J. DIFFERENTIAL GEOMETRY
68 (2004) 553-569

TOWARDS A CLASSIFICATION OF HOMOGENEOUS TUBE DOMAINS IN $\mathbb{C}^4$

Michael Eastwood, Vladimir Ezhov & Alexander Isaev

Abstract

We classify the tube domains in $\mathbb{C}^4$ with affinely homogeneous base whose boundary contains a non-degenerate affinely homogeneous hypersurface. It follows that these domains are holomorphically homogeneous and amongst them, there are four new examples of unbounded homogeneous domains (that do not have bounded realisations). These domains lie to either side of a pair of Levi-indefinite hypersurfaces. Using the geometry of these two hypersurfaces, we find the automorphism groups of the domains.

0. Introduction

The study of holomorphically homogeneous domains in complex space goes back to Cartan [**3**] who determined all bounded symmetric domains in $\mathbb{C}^n$ as well as all bounded homogeneous domains in $\mathbb{C}^2$ and $\mathbb{C}^3$. Due to the fundamental theorem of Vinberg, Gindikin, and Pyatetskii–Shapiro, every bounded homogeneous domain can be realised as a Siegel domain of the second kind (see [**21**]). Although this result does not immediately imply a complete classification of bounded homogeneous domains, it reduces the classification problem to that for domains of a very special form. A generalisation of the above theorem to the case of unbounded domains for the class of rational homogeneous domains was obtained by Penney in his remarkable paper [**19**], where the role of models is played by so-called Siegel domains of type $N-P$. Furthermore, all homogeneous connected complex manifolds of dimension not exceeding 3 have been classified in [**11**, **17**, **23**]. Nevertheless, the classification problem for unbounded domains is far from fully understood.

In [**14**], Loboda shows that any holomorphically homogeneous non-spherical tube hypersurface in $\mathbb{C}^2$ has an affinely homogeneous base.

The first author is supported by the Australian Research Council.
The second author gratefully acknowledges ARC Discovery grant DP0450725.
Received 08/18/2004.

The generalisation of this result to higher dimensions is an open problem. Loboda's result motivates our study. We shall consider tubes over affinely homogeneous bases and suppose (as does Loboda) that the boundary of the base contains an affinely homogeneous hypersurface. In these circumstances, we can classify the resulting tube domains in low dimensions and determine their automorphism groups. The first interesting examples occur in $\mathbb{C}^4$.

A simple natural class of unbounded homogeneous domains comes from generalising the unbounded form of the unit ball

$$B^n := \left\{ z \in \mathbb{C}^n : \operatorname{Re} z_n > |z'|^2 \right\}$$

and its complement $\left\{ z \in \mathbb{C}^n : \operatorname{Re} z_n < |z'|^2 \right\}$, where $z = (z_1, \ldots, z_n)$ are coordinates in $\mathbb{C}^n$ and $z' := (z_1, \ldots, z_{n-1})$. Indeed, the domains lying on either side of the quadric given by the equation

$$\operatorname{Re} z_n = \langle z', z' \rangle, \tag{0.1}$$

where $\langle z', z' \rangle$ is a Hermitian form in $\mathbb{C}^{n-1}$, are easily seen to be homogeneous. Among them, B^n is the only domain, up to a linear change of coordinates, that admits a bounded realisation. Domains of this kind with $\langle z', z' \rangle$ non-degenerate were generalised by Penney, who introduced and studied a special class of Siegel $N - P$ domains called (homogeneous) *nil-balls* (see [**18**, **20**]). A nil-ball is a certain $N - P$ domain on which a solvable algebraic group acts transitively and polynomially by biholomorphic transformations, with smooth connected algebraic Levi non-degenerate boundary on which a codimension 1 nil-radical of the group acts simply transitively.

The examples arising from Hermitian forms, as above, can also be written as tube domains, i.e., domains of the form $D_\Omega := \Omega + i\mathbb{R}^n$, where the base Ω is a domain in $\mathbb{R}^n \subset \mathbb{C}^n$ (see, e.g., [**12**] for an explicit change of coordinates). Tube domains are clearly unbounded and often do not have bounded realisations. If Ω is affinely homogeneous, then D_Ω is holomorphically homogeneous since every affine mapping of $\mathbb{R}^n$ can be lifted to an affine mapping of $\mathbb{C}^n$ and since D_Ω is invariant under translations in imaginary directions. In [**18**], a large class of nil-balls of this kind is introduced.

The new examples of tube domains in $\mathbb{C}^4$ that we give in this paper have affinely homogeneous bases, possess no bounded realisations, and are not equivalent to nil-balls. They arise by the following construction/classification. Start with an affinely homogeneous hypersurface $\Gamma \subset \mathbb{R}^n$. We might find that the orbits of the affine symmetry group of Γ are, in addition to Γ itself, some domain or domains Ω to either side of it. Of course, in this case, the action of the symmetry group on

Γ must have isotropy. A classification of suitable Γ and Ω leads to a classification of corresponding homogeneous tubes D_Ω.

Affinely homogeneous hypersurfaces $\Gamma \subset \mathbb{R}^n$ have been classified for $n = 2$ (see, e.g., [**16**]) and $n = 3$ in [**5**, **6**]. We are interested in hypersurfaces with non-trivial affine isotropy subgroup, and when $n = 4$, such Γ have also been classified (see [**7**]) under the additional hypothesis of non-degeneracy (equivalently, under the hypothesis that $\Gamma + i\mathbb{R}^n$ is Levi non-degenerate). In this paper, we concentrate on the case of non-degenerate Γ for $n = 4$. This leads to a relatively small number of affinely homogeneous domains: most of the automorphism groups from [**7**] do not have open orbits. This, in turn, leads to the following classification in $\mathbb{C}^4$:

Theorem 0.1. *Suppose $D_\Omega = \Omega + i\mathbb{R}^4$ is a tube with affinely homogeneous base, having as part of its boundary an affinely homogeneous non-degenerate hypersurface. Then, up to biholomorphism, D_Ω must be exactly one of the following:*

$$\begin{aligned}
B_+^> &:= \left\{z \in \mathbb{C}^4 : x_4 > x_1^2 + x_2^2 + x_3^2\right\},\\
B_+^< &:= \left\{z \in \mathbb{C}^4 : x_4 < x_1^2 + x_2^2 + x_3^2\right\},\\
B_-^> &:= \left\{z \in \mathbb{C}^4 : x_4 > x_1^2 + x_2^2 - x_3^2\right\},\\
B_-^< &:= \left\{z \in \mathbb{C}^4 : x_4 < x_1^2 + x_2^2 - x_3^2\right\},\\
H^> &:= \left\{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_3^2 \text{ and } x_1 > 0\right\},\\
H^< &:= \left\{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_3^2 \text{ and } x_1 > 0\right\},\\
N_+^> &:= \left\{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_3^2 + x_1^2x_3 + x_1^4\right\},\\
N_+^< &:= \left\{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_3^2 + x_1^2x_3 + x_1^4\right\},\\
N_-^> &:= \left\{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_3^2 + x_1^2x_3 - x_1^4\right\},\\
N_-^< &:= \left\{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_3^2 + x_1^2x_3 - x_1^4\right\},\\
C^> &:= \left\{z \in \mathbb{C}^4 : x_4 > x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\right\},\\
C^< &:= \left\{z \in \mathbb{C}^4 : x_4 < x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\right\},\\
D^> &:= \left\{z \in \mathbb{C}^4 : x_4^2 > x_1x_2 + x_1^2x_3 \text{ and } x_1 > 0\right\},\\
D^< &:= \left\{z \in \mathbb{C}^4 : x_4^2 < x_1x_2 + x_1^2x_3 \text{ and } x_1 > 0\right\},
\end{aligned}$$

where $x_j = \operatorname{Re} z_j$ for $j = 1, \ldots, 4$.

The domain $B_+^>$ is the ordinary unit ball written in tube form. The domains $B_+^<$, $B_-^>$, $B_-^<$ are the tube realisations of domains lying to either side of a quadric (0.1) as mentioned earlier. They are the (pseudo-)

balls and, in the indefinite case, there are also affinely homogeneous domains $H^{>}$ and $H^{<}$, which can be thought of as 'half-pseudo-balls'. The domains $N_{+}^{\geq}, \ldots, N_{-}^{\leq}$ are nil-balls in the sense of Penney [**18**, **20**]. This leaves $C^{>}$, $C^{<}$, $D^{>}$, $D^{<}$, which are new. Remarkably, for each of these domains, the Levi non-degenerate part of the boundary admits an explicit rational transformation to Chern–Moser normal form. In this normal form, the CR isotropy becomes linear. This enables us to determine the full holomorphic automorphism group of the domains in question.

The only pseudo-convex domain in Theorem 0.1 is the unit ball $B_{+}^{\geq}$. Its complement, $B_{+}^{\leq}$ is pseudo-concave. The remaining domains naturally fall into two classes

$$B_{-}^{\geq},\ H^{>},\ N_{+}^{\geq},\ N_{-}^{\geq},\ C^{>},\ D^{>} \quad \text{versus} \quad B_{-}^{\leq},\ H^{<},\ N_{+}^{\leq},\ N_{-}^{\leq},\ C^{<},\ D^{<}$$

according to whether the Levi form of the non-degenerate part of the boundary has type $++-$ or $+--$ (in some cases, there is also a Levi flat part to the boundary).

The paper is organised as follows. In Section 1, we examine the list of [**7**] and use it to prove Theorem 0.1. The new domains $D^{>}$ and $D^{<}$ are the main subject of the remainder of the article. We verify that they do not have bounded realisations. In Section 2, we determine the automorphism groups of these domains by means of exploiting the properties of Γ. These automorphism groups turn out to be Lie groups of dimension 10, and their structure shows that the domains $D^{>}$ and $D^{<}$ are not equivalent to any nil-balls. The new domains $C^{>}$ and $C^{<}$ are similar but slightly easier. In Section 3, we sketch their analysis along the lines already done in Section 2 for $D^{>}$ and $D^{<}$.

1. Classification and new examples

The non-degenerate affinely homogeneous hypersurfaces with isotropy in $\mathbb{R}^4$ are classified in [**7**], where they are grouped according to their complexification into 20 different types, some of which have more than one real form. Sometimes, there is a parameter, as in hypersurface #4:

$$\{x \in \mathbb{R}^4 : x_4 = x_1x_2 + x_3^2 + x_1^2x_3 + \alpha x_1^4\}, \quad \text{for } \alpha \in \mathbb{R},$$

where $x := (x_1, x_2, x_3, x_4)$. There is a list of explicit defining equations in [**7**] and it is an elementary matter to verify that each hypersurface Γ from the list is, indeed, affinely homogeneous with isotropy. This is best done by infinitesimal means: attached to any hypersurface is its symmetry algebra, namely the affine vector fields on $\mathbb{R}^4$ that are tangent to Γ along Γ, and the question is whether this algebra is acting transitively near Γ's chosen basepoint and has dimension at least 4. This is mere computation and in many cases, the symmetry algebra is exactly 4-dimensional. In these cases, we may manufacture a 4×4 matrix from any chosen basis of affine vector fields and compute its determinant: it will be a polynomial on $\mathbb{R}^4$. Where this polynomial does not vanish, the symmetry algebra contains linearly independent vector fields and so there are open orbits. Otherwise, there cannot be open orbits. All of these computations are easily carried out with computer algebra and a Maple program, with the explicit defining equations already included, is available (ftp://ftp.maths.adelaide.edu.au/pure/meastwood/maple7/orbits). It turns out that this determinant is, in many cases, identically zero. Hypersurfaces #7–#20 are thereby eliminated and we are left with (after some minor affine changes of variables from [7]):

	Equation	Basepoint
#1	$x_4 = x_1^2 + x_2^2 \pm x_3^2$	$(0,0,0,0)$
#2	$x_1^2 + x_2^2 + x_3^3 \pm x_4^2 = 1 \quad \text{or} \quad x_1^2 - x_2^2 - x_3^3 \pm x_4^2 = 1$	$(1,0,0,0)$
#3	$x_4 = x_1x_2 + x_3^2 + x_1^3$	$(0,0,0,0)$
#4	$x_4 = x_1x_2 + x_3^2 + x_1^2x_3 + \alpha x_1^4$	$(0,0,0,0)$
#5	$x_4 = x_1x_2 + x_1x_3^2$	$(1,0,0,0)$
#6	$x_4^2 = x_1x_2 + x_1^2x_3$	$(1,0,1,1)$

Proof of Theorem 0.1. We have just identified the only possible affinely homogeneous hypersurfaces satisfying the hypotheses of Theorem 0.1. That is not to say that all these possibilities actually occur and, in fact, #2 does not. The affine symmetry algebra of the sphere, for example, is spanned by the six vector fields $x_j\partial/\partial x_k - x_k\partial/\partial x_j$ for $j < k$ and these fields are nowhere linearly independent. In fact, the associated group is SO(4) with orbits concentric spheres and the origin. A similar conclusion holds for other signatures.

All the remaining cases do, in fact, give rise to affinely homogeneous domains as follows. Case #1, with either sign, has a 7-dimensional symmetry algebra $\mathcal{S}$ and outside of Γ itself, these fields have maximal rank: we obtain $B_+^{\geq}$, $B_+^{\leq}$, $B_-^{\geq}$, $B_-^{\leq}$. There is also the possibility of having a Lie subalgebra $\mathcal{R} \subset \mathcal{S}$ with $\dim \mathcal{R} = 4, 5, 6$ and such that the fields in $\mathcal{R}$ have maximal rank somewhere but not everywhere outside Γ. This possibility is easily investigated with computer algebra. First, one finds the possible subalgebras by parameterising the Grassmannians $\mathrm{Gr}_k(\mathcal{S})$ for $k = 4, 5, 6$ with affine coordinate patches in the usual way, in each case, solving the equations that these subspaces be closed under Lie bracket. Then, one computes the determinants of the 4×4 minors for each of these subalgebras. Up to affine change of coordinates, we obtain just two possible affinely homogeneous 'half-pseudo-balls' $H^>$ and $H^<$, these domains having 5-dimensional affine symmetry.

Case #3 has a 5-dimensional symmetry algebra and the domains to either side of Γ are easily verified to be affinely homogeneous. It is a simple computation to check that all of its 4-dimensional subalgebras consist of fields that are nowhere linearly independent. So, case #3 gives precisely the following two holomorphically homogeneous domains

$$\{x \in \mathbb{R}^4 : x_4 > x_1x_2+x_3^2+x_1^3\}+i\mathbb{R}^4,\ \{x \in \mathbb{R}^4 : x_4 < x_1x_2+x_3^2+x_1^3\}+i\mathbb{R}^4.$$

However, the polynomial transformation

$$z_2 \mapsto z_2 - 3z_1^2/2 \qquad z_4 \mapsto z_4 - z_1^3/2$$

takes these two domains biholomorphically onto domains that we have already listed, namely $B_-^{\geq}$ and $B_-^{\leq}$, respectively.

Case #4 is the subject of [8] and will appear elsewhere. In brief, there are three subcases according to whether $\alpha > 1/12$, $\alpha = 1/12$, or $\alpha < 1/12$. The case $\alpha = 1/12$ again yields $B_-^{\geq}$ and $B_-^{\leq}$ by an explicit polynomial change of coordinates. The equations of the other hypersurfaces may be placed in Chern–Moser normal form [4]:

$$2\mathrm{Im}\, w_4 = w_1\overline{w_2} + w_2\overline{w_1} + |w_3|^2 \pm |w_1|^4,$$

where the sign before $|w_1|^4$ is the sign of $\alpha - 1/12$. This sign is a CR invariant (as observed, for example, in [2]) and we obtain the domains $N_+^>$, $N_+^>$, $N_-^{\geq}$, $N_-^{\leq}$. These were also found by Penney [20] and are examples of his *nil-balls*.

In cases #4, #5, and #6, the symmetry algebra is 4-dimensional and so, there is no possibility of a Lie subalgebra giving rise to another domain. The Maple computer program 'orbits' mentioned early in this section finds the only open orbits and these account for the remaining domains $C^>$, $C^<$, $D^>$, $D^<$ in the statement of Theorem 0.1. That they are holomorphically distinct and not nil-balls is proved in Section 2, concerned with $D^>$ and $D^<$, and Section 3, concerned with $C^>$ and $C^<$. With these matters postponed, the theorem is proved. q.e.d.

We conclude this section with some preliminary discussion of the cases $D^>$ and $D^<$ before a detailed analysis of their holomorphic automorphisms in Section 2. So, consider the following affinely homogeneous hypersurface in $\mathbb{R}^4$:

$$\Gamma := \left\{x \in \mathbb{R}^4 : x_4^2 = x_1x_2 + x_1^2x_3, x_1 > 0\right\},$$

and let

$$\begin{aligned}\Omega^{>} &:= \left\{x \in \mathbb{R}^4 : x_4^2 > x_1x_2 + x_1^2x_3, x_1 > 0\right\},\\ \Omega^{<} &:= \left\{x \in \mathbb{R}^4 : x_4^2 < x_1x_2 + x_1^2x_3, x_1 > 0\right\}.\end{aligned}$$

Clearly, $\Gamma \subset \partial\Omega^{>} \cap \partial\Omega^{<}$. In fact, these domains are not as similar as they might first appear. Part of the boundary of $\Omega^{>}$, namely $\{x \in \mathbb{R}^4 \colon x_1 = 0 \text{ and } x_4 \neq 0\}$, is Levi flat. In the case of $\Omega^{<}$, however, the inequality $x_1 > 0$ serves only to specify one of the two connected components of $\{x \in \mathbb{R}^4 : x_4^2 < x_1x_2 + x_1^2x_3\}$.

One can verify that $\Omega^{>}$ and $\Omega^{<}$ are invariant under the group G that consists of the following affine transformations of $\mathbb{R}^4$

$$\begin{aligned}x_1 &\mapsto qx_1,\\ x_2 &\mapsto qr^2(s+t^2)x_1 + qr^2x_2 + 2qr^2tx_4,\\ x_3 &\mapsto r^2x_3 - r^2s,\\ x_4 &\mapsto qrtx_1 + qrx_4,\end{aligned}$$

where $q > 0$, $r \in \mathbb{R}^*$, $s, t \in \mathbb{R}$. We will now show that G acts transitively on each of $\Omega^{>}$ and $\Omega^{<}$. Take the point $(1, 0, 0, 1) \in \Omega^{>}$ and apply a mapping from G to it. The result is the point

$$\Big(q, qr^2(t^2 + 2t + s), -r^2s, qr(t+1)\Big).$$

Let $(x_1^0, x_2^0, x_3^0, x_4^0)$ be any other point in $\Omega^{>}$. Then, setting

$$q = x_1^0, \qquad r = \frac{\sqrt{(x_4^0)^2 - x_1^0x_2^0 - (x_1^0)^2x_3^0}}{x_1^0},$$

$$t = \frac{x_4^0}{\sqrt{(x_4^0)^2 - x_1^0x_2^0 - (x_1^0)^2x_3^0}} - 1, \quad s = -\frac{(x_1^0)^2x_3^0}{(x_4^0)^2 - x_1^0x_2^0 - (x_1^0)^2x_3^0},$$

we obtain an element of G that maps $(1, 0, 0, 1)$ into $(x_1^0, x_2^0, x_3^0, x_4^0)$. This proves that $\Omega^{>}$ is affinely homogeneous. A similar argument shows that $\Omega^{<}$ is affinely homogeneous as well. It can be shown that G is in fact the full group of affine automorphisms of each of $\Omega^{>}$ and $\Omega^{<}$.

It then follows that the corresponding tube domains $D^{>} := D_{\Omega^{>}}$ and $D^{<} := D_{\Omega^{<}}$ are holomorphically homogeneous since the group $\tilde{G}$ generated by G (viewed as a group of affine transformations of $\mathbb{C}^4$) and translations in the imaginary directions in $\mathbb{C}^4$, acts transitively on each of them.

We note that neither of these domains has a bounded realisation. In fact, neither of them is Kobayashi-hyperbolic (we remark here that it is shown in [**15**] that any connected homogeneous Kobayashi-hyperbolic manifold is biholomorphically equivalent to a bounded domain in complex space). Indeed, the domains $D^{>}$ and $D^{<}$ contain the complex affine lines

$$\{z \in \mathbb{C}^4 : z_1 = 1, z_2 + z_3 = 0, z_4 = 1\} \quad \text{and}$$
$$\{z \in \mathbb{C}^4 : z_1 = 1, z_2 + z_3 = 1, z_4 = 0\},$$

respectively.

2. The automorphism groups of $D^>$ and $D^<$

Denote by $G^>$ and $G^<$ the groups of holomorphic automorphisms of $D^>$ and $D^<$ respectively, equipped with the compact-open topology. In this section, we will determine these groups.

Let $\tilde{\Gamma} := \Gamma + i\mathbb{R}^4$. Clearly, $\tilde{\Gamma}$ is a connected Levi non-degenerate hypersurface contained in $\partial D^> \cap \partial D^<$. Since the Levi form of $\tilde{\Gamma}$ at every point has a positive and a negative eigenvalue, every element of $G^>$ and $G^<$ extends past $\tilde{\Gamma}$ to a biholomorphic mapping preserving $\tilde{\Gamma}$. Let H be the subgroup of $\tilde{G}$ given by the condition $r = 1$. It is straightforward to verify that H acts simply transitively on $\tilde{\Gamma}$. Therefore, every element g of either $G^>$ or $G^<$, for a fixed $p \in \tilde{\Gamma}$, can be uniquely represented as $g = h \circ t$, where $h \in H$ and t is an element of either $I_p^>$ or $I_p^<$, and $I_p^>$ and $I_p^<$ denote the isotropy subgroups of p in $G^>$ and $G^<$ respectively.

Thus, in order to determine $G^>$ and $G^<$, we must find $I_p^>$ and $I_p^<$ for some $p \in \tilde{\Gamma}$. Let $p_0 := (1, 0, 1, 1)$ and I_{p_0} be the group of all *local* holomorphic automorphisms of $\tilde{\Gamma}$ defined near p_0 and preserving p_0. Clearly, $I_{p_0}^> \subset I_{p_0}$ and $I_{p_0}^< \subset I_{p_0}$. Equipped with the topology of uniform convergence of the derivatives of all orders of the component functions on compact subsets of p_0, the group I_{p_0} is known to carry the structure of a real algebraic group (see, e.g., [**1**, **13**, **22**]).

It is straightforward to show that $I_{p_0} \cap \tilde{G}$ consists of the mappings

$$\begin{array}{lcl} z_1 & \mapsto & z_1, \\ z_2 & \mapsto & 2(r^2 - r)z_1 + r^2 z_2 + 2(r - r^2)z_4, \\ z_3 & \mapsto & r^2 z_3 + 1 - r^2, \\ z_4 & \mapsto & (1 - r)z_1 + r z_4, \end{array}$$

where $r \in \mathbb{R}^*$. As the following proposition shows, I_{p_0} is in fact 3-dimensional and admits an explicit description.

Proposition 2.1. *The group I_{p_0} consists of all mappings of the form*

$$
\begin{array}{rcl}
z_1 & \mapsto & z_1, \\
z_2 & \mapsto & -2v^2z_1^3 + i(-2v+4vr+u)z_1^2 - 4ivrz_1z_4 \\
 & & \quad +2(v^2+r^2-r)z_1 + r^2z_2 + 2(r-r^2)z_4 + 2iv - iu, \\
z_3 & \mapsto & 2v^2z_1^2 - i(2u+4vr)z_1 + r^2z_3 + 4ivrz_4 - r^2 \\
 & & \quad +1 - 2v^2 + 2iu, \\
z_4 & \mapsto & -ivz_1^2 + (1-r)z_1 + rz_4 + iv,
\end{array}
\tag{2.1}
$$

where $r \in \mathbb{R}^*$, $u, v \in \mathbb{R}$.

Proof. First, we will write the equation of $\tilde{\Gamma}$ near p_0 in Chern–Moser normal form [4]. Consider the change of coordinates

$$
\begin{aligned}
z_1 &= \frac{11w_1+4}{w_1+4}, \\
z_2 &= -\frac{96i\,(4iw_2 - 5iw_3 + 11w_4)}{5(w_1+4)} \\
&\quad + \frac{64i\left(4iw_2 - 5iw_3 - 6iw_3^2 + 6w_4\right)}{(w_1+4)^2} - \frac{1280w_3^2}{(w_1+4)^3} + 24iw_4, \\
z_3 &= \frac{32i\,(2\,iw_2 + 3w_4)}{5(w_1+4)} - \frac{32w_3^2}{(w_1+4)^2} - 4iw_4 + 1, \\
z_4 &= -\frac{8(6w_3+5)}{w_1+4} + \frac{160w_3}{(w_1+4)^2} + 11.
\end{aligned}
$$

In the w-coordinates, the point p_0 becomes the origin, and the equation of $\tilde{\Gamma}$ near the origin takes the form

$$
\operatorname{Im} w_4 = F\left(w', \overline{w'}\right) := \frac{N\left(w', \overline{w'}\right)}{D\left(w', \overline{w'}\right)},
\tag{2.2}
$$

where $w' := (w_1, w_2, w_3)$ and

$$
\begin{aligned}
D\left(w', \overline{w'}\right) &= \left(11|w_1|^2 + 24w_1 + 24\overline{w_1} + 16\right)\left(5|w_1|^2 + 16\right), \\
N\left(w', \overline{w'}\right) &= 8\mathrm{Re}\Big(48w_1|w_3|^2 + 25w_1^2\overline{w_3}^2 + 16|w_3|^2 + 36|w_1|^2|w_3|^2 \\
&\qquad + 2(11|w_1|^2 + 24w_1 + 24\overline{w_1} + 16)w_1\overline{w_2}\Big).
\end{aligned}
$$

Denote by $F_{k\bar{l}}$ the polynomial of degree k in w' and degree l in $\overline{w'}$ in the power series expansion of the function $F\left(w', \overline{w'}\right)$ about the origin.

A straightforward calculation gives

$$
\begin{aligned}
F_{1\overline{1}} &= \frac{|w_3|^2}{2} + \operatorname{Re} w_1\overline{w_2},\\
F_{2\overline{2}} &= 2\operatorname{Re}\left(-\frac{5}{32}w_1^2\overline{w_1 w_2} + \frac{25}{64}w_1^2\overline{w_3}^2 + \frac{5}{16}|w_1|^2|w_3|^2\right),\\
F_{3\overline{2}} &= -\frac{75}{128}w_1^3\overline{w_3}^2 - \frac{75}{64}w_1^2 w_3\overline{w_1 w_3} - \frac{75}{128}w_1 w_3^2\overline{w_1}^2,\\
F_{3\overline{3}} &= 2\operatorname{Re}\left(\frac{25}{512}w_1^3\overline{w_1}^2\overline{w_2} + \frac{175}{128}w_1^3\overline{w_1 w_3}^2 + \frac{1425}{1024}|w_1|^4|w_3|^2\right).
\end{aligned}
\tag{2.3}
$$

It follows from the first formula in (2.3) that the operator tr defined in [**4**] in this case is

$$
\operatorname{tr} = 2\left(\frac{\partial^2}{\partial w_1\partial\overline{w_2}} + \frac{\partial^2}{\partial w_2\partial\overline{w_1}} + \frac{\partial^2}{\partial w_3\partial\overline{w_3}}\right).
$$

Equation (2.2) and the last three formulae in (2.3) now yield

$$
F(w',0) = \frac{\partial F}{\partial\overline{w'}}(w',0) = \operatorname{tr} F_{2\overline{2}} = \operatorname{tr}^2 F_{3\overline{2}} = \operatorname{tr}^2 F_{3\overline{3}} = 0,
$$

which shows that equation (2.2) is indeed in Chern–Moser normal form.

We will now find I_{p_0} in the w-coordinates. Let I denote the group of mappings of the form

$$
\begin{array}{rcl}
w_1 & \mapsto & w_1,\\
w_2 & \mapsto & \left(i\mu - \dfrac{\nu^2}{2}\right)w_1 + r^2 w_2 + i\nu r w_3,\\
w_3 & \mapsto & i\nu w_1 + r w_3,\\
w_4 & \mapsto & r^2 w_4,
\end{array}
$$

where $r \in \mathbb{R}^*$, $\mu, \nu \in \mathbb{R}$. One can directly verify that I is a subgroup of I_{p_0}. We will show that in fact $I_{p_0} = I$.

Suppose that $I_{p_0} \neq I$ and let $f \in I_{p_0} \setminus I$. It then easily follows from the condition that the term $F_{2\overline{2}}$ does not change when f is applied to equation (2.2) that, without loss of generality, the Jacobian matrix of f at the origin can be assumed to have the form

$$
\begin{pmatrix}
\beta & 0 & 0 & 0\\
0 & \beta & 0 & 0\\
0 & 0 & \beta & 0\\
0 & 0 & 0 & 1
\end{pmatrix},
\tag{2.4}
$$

where $|\beta| = 1$, $\beta \neq 1$. Since the local isotropy group of any Levi non-degenerate hypersurface in $\mathbb{C}^4$ is linearisable (see [**9**, **10**]), there exist normal coordinates in which transformations from I remain linear and f coincides with the linear transformation given by matrix (2.4). Since the term $F_{3\overline{2}}$ in these new coordinates has to be preserved by f, it follows that $F_{3\overline{2}} = 0$. On the other hand, one can show that the term $F_{3\overline{2}}$ in formula (2.3) cannot be eliminated by any transformation to the normal form that leaves all mappings from I linear. This follows from the fact that the family of Chern–Moser chains invariant under the action of I is characterised by the condition that the tangent vector of a chain at the origin is a multiple of a vector of the form $(0, a, 0, 1)$. Thus, for a normalisation $w \mapsto w^*$ that transforms such a chain into the line $\{w^{*'} = 0, \operatorname{Im} w_4^* = 0\}$, we have $\partial w^{*'}/\partial w_4(0) = (0, a, 0, 1)$. With this single parameter freedom, one cannot eliminate the entire term $F_{3\overline{2}}$ in formula (2.3). This contradiction shows that $I_{p_0} = I$.

Recomputing the group I_{p_0} in the z-coordinates, we obtain formulae (2.1) with $u = \frac{16}{25}\mu$ and $v = \frac{2}{5}\nu$. The proposition is proved. q.e.d.

Since all elements of I_{p_0} are polynomial mappings, we have $I_{p_0} \subset I_{p_0}^{>}$ and $I_{p_0} \subset I_{p_0}^{<}$, which implies that $I_{p_0}^{>} = I_{p_0}^{<} = I_{p_0}$. Hence, $G^{>} = G^{<} = H \circ I_{p_0}$. A straightforward calculation now leads to the following:

Theorem 2.2. *The automorphism groups of $D^{>}$ and $D^{<}$ coincide and consist of all mappings of the form*

$$
\begin{array}{rcl}
z_1 & \mapsto & q z_1 + i\alpha_1, \\
z_2 & \mapsto & -2qv^2 z_1^3 + iq(-2v + 4vr - 2vt + u) z_1^2 - 4iqvr z_1 z_4 \\
 & & \quad + q(2v^2 + 2r^2 - 2r - 2tr + 2t + s + t^2) z_1 + qr^2 z_2 \\
 & & \quad + 2q(r - r^2 + tr) z_4 + i\alpha_2, \\
z_3 & \mapsto & 2v^2 z_1^2 - 2i(2vr + u) z_1 + r^2 z_3 + 4ivr z_4 \\
 & & \quad - r^2 - 2v^2 - s + 1 + i\alpha_3, \\
z_4 & \mapsto & -iqv z_1^2 + q(t - r + 1) z_1 + qr z_4 + i\alpha_4,
\end{array}
\tag{2.5}
$$

where $s, t, u, v, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, $q > 0$, $r \in \mathbb{R}^*$.

We will now show that neither of $D^{>}$, $D^{<}$ is equivalent to a nil-ball. Otherwise, one of these domains would be holomorphically equivalent to a domain D that has a Levi non-degenerate boundary and admits an action of a connected nilpotent Lie group N by holomorphic transformations defined in a neighbourhood of $\overline{D}$, such that the induced action of N on ∂D is transitive (see [**18**, **20**]). Let F denote the equivalence mapping. Since the Levi form of $\tilde{\Gamma}$ at every point has a positive and a negative eigenvalue, F extends to a neighbourhood of $\tilde{\Gamma}$ to a biholomorphic mapping onto a neighbourhood of $\overline{D}$, and we have $F(\tilde{\Gamma}) = \partial D$ (the values of F near the part of the boundary intersecting the hyperplane $\{\operatorname{Re} z_1 = 0\}$ necessarily approach infinity). Therefore, N acts on either

$D^>$ or $D^<$ by biholomorphic transformations defined in a neighbourhood of $\tilde{\Gamma}$ in such a way that the induced action of N on $\tilde{\Gamma}$ is transitive. This implies that the Lie algebra $\mathfrak{g}$ of $G^> = G^<$ has a nilpotent subalgebra that acts transitively on $\tilde{\Gamma}$. We will now show that such a subalgebra in fact does not exist.

The algebra $\mathfrak{g}$ is spanned by the following holomorphic vector fields

$$
\begin{aligned}
Z_1 &:= z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_4}, \qquad Z_2 := z_1\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3},\\
Z_3 &:= 2z_4\frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_4}, \qquad Z_4 := i\frac{\partial}{\partial z_1} + i\frac{\partial}{\partial z_4},\\
Z_5 &:= i\frac{\partial}{\partial z_2}, \qquad Z_6 := i\frac{\partial}{\partial z_3}, \qquad Z_7 := 2i\frac{\partial}{\partial z_2} + i\frac{\partial}{\partial z_4},\\
Z_8 &:= 2(z_1 + z_2 - z_4)\frac{\partial}{\partial z_2} + 2(z_3 - 1)\frac{\partial}{\partial z_3} + (z_4 - z_1)\frac{\partial}{\partial z_4},\\
Z_9 &:= iz_1^2\frac{\partial}{\partial z_2} - 2iz_1\frac{\partial}{\partial z_3},\\
Z_{10} &:= 2i(z_1^2 - 2z_1z_4)\frac{\partial}{\partial z_2} - 4i(z_1 - z_4)\frac{\partial}{\partial z_3} - iz_1^2\frac{\partial}{\partial z_4}.
\end{aligned}
$$

The commutation relations for the vector fields above are as follows

$[Z_i, Z_j]$	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
Z_1		0	0	$-Z_4$	$-Z_5$	0	$-Z_7$	0	Z_9	Z_{10}
Z_2			0	$-Z_5$	0	0	0	$2Z_2$	0	0
Z_3				$-Z_7$	0	0	$-2Z_5$	Z_3	0	$-2Z_9$
Z_4					0	0	0	0	$-2Z_2$	$2Z_3$
Z_5						0	0	$2Z_5$	0	0
Z_6							0	$2Z_6$	0	0
Z_7								Z_7	0	$4Z_2$
Z_8									$-2Z_9$	$-Z_{10}$
Z_9										0
Z_{10}										

If a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts transitively on $\tilde{\Gamma}$, it must contain vector fields $Z_1' := Z - Z_1$, $Z_4' := Z_4 + W$, with some $Z, W \in \mathfrak{i}$, where $\mathfrak{i}$ is the Lie algebra of I_{p_0} (observe that $\mathfrak{i}$ is spanned by Z_8, $Z_9 - Z_5 + 2Z_6$, $Z_{10} + Z_7$). It follows from the commutation relations above that $[Z_1', Z_4']$ is a vector field of the form $Z_4 + L$, where L is a linear combination of Z_2, Z_3, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}. Thus, by induction, all the higher-order commutators $[Z_1', [Z_1', \dots, [Z_1', [Z_1', Z_4']] \dots]]$ have this form and hence are non-zero. Therefore, $\mathfrak{h}$ is not nilpotent.

Thus, we have proved the following:

Theorem 2.3. *Each of the domains $D^>, D^< \subset \mathbb{C}^4$ is holomorphically homogeneous, does not have a bounded realisation, and is not equivalent to a nil-ball.*

3. The automorphism groups of $C^>$ and $C^<$

The discussion for $C^>$ and $C^<$ largely parallels that for $D^>$ and $D^<$ just given in the previous section. In this section, we omit all details.

Its is easily verified that affine transformations of the form

$$\begin{array}{lll} x_1 & \mapsto & qx_1, \\ x_2 & \mapsto & r^2(x_2 - 2sx_3 + t), \\ x_3 & \mapsto & r(x_3 + s), \\ x_4 & \mapsto & qr^2(x_4 + s^2x_1 + tx_1), \end{array} \tag{3.1}$$

where $q > 0$, $r \in \mathbb{R}^*$, $s, t \in \mathbb{R}$, form a group and that this group acts transitively on

$$\{x \in \mathbb{R}^4 : x_4 > x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}$$

and

$$\{x \in \mathbb{R}^4 : x_4 < x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}.$$

It follows that $C^>$ and $C^<$ are holomorphically homogeneous. Since the complex lines $\{z \in \mathbb{C}^4 : z_2 = z_3 = 0, z_4 = \pm 1\}$ lie entirely in $C^>$ and $C^<$, respectively, neither domain has a bounded realisation.

As in Section 2, to determine the full holomorphic automorphism groups of these domains, it suffices to know the local CR isotropy of the real Levi-indefinite hypersurface

$$\tilde{\Gamma} := \{z \in \mathbb{C}^4 : x_4 = x_1x_2 + x_1x_3^2 \text{ and } x_1 > 0\}$$

near the basepoint $(1, 0, 0, 0)$. Already, we have an isotropy transformation coming from the affine transformations (3.1) of $\mathbb{R}^4$, namely

$$\begin{array}{lll} z_1 & \mapsto & z_1, \\ z_2 & \mapsto & r^2z_2, \\ z_3 & \mapsto & rz_3, \\ z_4 & \mapsto & r^2z_4, \end{array}$$

for $r \in \mathbb{R}^*$. We claim that the CR isotropy is generated, in addition, by

$$\begin{array}{lll} z_1 & \mapsto & z_1, \\ z_2 & \mapsto & z_2 + iu(z_1 - 1), \\ z_3 & \mapsto & z_3, \\ z_4 & \mapsto & z_4 + \dfrac{iu(z_1^2 - 1)}{2} \end{array}$$

and

$$
\begin{array}{rcl}
z_1 & \mapsto & z_1, \\
z_2 & \mapsto & z_2 + \dfrac{(e^{2i\theta}-1)z_3^2}{2}, \\
z_3 & \mapsto & e^{i\theta} z_3, \\
z_4 & \mapsto & z_4,
\end{array}
\tag{3.2}
$$

where $u, \theta \in \mathbb{R}$. To see this, we check that the holomorphic change of variables

$$
\begin{aligned}
z_1 &= w_1 + 1, \\
z_2 &= w_2 - \frac{iw_1w_4}{10} - \frac{2w_3^2}{(w_1+2)^2}, \\
z_3 &= \frac{2w_3}{w_1+2}, \\
z_4 &= -iw_4 + w_2 + \frac{w_1(10w_2 - i(w_1+2)w_4)}{20}
\end{aligned}
$$

takes the equation of the surface into Chern–Moser normal form:

$$
\operatorname{Im} w_4 = 10\operatorname{Re} \frac{4w_3\overline{w_3}(1+w_1) + 2w_2w_1\overline{w_1} + \overline{w_2}w_1^2\overline{w_1} + 4\overline{w_2}w_1 + 2\overline{w_2}w_1^2}{(2+w_1)(2+\overline{w_1})(20 - w_1\overline{w_1})}
$$

and that, in these coordinates, the CR isotropy is linear:

$$
\begin{array}{rcl}
w_1 & \mapsto & w_1, \\
w_2 & \mapsto & r^2(w_2 + iuw_1), \\
w_3 & \mapsto & re^{i\theta} w_3, \\
w_4 & \mapsto & r^2 w_4,
\end{array}
$$

for $r \in \mathbb{R}^*$, $u, \theta \in \mathbb{R}$. In the original coordinates, we obtain precisely the stated isotropy. With the remaining affine symmetries ((3.1) with $r = 1$) and imaginary translations, we have now found the full holomorphic automorphism group. As a convenient basis for the corresponding Lie algebra of holomorphic vector fields, we may take

$$
\begin{aligned}
&Z_1 := z_1\frac{\partial}{\partial z_1} + z_4\frac{\partial}{\partial z_4}, \quad Z_2 := -2z_3\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}, \quad Z_3 := \frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_4}, \\
&Z_4 := i\frac{\partial}{\partial z_1}, \quad Z_5 := i\frac{\partial}{\partial z_2}, \quad Z_6 := i\frac{\partial}{\partial z_3}, \quad Z_7 := i\frac{\partial}{\partial z_4}, \\
&Z_8 := 2z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} + 2z_4\frac{\partial}{\partial z_4}, \quad Z_9 := 2iz_1\frac{\partial}{\partial z_2} + iz_1^2\frac{\partial}{\partial z_4}, \\
&Z_{10} := -iz_3^2\frac{\partial}{\partial z_2} + iz_3\frac{\partial}{\partial z_3}
\end{aligned}
$$

with commutation relations

$[Z_i, Z_j]$	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
Z_1		0	0	$-Z_4$	0	0	$-Z_7$	0	Z_9	0
Z_2			0	0	0	$2Z_5$	0	Z_2	0	Z_6
Z_3				$-Z_7$	0	0	0	$2Z_3$	0	0
Z_4					0	0	0	0	$-2Z_3$	0
Z_5						0	0	$2Z_5$	0	0
Z_6							0	Z_6	0	$-Z_2$
Z_7								$2Z_7$	0	0
Z_8									$-2Z_9$	0
Z_9										0
Z_{10}										

The rest of the discussion follows exactly the corresponding discussion in Section 2 (except that $\mathfrak{i}$ is spanned by Z_8, $Z_9 - 2Z_5 - Z_7$, Z_{10}) and we have proved:

Theorem 3.1. *Each of the domains $C^>, C^< \subset \mathbb{C}^4$ is holomorphically homogeneous, does not have a bounded realisation, and is not equivalent to a nil-ball.*

Finally, notice that $C^>$ and $C^<$ are holomorphically distinguished from $D^>$ and $D^<$, respectively, by their admitting a holomorphic circle action (3.2).

4. Acknowledgements

We would like to acknowledge that this work started while the third author was visiting the University of Adelaide and continued during his visit to CIAM at the University of South Australia. We would also like to thank Gerd Schmalz for several useful conversations.

References

[1] V.K. Beloshapka, *On the dimension of the group of automorphisms of an analytic hypersurface* (translated from Russian), Math. USSR-Izv. **14** (1980) 223–245, MR 0534592, Zbl 0456.32015.

[2] D. Burns, Jr., S. Shnider, & R.O. Wells, Jr., *Deformations of strictly pseudoconvex domains*, Invent. Math. **46** (1978) 237–253, MR 0481119, Zbl 0412.32022.

[3] É. Cartan, *Sur les domaines bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg **11** (1935) 116–162, Zbl 0011.12302.

[4] S.S. Chern & J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974) 219–271, MR 0709144, Zbl 0302.32015.

[5] B.M. Doubrov, B.P. Komrakov, & M. Rabinovich, *Homogeneous surfaces in the three-dimensional affine geometry*, Geometry and Topology of Submanifolds, **VIII** (F. Dillen et al., eds.), World Scientific, 1996, 168–178, MR 1434565, Zbl 0934.53007.

[6] M.G. Eastwood & V.V. Ezhov, *On affine normal forms and a classification of homogeneous surfaces in affine three-space*, Geom. Dedicata **77** (1999) 11–69, MR 1706516, Zbl 0999.53008.

[7] M.G. Eastwood & V.V. Ezhov, *Homogeneous hypersurfaces with isotropy in affine four-space*, Proc. Steklov Inst. Math. **235** (2001) 57–70, MR 1886573.

[8] M.G. Eastwood & A.V. Isaev, *Examples of unbounded homogeneous domains in complex space*, arXiv:math.CV/0304102.

[9] V.V. Ezhov, *Linearization of stability group for a class of hypersurfaces* (translated from Russian), Russian Math. Surveys **41** (1986) 203–204, MR 0854251, Zbl 0611.32016.

[10] V.V. Ezhov, *On the linearization of automorphisms of a real analytic hypersurface* (translated from Russian), Math. USSR-Izv. **27** (1986) 53–84, MR 0806683, Zbl 0595.32013.

[11] A.T. Huckleberry & E.L. Livorni, *A classification of homogeneous surfaces*, Canad. J. Math. **33** (1981) 1097–1110, MR 0638369, Zbl 0504.32025.

[12] A.V. Isaev, *Global properties of spherical tube hypersurfaces*, Indiana Univ. Math. J. **42** (1993) 179–213, MR 1218712, Zbl 0797.32014.

[13] A.V. Loboda, *On local automorphisms of real-analytic hypersurfaces* (translated from Russian), Math. USSR-Izv. **18** (1982) 537–559, MR 0623353, Zbl 0488.32014.

[14] A.V. Loboda, *Each holomorphically homogeneous tube in* $\mathbb{C}^2$ *has an affine-homogeneous base* (translated from Russian), Siberian Math. J. **42** (2001) 1111–1114, MR 1876819, Zbl 0987.32018.

[15] K. Nakajima, *Homogeneous hyperbolic manifolds and homogeneous Siegel domains*, J. Math. Kyoto Univ. **25** (1985) 269–291, MR 0794987, Zbl 0583.32066.

[16] K. Nomizu & T. Sasaki, *Affine Differential Geometry*, Cambridge University Press, 1994, MR 1311248, Zbl 0834.53002.

[17] K. Oeljeklaus & W. Richthofer, *Homogeneous complex surfaces*, Math. Ann. **268** (1984) 273–292, MR 0751730, Zbl 0551.32023.

[18] R. Penney, *Harmonic analysis on homogeneous domains in* $\mathbb{C}^n$, Lie Group Representations III (R. Herb et al., eds.), Lecture Notes in Math., **1077**, Springer, 1984, 359–374, Zbl 0554.22003.

[19] R. Penney, *The structure of rational homogeneous domains in* $\mathbb{C}^n$, Ann. of Math. **126** (1987) 389–414, MR 0908151, Zbl 0655.32027.

[20] R. Penney, *Homogeneous Koszul manifolds in* $\mathbb{C}^n$, J. Differential Geom. **36** (1992) 591–631, MR 1189497, Zbl 0770.32017.

[21] I. Pyatetskii-Shapiro, *Automorphic Functions and the Geometry of Classical Domains* (translated from Russian), Gordon and Breach, 1969, MR 0252690, Zbl 0196.09901.

[22] A.G. Vitushkin & N.G. Kruzhilin, *Description of the automorphism groups of real hypersurfaces of complex space* (Russian), Investigations in the theory of the approximation of functions, Akad. Nauk SSSR Bashkir. Filial, Otdel Fiz. Mat., Ufa, 1987, 26–69, MR 0950398, Zbl 0683.32015.

[23] J. Winkelmann, *The classification of three-dimensional homogeneous complex manifolds*, Lecture Notes Math. **1602**, 1995, Springer-Verlag, Berlin, MR 1439051, Zbl 0823.32017.

Department of Mathematics
University of Adelaide, SA 5005
Australia

E-mail address: meastwoo@maths.adelaide.edu.au

School of Mathematics and Statistics
University of South Australia
Mawson Lakes Blvd, Mawson Lakes, SA 5091
Australia

E-mail address: vladimir.ejov@unisa.edu.au

Department of Mathematics
The Australian National University
Canberra, ACT 0200
Australia

E-mail address: alexander.isaev@maths.anu.edu.au