitz and uniform embeddings (resp. chain maps, chain homotopies) of metric complexes.

**11.2. Coarse  $PD(n)$  spaces.** A coarse  $PD(n)$  space is a uniformly acyclic metric complex  $(X, C_*)$  equipped with chain maps

$$(X, C_c^*) \xrightarrow{P} (X, C_{n-*}) \text{ and } (X, C_*) \xrightarrow{\bar{P}} (X, C_c^{n-*})$$

over  $id_X$ , and chain homotopies  $\bar{P} \circ P \stackrel{\Phi}{\sim} id$  and  $P \circ \bar{P} \stackrel{\bar{\Phi}}{\sim} id$  over  $id_X$ .

As with metric simplicial complexes, we will assume implicitly that any group action  $G \curvearrowright (X, C_*)$  on a coarse  $PD(n)$  space commutes with  $P, \bar{P}, \Phi$ , and  $\bar{\Phi}$ .



**Remark 11.5.** Most of the results only require actions to commute with the operators  $P$  and  $\bar{P}$  up to chain homotopies with bounded displacement (in each dimension).

It follows from our assumptions that if  $G \curvearrowright (X, C_*)$  is a free action on a coarse  $PD(n)$  space, then the cohomological dimension of  $G$  is  $\leq n$ : for any  $\mathbb{Z}G$ -module  $M$ , we may compute  $H^*(G; M)$  using the cochain complex  $\text{Hom}_{\mathbb{Z}G}(C_*, M)$  which is  $\mathbb{Z}G$ -chain homotopy equivalent to the complex  $\text{Hom}_{\mathbb{Z}G}(C_c^{n-*}, M)$ , which vanishes in dimensions  $> n$ .

**Example 11.6.** Suppose  $G$  is a  $PD(n)$  group. Then (see [12]), there is a resolution

$$0 \leftarrow \mathbb{Z} \leftarrow A_0 \leftarrow A_1 \leftarrow \dots$$

of  $\mathbb{Z}$  by finitely generated free  $\mathbb{Z}G$ -modules,  $\mathbb{Z}G$ -chain mappings

$$A_* \xrightarrow{\bar{P}} \text{Hom}_{\mathbb{Z}G}(A_{n-*}, \mathbb{Z}G)$$

and  $\text{Hom}_{\mathbb{Z}G}(A_{n-*}, \mathbb{Z}G) \xrightarrow{P} A_*$ , and  $\mathbb{Z}G$ -chain homotopies  $P \circ \bar{P} \stackrel{\Phi}{\simeq} id$  and  $\bar{P} \circ P \stackrel{\bar{\Phi}}{\simeq} id$ . For each  $i$ , let  $\bar{\Sigma}_i$  be a free basis for the  $\mathbb{Z}G$ -module  $A_i$ , and let

$$\Sigma_i := \{g\tau \mid g \in G, \tau \in \bar{\Sigma}_i\} \subset A_i.$$

Define a  $G$ -equivariant map  $p_i : \Sigma_i \rightarrow G$  by sending  $g\tau \in \Sigma_i$  to  $g$ , for every  $g \in G, \tau \in \bar{\Sigma}_i$ . Then,  $(A_i, \Sigma_i, p_i)$  is a free module over  $G$  (equipped with a word metric and regarded here as a metric space) for each  $i$ , and the pair  $(G, A_*)$  together with the maps  $P, \bar{P}, \Phi, \bar{\Phi}$  define a coarse  $PD(n)$  space on which  $G$  acts freely cocompactly (recall that  $\text{Hom}_{\mathbb{Z}G}(A_*, \mathbb{Z}G) \simeq A_c^*$ ). Conversely, if  $G \curvearrowright (X, C_*)$  is a free cocompact action of a group  $G$  on a coarse  $PD(n)$  space, then  $G$  is  $FP_\infty$ ,  $c \dim(G) \leq n$  (by the remark above), and the existence of the duality operators implies that  $H^k(G, \mathbb{Z}G) = \{0\}$  for  $k \neq n$  and  $H^n(G, \mathbb{Z}G) \simeq \mathbb{Z}$ ; these conditions imply that  $G$  is a  $PD(n)$  group [12, Theorem 10.1]

**Remark 11.7.** If  $G \curvearrowright X$  is any group acting freely on a coarse  $PD(n)$  space  $(X, C_*)$ , then  $\dim(G) \leq n$ . To prove this, note that we can use the action  $G \curvearrowright C_*$  to compute the cohomology  $H^*(G; M)$  of  $G$ . Then, the  $\mathbb{Z}G$ -chain homotopy equivalence  $C_* \leftrightarrow C_c^*$  implies that  $H^k(G; M) = 0$  for  $k \geq n$ .

The material from Sections 6 and 7 now adapts in a straightforward way to the more general setting of coarse  $PD(n)$ -spaces, with the caveat that the displacement, distortion function, etc, may depend on the dimension (since the chain complexes will be infinite dimensional in general). For instance, we have the coarse Jordan separation theorem

**Theorem 11.8.** *Let  $(X, C_*)$  and  $(X', C_*)$  be coarse  $PD(n)$  and  $PD(n-1)$  spaces respectively, and let  $g : X' \rightarrow X$  be a uniform embedding. Then*

1.  $g(X')$  coarsely separates  $X$  into (exactly) two components.
2. For every  $R$ , each point of  $N_R(g(X'))$  lies within uniform distance from each of the deep components of  $Y_R := \overline{X - N_R(g(X'))}$ .
3. If  $Z \subset X'$ ,  $X' \not\subset N_R(Z)$  for any  $R$  and  $h : Z \rightarrow X$  is a uniform embedding, then  $h(Z)$  does not coarsely separate  $X$ . Moreover, for any  $R_0$ , there is an  $R_1 > 0$  depending only on  $R_0$  and the geometry of  $X, X'$ , and  $h$  such that precisely one component of  $X - N_{R_0}(h(Z))$  contains a ball of radius  $R_1$ .

**11.3. The proof of Theorem 1.1.** We now explain how to modify the main argument in Section 8 for metric complexes.

For simplicity, we will assume that  $\Sigma_0 = X$ . One can reduce to this case by replacing the  $X$  with  $\Sigma_0$ , and modifying the projection maps  $p_i$  accordingly (in a  $G$ -equivariant fashion).

The direct translation of the proof using the rules 1–5 above applies until Lemma 8.5. The only part of the lemma that is needed later is part 2, so we explain how to deduce this.

First, note that the system  $\{\tilde{H}_0(Y_{R,\alpha})\}$  is approximately zero as before. Likewise, for every  $k$ , the  $k$ -skeleton of the chain complex  $C_*(Y_R)$  decomposes as a direct sum  $\bigoplus_{\beta} [C_*(Y_{R,\beta})]_k$  for  $R$  sufficiently large, since the distance between the subsets  $Y_{R,\beta}$  for different  $\beta$  tends to infinity as  $R \rightarrow \infty$  by Lemma 11.3. This implies that as before,  $\{H_j(Y_{R,\alpha})\}$  is approximately zero for every  $j$ .

Let

$$r_0 := \text{displacement}(\partial_1 : (C_1, \Sigma_1, p_1) \rightarrow (C_0, \Sigma_0, p_0)).$$

We now claim that for each  $R$ , there is an  $R'$  such that  $N_R(C_\beta)$  is contained in  $C_\beta \cup N_{R'}(K)$ . (Here and below,  $C_\beta \subset X$  are the components of  $X - N_{R_0}(K)$  following the notation of Section 8.) To see this, pick  $x \in C_\beta$ ,  $x' \in X$  with  $d(x, x') \leq R$ , and apply part 1 of Lemma 11.3 to get a sequence  $x = x_1, \dots, x_j = x'$  with  $d(x_i, x_{i+1}) \leq r_0$  and  $j \leq M = M(R)$ . By Lemma 11.3, either  $x_j \in C_\beta$  (and we are done) or there is an  $i$  such that  $d(x_i, N_D(K)) < r = r(r_0)$ . In the latter case, we have  $x' \in N_{r+Mr_0}(K)$ , which proves the claim.

Following the proof of Lemma 8.5, there is an  $R_0$  such that for  $R \geq R_0$ , we have  $Z_{R,\alpha} = N_R(K) \cup (\bigcup_{\beta \neq \alpha} C_\beta)$ . From the claim in the previous paragraph, it now follows that for every  $R \geq R_0$ , there is an  $R'$  such that  $Z_{R,\alpha} \subset N_{R'}(Z_{R_0,\alpha})$  and  $N_R(Z_{R_0,\alpha}) \subset Z_{R',\alpha}$ . Therefore, the homology and compactly supported cohomology of the systems  $\{Z_{R,\alpha}\}$  and  $\{N_R(Z_{R_0,\alpha})\}$  are approximately isomorphic, and similar statements

also apply to the complements of these systems. Part 2 of Lemma 8.5 now follows from coarse Alexander duality.

The only issue in the remainder of the proof that requires different treatment for general metric complexes is the application of Mayer–Vietoris sequences for homology and compactly supported cohomology. If  $(X, C_*)$  is a metric complex, and  $X = A \cup B$ , then the Mayer–Vietoris sequences

$$\begin{aligned} &\rightarrow H_k[A \cap B] \rightarrow H_k[A] \oplus H_k[B] \rightarrow H_k(X) \xrightarrow{\partial} H_{k-1}[A \cap B] \rightarrow \\ &\rightarrow H_c^{k-1}[A \cap B] \xrightarrow{\delta} H_c^k[X] \rightarrow H_c^k[A] \oplus H_c^k[B] \rightarrow H_c^k[A \cap B] \rightarrow \end{aligned}$$

need not be exact in general. By the Barratt–Whitehead Lemma [21, Lemma 7.4], in order for the sequences to be exact through dimension  $k$ , it suffices for the inclusion of pairs  $(B, A \cap B) \rightarrow (X, A)$  to induce isomorphisms in homology and compactly supported cohomology through dimension  $k+2$ . One checks that there is a constant  $r = r(k)$  (depending on the displacements of the boundary operators  $\partial_1, \dots, \partial_{k+1}$ ) such that this will hold provided  $d(A - B, X - A) \geq r$ . So, the proof of Lemma 8.6 goes through provided one chooses the numbers  $R_1 \leq \dots \leq R_M$  to be well enough separated that the Mayer–Vietoris sequences hold through the relevant range of dimensions.

**11.4. Attaching metric complexes.** Suppose that  $Y \subset X$  is a pair of spaces of bounded geometry so that the inclusion  $Y \rightarrow X$  is a uniform embedding.

Let  $P, Q$  be metric complexes over  $X$  and  $Y$  respectively:

$$Q : 0 \leftarrow \mathbb{Z} \leftarrow Q_0 \leftarrow Q_1 \leftarrow \dots \leftarrow Q_n \leftarrow \dots,$$

the complex

$$P : 0 \leftarrow \mathbb{Z} \leftarrow P'_0 \oplus P''_0 \leftarrow P'_1 \oplus P''_1 \dots \leftarrow P'_n \oplus P''_n \leftarrow \dots$$

has the boundary maps  $\partial'_j \oplus \partial''_j : P_j \rightarrow P'_{j-1} \oplus P''_{j-1}$ , where

$$P' : 0 \leftarrow \mathbb{Z} \leftarrow P'_0 \leftarrow P'_1 \dots \leftarrow P'_n \leftarrow \dots$$

is a subcomplex over  $Y$ . Let  $\phi : P' \rightarrow Q, \phi_j : P'_j \rightarrow Q_j, j = 0, 1, \dots$ , be a chain map over  $Y$ , called the “attaching map.” We will define a complex  $R = \text{Att}(P, Q, \phi)$  determined by “attaching”  $P$  to  $Q$  via  $\phi$ ; the complex  $R$  will be a metric complex over  $X$ . This construction is similar to attaching a cell complex  $A$  to a complex  $B$  via an attaching map  $f : C \rightarrow B$ , where  $C$  is a subcomplex of  $A$ .

We let  $R_j := P''_j \oplus Q_j$ , this determines free generators for  $R_j$ ; the boundary map  $\partial_j : R_j \rightarrow R_{j-1} = P''_{j-1} \oplus Q_{j-1}$  is given by

$$\partial_j P'' := \partial'' \oplus (\phi \circ \partial'),$$

the restriction of  $\partial$  to  $Q$  is the boundary map  $\partial^Q$  of the complex  $Q$ . (It is clear that  $\partial \circ \partial = 0$ .) The control maps to  $X$  are defined by restricting the control map for  $P$  to the (free) generators of  $P'_j$  and using the control map of  $Q$  for the (free) generators of  $Q_j$ .

The following lemma is straightforward and is left to the reader.

**Lemma 11.9.** *Suppose that we are given a complex  $P$  over  $X$ , complexes  $Q, T$  over  $Y$ , a chain homotopy-equivalence  $h : Q \rightarrow T$  and attaching maps  $\phi : P' \rightarrow Q, \psi : P' \rightarrow T$  are such that  $\psi = h \circ \phi$ , where all the chain homotopies in question have bounded displacement  $\leq \text{Const}(j)$ . Then, the metric complexes  $\text{Att}(P, Q, \phi), \text{Att}(P, T, \psi)$  are chain homotopy-equivalent with bounds on the displacement of the chain homotopy depending only on  $\text{Const}(j)$ .*

**11.5. Coarse fibrations.** The goal of this section is to define a class of metric spaces  $W$  which are “coarsely fibered” over coarse  $PD(n)$  metric simplicial complexes  $X$  so that the “coarse fibers”  $Y_x$  are control spaces of  $PD(k)$  spaces. We will show that under a mild restriction on the base  $X$  and the fibers  $Y_x$ , the metric space  $W$  is the control space of a coarse  $PD(n+k)$  space.

Suppose that  $X$  is an  $n$ -dimensional metric simplicial complex equipped with an orientation of its 1-skeleton, and  $L, A \in \mathbb{R}$ . Assume that for each vertex  $x \in X^{(0)}$ , we are given a metric space  $Y_x$ , and  $(L, A)$ -quasi-isometries  $f_{pq} : Y_p \rightarrow Y_q$  for each positively oriented edge  $[pq]$  in  $X$ . We will assume that each  $Y_x$  is the control space of a metric complex  $(Y_x, Q_x)$  where the complexes  $Q_x$  are uniformly acyclic (with acyclicity function independent of  $x$ ); in particular, there exists  $C < \infty$  so that the  $C$ -Rips complex of each  $Y_x$  is connected.

**Remark.** For much of what follows, the assumption that the acyclicity function of  $Q_x$  is independent of  $x$  can be relaxed.

It follows that  $f_{pq}$  induce morphisms  $\hat{f}_{pq} : Q_p \rightarrow Q_q$  which are uniform embeddings and uniform chain homotopy-equivalences with the displacements independent of  $p, q$ .

The family of maps  $f_{pq} : Y_p \rightarrow Y_q$  together with the metric on  $X$  determine a metric space  $W = W(X, \{Y_p\}, \{f_{pq}\})$  which “coarsely fibers” over  $X$  with the fibers  $Y_p$ :

As a set,  $W$  is the disjoint union  $\sqcup_{x \in X^{(0)}} Y_x$ . Declare the distance between  $y, f_{pq}(y)$  (for each  $y \in Y_p$ ) equal 1 and then induce the quasi-path metric on  $W$  by considering chains where the distance between the consecutive points is at most  $\max(C, 1)$ . It is clear that  $W$  has bounded geometry.

The reader will verify that the maps  $Y_p \rightarrow W$  are uniform embeddings, where the distortion functions are independent of  $p$ . Let  $\text{proj}_X : W \rightarrow X$  denote the “coarse fibration”;  $\text{proj}_X : Y_x \rightarrow \{x\}$ .

**Example 11.10.** Suppose that we have a short exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

of finitely generated groups where the group  $H$  has type FP. This exact sequence determines a coarse fibration with the total space  $G$ , base  $K$  and fibers  $H \times \{k\}$ ,  $k \in K$ . (Each group is given a word metric.)

**Example 11.11.** The following example appears in [33]. Suppose that, we have a graph of groups  $\Gamma := \{G_v, h_{vw} : E_{e-} \rightarrow E_{e+}\}$ , where  $G_v$  are vertex groups,  $E_{e\pm}$  are the edge subgroups for the edge  $e$ ; we assume that each edge group  $E_{e\pm}$  has type FP and each edge group has finite index in the corresponding vertex group. Let  $G = \pi_1(\Gamma)$  be the fundamental group of this graph of groups,  $L \subset T$  be a geodesic in the tree  $T$  dual to the graph of groups  $\Gamma$ . There is a natural projection  $p : G \rightarrow T$ , let  $W := p^{-1}(L)$ . Then,  $W$  can be described as a coarse fibration whose base consists of the vertices of  $L$  and whose fibers are copies of the edge groups.

Examples of the above type as well as a question of Papasoglou motivate constructions and the main theorem of this section.

Our next goal is to define a metric complex  $R$  with the control space  $W$ . We define the complex  $R$  inductively.

Let  $R^0 := \bigoplus_{x \in X^{(0)}} Q_x$ . The (free) generators of  $R^0$  are the free generators of  $Q_x$ ,  $x \in X^{(0)}$ . Define the control map to  $W$  by sending generators of  $(Q_x)_0$  to the points of  $Y_x$  via the control map for the complex  $Q_x$ .

Orient each edge  $e \subset X^{(1)}$ ,  $e = [e_- e_+]$ . To construct  $R^1$ , first consider the complex  $P^1 := \bigoplus_{e \in X^{(1)}} C_*(e) \otimes Q_{e_-}$ . We have the attaching map  $\phi^1$

$$\phi^1 : \bigoplus_{e \in X^{(1)}} C_*(\partial e) \otimes Q_{e_-} \subset P^1 \rightarrow R^0$$

given by the identity maps

$$C_0(e_-) \otimes Q_{e_-} \rightarrow C_0(e_-) \otimes Q_{e_-} \subset R^0$$

and by

$$C_0(e_+) \otimes Q_{e_-} \rightarrow Q_{e_-} \xrightarrow{\hat{f}_{e_- e_+}} Q_{e_+}.$$

We then define  $R^1$  as  $\text{Att}(P^1, R^0, \phi^1)$  by attaching  $P^1$  to  $R^0$  via  $\phi^1$ , see Section 11.4. Note that  $\text{Att}(C_*(e) \otimes Q_{e_-}, R^0, \phi^1)$  is nothing but the mapping cone of the restriction of  $\phi^1$  to  $C_*(e) \otimes Q_{e_-}$ .

Let  $x_0$  be any point in  $X^{(0)}$ . Then, using uniform acyclicity of  $Q_x$ 's and Lemma 11.9, one constructs (inductively, by attaching one  $C_*(e) \otimes Q_{e_-}$  at a time) a proper chain homotopy-equivalence

$$R^1 \xrightarrow{h} C_*(X^{(1)}) \otimes Q_{x_0} \xrightarrow{\bar{h}} R^1$$

with uniform control of the displacement of  $h, \bar{h}$ ,  $h \circ \bar{h} \cong id$ ,  $\bar{h} \circ h \cong id$  as functions of the distance from  $\text{proj}_X(\text{supp}(\sigma))$  to  $x_0$ . These displacement functions are independent of  $x_0$ .

We continue inductively. Suppose that we have constructed  $R^m$ . We also assume that for each  $x_0 \in X^{(0)}$ , there is a proper chain homotopy-equivalence

$$R^m \xrightarrow{h} C_*(X^{(m)}) \otimes Q_{x_0} \xrightarrow{\bar{h}} R^m$$

with uniform control of the displacement for the chain homotopies  $h \circ \bar{h} \cong id$ ,  $\bar{h} \circ h \cong id$  as functions of the distance from  $\text{proj}_X(\text{supp}(\sigma))$  to  $x_0$ . (Here,  $h = h_{x_0}$ ,  $\bar{h} = \bar{h}_{x_0}$  depend on  $x_0$  and  $m$ .) These displacement functions are independent on  $x_0$ .

For each  $m+1$ -simplex  $\Delta^{m+1}$  in  $X$ , we choose a vertex  $v = v(\Delta^{m+1})$ . We define  $P^{m+1}$  as

$$\bigoplus_{\Delta^{m+1} \in X^{(m+1)}} C_*(\Delta) \otimes Q_{v(\Delta^{m+1})}.$$

Note that we have the maps  $C_*(\partial\Delta) \otimes Q_{v(\Delta^{m+1})} \rightarrow R^m$  constructed using the maps  $\bar{h}_v$ . These maps composed with  $\partial \otimes id$  define the attaching maps

$$\phi^{m+1} : P^{m+1} \rightarrow R^m.$$

Now, we define the complex  $R^{m+1}$  as

$$\text{Att}(P^{m+1}, R^m, \phi^{m+1}).$$

The proper chain homotopy-equivalences

$$R^{m+1} \xrightarrow{h} C_*(X^{(m+1)}) \otimes Q_{x_0} \xrightarrow{\bar{h}} R^{m+1}$$

are constructed using uniform acyclicity of  $Q_x$ 's, the induction hypothesis and Lemma 11.9.

As the result, we get the complex  $R := R^n$  which is a metric complex over  $W$ . We also get the proper chain homotopy-equivalences  $h_v, \bar{h}_v$  between  $R$  and  $C_*(X) \otimes Q_v$  ( $v \in X^{(0)}$ ) with uniform control over the displacement of the chain homotopies  $h_v \circ \bar{h}_v \cong id$ ,  $\bar{h}_v \circ h_v \cong id$  as functions of the distance from  $\text{proj}_X(\text{supp}(\sigma))$  to  $v$ . These functions in turn are independent of  $v$ .

**Lemma 11.12.** *Assume that the complexes  $X$ ,  $\text{Hom}_c(Q_x, \mathbb{Z})$  and  $\text{Hom}_c(C_*(X), \mathbb{Z})$  are uniformly acyclic. Then, the metric chain complexes  $R$  and  $\text{Hom}_c(R, \mathbb{Z})$  are also uniformly acyclic.*

*Proof.* The Künneth formula for  $C_*(X) \otimes Q_v$  implies the acyclicity of the chain and cochain complexes. Uniform estimates follow from uniform control on the chain homotopies  $h_v \circ \bar{h}_v \cong id, \bar{h}_v \circ h_v \cong id$  above. q.e.d.

Recall that if we have an exact sequence of groups

$$1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$$

where  $A$  and  $C$  are  $PD(n)$  and  $PD(k)$  groups respectively, then  $B$  is a  $PD(n+k)$  group. The following is a geometric analogue of this fact.

**Theorem 11.13.** *Assume that  $X$  is an  $n$ -dimensional metric simplicial complex which is a coarse  $PD(n)$ -space and that each  $Q_x$  is a coarse  $PD(k)$  metric complex of dimension  $k$ :*

$$0 \leftarrow \mathbb{Z} \leftarrow Q_{x,0} \leftarrow Q_{x,1} \leftarrow \cdots \leftarrow Q_{x,k} \leftarrow 0.$$

*Then, the metric complex  $R$ , whose control space is the coarse fibration*

$$W = W(X, \{Y_p\}, \{f_{pq}\}),$$

*is a  $PD(n+k)$  metric complex of dimension  $n+k$ .*

*Proof.* By construction, the complex  $R$  has dimension  $n+k$ . The complexes  $X$ ,  $C_c(X, \mathbb{Z})$ ,  $\text{Hom}_c(Q_x, \mathbb{Z})$  are uniformly acyclic. It now follows from Lemma 11.12 and Lemma 6.2 that  $R$  is a coarse  $PD(n+k)$  complex. We note that Lemma 6.2 was stated for metric simplicial complexes. The proof for metric complexes is the same. q.e.d.

**Remark 11.14.** A version of this theorem was proven in [33], where it was assumed that  $X$  is a contractible surface and the fibers  $Y_x$  are  $PD(n)$  groups each of which admits a compact Eilenberg–MacLane space. Under these conditions, Mosher, Sageev and Whyte [33] prove that  $W$  is quasi-isometric to a coarse  $PD(n+k)$  space.

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