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HAMILTONIAN 2-FORMS IN KÄHLER GEOMETRY, II GLOBAL CLASSIFICATION

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Abstract

We present a classification of compact Kähler manifolds admitting a hamiltonian 2-form (which were classified locally in part I of this work). This involves two components of independent interest.

The first is the notion of a rigid hamiltonian torus action. This natural condition, for torus actions on a Kähler manifold, was introduced locally in part I, but such actions turn out to be remarkably well behaved globally, leading to a fairly explicit classification: up to a blow-up, compact Kähler manifolds with a rigid hamiltonian torus action are bundles of toric Kähler manifolds.

The second idea is a special case of toric geometry, which we call orthotoric. We prove that orthotoric Kähler manifolds are diffeomorphic to complex projective space, but we extend our analysis to orthotoric orbifolds, where the geometry is much richer. We thus obtain new examples of Kähler–Einstein 4-orbifolds.

Combining these two themes, we prove that compact Kähler manifolds with hamiltonian 2-forms are covered by blow-downs of projective bundles over Kähler products, and we describe explicitly how the Kähler metrics with a hamiltonian 2-form are parameterized. We explain how this provides a context for constructing new examples of extremal Kähler metrics—in particular a subclass of such metrics which we call weakly Bochner-flat.

We also provide a self-contained treatment of the theory of compact toric Kähler manifolds, since we need it and find the existing literature incomplete.

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This paper is concerned with the construction of explicit Kähler metrics on compact manifolds, and has several interrelated motivations. The first is the notion of a hamiltonian 2-form, introduced in part I of this series [**4**].

Definition 1. Let ϕ be any (real) J-invariant 2-form on the Kähler manifold (M, g, J, ω) of dimension $2m$. We say ϕ is hamiltonian if

$$\nabla_X \phi = \tfrac{1}{2}(d \operatorname{tr}\phi \wedge JX - d^c \operatorname{tr}\phi \wedge X) \tag{1}$$

for any vector field X, where $\operatorname{tr}\phi = \langle \phi, \omega \rangle$ is the trace with respect to ω. When M is a Riemann surface ($m = 1$), this equation is vacuous and we require instead that $\operatorname{tr}\phi$ is a Killing potential, i.e., a hamiltonian for a Killing vector field $J \operatorname{grad}_g \operatorname{tr}\phi$.

A second motivation is the notion of a weakly Bochner-flat (WBF) Kähler metric, by which, we mean a Kähler metric whose Bochner tensor (which is part of the curvature tensor) is co-closed. By the differential Bianchi identity, this is equivalent (for $m \geq 2$) to the condition that $\rho - \frac{\mathrm{Scal}}{2(m+1)}\,\omega$ is a hamiltonian 2-form, where ρ is the Ricci form. WBF Kähler metrics are extremal in the sense of Calabi, i.e., the symplectic gradient of the scalar curvature is a Killing vector field, and provide a class of extremal Kähler metrics which include the Bochner-flat Kähler metrics studied by Bryant [**9**] and products of Kähler–Einstein metrics. The geometry of WBF Kähler metrics is tightly constrained, because the more specific the normalized Ricci form is, the closer the metric is to being Kähler–Einstein, while the more generic it is, the stronger the consequences of the hamiltonian property.

A hamiltonian 2-form ϕ induces an isometric hamiltonian ℓ-torus action on M for some $0 \leq \ell \leq m$, which we call the order of ϕ. This says nothing for $\ell = 0$, but for $\ell = m$, it means that M is toric. Toric Kähler manifolds are well understood, and a third motivation for our work is to extend this understanding to certain torus actions with $0 < \ell < m$. We introduce the notion of a rigid hamiltonian ℓ-torus action and prove that a compact Kähler manifold with such an action has a blow-up which is biholomorphic to a bundle of toric Kähler 2ℓ-manifolds.

We shall be particularly interested in the projective bundles of the form $M = P(\mathcal{L}_0 \otimes \mathbb{C}^{d_0+1} \oplus \cdots \oplus \mathcal{L}_\ell \otimes \mathbb{C}^{d_\ell+1}) \to S$, where $\mathcal{L}_0, \ldots \mathcal{L}_\ell$ are line bundles over a Kähler manifold S and the ℓ-torus action is induced by scalar multiplication on the vector bundles $\mathcal{L}_j \otimes \mathbb{C}^{d_j+1}$, with $d_j \geq 0$. The blow-up of M along the submanifolds determined by setting the jth fibrewise homogeneous coordinate (in $\mathcal{L}_j \otimes \mathbb{C}^{d_j+1}$) to zero, for $j = 0, \ldots \ell$, is a bundle of toric Kähler 2ℓ-manifolds: the projective bundle $P(\tilde{\mathcal{L}}_0 \oplus$

$\cdots \oplus \tilde{\mathcal{L}}_\ell) \to \mathbb{C}P^{d_0} \times \cdots \times \mathbb{C}P^{d_\ell} \times S$, where $\tilde{\mathcal{L}}_j = \mathcal{O}(0, \ldots 0, -1, 0, \ldots 0) \otimes \mathcal{L}_j$ (with $\mathcal{O}(-1)$ over the jth factor $\mathbb{C}P^{d_j}$).

When $\ell = 1$, projective line bundles have been well-used, since the seminal work of Calabi [**10**], in the construction of explicit examples of extremal Kähler metrics. The idea to consider blow-downs was introduced by Koiso and Sakane [**24**, **25**], who constructed Kähler–Einstein metrics in this way. Our fourth motivation is to provide a general framework for constructing extremal Kähler metrics on projective bundles, and their blow-downs, and in doing so, we obtain new examples.

The toric Kähler $2m$-manifolds arising from hamiltonian 2-forms of order m are of a special class, which we call orthotoric. Compact orthotoric Kähler manifolds are necessarily biholomorphic to complex projective space, but there are many more examples on orbifolds. Our final motivation is to study Kähler metrics on toric orbifolds, especially orthotoric orbifolds, and to obtain new examples.

The main goal of this paper is to show that, up to a covering, a compact Kähler manifold with a hamiltonian 2-form of order ℓ is necessarily biholomorphic to a projective bundle M of the form described above, and conversely to show precisely how to construct Kähler metrics with hamiltonian 2-forms of order ℓ on such bundles.

We hope, however, that with the various motivations discussed above, the Reader who does not share our enthusiasm for hamiltonian 2-forms will find something of interest in this paper. Hamiltonian 2-forms rather provide a device that unifies and underlies the above themes. The journey to our main result, and its consequences, yield a number of results of independent interest.

- We obtain necessary and sufficient first order boundary conditions for the compactification of compatible Kähler metrics on toric symplectic orbifolds, clarifying work of Abreu [**2**, **1**] (see Remark 3 and Section 1.4).
- We introduce and study rigid hamiltonian torus actions, and orthotoric Kähler manifolds and orbifolds.
- We construct new explicit Kähler–Einstein metrics on 4-orbifolds.
- We unify and extend constructions of Kähler metrics on projective bundles, obtaining new weakly Bochner-flat and extremal Kähler metrics on projective line bundles and on the projective plane bundle $P(\mathcal{O} \oplus \mathcal{O}(1) \otimes \mathbb{C}^2) \to \mathbb{C}P^1$.

We have attempted to make this paper as independent as possible from the first part [**4**]. However, we shall make essential use of the local classification of Kähler manifolds with a hamiltonian 2-form of order ℓ, so we recall the result here. The Reader who is not interested in hamiltonian 2-forms per se, could take this local classification result as a (rather complicated) definition of the class of Kähler metrics that, we wish to classify globally.

We define the *momentum polynomial* of a hamiltonian 2-form ϕ to be

$$p(t) := (-1)^m \operatorname{pf}(\phi - t\omega) = t^m - (\operatorname{tr}\phi)\, t^{m-1} + \cdots + (-1)^m \operatorname{pf}\phi \tag{2}$$

where the *pfaffian* is defined by $\phi \wedge \cdots \wedge \phi = (\operatorname{pf}\phi)\omega \wedge \cdots \wedge \omega$.

Theorem 1 ([4]). *Let (M, g, J, ω) be a connected Kähler $2m$-manifold with a hamiltonian 2-form ϕ. Then:*

(i) *the functions $p(t)$ on M (for each $t \in \mathbb{R}$) are Poisson-commuting hamiltonians for Killing vector fields $K(t) := J \operatorname{grad}_g p(t)$;*

(ii) *there is a monic polynomial $p_{\rm c}(t)$ with constant coefficients such that $p(t) = p_{\rm c}(t)p_{\rm nc}(t)$ and, if $p_{\rm nc}(t) = \sum_{r=0}^{\ell}(-1)^r\sigma_r t^{\ell-r}$ (with $0 \le \ell \le m$), then the Killing vector fields $K_r := J \operatorname{grad}_g \sigma_r$ ($r = 1, \dots \ell$) are linearly independent on a connected dense open subset M^0 of M. The integer ℓ is called the* order *of ϕ.*

On the open subset M^0, the roots $\xi_1, \dots \xi_\ell$ of $p_{\rm nc}(t)$ are smooth, functionally independent and everywhere pairwise distinct, and they extend continuously to M. Denote by η_a, $a = 1, \dots N$ ($N \le m - \ell$) the different *constant roots of $p_{\rm c}(t)$ and by d_a their multiplicities. Then, there are (positive or negative definite) Kähler metrics (g_a, ω_a) of real dimension $2d_a$, functions $F_1, \dots F_\ell$ of one variable, and 1-forms $\theta_1, \dots \theta_\ell$ with $\theta_r(K_s) = \delta_{rs}$ such that the Kähler structure on M^0 is of the form*

(3)

$$g = \sum_{a=1}^{N} p_{\rm nc}(\eta_a) g_a + \sum_{j=1}^{\ell} \frac{p'(\xi_j)}{F_j(\xi_j)} d\xi_j^2 + \sum_{j=1}^{\ell} \frac{F_j(\xi_j)}{p'(\xi_j)} \left(\sum_{r=1}^{\ell} \sigma_{r-1}(\hat{\xi}_j)\theta_r \right)^2,$$

$$\omega = \sum_{a=1}^{N} p_{\rm nc}(\eta_a)\omega_a + \sum_{r=1}^{\ell} d\sigma_r \wedge \theta_r, \qquad d\theta_r = \sum_{a=1}^{N} (-1)^r \eta_a^{\ell-r} \omega_a,$$

and the hamiltonian 2-form ϕ is given by

$$\phi = \sum_{a=1}^{N} \eta_a\, p_{\rm nc}(\eta_a)\omega_a + \sum_{r=1}^{\ell} (\sigma_r d\sigma_1 - d\sigma_{r+1}) \wedge \theta_r \tag{4}$$

with $\sigma_{\ell+1} = 0$. (Here $\sigma_{r-1}(\hat{\xi}_j)$ denote the elementary symmetric functions of the roots with ξ_j omitted. We remark also that $p'(\xi_j) = p_{\rm c}(\xi_j)\prod_{k \ne j}(\xi_j - \xi_k)$.)

We shall obtain our global description of compact Kähler manifolds admitting a hamiltonian 2-form of order ℓ by exploiting three aspects of the local geometry revealed by Theorem 1.

(i) The components $g(K_r, K_s)$ of the metric are constant on fibres of the momentum map $(\sigma_1, \dots \sigma_\ell)\colon M \to \mathbb{R}^\ell$. (This holds on all of M by continuity.)

(ii) The Kähler quotient metrics $\sum_{a=1}^N p_{\mathrm{nc}}(\eta_a) g_a$ are simultaneously diagonalizable (with respect to $\sum_{a=1}^N g_a$) with constant eigenvalues for each fixed $(\sigma_1, \dots \sigma_\ell)$.

(iii) The roots $\xi_1, \dots \xi_\ell$ of p_{nc} have orthogonal gradients.

In [**4**], these properties were interpreted by saying that (M, g, J, ω) is given locally by a rigid hamiltonian ℓ-torus action with semisimple Kähler quotient and orthotoric fibres. We shall see that this is not far from being true globally.

If M is compact, the closure of the group of hamiltonian isometries of M generated by $K_1, \dots K_\ell$ is a torus $\mathbb{T}$ (with $\ell \leq \dim \mathbb{T} \leq m$). When $\ell = m$, $K_1, \dots K_m$ generate a torus action, and M is a toric Kähler manifold. In the first section, we review the necessary background of toric Kähler geometry and introduce a suitable invariant language. Then, in section 2, we pursue a similar theory for $\ell < m$ when property (i) holds. In particular, we prove that $\dim \mathbb{T} = \ell$ so there is a global rigid ℓ-torus action. We provide a generalized Calabi construction for such actions which classifies them up to covering when the Kähler quotient is semisimple, i.e., when property (ii) is also satisfied. In section 3, we study toric Kähler manifolds (and orbifolds) satisfying property (iii) in general, and here we exhibit new explicit Kähler–Einstein metrics on compact 4-orbifolds. In section 4, we obtain a complete description of compact Kähler manifolds with hamiltonian 2-forms, which we use to construct new examples of compact weakly Bochner-flat and extremal Kähler manifolds. In subsequent work, we shall construct many more examples and classify weakly Bochner-flat Kähler metrics in dimension 6.

1. Hamiltonian actions and toric geometry

We begin by reviewing hamiltonian torus actions, paying particular attention to the theory of toric Kähler manifolds. Toric Kähler geometry can be studied either from the complex or symplectic viewpoint, and we adopt, primarily, the latter. Furthermore, with a view to applications, we do not restrict attention to manifolds, but also consider orbifolds: this is a natural context in toric symplectic geometry [**1**, **28**]. We refer to [**6**, **18**, **28**] for general information about torus actions on symplectic manifolds, and to [**2**, **11**, **13**, **16**, **17**] for further information about toric Kähler manifolds and orbifolds.

Our treatment has some novel features: in particular, we obtain first order boundary conditions for the compactification of compatible Kähler metrics on toric symplectic manifolds. Also, we present the theory in invariant language, because for the torus actions generated by hamiltonian 2-forms, the natural basis of the Lie algebra $\mathfrak{t}$ is not (in general) compatible with the lattice in $\mathfrak{t}$ defining the torus $\mathbb{T}$.

1.1. Hamiltonian torus actions. Let $\mathbb{T}$ be an ℓ-dimensional torus, with Lie algebra $\mathfrak{t}$, acting effectively on a symplectic $2m$-manifolds (M, ω), and for $\xi \in \mathfrak{t}$, denote by X_ξ the corresponding vector field on M. Then, we say that the action is hamiltonian if there is a $\mathbb{T}$-invariant smooth map $\mu\colon M \to \mathfrak{t}^*$, called a momentum map for the action, such that $\iota_{X_\xi}\omega = -\langle d\mu, \xi\rangle$ for any $\xi \in \mathfrak{t}$.

Remark 1. Note that our actions are hamiltonian in the strong sense that μ is $\mathbb{T}$-invariant (if μ has a critical point—as it does in the compact case—this is automatic). Since $\mathbb{T}$ is abelian, this implies that $\omega(X_\xi, X_\eta) = 0$ for any $\xi, \eta \in \mathfrak{t}$. We also remark that the action determines and is determined by μ up to a constant.

We shall normally be interested in the case that (M, ω, μ) has a compatible almost Kähler structure, i.e., a $\mathbb{T}$-invariant metric g and almost complex structure J with $\omega(X, Y) = g(JX, Y)$. Such compatible metrics always exist.

We shall make significant use of the symplectic slice theorem for $\mathbb{T}$-orbits in M, which we now recall. Let $\mathbb{T}\cdot x$ be such an orbit for $x \in M$. Since $\mathbb{T}\cdot x$ is isotropic with respect to ω, the isotropy representation of $\mathbb{T}_x$ on T_xM induces a $2(m-k)$-dimensional symplectic representation on $V_x := T_x(\mathbb{T}\cdot x)^0/T_x(\mathbb{T}\cdot x)$, where $T_x(\mathbb{T}\cdot x)^0$ denotes the annihilator with respect to ω_x of $T_x(\mathbb{T}\cdot x)$ in T_xM. This is called the symplectic isotropy representation.

Using the metric g_x, T_xM is an orthogonal direct sum of the subspaces

$$T_x(\mathbb{T}\cdot x) \cong \mathfrak{t}/\mathfrak{t}_x, \qquad JT_x(\mathbb{T}\cdot x) \cong (\mathfrak{t}/\mathfrak{t}_x)^* \cong \mathfrak{t}_x^0, \qquad V_x$$

where $\mathfrak{t}_x^0$ the annihilator of $\mathfrak{t}_x$ in $\mathfrak{t}^*$ (identified with $JT_x(\mathbb{T}\cdot x)$ using ω_x).

Lemma 1. *Let (M, g, J, ω) be an almost Kähler manifold with an isometric hamiltonian $\mathbb{T}$-action. Fix $x \in M$ and a splitting $\chi\colon \mathfrak{t} \to \mathfrak{t}_x$ of the inclusion.*

Then, the action of $\mathbb{T}_x$ on the symplectic isotropy representation V_x is effective, and there is a symplectic form ω_0 on the normal bundle $N = \mathbb{T} \times_{\mathbb{T}_x} (\mathfrak{t}_x^0 \oplus V_x) \to \mathbb{T}\cdot x$ and a symplectomorphism f from a neighbourhood of the zero section 0_N in N to a neighbourhood of $\mathbb{T}\cdot x$ in M such that:

- *the obvious* $\mathbb{T}$*-action on* N *by left multiplication is hamiltonian with momentum map* $\mu_0([\alpha, v]) = \alpha + \mu_V(v) \circ \chi$*, where* α *is an element of the fibre belonging to* $\mathfrak{t}_x^0 \subset \mathfrak{t}^*$*, and* $\mu_V\colon V_x \to \mathfrak{t}_x^*$ *is the momentum map of the symplectic isotropy representation*;
- f *is* $\mathbb{T}$*-equivariant, is equal to the bundle projection along* 0_N*, and its fibre derivative along* 0_N *is the natural identification of the vertical bundle with* N*.*

Proof. The normal exponential map provides a $\mathbb{T}$-equivariant diffeomorphism from a neighbourhood of $0_N \cong \mathbb{T} \cdot x$ of the normal bundle $N = \mathbb{T} \times_{\mathbb{T}_x} (\mathfrak{t}_x^0 \oplus V_x) \to \mathbb{T} \cdot x$ (with the natural $\mathbb{T}$ action induced by left multiplication on $\mathbb{T}$) to a neighbourhood of $\mathbb{T} \cdot x$ in M; then $\mathbb{T}$ acts effectively on N while $\mathbb{T}_x$ acts trivially on $\mathfrak{t}_x^0$, so $\mathbb{T}_x$ acts effectively on V_x.

The chosen projection $\chi\colon \mathfrak{t} \to \mathfrak{t}_x$ identifies the normal bundle N with the symplectic quotient of $T^*\mathbb{T} \times V_x$, by the diagonal action of $\mathbb{T}_x$ (since $T^*\mathbb{T} \cong \mathbb{T} \times \mathfrak{t}^*$). The induced symplectic form is $\mathbb{T}$-invariant with the given momentum map.

The pullback of ω by the normal exponential map gives another symplectic form ω_1 on a neighbourhood of 0_N in N, agreeing with ω_0 along 0_N (ω_1 and ω_0 both equal ω_x at $T_{(x,0)}N \cong T_xM$). By the equivariant relative Darboux theorem, there is a $\mathbb{T}$-equivariant diffeomorphism h of N fixing 0_N, with $dh = \mathrm{Id}$ there, and such that $h^*\omega_1 = \omega_0$ on a neighbourhood U of 0_N in N. Then, $f = \exp \circ h$ is the equivariant symplectomorphism we seek. q.e.d.

This result easily generalizes to orbifolds—see [**28**, Lemma 3.5 and Remark 3.7].

1.2. Toric manifolds and orbifolds. A connected $2m$-dimensional symplectic manifold or orbifold (M, ω) is said to be *toric* if it is equipped with an effective hamiltonian action of an m-torus $\mathbb{T}$ with momentum map $\mu\colon M \to \mathfrak{t}^*$. Compact toric symplectic manifolds were classified by Delzant [**13**], and this classification was extended to orbifolds by Lerman–Tolman [**28**]. Essentially, they are classified by the image of the momentum map μ, which is a compact convex polytope in $\mathfrak{t}^*$, but this statement requires some interpretation, particularly in the orbifold case.

Definition 2. Let $\mathfrak{t}$ be an m-dimensional real vector space. Then, a *rational Delzant polytope* $(\Delta, \Lambda, u_1, \ldots u_n)$ in $\mathfrak{t}^*$ is a compact convex polytope $\Delta \subset \mathfrak{t}^*$ equipped with *normals* belonging to a lattice Λ in $\mathfrak{t}$

$$u_j \in \Lambda \subset \mathfrak{t} \tag{5}$$

$(j = 1, \dots n,\ n > m)$ such that

$$\Delta = \{x \in \mathfrak{t}^* : L_j(x) \geq 0,\ j = 1, \dots n\} \tag{6}$$

with

$$L_j(x) = \langle u_j, x\rangle + \lambda_j$$

for some $\lambda_1, \dots \lambda_n \in \mathbb{R}$, and such that for any vertex $x \in \Delta$, the u_j with $L_j(x) = 0$ form a basis for $\mathfrak{t}$. If the normals form a basis for Λ at each vertex, then Δ is said to be *integral*, or simply a *Delzant polytope*.

The term *rational* refers to the fact that the normals span an m-dimensional vector space over $\mathbb{Q}$. A rational Delzant polytope is obviously m-valent, i.e., m codimension one faces and m edges meet at each vertex: by (6) the codimension one faces $F_1, \dots F_n$ are given by $F_j = \Delta \cap \{x \in \mathfrak{t}^* : L_j(x) = 0\}$, so that u_j is an inward normal vector to F_j. In the integral case, the u_j are necessarily primitive, and so are uniquely determined by (Δ, Λ). In general, the primitive inward normals are u_j/m_j for some positive integer labelling m_j of the codimension one faces F_j, so rational Delzant polytopes are also called *labelled polytopes* [**28**]. However, it turns out to be more convenient to encode the labelling in the normals. Note that $\lambda_1, \dots \lambda_n$ are uniquely determined by $(\Delta, \Lambda, u_1, \dots u_n)$.

The rational Delzant theorem [**13**, **28**] states that compact toric symplectic orbifolds are classified (up to equivariant symplectomorphism) by rational Delzant polytopes (with manifolds corresponding to integral Delzant polytopes). Given such a polytope, (M, ω) is obtained as a symplectic quotient of $\mathbb{C}^n$ by an $(n-m)$-dimensional subgroup G of the standard n-torus $(S^1)^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$: precisely, G is the kernel of the map $(S^1)^n \to \mathbb{T} = \mathfrak{t}/2\pi\Lambda$ induced by the map $(x_1, \dots x_n) \mapsto \sum_{j=1}^n x_j u_j$ from $\mathbb{R}^n$ to $\mathfrak{t}$, and the momentum level for the symplectic quotient is the image in $\mathfrak{g}^*$ of $(\lambda_1, \dots \lambda_n) \in \mathbb{R}^{n*}$ under the transpose of the natural inclusion of the Lie algebra $\mathfrak{g}$ in $\mathbb{R}^n$.

Conversely, a toric symplectic orbifold gives rise to a rational Delzant polytope as the image Δ of its momentum map μ, where Λ is the lattice of circle subgroups, and the positive integer labelling m_j of the codimension one faces F_j is determined by the fact that the local uniformizing group of every point in $\mu^{-1}(F_j^0)$ is $\mathbb{Z}/m_j\mathbb{Z}$. (Here, and elsewhere, for any face F, we denote by F^0 its interior.)

Remark 2. Toric symplectic manifolds and orbifolds are simply connected (as topological spaces—the inverse image of the union of the faces meeting a given vertex is contractible, and the complement has codimension two). However, one can consider orbifold coverings and quotients: a compact convex polytope with chosen normals (giving a basis for $\mathfrak{t}$ at each vertex) is a rational Delzant polytope with respect to *any* lattice satisfying (5). In particular, if Λ is a (finite index) sublattice of Λ', then the torus $\mathbb{T}' = \mathfrak{t}/2\pi\Lambda'$ is the quotient of $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$ by a finite abelian group $\Gamma \cong \Lambda'/\Lambda$. The corresponding toric symplectic orbifolds M and M' (under the tori $\mathbb{T}$ and $\mathbb{T}'$) are related by a regular orbifold covering: $M' = M/\Gamma$.

Clearly there is a 'smallest' lattice Λ satisfying (5), namely the lattice generated by the normals $u_1, \dots u_n$. This is a sublattice of any other lattice Λ' with $u_j \in \Lambda'$, so any toric symplectic orbifold M', corresponding to such a Λ', is a quotient of the toric symplectic orbifold M (corresponding to Λ) by a finite abelian group Γ.

In fact, M is the universal orbifold cover of M' in the sense of [**33**]. One may also characterize Λ as the unique lattice containing $u_1, \dots u_n$ for which G is connected, i.e., M is a symplectic quotient of $\mathbb{C}^n$ by a $(n-m)$-sub*torus* of $(S^1)^n$.

1.3. Compatible Kähler metrics: local theory. We turn now to the study of compatible Kähler metrics on toric symplectic orbifolds. On the union $M^0 := \mu^{-1}(\Delta^0)$ of the generic orbits, such metrics have an explicit description due to Guillemin [**16**, **17**]. Orthogonal to the orbits is a rank m distribution spanned by commuting holomorphic vector fields JX_ξ for $\xi \in \mathfrak{t}$. Hence, there is a function $t\colon M^0 \to \mathfrak{t}/2\pi\Lambda$, defined up to an additive constant, such that $dt(JX_\xi) = 0$ and $dt(X_\xi) = \xi$ for $\xi \in \mathfrak{t}$. The components of t are 'angular variables', complementary to the components of the momentum map $\mu\colon M^0 \to \mathfrak{t}^*$, and the symplectic form in these coordinates is simply

$$\omega = \langle d\mu \wedge dt \rangle, \tag{7}$$

where the angle brackets denote contraction of $\mathfrak{t}$ and $\mathfrak{t}^*$.

These coordinates identify each tangent space with $\mathfrak{t} \oplus \mathfrak{t}^*$, so any $\mathbb{T}$-invariant ω-compatible almost Kähler metric is given by

$$g = \langle d\mu, \mathbf{G}, d\mu \rangle + \langle dt, \mathbf{H}, dt \rangle, \tag{8}$$

where $\mathbf{G}$ is a positive definite $S^2\mathfrak{t}$-valued function on Δ^0, $\mathbf{H}$ is its inverse in $S^2\mathfrak{t}^*$—observe that $\mathbf{G}$ and $\mathbf{H}$ define mutually inverse linear maps $\mathfrak{t}^* \to \mathfrak{t}$ and $\mathfrak{t} \to \mathfrak{t}^*$ at each point—and $\langle \cdot, \cdot, \cdot \rangle$ denotes the pointwise contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^* \to \mathbb{R}$ or the dual contraction. The corresponding almost complex structure is defined by

$$J dt = -\langle \mathbf{G}, d\mu \rangle \tag{9}$$

from which it follows that J is integrable if and only if $\mathbf{G}$ is the hessian of a function on Δ^0 [**16**].

Remark 3. The description of $\mathbb{T}$-invariant ω-compatible Kähler metrics on M^0 shows that they are parameterized by functions on Δ^0 with positive definite hessian. There is a subtle point here, however, which is often overlooked in the literature, namely that the angular coordinates t depend on the (lagrangian) orthogonal distribution to the $\mathbb{T}$-orbits in M^0, and there is no reason for two metrics to have the same orthogonal distribution. This is not a problem on M^0, since the obvious map sending one set of angular coordinates to another is an equivariant symplectomorphism, but this symplectomorphism may not extend to M.

The Delzant construction realizes (M, ω) as a symplectic quotient of $\mathbb{C}^n$, so there is an obvious choice of a 'canonical' compatible Kähler metric g_0, namely the one induced by the flat metric on $\mathbb{C}^n$. An explicit formula for this Kähler metric in symplectic coordinates was obtained by Guillemin [16], and extended to the orbifold case by Abreu [1]: on M^0, the canonical metric is given by (8) with $\mathbf{G}$ equal to

$$\frac{1}{2}\mathrm{Hess}\left(\sum_{j=1}^{n} L_j(\mu)\log|L_j(\mu)|\right) = \frac{1}{2}\sum_{j=1}^{n}\frac{u_j\otimes u_j}{L_j(\mu)}. \tag{10}$$

Hence, the induced metric on Δ^0 is $\frac{1}{2}\sum_{j=1}^{n} d(L_j)^2/L_j$. (See also [11].)

1.4. Compatible Kähler metrics: compactification. On any compact toric symplectic manifold or orbifold, the canonical metric g_0 is globally defined on M—by construction. The study of other globally defined Kähler metrics is greatly facilitated by the following elementary lemma (see also [2] and Remark 4(ii) below).

Lemma 2. *Let (M, ω) be a toric symplectic $2m$-manifold or orbifold with momentum map $\mu\colon M \to \Delta \subset \mathfrak{t}^*$, and suppose that (g_0, J_0), (g, J) are compatible almost Kähler metrics on $M^0 = \mu^{-1}(\Delta^0)$ of the form (8)–(9), given by $\mathbf{G_0}, \mathbf{G}$ and the same angular coordinates, and such that (g_0, J_0) extends to an almost Kähler metric on M. Then, (g, J) extends to an almost Kähler metric on M provided that*

$$\mathbf{G} - \mathbf{G_0} \quad \textit{is smooth on } \Delta, \tag{11}$$

$$\mathbf{G_0 H G_0} - \mathbf{G_0} \quad \textit{is smooth on } \Delta. \tag{12}$$

Proof of Lemma 2. The key point is that it suffices to show g is smooth on M: it will then be non-degenerate because it is compatible with ω (equivalently if J extends smoothly to M, it is an almost complex structure on M by continuity). For the smoothness of g, we simply compute the difference

$$\begin{aligned} g - g_0 &= \langle d\mu, \mathbf{G} - \mathbf{G_0}, d\mu\rangle + \langle dt, \mathbf{H} - \mathbf{H_0}, dt\rangle \\ &= \langle d\mu, \mathbf{G} - \mathbf{G_0}, d\mu\rangle + \langle J_0 d\mu, \mathbf{G_0 H G_0} - \mathbf{G_0}, J_0 d\mu\rangle. \end{aligned}$$

Now μ, g_0 and J_0 are smooth on M, hence so is g by (11)–(12). q.e.d.

Remark 4.

(i) We use here the fact that any $\mathbb{T}$-invariant smooth function on M is the pullback by μ of a smooth function on Δ (this follows from the symplectic slice theorem and [**31**]: see [**28**]).

(ii) For generators X_ξ, X_η of the $\mathbb{T}$-action, $g_0(X_\xi, X_\eta)$ is a $\mathbb{T}$-invariant smooth function on M, hence the pullback of a smooth function on Δ. Thus $\mathbf{H_0}$ is a smooth $S^2\mathfrak{t}^*$-valued function on Δ (degenerating on $\partial\Delta$). Condition (11) thus implies that $\mathbf{H_0G}$ is smooth on Δ. We claim that in the presence of (11), (12) is equivalent to $\mathbf{H_0G}$ being non-degenerate on Δ. Indeed, if $\mathbf{H_0G}$ is non-degenerate, its inverse $\mathbf{HG_0}$ is smooth on Δ; now composing $\mathbf{G}-\mathbf{G_0}$ on the right by this, we obtain (12). Conversely, multiplying by $\mathbf{H_0}$, we deduce from (12) that $\mathbf{HG_0}$ is smooth on Δ, so $\mathbf{H_0G}$ is non-degenerate.

(iii) According to Abreu [**2**, **1**], when g_0 is the canonical (Guillemin) metric on (M,ω), these conditions are not only sufficient but necessary for the compactification of g. However, in our view there are some shortcomings in his (rather sketchy) proof. In particular, he does not address the issue of the dependence of the angular coordinates on the metric (see Remark 3). The following lemma partially resolves this issue. For a complete resolution, see Remark 5 below.

Lemma 3. *Let (M,ω) be a compact toric symplectic manifold with two compatible almost Kähler metrics which induce the same $S^2\mathfrak{t}$-valued function $\mathbf{G}$ on the interior of the Delzant polytope. Then, there is an equivariant symplectomorphism of M sending one metric to the other.*

Proof. By Remark 3, such a symplectomorphism exists on M^0. It extends uniquely to M, since M^0 is dense and (M,g) is a complete. The extension is a distance isometry by continuity, and is therefore smooth by a standard argument. q.e.d.

Note that this lemma makes essential use of the completeness of (M,g). It can, however, be extended to compact orbifolds, for instance by lifting the distance isometry to compatible uniformizing charts.

On the other hand, we learn nothing about the dependence of the angular coordinates on metrics which induce different $S^2\mathfrak{t}$-valued functions on the interior of the Delzant polytope. We shall therefore establish precise necessary and sufficient compactification conditions by a self-contained argument. Our proof also has the merit of being elementary and, modulo the above lemma, local, in contrast to [**2**, **1**], where the existence of a global biholomorphism is used. Indeed, compactification is about boundary conditions, so it is a local question. We shall present these boundary conditions in a form more closely analogous to the well-known conditions in complex dimension one. As a warm-up for the rest of the subsections, we first recall this case.

Let (M, ω) be a compact toric symplectic 2-orbifold. This must be an orbifold 2-sphere (i.e., equivariantly homeomorphic to $\mathbb{C}P^1$ with the standard circle action, but the two fixed points may be orbifold singularities), equipped with a rotation invariant area form. On M^0, which is diffeomorphic to $\mathbb{C}^\times$, a compatible Kähler metric takes the form

$$g = \frac{d\mu^2}{\Theta(\mu)} + \Theta(\mu)dt^2, \tag{13}$$

where $\omega = d\mu \wedge dt$. The rational Delzant polytope is an interval $[\alpha, \beta] \in \mathfrak{t}^*$ with normals $u_\alpha, u_\beta \in \mathfrak{t}$. If we identify a generator of the lattice Λ in $\mathfrak{t}$ with $1 \in \mathbb{R}$ (chosen so that u_α is positive), then $t\colon M^0 \to \mathfrak{t}/2\pi\Lambda$ becomes a coordinate of period 2π, and the orbifold singularities have cone angles $2\pi/m_\alpha$, $2\pi/m_\beta$ where $m_\alpha = u_\alpha, m_\beta = -u_\beta \in \mathbb{Z}^+$.

Since $\Theta(\mu)$ is the norm squared of the Killing vector field, Θ is smooth on $[\alpha, \beta]$, positive on the interior, and zero at the endpoints. On the other hand, μ is a Morse function (i.e., the two critical points are non-degenerate—this follows easily using a symplectic slice) and $dd^c\mu = \Theta'(\mu)\omega$, so that $\Theta'(\alpha)$ and $\Theta'(\beta)$ are non-zero.

Now, let $\hat{U} \subset \mathbb{R}^2$ be an orbifold chart covering an S^1-invariant neighbourhood $U = \hat{U}/\mathbb{Z}_{m_\alpha}$ of $\mu^{-1}(\alpha)$, where $\mathbb{Z}_{m_\alpha}$ acts in the standard way on $\mathbb{R}^2$ and the covering map π sends 0 to $\mu^{-1}(\alpha)$. The S^1-action on U lifts to one on $\hat{U}$, fixing 0 and commuting with $\mathbb{Z}_{m_\alpha}$. Now, $\hat{t} = t \circ \pi/m_\alpha$ is a coordinate of period 2π on $\hat{U} \setminus \{0\}$ while $\hat{\mu} = m_\alpha(\mu \circ \pi)$ is the momentum map of the S^1 action on $\hat{U}$, with respect to $\hat{\omega} = d\hat{\mu} \wedge d\hat{t} = \pi^*\omega$. The pull back of g to $\hat{U} \setminus \{0\}$ is

$$\hat{g} = \frac{d\hat{\mu}^2}{m_\alpha^2 \Theta(\hat{\mu}/m_\alpha)} + m_\alpha^2 \Theta(\hat{\mu}/m_\alpha) d\hat{t}^2.$$

If this metric compactifies smoothly at 0, we must have $m_\alpha\Theta'(\alpha) = 2$ (see [**21**]). With an analogous argument at $\mu^{-1}(\beta)$, we deduce that $u_\alpha\Theta'(\alpha) = 2 = u_\beta\Theta'(\beta)$.

To show that these conditions are sufficient for the smooth extension of g (in the orbifold sense) to M, we put $r^2/2 = \mu - \alpha$ and let t have period $2\pi/m_\alpha$. Since $\Theta(\alpha) = 0$, g differs from a multiple of $g_0 = dr^2 + \frac{1}{4}\Theta'(\alpha)^2 r^2 dt^2$ by a smooth bilinear form on M, vanishing at

$\mu = \alpha$. Clearly, the condition $\Theta'(\alpha) = 2/m_\alpha$ provides a smooth (orbifold) extension of g_0 to $\mu^{-1}(\alpha)$ by considering $(r, t/m_\alpha)$ to be the polar coordinates in a uniformising chart. The other endpoint is analogous.

To summarize, g given by (13) is globally defined on a toric orbifold whose rational Delzant polytope is $[\alpha, \beta] \subset \mathfrak{t}^*$, with normals $u_\alpha, u_\beta \in \mathfrak{t}$, if and only if Θ smooth on $[\alpha, \beta]$, with

$$\begin{gathered}\Theta(\alpha) = 0 = \Theta(\beta), \\ \Theta'(\alpha)u_\alpha = 2 = \Theta'(\beta)u_\beta\end{gathered} \tag{14}$$

and Θ positive on (α, β). The derivative conditions make invariant sense, since Θ takes values in $(\mathfrak{t}^*)^2$, so its derivative takes values in $\mathfrak{t}^*$. Also note that the conditions are manifestly independent of the choice of lattice (as they should be).

In order to generalize this criterion to the case $m > 1$, we introduce some notation. For any face $F \subset \Delta$, we denote by $\mathfrak{t}_F \subset \mathfrak{t}$, the vector subspace spanned by the inward normals $u_j \in \mathfrak{t}$ to all codimension one faces of Δ, containing F; thus the codimension of $\mathfrak{t}_F$ equals the dimension of F. Furthermore, the annihilator $\mathfrak{t}_F^0$ of $\mathfrak{t}_F$ in $\mathfrak{t}^*$ is naturally identified with $(\mathfrak{t}/\mathfrak{t}_F)^*$.

Proposition 1. *Let (M, ω) be a compact toric symplectic $2m$-manifold or orbifold with momentum map $\mu\colon M \to \Delta \subset \mathfrak{t}^*$ and $\mathbf{H}$ be a positive definite $S^2\mathfrak{t}^*$-valued function on Δ^0. Then, $\mathbf{H}$ comes from a $\mathbb{T}$-invariant, ω-compatible almost Kähler metric g via (8) if and only if it satisfies the following conditions*:

- [smoothness] $\mathbf{H}$ *is the restriction to Δ^0 of a smooth $S^2\mathfrak{t}^*$-valued function on Δ;*
- [boundary values] *for any point y on the codimension one face $F_j \subset \Delta$ with inward normal u_j, we have*

$$\mathbf{H}_y(u_j, \cdot) = 0 \qquad \text{and} \qquad (d\mathbf{H})_y(u_j, u_j) = 2u_j, \tag{15}$$

  where the differential $d\mathbf{H}$ is viewed as a smooth $S^2\mathfrak{t}^* \otimes \mathfrak{t}$-valued function on Δ;
- [positivity] *for any point y in interior of a face $F \subseteq \Delta$, $\mathbf{H}_y(\cdot, \cdot)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$.*

Proof. We first prove the necessity of these conditions. Let (M, ω, μ) be a compact toric symplectic orbifold with polytope Δ, and (g, J) a compatible Kähler metric. For any $x \in M$ and $\xi, \eta \in \mathfrak{t}$, we put $\mathbf{H}_{\mu(x)}(\xi, \eta) = g_x(X_\xi, X_\eta)$. Clearly, $\mathbf{H}$ is an $S^2\mathfrak{t}^*$-valued function on Δ and the smoothness and positivity properties follow immediately from the definition.

It remains to establish the boundary values (15) for $y = \mu(x)$ in a codimension one face F_j. The vanishing of $\mathbf{H}_y(u_j, \cdot) = 0$ is immediate from the definition (the Killing vector field corresponding to u_j vanishes on $\mu^{-1}(F_j)$). This implies in particular that $d\mathbf{H}_y(u_j, u_j)$ is proportional to u_j. To obtain the correct constant, we use a symplectic slice, as in Lemma 1, to pullback the metric g to the normal bundle N of the orbit $\mathbb{T} \cdot x$ for a point $x \in M$ with $\mu(x) = y$, and restrict to the symplectic isotropy representation V_x. By construction, the Killing vector field corresponding to u_j induces the generator X of the standard circle action on V_x, and the metric induced by g agrees to first order at 0 with the constant metric g_0 given by g_x. It is now straightforward to check that the constant is 2 (indeed, (V_x, g_0, ω_0) is a toric Kähler 2-orbifold, so we have already computed this above).

Now, we explain why the given conditions are sufficient to conclude $\mathbf{H}$ that comes from a smooth compatible metric on (M, ω).

We know that the function $\mathbf{H_0} = \mathbf{G_0}^{-1}$, with $\mathbf{G_0}$ defined by (10), does correspond to a globally defined invariant Kähler metric on (M, ω) (and so it satisfies the given conditions, as one can easily check directly). By virtue of Lemma 2, it is enough to show that for any $\mathbf{H} = \mathbf{G}^{-1}$ satisfying the given conditions, the sufficient conditions (11)–(12) are satisfied. As explained in Remark 4, we have to check that both $\mathbf{H}\mathbf{G_0}$ and $\mathbf{G} - \mathbf{G_0}$ are smoothly extendable about each point $y_0 \in \partial\Delta$. We shall establish this by a straightforward argument using Taylor's Theorem.

Suppose that y_0 belongs to the interior of a k-dimensional face F of Δ. Let us choose a vertex of F. Since Δ is a rational Delzant polytope, the affine functions $L_i(y) = \langle u_i, y \rangle + \lambda_i$ which vanish at this vertex form a coordinate system on Δ. By reordering the inward normals $u_1, \ldots u_n$, we can suppose that these coordinate functions are $L_1(y), \ldots L_m(y)$ (so $u_1, \ldots u_m$ form a basis for $\mathfrak{t}$) and that $L_1(y), \ldots L_{m-k}(y)$ vanish on F (so $u_1, \ldots u_{m-k}$ span $\mathfrak{t}_F$). We set $y_i = L_i(y) - L_i(y_0)$ for $i = 1, \ldots m$. These functions also form a coordinate system on Δ, with y_0 corresponding to the origin, and $y_1, \ldots y_{m-k}$ vanish on F.

We now let $H_{ij}(y) = \mathbf{H}_y(u_i, u_j)$ and let $(G_{ij}(y))$ be the inverse matrix to $(H_{ij}(y))$ (which is the matrix of $\mathbf{G}$ with respect to the dual basis). Similarly, we define inverse matrices $(H^0_{ij}(y))$ and $(G^0_{ij}(y))$. The conditions (i)–(iii) imply:

- $H_{ij}(y)$ are smooth functions on Δ;
- on any codimension one face F_i containing F (with inward normal u_i, $i = 1, \ldots m-k$), we have

(16)
$$H_{ij}(y) = H_{ji}(y) = 0 \qquad \text{for all } j = 1, \ldots m \qquad \text{and} \qquad \partial H_{ii}/\partial y_i = 2;$$

- the matrix $(H_{ij}(y))_{i,j=m-k+1}^{m}$ is positive definite on the interior of F;

We conclude from (16) that for $i = 1, \dots m-k$, $H_{ij}(y) = H_{ji}(y) = O(y_i)$ (for all $j = 1, \dots m$) and $H_{ii}(y) = 2y_i(1 + O(y_i))$, where $O(y_i)$ denotes the product of y_i with a smooth function of y.

Putting these conditions together, we then have:

$$\begin{aligned} &H_{ij}(y) = 2y_i\delta_{ij} + y_iy_jF_{ij}(y) && \text{for } i, j = 1, \dots m-k \\ &H_{ij}(y) = y_iF_{ij}(y) && \text{for } i = 1, \dots m-k \\ & && \text{and } j = m-k+1, \dots m, \end{aligned}$$

where F_{ij} are smooth functions. (Recall also that $H_{ij} = H_{ji}$.)

It follows that $\det(H_{ij}(y)) = 2^{m-k}y_1y_2\cdots y_{m-k}P(y)$ where the function $P(y) = \det(H_{ij}(y))_{i,j=m-k+1}^{m} + O(y_1) + O(y_2) + \cdots O(y_{m-k})$ is positive at the origin. Since the same holds for $H_{ij}^0(y)$, it follows that $\det(H_{ij}(y))/\det(H_{ij}^0(y))$ can be extended to the origin as a smooth and positive function.

On the other hand, $G_{pq}(y)$ is the determinant of a cofactor matrix of $(H_{ij}(y))$ divided by $\det(H_{ij}(y))$. This will be smooth if the determinant of the cofactor is $O(y_i)$ for each $i = 1, \dots m-k$. We see that this is true unless $1 \le p = q \le m-k$, in which case, we obtain $G_{pp}(y) = (1 + O(y_p))/2y_p$. The same holds for $G_{pq}^0(y)$.

We deduce that $\mathbf{G} - \mathbf{G_0}$ is smooth at y_0, and hence $\mathbf{H_0G}$ is smooth at y_0. Since it is non-degenerate there, its inverse $\mathbf{HG_0}$ is also smooth. q.e.d.

Remark 5. By continuity, it suffices that the boundary conditions (15) hold on the interior of the codimension one faces. However, they and their tangential derivatives imply that for a point y on *any* face $F \subset \Delta$, we have

$$\mathbf{H}_y(u_j, \cdot) = 0 \quad \text{and} \quad (d\mathbf{H})_y(u_j, u_k) = 2\delta_{jk}u_j \tag{17}$$

for any inward normals u_j, u_k in $\mathfrak{t}_F$.

The proof also shows that any $\mathbf{H}$ satisfying the given conditions defines an ω-compatible almost Kähler metric on M with the same angular coordinates as the Guillemin metric. Using Lemma 3, this shows that the group of equivariant symplectomorphisms of M acts transitively on the set of angular coordinates on M^0 which come from ω-compatible almost Kähler metrics on M. It also follows that the conditions (11)–(12) in Lemma 2 are necessary as well as sufficient. When g_0 is the Guillemin metric, this agrees with [**1**, Theorem 2].

1.5. Toric complex manifolds and orbifolds. We now turn briefly to the complex point of view on toric Kähler manifolds and orbifolds. Given a rational Delzant polytope $(\Delta, \Lambda, u_1, \dots u_n)$, we obtain a complex subgroup G^c of $(\mathbb{C}^\times)^n$ as the complexification of G. The relation between complex quotients and symplectic quotients then shows [6, 16, 22] that the canonical complex structure on the toric symplectic orbifold (M, ω) constructed from Δ is equivariantly biholomorphic to the quotient by G^c of a dense open subset $\mathbb{C}^n_s$ of $\mathbb{C}^n$ given by

$$\mathbb{C}^n_s = \bigcup\nolimits_F \mathbb{C}^n_F, \quad \mathbb{C}^n_F = \{(z_1, \dots z_n) \in \mathbb{C}^n : z_j = 0 \text{ iff } L_j(x) = 0 \text{ for } x \in F^0\}. \tag{18}$$

Thus, $\mathbb{C}^n_s$ is $\mathbb{C}^n$ with the coordinate subspaces removed that do not correspond to faces of Δ. Observe that the complex quotient only depends on the inward normals (which determine G^c) and the combinatorics of the faces (which determine $\mathbb{C}^n_s$), i.e., by specifying which sets of codimension one faces have non-empty intersection. These data can be encoded in a family of convex simplicial cones called a *fan*.

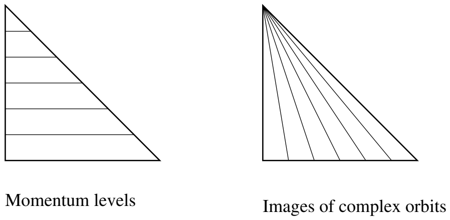
Furthermore, *any* $\mathbb{T}$-invariant ω-compatible complex structure on M is equivariantly biholomorphic to the standard one (see [28] for the result in the general orbifold case). Of course, this biholomorphism does not preserve ω in general. Thus, two toric Kähler manifolds (or orbifolds) are equivariantly biholomorphic if and only if they have the same fan.

1.6. Restricted toric manifolds. Toric Kähler manifolds can be used to provide examples of Kähler manifolds with non-toric isometric hamiltonian torus actions simply by restricting the action to a subtorus. These torus actions can be surprisingly complicated in general. However, the subtori generated by a subset of the normals to the Delzant polytope have much simpler actions.

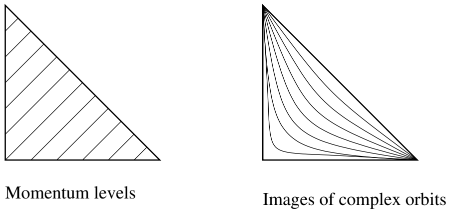
Example 1. We can illustrate this in the simplest non-trivial case of S^1 actions on $\mathbb{C}P^2$, which is toric under the action of $\mathbb{T} \cong S^1 \times S^1$ given by $(\lambda_1, \lambda_2)\colon [z_0, z_1, z_2] \mapsto [z_0, \lambda_1 z_1, \lambda_2 z_2]$. The 'tame' S^1 subgroups generated by the normals are given by $\lambda_1 = 0$, $\lambda_2 = 0$ or $\lambda_1 = \lambda_2$. The momentum map of the S^1 action is then the projection of the momentum map of $\mathbb{T}$ along the corresponding face of the Delzant polytope Δ (which is a simplex). The momentum map of 'wild' S^1 subgroups, such as $\lambda_1 = \lambda^2$, $\lambda_2 = \lambda^3$, are given by more general projections. We wish to draw attention to two distinctions between these two types of S^1 action.

(i) For tame actions, the momentum map of the S^1 action has no critical values on the interior of the momentum interval, whereas for wild actions it does.

(ii) For tame actions, the orbits of the complexified action (of $\mathbb{C}^\times$) have smooth closures, whereas for wild actions, they do not—for instance, they are singular cubics for the case $\lambda_1 = \lambda^2$, $\lambda_2 = \lambda^3$.



A tame circle action on the projective plane



A wild circle action on the projective plane

Figure 1. Circle actions on $\mathbb{C}P^2$.

The blow up of $\mathbb{C}P^2$ at a point is the first Hirzebruch surface $F_1 = P(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{C}P^1$. If this point is one of the three fixed points of the $\mathbb{T}$-action corresponding to a vertex of Δ, then the standard fibrewise S^1 action on F_1 descends to the tame S^1 action on $\mathbb{C}P^2$ corresponding to the opposite edge. Thus, a tame S^1 action realises $\mathbb{C}P^2$ as the blowdown of a toric bundle (of projective lines) over $\mathbb{C}P^1$.

We generalize this by considering torus actions on blowdowns of toric bundles (with fibre any toric Kähler manifold) over a product of complex projective spaces.

Let $\mathcal{V}$ be a toric Kähler 2ℓ-manifold, under a torus $\mathbb{T}$, with Delzant polytope Δ. By the Delzant construction $\mathcal{V}$ is ($\mathbb{T}$-equivariantly symplectomorphic to) a symplectic quotient of $\mathbb{C}^n$ by an $n - \ell$ dimensional subgroup G of the standard n-torus $\mathbb{T}^n$ (with $\mathbb{T} = \mathbb{T}^n/G$). From Section 1.5, $\mathcal{V}$ is also ($\mathbb{T}^c$-equivariantly biholomorphic to) the holomorphic quotient $\mathbb{C}^n_s/G^c$, where $\mathbb{C}^n_s$ is the set of stable points in $\mathbb{C}^n$.

Given integers $d_1, \ldots d_n \geq 0$, there are now two constructions we can make.

(i) Let $\mathbb{C}^D = \mathbb{C}^{d_1+1} \times \mathbb{C}^{d_2+1} \times \cdots \mathbb{C}^{d_n+1}$. Then, we have a block diagonal action of $\mathbb{T}^n$ on $\mathbb{C}^D$ as a subtorus of the standard torus $\mathbb{T}^D$: the ith circle in $\mathbb{T}^n$ acts by scalar multiplication on $\mathbb{C}^{d_i+1}$ and trivially on the other factors. Since G is a subtorus of $\mathbb{T}^n$, we can form the symplectic quotient of $\mathbb{C}^D$ by G and this will be diffeomorphic to the stable quotient by G^c. Let us denote the corresponding manifold by M.

The standard Kähler structure on $\mathbb{C}^D$ can be written in block diagonal momentum coordinates $(x_1, \dots x_n)$ of $\mathbb{T}^n$ as

$$\tilde{g}_0 = \sum_{j=1}^n \left(x_j g_j + \frac{dx_j^2}{2x_j} + 2x_j\theta_j^2 \right)$$

$$\tilde{\omega} = \sum_{j=1}^n \bigl(x_j\omega_j + dx_j \wedge \theta_j \bigr), \qquad d\theta_j = \omega_j$$

where $x_j = r_j^2/2$, for the radial coordinate r_j on $\mathbb{C}^{d_j+1}$, and g_j is the Fubini–Study metric on $\mathbb{C}P^{d_j}$, normalized so that θ_j is the connection 1-form of the Hopf fibration and $\frac{1}{2}g_j + \theta_j^2$ is the round metric on the unit sphere S^{2d_j+1}: we obtain the flat metric in spherical polar coordinates on each $\mathbb{C}^{d_j+1}$ factor by substituting $x_j = r_j^2/2$.

This induces a Kähler structure on M by writing the momentum coordinates $x_j = L_j(\mu)$ of $\mathbb{T}^n$ in terms of the momentum map μ of $\mathbb{T}$, where $L_1, \dots L_n$ are the affine functions defining Δ. The resulting Kähler metric, in the notation of Section 1.3, is

$$g'_0 = \sum_{j=1}^n L_j(\mu) g_j + \langle d\mu, \mathbf{G_0}, d\mu\rangle + \langle \boldsymbol{\theta}, \mathbf{H_0}, \boldsymbol{\theta}\rangle, \tag{19}$$

$$\omega' = \sum_{j=1}^n L_j(\mu)\omega_j + \langle d\mu \wedge \boldsymbol{\theta}\rangle, \qquad d\boldsymbol{\theta} = \sum_{j=1}^n \omega_j \otimes u_j$$

with $\mathbf{G_0}$ given by (10) and $\mathbf{H_0}$ is inverse to $\mathbf{G_0}$. This reduces to the *canonical* toric Kähler structure on $\mathcal{V}$ when $d_j = 0$ for all j.

Our aim is to show that there is a compatible Kähler structure on M generalizing the *given* toric Kähler structure of $\mathcal{V}$, which is determined by an arbitrary matrix $\mathbf{H}$ satisfying the necessary and sufficient compactification conditions of Proposition 1. To do this, and to understand better the holomorphic geometry of M, we consider another construction.

(ii) Let $\tilde{M} = \bigoplus_{j=1}^n \mathcal{O}(-1)_j \to \prod_{j=1}^n \mathbb{C}P^{d_j}$, where $\mathcal{O}(-1)_j = \mathcal{O}(0, \dots 0, -1, 0, \dots 0)$ is the line bundle which is $\mathcal{O}(-1)$ over $\mathbb{C}P^{d_j}$ and trivial over the other factors. Let $\tilde{M}^0 = \prod_{j=1}^n \mathcal{O}(-1)_j^\times$ be the associated holomorphic principal $(\mathbb{C}^\times)^n$-bundle (given by removing the zero section from each line bundle). Now, $\tilde{M} = \tilde{M}_0 \times_{(\mathbb{C}^\times)^n} \mathbb{C}^n$ admits a fibre-preserving holomorphic action of $(\mathbb{C}^\times)^n$.

Since G is a subtorus of $\mathbb{T}^n$, we can form the holomorphic stable quotient of $\tilde{M}$ by G^c to obtain a complex manifold $\hat{M}$. We see immediately that $\hat{M} = M^0 \times_{\mathbb{T}^c} \mathcal{V}$ where $M^0 = \tilde{M}^0/G^c$. Thus, $\hat{M}$ is a bundle of toric complex manifolds.

It is easy to see how constructions (i) and (ii) are related, since $\mathcal{O}(-1) \to \mathbb{C}P^{d_j}$ is the blow-up of $\mathbb{C}^{d_j+1}$ at the origin, so that $\tilde{M}$ is (equivariantly biholomorphic to) the blow-up of $\mathbb{C}^D = \prod_j \mathbb{C}^{d_j+1}$ along the union over j of the coordinate subspaces with zero in the jth factor. The stable quotients of $\tilde{M}$ and $\mathbb{C}^D$ that we consider are related by this blow-up (by construction), and so $\hat{M}$ is (equivariantly biholomorphic to) a blow-up of M.

The Kähler structure (19) on M therefore pulls back to give a Kähler structure on $\hat{M}$, except that the metric and symplectic form degenerate on the exceptional divisor. Again by construction, this induces the canonical toric Kähler structure of $\mathcal{V}$ on each fibre of $\hat{M}$.

Let $\mathbf{G} = \mathbf{H}^{-1}$ be the matrices inducing the given toric Kähler structure on $\mathcal{V}$. Then, we obtain a new Kähler structure on $\hat{M}$, degenerating on the exceptional divisor and inducing the given toric Kähler structure on each fibre:

$$g' = \sum_{j=1}^n L_j(\mu) g_j + \langle d\mu, \mathbf{G}, d\mu \rangle + \langle \boldsymbol{\theta}, \mathbf{H}, \boldsymbol{\theta} \rangle, \tag{20}$$
$$\omega' = \sum_{j=1}^n L_j(\mu)\omega_j + \langle d\mu \wedge \boldsymbol{\theta} \rangle, \qquad d\boldsymbol{\theta} = \sum_{j=1}^n \omega_j \otimes u_j.$$

There is no reason *a priori* why this should descend to M (in particular, the complex structure is different). Nevertheless, it does, because of the strong control over the boundary behaviour of $\mathbf{H}$ given by Proposition 1.

Proposition 2. *The degenerate Kähler structure* (20) *on* $\hat{M}$ *descends to give a* (*non-degenerate*) *Kähler structure on* M.

Proof. We know that (19) is globally defined smooth Kähler structure on M. We shall show that (20) defines a compatible Kähler metric on the same symplectic manifold (with the same angular coordinates). For this, it suffices to show that the difference $g' - g_0'$ is smooth on M. However, since the compatible Kähler metrics defined on $\mathcal{V}$ by $\mathbf{H}$ and $\mathbf{H_0}$ are smooth, Proposition 1 and Remark 5 show that $\mathbf{G}-\mathbf{G_0}$ and $\mathbf{G_0HG_0}-\mathbf{G_0}$ are smooth functions on the Delzant polytope Δ of $\mathcal{V}$. Now, the momentum map μ on (M,ω') is smooth, with image Δ. It therefore follows, as in the proof of Lemma 2, that $g'-g'_0$ is smooth. q.e.d.

2. Rigid hamiltonian torus actions

In this section, we introduce the notion of a *rigid hamiltonian torus action.* Toric Kähler manifolds automatically carry such an action: our goal is to extend some of the rigid properties of toric Kähler manifolds to rigid torus actions in general, and to classify them. In the first three subsections, we study respectively the differential topology, symplectic geometry and biholomorphism type of compact (smooth) Kähler manifolds with such an action, then we combine these threads to describe the Kähler geometry. In the final subsection, we specialize to the case that the torus action is 'semisimple' and give a generalized Calabi construction of all compact Kähler manifolds with a semisimple rigid torus action.

2.1. Stratification of the momentum polytope. Before defining the torus actions, we will consider, we establish a couple of basic facts. We shall make essential use of the convexity theorem of Atiyah and Guillemin–Sternberg [5, 18].

Lemma 4. *Let $\mathbb{T}$ be a torus in the group of hamiltonian isometries of a compact connected Kähler manifold (M,g,J,ω), which is the closure of the group generated by ℓ hamiltonian Killing vector fields $K_r = J\,\mathrm{grad}_g\,\sigma_r$ $(r=1,\dots\ell)$ that are independent on a dense open set. Suppose that $g(K_r,K_s)$ depends only on $(\sigma_1,\dots\sigma_\ell)$ for $r,s=1,\dots\ell$. Then:*

- (i) *the torus $\mathbb{T}$ has dimension ℓ;*
- (ii) *the image of the momentum map $\mu\colon M\to\mathfrak{t}^*$ of $\mathbb{T}$ is a compact convex polytope such that μ is regular (i.e., submersive) as a map to the interior of any of its faces.*

Proof. By the Atiyah–Guillemin–Sternberg convexity theorem [5, 18], the image of μ is a compact convex polytope Δ in $\mathfrak{t}^*$, the convex hull of the finite image I of the fixed point set of $\mathbb{T}$. The momentum coordinates $\boldsymbol{\sigma}=(\sigma_1,\dots\sigma_\ell)$ are related to μ by the natural inclusion

$$\mathbb{R}^\ell \cong \mathrm{span}(K_1,\dots K_\ell)\subseteq\mathfrak{t},$$

which in turn gives rise to a linear projection $\pi\colon\mathfrak{t}^*\to\mathbb{R}^{\ell*}$ such that $\boldsymbol{\sigma}=\pi\circ\mu$.

Let us first consider the image of Δ by π. We claim that π is injective on I. Indeed, since $K_1, \dots K_\ell$ generate $\mathbb{T}$, the fixed point set is precisely the set of common zeros of $K_1, \dots K_\ell$, and since $g(K_r, K_s)$ depends only on $\boldsymbol{\sigma}$, the preimage of an element of $\pi(\Delta)$, containing an element of I, consists entirely of elements of I. Now, I is finite and the preimages of π are convex, so each such preimage has just one point.

Second, we note that the set of regular values of $\boldsymbol{\sigma}$ is connected. Indeed, the critical point set of $\boldsymbol{\sigma}$ in M has codimension at least two—it is the set where the holomorphic ℓ-vector $K_1^{1,0} \wedge \cdots \wedge K_\ell^{1,0}$ vanishes—so the set of regular points U is connected. Now, as $g(K_r, K_s)$ depends only on $\boldsymbol{\sigma}$, the inverse image of a critical value consists entirely of critical points, so the set of regular values is $\boldsymbol{\sigma}(U)$.

Third, consider the orbits of the commuting vector fields $JK_1, \dots JK_\ell$—this is the gradient flow of $\boldsymbol{\sigma}$, and so the orbit of any regular point consists entirely of regular points and its boundary points are all critical. Now, regular points map to regular values and critical points to critical values, so by the connectivity of the regular values, all regular orbits have the same image—and the closure is the image of $\boldsymbol{\sigma}$ since regular values are dense.

These facts imply the conclusions of the lemma as follows.

(i) Suppose x is a regular point of $\boldsymbol{\sigma}$ and $\mu(x)$ belongs to a closed face F of Δ. Then, the $\mathbb{T}^c$ orbit of x also maps to F, where $\mathbb{T}^c$ is the complexification of $\mathbb{T}$. Since the orbit under $JK_1, \dots JK_\ell$ is contained in the $\mathbb{T}^c$ orbit, π maps F onto $\operatorname{Im}\boldsymbol{\sigma}$. Now, π is bijective on vertices, so $F = \Delta$. In other words, the inverse image (under π) of a regular value of $\boldsymbol{\sigma}$ meets no proper face of Δ: this clearly implies π is bijective, hence $\mu = \boldsymbol{\sigma}$ and $\dim \mathbb{T} = \ell$.

(ii) We have seen that the image of the closure C of any regular $\mathbb{T}^c$ orbit is the whole of Δ. Atiyah [5] shows that the inverse image in C of any open face F^0 is a single $\mathbb{T}^c$-orbit and μ is a submersion from this orbit to F^0. Since this is true for all regular orbits, and the union of the regular orbits is dense, the claim follows. q.e.d.

Definition 3. Let (M, g, J, ω) be a connected Kähler $2m$-manifold with an effective isometric hamiltonian action of an ℓ-torus $\mathbb{T}$ with momentum map $\mu \colon M \to \mathfrak{t}^*$. We say the action is *rigid* iff for all $x \in M$, $R_x^* g$ depends only on $\mu(x)$, where $R_x \colon \mathbb{T} \to \mathbb{T} \cdot x \subset M$ is the orbit map.

In other words, for any two generators X_ξ, X_η of the action ($\xi, \eta \in \mathfrak{t}$), $g(X_\xi, X_\eta)$ is constant on the levels of the momentum map μ. We remark that the inverse image of a critical value of μ can be approximated (to first order on a dense open subset) by inverse images of nearby regular values. Hence, it suffices to know that the generators have constant inner products on the generic level sets of μ. Thus, part (i) of Lemma 4 implies that on a compact manifold, a *local* rigid torus action (as in [**4**]) is necessarily a global one. In particular, on a compact Kähler manifold with a hamiltonian 2-form of order ℓ, the associated Killing vector fields $K_1, \dots K_\ell$ generate a rigid ℓ-torus action. Another example is any toric Kähler manifold.

Part (ii) of Lemma 4 has further consequences for compact Kähler manifolds with a rigid torus action.

Proposition 3. *Suppose (M, g, J, ω) is a compact connected Kähler manifold of dimension $2m$, with a rigid hamiltonian ℓ-torus action with momentum map μ whose image is a compact convex polytope Δ.*

- (i) *If F is a k-dimensional closed face $(0 \leq k \leq \ell)$ of Δ, then $M_F := \mu^{-1}(F)$ is a compact totally geodesic Kähler submanifold of M of dimension $2(m_F + k)$ $(0 \leq m_F \leq m - \ell)$ with a rigid hamiltonian action of a k-torus $\mathbb{T}/\mathbb{T}_F$, where $\mathbb{T}_F$ is the intersection of the isotropy subgroups of points in M_F.*
- (ii) *If F^0 is the interior of F, then $M_F^0 := \mu^{-1}(F^0) \cong F^0 \times P_F$ where P_F is a compact manifold of dimension $2m_F + k$ with a locally free action of $\mathbb{T}/\mathbb{T}_F$. Moreover, the levels of μ are compact connected submanifolds of M.*

Proof.

(i) Let $\mathbb{T}_F$ be the intersection of the isotropy subgroups of points in M_F. Then, the connected component of the identity in $\mathbb{T}_F$ is an $(\ell - k)$-dimensional subtorus of $\mathbb{T}$, and M_F is a connected component of its fixed point set. Since $\mathbb{T}_F$ acts on M effectively by hamiltonian isometries, M_F is a compact totally geodesic Kähler submanifold of M, of dimension at most $2m - 2(\ell - k)$. By definition, M_F carries an effective hamiltonian action of $\mathbb{T}/\mathbb{T}_F$ (which is connected, hence a k-torus), so it has dimension at least $2k$. The momentum map is essentially μ, viewed as a map from M_F to the affine span of F, so the action is rigid.

(ii) By Lemma 4, the critical values of μ, regarded in the above way, are precisely the boundary points of F, and μ is regular as a map from M_F^0 to F^0. The gradient flow of μ commutes with $\mathbb{T}$ and hence, provides an equivariant trivialization of M_F^0. Thus, M_F^0 is diffeomorphic to $F^0 \times P_F$ and the action of $\mathbb{T}/\mathbb{T}_F$ is given by an effective locally free action on P_F, with trivial action on F^0. The levels of μ are smooth since any point in the image of μ is in some open face; they are connected by [**5**]. q.e.d.

The absense of interior critical values for rigid actions shows that 'wild' S^1 actions on $\mathbb{C}P^2$ (as a symplectic manifold) of Example 1 cannot be rigid with respect to any compatible Kähler metric.

2.2. The symplectic isotropy representations. We now wish to obtain precise information about the symplectic isotropy representations of the torus action. If $\mu(x)$ belongs to an open k-dimensional face F^0, then the Lie algebra $\mathfrak{t}_x$ of the isotropy group $\mathbb{T}_x \geq \mathbb{T}_F$ of x is the vector subspace of elements of $\mathfrak{t}$, annihilated by the elements of the vector subspace of $\mathfrak{t}^*$ parallel to F: indeed, this is clearly the image of $d\mu_x$, and $\mathfrak{t}_x$ is the kernel of the transpose of $d\mu_x$.

Since the orbit $\mathbb{T} \cdot x$ is k-dimensional, the symplectic isotropy representation $V_x = T_x(\mathbb{T} \cdot x)^0/T_x(\mathbb{T} \cdot x)$ of $\mathbb{T}_x$ (and its Lie algebra $\mathfrak{t}_x$) has dimension $m-k$. Hence, it is an orthogonal direct sum of $m-k$ complex 1-dimensional representations with (not necessarily distinct) characters $\mathbb{T}_x \to S^1$. Differentiating this action gives the weights $\alpha_1, \dots \alpha_{m-k}$ of the action of $\mathfrak{t}_x$, which are integral elements of $\mathfrak{t}_x^*$.

Since the $\mathfrak{t}_x$ action is effective, the weights $\alpha_1, \dots \alpha_{m-k}$ span $\mathfrak{t}_x^*$, and we order them so that $\alpha_1, \dots \alpha_{\ell-k}$ form a basis for $\mathfrak{t}_x^*$.

Lemma 5. *Suppose $\mu(x)$ belongs to an open k-dimensional face F^0 of Δ and let V_x be the symplectic isotropy representation of $\mathbb{T}_x$ at $x \in M$.*

(i) *The induced $\mathfrak{t}_x$ action has exactly $\ell - k$ distinct non-zero weights.*
(ii) *$\mathbb{T}_x$ is connected.*

Proof.

(i) We choose a projection $\chi\colon \mathfrak{t} \to \mathfrak{t}_x$ and introduce a symplectic slice as in Lemma 1. Thus, there is a $\mathbb{T}$-equivariant symplectomorphism from a neighbourhood U of the zero section 0_N in the normal bundle $N \to \mathbb{T} \cdot x$ to a neighbourhood of $\mathbb{T} \cdot x$ in (M, ω), where the normal bundle $N = \mathbb{T} \times_{\mathbb{T}_x} (\mathfrak{t}_x^0 \oplus V_x) \to \mathbb{T} \cdot x$ is realised as a symplectic quotient of $T^*\mathbb{T} \times V_x$ by the diagonal action of $\mathbb{T}_x$. The symplectomorphism identifies 0_N with $\mathbb{T} \cdot x$ and its differential along the zero section is essentially the identity map. Let us denote the pullback of (g, J, ω) by (g_0, J_0, ω_0). We then have that g_0 agrees with g_x at x.

We now bring in the rigidity condition that the induced metric on $\mathbb{T}$ depends only on μ. This implies that for any vector fields X_ξ, X_η ($\xi, \eta \in \mathfrak{t}$) induced by the action of $\mathbb{T}_x$ on (U, g_0, J_0, ω_0), $g_0(X_\xi, X_\eta)$, as a function on U, depends only on the momentum map μ_0 of N, $\mu_0([\alpha, v]) = \alpha + \mu_V(v) \circ \chi$ with $\mu_V = \frac{1}{2}\sum_{i=1}^{m-k} |z_i|^2 \alpha_i$, where $z_1, \dots z_{m-k}$ are the standard complex coordinates on the weight spaces in V_x. It follows from [**31**] that (being smooth on U) $g_0(X_\xi, X_\eta)$ is a smooth function of μ_0. In particular, for $\alpha = 0 \in \mathfrak{t}_x^0$, $g_0(X_\xi, X_\eta)$ is a smooth function of μ_V. Thus, on $V_x \cap U$, $d(g_0(X_\xi, X_\eta))$ is a pointwise linear combination of the components of

$$d\mu_V = \frac{1}{2}\sum_{i=1}^{m-k}(z_i d\overline{z}_i + \overline{z}_i dz_i)\alpha_i. \tag{21}$$

In other words, (since it vanishes at the origin of V_x) it equals $\langle d\mu_V, B(\xi,\eta)\rangle$ for a smooth bilinear form $B\colon V_x \cap U \to S^2\mathfrak{t}_x^* \otimes \mathfrak{t}_x$. Now, since X_ξ and X_η vanish at the origin, $g_0(X_\xi, X_\eta)$ differs from $g_x(X_\xi, X_\eta) = \sum_{i=1}^{m-k}\alpha_i(\xi)\alpha_i(\eta)|z_i|^2$ by a smooth function vanishing to second order at the origin, so its exterior derivative on $V_x \cap U$ is, to first order, equal to

$$d(g_x(X_\xi, X_\eta)) = \sum_{i=1}^{m-k}(z_i d\overline{z}_i + \overline{z}_i dz_i)\alpha_i(\xi)\alpha_i(\eta). \tag{22}$$

If we differentiate $d(g_0(X_\xi, X_\eta)) = \langle d\mu_V, B(\xi,\eta)\rangle$ with respect to $\overline{z}_i$, using (21) and (22), and evaluate at the origin of V_x, the error terms and derivative of B go away. Equating coefficients of $dz_1, \ldots dz_{m-k}$ therefore gives

$$2\alpha_i(\xi)\alpha_i(\eta) = \alpha_i(B_0(\xi,\eta))$$

for all i, i.e., $B_0^*\alpha_i = 2\alpha_i \otimes \alpha_i$. (We remark that this generalizes the conditions (17) in the toric case.) Now, $\alpha_1, \ldots \alpha_{\ell-k}$ is a basis for $\mathfrak{t}_x^*$, so we may write $\alpha_{\ell-k+1}, \ldots \alpha_{m-k}$ as $\alpha_i = \sum_{j=1}^{\ell-k}\lambda_{ij}\alpha_j$. We then deduce from $B_0^*\alpha_i = 2\alpha_i \otimes \alpha_i$ that

$$\lambda_{ij}\lambda_{ik} = \delta_{jk}\lambda_{ij}.$$

Thus, for each i, λ_{ij} is non-zero for at most one j, and then equal to one, i.e., for any $i = \ell - k + 1, \ldots m - k$, the weight α_i is either zero, or it is one of $\alpha_1, \ldots \alpha_{\ell-k}$.

(ii) We prove that all isotropy groups of the $\mathbb{T}$-action are connected. Since the gradient flow of μ commutes with $\mathbb{T}$, it suffices to prove this near a fixed point y of the $\mathbb{T}$-action, where the symplectic slice gives a $\mathbb{T}$-equivariant symplectomorphism with a neighbourhood of the origin in a symplectic vector space V_y. Now, since the $\mathbb{T}$-action on V_y is effective, with ℓ-distinct non-zero weights, these form a basis for the dual lattice. This ensures the isotropy groups of points in V_y are connected. q.e.d.

Part (i) of Lemma 5 is the key to the theory of rigid hamiltonian torus actions. In particular, it allows us to refine Proposition 3.

Proposition 4. *Suppose (M, g, J, ω) is a compact connected Kähler manifold with a rigid hamiltonian ℓ-torus action, as in Proposition 3.*

(i) *If* F^0 *is an open* k*-dimensional face, then the isotropy group of all points in* M_F^0 *is an* $(\ell - k)$*-torus* $\mathbb{T}_F$*, and the isotropy representations are all equivalent, with the distinct non-zero weights in* $\mathfrak{t}_F^*$ *forming a basis for the lattice dual to the lattice of circle subgroups of* $\mathfrak{t}_F$*.*

(ii) *The image* Δ *of* μ *is a Delzant polytope.*

(iii) P_F *is a principal* k*-torus bundle (under* $\mathbb{T}/\mathbb{T}_F$*) over a compact manifold* S_F *of dimension* $2m_F$*, with a family of Kähler structures parameterized by* F^0*.*

Proof.

(i) This is immediate from Lemma 5: the distinct non-zero weights form a basis for $\mathfrak{t}_F$, the Lie algebra of the (connected) isotropy group of any point in M_F^0.

(ii) Applying this to a fixed point, observe that the directions of the distinct non-zero weights are the edges meeting the corresponding vertex of Δ. There are ℓ of these and the dual basis gives a basis for the lattice of circle subgroups of $\mathbb{T}$ consisting of normals to the faces meeting the vertex.

(iii) By Proposition 3, P_F has a locally free action of $\mathbb{T}/\mathbb{T}_F$, and by Lemma 5, the isotropy groups are connected, so the action is free. Hence, P_F is a principal $\mathbb{T}/\mathbb{T}_F$ bundle over a compact manifold S_F. Choosing a point v in F^0 identifies S_F with the Kähler quotient of M_F at momentum level v. q.e.d.

2.3. The complexified torus action. We now turn to the structure of the orbits of the complexified torus action. If the $\mathbb{T}$ action is generated by vector fields $K_1, \dots K_\ell$, then the complexified action of $\mathbb{T}^c$ is generated by the (real) holomorphic vector fields $K_1, \dots K_\ell, JK_1, \dots JK_\ell$. These are linearly independent on a dense open set (since the $\mathbb{T}$ action is hamiltonian) and generate a foliation of M by complex orbits, whose generic leaf is 2ℓ-dimensional. As we have already remarked in Section 2.1, $JK_1, \dots JK_\ell$ generate the gradient flow of μ, and therefore the momentum image of a $2k$-dimensional leaf is a k-dimensional open face F^0 of Δ; the isotropy group of any point in this leaf is the complexification $\mathbb{T}_F^c$ of $\mathbb{T}_F$ and the closure (in M) of the leaf maps onto the closed face F.

To understand the complex orbits further, we reinterpret V_x as the fibre of the normal bundle to $\mathbb{T}^c \cdot x$ at x, carrying the complex isotropy representation, and we linearize the $\mathbb{T}^c$ action using a holomorphic slice rather than a symplectic one.

In general, let G be a compact Lie group of hamiltonian isometries of a Kähler manifold M, and let G^c be the complexification, which acts holomorphically on M. Then, the *holomorphic slice theorem* [**19**, **30**] states that if $G^c \cdot x$ is the orbit through $x \in M$ with isotropy representation (G^c_x, V_x), then there is a G^c-equivariant biholomorphism from a neighbourhood of $G^c \cdot x$ in M to a neighbourhood of the zero section in $G^c \times_{G^c_x} V_x \to G^c \cdot x$.

Remark 6. For many purposes, it suffices to know that a neighbourhood of x is *locally* G^c-equivariantly biholomorphic to a neighbourhood of the zero section in $G^c \times_{G^c_x} V_x$. This is quite easy to establish. Indeed, let $\psi \colon U \to M$ be a holomorphic chart with $\psi(0) = x$ and $d\psi_0 = \mathrm{Id}$, where U is an open neighbourhood of the origin in $T_x M$. We can assume U and ψ are G_x-equivariant by averaging, since G_x is compact. Now, by acting with G, we obtain a G-equivariant biholomorphism $\tilde{\psi}$ from a neighbourhood $\tilde{U}$ of $G \cdot x$ in M to a neighbourhood of the zero section in $G \times_{G_x} \tilde{V}_x \to G \cdot x$. Here, $\tilde{V}_x$ is the orthogonal complement of $T_x(G \cdot x)$: note $\tilde{V}_x = V_x \oplus W_x$ where V_x is the orthogonal complement of $T_x(G^c \cdot x)$, and $W_x = J T_x(G \cdot x)$.

Now, since $\tilde{\psi}$ is holomorphic and G-equivariant, it is (locally) G^c-equivariant. This is only a local result, because the domain $\tilde{U}$ is *a priori* only G-invariant, not G^c-invariant. The hard part of the holomorphic slice theorem is to show such a 'local' slice can be analytically continued to a G^c-invariant neighbourhood of $G^c \cdot x$.

Lemma 6. *Suppose $\mu(x)$ belongs to an open k-dimensional face F^0 of Δ and let $\mathbb{T}^c_1, \mathbb{T}^c_2, \dots \mathbb{T}^c_{\ell-k}$ be the complexifications of the circle subgroups of the isotropy subgroup $\mathbb{T}_F$ dual to the basis of distinct non-zero weights in the symplectic isotropy representation of $\mathbb{T}_F$.*

Then, $\mathbb{T}^c_F = \mathbb{T}^c_1 \times \cdots \times \mathbb{T}^c_{\ell-k}$ and there is a $\mathbb{T}^c$-equivariant biholomorphism from a neighbourhood U of $\mathbb{T}^c \cdot x$ in M to a neighbourhood W of the zero section in

$$\mathbb{T}^c \times_{\mathbb{T}^c_F} \big(V_0 \oplus V_1 \oplus \cdots \oplus V_{\ell-k}\big) \to \mathbb{T}^c / \mathbb{T}^c_F$$

where V_0 is the trivial representation (possibly zero), while for $i = 1, \dots \ell - k$, V_i is a non-zero vector space carrying the standard action of $\mathbb{T}^c_i \cong \mathbb{C}^\times$ by scalar multiplication, with $\mathbb{T}^c_j$ acting trivially for $j \neq i$. Under this biholomorphism:

(i) *the p-dimensional faces F' meeting F correspond bijectively to $(p-k)$-element subsets $\mathcal{J}_{F'} \subseteq \{1, \dots \ell - k\}$ in such a way that*
  - *$M_{F'} \cap U$ is the intersection of W with those elements whose V_j component vanishes for $j \in \{1, \dots \ell - k\} \smallsetminus \mathcal{J}_{F'}$;*

(i) *if Y is a p-dimensional complex orbit with $x \in \overline{Y} \subseteq M_{F'}$, $\dim F' = p$, then there are one dimensional subspaces of V_j for $j \in \mathcal{J}_{F'}$ such that*

$$\overline{Y} \cap U \cong \mathbb{T}^c \times_{\mathbb{T}^c_F} \textstyle\bigoplus_{j \in \mathcal{J}_{F'}} L_j \tag{23}$$

under the obvious inclusion into $\mathbb{T}^c \times_{\mathbb{T}^c_F} \big(V_0 \oplus \cdots \oplus V_{\ell-k}\big)$.

Proof. By the holomorphic slice theorem, there is a $\mathbb{T}^c$-equivariant biholomorphism from a neighbourhood of $\mathbb{T}^c \cdot x$ to neighbourhood of the zero section in $\mathbb{T}^c \times_{\mathbb{T}^c_F} V_x$ where V_x is normal to $\mathbb{T}^c \cdot x$ at x. Equivalently, V_x is the symplectic isotropy representation of $\mathbb{T}_F$, now equipped with the natural complexified action of $\mathbb{T}^c_F$. By Lemma 5, the distinct non-zero weights of the $\mathfrak{t}_F$ action on V_x are dual to a basis for the lattice of circle subgroups of $\mathbb{T}_F$, and we take the V_i's to be the weight spaces (with V_0 the zero weight space). This gives what we want.

(i) It is clear that the faces F' containing F correspond to subsets $\mathcal{J}_{F'}$ of $\{1, \dots \ell - k\}$ with $\mathbb{T}^c_j$ acting non-trivially on $M_{F'}$ for $j \in \mathcal{J}_{F'}$. The biholomorphism identifies $M_{F'} \cap U$ with those elements of W whose isotropy group is contained in $\mathbb{T}^c_{F'}$. Since the latter is the product of the $\mathbb{T}^c_j$ for $j \in \{1, \dots \ell - k\} \smallsetminus \mathcal{J}_{F'}$, the result follows.

(ii) Under the biholomorpism, the complex orbits Y near $\mathbb{T}^c \cdot x$ are all of the form $\mathbb{T}^c \times_{\mathbb{T}^c_F} (v_0 + U_1 \times \cdots \times U_{\ell-k})$, where $v_0 \in V_0$ and either $U_j = L_j^\times := L_j \smallsetminus \{0\}$, where L_j is a one-dimensional subspace of V_j, or $U_j = \{0\} \subset V_j$.

If Y is a p-dimensional orbit in $M^0_{F'}$, then these two cases occur accordingly as $j \in \mathcal{J}_{F'}$ or not. Clearly, $x \in \overline{Y}$ if and only if $v_0 = 0$, and then the biholomorpism identifies $\overline{Y} \cap U$ with $\bigoplus_{j \in \mathcal{J}_{F'}} L_j$ as stated.

q.e.d.

Lemma 6 gives a lot of information about the equivariant holomorphic geometry of M. For instance, applying it at a fixed point gives a $\mathbb{T}^c$-equivariant chart from a neighbourhood of the fixed point to $U_0 + V_1 \oplus \cdots \oplus V_\ell$, where $V_1, \dots V_\ell$ are the non-trivial weight spaces associated to the corresponding vertex v of Δ, and U_0 is a neighbourhood of the origin in the trivial weight space V_0. In the toric case, $V_0 = 0$ and $\dim V_j = 1$ for all j, and we obtain the linear charts underlying the toric complex manifold. In the general case, such charts provide a finite atlas, since there are finitely many vertices v and they have compact preimages $S_v = \mu^{-1}(v)$.

Proposition 5. *Suppose (M, g, J, ω) is a compact connected Kähler manifold with a rigid hamiltonian ℓ-torus action, as in Proposition 3.*

(i) *The closure of a $2k$-dimensional complex orbit in M is a toric Kähler submanifold of M whose Delzant polytope is a k-dimensional face F of Δ.*

(ii) *For any k-dimensional face F of Δ, $M_F^0 = F^0 \times P_F$ is a holomorphic principal $\mathbb{T}^c/\mathbb{T}^c_F$-bundle over a complex manifold S_F.*

(iii) *The blow-up of M_F along the inverse images of the codimension one faces of F is equivariantly biholomorphic to the total space of $M_F^0 \times_{\mathbb{T}^c/\mathbb{T}^c_F} \mathcal{V}_F \to S_F$ for some smooth toric complex manifold $\mathcal{V}_F$.*

(iv) *If F is a k-dimensional face, with the $(k-1)$-dimensional face F' in its boundary, then S_F is a holomorphic $\mathbb{C}P^d$-bundle over $S_{F'}$ with $d = m_F - m_{F'} \geq 0$.*

Furthermore, if Q_F denotes the fibrewise Hopf fibration over the $\mathbb{C}P^d$-bundle $S_F \to S_{F'}$, then $P_F \to S_F$ is the pullback of $P_{F'} \to S_{F'}$ along the S^{2d+1}-bundle map $Q_F \to S_{F'}$ composed with the S^1-bundle map $Q_F \to S_F$.

Proof.

(i) For all $x \in M$, any complex orbit has a smooth closure along $\mathbb{T}^c \cdot x$ by Lemma 6. Hence, the closures of the complex orbits are smoothly embedded, and become toric Kähler manifolds under the induced metric. We have already remarked that μ maps any such orbit closure to a face F of Δ, and clearly, μ, viewed as a map to the affine span of F (with a choice of origin), is a momentum map for the induced toric action.

(ii) For convenience, we prove this result for $F = \Delta$: the general result follows by replacing M with M_F and $\mathbb{T}^c$ by $\mathbb{T}^c/\mathbb{T}^c_F$.

Since $\mathbb{T}^c$ acts freely on M^0, it defines a holomorphic fibration over S_Δ. To verify that the fibration is locally trivial, observe that a neighbourhood of a $\mathbb{T}^c$ orbit in M^0 is equivariantly biholomorphic to a neighbourhood of the zero section in $\mathbb{T}^c \times V_0 \to \mathbb{T}^c$. The latter, being $\mathbb{T}^c$-invariant, is of the form $\mathbb{T}^c \times U_0$, and the projection to U_0 gives the required local trivialization. Since $\mathbb{T}^c$ acts simply transitively on the fibers, M^0 is a principal $\mathbb{T}^c$-bundle over S_Δ.

(iii) We again prove the result when the face is the whole polytope Δ.

We first consider the blow-up $\hat{M}$ of M along all M_F with F codimension one in Δ. (Of course, the blow-up is trivial if M_F already has complex codimension one in M). Thus, $\hat{M}$ is the complex manifold obtained from M by replacing each M_F by its projectivized normal bundle $\hat{M}_F$; these become divisors (i.e., of complex codimension one) in $\hat{M}$, and the $\mathbb{T}^c$ action lifts naturally to $\hat{M}$. Lemma 6 shows that the generic $\mathbb{T}^c$ orbits for the lifted action have disjoint smooth closures in $\hat{M}$, and this gives a holomorphic fibration of $\hat{M}$ whose fibres are all toric Kähler manifolds with Delzant polytope Δ. In particular (forgetting the symplectic structure), they are all isomorphic toric complex manifolds [**17**, **28**].

Let $\mathcal{V}_\Delta$ be a toric complex manifold in this isomorphism class, and choose a basepoint on the generic orbit $\mathcal{V}^0_\Delta$ to identify it with $\mathbb{T}^c$. Then, there is an equivariant biholomorphism $M^0 \times_{\mathbb{T}^c} \mathcal{V}^0_\Delta \to M^0 = \hat{M}^0$ (here, $\hat{M}^0$ stands for the subset of points of $\hat{M}$ with generic $\mathbb{T}^c$ orbits; it is the same as M^0 because the blow-up is the identity on the complement of the exceptional divisor). Since $\mathcal{V}_\Delta$ has the same isotropy representations as the fibres of $\hat{M}$, this extends to an equivariant biholomorphism $M^0 \times_{\mathbb{T}^c} \mathcal{V}_\Delta \to \hat{M}$ (indeed, the holomorphic slices of Lemma 6 provide the extension).

(iv) Consider, as in (iii), the blow-up $\hat{M}$ of M along its codimension one faces. This is equivariantly biholomorphic to $M^0 \times_{\mathbb{T}^c} \mathcal{V}_\Delta$ and for *any* face F, the inverse image $\hat{M}_F$ of M_F in $\hat{M}$ is $M^0 \times_{\mathbb{T}^c} \mathcal{V}_F$ (where only $\mathbb{T}^c/\mathbb{T}^c_F$ acts effectively on $\mathcal{V}_F$, which is the inverse image of F in $\mathcal{V}_\Delta$).

Now, $\mathcal{V}^0_F$ is equivariantly biholomorphic to a $\mathbb{T}^c_{F'}/\mathbb{T}^c_F$ bundle over $\mathcal{V}^0_{F'}$, namely the punctured normal bundle of $\mathcal{V}^0_{F'}$ in $\mathcal{V}_F$, so it follows that the same is true for $\hat{M}^0_F$: it is equivariantly biholomorphic to the punctured normal bundle of $\hat{M}^0_{F'}$ in $\hat{M}_F$. Passing to the blow-down, we deduce that M^0_F is equivariantly biholomorphic to the punctured normal bundle of $M^0_{F'}$ in M_F, which is a $\mathbb{T}^c/\mathbb{T}^c_F$-equivariant bundle with $\mathbb{T}^c_{F'}/\mathbb{T}^c_F$ acting by scalar multiplication on the fibres.

The quotient by $\mathbb{T}^c/\mathbb{T}^c_F$ identifies S_F biholomorphically with a bundle over S_F. To describe this bundle, we first divide the punctured normal bundle of $M^0_{F'}$ by $\mathbb{T}^c_{F'}/\mathbb{T}^c_F$ to obtain the projectivized normal bundle as a $\mathbb{T}^c/\mathbb{T}^c_{F'}$-equivariant $\mathbb{C}P^d$ bundle over $M^0_{F'}$ with trivial action on the fibres. Now, the quotient by $\mathbb{T}^c/\mathbb{T}^c_{F'}$ shows that $S_F \to S_{F'}$ is a holomorphic $\mathbb{C}P^d$-bundle.

The unit normal bundle of $M^0_{F'}$ is the sphere bundle induced by the Hopf fibtration over the projectivized normal bundle and the result follows. q.e.d.

One can easily check that the 'tame' S^1 actions on $\mathbb{C}P^2$ given in Example 1 are rigid with respect to the Fubini–Study metric, and the complex orbits do indeed have smooth closures. On the other hand, we again see that the 'wild' actions cannot be rigid with respect to any compatible Kähler metric.

2.4. Kähler geometry of rigid hamiltonian torus actions. Given a Kähler $2m$-manifold M with a rigid hamiltonian action of an ℓ-torus $\mathbb{T}$, we have obtained a description of the equivariant biholomorphism type of M, stratified by the inverse images of the faces of the momentum polytope Δ: M^0 is a principal $\mathbb{T}^c$-bundle over a complex manifold S_Δ of dimension $2m_\Delta$, with $m_\Delta = m-\ell$, and there is a toric complex manifold $\mathcal{V}_\Delta$ such that the blow up of M along the codimension one faces of Δ is biholomorphic to $M^0 \times_{\mathbb{T}^c} \mathcal{V}_\Delta \to S_\Delta$; *mutatis mutandis*, the inverse image $M_F = \mu^{-1}(F)$ of a face of Δ has the same structure; further, if $F_1, \dots F_n$ denote the codimension one faces of Δ, then S_{F_j} has dimension $2m_{F_j} \leq 2m_\Delta$ and S_Δ is a $\mathbb{C}P^{d_j}$-bundle over S_{F_j} with $d_j = m_\Delta - m_{F_j}$, and we say a *blow-down occurs* over F_j if $d_j > 0$. (We remark that if F' is a codimension one face of F, it must be $F \cap F_j$ for some codimension one face F_j of Δ. We then have $m_F - m_{F'} = d_j = m_\Delta - m_{F_j}$.)

It remains to descibe the Kähler structure of M in terms of this equivariant biholomorphism type. To do that, we first recall some equivalent formulations of the rigidity condition established (locally) in [4].

Suppose, generally, that M is a Kähler manifold endowed with an isometric hamiltonian action of an ℓ-torus $\mathbb{T}$ with momentum map μ. For a contractible open subset U of the regular values of μ, the gradient flow of μ identifies $\mu^{-1}(U)$ with $\mu^{-1}(v) \times U$ for any v in U, and hence $\mu^{-1}(U)/\mathbb{T} \cong S \times U$ for a complex manifold S, with a family ω_h of compatible symplectic forms on the fibres of $S \times U \to U$. We can, therefore, define the derivative $d_\mu \omega_h$ with respect to μ, and this will be a 2-form on S with values in $\mathfrak{t}$. Now, $\mu^{-1}(U)$ is a principal $\mathbb{T}$-bundle with connection over $S \times U$, so it has a curvature form $\boldsymbol{\Omega}$, which is also a closed 2-form with values in $\mathfrak{t}$. If $d_\mu \omega_h = \boldsymbol{\Omega}$ on $S \times U$, we say that the *rigid Duistermaat–Heckman property* holds (so-called because it holds in cohomology by work of Duistermaat and Heckman). We then have the following global version of [4, Proposition 8].

Lemma 7. *For an isometric hamiltonian $\mathbb{T}$-action, the following are equivalent.*

(i) *The action is rigid.*
(ii) *The $\mathbb{T}^c$-orbits are totally geodesic.*
(iii) *The orthogonal distribution to the $\mathbb{T}^c$-orbits is $\mathbb{T}^c$-invariant.*
(iv) *The rigid Duistermaat–Heckman property holds.*

Proof. This is essentially the same as [4, Proposition 8]. Let X denote a vector field which is orthogonal to a $\mathbb{T}^c$-orbit. The rigidity condition is equivalent to the statement that $\partial_X(g(K_r, K_s)) = -2g(\nabla_{K_r} K_s, X)$ vanishes along the given orbit for all such vector fields X. Since J is parallel and K_s is holomorphic, this is equivalent to the fact that the

$\mathbb{T}^c$ orbit is totally geodesic. It is easy to compute that this condition is equivalent to the fact that $\mathcal{L}_{K_r}X$ and $\mathcal{L}_{JK_r}X$ are orthogonal to the given $\mathbb{T}^c$ orbit for all X, $r = 1, \dots \ell$, i.e., the orthogonal distribution is $\mathbb{T}^c$-invariant.

(iv) is equivalent to the local rigidity of the action on M^0 by the Pedersen–Poon construction (see [**29**, **4**]); this implies rigidity on M by continuity. q.e.d.

We next show that a compact Kähler manifold with a rigid hamiltonian action of a torus gives rise in a natural way to the following data.

Definition 4. Let $\mathcal{V}$ be a compact toric Kähler manifold under an ℓ-torus $\mathbb{T}$ with Delzant polytope Δ. Then, *rigid hamiltonian data* for $\mathcal{V}$ consists of a quadruple $(\mathcal{V}_F, S_F, P_F, \omega_F)$ for each face F of Δ, where:

- (i) $\mathcal{V}_F$ is the inverse image of F in $\mathcal{V}$, which is a compact toric Kähler manifold under $\mathbb{T}/\mathbb{T}_F$, where $\mathbb{T}_F$ is the isotropy subgroup of $\mathbb{T}$ associated to F;
- (ii) S_F is a compact complex manifold which is a holomorphic projective space bundle over $S_{F'}$ for any codimension one face F' of F;
- (iii) $\pi\colon P_F \to S_F$ is a principal $\mathbb{T}/\mathbb{T}_F$-bundle with connection $\boldsymbol{\theta}_F\colon TP_F \to \mathfrak{t}/\mathfrak{t}_F$, whose curvature $\boldsymbol{\Omega}_F \in C^\infty(S_F, \Lambda^{1,1}S_F \otimes \mathfrak{t}/\mathfrak{t}_F)$ pulls back to the fibres of $S_F \to S_{F'}$ to give the Fubini–Study metric in $2\pi c_1(\mathcal{O}(1))$ tensored with the (primitive inward) normal to the codimension one face F';
- (iv) ω_F is a section of (the pullback of) $\Lambda^{1,1}S_F$ over $S_F \times F$, which
  - is positive on $S_F \times F^0$,
  - satisfies $d_\mu \omega_F = \boldsymbol{\Omega}_F$ on $S_F \times \{v\}$ for all $v \in F^0$,
  - and whose restriction to $S_F \times F'$, for any codimension one face F' of F, is the pullback of $\omega_{F'}$ along the map $S_F \times F' \to S'_F \times F'$.

Proposition 6. *Let M be a compact connected Kähler $2m$-manifold with a rigid hamiltonian action of an ℓ-torus $\mathbb{T}$ and momentum map $\mu\colon M \to \Delta$. Then, there are rigid hamiltonian data $(\mathcal{V}_F, S_F, P_F, \omega_F)$ (for the faces F of Δ) associated to a toric Kähler manifold $\mathcal{V}$ with Delzant polytope Δ such that*:

- *the pullback of the Kähler metric on $M_F = \mu^{-1}(F)$ to the fibres of the blow-up $\hat{M}_F \cong P_F \times_{\mathbb{T}} \mathcal{V}_F$ (see Proposition 5) is induced by the Kähler metric on $\mathcal{V}_F$;*
- *S_F is the Kähler quotient of M_F by $\mathbb{T}/\mathbb{T}_F$ and the Kähler quotient metric at momentum level $v \in F^0$ is induced by ω_F on $S_F \times \{v\}$;*

- *the orthogonal distribution to the generic* $\mathbb{T}^c/\mathbb{T}^c_F$ *orbits in* M_F *is the joint kernel of* $\boldsymbol{\theta}_F$ *and* $d\mu$.

In particular, on $M^0 \cong P_\Delta \times_{\mathbb{T}} \mathcal{V}^0_\Delta$, *the Kähler structure is given by*

$$\begin{aligned} g &= h_0 + \langle \mu, \boldsymbol{h} \rangle + \langle d\mu, \mathbf{G}, d\mu \rangle + \langle \boldsymbol{\theta}, \mathbf{G}^{-1}, \boldsymbol{\theta} \rangle, \\ \omega &= \Omega_0 + d\langle \mu, \boldsymbol{\theta} \rangle = \Omega_0 + \langle \mu, \boldsymbol{\Omega} \rangle + \langle d\mu \wedge \boldsymbol{\theta} \rangle, \end{aligned} \tag{24}$$

where $\omega_\Delta = \Omega_0 + \langle \mu, \boldsymbol{\Omega} \rangle$, $h_0 + \langle \mu, \boldsymbol{h} \rangle$ *is the corresponding family of hermitian metrics,* $\boldsymbol{\theta} = \boldsymbol{\theta}_\Delta$, $\boldsymbol{\Omega} = \boldsymbol{\Omega}_\Delta$, *the toric Kähler metric on* $\mathcal{V}^0_\Delta$ *is given by* (8), *for* $\mathbf{G} \colon \Delta^0 \to S^2\mathfrak{t}$, *and* (*as before*) *angled brackets denote pointwise contractions.*

Proof. It suffices to prove the result for the whole polytope Δ. We know by Propositions 4 and 5 that the blow-up $\hat{M}$ is equivariantly biholomorphic to $M^0 \times_{\mathbb{T}^c} \mathcal{V}_\Delta \cong P_\Delta \times_{\mathbb{T}} \mathcal{V}_\Delta$ for a toric complex manifold $\mathcal{V}_\Delta$, and the fibres of $P_\Delta \times_{\mathbb{T}} \mathcal{V}_\Delta \to S_\Delta$ map biholomorphically onto the complex orbit closures in M. The Kähler metric of M induces a Kähler structure on each complex orbit closure, which depends only on the momentum map μ. Since μ is $\mathbb{T}$-invariant, there is a toric Kähler structure on $\mathcal{V}_\Delta$, with Delzant polytope Δ, such that the fibres of $P_\Delta \times_{\mathbb{T}} \mathcal{V}_\Delta$, with the metric induced from $\mathcal{V}_\Delta$, map *isometrically* onto the complex orbit closures in M.

The Kähler metric on M^0 induces a principal $\mathbb{T}$-connection on $M^0 \to B_\Delta = S_\Delta \times \Delta^0$ (the orthogonal distribution to the fibres), and by Lemma 7, this is the pullback of a principal $\mathbb{T}$-connection $\boldsymbol{\theta}$ on $\pi \colon P_\Delta \to S_\Delta$. The lemma also shows that the family ω_Δ of Kähler forms induced on S_Δ depends affinely on $\mu \in \Delta^0$ and $\pi^* d_\mu \omega_\Delta = d\boldsymbol{\theta}$, so the linear part is the curvature $\boldsymbol{\Omega}$ of the connection $\boldsymbol{\theta}$ for all $\mu \in \Delta^0$; ω_Δ is therefore, smoothly defined for all μ.

The Kähler form on M pulls back to the blow-up $\hat{M}$ to give a 2-form which degenerates on the exceptional divisor. Using the description of this divisor given in Proposition 5 and the smooth dependence of the Kähler form on μ, it follows that the Kähler form ω_Δ approaches to the pullback of ω_F along $S_\Delta \to S_F$ as $\mu \rightsquigarrow F^0 \subset F$, for a codimension one face F of Δ. We then deduce that the pullback of $d_\mu \omega_\Delta$ to a fibre of $S_\Delta \to S_F$ takes values, for $\mu \in F^0$, in the annihilator $\mathfrak{t}_F$ of $T_\mu F$, i.e., is of the form $\Omega \otimes u_F$, where u_F is the primitive inward normal to F, and Ω is a $(1,1)$-form on S_F. Since the normal bundle to the divisor $\hat{M}_F$ in $\hat{M}$ must have degree -1 on each fibre of $S_\Delta \to S_F$ and Ω is the curvature of a connection on this degree -1 line bundle, we must have $[-\Omega/2\pi] \in c_1(\mathcal{O}(-1))$.

To show that Ω is the Fubini–Study metric in its Kähler class, we take $v \in F^0$, the interior of a codimension one face of Δ, and construct a symplectic slice, as in Lemma 1, to a point x in $\mu^{-1}(v)$ projecting to the given fibre of $S_\Delta \to S_F$. Thus, a neighbourhood of $\mathbb{T} \cdot x$ in M is equivariantly symplectomorphic to a neighbourhood U of the zero section $0_N \cong \mathbb{T}\cdot x$ of the normal bundle $N = \mathbb{T} \times_{\mathbb{T}_F} (\mathfrak{t}^0_x \oplus V_x) \to \mathbb{T}\cdot x$, with the obvious $\mathbb{T}$-action, and canonical symplectic form ω_0. Pulling back the Kähler structure of M, and restricting to the fibre V_x at x, gives a Kähler metric on a neighbourhood of the origin in V_x with a rigid hamiltonian circle action of $\mathbb{T}_F$ and constant symplectic form. Observe that the Kähler quotient $P(V_x)$ of $V_x \setminus \{0\}$ by $\mathbb{T}_F$ is a fibre of $S_\Delta \to S_F$.

Let $z = r^2/2$ be half the distance squared to the origin in V_x—which is the momentum map of the $\mathbb{T}_F$ action contracted with $u_F \in \mathfrak{t}_F$. Then, the Kähler structure on V_x may be written

$$g = zh + \frac{dz^2}{H(z)} + H(z)\theta^2, \qquad\qquad \omega = z\Omega + dz \wedge \theta,$$

for some function $H(z)$, where $d\theta = \Omega$ and Ω is as before, and (h, Ω) is independent of z (the Kähler quotient depends affinely on z and degenerates at $z = 0$). The vector field dual to θ generates the S^1 action, and this preserves z, so it is tangent to the level surfaces of z (which are spheres), and generates a (topological) Hopf fibration of them. Now, z is a function of the geodesic distance to $z = 0$ (the geodesic distance is obtained by integrating $1/\sqrt{H(z)}$). For smooth compactification at $z = 0$, the metric on geodesic spheres must have constant curvature when $z \to 0$. Hence, (h, Ω) must tend to the Fubini–Study metric, so that θ tends to the standard connection as $z \to 0$. Since (h, Ω) is independent of z, it is the Fubini–Study metric.

The explicit form of the metric on M^0 easily follows from Lemma 7 and Proposition 6. Note that a similar formula can be established on $M^0_F = \mu^{-1}(F^0)$ for any face F, but an origin needs to be chosen in F so that $\mu|_{M_F}$ can be considered to take values in $(\mathfrak{t}/\mathfrak{t}_F)^* = \mathfrak{t}^0_F$. q.e.d.

Remark 7. In the absense of blow-downs, $\Omega_0 + \langle \mu, \boldsymbol{\Omega} \rangle$ is positive for all μ in Δ, and for all F, $S_F = S_\Delta$, $P_F = P_\Delta/\mathbb{T}_F$, with the induced Kähler metrics and connections; then the data of this proposition clearly *do* define (uniquely) a Kähler metric on M with a rigid hamiltonian action of $\mathbb{T}$. However, the existence of the connection $\boldsymbol{\theta}$ implies *integrality conditions* on the curvature form $\boldsymbol{\Omega}$, and the compactification of the toric Kähler metric on $\mathcal{V}_\Delta$ implies *boundary conditions* on $\mathbf{G}$.

When there are blow-downs, it is difficult to describe the data needed to construct the Kähler metric on M, because of the family of fibrations $S_{F'} \to S_F$: the Kähler quotient metrics are related by pullback, and the fibrations and pullbacks must commute. Rather than attempt this in full generality, we restrict attention to a special case, which is all we shall need for the application to hamiltonian 2-forms.

2.5. Semisimple actions and the generalized Calabi construction.

Definition 5. A hamiltonian torus action is *semisimple* if for any regular value v of the momentum map μ, the derivative with respect to μ of the family ω_h of Kähler forms on the complex quotient S is parallel and diagonalizable with respect to ω_h at $\mu = v$. (Observe that S is well defined, as a complex orbifold at least, for μ in the connected component U_v of v in the regular values, since the gradient flow of μ is transitive on U_v.)

Integrating this condition, we deduce that on any connected component of the regular values of μ, the corresponding Kähler quotient metrics ω_h are simultaneously diagonal with the same Levi–Civita connections. Thus, for a semisimple rigid hamiltonian torus action, there is a symplectic $(1,1)$-form Ω_S on S_Δ such that the family of Kähler forms induced by $\mu \in \Delta^0$ are parallel and simultaneously diagonalizable with respect to Ω_S.

Definition 6. By *generalized Calabi data* of dimension m, rank ℓ, we mean:

(i) a $2(m-\ell)$-dimensional product S of $N \geq 0$ Kähler manifolds $(S_a, \pm g_a, \pm\omega_a)$ of dimension $2m_a > 0$ (if $\ell = m$, $N = 0$);
(ii) a compact toric 2ℓ-dimensional Kähler manifold $\mathcal{V}$ with Delzant polytope $\Delta \subset \mathfrak{t}^*$ and momentum map $\mu_{\mathcal{V}} \colon \mathcal{V} \to \Delta$;
(iii) a principal $\mathbb{T}$-bundle $P \to S$, with a principal connection of curvature $\boldsymbol{\Omega} \in C^\infty(S, \Lambda^{1,1}S \otimes \mathfrak{t})$, where $\mathbb{T}$ is the ℓ-torus acting on $\mathcal{V}$;
(iv) a $(1,1)$-form Ω_0 on S such that $\Omega_0 + \langle v, \boldsymbol{\Omega}\rangle$ is positive for $v \in \Delta^0$;
(v) constants $c_{a0} \in \mathbb{R}$ and $\boldsymbol{c}_a \in \mathfrak{t}$ such that $\Omega_0 = \sum_{a=1}^N c_{a0}\omega_a$ and $\boldsymbol{\Omega} = \sum_{a=1}^N \boldsymbol{c}_a\omega_a$;
(vi) a subset $\mathcal{C} \subset \{1, \dots N\}$ such that for $a \notin \mathcal{C}$, $\{v \in \Delta : c_{a0} + \langle v, \boldsymbol{c}_a\rangle = 0\}$ is empty, while for $a \in \mathcal{C}$ they are distinct codimension one faces of Δ with (primitive) inward normals $u_a \in \mathfrak{t}$, and $S_a = \mathbb{C}P^{d_a}$ with $d_a > 0$, $\pm g_a$ is a Fubini–Study metric and $\boldsymbol{c}_a \otimes \omega_a/2\pi \in u_a \otimes c_1(\mathcal{O}(-1))$.

Given these data, we define the manifold $\hat{M} = P \times_{\mathbb{T}} \mathcal{V} = M^0 \times_{\mathbb{T}^c} \mathcal{V} \to S$, where $M^0 = P \times_{\mathbb{T}} \mu_{\mathcal{V}}^{-1}(\Delta^0)$. Since the curvature 2-form of P has type $(1,1)$, M^0 becomes a holomorphic principal $\mathbb{T}^c$-bundle with connection and $\hat{M}$ is a complex manifold. The toric Kähler structure on $\mathcal{V}$ endows $\hat{M}$ with a fibrewise metric and 'momentum map' $\hat{\mu}\colon \hat{M} \to \Delta$: indeed, being $\mathbb{T}$ invariant, the momentum map $\mu_{\mathcal{V}}$ of $\mathcal{V}$ can be defined on $\hat{M} = P \times_{\mathbb{T}} \mathcal{V}$.

According to (vi), the set $\mathcal{C}$ corresponds bijectively to a subset $\mathcal{B}$ of the codimension one faces of Δ, and for $F \in \mathcal{B}$ corresponding to $a \in \mathcal{C}$, the connection on $\hat{M}_F := \hat{\mu}^{-1}(F)$ is flat over each fibre of $S \to \prod_{b \neq a} S_b$. This gives a $\mathbb{C}P^{d_a}$ fibration of $\hat{M}_F$ such that the normal bundle to $\hat{M}_F$ in $\hat{M}$ is a line bundle which has degree -1 on each $\mathbb{C}P^{d_a}$ fibre. Since a tubular neighbourhood of $\hat{M}_F$ in $\hat{M}$ is diffeomorphic to a neighbourhood of the zero section in the normal bundle, it follows that the topological space M, obtained by contracting $\hat{M}$ along the $\mathbb{C}P^{d_a}$ fibration of each such $\hat{M}_F$, is a smooth manifold and M^0 is an open dense submanifold.

If the Kähler structure given by (24) (which pulls back to the fibrewise metric on the fibres of $\hat{M} \to S$) extends smoothly to M, then we say that this Kähler manifold (M, g, J, ω) is given by *the generalized Calabi construction* (with blow-downs).

We shall see that the contraction $\hat{M} \to M$ realises $\hat{M}$ (with the complex structure described above) as a blow-up of M. We therefore refer to this contraction as a blow-down. Our main result shows that all generalized Calabi data give rise to a generalized Calabi construction, and that this classifies compact Kähler manifolds with a semisimple rigid hamiltonian torus actions up to a covering.

Theorem 2. *Let M be a compact connected Kähler $2m$-manifold with a semisimple rigid hamiltonian action of an ℓ-torus $\mathbb{T}$ and momentum map $\mu\colon M \to \Delta \subset \mathfrak{t}^*$. Then, some cover of M is given by the generalized Calabi construction.*

Conversely, for any generalized Calabi data (i)–(vi)*, the generalized Calabi construction produces a smooth Kähler manifold with a semisimple rigid hamiltonian action of an ℓ-torus.*

Proof. We construct the generalized Calabi data from Proposition 6, imposing the condition that the action is semisimple. As remarked in [**4**, Section 3.3], the condition that Ω_0 and the components of $\boldsymbol{\Omega}$ are simultaneously diagonalizable and parallel (with respect to some Kähler metric Ω_S) implies that the (distinct) eigendistributions $\mathcal{H}_a$ $(a = 1, \ldots N)$ are parallel. By the deRham decomposition theorem, some cover of S_Δ (for instance, the universal cover), is a Kähler product $(S, \Omega_S) = \prod_{a=1}^{N} (S_a, \omega_a)$ (note that S may not be compact). The generalized Calabi data (i)–(v) are then obtained from Proposition 6 by setting $\mathcal{V} = \mathcal{V}_\Delta$, pulling back P_Δ, θ_Δ, Ω_0 and $\boldsymbol{\Omega}$ to give a principal bundle P with connection over S, and defining the constants c_{a0} and $\boldsymbol{c}_a$ by (v).

Let $\mathcal{B}$ be the set of codimension one faces F of Δ such that a blow-down occurs (i.e., M_F is not a divisor); then, $\hat{M}$ is the blow-up of M along M_F with $F \in \mathcal{B}$. The pullback of the metric to $\hat{M}$ degenerates on the fibres of a $\mathbb{C}P^d$-bundle $S_\Delta \to S_F$ for some $d > 0$. Now, $\mathbb{C}P^d$ is simply connected, so this is covered by a $\mathbb{C}P^d$-bundle with total space S, whose base is a cover of S_F. Hence, there must be at least one a such that $c_{a0} + \langle v, \boldsymbol{c}_a\rangle = 0$ for $v \in F^0$; since $\mathbb{C}P^d$ does not admit a Kähler product metric, this a is unique, and S_F is covered by $\prod_{b\neq a} S_b$, while $S_a = \mathbb{C}P^{d_a}$ with $d_a = d$. On the other hand, $c_{a0} + \langle v, \boldsymbol{c}_a\rangle$ is an affine function of v, so it can vanish on at most one codimension one face of the Delzant polytope Δ. Thus, $\mathcal{B}$ corresponds bijectively to a subset $\mathcal{C} \subset \{1, \dots N\}$. Now, note that for any face F, with $v \in F^0$, the metric induced on S_F is non-degenerate, so $c_{a0} + \langle v, \boldsymbol{c}_a\rangle$ does not vanish on Δ for $a \notin \mathcal{C}$. This establishes (vi).

The pullback of $\hat{M}$ to S is a cover of $\hat{M}$, and by construction, this descends to M. Hence, up to a cover, M is obtained from the generalized Calabi construction.

Conversely, given the data of Definition 6, we will prove that there exists a smooth compact Kähler manifold (M, g, J, ω) with a semisimple rigid hamiltonian action of the ℓ-torus $\mathbb{T}$ given by the generalized Calabi construction. The main difficulty is to deal with the blow-downs.

Let us suppose there are $k \geq 0$ blow-downs: then, after reordering, we may assume $\mathcal{C} = \{1, \dots k\}$ and that $S = \mathbb{C}P^{d_1} \times \cdots \times \mathbb{C}P^{d_k} \times S''$ for some Kähler product S''. The conditions (iii) and (v) of Definition 6 imply that $\boldsymbol{\Omega}'' := \sum_{a=k+1}^{N} \boldsymbol{c}_a \omega_a$ is the curvature of a principal $\mathbb{T}$-bundle $P'' \to S''$. We are going to let M be of the form $P'' \times_{\mathbb{T}} M'$, where M' is a $2(\ell + d_1 + \cdots + d_k)$ dimensional Kähler manifold with a rigid semisimple isometric hamiltonian action of $\mathbb{T}$, obtained from the generalized Calabi construction with respect to the following data:

(i) $S' = \mathbb{C}P^{d_1} \times \cdots \times \mathbb{C}P^{d_k}$;
(ii) $(\mathcal{V}, \omega, \mu, \Delta)$, with the given compatible toric Kähler metric;
(iii) a principal $\mathbb{T}$-bundle P' with curvature form $\boldsymbol{\Omega}' = \sum_{a=1}^{k} \boldsymbol{c}_a \omega_a$;
(iv) $\Omega'_0 = \sum_{a=1}^{k} c_{a0} \omega_a$;
(v) the given constants c_{a0} and $\boldsymbol{c}_a$ for $a = 1, \dots k$;
(vi) $\mathcal{C} = \{1, \dots k\}$.

Since the data for M are generalized Calabi data, so are these data for M'. If M' can be constructed with these data, it follows from Proposition 6 and Remark 7 that M is equipped with a Kähler metric and a semisimple rigid action of $\mathbb{T}$; using the first part of the theorem, we also see that M is given by the generalized Calabi construction associated to the initial data.

Thus, it remains only to establish the generalized Calabi construction for M'. However, such an M' is obtained as a restricted toric Kähler manifold, the construction of which we discussed already in Section 1.6. q.e.d.

Just as toric complex manifolds may be described in terms of linear charts, i.e., in terms of a family of vector spaces, each with a decomposition into one dimensional subspaces, glued together by Laurent monomials, so bundles of toric complex manifolds (arising in the generalized Calabi construction without blow-downs) may be decribed (by the holomorphic slice theorem) in terms of families of vector bundles, each a direct sum of line bundles, glued together in a similar way. The simplest case is the case of projective bundles $P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \to S$, which are obtained by gluing together the vector bundles $\mathcal{L}_j^{-1} \otimes \big(\bigoplus_{k\neq j} \mathcal{L}_k\big)$ for $j = 0, \dots \ell$. This is the only case we shall need in the sequel.

3. Orthotoric geometry

We now return to our primary aim: the classification of compact Kähler manifolds endowed with a hamiltonian 2-form. In this section, we treat the case when the order of the hamiltonian 2-form is maximal, and therefore, the corresponding Kähler manifolds are toric. Motivated by the orthogonality of the gradients of the roots of the momentum polynomial, see Theorem 1, we define orthotoric Kähler manifolds and orbifolds, and classify the compact ones.

3.1. The polytope of an orthotoric orbifold.

Definition 7. An *orthotoric* Kähler manifold (or orbifold) M is a toric Kähler $2m$-manifold (or orbifold) with a momentum map $\boldsymbol{\sigma} = (\sigma_1, \dots \sigma_m)$ and (rational) Delzant polytope $\Delta = \boldsymbol{\sigma}(M)$, such that on the dense open set $M^0 = \boldsymbol{\sigma}^{-1}(\Delta^0)$ of regular points of $\boldsymbol{\sigma}$, the roots $\xi_1, \dots \xi_m$ of the momentum polynomial $\sum_{r=0}^{m} (-1)^r \sigma_r t^{m-r}$ $(\sigma_0 = 1)$ are smoothly defined, pairwise distinct and functionally independent, and the Kähler metric has the explicit form

$$(25) \qquad g = \sum_{j=1}^{m} \frac{\Delta_j}{\Theta_j(\xi_j)}\, d\xi_j^2 + \sum_{j=1}^{m} \frac{\Theta_j(\xi_j)}{\Delta_j} \left(\sum_{r=1}^{m} \sigma_{r-1}(\hat{\xi}_j)\, dt_r \right)^2$$

$$= \sum_{r,s,j=1}^{m} \left(\frac{(-1)^{r+s} \Delta_j \xi_j^{2m-r-s}}{\Theta_j(\xi_j)} d\sigma_r\, d\sigma_s \right.$$

$$+\frac{\Theta_j(\xi_j)\sigma_{r-1}(\hat{\xi}_j)\sigma_{s-1}(\hat{\xi}_j)}{\Delta_j}dt_r\,dt_s\Bigg)$$

$$\omega=\sum_{j=1}^m d\xi_j\wedge\left(\sum_{r=1}^m\sigma_{r-1}(\hat{\xi}_j)\,dt_r\right)=\sum_{r=1}^m d\sigma_r\wedge dt_r,$$

for functions $\Theta_1,\dots\Theta_m$ of one variable. Here, $\Delta_j=\prod_{k\neq j}(\xi_j-\xi_k)$.

Clearly, the gradients of $\xi_1,\dots\xi_m$ are orthogonal with respect to g. Conversely, it was shown in [4, Section 3.4] that this property characterizes orthotoric Kähler manifolds (and the result applies equally to orbifolds).

Note that the basis $K_1,\dots K_m$ of the Lie algebra of the torus identifies it with $\mathbb{R}^m$, and we view the invariant 1-forms $dt_1,\dots dt_m$ as the dual basis of $\mathbb{R}^{m*}$.

Proposition 7. *Let M be a compact orthotoric Kähler $2m$-manifold or orbifold with momentum map $\boldsymbol{\sigma}=(\sigma_1,\dots\sigma_m)$ and rational Delzant polytope Δ.*

(i) *Δ is the (one to one) image under the elementary symmetric functions of a domain of the form*

$$D=\{(\xi_1,\dots\xi_m)\in\mathbb{R}^m:\alpha_j\le\xi_j\le\beta_j\}$$

where

$$\alpha_1<\beta_1\le\alpha_2<\beta_2\le\cdots<\beta_{m-1}\le\alpha_m<\beta_m.$$

Thus, setting $\sigma_0=1$, $\Delta=\{(\sigma_1,\dots\sigma_m):(-1)^{m-j}\sum\limits_{r=0}^m(-1)^r\sigma_r\alpha_j^{m-r}$ ≤ 0 and $(-1)^{m-j}\sum_{r=0}^m(-1)^r\sigma_r\beta_j^{m-r}\ge 0$ for $j=1,\dots m\}$. This is a simplex if and only if $\alpha_{j+1}=\beta_j$ for $j=1,\dots m-1$.

(ii) *If M is non-singular (i.e., a manifold), then Δ is a simplex.*

Proof. (i) $\sigma_1,\dots\sigma_m$ are the elementary symmetric functions of the roots $\xi_1,\dots\xi_m$ of the momentum polynomial, and we want to find the domain D in the ξ_j coordinates corresponding to Δ. We first remark that this domain must be bounded. Also, the functions $\Theta_j(\xi_j)$ must be non-zero on the interior D^0 of D in order that the metric be finite and non-degenerate.

Now, consider in particular the metric on the torus given by

$$g(K_r,K_s)=\sum_{j=1}^m\frac{\Theta_j(\xi_j)\,\sigma_{r-1}(\hat{\xi}_j)\sigma_{s-1}(\hat{\xi}_j)}{\Delta_j}.$$

The determinant of this matrix is (up to a sign) $\prod_{j=1}^m\Theta_j(\xi_j)$. As we approach a special orbit of the m-torus action, i.e., as $\boldsymbol{\sigma}$ approaches the boundary of Δ, this must tend to zero, i.e., at least one of the functions Θ_j of one variable must tend to zero. Since these functions are non-vanishing on D^0, it follows that D^0 is a domain of the form $\prod_{j=1}^m(\alpha_j, \beta_j)$, where Θ_j is non-vanishing on the interval (α_j, β_j) and tends to zero at the endpoints. Now, $\xi_1, \dots \xi_m$ must be pairwise distinct on D^0, so we may assume (after reordering) that $\xi_1 < \cdots < \xi_m$ on D^0. Hence,

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots < \beta_{m-1} \leq \alpha_m < \beta_m.$$

Noting that the elementary symmetric functions are affine in each variable, we readily check that this domain does indeed map bijectively to a convex polytope. Indeed, any $(\xi_1, \dots \xi_m)$ in D satisfy

$$(-1)^{m-j}\prod_{k=1}^m(\alpha_j - \xi_k) \leq 0, \qquad (-1)^{m-j}\prod_{k=1}^m(\beta_j - \xi_k) \geq 0 \tag{26}$$

for all $j = 1, \dots m$; equality is attained in one of these expressions on any face, and in any of these expressions on some face. Expanding in terms of the elementary symmetric functions of $(\xi_1, \dots \xi_m)$ gives the explicit description of Δ.

A compact convex polytope in $\mathbb{R}^{m*}$ is a simplex if and only if it has $m+1$ vertices. The vertices of D are the points where $\xi_j \in \{\alpha_j, \beta_j\}$ for all $j = 1, \dots m$. Now, observe that a vertex of D maps to a vertex of Δ if and only if it does not lie on one of the diagonals $\xi_j = \xi_k$ for $j \neq k$.

(ii) We shall show that $\alpha_{j+1} = \beta_j$ for $j = 1, \dots m-1$. Suppose for contradiction that this does not hold for some $j \in \{1, \dots m-1\}$ and consider the four vertices

$$\begin{aligned}
&(\alpha_1, \dots \alpha_{j-1}, \alpha_j, \alpha_{j+1}, \alpha_{j+2}, \dots \alpha_m),\\
&(\alpha_1, \dots \alpha_{j-1}, \alpha_j, \beta_{j+1}, \alpha_{j+2}, \dots \alpha_m),\\
&(\alpha_1, \dots \alpha_{j-1}, \beta_j, \alpha_{j+1}, \alpha_{j+2}, \dots \alpha_m),\\
&(\alpha_1, \dots \alpha_{j-1}, \beta_j, \beta_{j+1}, \alpha_{j+2}, \dots \alpha_m).
\end{aligned}$$

Since $\alpha_j < \beta_j < \alpha_{j+1} < \beta_{j+1}$, these four points map to four distinct vertices spanning a two dimensional face of Δ ($\xi_k = \alpha_k$ defines a hyperplane). Now, any face of a Delzant polytope is Delzant (as one easily checks) and the Delzant property is invariant under affine transformation. Hence, we may as well map this 2-dimensional face into $\mathbb{R}^2$ by sending $(\sigma_1, \dots \sigma_m)$ to $(\sigma_1 - a_1, \sigma_2 - a_1\sigma_1 + a_1^2 - a_2)$, where a_1 and a_2 are the first two elementary symmetric functions of $\{\alpha_k : k \neq j, j+1\}$: in terms of ξ_j, ξ_{j+1} (fixing $\xi_k = \alpha_k$ for $k \neq j, j+1$), this formula gives $(\xi_j + \xi_{j+1}, \xi_j\xi_{j+1})$, and so our face gets mapped to the quadrilateral with vertices

$$(\alpha_j + \alpha_{j+1}, \alpha_j\alpha_{j+1}), \quad (\alpha_j + \beta_{j+1}, \alpha_j\beta_{j+1}),$$
$$(\beta_j + \alpha_{j+1}, \beta_j\alpha_{j+1}), \quad (\beta_j + \beta_{j+1}, \beta_j\beta_{j+1})$$

and normals (up to scale)

$$(\alpha_j, -1), \quad (\beta_j, -1), \quad (\alpha_{j+1}, -1), \quad (\beta_{j+1}, -1).$$

Again, $\alpha_j < \beta_j < \alpha_{j+1} < \beta_{j+1}$, so these four normals point in distinct directions, and so cannot be scaled to form a basis for the same lattice at each vertex. Our quadrilateral is therefore not Delzant, hence neither is Δ, a contradiction. q.e.d.

For the rest of this subsection, we suppose Δ is a simplex: the above proposition shows that this is necessarily true if M is non-singular.

By the Delzant construction, any symplectic orbifold M whose rational Delzant polytope is a simplex is a symplectic quotient of $\mathbb{C}^{m+1}$ by a one dimensional subgroup G of $(S^1)^{m+1}$. From the relation between complex and symplectic quotients, cf. (18), it follows that M is a quotient of a *weighted projective space* $\mathbb{C}P^m_{a_0,\dots a_m}$—here $a_0, \dots a_m \in \mathbb{Z}^+$ have highest common factor 1 and $\mathbb{C}P^m_{a_0,\dots a_m}$ is the quotient of $\mathbb{C}^{m+1} \setminus \{0\}$ by the holomorphic action

$$(z_0, \dots z_m) \to (\zeta^{a_0} z_0, \dots \zeta^{a_m} z_m) \quad \text{for} \quad \zeta \in \mathbb{C}^\times;$$

note that $\mathbb{C}P^m_{1,\dots 1}$ is the usual (non-singular) $\mathbb{C}P^m$.

We want to describe Δ more explicitly as a rational Delzant simplex. We put $\beta_0 = \alpha_1$, so Δ is the image under the elementary symmetric functions of the domain

$$D = \{(\xi_1, \dots \xi_m) \in \mathbb{R}^m : \beta_{j-1} \le \xi_j \le \beta_j\} \tag{27}$$

where $\beta_0 < \beta_1 < \cdots < \beta_{m-1} < \beta_m$.

Proposition 8. *Let M be a compact orthotoric Kähler $2m$-orbifold whose Delzant polytope Δ is the image of (27) under the elementary symmetric functions.*

(i) $\Delta = \{\boldsymbol{\sigma} : \langle v_j, \boldsymbol{\sigma}\rangle + \kappa_j \ge 0\}$, *where* $\kappa_j = \beta_j^m / \prod_{k\ne j}(\beta_j - \beta_k)$ *and*

$$v_j = \left(\frac{-\beta_j^{m-1}}{\prod_{k\ne j}(\beta_j - \beta_k)}, \dots \frac{(-1)^r \beta_j^{m-r}}{\prod_{k\ne j}(\beta_j - \beta_k)}, \dots \frac{(-1)^m}{\prod_{k\ne j}(\beta_j - \beta_k)} \right). \tag{28}$$

The codimension one faces of Δ are $F_0, \dots F_m$, where

- *F_0 is the image of the boundary component $\xi_1 = \beta_0$ of D,*
- *F_m is the image of the boundary component $\xi_m = \beta_m$ of D, and*

- *F_j, for $j = 1, \dots m-1$, is the union of the images of the boundary components $\xi_j = \beta_j$ and $\xi_{j+1} = \beta_j$ of D.*

(ii) *The normals are of the form $u_j = 2n_j v_j/c$, where $c > 0$ and $n_j \in \mathbb{Z}^+$ $(j = 0, \dots m)$ have highest common factor 1; then, M is equivariantly biholomorphic to an orbifold quotient of $\mathbb{C}P^m_{a_0,\dots a_m}$, where $n_j = \prod_{k\neq j} a_k$.*

(iii) *M is non-singular if and only if it is biholomorphic to $\mathbb{C}P^m$ if and only if $n_j = 1$ (for all j) and the lattice of circle subgroups is generated by $u_0, \dots u_m$. The dual lattice in $\mathbb{R}^{m*}$ is then generated by*

$$\theta_{p,q} = \sum_{r=0}^{m} \tfrac{1}{2} c\big(\sigma_r^\beta(\hat\beta_q) - \sigma_r^\beta(\hat\beta_p)\big) dt_r \tag{29}$$

where $\sigma_r^\beta(\hat\beta_p)$ denotes the rth elementary symmetric function of the m variables $\{\beta_j : j = 0, \dots m,\, j \neq p\}$.

Proof. (i) When Δ is a simplex, the inequalities in (26) may be written

$$\frac{\prod_{k=1}^m (\beta_j - \xi_k)}{\prod_{k\neq j}(\beta_j - \beta_k)} \geq 0$$

for all $j = 0, \dots m$, which immediately gives the stated form of Δ. (Note that the apparent codimension two face $\xi_j = \beta_j = \xi_{j+1}$ is 'straightened out' by the elementary symmetric functions; this is why Δ has only $m+1$ faces, not $2m$.)

(ii) From the form of the simplex Δ, it is immediate that the normals $u_0, \dots u_m$ are positive multiples of $v_0, \dots v_m$. They belong to a common lattice if and only if the linear dependence relation among them can be written $\sum_{j=0}^m u_j/n_j = 0$, where $n_0, \dots n_m$ are non-zero rational numbers. We now observe that the v_j's already satisfy $\sum_{j=0}^m v_j = 0$ by the Vandermonde identity (cf. [**4**, Appendix B]). Hence, we must have $u_j = C n_j v_j$ for some non-zero constant C and without loss of generality, we can take C and n_j's to be positive and suppose $n_0, \dots n_m$ are integers with highest common factor 1. We then put $C = 2/c$.

We have already seen that any toric Kähler orbifold with polytope a simplex is equivariantly biholomorphic to an orbifold quotient of a weighted projective space. It remains to show that the integers n_j are related to the weights a_k by $n_j = \prod_{k\neq j} a_k$. For this, we note [**1**] that any weighted projective space has an orbifold quotient whose simplex is standard with respect to the lattice Λ, i.e., the primitive normals sum to zero. The primitive normals are u_j/m_j and Abreu shows that the labels (in this case) are given by $m_j = \prod_{k\neq j} a_k$. Since $\sum_{j=0}^m v_j = 0$, and the m_j have highest common factor 1, we have $m_j = n_j$.

(iii) The only (orbifold quotient of a) weighted projective space which is non-singular is $\mathbb{C}P^m$. Clearly, M is equivariantly biholomorphic to $\mathbb{C}P^m$ if and only if the n_j all equal 1 and the lattice Λ of circle subgroups is the minimal one. In terms of the vector fields $K_1, \dots K_m$, it follows that vector fields generating Λ are

$$X_j = \frac{2}{c}\sum_{r=1}^m \frac{(-1)^r \beta_j^{m-r} K_r}{\prod_{k\neq j}(\beta_j - \beta_k)} = J\,\mathrm{grad}_g \frac{2\prod_{k=1}^{\ell}(\beta_j - \xi_k)}{c\prod_{k\neq j}(\beta_j - \beta_k)}, \tag{30}$$

with $\sum_{j=0}^m X_j = 0$. To see that (29) generate the dual lattice, we note that

$$\theta_{p,q} = \sum_{r=1}^m \tfrac{1}{2}c\,\sigma^\beta_{r-1}(\hat\beta_p, \hat\beta_q)(\beta_p - \beta_q)dt_r$$

for $0 \le p < q \le m$, where $\sigma^\beta_{r-1}(\hat\beta_p, \hat\beta_q)$ is the $(r-1)$st elementary symmetric function of the $m-1$ variables $\{\beta_j : j = 0, \dots m, j \neq p, q\}$. We compute

$$\begin{aligned}\theta_{p,q}(X_j) &= \sum_{r=1}^m \frac{(-1)^r \sigma^\beta_{r-1}(\hat\beta_p, \hat\beta_q)\,(\beta_p - \beta_q)\beta_j^{m-r}}{\prod_{k\neq j}(\beta_j - \beta_k)}\\ &= \frac{\prod_{k\neq p,q}(\beta_j - \beta_k)}{\prod_{k\neq j}(\beta_j - \beta_k)}(\beta_q - \beta_p) = \delta_{jq} - \delta_{jp}\end{aligned}$$

and the result follows. q.e.d.

The constant c determines the scale of M: the symplectic volume is proportional to $1/c$. The other constants $\beta_0, \dots \beta_m$ are related to the fact that the Killing vector fields $K_1, \dots K_m$ do not necessarily form an integral basis.

We remark that all simplices are equivalent under affine transformation, and so for any $\beta_0 < \cdots < \beta_m$, any rational Delzant simplex is equivalent to the simplex of this proposition for some lattice Λ in $\mathbb{R}^m$ and some normals $u_j = 2n_j v_j/c \in \Lambda$.

3.2. Compactification of orthotoric Kähler metrics. We next establish necessary and sufficient conditions for the compactification of the orthotoric Kähler metric (25) on a compact $2m$-orbifold M. We obtain these conditions by specializing those of Proposition 1 to the orthotoric case.

Proposition 9. *Let M be a compact symplectic $2m$-orbifold such that the rational Delzant polytope $\Delta \subset \mathbb{R}^{m*}$ is the image of $\prod_{j=1}^m [\alpha_j, \beta_j]$ under the elementary symmetric functions, where*

$$\alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots < \beta_{m-1} \le \alpha_m < \beta_m.$$

Let $L_j^\alpha(\boldsymbol{\sigma}) = \langle u_j^\alpha, \boldsymbol{\sigma}\rangle + \lambda_j^\alpha$ and $L_j^\beta(\boldsymbol{\sigma}) = \langle u_j^\beta, \boldsymbol{\sigma}\rangle + \lambda_j^\beta$ where

$$\begin{aligned} \lambda_j^\alpha &= -c_j^\alpha \alpha_j^m, \qquad u_j^\alpha = c_j^\alpha \big(\alpha_j^{m-1}, \ldots (-1)^{r-1}\alpha_j^{m-r}, \ldots (-1)^{m-1}\big), \\ \lambda_j^\beta &= -c_j^\beta \beta_j^m, \qquad u_j^\beta = c_j^\beta \big(\beta_j^{m-1}, \ldots (-1)^{r-1}\beta_j^{m-r}, \ldots (-1)^{m-1}\big), \end{aligned}$$

and the constants $c_j^\alpha, c_j^\beta \in \mathbb{R}$ are such that the normals of Δ are the distinct elements among u_j^α, u_j^β, i.e., $\Delta = \{\boldsymbol{\sigma} \in \mathbb{R}^{m*} : L_j^\alpha(\boldsymbol{\sigma}) \ge 0$ and $L_j^\beta(\boldsymbol{\sigma}) \ge 0$ for $j = 1, \ldots m\}$, but if $\alpha_{j+1} = \beta_j$, we have $c_{j+1}^\alpha = c_j^\beta$ as the normals $u_{j+1}^\alpha, u_j^\beta$ are then not distinct.

Then, the Kähler metric (25), defined for $\xi_j \in (\alpha_j, \beta_j)$, extends to an orthotoric Kähler metric on M if and only if for $j = 1, \ldots m$, Θ_j is the restriction to (α_j, β_j) of a smooth function Θ on $\bigcup_{j=1}^m [\alpha_j, \beta_j]$ satisfying (for $j = 1, \ldots m$):

$$\begin{aligned} \Theta(\alpha_j) &= 0 = \Theta(\beta_j), \\ \Theta'(\alpha_j) c_j^\alpha &= 2 = \Theta'(\beta_j) c_j^\beta; \end{aligned} \tag{31}$$

$$(-1)^{m-j}\Theta > 0 \quad \textit{on} \quad (\alpha_j, \beta_j). \tag{32}$$

Proof. By (25), $\mathbf{H}$ is given by

$$H_{rs} = \sum_{j=1}^m \frac{\Theta_j(\xi_j)\, \sigma_{r-1}(\hat{\xi}_j)\sigma_{s-1}(\hat{\xi}_j)}{\Delta_j}.$$

This is a smooth and symmetric function of $\xi_1, \ldots \xi_m$, so by Glaeser [**14**], it is a smooth function of $\sigma_1, \ldots \sigma_m$. The positivity condition is clear, so it remains to consider the boundary conditions (31). We must show these are equivalent to (15).

The form of the normals shows that $\mathbf{H}(u_i^\alpha, \cdot)$ is given by

$$\begin{aligned} \sum_{r=1}^m H_{rs}(u_i^\alpha)_r &= \sum_{j,r=1}^m \frac{c_i^\alpha \Theta_j(\xi_j)\, (-1)^{r-1}\sigma_{r-1}(\hat{\xi}_j)\alpha_i^{m-r}\sigma_{s-1}(\hat{\xi}_j)}{\Delta_j} \\ &= \sum_{j=1}^m \frac{c_i^\alpha \Theta_j(\xi_j)\sigma_{s-1}(\hat{\xi}_j)\prod_{k\ne j}(\alpha_i - \xi_k)}{\Delta_j}. \end{aligned}$$

On the codimension one face $\xi_i = \alpha_i$, this reduces to $c_i^\alpha \Theta_i(\alpha_i)\sigma_{s-1}(\hat{\xi}_i)$, which vanishes for all s if and only if $\Theta_i(\alpha_i) = 0$. For the derivative conditions, we differentiate

$$\mathbf{H}(u_i^\alpha, u_i^\alpha) = \sum_{j=1}^m \frac{(c_i^\alpha)^2 \Theta_j(\xi_j) \prod_{k\neq j}(\alpha_i - \xi_k)^2}{\Delta_j}$$

and evaluate along $\xi_i = \alpha_i$ to obtain

$$(c_i^\alpha)^2 \Theta_i'(\alpha_i) \prod_{k\neq i}(\alpha_i - \xi_k)\, d\xi_i = c_i^\alpha \Theta_i'(\alpha_i) d \prod_{k=1}^m c_i^\alpha(\alpha_i - \xi_k)\Big|_{\xi_i=\alpha_i}.$$

This equals $2u_i^\alpha$ if and only if $c_i^\alpha \Theta_i'(\alpha_i) = 2$. The boundary conditions at the β endpoints are analogous. q.e.d.

Note that (31) could be taken as the definition of the constants c_j^α and c_j^β. However, these are then required to satisfy positivity and integrality conditions, since $(-1)^{m-j}c_j^\alpha$ and $(-1)^{m-j+1}c_j^\beta$ must be positive for the normals to be inward pointing, while u_j^α and u_j^β must belong to a common lattice in $\mathbb{R}^m$.

We summarize our results for the case that the rational Delzant polytope is a simplex. The following is immediate from Propositions 7, 8 and 9.

Theorem 3. *Let M be a compact orthotoric $2m$-manifold or orbifold with momentum map $\boldsymbol{\sigma}$, whose rational Delzant polytope is a simplex Δ with normals $u_0, u_1, \dots u_m \in \mathbb{R}^m$, where the Kähler metric is given by (25) on $M^0 = \boldsymbol{\sigma}^{-1}(\Delta^0)$.*

(i) *M is equivariantly biholomorphic to a toric orbifold quotient of $\mathbb{C}P^m_{a_0,\dots a_m}$ and, with $n_j = \prod_{k\neq j} a_k$, there are constants $\beta_0 < \beta_1 < \cdots < \beta_m$, $c > 0$, and a smooth function Θ on $[\beta_0, \beta_m]$, such that for $j = 0, \dots m$:*

(33)
$$u_j = \frac{2n_j}{c}\left(\frac{-\beta_j^{m-1}}{\prod_{k\neq j}(\beta_j - \beta_k)}, \dots \frac{(-1)^r \beta_j^{m-r}}{\prod_{k\neq j}(\beta_j - \beta_k)}, \dots \frac{(-1)^m}{\prod_{k\neq j}(\beta_j - \beta_k)}\right);$$

(34) $$\Theta_j = \Theta \quad on \quad [\beta_{j-1}, \beta_j];$$

(35) $$(-1)^{m-j}\Theta > 0 \quad on \quad (\beta_{j-1}, \beta_j);$$

(36) $$\Theta(\beta_j) = 0, \qquad \Theta'(\beta_j) = -\frac{c}{n_j}\prod_{k\neq j}(\beta_j - \beta_k).$$

(ii) *Conversely, given constants $\beta_0 < \beta_1 < \cdots < \beta_m$, $c > 0$ and a smooth function Θ on $[\beta_0, \beta_m]$ satisfying (35)–(36), the Kähler metric given by (25) and (34) defines an orthotoric structure on $\mathbb{C}P^m_{a_0,\dots a_m}$ and its toric orbifold quotients, such that the rational*

Delzant polytope is the image of $[\beta_0,\beta_1]\times[\beta_1,\beta_2]\times\cdots\times[\beta_{m-1},\beta_m]$ *under the elementary symmetric functions, with normals given by* (33)*.*

(iii) *Any* (non-singular) *compact orthotoric Kähler* $2m$*-manifold* M *arises in this way* (*with* $n_j=1$ *for* $j=0,\dots m$) *and is equivariantly biholomorphic to* $\mathbb{C}P^m$*.*

3.3. Examples on weighted projective spaces.

3.3.1. The Fubini–Study metric. We recall from [**4**, Section 5.4] that an orthotoric Kähler metric has constant holomorphic sectional curvature c if and only if $\Theta_j=\Theta_0$ for all $j=1,\dots m$, where Θ_0 is a polynomial of degree $m+1$ with distinct roots and leading coefficient $-c$. We then have $\Theta_0(t)=-c\prod_{j=0}^m(t-\beta_j)$ with $\beta_0<\cdots<\beta_m$, which clearly satisfies (35)–(36). Thus, we see directly that this orthotoric metric is defined on $\mathbb{C}P^m$, in accordance with [**4**, Section 2.4], where it was shown more generally that the Fubini–Study metric on $\mathbb{C}P^m$ admits hamiltonian 2-forms of arbitrary order $\le m$, in one to one correspondence with Killing potentials.

This form of the Fubini–Study metric is familiar for $m=1$, when (25) yields

$$g=\frac{d\xi^2}{c(\xi-\beta_0)(\beta_1-\xi)}+c(\xi-\beta_0)(\beta_1-\xi)dt^2.$$

Setting $2\xi=(\beta_1-\beta_0)z+\beta_0+\beta_1$, $t=2\psi/c(\beta_1-\beta_0)$ and rescaling g by c, we get

$$g_{FS}=\frac{dz^2}{1-z^2}+(1-z^2)d\psi^2.$$

In arbitrary dimension m, the Fubini–Study metric is the 'canonical' metric associated to its simplex, hence is given here by (8) with

$$\mathbf{G}=\frac{1}{2}\mathrm{Hess}\left(\sum_{j=0}^m L_j(\boldsymbol{\sigma})\log|L_j(\boldsymbol{\sigma})|\right)$$

where $$L_j(\boldsymbol{\sigma})=\langle u_j,\boldsymbol{\sigma}\rangle+\lambda_j=\frac{2}{c}\frac{\prod_{k=1}^m(\beta_j-\xi_k)}{\prod_{k\neq j}(\beta_j-\beta_k)}.$$

It follows that

$$\sum_{r,s=1}^m \mathbf{G}_{rs}d\sigma_r\,d\sigma_s$$
$$=\frac{1}{2}\sum_{j=0}^m L_j\left(\frac{dL_j}{L_j}\right)^2$$

$$= \frac{1}{c} \sum_{j=0}^{m} \frac{\prod_{k=1}^{m}(\beta_j - \xi_k)}{\prod_{k \neq j}(\beta_j - \beta_k)} \left(\sum_{k=1}^{m} \frac{d\xi_k}{\xi_k - \beta_j} \right)^2$$

$$= \sum_{p,q=1}^{m} \frac{1}{\Theta_0(\xi_p)} \sum_{j=0}^{m} \left(\prod_{k \neq q} (\beta_j - \xi_k) \right) \left(\prod_{k \neq j} \frac{\xi_q - \beta_k}{\beta_j - \beta_k} \right) d\xi_p \, d\xi_q$$

which immediately yields the orthotoric description (25), since the inner sum over j is $\Delta_q \, \delta_{pq}$ by the Lagrange interpolation formula.

3.3.2. Bochner-flat metrics. More generally, any *extremal* orthotoric metric (25) for which $\Theta_j = \Theta$ is necessarily Bochner-flat [**4**, Section 5.4]; in this case, Θ must be a polynomial of degree $\leq m+2$, and the boundary conditions (31) imply that Θ has $m+1$ or $m+2$ distinct roots. The former case gives the Fubini–Study metric and its orbifold quotients, while the latter recovers the Bochner-flat examples of [**9**], which are defined on $\mathbb{C}P^m_{a_0,\ldots a_m}$ for distinct weights $a_0, \ldots a_m$. Indeed, for any positive integers $a_0 > \cdots > a_m$ we take the metric (25) with

$$\Theta_j(t) = \Theta(t) = -(t-\beta)\Theta_0(t) = c(t-\beta) \prod_{j=0}^{m} (t - \beta_j),$$

where $c > 0$ is a homothety factor for the metric, and we deduce from (36) that the real numbers $\beta_0 < \cdots < \beta_m < \beta$ and $c > 0$ satisfy

$$\beta_j = \beta - \frac{a_j}{\prod_{k=0}^{m} a_k}. \tag{37}$$

This metric is Bochner-flat (see [**4**, Proposition 16]) and compactifies on $\mathbb{C}P^m_{a_0,\ldots a_m}$ (see Theorem 3). As shown by Bryant [**9**], there are actually Bochner-flat metrics on $\mathbb{C}P^m_{a_0,\ldots a_m}$ for *any* choice of weights. An alternative, easy way to see this [**12**] uses the relation between Bochner-flat metrics and flat CR structures found by Webster [**35**]. Indeed, $\mathbb{C}P^m_{a_0,\ldots a_m}$ is a quotient of S^{2m+1} by a weighted S^1-action by CR automorphisms of the flat CR structure, and the Sasakian structure induced by the associated Reeb field gives rise to a Bochner-flat Kähler metric on the quotient.

The Bochner-flat metrics on weighted projective spaces are all toric (see [**1**] for the general form in momentum coordinates). However, when the weights are not distinct, they are not orthotoric (apart from the Fubini–Study metric): from our point of view, the Bochner-flat metric is endowed with a natural hamiltonian 2-form which is (an affine deformation of) the normalized Ricci form [**4**] and it has order m if and only if the weights a_j are distinct.

Remark 8. Note that the orthotoric Bochner-flat Kähler metric on a weighted projective space is unique (up to isomorphism and scale): $\beta_0, \dots \beta_m$ are determined as above (the choice of β can be absorbed in the coordinate freedom). In fact, a stronger uniqueness result is true: the Bochner-flat metric is the unique *extremal* Kähler metric (up to isomorphism and scale) on *any* weighted projective space. To see this, recall that the second deRham cohomology group of $\mathbb{C}P^m_{a_0,\dots a_m}$ is one dimensional, so there is only one Kähler class up to scale (this follows, for instance, by the Smith–Gysin sequence for the space of orbits, $\mathbb{C}P^m_{a_0,\dots a_m}$, of the weighted S^1-action on the $(2m+1)$-sphere); therefore, the uniqueness result of Guan [**15**] (which readily generalizes to orbifolds) applies to the Kähler class of $\mathbb{C}P^m_{a_0,\dots a_m}$.

The uniqueness implies that any toric $2m$-orbifold of constant scalar curvature, whose rational Delzant polytope is a simplex, is an orbifold quotent of $\mathbb{C}P^m$. Note that the Futaki invariant of $\mathbb{C}P^m_{a_0,\dots a_m}$ vanishes if and only if $a_0 = a_1 = \cdots = a_m$.

3.4. Kähler–Einstein orthotoric surfaces. In this subsection, we present new examples of Kähler–Einstein metrics on compact orbifolds. As we have seen in the previous subsection, we have to work beyond the context of weighted projective spaces, so we consider polytopes with more than $m+1$ codimension one faces. We restrict attention to complex orbifold surfaces ($m = 2$) in order to make the construction completely explicit. In this case, a polytope with more than $m + 1 = 3$ faces necessarily has $2m = 4$ faces and we are in the 'generic' case where the roots ξ_1, ξ_2 are *everywhere* distinct on Δ.

According to [**4**, Section 5.3], an orthotoric Kähler metric on a 4-orbifold is Kähler–Einstein if and only if $\Theta_j(t) = -P_j(t)/C$, $j = 1, 2$ for some positive constant C, and some $\pm$-monic polynomials P_j of degree 3, such that $P_1(t) - P_2(t) = c$ where c is a constant. The Bochner tensor vanishes precisely when $c = 0$, and the metric is then the Fubini–Study metric. We, therefore, assume that $c \neq 0$ in order to obtain new examples. Also, for compactness, the scalar curvature must be positive (otherwise the Ricci tensor would be non-positive, contradicting the existence of Killing vector fields with zeros), which implies that the polynomials P_j are monic.

It remains to solve the compactification conditions. For simplicity, we shall take the lattice $\Lambda \subset \mathbb{R}^2$ to be $\mathbb{Z}^2$ or a sublattice. The conditions of Proposition 9 can then be satisfied by supposing that P_j has integer roots (including the endpoints α_j and β_j) and C is chosen so that $2/\Theta'(\alpha_j) = c^\alpha_j = -2C/P'_j(\alpha_j)$ and $2/\Theta'(\beta_j) = c^\beta_j = -2C/P'_j(\beta_j)$ are all integers for $j = 1, 2$.

The condition (31) implies that P_1 and P_2 have three distinct roots, $p_1 < q_1 < r_1$ and $p_2 > q_2 > r_2$, respectively. The condition $P_1 - P_2 = c$ reads

$$p_1 + q_1 + r_1 = p_2 + q_2 + r_2,$$
$$p_1^2 + q_1^2 + r_1^2 = p_2^2 + q_2^2 + r_2^2.$$

Positivity and (31) give $\alpha_1 = p_1$, $\beta_1 = q_1$, $\alpha_2 = q_2$, $\beta_2 = p_2$ and hence (without loss) $q_1 < q_2$. Taking the roots to be all integral, we note that, up to an affine deformation of the hamiltonian 2-form and orbifold coverings/quotients, we can also assume that $\gcd(p_1, q_1, r_1, p_2, q_2, r_2) = 1$ and

$$p_1 + q_1 + r_1 = 0 = p_2 + q_2 + r_2.$$

A class of solutions to this problem is obtained by taking any coprime positive integers (p, q) with $p > q$ and putting

$$p_1 = -p, \quad q_1 = -q, \quad r_1 = p + q, \quad p_2 = p, \quad q_2 = q, \quad r_2 = -p - q.$$

With these assumptions, we have

$$\alpha_1 = -p, \quad \beta_1 = -q, \quad \alpha_2 = q, \quad \beta_2 = p;$$

$$\Theta_1(\xi) = -\frac{(\xi + p)(\xi + q)(\xi - p - q)}{C}; \tag{38}$$

$$\Theta_2(\xi) = -\frac{(\xi - p)(\xi - q)(\xi + p + q)}{C}. \tag{39}$$

The corresponding Delzant polytope Δ is the quadrilateral with vertices

$$(0, -p^2), \quad (0, -q^2), \quad (p - q, -pq), \quad (q - p, -pq)$$

and one-dimensional faces F_j^α, F_j^β, $j = 1, 2$ determined by the lines

$$\{\boldsymbol{\sigma} : \ell_j^\alpha(\boldsymbol{\sigma}) = 0\}, \quad \{\boldsymbol{\sigma} : \ell_j^\beta(\boldsymbol{\sigma}) = 0\},$$

where

$$\ell_1^\alpha(\boldsymbol{\sigma}) = p^2 + p\sigma_1 + \sigma_2, \qquad \ell_1^\beta(\boldsymbol{\sigma}) = q^2 + q\sigma_1 + \sigma_2,$$
$$\ell_2^\alpha(\boldsymbol{\sigma}) = q^2 - q\sigma_1 + \sigma_2, \qquad \ell_2^\beta(\boldsymbol{\sigma}) = p^2 - p\sigma_1 + \sigma_2.$$

Furthermore, letting $2C = (p - q)(2q + p)(2p + q)$ in (38) and (39), we get

$$\Theta_1'(\alpha_1) = \Theta_2'(\beta_2) = 2/(2p + q), \quad \Theta_1'(\beta_1) = \Theta_2'(\alpha_2) = -2/(2q + p),$$

so the conditions of Proposition 9 are satisfied with

$$c_1^\alpha = c_2^\beta = 2q + p, \quad c_2^\alpha = c_1^\beta = 2p + q.$$

Thus, according to Proposition 9, the corresponding Kähler–Einstein orthotoric metric $g_{p,q}$ compactifies on the toric orbifold Kähler surface $M(p,q)$ classified by

$$(\Delta, \Lambda, c_1^\alpha, c_1^\beta, c_2^\alpha, c_2^\beta),$$

where Λ is the standard lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ (in which case c_j^α, c_j^β are nothing but the integer labels corresponding to the 1-dimensional faces of Δ, see Section 1.2).

We claim that two orbifold surfaces $M(p,q)$ and $M(p',q')$ are biholomorphically equivalent iff $p = p'$ and $q = q'$. Indeed, in order to be biholomorphic as complex orbifolds, $M(p,q)$ and $M(p',q')$ must be isomorphic as toric varieties. Therefore, the cooresponding polytopes Δ and Δ' must determine congruent fans [**28**, Thm.9.4]. One easily checks that the latter happens iff $(p,q) = (p',q')$; alternatively, using the uniqueness of the hamiltonian 2-form established in Propostion 10 below, one can see that the Kähler–Einstein metrics $g_{p,q}$ and $g_{p',q'}$ are locally isometric if and only if $(p,q) = (p',q')$.

We summarize our construction as follows.

Theorem 4. *There is a family of non-equivalent compact Kähler–Einstein orthotoric orbifold surfaces* $(M(p,q), g_{p,q})$*, depending on co-prime positive integers* $q < p$*.*

Remark 9.

(i) According to the results of [**3**], the primitive part of the hamiltonian 2-form ϕ associated to $g_{p,q}$ defines an integrable almost-complex structure I on $M(p,q)$, which is compatible with $g_{p,q}$, but induces the opposite orientation to the one of $M(p,q)$. With respect to this structure, $(M(p,q), g_{p,q}, I)$ become a compact, Einstein, non-Kähler hermitian complex orbifold surface (see [**27**] for a classification in the smooth case).

(ii) A similar construction yields a countable family of compact orbifold complex surfaces supporting orthotoric weakly Bochner-flat metrics which are neither Bochner-flat nor Kähler–Einstein (see [**3**] for a classification in the smooth case).

(iii) According to [**8**], any Kähler–Einstein orbifold (M, g, J, ω) of complex dimension m gives rise to a Sasaki–Einstein structure on the total space S of a principal S^1 V-bundle over M (which is suitably associated to the canonical bundle of M). In general, S is a $(2m+1)$-dimensional orbifold rather than a manifold, but it may happen that S is non-singular even though M is singular [**8**]: in fact, S is non-singular if and only if all local uniformizing groups of M inject into the structure group S^1 (see [**8**, Theorem 2.3]). In the case of *toric* Kähler orbifolds, all local uniformizing groups are abelian [28] so that if S is non-singular, then all local uniformizing groups of M must be cyclic. Using this observation, one can show that the universal orbifold covers $\widehat{M}(p,q)$ of $M(p,q)$ (the one which corresponds to the lattice generated by the normals of Δ, see Remark 2) give rise only to *singular* 5-dimensional Sasaki–Einstein orbifolds.

4. Compact Kähler manifolds with hamiltonian 2-forms

We now combine the work of the previous three sections to classify, up to a covering, compact Kähler manifolds with a hamiltonian 2-form. In the case of Kähler surfaces, we can refine the classification. We end by giving some examples of extremal and weakly Bochner-flat Kähler metrics with hamiltonian 2-forms.

4.1. General classification. There are two parts to a general classification of compact Kähler manifolds with a hamiltonian 2-form. First, we must classify the possible equivariant biholomorphism types of manifolds which can admit a hamiltonian 2-form. Second, we describe the compatible Kähler structures on such manifolds which *do* admit hamiltonian 2-forms.

The equivariant biholomorphism type is described in parts (ii)–(iv) of the following theorem: we show that, up to a blow-up and a covering, a compact Kähler manifold with a hamiltonian 2-form of order ℓ is biholomorphic to a projective bundle of the form $P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \to S$ where $\mathcal{L}_j$ are holomorphic line bundles over a product S of Kähler manifolds S_a. Such a bundle admits an action of a complex ℓ-torus $\mathbb{T}^c$, defined by scalar multiplication in each line bundle (an $(\ell+1)$-torus action on $\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_\ell$) modulo overall scalar multiplication (which acts trivially on the projectivization). In part (v) of the theorem, we show that the relevant Kähler structures are given by a special case of the generalized Calabi construction with $\mathcal{V} = \mathbb{C}P^\ell$. Conversely, we show that this construction produces compact Kähler manifolds with a hamiltonian 2-form.

Theorem 5. *Let (M, g, J, ω) be a compact connected Kähler $2m$-manifold with a hamiltonian 2-form ϕ of order $\ell \geq 0$, with non-constant roots $\xi_1, \ldots \xi_\ell$ and (distinct) constant roots $\eta_1, \ldots \eta_N$, $N \geq 0$.*

(i) *The elementary symmetric functions $(\sigma_1, \ldots \sigma_\ell)$ of $(\xi_1, \ldots \xi_\ell)$ are the components of the momentum map $\boldsymbol{\sigma}\colon M \to \mathbb{R}^{\ell *}$ of an ℓ-torus $\mathbb{T} \leq \mathrm{Isom}(M,g)$. The image Δ of $\boldsymbol{\sigma}$ is a Delzant simplex in $\mathbb{R}^{\ell *}$, whose interior is the image under the elementary symmetric* *functions of a domain* $D = \prod_{j=1}^{\ell}(\beta_{j-1}, \beta_j)$ *with* $\beta_0 < \beta_1 < \cdots < \beta_\ell$.

(ii) *Let* S_Δ *be the stable quotient of* M *by the complex torus* $\mathbb{T}^c$ *and let* $\hat{M}$ *be the blow-up of* M *along the inverse image of the codimension one faces* $F_0, F_1, \ldots F_\ell$ *of* Δ. *Then, there are holomorphic line bundles* $\mathcal{L}_0, \mathcal{L}_1, \ldots \mathcal{L}_\ell$ *over* S_Δ *(uniquely determined up to overall tensor product with a holomorphic line bundle) such that* $\hat{M}$ *is* $\mathbb{T}^c$*-equivariantly biholomorphic to* $P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \to S_\Delta$.

(iii) S_Δ *is covered by a product* S *of* N *Hodge Kähler manifolds* $(S_a, \pm g_a, \pm\omega_a)$ *of dimension* $2d_a$, *indexed by the constant roots* η_a *(*S *is a point if* $N = 0$*). There are constants* $c > 0, C_1, \ldots C_N$ *such that for* $j = 0, \ldots \ell$, $a = 1, \ldots N$

$$\tfrac{1}{2}c\left(\prod_{k\neq j}(\eta_a - \beta_k)\right)\big(C_a(\eta_a - \beta_j) - 1\big)[\omega_a/2\pi] \tag{40}$$

is an integral cohomology class on S_a, *and the pullback of* $\mathcal{L}_j$ *to* S *is a tensor product* $\bigotimes_{a=1}^{N} \pi_a^*\mathcal{L}_{j,a}$, *where* π_a *is the projection of* S *to* S_a *and* $\mathcal{L}_{j,a} \to S_a$ *is a holomorphic line bundle with first Chern class given by* (40).

(iv) *The subset* $\mathcal{B}$ *of those* $j \in \{0, \ldots \ell\}$ *for which the blow up over the face* F_j *is non-trivial corresponds bijectively to a subset* $\mathcal{C}$ *of* $\{1, \ldots N\}$ *such that for* $j \in \mathcal{B}$ *corresponding to* $a \in \mathcal{C}$, $\eta_a = \beta_j$, $S_a = \mathbb{C}P^{d_a}$, $\pm g_a$ *is the Fubini–Study metric on* S_a *of constant holomorphic sectional curvature* $\pm c\prod_{k\neq j}(\beta_j - \beta_k)$, *and (without loss)* $\mathcal{L}_{j,a} = \mathcal{O}(-1)$ *and* $\mathcal{L}_{k,a} = \mathcal{O}$ *for* $k \neq j$.

For $a \notin \mathcal{C}$ *either* $\eta_a < \beta_0$ *or* $\eta_a > \beta_\ell$.

(v) *The Kähler metric on* M *and its pullback to* $\hat{M}$ *are determined by the explicit metric* (3) *on* M^0, *where:*

- *the pullback to* $S = \prod_{a=1}^{N} S_a$ *of the Kähler quotient metric on* S_Δ *induced by* $\boldsymbol{\sigma}(\xi_1, \ldots \xi_\ell) \in \Delta^0$ *is the Kähler product metric*

$$\sum_{a=1}^{N}\left(\prod_{j=1}^{\ell}(\eta_a - \xi_j)\right) g_a; \tag{41}$$

- $\theta_1, \ldots \theta_\ell$ *are the components of a connection on* $\hat{M} \to S_\Delta$ *associated to a principal* $\mathbb{T}$*-connection;*
- *for* $j = 1, \ldots \ell$, $F_j(t) = p_c(t)\Theta(t)$, *where* $p_c(t) = \prod_{a=1}^{N}(t - \eta_a)^{d_a}$,

$$(-1)^{\ell-j}\Theta > 0 \quad \text{on} \quad (\beta_{j-1}, \beta_j), \tag{42}$$

$$\Theta(\beta_j) = 0, \qquad \Theta'(\beta_j) = -c\prod_{k\neq j}(\beta_j - \beta_k), \tag{43}$$

and the metric on the $\mathbb{C}P^\ell$*-fibres of* $\hat{M} \to S_\Delta$ *is the orthotoric Kähler metric* (25) *with* $\Theta_j(t) = \Theta(t)$*;*

Conversely, suppose S *is a product of Hodge manifolds* $(S_a, \pm g_a, \pm\omega_a)$ *and constants* $\beta_0, \dots \beta_\ell$*,* $\eta_1, \dots \eta_N$*,* $c, C_1, \dots C_N$ *satisfying the conditions in* (i)–(iv) *above and such that* (41) *is positive for* $\boldsymbol{\sigma}(\xi_1, \dots \xi_\ell) \in \Delta^0$*.*

Then, there is a complex manifold M *obtained by a blow-down of a projective bundle* $\hat{M} = P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \to S$ *which gives rise to these data. Further, for any smooth function* Θ *on* $[\beta_0, \beta_\ell]$ *satisfying* (42)*–*(43)*, a Kähler metric of the form* (3)*, with* $F_j(t) = p_{\rm c}(t)\Theta(t)$*, is globally defined on* M *and admits a hamiltonian* 2*-form of order* ℓ*.*

Proof. Consider the explicit form (3) of the metric on the open subset M^0 of M. By Lemma 4, the map $\boldsymbol{\sigma} = (\sigma_1, \dots \sigma_\ell)\colon M \to \mathbb{R}^\ell$ generates a rigid hamiltonian torus action, and the Kähler quotient metric (i.e., (41)) is clearly semisimple, so that by Theorem 2, there is a cover of M which is given by the generalized Calabi construction. The covering is straightforward: there is a discrete group Γ of holomorphic isometries of S which lifts to the bundle $M_0 \times_{\mathbb{T}^c} \mathcal{V}$ and $\hat{M}$ is the quotient. We shall therefore, suppose Γ is trivial in the following.

The Kähler metrics $\pm\omega_a$ are determined by (41) and the constants c_{a0} and $\boldsymbol{c}_a = (c_{a1}, \dots c_{a\ell})$ appearing in the generalized Calabi data are

$$c_{a0} = \eta_a^\ell, \quad c_{ar} = (-1)^r \eta_a^{\ell - r}, \quad r = 1, \dots \ell. \tag{44}$$

Since the roots $\xi_1, \dots \xi_\ell$ of the momentum polynomial $p_{\rm nc}(t)$ are smooth, functionally independent and pairwise distinct on M^0, with orthogonal gradients, the toric Kähler manifold $\mathcal{V}$ appearing in the generalized Calabi data is orthotoric.

(i) $\boldsymbol{\sigma}$ is a momentum map by definition, and by Proposition 7, its image Δ is a simplex as stated. In particular, the codimension one faces $F_0, F_1, \dots F_\ell$ correspond to the boundary points $\beta_0 < \beta_1 < \cdots < \beta_\ell$ of D, and $\mathcal{V}$ is biholomorphic to $\mathbb{C}P^\ell$.

(ii) M^0 is a holomorphic principal $\mathbb{T}^c$-bundle over S and the blow up of $\hat{M}$ along the inverse image of the codimension one faces is equivariantly biholomorphic to a projective bundle $M^0 \times_{\mathbb{T}^c} \mathbb{C}P^\ell$ with a global fibre preserving $\mathbb{T}^c$ action. This action identifies $\hat{M}$ with $P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell)$ for holomorphic line bundles $\mathcal{L}_0, \mathcal{L}_1 \dots \mathcal{L}_\ell$ over S (uniquely determined as stated) in such a way that the $2(\ell-1)$-dimensional orbits of $\mathbb{T}^c$ in each fibre are orbits of elements of $P(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_\ell)$ with one homogeneous coordinate vanishing. We label the line bundles so that the codimension one face F_j corresponds to the orbit of $\mathbb{T}^c$ with $\mathcal{L}_j$ component vanishing.

(iii) We only need to construct the line bundles $\mathcal{L}_{j,a}$ and establish the formula (40) for their first Chern classes. The explicit form (3) of the metric on the principal bundle M^0 shows that the connection 1-forms θ_r, with $\theta_r(K_s) = \delta_{rs}$, satisfy

$$d\theta_r = \sum_{a=1}^{N} (-1)^r \eta_a^{\ell-r} \omega_a, \tag{45}$$

where $\pm\omega_a$ are the Kähler forms of the globally defined metrics $\pm g_a$ on S_a. Note that the θ_r are not necessarily integral. The integral principal connection forms are those which evaluate to integers on the Euler fields $X_0, \dots X_\ell$, which, according to Proposition 8, are given by (29):

$$\theta_{p,q} = \sum_{r=0}^{\ell} \tfrac{1}{2} c\big(\sigma_r^\beta(\hat\beta_q) - \sigma_r^\beta(\hat\beta_p)\big)\theta_r,$$

More specifically, this is the connection form of the line bundle $\mathcal{L}_p^{-1} \otimes \mathcal{L}_q$. The curvature form of $\mathcal{L}_p^{-1} \otimes \mathcal{L}_q$ is therefore, (up to a sign convention)

$$\begin{aligned} d\theta_{p,q} &= \sum_{a=1}^{N} \sum_{r=0}^{\ell} \tfrac{1}{2} c (-1)^r \big(\sigma_r^\beta(\hat\beta_q) - \sigma_r^\beta(\hat\beta_p)\big) \eta_a^{\ell-r} \omega_a \\ &= \sum_{a=1}^{N} \tfrac{1}{2} c \Big(\prod_{k\neq q} (\eta_a - \beta_k) - \prod_{k \neq p} (\eta_a - \beta_k) \Big) \omega_a. \end{aligned} \tag{46}$$

It follows that for each $a = 1, \dots N$, the corresponding 2-form in this sum is *integral* in the sense that the cohomology class

$$\tfrac{1}{2} c \left(\prod_{k\neq q} (\eta_a - \beta_k) - \prod_{k \neq p} (\eta_a - \beta_k) \right) [\omega_a / 2\pi] \tag{47}$$

is in the image of $H^2(S_a, \mathbb{Z})$ in $H^2(S_a, \mathbb{R})$. If $\eta_a = \beta_j$ for some j, we deduce (by taking $p = j$, $q \neq j$) that $\frac{1}{2} c \big(\prod_{k \neq j} (\eta_a - \beta_k) \big) \omega_a$ is integral. Otherwise, this will differ from an integral class by a constant. Hence, there are constants $C_1, C_2, \dots C_N$ such that for each $j = 0, \dots \ell$ and $a = 1, \dots N$, the 2-form

$$\tfrac{1}{2} c \left(\prod_{k \neq j} (\eta_a - \beta_k) \right) \big(1 - C_a(\eta_a - \beta_j)\big)\, \omega_a \tag{48}$$

is also integral. The Lefschetz Theorem for $(1,1)$-classes implies there are holomorphic line bundles $\mathcal{L}_{j,a}$ with connection over S_a with curvature forms given by (48): with our sign convention, the first Chern classes are then as stated in (40). It follows that $\mathcal{L}_j$ is the tensor product of $\bigotimes_{a=1}^N \pi_a^* \mathcal{L}_{j,a}$ by a flat line bundle $\mathcal{F}_j$. Since any flat line bundle on S is a tensor product of flat line bundles pulled back from the factors S_a, we may use the freedom in the choice of $\mathcal{L}_{j,a}$ to make $\mathcal{F}_j$ trivial.

Finally, note that for each a, the Chern classes $c_1(\mathcal{L}_{j,a})$ cannot vanish for all j. It follows that the manifold S_a is Hodge, i.e., admits a Kähler metric whose Kähler class is integral in cohomology.

(iv) For any $\boldsymbol{\sigma}$ in Δ^0, the Kähler quotient metric (41) is global on S, so that $p_{\mathrm{nc}}(\eta_a) = \prod_{j=1}^{\ell}(\xi_j - \eta_a)$ does not vanish on Δ^0. Hence, no η_a can belong to any of the open intervals (β_{j-1}, β_j). Clearly, when $\eta_a = \beta_j$ for some $j = 0, \dots \ell$, $p_{\mathrm{nc}}(\eta_a)$ vanishes on the codimension one face F_j of Δ and this is precisely the condition that a blow-down occurs (over the factor S_a). The rest is immediate from the definition of generalized Calabi data apart from the normalization of the Fubini–Study metric on S_a. For this, we note that the formula (40) gives

$$c_1(\mathcal{L}_{j,a}) = -\tfrac{1}{2}c \left(\prod_{k \neq j} (\beta_j - \beta_k) \right) [\omega_a / 2\pi] \tag{49}$$

(and $c_1(\mathcal{L}_{k,a}) = 0$ for $k \neq j$). Since $\mathcal{L}_{j,a}$ has to be $\mathcal{O}(-1)$, we must have

$$[\rho_a/2\pi] = -(d_a+1)c_1(\mathcal{L}_{j,a}) = \tfrac{1}{2}c(d_a+1) \left(\prod_{k \neq j} (\beta_j - \beta_k) \right) [\omega_a/2\pi],$$

where $\rho_a = \frac{\mathrm{Scal}_a}{2d_a}\omega_a$ is the Ricci form of the Fubini–Study metric $\pm g_a$. The holomorphic sectional curvature $\pm \frac{1}{d_a(d_a+1)} \mathrm{Scal}_a$ is therefore, as stated. (Note that this can be always achieved by rescaling ω_a.)

(v) This is immediate from the explicit form of the metric, the generalized Calabi construction, and the necessity of the conditions of Theorem 3 for the compactification of orthotoric Kähler metrics on $\mathbb{C}P^\ell$.

For the converse, observe first that the integrality conditions ensure (by the Lefschetz Theorem for (1,1)-classes) that there are holomorphic line bundles $\mathcal{L}_{j,a}$ over S_a with first Chern classes given by (40), equipped with compatible connections whose curvatures are given by (48), and we define $\mathcal{L}_j = \bigotimes_{a=1}^N \pi_a^* \mathcal{L}_{j,a}$.

Let $\hat{M} = P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_\ell) \cong M^0 \times_{\mathbb{T}^c} \mathbb{C}P^\ell$, where $\mathbb{T}^c$ acts by scalar multiplication on each line bundle $\mathcal{L}_j$ modulo overall scalar multiplication on the direct sum (which acts trivially on the projective bundle), M^0 is the union of the open $\mathbb{T}^c$ orbits in each fibre of $\hat{M} \to S$, and $\mathbb{C}P^\ell$ is toric under $\mathbb{T}^c$.

By Section 1.6, (see also Theorem 2) $\hat{M}$ has a blow-down M, which collapses a family of divisors (which are closures of complex codimension one $\mathbb{T}^c$-orbits) corresponding to $\eta_a \in \mathcal{C}$ along the $\mathbb{C}P^{d_a}$ fibrations induced by the connection on $\hat{M}$.

Because of the sufficiency of the conditions of Theorem 3 for the compactification of orthotoric Kähler metrics on $\mathbb{C}P^\ell$, we have generalized Calabi data for the construction of a Kähler metric on M using Theorem 2, where $\mathcal{V} = \mathbb{C}P^\ell$ equipped with this orthotoric structure, the connection has curvature (48), and the constants are given by (44). On M^0, the Kähler structure is given by (3).

The hamiltonian 2-form $\phi = \sum_{r=1}^{\ell}(\sigma_r d\sigma_1 - d\sigma_{r+1}) \wedge dt_r$ (defined on M^0) also extends on M. Indeed, it follows from [**4**, Section 2.2] that the 2-jet of ϕ is a parallel section (over M^0) of a vector bundle with linear connection globally defined on M. Since $M \smallsetminus M^0$ has codimension at least two in M, ϕ extends to the whole of M. q.e.d.

Remark 10. It follows from the proof of Theorem 2 that M is covered by a bundle of restricted toric Kähler manifolds over $\prod_{a \notin \mathcal{C}} S_a$. The typical fibre $\mathcal{X}$ is a toric Kähler manifold of dimension $2k$, $k = \ell + \sum_{a \in \mathcal{C}} d_a$, obtained as a blow-down of a $\mathbb{C}P^\ell$ bundle over a product of $\#\mathcal{C} < \ell + 1$ projective spaces as in Section 1.6, and admits a hamiltonian 2-form of order ℓ. However, by [**4**, Section 2.4, Section 5.4], the Fubini–Study metric on $\mathbb{C}P^k$ admits a hamiltonian 2-form of order ℓ with any number of distinct constant roots between 0 and $\ell + 1$, with all factors in the Kähler quotient being blown down over some face of the Delzant polytope of $\mathbb{C}P^\ell$. It follows that $\mathcal{X}$ is biholomorphic to $\mathbb{C}P^k$, and M is covered by a bundle of projective spaces over $\prod_{a \notin \mathcal{C}} S_a$. However, the metric on the fibres need not be orthotoric unless $k = \ell$. In fact, it is not hard to see directly that the blow-down of $P(\mathcal{L}_0 \otimes \mathcal{O} \oplus \mathcal{L}_1 \otimes \mathcal{O} \oplus \cdots \oplus \mathcal{L}_j \otimes \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{L}_\ell \otimes \mathcal{O}) \to S' \times \mathbb{C}P^d$ is biholomorphic to $P(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \mathcal{L}_j \otimes \mathbb{C}^{d+1} \oplus \cdots \oplus \mathcal{L}_\ell) \to S'$.

We also note that any two compact Kähler metrics of the form (3), corresponding to the same data $(S_a, \pm g_a, \pm\omega_a), \beta_0, \ldots \beta_\ell, \eta_1, \ldots \eta_N, c$, $C_1, \ldots C_N, \Theta$ are isometric on M^0 and hence also on M, by the argument in Lemma 3. If we fix all the data except the smooth function Θ, we get Kähler metrics on the same smooth manifold M, parametrized by smooth functions Θ satisfying (42)–(43). In the context of our construction, these metrics are compatible with the same symplectic form ω. However, the corresponding complex structures are isomorphic under diffeomorphisms in the connected component of the identity [**2**], so that one can equally think of these as Kähler metrics on a fixed complex manifold (M, J), which belong to the same Kähler class $[\omega]$.

4.2. Kähler surfaces with hamiltonian 2-forms. In this subsection, we specialize to the case that (M, g, J, ω) is a smooth compact Kähler surface with a non-trivial hamiltonian 2-form; here non-trivial means that ϕ is not a constant multiple of ω. We obtain a complete classification, overcoming the issue of coverings raised in the previous subsection. We first recall that if ϕ is a non-trivial hamiltonian 2-form, then for any real numbers a, b ($a \neq 0$), the affine deformation $a\phi + b\omega$ is again a non-trivial hamiltonian 2-form (of the same order as ϕ).

Proposition 10. *Let (M, g, J) be a connected Kähler surface not of constant holomorphic sectional curvature. Then, (M, g, J) admits at most one (up to an affine deformation) non-trivial hamiltonian 2-form, even locally.*

Proof. According to [3, Lemmas 2 and 6], the primitive part ϕ_0 of a non-trivial hamiltonian 2-form ϕ defines (on the open dense subset U where $\phi_0 \neq 0$) a conformally Kähler hermitian structure (g, I), such that I and J induce opposite orientations on U; then the antiselfdual tensor W^- of g, with respect to orientation induced by J, has degenerate spectrum on U, hence on M; moreover, ϕ_0 is an eigenform of W^-, whose eigenvalue, at each point where W^- does not vanish, is the (unique) simple eigenvalue of W^-. We also know that ϕ commutes with the Ricci form ρ—see [4, Section 2.2]—so on an open subset where $\rho_0 \neq 0$, ϕ_0 is proportional to ρ_0.

It follows that on any open subset where $W^- \neq 0$ or $\rho_0 \neq 0$, the primitive parts of two non-trivial hamiltonian 2-forms, ϕ and ϕ', are related by $\phi_0 = f\phi'_0$ for a smooth function f; since the primitive part of any hamiltonian 2-form satisfies $d(\phi_0/|\phi_0|^3) = 0$ (see [3, Lemma 2]), f must be a constant, i.e., $\phi_0 = a\phi'_0$. By unique continuation [4, Section 2.2], this equality holds everywhere on M and so $\phi - a\phi'$ is a hamiltonian 2-form with vanishing primitive part, hence a multiple of ω. Thus, ϕ and ϕ' are affinely equivalent unless W^- and ρ_0 are identically zero. q.e.d.

Remark 11. The above result is optimal: according to [4, Section 2.3], each of the manifolds $\mathbb{C}P^2$, $\mathbb{C}^2$ and $\mathbb{C}\mathcal{H}^2$ endowed with its canonical Kähler structure admits a 9-dimensional family of non-trivial hamiltonian 2-forms.

Theorem 6. *Let (M, J) be a compact complex surface which supports a Kähler metric g with a non-trivial hamiltonian 2-form ϕ. Then, the following cases occur.*

(i) *ϕ is of order zero; then, (M, J) is biholomorphic to a compact locally symmetric Kähler surface of reducible type.*
(ii) *ϕ is of order one; then, (M, J) is biholomorphic to either $\mathbb{C}P^2$ or to a ruled surface of the form $P(\mathcal{O} \oplus \mathcal{L}) \to S$ where S is a compact complex curve and $\mathcal{L}$ is a holomorphic line bundle over S of positive degree.*
(iii) *ϕ is of order two; then, (M, J) is biholomorphic to $\mathbb{C}P^2$.*

Each complex surface listed in (i)–(iii) *above admits (infinitely many) Kähler metrics with non-trivial hamiltonian 2-forms of the corresponding order.*

Proof.
(i) If the order of ϕ is zero, i.e., if ϕ is parallel, then, by the deRham decomposition theorem, the universal cover $(\tilde{M}, \tilde{g})$ of (M, g) is a Kähler product $(\mathbb{U}_1 \times \mathbb{U}_2, g_1 \times g_2)$ where each $\mathbb{U}_i$ biholomorphic to $\mathbb{C}P^1$, $\mathbb{C}\mathcal{H}^1$ or $\mathbb{C}$, equipped with a Kähler metric g_i. Taking the conjugate complex structure on one of the factors defines a Kähler structure (g, I) on M, with the opposite orientation to (g, J). By a result of Kotschick [**26**], (M, J) is either a geometric complex surface [**33**] or is a minimal ruled surface.

If (M, J) is geometric Kähler surface, the fundamental group acts biholomorphically and isometrically with respect to the product of constant curvature metrics on $\mathbb{U}_i$, i.e., (M, J) carries a reducible locally symmetric Kähler structure [**34**].

If (M, J) is a minimal ruled surface, then it is biholomorphic to the total space of the projectivization $P(E)$ of a rank 2 holomorphic vector bundle E over a compact complex curve S (see, for instance, [**7**]) and so, without loss, $\mathbb{U}_1$ is the universal cover of S and $\mathbb{U}_2 = \mathbb{C}P^1$: the Kähler product metric $\tilde{g} = g_1 \times g_2$ must be compatible with the holomorphic splitting. If $\mathbb{U}_1 = \mathbb{C}P^1$ as well, then $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ so it admits a product symmetric structure. Suppose $\mathbb{U}_1 = \mathbb{C}$ or $\mathbb{C}\mathcal{H}^1$; by Liouville's Theorem, any holomorphic isometry of $(\tilde{M}, \tilde{g})$ has the form $\Psi(z, w) = (\psi_1(z), \psi_2(z, w))$, where ψ_1 is a holomorphic isometry of $(\mathbb{U}_1, g_1)$ and (for any fixed z) $w \mapsto \psi_2(z, w)$ is a holomorphic isometry of $(\mathbb{C}P^1, g_2)$. Since ψ_1 is a holomorphic automorphism of $\mathbb{U}_1$, it preserves a constant curvature metric on $\mathbb{U}_1$; similarly, since $\mathrm{Isom}(g_2)$ is a compact subgroup of $PSL(2, \mathbb{C})$, it lies in a conjugate of $PSU(2)$ and hence preserves a constant curvature metric on $\mathbb{C}P^1$. Thus, the fundamental group preserves the product of constant curvature metrics on $\mathbb{U}_1$ and $\mathbb{C}P^1$, so (M, J) is again a geometric complex surface supporting a reducible locally symmetric Kähler structure.

(ii) Suppose now that ϕ has order 1. By Theorem 5, after blowing up M at most once, we get a compact complex surface $\hat{M}$ which is a holomorphic $\mathbb{C}P^1$-bundle over a compact complex curve S, i.e., M is a ruled complex surface [**7**]. If $S \cong \mathbb{C}P^1$, then M is either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or a Hirzebruch surface $F_k = P(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{C}P^1$. Of these surfaces, only F_1 is not minimal: it is the blow-up $\mathbb{C}P^2$ at one point. We conclude that M is either $\mathbb{C}P^2$ or can be written as $P(\mathcal{O} \oplus \mathcal{O}(k)) \to \mathbb{C}P^1$, $k \in \mathbb{Z}$. If S has genus $\mathbf{g}(S) \geq 1$, by using again Theorem 5, we have $M = \hat{M}$ and therefore M is a (minimal) ruled surface $P(E)$ over a compact complex curve S, with the induced $\mathbb{C}^\times$-action tangent to the projective fibers. Clearly, in the latter case, E must be split, and so without loss, $E = \mathcal{O} \oplus \mathcal{L}$.

As a final point, we have to show that we can assume $\deg \mathcal{L} > 0$ (or $k > 0$ in the case of F_k). But this is an immediate consequence of Theorem 5: the formula (40) specializes to give (see also (48)) $c_1(\mathcal{L}) = -\frac{1}{2}c(\beta_1 - \beta_0)[\omega_S/2\pi]$, where $\beta_0 < \beta_1$ and $c > 0$, while $\pm\omega_S$ is the Kähler structure induced on stable quotient S. Thus, $\deg \mathcal{L} \neq 0$ and since $P(E) \cong P(E \otimes \mathcal{L}^*)$, we can assume that $\deg \mathcal{L} > 0$.

(iii) This is an immediate consequence of Proposition 7.

It follows from Theorem 5 that each complex surface listed in Theorem 6 does admit infinitely many (non-isometric) Kähler metrics with non-trivial hamiltonian 2-forms of the corresponding order. q.e.d.

Remark 12. The complex surfaces in Theorem 6 also admit *extremal* Kähler metrics with non-trivial hamiltonian 2-forms, see [**10**, **32**].

4.3. Examples: extremal and weakly Bochner-flat Kähler metrics.

We turn now to the construction of particular types of Kähler metrics with hamiltonian 2-forms. From this point of view, the notion of a hamiltonian 2-form is simply a device which provides constructions of interesting Kähler manifolds, and this uses very little of the theory that we have developed: the converse part of Theorem 5, which essentially amounts to the sufficiency of the conditions for the compactification of a toric Kähler metric and for the construction of Kähler metrics on blow-downs. In fact, we shall mainly restrict attention here to metrics on projective line bundles (with no blow-downs) where these issues are trivial.

We recall from [**4**] how Bochner-flat, weakly Bochner-flat and extremal Kähler metrics with hamiltonian 2-forms arise. A Kähler manifold M is *Bochner-flat* if the Bochner tensor (a component of the Kähler curvature) vanishes, *weakly Bochner-flat* (WBF) if the Bochner tensor is co-closed, and *extremal* if the scalar curvature is a Killing potential (i.e., its symplectic gradient is a Killing vector field).

By the differential Bianchi identity, a Kähler metric is WBF if and only if its normalized Ricci form $\tilde{\rho} = \rho - \frac{\mathrm{Scal}}{2(m+1)}\,\omega$ is a hamiltonian 2-form. It follows that a WBF Kähler manifold is extremal. Any Kähler–Einstein manifold is WBF, since $\tilde{\rho}$ a constant multiple of ω; however, the hamiltonian 2-form in this case is trivial. To deal with this, and the case of extremal Kähler metrics, we shall suppose that there is a non-trivial hamiltonian 2-form ϕ on M such that $\tilde{\rho} = a\phi + b\omega$ in the case of WBF Kähler metrics, and such that the scalar curvature $\mathrm{Scal} = a\,\mathrm{tr}_\omega\,\phi + b$ in the case of extremal Kähler metrics (for constants a, b).

Suppose that we have a Kähler manifold (M, g, J, ω) with a hamiltonian 2-form ϕ of order ℓ where the Kähler quotient is a product of N Kähler manifolds S_a of dimension $2d_a$, corresponding to the constant roots η_a of ϕ. The Kähler metric then has the explicit form (3) and there is the following local classification result [**4**].

(i) g is extremal, with Scal as above, if and only if
  - for all j, $F_j''(t) = \check{p}_{\mathrm{c}}(t)q(t)$, where $\check{p}_{\mathrm{c}}(t) = \prod_{a=1}^{N}(t-\eta_a)^{d_a-1}$ and q is a polynomial of degree $\ell + N$ independent of j;
  - for all a, $\pm g_a$ has constant scalar curvature $\mp q(\eta_a)/\prod_{b\neq a}(\eta_a - \eta_b)$.

  g then has constant scalar curvature if and only if q has degree $\ell + N - 1$.

(ii) g is weakly Bochner-flat, with $\tilde{\rho}$ as above, if and only if
  - for all j, $F_j'(t) = p_{\mathrm{c}}(t)q(t)$, where $p_{\mathrm{c}}(t) = \prod_{a=1}^{N}(t-\eta_a)^{d_a}$ and q is a polynomial of degree $\ell + 1$ independent of j;
  - for all a, $\pm g_a$ is Kähler–Einstein with scalar curvature $\mp d_a q(\eta_a)$.

  g is then Kähler–Einstein if and only if q has degree ℓ.

(iii) [**9**] g is Bochner-flat, with $\tilde{\rho}$ as above, if and only if
  - for all j, $F_j(t) = \hat{p}_{\mathrm{c}}(t)q(t)$ where $\hat{p}_{\mathrm{c}}(t) = \prod_{a=1}^{N}(t-\eta_a)^{d_a+1}$ and q is a polynomial of degree $\ell + 2 - N$ independent of j;
  - for all a, $\pm g_a$ has constant holomorphic sectional curvature with scalar curvature $\mp d_a(d_a+1)q(\eta_a)\prod_{b\neq a}(\eta_a - \eta_b)$.

  g has constant holomorphic sectional curvature if and only if q has degree $\ell + 1 - N$.

We want to combine this local classification with the global construction of Theorem 5. To do this, we have to satisfy the *boundary conditions* of (43), and the *integrality conditions* for the first Chern classes $c_1(\mathcal{L}_{j,a})$ given by (40).

Remark 13. In practice, we need enough freedom in the choice of $F(t)$ and the constants both to satisfy these boundary conditions and to prescribe the first Chern classes freely (up to some open conditions), since otherwise, we face potentially non-trivial diophantine problems on our data. Let us analyse the implications of this in the case that there are no blow-downs, i.e., $M = P(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_\ell) \to S$ where S is the product of compact Hodge manifolds $S_1, \dots S_N$. Thus, we have $N(\ell+1)$ integrality conditions, together with $2(\ell+1)$ boundary conditions for the function $F(t) = p_c(t)\Theta(t)$, giving $(N+2)(\ell+1)$ constraints on $F(t)$ and the constants $\beta_0, \dots \beta_\ell$, $\eta_1, \dots \eta_N$ and $c, C_1, \dots C_N$. Three of these constants, say c, β_0, β_ℓ are useless for satisfying the constraints, since there is a homethety freedom $g \mapsto kg$ in the Kähler metric and an affine freedom $\xi_j \mapsto a\xi_j + b$ in the orthotoric coordinates. We therefore have $2N+\ell-1$ effective constants. This leaves $N(\ell+1)+2(\ell+1)-2N-\ell+1 = (N+1)(\ell-1)+4$ constraints on $F(t)$. Subtracting this from the number of coefficients defining $F(t)$ gives the expected dimension of the moduli space of solutions, which we require to be non-negative.

(i) In the extremal case, $F(t)$ is determined by $\ell + 3 + N$ constants, giving $N(2-\ell)$ dimensional moduli and forcing $\ell \leq 2$.

(ii) In the WBF case, $F(t)$ is determined by $\ell + 3$ constants, giving $N(1-\ell)$ dimensional moduli and forcing $\ell \leq 1$

(iii) In the Bochner-flat case, $F(t)$ is determined by $\ell + 3 - N$ constants, giving $-N\ell$ dimensional moduli and forcing $\ell = 0$, $N \leq 2$. This paremeter count agrees with the classification of Bryant [**9**]: the only compact Bochner-flat Kähler manifolds are products of at most two constant holomorphic sectional curvature manifolds (which corresponds to the case when the normalized Ricci form is a hamiltonian form of order zero). Note that Bryant's result can be derived from our classification as follows. If, for a smooth compact Bochner-flat manifold, the normalized Ricci form is a hamiltonian 2-form of order $\ell > 0$, it would define a hamiltonian action of an ℓ-torus $\mathbb{T}$ whose complexified action has totally geodesic orbits (Lemma 7) with smooth closures biholomorphic to $\mathbb{C}P^\ell$ (Theorem 5). Since the restriction of the metric to such an orbit is Bochner-flat[**4**, **9**], it must be a Fubini-Study metric. This shows that $F(t)$ is a polynomial of degree $m+1$ (rather than $m+2$) which in turn implies $\ell = 0$, a contradiction.

We concentrate here on WBF Kähler metrics on projective line bundles, by assuming the existence of a hamiltonian 2-form ϕ of order 1. In this case, once we fix the base manifolds S_a and the line bundles $\mathcal{L}_{j,a}$, the moduli are zero dimensional.

4.3.1. The general setting. In order to render our discussion as self-contained as possible, we first recall our notations. Let $(S_a, \pm g_a, \pm\omega_a)$, $a = 1, \dots N$, be compact connected Kähler manifolds of real dimension

$2d_a$, associated to the distinct constant roots η_a of the hamiltonian 2-form. A Kähler metric with a hamiltonian 2-form of order 1 is defined on a projective line bundle over $S = S_1 \times \cdots \times S_N$, using a metric of the form

$$
\begin{aligned}
(50) \qquad g &= \sum_{a=1}^{N} (z - \eta_a) g_a + \frac{\prod_{a=1}^{N} (z - \eta_a)^{d_a}}{F(z)}\, dz^2 + \frac{F(z)}{\prod_{a=1}^{N}(z - \eta_a)^{d_a}}\, \theta^2, \\
\omega &= \sum_{a=1}^{N} (z - \eta_a)\omega_a + dz \wedge \theta, \qquad\qquad d\theta = \sum_{a=1}^{N} \omega_a,
\end{aligned}
$$

where we normalize the momentum interval for z to $[-1, 1]$ and require $|\eta_a| > 1$. Note that each Kähler metric g_a can be positive or negative definite, depending on the sign of η_a, and for convenience, it is taken here with the opposite sign to the one used in equation (3)—observe that $p_{\mathrm{nc}}(\eta_a) = \eta_a - z$ rather than $z - \eta_a$). It is convenient to set $\eta_a = -1/x_a$: now, the sign of g_a is the sign of x_a.

The projective line bundle is $M = P(\mathcal{O} \oplus \mathcal{L}) \cong P(\mathcal{O} \oplus \mathcal{L}^{-1})$, where up to a sign convention, θ is a connection form on the principal S^1-bundle associated to $\mathcal{L}$ with curvature $d\theta$. By Theorem 5, g compactifies on M when $F(z)$ satisfies the following boundary conditions (for the fibrewise compactification on $\mathbb{C}P^1$):

$$
(51) \qquad F(\pm 1) = 0, \qquad F'(\pm 1) = \mp 2 p_{\mathrm{c}}(\pm 1).
$$

For the existence of $\mathcal{L}$, we require that ω_a is integral, i.e., $[\omega_a/2\pi]$ is in the image of $H^2(S_a, \mathbb{Z})$ in $H^2(S_a, \mathbb{R})$, and we write $\mathcal{L} = \bigotimes_a \mathcal{L}_a$, where $\mathcal{L}_a$ is (the pullback to M of) a line bundle on S_a with $c_1(\mathcal{L}_a) = [\omega_a/2\pi]$.

In order to obtain WBF Kähler metrics, the S_a must be Kähler–Einstein, i.e., with Ricci form $\rho_a = s_a \omega_a$. Since $[\rho_a/2\pi]$ is an integral class, the first Chern class of the anti-canonical bundle, $s_a = p_a/q_a$ for integers p_a, q_a. If $s_a \neq 0$, we take p_a maximal so that the anti-canonical bundle has a p_ath root (i.e., $[\rho_a/2\pi p_a]$ is a primitive class); then, $\mathcal{L}_a$ is $\mathcal{K}_a^{-q_a/p_a}$ twisted by a flat line bundle. Note that any flat bundle is (holomorphically) trivial if $H^1(S_a, \mathbb{R}) = \{0\}$. In particular, such flat factors do not appear when S_a is a compact positive Kähler–Einstein manifold.

Remark 14. If S_a is a Riemann surface $\Sigma_{\mathbf{g}}$ of genus $\mathbf{g}$, then $p_a = 2|\mathbf{g} - 1|$, while if $S_a = \mathbb{C}P^{d_a}$, then $p_a = d_a + 1$ so that $\mathcal{K}^{-1/p_a} = \mathcal{O}(1)$. More generally, if the scalar curvature of S_a is positive, then $p_a \leq d_a + 1$ by Kobayashi–Ochiai [**23**].

The remaining conditions to obtain a WBF metric (as in Section 4.3(ii) above) are

$$F'(z) = p_{\mathrm{c}}(z)(b_{-1}z^2 + b_0 z + b_1), \tag{52}$$

where $p_{\mathrm{c}}(z) = \prod_{a=1}^{N}(z-\eta_a)^{d_a}$, and

$$2s_a = b_{-1}\eta_a^2 + b_0\eta_a + b_1. \tag{53}$$

Using the boundary conditions (51) and the equation (52) for F', we deduce that $b_0 = -2$ and $b_1 = -b_{-1}$. So, re-naming b_{-1} to B, equation (52) becomes

$$F'(z) = p_{\mathrm{c}}(z)\big(B(z^2-1) - 2z\big) \tag{54}$$

and (53) gives

$$B(1-x_a^2) = 2x_a(x_a s_a - 1). \tag{55}$$

g is Kähler–Einstein if and only if $B = 0$, which holds if and only if $s_a = 1/x_a$ for all a. (This implies in particular that the base factors have positive scalar curvature.)

On the other hand, given the above, then (51) is satisfied if and only if we set $F(z) = \int_{-1}^{z} p_{\mathrm{c}}(t)\big(B(t^2-1) - 2t\big)dt$ and

$$\int_{-1}^{1} p_{\mathrm{c}}(t)\big(B(t^2-1) - 2t\big)dt = 0. \tag{56}$$

Since $F'(z)$ only changes sign once on the interval $(-1,1)$, $F(z)$ as defined above will not have any zeroes between $z=-1$ and $z=1$. Therefore, as the sign of $F(z)$ equals the sign of $p_{\mathrm{c}}(z)$ between -1 and 1, the metric g will be positive definite.

So, in conclusion, the problem of constructing a WBF Kähler metric on M (for given Kähler–Einstein manifolds S_a with $s_a = p_a/q_a$) reduces to finding solutions $B, x_1, \dots x_N$ to (55) and (56). However, $p_{\mathrm{c}}(t)(1-t^2)$ has constant sign on $(-1,1)$, so B is uniquely determined by (56): substituting for B from (55) (for each a), it suffices to show that there exist distinct $(x_1, \dots x_N)$ with $0 < |x_a| < 1$ such that

$$h_a(x_1, \dots x_N) := \int_{-1}^{1} \widetilde{p_{\mathrm{c}}}(t) H_a(t)dt \tag{57}$$

vanishes for $a = 1, \dots N$, where $\widetilde{p_{\mathrm{c}}}(t) = \prod_{b=1}^{N}(x_b t + 1)^{d_b}$ and

$$H_a(t) = x_a(x_a s_a - 1)(1-t^2) + t(1-x_a^2) = x_a^2 s_a(1-t^2) + (t-x_a)(x_a t+1)$$

Remark 15. If $s_b \neq s_a$, x_b cannot equal x_a, again since $p_{\mathrm{c}}(t)(1-t^2)$ has constant sign on $(-1,1)$. Hence if $x_a = x_b$, then $s_a = s_b$ and $S_a \times S_b$ is Kähler–Einstein. Thus, we do not actually need to check that

$x_1, \dots x_N$ are distinct: if $x_a = x_b$, we still get a WBF Kähler metric, but the hamiltonian 2-form has fewer distinct constant roots.

4.3.2. WBF Kähler metrics over Kähler–Einstein manifolds. We consider the simplest case $N = 1$, when $S = S_1$ is a Kähler–Einstein manifold. Replacing the momentum coordinate z by $-z$ if necessary (and dropping the 1 subscripts), we may suppose that we have to find $0 < x < 1$ such that $h(x) = 0$, where

$$h(x) := \int_{-1}^{1} (xt + 1)^d \big(x(xs - 1)(1 - t^2) + t(1 - x^2)\big) dz.$$

Since $h(0) = 0$, $h'(0) = 2(d - 2)/3$ and the sign of $h(1)$ is equal to the sign of $s-1$, we certainly have a solution $0 < x < 1$ to $h(x) = 0$ if $d > 2$ and $s < 1$.

For the case $d = 2$, we calculate directly that

$$h(x) = \frac{4x^2}{15}\big(s(x^2 + 5) - 6x\big) \tag{58}$$

and there is a solution $0 < x < 1$ to $h(x) = 0$ if and only if $0 < s < 1$.

Theorem 7. *There are WBF Kähler metrics of the form* (50) *on:*

- $P(\mathcal{O} \oplus \mathcal{L}) \to S$, *where* S *is a compact Ricci-flat Kähler manifold of complex dimension* ≥ 3 *whose Kähler form* ω_S *is integral, and* $\mathcal{L}$ *is a holomorphic line bundle with* $c_1(\mathcal{L}) = [\omega_S/2\pi]$;
- $P(\mathcal{O} \oplus \mathcal{K}^{-q/p} \otimes \mathcal{L}_0) \to S$, *where* S *is a compact negative Kähler–Einstein manifold of complex dimension* ≥ 3, $q \in \mathbb{Z}$ *with* $q < 0$, $\mathcal{K}$ *is the canonical bundle on* S, *and* $\mathcal{L}_0$ *is a flat line bundle on* S;
- $P(\mathcal{O} \oplus \mathcal{K}^{-q/p}) \to S$, *where* S *is a compact positive Kähler–Einstein manifold of complex dimension* ≥ 2, $q \in \mathbb{Z}$ *with* $q > p$, *and* $\mathcal{K}$ *is the canonical bundle on* S.

For the case $d = 1$, we compute that

$$h(x) = -\frac{2x}{3}\big(x^2 + 1 - 2sx\big) \tag{59}$$

and there is a solution $0 < x < 1$ to $h(x) = 0$ if and only if $s > 1$. Since S in this case is $\mathbb{C}P^1$, $\mathcal{K} = \mathcal{O}(2)$ and the only possibility is $s = 2$, $\mathcal{L} = \mathcal{O}(1)$, in accordance with the classification of [**3**].

4.3.3. WBF Kähler metrics over products of two Kähler–Einstein manifolds. In this section, we give a taste of the case $N = 2$, but we postpone a more thorough analysis to a subsequent paper. In this case, we are looking for common zeros of the functions

$$h_a(x_1, x_2) := \int_{-1}^{1} (x_1 t + 1)^{d_1} (x_2 t + 1)^{d_2} \big(x_a(x_a s_a - 1)(1 - t^2) + t(1 - x_a^2)\big) dt$$

(for $a = 1, 2$) with $0 < |x_a| < 1$. Analysing this problem in general involves some delicate calculus arguments, but there are some special cases which are straightforward. One of the simplest is the case that $d_1 = d_2$, and $s_1 = -s_2$, when symmetry solves the problem for us, and we recover some of the Kähler–Einstein metrics of Koiso and Sakane.

Theorem 8. [24, 25] *On the total space of* $P(\mathcal{O} \oplus \mathcal{O}(k, -k)) \to \mathbb{C}P^d \times \mathbb{C}P^d$, *with* $1 \le k \le d$, *there is a Kähler–Einstein metric, given* (*on a dense open set*) *by*

$$g = \left(\frac{d+1}{k} + z\right) g_1 + \left(\frac{d+1}{k} - z\right) g_2 + \frac{z^2 - \frac{(d+1)^2}{k^2}}{F(z)}\, dz^2 + \frac{F(z)}{z^2 - \frac{(d+1)^2}{k^2}}\, \theta^2,$$

where (g_1, ω_1) *and* (g_2, ω_2) *are Fubini–Study metrics on the* $\mathbb{C}P^d$ *factors with holomorphic sectional curvature* $2/k$, $d\theta = \omega_1 - \omega_2$ *and* $F(z) = \int_{-1}^{z} 2t\big(\frac{(d+1)^2}{k^2} - t^2\big)\, dt = -\frac{(d+1)^2}{k^2}(1 - z^2) + \frac{1}{2}(1 - z^4)$.

Proof. Let $s_1 = -s_2 = \frac{d+1}{k}$ and $x_1 = -x_2 = \frac{k}{d+1}$. Then clearly, $0 < |x_a| < 1$ and $h_a(x_1, x_2) = 0$ for $a = 1, 2$. Further, $x_a = 1/s_a$ so the WBF metric is Kähler–Einstein. q.e.d.

In a subsequent paper, we generalize these metrics by proving the following.

Theorem 9. *There is a WBF Kähler metric on the total space* $P(\mathcal{O} \oplus \mathcal{O}(k_1, k_2)) \to \mathbb{C}P^{d_1} \times \mathbb{C}P^{d_2}$ *in the following cases*:

- $k_1 > d_1 + 1$ *and* $k_2 > d_2 + 1$;
- $1 \le k_1 \le d_1$ *and* $1 \le -k_2 \le d_2$.

We illustrate this with the case $P(\mathcal{O} \oplus \mathcal{O}(1, -2)) \to \mathbb{C}P^2 \times \mathbb{C}P^3$, where $d_1 = 2$, $d_2 = 3$, $s_1 = 3$, $s_2 = 4/(-2) = -2$. The graphs of $h_1 = 0$ (solid) and $h_2 = 0$ (dashed) for $0 < x_1 < 1$ and $-1 < x_2 < 0$ are plotted below. Proving that the graphs do cross as shown is a tedious calculus exercise.

We end this section by giving an example with a blow-down. Consider again $P(\mathcal{O} \oplus \mathcal{O}(1, -1)) \to \mathbb{C}P^1 \times \mathbb{C}P^1$. This carries a Koiso–Sakane Kähler–Einstein metric by setting $x_1 = -x_2 = 1/2$, but it also admits two blow-downs in which a $\mathbb{C}P^1$ factor collapses at an endpoint of the momentum interval $[-1, 1]$. Such a collapse corresponds to setting $x_1 = 1$ and/or $x_2 = -1$. If we carry out both blow-downs, the resulting manifold is $\mathbb{C}P^3$, which admits a WBF metric, namely the Fubini–Study metric, so let us consider the case of a single blow-down. The two

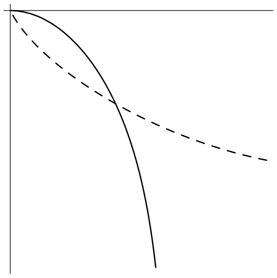


Figure 2. $d_1 = 2$, $d_2 = 3$, $s_1 = 3$, $s_2 = -2$.

complex manifolds we obtain are both isomorphic to $P(\mathcal{O}\oplus\mathcal{O}(1)\otimes\mathbb{C}^2) \to \mathbb{C}P^1$, so without loss, we suppose $x_1 = 1$, $-1 < x_2 < 0$.

This changes the boundary condition for $F'(z)$ at $z = -1$, since $p_{\mathrm{c}}(z) = (z+1)(z+1/x_2)$ vanishes at one of the endpoints. By a straightforward application of L'Hôpital's rule, we obtain

$$(F'/p_{\mathrm{c}})(-1) = 4$$

Thus, (54) is replaced by:

$$F'(z) = (z+1)(z+1/x_2)\big(B(z^2-1) - 2z + z(z-1)\big).$$

Setting $x_1 = 1$ automatically solves one of the integrality constraints with $s_1 = 2$, so it remains to show that we can find x_2 to satisfy the second constraint, with $s_2 = -2$. Proceeding as in the case of no blow-downs, this reduces to showing there is $-1 < x < 0$ with $f(x) = 0$ where

$$f(x) = \int_{-1}^{1} (t+1)(xt+1)\Big(-x(2x+1)(1-t^2) + t(1-x^2) + \tfrac{1}{2}(x+1)(t-1)(xt+1)\Big)\,dt.$$

This holds because $f(-1)$ is negative, while $f(0) = 0$ and $f'(0)$ is negative. Further, $f(-1/2)$ is non-zero. So, the solution $x = x_2$ does not equal $1/s_2$, and the metric is not Kähler–Einstein.

Theorem 10. *There is a WBF Kähler metric on* $P(\mathcal{O}\oplus\mathcal{O}(1)\otimes\mathbb{C}^2) \to \mathbb{C}P^1$ *whose normalized Ricci form is a hamiltonian* 2*-form of order one. In particular, this is an extremal Kähler metric with non-constant scalar curvature.*

4.3.4. Further extremal Kähler metrics. Since any WBF metric is extremal, the results presented so far provide new examples of extremal Kähler metrics on projective line bundles and their blow-downs. Furthermore, to obtain an extremal Kähler metric, it suffices that the base manifolds S_a are Hodge manifolds of constant scalar curvature, giving examples which are not WBF in general.

On the other hand, such an approach is not very satisfactory, since it produces only one extremal Kähler metric in each case, whereas the parameter count of Remark 13 suggests that these metrics should come in N dimensional families (parameterized by admissible Kähler classes on M). When the base manifolds have non-negative scalar curvatures, we can obtain such N dimensional families.

Theorem 11. *For $a = 1, \dots N$, let $(S_a, \pm\omega_a)$ be Hodge Kähler manifolds of constant non-negative scalar curvature, let $\mathcal{L}_a$ be a holomorphic line bundles on each S_a with $c_1(\mathcal{L}_a) = [\omega_a/2\pi]$ and let $\mathcal{L} = \bigotimes_{a=1}^N \mathcal{L}_a$. Then $M = P(\mathcal{O} \oplus \mathcal{L})$ admits an N parameter family of extremal Kähler metrics. Furthermore, if the Kähler forms $\pm\omega_a$ do not all have the same sign (i.e., if $c_1(\mathcal{L})$ is strictly indefinite) there is an $N-1$ dimensional subfamily of constant scalar curvature Kähler metrics on M.*

This Theorem generalizes results of Hwang [**20**] and Hwang–Singer [**21**], and the proof is not materially different. The first of these two papers considers the case that the base manifold has constant eigenvalues of the Ricci tensor (e.g., a product of Kähler–Einstein manifolds) and the idea to weaken this condition is explored in the second paper. However, it is the notion of a hamiltonian 2-form that has selected for us a more general hypothesis for the base. We shall discuss this, and further results, in more detail in a subsequent paper.

Finally, we note that the parameter count of Remark 13 suggests that one should be able to construct examples of extremal Kähler metrics on projective plane bundles (and their blow downs) over products of constant scalar curvature manifolds. Unfortunately, the existence problem here is considerably less tractible than in the case of WBF metrics on projective line bundles. Nevertheless, we hope to be able obtain examples in subsequent work.

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