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CONSTANT SCALAR CURVATURE KÄHLER METRICS ON FIBRED COMPLEX SURFACES

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Abstract

This article finds constant scalar curvature Kähler metrics on certain compact complex surfaces. The surfaces considered are those admitting a holomorphic submersion to curve, with fibres of genus at least 2. The proof is via an adiabatic limit. An approximate solution is constructed out of the hyperbolic metrics on the fibres and a large multiple of a certain metric on the base. A parameter dependent inverse function theorem is then used to perturb the approximate solution to a genuine solution in the same cohomology class. The arguments also apply to certain higher dimensional fibred Kähler manifolds.

1. Introduction

This article proves the existence of constant scalar curvature Kähler metrics on certain complex surfaces. Let $\pi\colon X \to \Sigma$ be a holomorphic submersion from a compact connected complex surface to a curve, with fibres of genus at least 2. Topologically, X is a locally trivial surface bundle; analytically, however, the fibrewise complex structure may vary. Examples of such surfaces appear in, for example, [**1**, **9**].

If Σ has genus greater than one, then X is polarised by $\kappa_r = -c_1(V) - r\pi^* c_1(\Sigma)$, where V is the vertical tangent bundle and the integer r is sufficiently large. More generally, this is a Kähler class when r is non-integral.

The precise result proved here is as follows.

Theorem 1.1. *If X is a compact connected complex surface admitting a holomorphic submersion onto a high genus complex curve with fibres of genus at least two, then, for all large r, the Kähler class κ_r contains a constant scalar curvature metric.*

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In fact, the arguments apply to certain higher dimensional fibred Kähler manifolds, although the conditions are more awkward to state; see Theorem 8.1.

1.1. Stability of polarised varieties. There are many reasons to be interested in constant scalar curvature Kähler (cscK) metrics. Perhaps the most important comes from a well known conjecture relating the existence of a cscK metric to the stability of the underlying polarised variety. This is analogous to the Hitchin–Kobayashi correspondence and is due, in both chronological order and degree of generality, to Yau [**12**], Tian [**11**] and Donaldson [**3**].

The surfaces covered by Theorem 1.1 are each a family of stable curves parameterised by a stable base, and so stand a good chance of being stable, at least for a polarisation in which the base is large. This is certainly true in the analogous situation for bundles.

The algebro-geometric view point is not explored here, except for a few comments relating analytic observations to stability conditions.

1.2. Projectivised vector bundles. A result analogous to Theorem 1.1, but concerning projectivised vector bundles, is proved by Hong in [**6**] and [**7**]. The techniques used there are similar to those applied here. There are two important differences between the situations, however.

In both cases, the fibres have cscK metrics. However, in Theorem 1.1, the fibres have no non-zero holomorphic vector fields, whilst the fibres of a projectivised vector bundle have holomorphic vector fields which are not Killing. If (F, ω) is a compact Kähler manifold with a cscK metric, and ξ is a holomorphic vector field on F which is not Killing, then flowing ω along ξ gives a non-trivial family of cohomologous cscK metrics.

From the point of view of the analysis, this situation leads to problems, as is explained later. It is avoided in [**6**] and [**7**] by the additional assumption that, whilst there are holomorphic vector fields on the fibres, they are not induced by a global holomorphic vector field. From the point of view of the algebraic geometry, this is related to the stability of the vector bundle which is projectivised.

The other important difference between the two situations is that the fibres of a ruled manifold are rigid, whilst the fibres in Theorem 1.1 have moduli. This leads to extra considerations in the proof, which are not required in Hong's work.

1.3. Base genus 0 or 1. Let X satisfy the hypotheses of Theorem 1.1 apart from the condition on the genus of Σ. When Σ has genus 0 or 1, a direct proof of the existence of cscK metrics can be given.

Theorem 1.2. *Let X be a compact connected complex surface admitting a holomorphic submersion $\pi\colon X \to \Sigma$ to a surface of genus 0 or 1, for which all the fibres are of genus at least 2. Then, X admits cscK metrics.*

Proof. Mapping each fibre of π to its Jacobian determines a map $j\colon \Sigma \to \mathscr{A}$, where $\mathscr{A}$ denotes the moduli space of principally polarised abelian varieties (of dimension equal to the genus of the fibres of π). The universal cover of $\mathscr{A}$ is the Siegel upper half space $\mathscr{S}$ which can be realised as a bounded domain in $\mathbb{C}^N$.

The map j lifts to a holomorphic map from the universal cover of Σ to $\mathscr{S}$. Since $g(\Sigma) = 0$ or 1, this map must be constant: when $g(\Sigma) = 0$ the map is a holomorphic map from a compact manifold; when $g(\Sigma) = 1$ the map is a bounded holomorphic map from $\mathbb{C}$. Hence, j is constant. By Torelli's theorem, all the fibres of π are biholomorphic.

As the model fibre S of $X \to \Sigma$ has genus at least 2, its group of biholomorphisms Γ is finite. Define a principal Γ bundle $P \to \Sigma$ by setting the fibre over σ to be the group of biholomorphisms from $\pi^{-1}(\sigma)$ to S. Since P is a cover of Σ, it arises from some representation $\pi_1(\Sigma) \to \Gamma$. Using this representation,

$$X = P \times_{\pi_1(\Sigma)} S.$$

In the case $g(\Sigma) = 0$, this gives $X = S \times \mathbb{P}^1$ which admits product cscK metrics. If $g(\Sigma) = 1$, X is a quotient of $S \times \mathbb{C}$ by $\pi_1(\Sigma)$. The product $S \times \mathbb{C}$ admits a natural cscK metric, and with respect to this metric $\pi_1(\Sigma)$ acts by isometries. Hence, the metric descends to a metric on X which also has constant scalar curvature. q.e.d.

From now on, in the proof of Theorem 1.1, only the case $g(\Sigma) \geq 2$ is considered.

1.4. Outline of proof. This section gives an outline of the proof of Theorem 1.1. The main technique used is an adiabatic limit. This involves studying a family of metrics on X for which the base becomes increasingly large.

The first step is to construct a family of approximate solutions. This is done in Section 3. The motivating idea is that in an adiabatic limit, the local geometry is dominated by that of the fibre. The approximate solutions are constructed by fitting together the cscK metrics on the fibres of X and a large multiple of a metric on the base. If the base is scaled by a factor r, the metrics on X have scalar curvature which is $O(r^{-1})$ from being constant. Section 3 also discusses adjusting these metrics to decrease the error to $O(r^{-n})$ for any positive integer n.

The remaining sections carry out the analysis necessary to show that a genuine solution lies nearby. Doing this involves solving a parameter dependent inverse function theorem. As is explained in Section 4, such arguments hinge on certain estimates. Two sorts of estimate are used. The first kind involves local analysis; the various constants appearing in Sobolev inequalities are shown to be independent of r (see Section 5). The second kind of estimate involves global analysis, specifically a lower bound for the first eigenvalue of the linearisation of the scalar curvature equation as is found in Section 6.

Before the start of the proof itself, Section 2 collates some background information concerning straight forward properties of scalar curvature and its dependence on the Kähler structure. In particular, it states precisely how various geometric objects are uniformly continuous with respect to the metric used to define them. These results are standard and widely known, although not stated in the literature in quite the form used here.

2. General Properties of Scalar Curvature

2.1. The scalar curvature map. Kähler metrics on a compact Kähler manifold X and in a fixed cohomology class are parametrised by Kähler potentials: any other Kähler form cohomologous to a given one ω is of the form $\omega_\phi = \omega + i\bar{\partial}\partial\phi$ for some real valued $\phi \in C^\infty$.

The scalar curvature of a Kähler metric can be expressed as the trace of the Ricci form with respect to the Kähler form: $\mathrm{Scal}(\omega)\,\omega^n = n\rho \wedge \omega^{n-1}$, where n is the complex dimension of X. The equation studied in this article is

$$\mathrm{Scal}(\omega_\phi) = \mathrm{const}.$$

This is a fourth order, fully non-linear, partial differential equation for the function ϕ.

Scalar curvature determines a map $S\colon \phi \mapsto \mathrm{Scal}(\omega_\phi)$ defined, initially, on an open set $U \subset C^\infty$.

Lemma 2.1. *Let V denote the L^p_{k+4}-completion of U. S extends to a smooth map $S : V \to L^p_k$ whenever $(k+2)p - 2n > 0$. Its derivative at 0 is given by*

$$L(\phi) = \left(\Delta^2 - S(0)\Delta\right)\phi + n(n-1)\frac{i\bar{\partial}\partial\phi \wedge \rho \wedge \omega^{n-2}}{\omega^n}. \tag{2.1}$$

Proof. Scalar curvature is analytic in the metric, and hence, S extends to a smooth map from L^p_{k+4} provided that $i\bar{\partial}\partial\phi$ is continuous, i.e., for $\phi \in L^2_{k+4}$ where $(k+2)p - 2n > 0$.

To compute its derivative, let $\omega_t = \omega + ti\bar{\partial}\partial\phi$. The corresponding metric on the tangent bundle is $g_t = g + t\Phi$ where Φ is the real symmetric tensor corresponding to $i\bar{\partial}\partial\phi$. The Ricci form is given locally by

$$\rho_t = \rho + i\bar{\partial}\partial \log\det\left(1 + tg^{-1}\Phi\right).$$

Now, $\operatorname{tr}(g^{-1}\Phi) = \Lambda(i\bar{\partial}\partial\phi) = \Delta\phi$. Hence, at $t = 0$,

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = i\bar{\partial}\partial\phi, \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = i\bar{\partial}\partial(\Delta\phi).$$

The result follows from differentiating the equation $S\omega^n = n\rho\wedge\omega^{n-1}$. q.e.d.

Remarks 2.2. In flat space $L = \Delta^2$; in general, the leading order term of L is Δ^2. It follows immediately that L is elliptic, with index zero.

It follows, either from the fact that $\int S(\phi)\omega_\phi^n$ is constant, or from the formula for L, that

$$\int L(\phi)\,\omega^n = -\int \phi\Delta S(0)\,\omega^n.$$

In particular, for cscK metrics, $\operatorname{im} L$ is L^2-orthogonal to the constant functions.

Lemma 2.3. *Let $k \geq 2$. If $\phi \in C^{k,\alpha}$ satisfies $S(\phi) \in C^{k,\alpha}$, then $\phi \in C^{k+4,\alpha}$.*

Proof. $S(\phi) = \Delta_\phi V$ where Δ_ϕ is the ω_ϕ-Laplacian and

$$V = -\log\det(g + \Phi),$$

where Φ is the real symmetric tensor associated to the $(1,1)$-form $i\bar{\partial}\partial\phi$.

Since $\phi \in C^{k,\alpha}$, Δ_ϕ is a linear second order elliptic operator with coefficients in $C^{k-2,\alpha}$. By elliptic regularity (see e.g., [2], p. 87) and assumption on $S(\phi)$, $V \in C^{k,\alpha}$.

The map $\phi \mapsto -\log\det(g+\Phi)$ is non-linear, second order and elliptic. Such maps also satisfy a regularity result (see e.g., [2], p. 86), hence, $\phi \in C^{k+2,\alpha}$.

This implies that Δ_ϕ has $C^{k,\alpha}$ coefficients meaning that in fact $V \in C^{k+2,\alpha}$ and so $\phi \in C^{k+4,\alpha}$. q.e.d.

Example 2.4 (High genus curves)**.** For the hyperbolic metric on a high genus curve, $S(0) = -1$. Hence, the above Lemma gives $L = \Delta^2 + \Delta$. The kernel of L is precisely the constant functions. As L is self-adjoint, it is an isomorphism when considered as a map between spaces of functions with mean value zero.

In the above discussion of scalar curvature, the underlying complex manifold (X, J) is regarded as fixed, whilst the Kähler form ω is varying. An alternative point of view is described in [**3**]. There the symplectic manifold (X, ω) is fixed, whilst the complex structure varies (through complex structures compatible with ω). The two points of view are related as follows.

Calculation shows that, on a Kähler manifold (X, J, ω),

$$i\bar{\partial}\partial\phi = \mathscr{L}_{\nabla\phi}\omega.$$

Hence, infinitesimally, the change in ω due to the Kähler potential ϕ is precisely that caused by flowing ω along $\nabla\phi$. Flowing the Kähler structure back along $-\nabla\phi$ restores the original symplectic form, but changes the complex structure by $-\mathscr{L}_{\nabla\phi}J$. This means that the two points of view (varying ω versus varying J) are related by the diffeomorphism generated by $\nabla\phi$.

The following result, on the first order variation of scalar curvature under changes in complex structure, is proved in [**3**]. The operator

$$\mathscr{D} : C^{\infty}(X) \to \Omega^{0,1}(TX)$$

is defined by $\mathscr{D}\phi = \bar{\partial}\nabla\phi$, where $\bar{\partial}$ is the $\bar{\partial}$-operator of the holomorphic tangent bundle. The operator $\mathscr{D}^*$ is the formal adjoint of $\mathscr{D}$ with respect to the L^2-inner product determined by the Kähler metric.

Lemma 2.5. *An infinitesimal change of $-\mathscr{L}_{\nabla\phi}J$ in the complex structure J causes an infinitesimal change of $\mathscr{D}^*\mathscr{D}\phi$ in the scalar curvature of (X, J, ω).*

Taking into account the diffeomorphism required to relate this point of view to that in which ω varies gives the following formula for the linearisation of scalar curvature with respect to Kähler potentials:

Lemma 2.6.

$$L(\phi) = \mathscr{D}^*\mathscr{D}\phi + \nabla \operatorname{Scal} \cdot \nabla\phi \tag{2.2}$$

Alternatively, this can be deduced from (2.1) and the Weitzenböck-type formula relating $\mathscr{D}^*\mathscr{D}$ and Δ^2.

If the scalar curvature is constant, then $L = \mathscr{D}^*\mathscr{D}$. In particular, $\ker L = \ker \mathscr{D}$ consists of functions with holomorphic gradient. If X has constant scalar curvature and no holomorphic vector fields, then $\ker L = \mathbb{R}$. Since L is also self-adjoint, it has index zero and so is an isomorphism between spaces of functions with mean value zero. This generalises Example 2.4.

2.2. Dependence on the Kähler structure. This section proves that the scalar curvature map is uniformly continuous under changes of the Kähler structure. The arguments are straight forward, this section simply serves to give a precise statement of estimates that will be used later.

2.2.1. C^k**-topology.** The results are first proved in the C^k-topology. The Leibniz rule implies that there is a constant C such that for tensors T, $T' \in C^k$

$$\|T \cdot T'\|_{C^k} \leq C\|T\|_{C^k}\|T'\|_{C^k}. \tag{2.3}$$

The dot stands for any algebraic operation involving tensor product and contraction. The constant C depends only on k, not on the metric used to calculated the norms (in contrast to the Sobolev analogue, which is discussed in Section 2.2.2).

Lemma 2.7. *There exist positive constants c, K, such that whenever g, g' are two different metrics on the same compact manifold, satisfying*

$$\|g' - g\|_{C^{k+2}} \leq c,$$

with corresponding curvature tensors R, R', then,

$$\|R' - R\|_{C^k} \leq K\|g' - g\|_{C^{k+2}}.$$

All norms are taken with respect to the metric g'. K depends only on c and k (and not on g or g').

Proof. Let $g = g' + h$. If the corresponding Levi–Civita connections are denoted ∇ and ∇', then $\nabla = \nabla' + a$, where a corresponds to $-\nabla' h$ under the isomorphism $T^* \otimes \operatorname{End} T \cong T^* \otimes T^* \otimes T^*$ defined by g. That is,

$$a \cdot (g' + h) = -\nabla' h,$$

where the dot is an algebraic operation. Hence

$$\begin{aligned} \|a\|_{C^{k+1}} &= \|a \cdot g'\|_{C^{k+1}}, \\ &\leq \|\nabla' h\|_{C^{k+1}} + C\|a\|_{C^{k+1}}\|h\|_{C^{k+1}}, \end{aligned}$$

(using inequality (2.3) above). Taking $c < C^{-1}$ gives

$$\|a\|_{C^{k+1}} \leq \|h\|_{C^{k+2}} \left(1 - c^{-1}\|h\|_{C^{k+2}}\right)^{-1}.$$

The difference in curvatures is given by $R - R' = \nabla' a + a \wedge a$. The result now follows from (2.3). q.e.d.

Lemma 2.8. *Given* k *and* $M > 0$*, there exist positive constants* c *and* K *such that, whenever,* g *and* g' *are two different metrics on the same compact manifold, satisfying*

$$\begin{aligned} \|g' - g\|_{C^{k+2}} &\leq c, \\ \|R'\|_{C^k} &\leq M, \end{aligned}$$

where R' *is the curvature tensor of* g'*, then*

$$\begin{aligned} \|\operatorname{Ric}' - \operatorname{Ric}\|_{C^k} &\leq K\|g' - g\|_{C^{k+2}}, \\ \|\operatorname{Scal}' - \operatorname{Scal}\|_{C^k} &\leq K\|g' - g\|_{C^{k+2}}. \end{aligned}$$

Here, Ric *and* Ric$'$*,* Scal *and* Scal$'$ *are the Ricci and scalar curvatures of* g *and* g' *respectively.*

Proof. The Ricci tensor is given by $\operatorname{Ric} = R \cdot g$ where the dot denotes contraction with the metric. Simple algebra gives

$$\operatorname{Ric}' - \operatorname{Ric} = (R' - R) \cdot g' - (R' - R) \cdot (g' - g) + R' \cdot (g' - g).$$

It follows from inequality (2.3) that $\|\operatorname{Ric}' - \operatorname{Ric}\|_{C^k}$ is controlled by a constant multiple

$$\|R' - R\|_{C^k}\|g'\|_{C^k} + \|R' - R\|_{C^k}\|g' - g\|_{C^k} + \|R'\|_{C^k}\|g' - g\|_{C^k}.$$

Since the C^k-norm of g' is constant, the result follows from Lemma 2.7.

A similar argument applies to $\operatorname{Scal} = \operatorname{Ric} \cdot g$. q.e.d.

Lemma 2.9. *Given* k *and* $M > 0$*, there exist positive constants* c *and* $K > 0$ *such that whenever* (J, ω) *and* (J', ω') *are two different Kähler structures on the same compact manifold satisfying*

$$\begin{aligned} \|(J', \omega') - (J, \omega)\|_{C^{k+2}} &\leq c, \\ \|R'\|_{C^k} &\leq M, \end{aligned}$$

where R' *is the curvature tensor of* (J', ω')*, then the linearisations* L *and* L' *of the corresponding scalar curvature maps satisfy*

$$\|(L' - L)\phi\|_{L^p_k} \leq K\|(J', \omega') - (J, \omega)\|_{C^{k+2}}\|\phi\|_{L^p_{k+4}}$$

for all $\phi \in L^p_{k+4}$*. All norms are computed with respect to the primed Kähler structure.*

Proof. The formula (2.1) shows that L is a sum of compositions of the operators Δ, $i\bar{\partial}\partial$, of multiplication by Scal, ρ and ω, and of division of top degree forms by ω^n. It suffices, then, to show that these operations satisfy inequalities analogous to that in the statement of the lemma.

For multiplication by ω, this is immediate. Since dividing top degree forms by ω^n is the same as taking the inner product with $\omega^n/n!$, it holds for this operation too. For multiplication by Scal and ρ, the inequalities follow from Lemma 2.8 and the inequality $\|uv\|_{L^p_k} \leq C\|u\|_{C^k}\|v\|_{L^p_k}$ for some C (depending only on k).

On functions, Δ is the trace of $i\bar{\partial}\partial$, so to prove the lemma it suffices to prove that $i\bar{\partial}\partial$ satisfies the relevant inequality (cf. the proof of Lemma 2.8). Since $\pi^{1,0} = \frac{1}{2}(1 - iJ)$, the operator $\partial = \pi^{1,0}\mathrm{d}$ satisfies the required inequality, similarly for $\bar{\partial}$. Hence, $i\bar{\partial}\partial$ and Δ do too. q.e.d.

2.2.2. L^p_k-topology. In the above discussion of continuity, it is possible to work with Sobolev rather than C^k norms. The same arguments apply with one minor modification. Inequality (2.3) is replaced by

$$\|T \cdot T'\|_{L^p_k} \leq C\|T\|_{L^p_k}\|T'\|_{L^p_k},$$

which holds provided $kp > 2n$. Moreover, the constant C depends on the metric through the constants appearing in the Sobolev inequalities

$$\|T\|_{C^0} \leq C'\|T\|_{L^p_k} \quad \text{for } kp > 2n, \tag{2.4}$$

$$\|T\|_{L^p_k} \leq C''\|T\|_{L^q} \quad \text{for } kp > 2n. \tag{2.5}$$

With this in mind, the same chain of reasoning which leads to Lemma 2.9 also proves.

Lemma 2.10. *Let $kp - 2n > 0$, and $M > 0$. There exist positive constants c, K, such that whenever (J, ω), (J', ω') are two different Kähler structures on the same compact complex n-dimensional manifold, satisfying*

$$\begin{aligned} \|(J', \omega') - (J, \omega)\|_{L^p_{k+2}} &\leq c, \\ \|R'\|_{L^p_k},\, C',\, C'' &\leq M, \end{aligned}$$

where R' is the curvature tensor and C', C'' are the Sobolev constants from inequalities (2.4) and (2.5) for the primed Kähler structure, then the linearisations L and L' of the corresponding scalar curvature maps satisfy

$$\|(L' - L)\phi\|_{L^p_k} \leq K\|(J', \omega') - (J, \omega)\|_{L^p_{k+2}}\|\phi\|_{L^p_{k+4}}$$

for all $\phi \in L^p_{k+4}$. All norms are computed with respect to the primed Kähler structure.

3. Approximate Solutions

Return now to the case where X is a compact connected complex surface and $\pi\colon X \to \Sigma$ is a holomorphic submersion onto a smooth high genus curve with fibres of genus at least 2.

This section constructs families of metrics on X, each depending on a parameter r, which have approximately constant scalar curvature. During this procedure, various power series expansions in negative powers of r will be used. Questions of convergence with respect to various Banach space norms will be addressed later. For now, the expression $f(r) = O(r^{-n})$ is to be interpreted as holding pointwise. When such an expression is used to describe an operator, it should be interpreted as holding after the operator acts on a function.

The ultimate aim of this chapter is to construct, for each non-negative integer n, a family of metrics $\omega_{r,n}$ parametrised by r, satisfying

$$\mathrm{Scal}(\omega_{r,n}) = -1 + \sum_{i=1}^{n} c_i r^{-i} + O\left(r^{-n-1}\right),$$

where $c_1, \ldots, c_n$ are constants. This is accomplished in Theorem 3.14.

3.1. The first order approximate solution.

Recall the classes

$$\kappa_r = -2\pi\left(c_1(V) + r c_1(\Sigma)\right)$$

mentioned (up to a factor of 2π) in the introduction. Here, V denotes the vertical tangent bundle over X and r is a positive real number.

Lemma 3.1. *For all sufficiently large r, κ_r is a Kähler class. Moreover, it contains a Kähler representative ω_r whose fibrewise restriction is the canonical hyperbolic metric on that fibre.*

Proof. Each fibre has a canonical hyperbolic metric. These metrics define a Hermitian structure in the holomorphic bundle $V \to X$. Denote the corresponding curvature form by F_V, and define a closed real $(1,1)$-form by $\omega_0 = -iF_V$. Notice that $[\omega_0] = -2\pi c_1(V)$, and the fibrewise restriction of ω_0 is the hyperbolic metric of that fibre.

Since the fibrewise restriction of ω_0 is non-degenerate, it defines a splitting $TX = V \oplus H$, where

$$H_x = \{u \in T_x X : \omega_0(u,v) = 0 \quad \text{for all } v \in V_x\}.$$

Let ω_Σ be any Kähler form on the base, scaled so that $[\omega_\Sigma] = -2\pi c_1(\Sigma)$. The form ω_Σ (pulled back to X) is a pointwise basis for the purely horizontal $(1,1)$-forms. This means that, with respect to the vertical–horizontal decomposition,

$$\omega_0 = \omega_\sigma \oplus \theta\omega_\Sigma$$

for some function $\theta \colon X \to \mathbb{R}$, where ω_σ is the hyperbolic Kähler form on the fibre S_σ over σ.

For $r > -\inf \theta$, the closed real $(1,1)$-form

$$\omega_r = \omega_0 + r\pi^* \omega_\Sigma$$

is positive, and hence Kähler, with $[\omega_r] = \kappa_r$. Its restriction to S_σ is ω_σ as required. q.e.d.

Definition 3.2. The vertical Laplacian, denoted Δ_V, is defined by

$$(\Delta_V \phi)\, \omega_\sigma = i(\bar{\partial}\partial\phi)_{VV},$$

where $(\alpha)_{VV}$ denotes the purely vertical component of a $(1,1)$-form α. The fibrewise restriction of Δ_V is the Laplacian determined by ω_σ.

The horizontal Laplacian, denoted Δ_H, is defined by

$$(\Delta_H \phi)\, \omega_\Sigma = (i\bar{\partial}\partial\phi)_{HH},$$

where $(\alpha)_{HH}$ denotes the purely horizontal component of a $(1,1)$-form α. On functions pulled up from the base Δ_H is the Laplacian determined by ω_Σ.

Lemma 3.3.

$$\operatorname{Scal}(\omega_r) = -1 + r^{-1}\left(\operatorname{Scal}(\omega_\Sigma) - \theta + \Delta_V \theta\right) + O(r^{-2}). \tag{3.1}$$

Proof. The short exact sequence of holomorphic bundles

$$0 \to V \to TX \to H \to 0$$

induces an isomorphism $K_X \cong V^* \otimes H^*$. This implies that the Ricci form of ω_r is given by $\rho_r = i(F_V + F_H)$, where F_V and F_H are the curvature forms of V and H, respectively.

The metric on the horizontal tangent bundle is $(r+\theta)\omega_\Sigma$. Its curvature is given by

$$iF_H = \rho_\Sigma + i\bar{\partial}\partial \log(1 + r^{-1}\theta),$$

where ρ_Σ is the Ricci form of ω_Σ. The curvature of the vertical tangent bundle has already been considered in the definition $\omega_0 = -iF_V$. Hence,

$$\rho_r = -\omega_\sigma - \theta\omega_\Sigma + \rho_\Sigma + i\bar{\partial}\partial \log(1 + r^{-1}\theta). \tag{3.2}$$

Taking the trace gives

$$\operatorname{Scal}(\omega_r) = -1 + \frac{\operatorname{Scal}(\omega_\Sigma) - \theta}{r + \theta} + \Delta_r \log(1 + r^{-1}\theta).$$

Where Δ_r is the Laplacian determined by ω_r. Using the formula

$$\Delta_r = \Delta_V + \frac{\Delta_H}{r + \theta} \tag{3.3}$$

and expanding out in powers of r^{-1} proves the result. q.e.d.

Since $\mathrm{Scal}(\omega_r) = -1 + O(r^{-1})$, setting $\omega_{r,0} = \omega_r$ gives the first family of approximate solutions.

3.2. The second order approximate solution. Let L_r denote the linearisation of the scalar curvature map on Kähler potentials determined by ω_r. The r dependence of L_r will be of central importance in the proof of Theorem 1.1. Its study will essentially occupy the remainder of this article. A first step in this direction is provided by the following lemma.

Lemma 3.4.

$$L_r = \Delta_V^2 + \Delta_V + O(r^{-1}).$$

Proof. Recall the formula (2.1) for L_r:

$$L_r(\phi) = \Delta_r^2\phi - \mathrm{Scal}(\omega_r)\Delta_r\phi + \frac{2i\bar{\partial}\partial\phi \wedge \rho_r}{\omega_r^2}.$$

Equations (3.1)–(3.3) give the r dependence of $\mathrm{Scal}(\omega_r)$, ρ_r and Δ_r, respectively. Direct calculation gives the result. q.e.d.

Remark. Notice that the $O(1)$ term of L_r is the first order variation in the scalar curvature of the fibres (see Example 2.4). This can be seen as an example of the dominance of the local geometry of the fibre in an adiabatic limit.

Instead of using a calculation as above, this result can be seen directly from formula (3.1). The $O(1)$ term in $\mathrm{Scal}(\omega_r)$ is $\mathrm{Scal}(\omega_\sigma)$. Rather than considering a Kähler potential as a change in ω_r, it can be thought of as a change in ω_0. This gives a corresponding change in ω_σ and the $O(1)$ effect on $\mathrm{Scal}(\omega_r)$ is precisely that claimed.

Given a function $\phi \in C^\infty(X)$, taking the fibrewise mean value gives a function $\pi_\Sigma\phi \in C^\infty(\Sigma)$. The projection maps and π_Σ and $1 - \pi_\Sigma$ determine a splitting

$$C^\infty(X) = C_0^\infty(X) \oplus C^\infty(\Sigma),$$

where $C_0^\infty(X)$ denotes functions with fibrewise mean value zero.

The previous lemma implies that, at least to $O(r^{-1})$, functions in the image of L_r have fibrewise mean value zero. It is because of this that the $C_0^\infty(X)$ and $C^\infty(\Sigma)$ components of the errors in $\mathrm{Scal}(\omega_r)$ must be dealt with differently.

3.2.1. The correct choice of ω_Σ.

Notice that the definition of ω_r has, so far, involved an arbitrary metric on Σ.

Theorem 3.5. *Each conformal class on Σ contains a unique representative ω_Σ such that*

$$\pi_\Sigma \operatorname{Scal}(\omega_r) = -A - r^{-1} + O\left(r^{-2}\right),$$

where A is the area of a fibre in its hyperbolic metric.

Proof. By Lemma 3.3, the $C^\infty(\Sigma)$ component of the $O(r^{-1})$ term in $\operatorname{Scal}(\omega_r)$ is $\operatorname{Scal}(\omega_\Sigma) - \pi_\Sigma\theta$. It follows from the definition of θ (as the horizontal part of $\omega_0 = -iF_V$ divided by ω_Σ) that

$$\pi_\Sigma\theta = -A^{-1}\Lambda_\Sigma\pi_*(F_V^2),$$

where Λ_Σ is the trace on $(1,1)$-forms on Σ determined by ω_Σ.

Write $\alpha = -\pi_*(F_V^2)$. The surface X determines a map to the moduli space of curves. The form α is a representative for the pull back of the first tautological class via this map. (See, e.g., [**5**].) Since the first tautological class is ample, $\int \alpha \geq 0$. The results of [**8**] (which discusses prescribing curvature on curves) can now be applied. They prove the existence of a unique metric in each conformal class with $\operatorname{Scal} - A^{-1}\Lambda\alpha = -1$. q.e.d.

From now on, this choice of metric is assumed to be included in the definition of ω_r.

3.2.2. The correct choice of Kähler potential ϕ_1.

Let Θ_1 denote the $C_0^\infty(X)$ component of the $O(r^{-1})$ term in $\operatorname{Scal}(\omega_r)$. This means that

$$\operatorname{Scal}(\omega_r) = -1 + r^{-1}(\Theta_1 - 1) + O(r^{-2}).$$

It follows from Lemma 3.4 that

$$\operatorname{Scal}(\omega_r + ir^{-1}\bar\partial\partial\phi) = \operatorname{Scal}(\omega_r) + r^{-1}(\Delta_V^2 + \Delta_V)\phi + O(r^{-2}). \tag{3.4}$$

Lemma 3.6. *Let $\Theta \in C_0^\infty(X)$. There exists a unique $\phi \in C_0^\infty(X)$ such that*

$$\left(\Delta_V^2 + \Delta_V\right)\phi = \Theta.$$

Proof. Given a function $\phi \in C^\infty(X)$, let ϕ_σ denote the restriction of ϕ to S_σ. The fibrewise restriction of the operator $\Delta_V^2+\Delta_V$ is the first order variation of the scalar curvature of the fibre. Applying Example 2.4 fibrewise certainly gives a unique function ϕ on X such that ϕ has fibrewise mean value zero; for each σ, $\phi_\sigma \in C^\infty(S_\sigma)$ and $L_\sigma\phi_\sigma = \Theta_\sigma$, i.e., $(\Delta_V^2 + \Delta_V)\phi = \Theta$. It only remains to check that ϕ is smooth transverse to the fibres. (The operator $\Delta_V^2 + \Delta_V$ is only elliptic in the fibre directions, so regularity only follows automatically in those directions.)

In fact, this is straight forward. Since $\phi_\sigma = L_\sigma^{-1}\Theta_\sigma$, the required differentiability follows from that of Θ and the fact that L_σ is a smooth family of differential operators. q.e.d.

Applying this lemma to $\Theta = -\Theta_1$ and using Equation (3.4) shows that there exists a unique $\phi_1 \in C_0^\infty(X)$ such that the metric $\omega_{r,1} = \omega_r + i\bar\partial\partial r^{-1}\phi_1$ is an $O(r^{-2})$ approximate solution to the constant scalar curvature equation:

$$\operatorname{Scal}(\omega_{r,1}) = -1 - r^{-1} + O(r^{-2}).$$

3.3. The third order approximate solution. Now that the correct metric has been found on the base, the higher order approximate solutions are constructed recursively. In order to demonstrate the key points clearly, however, this section does the first step in detail.

The strategy is straight forward, even if the notation sometimes is not. The first step is to find a Kähler potential f_1 on the base to deal with the $C^\infty(\Sigma)$ component of the $O(r^{-2})$ error; that is, so that

$$\operatorname{Scal}(\omega_{r,1} + i\bar\partial\partial f_1) = -1 - r^{-1} + (c + \Theta_2')r^{-2} + O(r^{-3})$$

for some constant c, where Θ_2' has fibrewise mean value zero.

The second step is to find a Kähler potential ϕ_2 to deal with the remaining $O(r^{-2})$ error Θ_2'; that is, so that

$$\operatorname{Scal}\left(\omega_{r,1} + i\bar\partial\partial(f_1 + r^{-2}\phi_2)\right) = -1 - r^{-1} + cr^{-2} + O(r^{-3}).$$

Both of the potentials f_1 and ϕ_2 are found as solutions to linear partial differential equations. To find the relevant equations, it is important to understand the linearisation of the scalar curvature map on Kähler potentials determined by $\omega_{r,1}$ (and the operators determined by the later, higher order, approximate solutions). To this end, the first lemma in this section deals with the r dependence of such an operator when the fibrewise metrics are not necessarily the canonical constant curvature ones. First, some notation.

Notation for Lemma 3.11. Let Ω_0 be any closed real $(1,1)$-form whose fibrewise restriction is Kähler. Let Ω_σ be the Kähler form on S_σ induced by Ω_0. Let Ω_Σ be any choice of metric on the base. As in Lemma 3.1, for large enough r, the form $\Omega_r = \Omega_0 + r\Omega_\Sigma$ is Kähler; the vertical–horizontal decomposition of the tangent bundle determined by Ω_r depends only on Ω_0.

Definition 3.7. The form Ω_Σ is a pointwise basis for the horizontal $(1,1)$-forms. Define a function ξ as follows: write the vertical–horizontal decomposition of Ω_0 (with respect to Ω_r) as $\Omega_0 = \Omega_\sigma \oplus \xi\Omega_\Sigma$.

Definition 3.8. The family of fibrewise Kähler metrics Ω_σ determines a Hermitian structure in the vertical tangent bundle. Denote the curvature of this bundle as F_V. Define a function η as follows: write the horizontal–horizontal component of iF_V (with respect to Ω_r) as $\eta\Omega_\Sigma$.

Remark. Since the fibrewise metrics are not the canonical constant curvature ones, this curvature form is not the same as that appearing earlier. If, instead of any old Ω_0 and Ω_Σ, the forms ω_0 and ω_Σ from before are used in both of these definitions, then $\xi = -\eta = \theta$.

Definition 3.9. Taking the fibrewise mean value of η gives a function $\pi_\Sigma\eta$ on the base. Using this, define a fourth order differential operator

$$D_\Sigma : C^\infty(\Sigma) \to C^\infty(\Sigma),$$

$$D_\Sigma(f) = \Delta_\Sigma^2 f - (\mathrm{Scal}(\Omega_\Sigma) + \pi_\Sigma\eta)\Delta_\Sigma f,$$

where Δ_Σ is the Ω_Σ-Laplacian.

Remark 3.10. The operator D_Σ is the linearisation of a non-linear map on functions, which is now described. Taking the fibrewise mean value of $\eta\Omega_\Sigma$ defines a 2-form on the base Σ. Notice this is independent of the choice of Ω_Σ. The trace of this form with respect to Ω_Σ is precisely the fibrewise mean value of η.

Next, consider varying Ω_Σ by a Kähler potential $f \in C^\infty(\Sigma)$. Denote by $\Lambda_{\Sigma,f}$ the trace operator determined by $\Omega_\Sigma + i\bar\partial\partial f$. The equation

$$\Lambda_{\Sigma,f} = \frac{\Lambda_\Sigma}{1 + \Delta_\Sigma f}$$

shows that the linearisation at 0 of the map $f \mapsto \pi_\Sigma\eta$ is $-\pi_\Sigma\eta\Delta_\Sigma$. Combining this with the formula for the linearisation of the scalar curvature map on curves derived in Example 2.4, shows that D_Σ is the linearisation, at 0, of the map

$$F\colon f \mapsto \mathrm{Scal}(\Omega_\Sigma + i\bar\partial\partial f) + \pi_\Sigma\eta.$$

If, instead of using any old Ω_0, the definitions were made using ω_0 from the earlier, then the map F is one which has been described before. It is precisely the map which was shown to take the value -1 at ω_Σ (see Section 3.2.1). Notice that using ω_0 and ω_Σ to define D_Σ gives $D_\Sigma = \Delta_\Sigma^2 + \Delta_\Sigma$. As in Example 2.4, this operator is an isomorphism on functions of mean value zero (when considered as a map between the relevant Sobolev spaces).

The vertical and horizontal Laplacians are defined just as before, with Ω_0 and Ω_Σ replacing ω_0 and ω_Σ respectively (see Definition 3.2). To indicate that they are defined with respect to different forms (and also a different vertical–horizontal decomposition of the tangent bundle, notice), the vertical and horizontal Laplacians determined by Ω_0 and Ω_Σ are denoted Δ'_V and Δ'_H. The un-primed symbols are reserved for the vertical and horizontal Laplacians determined by ω_0 and ω_Σ. Let $L(\Omega_r)$ denote the linearisation of the scalar curvature map on Kähler potentials defined by Ω_r.

Lemma 3.11.

$$L(\Omega_r) = (\Delta'^2_V - \mathrm{Scal}(\Omega_\sigma)\Delta'_V) + r^{-1}D_1 + r^{-2}D_2 + O(r^{-3}),$$

where the operators D_1 and D_2 have the following behaviour: if f is a function pulled back from Σ,

$$D_1(f) = 0, \tag{3.5}$$

$$\pi_\Sigma D_2(f) = D_\Sigma(f). \tag{3.6}$$

Proof. The proof given here is a long calculation. A slightly more conceptual proof is described in a following remark. Recall the formula (2.1) for the linearisation of the scalar curvature map. It involves the Laplacian, the scalar curvature and the Ricci form of Ω_r. Repeating the calculations that were used when the fibres had cscK metrics gives formulae for these objects. They are, respectively,

$$\Delta_{\Omega_r} = \Delta'_V + \frac{\Delta'_H}{r+\xi}, \tag{3.7}$$

$$\mathrm{Scal}(\Omega_r) = \mathrm{Scal}(\Omega_\sigma) + \frac{\mathrm{Scal}(\Omega_\Sigma)+\eta}{r+\xi} + \Delta_{\Omega_r}\log(1+r^{-1}\xi), \tag{3.8}$$

$$\rho(\Omega_r) = \rho(\Omega_\sigma) + (\mathrm{Scal}(\Omega_\Sigma)+\eta)\,\Omega_\Sigma + i\bar\partial\partial\log(1+r^{-1}\xi). \tag{3.9}$$

The result now follows from routine manipulation and expansion of power series; the following formulae can be verified for D_1 and D_2:

$$\begin{aligned} D_1 &= 2\Delta'_V\Delta'_H - (\Delta'_V\xi)\Delta'_V,\\ D_2 &= \Delta'^2_H - (\mathrm{Scal}(\Omega_\Sigma)+\eta)\Delta'_H - \eta\Delta'_V\Delta'_H\\ &\quad + \frac{1}{2}(\Delta'_V(\xi^2))\Delta'_V + (\Delta'_V\eta)\Delta'_H. \end{aligned}$$

The statements about $D_1(f)$ and $\pi_\Sigma D_2(f)$ for f pulled up from the base follow from these equations. q.e.d.

Remark. The actual equations for D_1 and D_2 will not be needed in what follows. All that will be used is their stated behaviour on functions on the base as stated in Lemma 3.11. This behaviour can be understood, without laborious calculation, as follows.

The potential f can be thought of as altering Ω_Σ, rather than Ω_r. Since Ω_Σ is scaled by r in the definition of Ω_r, the effect is equivalent to adding the potential $r^{-1}f$ to Ω_Σ. The analogue of Equation (3.1) shows that the lowest order effect of Ω_Σ on $\operatorname{Scal}(\Omega_r)$ occurs at $O(r^{-1})$. Hence, the combined effect is $O(r^{-2})$: for potentials f pulled up from the base, $D_1(f) = 0$.

The fibrewise mean value of the $O(r^{-1})$ term in $\operatorname{Scal}(\Omega_r)$ is

$$\operatorname{Scal}(\Omega_\Sigma) + \pi_\Sigma \eta.$$

So, after taking the fibrewise mean value, a change of $r^{-1}f$ in Ω_Σ gives a change in $\pi_\Sigma \operatorname{Scal}(\Omega_r)$ whose $O(r^{-2})$ term is given by the derivative of the above expression with respect to Kähler potentials on the base. Hence, $\pi_\Sigma D_2(f) = D_\Sigma(f)$.

3.3.1. The correct choice of Kähler potential f_1.

Let $L_{r,1}$ be the linearisation of the scalar curvature map on Kähler potentials determined by $\omega_{r,1}$.

Lemma 3.12. *Let $f \in C^\infty(\Sigma)$. Then*

$$\pi_\Sigma L_{r,1}(f) = r^{-2}(\Delta_\Sigma^2 + \Delta_\Sigma)f + O(r^{-3}).$$

Proof. Begin by applying Lemma 3.11 with

$$\begin{aligned} \Omega_0 &= \omega_0 + i\bar\partial\partial r^{-1}\phi_1, \\ \Omega_\Sigma &= \omega_\Sigma. \end{aligned}$$

There is a slight difficultly in interpreting the expansion given in Lemma 3.11. The r-dependence of Ω_0 means that the coefficients in the $O(r^{-3})$ piece of that expansion will be r-dependent, a priori making them of higher order overall.

In fact, this cannot happen. The reason is that all such coefficients come ultimately from analytic expressions in the fibrewise metrics induced by Ω_0 (as is shown, for example, by the calculation described in the proof of Lemma 3.11). These metrics have the form

$$\Omega_\sigma = (1 + r^{-1}\Delta_V \phi_1)\omega_\sigma.$$

(Here, Δ_V is the vertical Laplacian determined by ω_0.) Since the fibrewise metric is algebraic in r^{-1}, the coefficients in the expression form Lemma 3.11 are analytic in r^{-1}, i.e., they have expansions involving only non-positive powers of r.

This means that the $O(r^{-2})$ term can simply be read off from the formula given in Lemma 3.11. This gives

$$\pi_\Sigma L_{r,1}(f) = r^{-2}D_\Sigma(f) + O(r^{-3}).$$

As is pointed out in Remark 3.10, for the choice of ω_Σ that was determined whilst finding the $O(r^{-2})$ approximate solution, $D_\Sigma = \Delta_\Sigma^2 + \Delta_\Sigma$ as required. q.e.d.

Denote the $C^\infty(\Sigma)$ component of the $O(r^{-2})$ term of $\mathrm{Scal}(\omega_{r,1})$ by Θ_2:

$$\pi_\Sigma \,\mathrm{Scal}(\omega_{r,1}) = -1 - r^{-1} + r^{-2}\Theta_2 + O(r^{-3}).$$

Let c be the mean value of Θ_2 and let f_1 solve $(\Delta_\Sigma^2 + \Delta_\Sigma)f_1 = c - \Theta_2$. (Existence of f_1 follows from Example 2.4.) Elliptic regularity ensures that f_1 is smooth, completing the first step in finding the $O(r^{-3})$ approximate solution: by Lemma 3.12,

$$\mathrm{Scal}(\omega_{r,1} + i\bar{\partial}\partial f_1) = -1 - r^{-1} + r^{-2}(c + \Theta_2') + O(r^{-3}),$$

where Θ_2' has fibrewise mean value zero.

3.3.2. The correct choice of Kähler potential ϕ_2.

Let $L'_{r,1}$ be the linearisation of the scalar curvature map on Kähler potentials determined by the metric $\omega_{r,1} + i\bar{\partial}\partial f_1$.

Lemma 3.13.

$$L'_{r,1} = \Delta_V^2 + \Delta_V + O(r^{-1}).$$

Remark. Again, the symbol Δ_V means the vertical Laplacian determined by the form ω_0. Compare this with Lemma 3.4.

Proof. Apply Lemma 3.11 with

$$\begin{aligned} \Omega_0 &= \omega_0 + i\bar{\partial}\partial r^{-1}\phi_1, \\ \Omega_\Sigma &= \omega_\Sigma + i\bar{\partial}\partial r^{-1} f_1. \end{aligned}$$

As in the proof of Lemma 3.12, there is a problem with interpreting the expansion in Lemma 3.11, namely that the r-dependence of Ω_0 and Ω_Σ means that the coefficients in the expansion are also r-dependent. As in the proof of Lemma 3.12, however, this actually causes no difficulty. Both forms are algebraic in r^{-1}, hence, the coefficients in the expansion are analytic in r^{-1}; the r-dependence of the coefficient of r^{-n} causes changes only at $O(r^{-n-k})$ for $k \geq 0$. This means that the genuine $O(1)$ behaviour of $L'_{r,1}$ is the same as the $O(1)$ behaviour of $\Delta_V'^2 - \mathrm{Scal}(\Omega_\sigma)\Delta_V'$. Here, Ω_σ is the metric on S_σ determined by Ω_0, i.e., $\Omega_\sigma = (1 + r^{-1}\Delta_V\phi_1)\omega_\sigma$.

Since, to $O(1)$, Ω_σ and ω_σ agree,

$$\begin{aligned} \mathrm{Scal}(\Omega_\sigma) &= \mathrm{Scal}(\omega_\sigma) + O(r^{-1}), \\ \Delta_V' &= \Delta_V + O(r^{-1}). \end{aligned}$$

Hence,

$$\Delta_V'^2 - \mathrm{Scal}(\Omega_\sigma)\Delta_V' = \Delta_V^2 + \Delta_V + O(r^{-1}).$$

q.e.d.

Let $\omega_{r,2} = \omega_{r,1} + i\bar{\partial}\partial(f_1 + r^{-2}\phi_2)$, where ϕ_2 solves $(\Delta_V^2 + \Delta_V)\phi_2 = -\Theta_2'$. (Existence of ϕ_2 follows from Lemma 3.6.) Then,

$$\mathrm{Scal}(\omega_{r,2}) = -1 - r^{-1} + cr^{-2} + O(r^{-3}).$$

3.4. The higher order approximate solutions.

Theorem 3.14 (Approximately cscK metrics)**.** *Let n be a positive integer. There exist functions $f_1, \dots, f_{n-1} \in C^\infty(\Sigma)$, $\phi_1, \dots, \phi_n \in C_0^\infty(X)$ and constants $c_1, \dots c_n$ such that the metric*

$$\omega_{r,n} = \omega_r + i\bar{\partial}\partial \sum_{i=1}^{n-1} r^{-i+1} f_i + i\bar{\partial}\partial \sum_{i=1}^{n} r^{-i}\phi_i$$

satisfies

$$\mathrm{Scal}(\omega_{r,n}) = -1 + \sum_{i=1}^{n} c_i r^{-i} + O\left(r^{-n-1}\right).$$

Proof. The proof is by induction, the inductive step being the same as that used to construct the third order approximate solution above. q.e.d.

In fact, it is straight forward to show that the functions f_i, ϕ_i are unique subject to the constraints

$$\int_\Sigma f_i\,\omega_\Sigma = 0 \qquad \int_{S_\sigma} \phi_i\,\omega_\sigma = 0.$$

This uses the injectivity of the operator discussed in Example 2.4 (acting on functions with mean value zero).

It is also possible to calculate the exact values of the c_i by considering the mean value of $\mathrm{Scal}(\omega_r)$. Let $\mathrm{vol}_r = \int \omega_r^2/2 = rA + B$ where $A = [\omega_0].[\omega_\Sigma]$ and $B = \frac{1}{2}[\omega_0]^2$, and let $\int \mathrm{Scal}(\omega_r)\,\omega_r^2/2 = 2\pi c_1(X).[\omega_r] = rC + D$ where $C = 2\pi c_1(X).[\omega_0]$ and $D = 2\pi c_1(X).[\omega_\Sigma]$. Then, the mean value of the scalar curvature is $(rC+D)(rA+B)^{-1} = -1 + \sum c_i r^{-i}$ where

$$c_i = (-1)^i A^{-i} B^i (A^{-1}C - B^{-1}D).$$

3.5. Summary. Four essential facts were used in the construction of the approximate solutions in Theorem 3.14.

1. The non-linear partial differential equation $\mathrm{Scal}(\omega_\sigma) = $ constant in the fibre directions has a solution. This enables $\omega_{r,0}$ to be constructed.
2. The linearisation of this equation, at a solution, is surjective onto functions with mean value zero. This enables the $C_0^\infty(X)$ components of error terms to be eliminated.

(3) The non-linear partial differential equation $\mathrm{Scal}(\omega_\Sigma) - \Lambda_\Sigma \alpha =$ constant on the base has a solution. This enables $\omega_{r,1}$ to be constructed.
(4) The linearisation of this equation, at a solution, is surjective onto functions with mean value zero. This enables the $C^\infty(\Sigma)$ components of error terms to be made constant.

The equation in (3) and (4) is needed only because the linearisation of (2) does not map onto functions pulled up from the base. The surjectivity of the linear operators in (2) and (4) can be viewed in terms of automorphisms of the solutions in (1) and (3), respectively. This is because both operators are elliptic with index zero; they are surjective if and only if they have no kernel (thought of as maps between spaces of functions with mean value zero). The absence of any kernel is equivalent to there being no non-trivial family of solutions to the equations in (1) and (3).

Finally, it should be noted that the two parameters r and n appearing in the approximate solutions are of a very different nature. In particular, whilst the perturbation to a genuine solution is carried out, n is considered as fixed, whilst r tends to infinity.

4. Applying the Inverse Function Theorem

First, some notation. The integer n is considered as fixed throughout and so is often omitted. Write g_r for the metric tensor corresponding to the Kähler form $\omega_{r,n}$. Each metric defines Sobolev spaces $L^2_k(g_r)$ of functions over X. Since the Sobolev norms determined by g_r, for different values of r, are equivalent, the spaces $L^2_k(g_r)$ contain the same functions. The constants of equivalence, however, will depend on r. Similarly, for the Banach spaces $C^k(g_r)$. When the actual norms themselves are not important, explicit reference to the metrics will be dropped, and the spaces referred to simply as C^k or L^2_k.

Statements like "$a_r \to 0$ in $C^k(g_r)$ as $r \to \infty$," mean "$\|a_r\|_{C^k(g_r)} \to 0$ as $r \to \infty$." Notice that both the norm and the object whose norm is being measured are changing with r. Similarly, statements such as "a_r is $O(r^{-1})$ in $L^2_k(g_r)$ as $r \to \infty$," mean "$\|a_r\|_{L^2_k(g_r)}$ is $O(r^{-1})$ as $r \to \infty$."

Let L_r denote the linearisation of the scalar curvature map on Kähler potentials determined by $\omega_{r,n}$. As with Example 2.4, this derivative will be shown to be an isomorphism when considered modulo the constant functions.

Recall from Remark 2.2 that when the scalar curvature of a Kähler metric ω is constant, the corresponding linear operator maps into functions with ω-mean value zero. In the situation considered here, $\omega_{r,n}$ has nearly constant scalar curvature. So it makes sense to try and show that L_r gives an isomorphism after composing with the projection p onto functions with $\omega_{r,n}$-mean value zero.

Let $L^2_{k,0}$ denote functions in L^2_k with $\omega_{r,n}$-mean value zero. Composing the scalar curvature map with the projection p gives, for $k > 1$, a map

$$S_r \colon L^2_{k+4,0} \to L^2_{k,0}, \qquad S_r(\phi) = p\,\mathrm{Scal}(\omega_{r,n} + i\bar{\partial}\partial\phi).$$

To complete the proof of Theorem 1.1, it will be shown that for each k and sufficiently large r, there is a unique $\phi \in L^2_{k+4,0}$ with $S_r(\phi) = 0$.

In order to apply the inverse function theorem to find ϕ, it is necessary to know that $S_r(0)$ is sufficiently close to zero. Assume, for now, that $S_r(0) \to 0$ in $L^2_k(g_r)$. (Section 3 shows only that $S_r(0)$ converges to zero pointwise. Convergence in $L^2_k(g_r)$ is proved in Lemma 5.7.) Assume also that pL_r, the linearisation of S_r, is an isomorphism between spaces of functions with mean value zero. (This is proved in Theorem 6.1.) These two facts alone, however, are not sufficient to be able to apply the inverse function theorem. A close look at the statement of the inverse function theorem may clarify why.

Theorem 4.1 (Quantitative inverse function theorem)**.**

- *Let $F : B_1 \to B_2$ be a differentiable map of Banach spaces, whose derivative at 0, DF, is an isomorphism of Banach spaces, with inverse P.*
- *Let δ' be the radius of the closed ball in B_1, centred at 0, on which $F - DF$ is Lipschitz, with constant $1/(2\|P\|)$.*
- *Let $\delta = \delta'/(2\|P\|)$.*

Then, whenever $y \in B_2$ satisfies $\|y - F(0)\| < \delta$, there exists $x \in B_1$ with $F(x) = y$. Moreover, such an x is unique subject to the constraint $\|x\| < \delta'$.

(This quantitative statement of the inverse function theorem follows from the standard proof; see, for example, [**10**].)

Applying this to the map $S_r \colon L^2_{k+4,0} \to L^2_{k,0}$ (and assuming its derivative is an isomorphism) gives the existence of a δ_r such that if $\|S_r(0)\|_{L^2_k(g_r)} < \delta_r$, then there exists a ϕ with $S_r(\phi) = 0$. The proof will be completed by showing that (for any choice of $n \geq 6$) $S_r(0)$ converges to zero more quickly than δ_r.

5. Local Analysis

To control the constants appearing in the local analytic estimates, this section constructs a local (over the base) model for the metrics $\omega_{r,n}$.

The notation used here is all defined in Section 3. In particular, the forms ω_0, ω_r and the function θ are defined in the proof of Lemma 3.1, whilst the higher order approximate solutions $\omega_{r,n}$ are constructed in Theorem 3.14.

5.1. Constructing the local model. Let $D \subset \Sigma$ be a holomorphic disc centred at σ_0. Since D is contractible, $X|_D$ is diffeomorphic to $S \times D$. The horizontal distribution of ω_0 is trivial when restricted to the central fibre S_{σ_0}. By applying a further diffeomorphism, if necessary, the identification $X|_D \cong S \times D$ can be arranged so that the horizontal distribution on S_{σ_0} coincides with the restriction to S_{σ_0} of the TD summand in the splitting

$$T(S \times D) \cong TS \oplus TD. \tag{5.1}$$

For each value of r, there are two Kähler structures on $S \times D$ of interest. The first comes from simply restricting the Kähler structure $(X, J, \omega_{r,n})$ to $X|_D$. The complex structure has the form $J = J_\sigma \oplus J_D$ with respect to the splitting (5.1), where J_D is the complex structure on D and J_σ is the varying complex structure on the fibres.

The second is the natural product structure. With respect to (5.1), let

$$\begin{aligned} J' &= J_S \oplus J_D, \\ \omega'_r &= \omega_S \oplus r\omega_D. \end{aligned}$$

where ω_D is the flat Kähler form on D agreeing with ω_Σ at the origin, and J_S, ω_S are the complex structure and Kähler form on the central fibre $S = S_{\sigma_0}$. Denote by g'_r the corresponding metric on $S \times D$.

It is useful to have the following result stated explicitly.

Lemma 5.1. *Let $\alpha \in C^k(T^*X^{\otimes i})$. Over $X|_D$, $\|\alpha\|_{C^k(g'_r)} = O(1)$. Moreover, if α is pulled up from the base, $\|\alpha\|_{C^k(g'_r)} = O\left(r^{-i/2}\right)$.*

Proof. These statements follow from the fact that g'_r is a product metric, scaled by r in the D directions. q.e.d.

Theorem 5.2. *For all $\varepsilon > 0$, $\sigma_0 \in \Sigma$, there exists a holomorphic disc $D \subset \Sigma$, centred at σ_0, such that for all sufficiently large r, over $X|_D$,*

$$\|(J', \omega'_r) - (J, \omega_{r,n})\|_{C^k(g'_r)} < \varepsilon.$$

Proof. First notice that, by Lemma 5.1, for any holomorphic disc $D \subset \Sigma$, over $X|_D$,

$$\|\omega_{r,n} - \omega_{r,0}\|_{C^k(g'_r)} = O(r^{-1}).$$

Since $\omega_{r,n} - \omega_{r,0}$ is $O(r^{-1})$ in $C^k(g'_r)$, it suffices to prove the result just for $n = 0$.

Choose a holomorphic disc D centred at σ_0. The splitting (5.1) is parallel, implying that

$$\nabla^i(J - J') \in \mathrm{End}(TS) \otimes T^*(S \times D)^{\otimes i}.$$

The only changes in length as r varies come from the T^* factor. Write $\nabla^i(J - J')$ as $\alpha_i + \beta_i$ with respect to the splitting

$$T^*(S \times D)^{\otimes i} \cong T^*S^{\otimes i} \oplus \Big(\big(T^*S^{\otimes i-1} \otimes T^*D\big) \oplus \cdots \oplus T^*D^{\otimes i}\Big).$$

$$\alpha_i \in T^*S^{\otimes i}, \qquad \beta_i \in \big(T^*S^{\otimes i-1} \otimes T^*D\big) \oplus \cdots \oplus T^*D^{\otimes i}.$$

The metric g'_r does not change in the S-directions, so $|\alpha_i|_{g'_r}$ is independent of r. Since $J = J'$ on the central fibre, reducing the size of D ensures that $|\alpha_i|_{g'_r}$ is less than $\varepsilon/(2k + 2)$.

The metric g'_r scales lengths of cotangent vectors by $r^{-1/2}$ in the base directions. So, $|\beta_i|_{g'_r} = O(r^{-1/2})$; for large enough r, $|\beta_i|_{g'_r}$ is less than $\varepsilon/(2k + 2)$. Hence,

$$\|\nabla^i(J - J')\|_{C^0(g'_r)} < \varepsilon/(k + 1).$$

Summing from $i = 0, \dots, k$ proves $\|J' - J\|_{C^k(g'_r)} < \varepsilon$.

To prove $\|\omega'_r - \omega_r\|_{C^k(g'_r)} < \varepsilon$, it is enough to prove the same result for the metrics g_r, g'_r (since the Kähler forms can be recovered algebraically from the metric tensors via the complex structures).

Let u_1, u_2 be a local g_S-orthonormal frame for TS and v_1, v_2 be a local g_D-orthonormal frame for TD. Recall g_r induces a different horizontal–vertical splitting of the tangent bundle of $S \times D$, which is independent of r. With respect to this splitting, $g_r = g_\sigma \oplus (r + \theta) g_\Sigma$, where g_σ is the hyperbolic metric on the fibre S_σ, and g_Σ the metric on the base. Write $v_j = \eta_j + \xi_j$ with respect to the horizontal–vertical splitting induced by g_r (η_j is horizontal, ξ_j is vertical). With respect to the g'_r-orthonormal frame $u_1, u_2, r^{-1/2}v_1, r^{-1/2}v_2$, the matrix representative for g_r is

$$\begin{pmatrix} g_\sigma(u_i, u_j) & r^{-1/2} g_\sigma(u_i, \xi_j) \\ & \\ r^{-1/2} g_\sigma(u_j, \xi_i) & \big(1 + r^{-1}\theta\big)\, g_\Sigma(\eta_i, \eta_j) + r^{-1} g_\sigma(\xi_i, \xi_j) \end{pmatrix}$$

This means that, in a g'_r-orthonormal frame, $g_r - g'_r$ has the matrix representative

$$\begin{pmatrix} g_\sigma(u_i, u_j) - \delta_{ij} & 0 \\ & \\ 0 & g_\Sigma(\eta_i, \eta_j) - \delta_{ij} \end{pmatrix} + r^{-1/2} A + r^{-1} B.$$

for fixed matrices A and B.

The top left corner of the first term vanishes along the central fibre. Just as in the proof of $\|J' - J\| < \varepsilon$, this can be made arbitrarily small in $C^k(g'_r)$ by shrinking D and taking r large.

The bottom right corner of the first term is a function of the D-variables only. By construction, it vanishes at the origin. The $C^0(g'_r)$-norm of this piece is just the conventional C^0-norm of the function $g_\Sigma(\eta_i, \eta_j) - \delta_{ij}$ and hence can be made arbitrarily small by shrinking D.

The derivatives of this piece are all in the D-directions. The length of the i-th derivative is $O(r^{-i/2})$ due to the scaling of g'_r in the D-directions. Hence, the $C^k(g'_r)$ norm of this piece can be made arbitrarily small by taking r large (once D has been shrunk to deal with the C^0 term).

Finally, since A and B are independent of r, the $C^k(g'_r)$-norms of the tensors they represent are bounded as $r \to \infty$. So $r^{-1/2}A$, $r^{-1}B \to 0$ in $C^k(g'_r)$, which proves the theorem. q.e.d.

5.2. Analysis in the local model. This section states various analytic results concerning the Kähler product $S \times \mathbb{C}$. Since this manifold is not compact, the results are not standard per se. The situation is almost identical, however, to that of "tubes" considered in instanton Floer homology. A tube is the four-manifold $Y \times \mathbb{R}$, where Y is a compact three-manifold. The proofs of the following Sobolev inequalities (as given in Chapter 3 of [4]) carry over almost verbatim.

Lemma 5.3. *For indices k,l, and $q \geq p$ satisfying $k - 4/p \geq l - 4/q$ there is a constant c (depending only on p, q, k and l) such that for all $\phi \in L^p_k(S \times \mathbb{C})$,*

$$\|\phi\|_{L^q_l} \leq c\|\phi\|_{L^p_k}.$$

For indices p, k satisfying $k - 4/p > 0$, there is a constant c depending only on k and p, such that for all $\phi \in L^p_k(S \times \mathbb{C})$,

$$\|\phi\|_{C^0} \leq c\|\phi\|_{L^p_k}.$$

The Kähler structure on $S \times \mathbb{C}$ determines a scalar curvature map on Kähler potentials. Denote the linearisation of this map by L'. Again, using the same arguments as in Chapter 3 of [4] gives the following elliptic estimate for L'. (The elliptic operator considered in [4] is not L'; the arguments given there apply, however, to any elliptic operator determined by the local geometry.)

Lemma 5.4. *There exists a constant A such that for all $\phi \in L^2_{k+4}(S \times \mathbb{C})$,*

$$\|\phi\|_{L^2_{k+4}} \leq A\left(\|\phi\|_{L^2} + \|L'(\phi)\|_{L^2_k}\right).$$

Lemma 5.5. *There exists a constant* C *such that for any compactly supported* $u \in C^{k+4}(\mathbb{C})$*, and any* $\phi \in L^2_{k+4}(S \times \mathbb{C})$*,*

$$\|L'(u\phi) - uL'(\phi)\|_{L^p_k} \quad \leq \quad C \sum_{j=1}^{k+4} \|\nabla^j u\|_{C^0} \, \|\phi\|_{L^p_{k+4}},$$

Proof. This follows from the fact that the coefficients of L' are constant in the $\mathbb{C}$ directions. q.e.d.

5.3. Local analysis for $\omega_{r,n}$. This section explains how to use Theorem 5.2 to convert the results over $S\times\mathbb{C}$ from above to uniform estimates over $(X, J, \omega_{r,n})$.

First, notice that by Theorem 5.2, with $\varepsilon < 1$, over $X|_D$, $g_r - g'_r$ is uniformly bounded in $C^k(g'_r)$. Moreover, the choice of ε ensures the metrics are sufficiently close that the difference $g^{-1} - g'^{-1}$ in induced metrics on the cotangent bundle is also uniformly bounded. This means that the Banach space norms on tensors determined by g_r and g'_r are uniformly equivalent. An immediate application of this is the following.

Lemma 5.6. *For a tensor* $\alpha \in C^k(T^*X^{\otimes i})$, $\|\alpha\|_{C^k(g_r)} = O(1)$*. Moreover, if* α *is pulled up from the base,* $\|\alpha\|_{C^k(g_r)} = O\left(r^{-i/2}\right)$*.*

Proof. By Lemma 5.1, the result is true for the local model. Let D be a disc over which Theorem 5.2 applies for, say, $\varepsilon = 1/2$. Since $C^k(g_r)$ and $C^k(g'_r)$ are uniformly equivalent over $X|_D$, the result holds for $C^k(g_r)$ over $X|_D$. Cover Σ with finitely many discs D_i. The result holds for $C^k(g_r)$ over each $X|_{D_i}$ and hence over all of X. q.e.d.

Lemma 5.7.

$$\begin{aligned} \mathrm{Scal}(\omega_{r,n}) &= O\left(r^{-n-1}\right) && \text{in } C^k(g_r) \text{ as } r \to \infty, \\ \mathrm{Scal}(\omega_{r,n}) &= O\left(r^{-n-1/2}\right) && \text{in } L^2_k(g_r) \text{ as } r \to \infty. \end{aligned}$$

Proof. The expansions in negative powers of r in Chapter 3 all arise via absolutely convergent power series and algebraic manipulation. This means that with respect to a *fixed* metric g,

$$\mathrm{Scal}(\omega_{r,n}) = O\left(r^{-n-1}\right) \quad \text{in } C^k(g) \text{ as } r \to \infty.$$

For example, $\log(1 + r^{-1}\theta)$ is $O(r^{-1})$ in $C^k(g)$ because

$$\begin{aligned} \|\log(1 + r^{-1}\theta)\|_{C^k} \quad &\leq \quad \sum_{j\geq 1} r^{-(j+1)} C^j \frac{\|\theta\|^j_{C^k}}{j}, \\ &= \quad \log\left(1 + Cr^{-1}\|\theta\|_{C^k}\right). \end{aligned}$$

where C a constant such that $\|\phi\psi\|_{C^k} \leq C\|\phi\|_{C^k}\|\psi\|_{C^k}$.

The same is true with respect to the $C^k(g_r)$-norm provided that the $C^k(g_r)$-norm of a fixed function is bounded as $r \to \infty$. (Notice that the constant C above does not depend on g.) The C^k result now follows from Lemma 5.6.

To deduce the result concerning L^2_k-norms, notice that the g'_r-volume form is r times a fixed form. Hence, over a disc D where Theorem 5.2 applies with $\varepsilon = 1/2$, the g_r-volume form is $O(r)$ times a fixed form. So, the volume of $X|_D$ with respect to g_r is $O(r)$. Cover Σ with finitely many such discs, D_i. The volume vol_r of X, with respect to g_r, satisfies $\mathrm{vol}_r \leq \sum \mathrm{vol}(X|_{D_i}) = O(r)$. The result follows from this, the C^k result and the fact that $\|\phi\|_{L^2_k(g_r)} \leq (\mathrm{vol}_r)^{1/2}\|\phi\|_{C^k(g_r)}$. q.e.d.

To transfer other results from the product to X, a slightly more delicate patching argument is required. Fix $\varepsilon < 1$ and cover Σ in discs $D_1, \ldots, D_N$ satisfying the conclusions of Theorem 5.2. Let χ_i be a partition of unity subordinate to the cover D_i.

Let $\phi \in L^p_k$. Then, by the Leibniz rule and the boundedness of $\|\chi_i\|_{C^k(g_r)}$, there exists a constant a such that, for any $i = 1, \ldots, N$,

$$\|\chi_i\phi\|_{L^p_k(g_r)} \leq a\|\phi\|_{L^p_k(g_r)}. \tag{5.2}$$

Everything is now in place to transfer the estimates from $S \times \mathbb{C}$ to $(X, J, \omega_{r,n})$.

Lemma 5.8. *For indices k, l, and $q \geq p$ satisfying $k - 4/p \geq l - 4/q$, there is a constant c (depending only on p, q, k and l) such that for all $\phi \in L^p_k$ and all sufficiently large r,*

$$\|\phi\|_{L^q_l(g_r)} \leq c\|\phi\|_{L^p_k(g_r)}.$$

For indices p, k satisfying $k - 4/p \geq 0$, there is a constant c (depending only on k and p) such that for all $\phi \in L^p_k$ and all sufficiently large r,

$$\|\phi\|_{C^0} \leq c\|\phi\|_{L^p_k(g_r)}.$$

Proof. Recall the analogous result for $S \times \mathbb{C}$ (Lemma 5.3). Using the partition of unity χ_i from above,

$$\|\phi\|_{L^q_l(g_r)} \leq \sum \|\chi_i\phi\|_{L^q_l(g_r)} \leq \text{const.} \sum \|\chi_i\phi\|_{L^q_l(g'_r)},$$

(using the uniform equivalence of the g_r- and g'_r-Sobolev norms).

Considering $\chi_i\phi$ as a function over $S \times \mathbb{C}$, Lemma 5.3 gives

$$\|\chi_i\phi\|_{L^q_l(g'_r)} \leq \text{const.}\|\chi_i\phi\|_{L^p_k(g'_r)}.$$

Using the uniform equivalence of the g_r- and g'_r-Sobolev norms again gives

$$\|\chi_i\phi\|_{L^p_k(g'_r)} \leq \text{const.}\|\chi_i\phi\|_{L^p_k(g_r)}.$$

Finally, combining these inequalities and inequality (5.2) gives

$$\|\phi\|_{L^q_l(g_r)} \leq \text{const.}\sum \|\chi_i\phi\|_{L^p_k(g_r)} \leq \text{const.}\|\phi\|_{L^p_k(g_r)}.$$

The second Sobolev inequality is proved similarly. q.e.d.

Lemma 5.9. *There is a constant A, depending only on k, such that for all $\phi \in L^2_{k+4}$ and all sufficiently large r,*

$$\|\phi\|_{L^2_{k+4}(g_r)} \leq A\left(\|\phi\|_{L^2(g_r)} + \|L_r(\phi)\|_{L^2_k(g_r)}\right).$$

Proof. Recall the analogous result for L' over $S \times \mathbb{C}$ (Lemma 5.4). This time, the patching argument must be combined with Lemma 2.9 on the uniform continuity of the linearisation of the scalar curvature map. To apply this result, it is necessary to observe that the curvature tensor of g'_r is bounded in $C^k(g'_r)$. Also, ε must be taken suitably small in Theorem 5.2.

Using the uniform equivalence of g'_r- and g_r-Sobolev norms,

$$\begin{aligned}\|\phi\|_{L^2_{k+4}(g_r)} &\leq \text{const.}\sum \|\chi_i\phi\|_{L^2_{k+4}(g'_r)},\\ &\leq \text{const.}\sum\left(\|\phi\|_{L^2(g'_r)} + \|L'(\chi_i\phi)\|_{L^2_k(g'_r)}\right).\end{aligned}$$

Since the χ_i are functions on the base, by Lemmas 5.1 and 5.5,

$$\|L'(\chi_i\phi) - \chi_i L'(\phi)\|_{L^2_k(g'_r)} \leq \text{const.}r^{-1/2}\|\phi\|_{L^2_k(g'_r)}.$$

Using this, the uniform equivalence of g'_r- and g_r-Sobolev norms, and Lemma 2.9 to replace L' with L_r gives

$$\|\phi\|_{L^2_{k+4}(g_r)} \leq \text{const.}\left(\|\phi\|_{L^2(g_r)} + \|\phi\|_{L^2_k(g'_r)} + \|L_r(\phi)\|_{L^2_k(g_r)}\right).$$

This proves the result for $k = 0$. It also provides the inductive step giving the result for all k. q.e.d.

6. Global Analysis

Recall that $L^2_{k,0}$ is the Sobolev space of functions with g_r-mean value zero, whilst p is projection onto such functions. The aim of this section is to prove the following result.

Theorem 6.1. *For all large r and $n \geq 3$, the operator $pL_r : L^2_{k+4,0} \to L^2_{k,0}$ is a Banach space isomorphism. There exists a constant C, such that for all large r and all $\psi \in L^2_{k,0}$, the inverse operator P_r satisfies*

$$\|P_r \psi\|_{L^2_{k+4}(g_{r,n})} \leq C r^3 \|\psi\|_{L^2_k(g_{r,n})}.$$

Unlike the uniform local results of the previous section, controlling the inverse P_r is a global issue; indeed it is only because of global considerations (compactness of X, no holomorphic vector fields) that such an inverse exists. This means that the local model used in the previous chapter is not directly useful. Instead, a global model is used to make calculations more straight forward.

6.1. The global model. Define a Riemannian metric h_r on X by using the fibrewise metrics determined by ω_0 on the vertical vectors, and the metric $r\omega_\Sigma$ on the horizontal vectors. The metric h_r is a Riemannian submersion on $X \to (\Sigma, r\omega_\Sigma)$.

By construction, $g_{r,0} = h_r + a$ for some purely horizontal tensor $a \in s^2(T^*X)$, independent of r (which is essentially given by the horizontal components of ω_0). Since horizontal 1-forms scale by $r^{-1/2}$ in the metric h_r, it follows immediately that for all r sufficiently large,

$$\|g_{r,0} - h_r\|_{C^0(h_r)} \leq 1/2. \tag{6.1}$$

Moreover, since $\|g_r - g_{r,0}\|_{C^0(h_r)} = O\left(r^{-1}\right)$, inequality (6.1) holds with $g_{r,0}$ replaced by g_r. In particular, this means that the difference in the induced metrics on the cotangent bundle is uniformly bounded and so the L^2-norms on tensors determined by h_r and g_r are uniformly equivalent:

Lemma 6.2. *Let E denote any bundle of tensors. There exist positive constants k and K such that for all $t \in \Gamma(E)$ and all sufficiently large r,*

$$k\|t\|_{L^2(h_r)} \leq \|t\|_{L^2(g_r)} \leq K\|t\|_{L^2(h_r)}.$$

6.2. The lowest eigenvalue of $\mathscr{D}^*\mathscr{D}$. It is more convenient to work first with the positive self-adjoint elliptic operator $\mathscr{D}^*\mathscr{D}$. Here, $\mathscr{D} = \bar\partial \circ \nabla$ where $\bar\partial$ is the $\bar\partial$-operator of the holomorphic tangent bundle and $\mathscr{D}^*$ is the L^2-adjoint of $\mathscr{D}$. Recall Equation (2.2) which relates $\mathscr{D}^*\mathscr{D}$ to L.

Notice that $\mathscr{D}^*\mathscr{D}$ depends on $\omega_{r,n}$, and so on r. This section finds a lower bound for its first non-zero eigenvalue.

Lemma 6.3. *There are no non-zero holomorphic vector fields on X.*

Proof. The fibres and base of X have high genus. The short exact sequence of holomorphic bundles

$$0 \to V \to TX \to \pi^* T\Sigma \to 0$$

gives a long exact sequence in cohomology

$$0 \to H^0(X, V) \to H^0(X, TX) \to H^0(X, \pi^* T\Sigma) \to \cdots$$

$H^0(X, V) = 0$ as the fibres admit no non-zero holomorphic vector fields. Similarly, $H^0(X, \pi^* T\Sigma) = H^0(\Sigma, \pi_* \pi^* T\Sigma) = H^0(\Sigma, T\Sigma) = 0$. The result now follows from the long exact sequence. q.e.d.

Corollary 6.4. $\ker \mathscr{D}^*\mathscr{D} = \mathbb{R}$. *Equivalently,* $\mathscr{D}^*\mathscr{D} \colon L^2_{k+4,0} \to L^2_{k,0}$ *is an isomorphism.*

Proof. $\ker \mathscr{D}^*\mathscr{D} = \ker \mathscr{D}$ is those functions with holomorphic gradient. The previous lemma implies such functions must be constant. The second statement now follows from the fact that $\mathscr{D}^*\mathscr{D}$ is a self-adjoint index zero operator. q.e.d.

To find a lower bound for the first non-zero eigenvalue of $\mathscr{D}^*\mathscr{D}$, similar bounds are first found for the Hodge Laplacian and for the $\bar{\partial}$-Laplacian on sections of the holomorphic tangent bundle.

Lemma 6.5. *There exists a positive constant* C_1 *such that for all* ϕ *with* g_r*-mean value zero and all sufficiently large* r*,*

$$\|\mathrm{d}\phi\|^2_{L^2(g_r)} \geq C_1 r^{-1} \|\phi\|^2_{L^2(g_r)} .$$

Proof. There exists a constant m such that $\phi - m$ has h_1-mean value zero and, of course, $\mathrm{d}\phi = \mathrm{d}(\phi - m)$. Using Lemma 6.2, $\|\mathrm{d}\phi\|_{L^2(g_r)} \geq$ constant $\|\mathrm{d}(\phi - m)\|_{L^2(h_r)}$.

Let $|\cdot|_{h_r}$ denote the pointwise inner product defined by h_r. By definition of h_r, it follows that $|\mathrm{d}(\phi - m)|^2_{h_r} \geq r^{-1} |\mathrm{d}(\phi - m)|^2_{h_1}$. Moreover, the volume forms satisfy $\mathrm{dvol}(h_r) = r\mathrm{dvol}(h_1)$. Hence,

$$\|\mathrm{d}(\phi - m)\|^2_{L^2(h_r)} \geq \|\mathrm{d}(\phi - m)\|^2_{L^2(h_1)} .$$

Now, $\phi - m$ has h_1-mean value zero. Let c be the first eigenvalue of the h_1-Laplacian. Then

$$\|\mathrm{d}(\phi - m)\|^2_{L^2(h_1)} \geq c \|\phi - m\|^2_{L^2(h_1)} = c r^{-1} \|\phi - m\|^2_{L^2(h_r)} .$$

Using Lemma 6.2 again gives

$$\|\phi - m\|^2_{L^2(h_r)} \geq \text{constant } \|\phi - m\|^2_{L^2(g_r)} \geq \text{constant } \|\phi\|^2_{L^2(g_r)} ,$$

where the second inequality follows from the fact that ϕ has g_r-mean value zero.

Putting the pieces together completes the proof. q.e.d.

Lemma 6.6. *There exists a positive constant* C_2 *such that for all* $\xi \in \Gamma(TX)$ *and all sufficiently large* r*,*

$$\left\|\bar{\partial}\xi\right\|^2_{L^2(g_r)} \geq C_2 r^{-2} \left\|\xi\right\|^2_{L^2(g_r)} .$$

Proof. The proof is similar to that of Lemma 6.5 above. By Lemma 6.2,

$$\left\|\bar{\partial}\xi\right\|^2_{L^2(g_r)} \geq \text{constant} \left\|\bar{\partial}\xi\right\|^2_{L^2(h_r)} .$$

By definition of h_r, $\left|\bar{\partial}\xi\right|^2_{h_r} \geq r^{-1}\left|\bar{\partial}\xi\right|^2_{h_1}$. Using this and $\mathrm{dvol}(h_r) = r\mathrm{dvol}(h_1)$ gives $\left\|\bar{\partial}\xi\right\|^2_{L^2(h_r)} \geq \left\|\bar{\partial}\xi\right\|^2_{L^2(h_1)}$.

Let c be the first eigenvalue of the $\bar{\partial}$-Laplacian determined by the metric h_1. Then, $\left\|\bar{\partial}\xi\right\|^2_{L^2(h_1)} \geq c\left\|\xi\right\|^2_{L^2(h_1)}$. By definition of h_r, $\left|\xi\right|^2_{h_1} \geq r^{-1}\left|\xi\right|^2_{h_r}$. Hence, $\left\|\xi\right\|^2_{L^2(h_1)} \geq r^{-2}\left\|\xi\right\|^2_{L^2(h_r)}$. Finally, using Lemma 6.2 to convert back to the $L^2(g_r)$-norm of ξ, and putting all the pieces together gives the result. q.e.d.

Lemma 6.7. *There exists a constant* C *such that for all* ϕ *with* g_r*-mean value zero and all sufficiently large* r*,*

$$\left\|\mathscr{D}\phi\right\|^2_{L^2(g_r)} \geq C r^{-3} \left\|\phi\right\|^2_{L^2(g_r)} .$$

Proof. Combining Lemmas 6.5 and 6.6 shows that whenever ϕ has g_r-mean value zero,

$$\begin{aligned} \left\|\bar{\partial}\nabla\phi\right\|^2_{L^2(g_r)} &\geq C_2 r^{-2}\|\nabla\phi\|^2_{L^2(g_r)}, \\ &= C_2 r^{-2}\|\mathrm{d}\phi\|^2_{L^2(g_r)}, \\ &\geq C_1C_2 r^{-3}\|\phi\|^2_{L^2(g_r)}. \end{aligned}$$

q.e.d.

6.3. A uniformly controlled inverse.

Lemma 6.8. *There is a constant* A*, depending only on* k*, such that for all* $\phi \in L^2_{k+4}$ *and sufficiently large* r*,*

$$\|\phi\|_{L^2_{k+4}(g_r)} \leq A\left(\|\phi\|_{L^2(g_r)} + \|\mathscr{D}^*\mathscr{D}(\phi)\|_{L^2_k(g_r)}\right).$$

Proof. Recall Equation (2.2): $L_r(\phi) = \mathscr{D}^*\mathscr{D}(\phi) + \nabla\,\mathrm{Scal}(\omega_{r,n})\cdot\nabla\phi$. Since $\mathrm{Scal}(\omega_{r,n})$ tends to zero in $C^k(g_r)$, $L_r - \mathscr{D}^*\mathscr{D}$ converges to zero in operator norm calculated with respect to the $L^2_k(g_r)$-Sobolev norms. Hence, the estimate follows from the analogous result for L_r (Lemma 5.9). q.e.d.

Theorem 6.9. *The operator* $\mathscr{D}^*\mathscr{D} : L^2_{k+4,0} \to L^2_{k,0}$ *is a Banach space isomorphism. There exists a constant* K*, such that for all large* r *and all* $\psi \in L^2_{k,0}$*, the inverse operator* Q_r *satisfies*

$$\|Q_r\psi\|_{L^2_{k+4}(g_r)} \leq Kr^3 \|\psi\|_{L^2_k(g_r)}.$$

Proof. The inverse Q_r exists by Corollary 6.4. It follows from Lemma 6.7 applied to $\phi = Q_r\psi$ that there is a constant C such that for all $\psi \in L^2_{k,0}$,

$$\|Q_r\psi\|_{L^2(g_r)} \leq Cr^3 \|\psi\|_{L^2(g_r)}.$$

Applying Lemma 6.8 to $\phi = Q_r\psi$ extends this bound to the one required. q.e.d.

Next, recall the following standard result (proved via a geometric series).

Lemma 6.10. *Let* $D\colon B_1 \to B_2$ *be a bounded invertible linear map of Banach spaces with bounded inverse* Q*. If* $L\colon B_1 \to B_2$ *is another linear map with*

$$\|L - D\| \leq (2\|Q\|)^{-1},$$

then L *is also invertible with bounded inverse* P *satisfying* $\|P\| \leq 2\|Q\|$*.*

The pieces are now in place to prove Theorem 6.1 (which is stated at the start of this chapter).

Proof of Theorem 6.1. Since $(L_r - \mathscr{D}^*\mathscr{D})\phi = \nabla \operatorname{Scal}(\omega_{r,n}) \cdot \nabla\phi$, there exists a constant c such that, in operator norm computed with respect to the g_r-Sobolev norms, $\|pL_r - \mathscr{D}^*\mathscr{D}\| \leq cr^{-n-1}$.

So, for $n \geq 3$, and for large enough r, $\|pL_r - \mathscr{D}^*\mathscr{D}\| \leq (2\|Q_r\|)^{-1}$. Lemma 6.10 shows that pL_r is invertible and gives the upper bound

$$\|P_r\| \leq 2\|Q_r\| \leq Cr^3$$

for some C. q.e.d.

6.4. An improved bound.

It should be possible to improve on this estimate. For example, over a product of two curves, a simple separation of variables argument shows that $\|P_r\| = Cr^2$.

The above proof of Theorem 6.1 concatenates two eigenvalue estimates, each of which is saturated only when applied to an eigenvector corresponding to the first eigenvalue. Certainly over a product, the functions which get closest to saturating the first estimate (Lemma 6.5) have gradients which can be controlled more efficiently than is done in the proof of the second estimate (Lemma 6.6).

In general, it should be possible to obtain a better bound for $\|P_r\|$ by examining this interplay between the two eigenvalue estimates. However, the bound proved above is sufficient to complete the proof of Theorem 1.1.

7. Loose Ends

7.1. Controlling the non-linear terms. Denote by Scal_r the scalar curvature map on Kähler potentials determined by $\omega_{r,n}$: $\mathrm{Scal}_r(\phi) = \mathrm{Scal}(\omega_r + i\bar{\partial}\partial\phi)$. Recall that $S_r = p\,\mathrm{Scal}_r$. Denote by $N_r = S_r - pL_r$ the non-linear terms of S_r

Lemma 7.1. *Let $k \geq 3$. There exists positive constants c and K, such that for all ϕ, $\psi \in L^2_{k+4}$ with $\|\phi\|_{L^2_{k+4}}$, $\|\psi\|_{L^2_{k+4}} \leq c$ and for sufficiently large r,*

$$\|N_r(\phi) - N_r(\psi)\|_{L^2_k} \leq K \max\left\{\|\phi\|_{L^2_{k+4}}, \|\psi\|_{L^2_{k+4}}\right\} \|\phi - \psi\|_{L^2_{k+4}}$$

where g_r-Sobolev norms are used throughout.

Proof. By the mean value theorem,

$$\|N_r(\phi) - N_r(\psi)\|_{L^2_k(g_r)} \leq \sup_{\chi\in[\phi,\psi]} \|(DN_r)_\chi\| \|\phi - \psi\|_{L^2_{k+4}(g_r)}$$

where $(DN_r)_\chi$ is the derivative of N_r at χ.

Now, $DN_r = p(L_r)_\chi - pL_r$ where $(L_r)_\chi$ is the linearisation of Scal_r at χ. In other words, $(L_r)_\chi$ is the linearisation of the scalar curvature map determined by the metric $\omega_{r,n} + i\bar{\partial}\partial\chi$. Applying Lemma 2.10 to this metric and $\omega_{r,n}$ gives $\|(L_r)_\chi - L_r\| \leq \text{constant}\|\chi\|_{L^2_{k+4(g_r)}}$. As $k \geq 3$, the condition on the indices in Lemma 2.10 is met. Notice also that Lemma 2.10 requires the constants in the g_r-Sobolev inequalities to be uniformly bounded — which is proved in Lemma 5.8 — and the $C^k(g_r)$-norm of the curvature of $\omega_{r,n}$ to be bounded above — which follows from Theorem 5.2 and Lemma 2.7.

Since p is uniformly bounded (an $L^2_k(g_r)$-orthogonal projection even) and since, for all $\chi \in [\phi, \psi]$, $\|\chi\|_{L^2_{k+4}} \leq \max\{\|\phi\|_{L^2_{k+4}}, \|\psi\|_{L^2_{k+4}}\}$ the result follows. q.e.d.

7.2. Completing the proof.

Proof of Theorem 1.1. For all large r and $n \geq 3$, the map

$$S_r \colon L^2_{k+4,0}(g_r) \to L^2_{k,0}(g_r)$$

has the following properties:

(1) $S_n(0) = O\left(r^{-n-1/2}\right)$ in $L^2_k(g_r)$, by Lemma 5.7 and the fact that p has operator norm 1.
(2) The derivative of S_r at the origin is an isomorphism with inverse P_r which is $O(r^3)$. This is proved in Theorem 6.1.
(3) There exists a constant K such that for all sufficiently small M, the non-linear piece N_r of S_r is Lipschitz with constant M on a ball of radius KM. This follows directly from Lemma 7.1.

Recall the statement of the inverse function theorem (Theorem 4.1). The second and third of the above properties imply that the radius δ'_r of the ball about the origin on which N_r is Lipschitz with constant $(2\|P_r\|)^{-1}$ is bounded below by Cr^{-3} for some positive C. As $\delta_r = \delta'_r(2\|P_r\|)^{-1}$, it follows that δ_r is bounded below by Cr^{-6} for some positive C.

Hence, for $\psi \in L^2_k$ with $\|S_r(0) - \psi\|_{L^2_k(g_r)} \leq Cr^{-6}$, the equation $S_r(\phi) = \psi$ has a solution. In particular, the first of the above properties implies that, for $n \geq 6$ and sufficiently large r, the equation $S_r(\phi) = 0$ has a solution.

Since Scal differs from S_r by a constant, the metric $\omega_{r,n} + i\bar{\partial}\partial\phi$ has constant scalar curvature. Iteratively applying the regularity Lemma 2.3 (which can be done provided k is high enough to ensure that $L^2_{k+4} \hookrightarrow C^{2,\alpha}$) gives that ϕ is smooth. q.e.d.

8. Higher Dimensional Varieties

Theorem 1.1 extends to certain higher dimensional fibrations. The required conditions are set out below. To understand their relevance, compare the summary at the end of Chapter 3.

(A) *Let X be a compact connected Kähler manifold with no non-zero holomorphic vector fields and $\pi\colon X \to B$ a holomorphic submersion.*

Let κ_0 be a Kähler class on X; denote by κ_b the Kähler class on the fibre F_b over b obtained by restricting κ_0.

(B) *For every $b \in B$, κ_b contains a unique cscK metric ω_b; the form ω_b depends smoothly on b.*

Let ω be a Kähler form representing κ_0. For each b, there is a unique function $\phi_b \in C^\infty(F_b)$ with ω_b-mean value zero such that the fibrewise restriction of ω plus $i\bar{\partial}\partial\phi_b$ is ω_b. The smoothness assumption in (B) implies that the ϕ_b fit together to give a smooth function $\phi \in C^\infty(X)$; so, $\omega_0 = \omega + i\bar{\partial}\partial\phi$ is a Kähler metric in κ_0 whose fibrewise restriction has constant scalar curvature.

The metric ω_0 gives a Hermitian structure in the vertical tangent bundle V and hence, also in the line bundle $\Lambda^{\max} V^*$; denote its curvature F. Taking the fibrewise mean value of the horizontal–horizontal component of iF (with respect to the metric ω_0) defines a form $\alpha \in \Omega^{1,1}(B)$.

(C) *There is a metric ω_B on the base solving $\operatorname{Scal}(\omega_B) - \Lambda\alpha = \text{const}$; there are no non-trivial deformations of ω_B through cohomologous solutions to this equation.*

Theorem 8.1. *Let X satisfy* (A), (B) *and* (C). *Then, for all large r the Kähler class $\kappa_r = \kappa_0 + r[\omega_B]$ contains a constant scalar curvature Kähler metric.*

Proof. The proof follows the same lines as that of Theorem 1.1. Using the same notation as in the preceding paragraphs, let $\omega_r = \omega_0 + r\omega_B$, where ω_B is the solution to the PDE mentioned in (C). Notice that $[\omega_r] = \kappa_r$.

The first step is to construct approximate solutions of arbitrary accuracy. An identical calculation to that in Lemma 3.3 shows that $\operatorname{Scal}(\omega_r) = \operatorname{Scal}(\omega_b) + \sum r^{-j}\psi_j$ for some functions ψ_j; moreover, the fibrewise mean value of of ψ_1 is $\operatorname{Scal}(\omega_B) - \Lambda_{\omega_B}\alpha$. By assumptions (B) and (C), then the fibrewise mean value of $\operatorname{Scal}(\omega_r)$ is constant to $O(r^{-2})$.

The same argument which proves Lemma 3.4 gives that the linearisation of the scalar curvature map determined by ω_r satisfies $L_r = L_b + O(r^{-1})$, where L_b is the linearisation of the scalar curvature map on the fibre over b determined by ω_b. By assumption (B), ω_b is the unique constant scalar curvature metric in κ_b, hence, $\ker L_b$ is the constant functions on F_b. This, elliptic regularity and the fact that L_b is self adjoint mean that for any $\Theta \in C^\infty(F_b)$ with ω_b-mean value zero, there exists a unique $\phi \in C^\infty(F_b)$ with ω_b-mean value zero satisfying $L_b\phi = \Theta$.

Applying this argument fibrewise, as in the proof of Lemma 3.6, shows that L_b is a bijection on $C^\infty_0(X)$ (i.e., on functions with fibrewise mean value zero). This guarantees the existence of a function $\phi_1 \in C^\infty_0(X)$ with

$$\operatorname{Scal}\left(\omega_r + i\bar{\partial}\partial r^{-1}\phi_1\right) = c_0 + c_1 r^{-1} + \sum_{j\geq 2} r^{-j}\chi_j$$

for some constants c_0 and c_1 and functions χ_j.

The higher order approximate solutions are constructed exactly as in Sections 3.3 and 3.4. The argument hinges on the surjectivity of two particular linear differential operators. The first is the operator $L_b \colon C^\infty_0(X) \to C^\infty_0(X)$ whose surjectivity is justified above. The second is the linearisation of the map $C^\infty(B) \to C^\infty(B)$ given by

$$G\colon f \mapsto \mathrm{Scal}(\omega_B + i\bar\partial\partial f) + \Lambda_{\omega_B + i\bar\partial\partial f}\alpha.$$

Notice that $\int G(f)(\omega_B + i\bar\partial\partial f)^{\dim B}$ is independent of f; by assumption (C), $G(0)$ is constant; hence, differentiating gives that $\operatorname{im} DG$ is L^2-orthogonal to the constants. By assumption (C), $\ker DG$ is the constants. This, elliptic regularity and the fact that DG has index zero implies that DG is surjective onto smooth functions on B with mean value zero.

In conclusion, for any integer n, there exist functions $f_1, \dots, f_{n-1} \in C^\infty(B)$, $\phi_1, \dots, \phi_n \in C_0^\infty(X)$ and constants $c_0, \dots, c_n$ such that the metric

$$\omega_{r,n} = \omega_r + i\bar\partial\partial \sum_{j=1}^{n-1} r^{-j} f_j + i\bar\partial\partial \sum_{j=1}^{n} r^{-j}\phi_j$$

satisfies

$$\mathrm{Scal}(\omega_{r,n}) = \sum_{j=0}^{n} c_j r^{-j} + O\left(r^{-n-1}\right).$$

The remainder of the proof of Theorem 1.1 is not dimension specific. Similar to Section 5, the local model is given by $(F \times \mathbb{C}^{\dim B}, \omega_F \oplus \omega_{\mathrm{flat}})$ with ω_F a constant scalar curvature metric on F. The same Sobolev inequalities (modulo the obvious changes regarding the indices) hold over this space as over $S \times \mathbb{C}$. (Indeed, they hold over any manifold with uniformly bounded geometry, as is remarked in [**4**].) These estimates transfer to X in an identical way to before.

The global model is, as in Section 6, a Riemannian submersion constructed by ignoring the horizontal contribution of ω_0. The remaining steps in the proof now proceed identically. q.e.d.

It is, perhaps, suprising that condition (C) is more complicated than just the existence of a constant scalar curvature metric on B. Indeed, the equation in (C) involves the whole of X and is not just a condition on the base. Bearing in mind the conjectured correspondence with stably polarised varieties, this may have an algebro-geometric interpretation. Namely, X may be stable with respect to the polarisation $\kappa_0 + r[\omega_B]$ if the fibres are stably polarised by the restriction of κ_0 and if the base is also stably polarised, not with respect to $[\omega_B]$, but rather some other polarisation constructed from $[\omega_B]$ and the push down of the top exterior power of the vertical cotangent bundle of X.

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