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FACTORIZATION THEOREM FOR PROJECTIVE VARIETIES WITH FINITE QUOTIENT SINGULARITIES

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Abstract

In this paper, we prove that any two birational projective varieties with finite quotient singularities can be realized as two geometric GIT quotients of a non-singular projective variety by a reductive algebraic group. Then, by applying the theory of Variation of Geometric Invariant Theory Quotients ([3]), we show that they are related by a sequence of GIT wall-crossing flips.

1. Statements of results

In this paper, we will assume that the ground field is $\mathbb{C}$.

Theorem 1.1. *Let $\phi : X \to Y$ be a birational morphism between two projective varieties with at worst finite quotient singularities. Then there is a smooth polarized projective $(\mathrm{GL}_n \times \mathbb{C}^*)$-variety $(M, \mathcal{L})$ such that*

1) *$\mathcal{L}$ is a very ample line bundle and admits two (general) linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ with $M^{ss}(\mathcal{L}_1) = M^s(\mathcal{L}_1)$ and $M^{ss}(\mathcal{L}_2) = M^s(\mathcal{L}_2)$.*
2) *The geometric quotient $M^s(\mathcal{L}_1)/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to X and the geometric quotient $M^s(\mathcal{L}_2)/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to Y.*
3) *The two linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ differ only by characters of the $\mathbb{C}^*$-factor, and $\mathcal{L}_1$ and $\mathcal{L}_2$ underly the same linearization of the GL_n-factor. Let $\underline{\mathcal{L}}$ be this underlying GL_n-linearization. Then, we have $M^{ss}(\underline{\mathcal{L}}) = M^s(\underline{\mathcal{L}})$.*

As a consequence, we obtain

Theorem 1.2. *Let X and Y be two birational projective varieties with at worst finite quotient singularities. Then, Y can be obtained from X by a sequence of GIT weighted blowups and weighted blowdowns.*

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The factorization theorem for *smooth* projective varieties was proved by Wlodarczyk and Abramovich–Karu–Matsuki–Wlodarczyka a few years ago ([**1**], [**12**], [**13**]). Hu and Keel, in [**7**], gave a short proof by interpreting it as VGIT wall-crossing flips of $\mathbb{C}^*$-action. My attention to varieties with finite quotient singularities was brought out by Yongbin Ruan. The proof here uses the same idea of [**7**] coupled with a key suggestion of Dan Abramovich which changed the route of my original approach. Only the first paragraph of Section 2 uses a construction of [**7**] which we reproduce for completeness. The rest is independent. Theorem 1.1 reinforces the philosophy that began in [**6**]: Birational geometry of $\mathbb{Q}$-factorial projective varieties is a special case of VGIT.

2. Proof of Theorem 1.1

By the construction of [**7**] (cf. Section 2 of [**6**]), there is a polarized $\mathbb{C}^*$-projective normal variety (Z, L) such that L admits two (general) linearizations L_1 and L_2 such that

1) $Z^{ss}(L_1) = Z^s(L_1)$ and $Z^{ss}(L_2) = Z^s(L_2)$.
2) $\mathbb{C}^*$ acts freely on $Z^s(L_1) \cup Z^s(L_2)$.
3) The geometric quotient $Z^s(L_1)/\mathbb{C}^*$ is isomorphic to X and the geometric quotient $Z^s(L_2)/\mathbb{C}^*$ is isomorphic to Y.

The construction of Z is short, so we reproduce it here briefly. Choose an ample cartier divisor D on Y. Then, there is an effective divisor E on X whose support is exceptional such that $\phi^*D = A + E$ with A ample on X. Let C be the image of the injection $\mathbb{N}^2 \to N^1(X)$ given by $(a, b) \to aA + bE$. The edge generated by ϕ^*D divides C into two chambers: the subcone C_1 generated by A and ϕ^*D, and the subcone C_2 generated by ϕ^*D and E. The ring $R = \oplus_{(a,b)\in\mathbb{N}^2} H^0(X, aA + bE)$ is finitely generated and is acted upon by $(\mathbb{C}^*)^2$ with weights (a, b) on $H^0(X, aA+bE)$. Let $Z = \mathrm{Proj}(R)$ with R graded by total degree $(a+b)$. Then, a subtorus $\mathbb{C}^*$ of $(\mathbb{C}^*)^2$ complementary to the diagonal subgroup Δ acts naturally on Z. The very ample line bundle $L = \mathcal{O}_Z(1)$ has two linearizations L_1 and L_2 descended from two interior integral points in the chambers C_1 and C_2, respectively. One verifies (1), (2) by algebra, and (3) by algebra and the projection formula.

Now, since $\mathbb{C}^*$ acts freely on $Z^s(L_1)\cup Z^s(L_2)$, we deduce that $Z^s(L_1)\cup Z^s(L_2)$ has at worse finite quotient singularities. By Corollary 2.20 and Remark 2.11 of [**4**], there is a *smooth* GL_n-algebraic space U such that the geometric quotient $\pi : U \to U/\mathrm{GL}_n$ exists and is isomorphic to $Z^s(L_1) \cup Z^s(L_2)$ for some $n > 0$. Since $Z^s(L_1) \cup Z^s(L_2)$ is quasi-projective, we see that so is U. In fact, since $Z^s(L_1) \cup Z^s(L_2)$ admits a $\mathbb{C}^*$-action, all of the above statements can be made $\mathbb{C}^*$-equivariant. In other words, U admits a $\mathrm{GL}_n \times \mathbb{C}^*$ action and a very ample line bundle $L_U = \pi^*(L^k|_{Z^s(L_1)\cup Z^s(L_2)})$ (for some fixed sufficiently large k) with two $(\mathrm{GL}_n \times \mathbb{C}^*)$-linearizations $L_{U,1}$ and $L_{U,2}$ such that

1) $U^{ss}(L_{U,1}) = U^s(L_{U,1})$ and $U^{ss}(L_{U,2}) = U^s(L_{U,2})$.
2) The geometric quotient $U^s(L_{U,1})/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to X and the geometric quotient $U^s(L_{U,2})/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to Y. Moreover,
3) the two linearizations $L_{U,1}$ and $L_{U,2}$ differ only by characters of the $\mathbb{C}^*$ factor.

Since we assume that L_U is very ample, we have an $(\mathrm{GL}_n \times \mathbb{C}^*)$-equivariant embedding of U in a projective space such that the pullback of $\mathcal{O}(1)$ is L_U. Let $\overline{U}$ be the compactification of U which is the closure of U in the projective space. Let $L_{\overline{U}}$ be the pullback of $\mathcal{O}(1)$ to $\overline{U}$. This extends L_U and in fact extends the two linearizations $L_{U,1}$ and $L_{U,2}$ to $L_{\overline{U},1}$ and $L_{\overline{U},2}$, respectively, such that

$$\overline{U}^{ss}(L_{\overline{U},1}) = \overline{U}^s(L_{\overline{U},1}) = U^{ss}(L_{U,1}) = U^s(L_{U,1})$$

and

$$\overline{U}^{ss}(L_{\overline{U},2}) = \overline{U}^s(L_{\overline{U},2}) = U^{ss}(L_{U,2}) = U^s(L_{U,2}).$$

It follows that the geometric quotient $\overline{U}^s(L_{\overline{U},1})/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to X and the geometric quotient $\overline{U}^s(L_{\overline{U},2})/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to Y.

Resolving the singularities of $\overline{U}$, $(\mathrm{GL}_n \times \mathbb{C}^*)$-equivariantly, we will obtain a smooth projective variety M. Notice that $\overline{U}^s(L_{\overline{U},1}) \cup \overline{U}^s(L_{\overline{U},2}) = U^s(L_{U,1}) \cup U^s(L_{U,2}) \subset U$ is smooth, hence we can arrange the resolution so that it does not affect this open subset. Let $f : M \to \overline{U}$ be the resolution morphism and Q be any relative ample line bundle over M. Then, by the relative GIT (Theorem 3.11 of [**5**]), there is a positive integer m_0 such that for any fixed integer $m \geq m_0$, we obtain a very ample line bundle over M, $\mathcal{L} = f^* L_{\overline{U}}^m \otimes Q$, with two linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ such that

1) $M^{ss}(\mathcal{L}_1) = M^s(\mathcal{L}_1) = f^{-1}(\overline{U}^s(L_{\overline{U},1}))$ and $M^{ss}(\mathcal{L}_2) = M^s(\mathcal{L}_2) = f^{-1}(\overline{U}^s(L_{\overline{U},2}))$.
2) The geometric quotient $M^s(\mathcal{L}_1)/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to $\overline{U}^s(L_{\overline{U},1})/(\mathrm{GL}_n \times \mathbb{C}^*)$ which is isomorphic to X, and, the geometric quotient $M^s(\mathcal{L}_2)/(\mathrm{GL}_n \times \mathbb{C}^*)$ is isomorphic to $\overline{U}^s(L_{\overline{U},2})/(\mathrm{GL}_n \times \mathbb{C}^*)$ which is isomorphic to Y.

Finally, we note from the construction that the two linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ differ only by characters of the $\mathbb{C}^*$-factor, and $\mathcal{L}_1$ and $\mathcal{L}_2$ underly the same linearization of the GL_n-factor. Let $\underline{\mathcal{L}}$ be this underlying GL_n-linearization. It may happen that $M^{ss}(\underline{\mathcal{L}}) \neq M^s(\underline{\mathcal{L}})$. But if this is the case, we can then apply the method of Kirwan's canonical desingularization ([**9**]), but we need to blow up $(\mathrm{GL}_n \times \mathbb{C}^*)$-equivariantly instead of just GL_n-equivariantly. More precisely, if $M^{ss}(\underline{\mathcal{L}}) \neq M^s(\underline{\mathcal{L}})$, then there exists a reductive subgroup R of GL_n of dimension at least 1 such that

$$M^{ss}_R(\underline{\mathcal{L}}) := \{m \in M^{ss}(\underline{\mathcal{L}}) : m \text{ is fixed by } R\}$$

is not empty. Now, because the action of $\mathbb{C}^*$ and the action of GL_n commute, using the Hilbert–Mumford numerical criterion (or by manipulating invariant sections, or by other direct arguments), we can check that

$$\mathbb{C}^* M^{ss}(\underline{\mathcal{L}}) = M^{ss}(\underline{\mathcal{L}}),$$

in particular,

$$\mathbb{C}^* M^{ss}_R(\underline{\mathcal{L}}) = M^{ss}_R(\underline{\mathcal{L}}).$$

Hence, we have

$$(\mathrm{GL}_n \times \mathbb{C}^*) M^{ss}_R = \mathrm{GL}_n\, M^{ss}_R \subset M \setminus M^s(\underline{\mathcal{L}}).$$

Therefore, we can resolve the singularities of the closure of the union of $\mathrm{GL}_n\, M^{ss}_R$ in M for all R with the maximal $r = \dim R$ and blow M up along the proper transform of this closure. Repeating this process at most r times gives us a desired non-singular $(\mathrm{GL}_n \times \mathbb{C}^*)$-variety with GL_n-semistable locus coincides with the GL_n-stable locus (see pages 157–158 of [**10**]). Obviously, Kirwan's process will not affect the open subset $M^{ss}(\mathcal{L}_1) \cup M^{ss}(\mathcal{L}_2) = M^s(\mathcal{L}_1) \cup M^s(\mathcal{L}_2) \subset M^s(\underline{\mathcal{L}})$. Hence, this will allow us to assume that $M^{ss}(\underline{\mathcal{L}}) = M^s(\underline{\mathcal{L}})$.

This completes the proof of Theorem 1.1.

The proof implies the following.

Corollary 2.1. *Let $\phi : X \to Y$ be a birational morphism between two projective varieties with at worst finite quotient singularities. Then, there is a polarized projective $\mathbb{C}^*$-variety $(\underline{M}, \underline{L})$ with at worst finite quotient singularities such that X and Y are isomorphic to two geometric GIT quotients of $(\underline{M}, \underline{L})$ by $\mathbb{C}^*$.*

3. Proof of Theorem 1.2

Let $\phi : X \longrightarrow Y$ be the birational map. By passing to the (partial) desingularization of the graph of ϕ, we may assume that ϕ is a birational morphism. This reduces to the case of Theorem 1.1.

We will then try to apply the proof of Theorem 4.2.7 of [**3**] (see also [**11**]). Unlike the torus case for which Theorem 4.2.7 applies almost automatically, here, because $(\mathrm{GL}_n \times \mathbb{C}^*)$ involves a non-Abelian group, the validity of Theorem 4.2.7 must be verified.

From the last section, the two linearizations $\mathcal{L}_1$ and $\mathcal{L}_2$ differ only by characters of the $\mathbb{C}^*$-factor, and $\mathcal{L}_1$ and $\mathcal{L}_2$ underly the same linearization of the GL_n-factor. We denote this common GL_n-linearized line bundle by $\underline{\mathcal{L}}$. For any character χ of the $\mathbb{C}^*$ factor, let $\mathcal{L}_\chi$ be the corresponding $(\mathrm{GL}_n \times \mathbb{C}^*)$-linearization. Note that $\mathcal{L}_\chi$ also underlies the GL_n-linearization $\underline{\mathcal{L}}$. From the constructions of the compactification $\overline{U}$ and the resolution M, we know that $M^{ss}(\underline{\mathcal{L}}) = M^s(\underline{\mathcal{L}})$. In particular, GL_n acts with only finite isotropy subgroups on $M^{ss}(\underline{\mathcal{L}}) = M^s(\underline{\mathcal{L}})$. Now, to go from $\mathcal{L}_1$ to $\mathcal{L}_2$, we will (only) vary the characters of the $\mathbb{C}^*$-factor, and we will encounter a "wall" when a character χ gives $M^{ss}(\mathcal{L}_\chi) \setminus M^s(\mathcal{L}_\chi) \neq \emptyset$. In such a case, since $M^{ss}(\mathcal{L}_\chi) \subset M^{ss}(\underline{\mathcal{L}}) = M^s(\underline{\mathcal{L}})$ which implies that GL_n operates on $M^{ss}(\mathcal{L}_\chi)$ with only finite isotropy subgroups, the only isotropy subgroups of $(\mathrm{GL}_n \times \mathbb{C}^*)$ of positive dimensions have to come from the factor $\mathbb{C}^*$, and hence, we conclude that such isotropy subgroups of $(\mathrm{GL}_n \times \mathbb{C}^*)$ on $M^{ss}(\mathcal{L}_\chi)$ have to be one-dimensional (possibly disconnected) diagonalizable subgroups. This verifies the condition of Theorem 4.2.7 of [**3**] and hence, its proof goes through without changes (Theorem 4.2.7 of [**3**] assumes that the isotropy subgroup corresponding to a wall is a one-dimensional (possibly disconnected) diagonalizable group. The main theorems of [**11**] assume that the isotropy subgroup is $\mathbb{C}^*$ (see his Hypothesis (4.4), p. 708)).

4. GIT on projective varieties with finite quotient singularites

The proof in Section 2 can be modified slightly to imply the following.

Theorem 4.1. *Assume that a reductive algebraic group G acts on a polarized projective variety (X, L) with at worst finite quotient singularities. Then, there exists a* <u>*smooth*</u> *polarized projective variety $(M, \mathcal{L})$ which is acted upon by $(G \times \mathrm{GL}_n)$ for some $n > 0$ such that for any linearization L_χ on X, there is a corresponding linearization $\mathcal{L}_\chi$ on M such that $M^{ss}(\mathcal{L}_\chi)//(G\times \mathrm{GL}_n)$ is isomorphic to $X^{ss}(L_\chi)//G$. Moreover, if $X^{ss}(L_\chi) = X^s(L_\chi)$, then $M^{ss}(\mathcal{L}_\chi) = M^s(\mathcal{L}_\chi)$.*

This is to say that all GIT quotients of the *singular* (X, L) (L is fixed) by G can be realized as GIT quotients of the *smooth* (M, L) by $G \times GL_n$. In general, this realization is a strict inclusion as $(M, \mathcal{L})$ may have more GIT quotients than those coming from (X, L).

When the underlying line bundle L is changed, the compatification $\overline{U}$ is also changed, so will M. Nevertheless, it is possible to have a similar construction to include a finitely many different underlying ample line bundles. However, Theorem 4.1 should suffice in most practical problems because: (1) in most natural quotient and moduli problems, one only needs to vary linearizations of a fixed ample line bundle; (2) Variation of the underlying line bundle often behaves so badly that the condition of Theorem 4.2.7 of [**3**] cannot be verified.

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