J. DIFFERENTIAL GEOMETRY
66 (2004) 289-301

CONFORMALLY FLAT METRICS ON 4-MANIFOLDS

Michael Kapovich

Abstract

We prove that for each closed smooth spin 4-manifold M there exists a closed smooth 4-manifold N such that $M\#N$ admits a conformally flat Riemannian metric.

1. Introduction

The goal of this paper is to prove:

Theorem 1.1. *Let M^4 be an closed connected smooth spin* 4*-manifold. Then there exists a closed orientable* 4*-manifold N such that $M\#N$ admits a conformally flat Riemannian metric. The manifold N is* (*in principle*) *computable in terms of triangulation of M.*

Recall that there are many closed 4-dimensional spin manifold which admit no flat conformal structure; for instance, simply-connected manifolds (not diffeomorphic to S^4), manifolds with simple infinite fundamental group. (First examples of first 3-manifolds not admitting flat conformal structure were constructed by W. Goldman in [**5**].) The above theorem shows that if M admits a flat conformal structure, it does not imply that all components of its connected sum decomposition are also conformally flat.

Our motivation comes from the following theorem of C. Taubes [**15**]:

Theorem 1.2. *Let M be a smooth closed oriented* 4*-manifold. Then there exists a number k so that the connected sum of M with k copies of $\overline{\mathbb{CP}^2}$ admits a half-conformally flat structure.*

Here $\overline{\mathbb{CP}^2}$ is the complex-projective plane with the reversed orientation. Recall that a Riemannian metric g on M is *anti self-dual* (or half-conformally flat) if the self-dual part W_+ of the Weyl tensor vanishes. Vanishing of both self dual and anti self-dual parts of the Weyl

During the work on this paper the author was supported by the NSF grants DMS-99-71404 and DMS-02-03045.

Received 08/22/2003.

tensor (i.e., vanishing of the entire Weyl tensor) is equivalent to local conformal flatness of the metric g.

Note that the assumption that M^4 is spin is equivalent to vanishing of all Stiefel–Whitney classes, which is equivalent to triviality of the tangent bundle of $M' = M \setminus \{p\}$. According to the Hirsch-Smale theory (see for instance [**6**, Theorem 4.7] or [**13**]), $M' := M \setminus \{p\}$ is parallelizable iff M' admits an immersion into $\mathbb{R}^4$. Thus, by taking M to be simply-connected with nontrivial 2-nd Stiefel–Whitney class, one sees that $M\#N$ does not admit a flat conformal structure for *any* N: otherwise the developing map would immerse M' into S^4. Therefore the vanishing condition is, to some extent, necessary. Note also that (unlike in Taubes' theorem) one cannot expect N to be simply-connected since the only closed conformally flat simply-connected Riemannian manifold is the sphere with the standard conformal structure.

Sonjong Hwang in his thesis [**7**], has verified that for 3-manifolds an analogue of Theorem 1.1 holds, moreover, one can use a connected sum of Haken manifolds as the manifold N. Similar arguments can be used to prove an analogous theorem in the context of locally spherical CR structures on 3-manifolds.

The arguments in both 3-dimensional and 4-dimensional cases, in spirit (although, not in the technique), are parallel to Taubes': we start with a singular conformally-flat metric on M, where the singularity is localized in a ball $B \subset M$. The singular metric is obtained by pullback of the standard metric on the 4-sphere under a branched covering $M \to S^4$. Then we would like to "resolve the singularity". To do so we remove an open tubular neighborhood U of the singular locus. Afterwards, use the "orbifold trick" (cf. for instance [**3**]) to eliminate the boundary of $M \setminus U$: introduce a Möbius reflection orbifold structure on $M \setminus U$ to get a closed Möbius orbifold O which is a connected sum of M with an orbifold. After passing to an appropriate finite manifold cover over O we get a conformally-flat manifold which has M as a connected summand.

Lack of reflection groups in higher-dimensional hyperbolic spaces limits this strategy to low dimensions. Using arguments somewhat similar to the *strict hyperbolization* of Charney and Davis (see [**2**]) one can generalize Theorem 1.1 to higher-dimensional almost parallelizable manifolds. We will discuss this issue elsewhere.

Acknowledgments. Part of this work was done during the author's stay at Max-Plank-Instutit für Mathematik in Bonn, whose hospitality is gratefully acknowledged. The author is also grateful to Tadeusz Januszkiewicz who had pointed out at a flaw in the original version of the paper and to the referee of this paper for several useful remarks.

2. Definitions and notation

We let $\text{Möb}(S^4)$ denote the full group of Möbius transformations of S^4, i.e., the group generated by inversions in round spheres. Equivalently, $\text{Möb}(S^4)$ is the restriction of the full group of isometries $\text{Isom}(\mathbb{H}^5)$ to the 4-sphere S^4 which is the ideal boundary of $\mathbb{H}^5$. We will regard S^4 as 1-point compactification $\mathbb{R}^4 \cup \{\infty\}$ of then Euclidean 4-space.

Definition 2.1. Let Q be a unit cube in $\mathbb{R}^4$. We define the PL *inversion* J in the boundary of Q as follows: let $h : S^4 \to S^4$ be a PL homeomorphism which sends $\Sigma = \partial Q$ onto the round sphere $S^3 \subset \mathbb{R}^4$ and $h(\infty) = \infty$. Let $j : S^4 \to S^4$ be the ordinary inversion in S^3. Then $J := h^{-1} \circ j \circ h$.

Definition 2.2. A *Möbius* or a *flat conformal structure* on a smooth 4-manifold M is an atlas $\{(V_\alpha, \varphi_\alpha), \alpha \in A\}$ which consists of diffeomorphisms $\varphi_\alpha : V_\alpha \to U_\alpha \subset S^4$ so that the transition mappings $\varphi_\alpha \circ \varphi_\beta^{-1}$ are restrictions of Möbius transformations.

Equivalently, one can describe Möbius structures on M as conformal classes of conformally-Euclidean Riemannian metrics on M. Each conformal structure on M gives rise to a local conformal diffeomorphism, called a *developing map*, $d : \widetilde{M} \to S^4$, where $\widetilde{M}$ is the universal cover of M. If M is connected, the mapping d is equivariant with respect to a *holonomy representation* $\rho : \pi_1(M) \to \text{Möb}(S^4)$, where $\pi_1(M)$ acts on $\widetilde{M}$ as the group of deck-transformations. Given a pair (d, ρ), where ρ is a representation of $\pi_1(M)$ into $\text{Möb}(S^4)$ and d is a ρ-equivariant local diffeomorphism from $\widetilde{M}$ to S^4, one constructs the corresponding Möbius structure on M by taking a pullback of the standard flat conformal structure on S^4 to $\widetilde{M}$ via d and then projecting the structure to M.

Analogously, one defines a *complex-projective structure* on complex 3-manifold Z, as a $\mathbb{CP}^3$-valued holomorphic atlas on Z so that the transition mappings belong to $\text{PGL}(3, \mathbb{C})$.

The concept of Möbius structure generalizes naturally to the category of orbifolds:

A 4-dimensional *Möbius orbifold* O is a pair $(X, \mathcal{A})$, where X is a Hausdorff topological space, the *underlying space of the orbifold*, $\mathcal{A}$ is a family of local parameterizations $\psi_\alpha : U_\alpha \to U_\alpha/\Gamma_\alpha = V_\alpha$, where $\{V_\alpha, \alpha \in A\}$ is an open covering of X, U_α are open subsets in S^4, Γ_α

are finite groups of Möbius automorphisms of U_α and the mappings ψ_α satisfy the usual compatibility conditions:

If $V_\alpha \to V_\beta$ is the inclusion map then we have a Möbius embedding $U_\alpha \to U_\beta$ which is equivariant with respect to a monomorphism $\Gamma_\alpha \to \Gamma_\beta$, so that the diagram

$$\begin{array}{ccc} U_\alpha & \to & U_\beta \\ \downarrow & & \downarrow \\ V_\alpha & \to & V_\beta \end{array}$$

is commutative. The groups Γ_α are the *local fundamental groups* of the orbifold O.

We refer the reader to [**11**] for a more detailed discussion of geometric structures on orbifolds.

For a Möbius orbifold one defines a developing mapping $d : \widetilde{O} \to S^4$ (which is a local homeomorphism from the universal cover $\widetilde{O}$ of O) and, if O is connected, a holonomy homeomorphism $\rho : \pi_1(O) \to \text{Möb}(S^4)$, which satisfy the same equivariance condition as in the manifold case. Again, given a pair (d, ρ), where d is a ρ-equivariant homeomorphism, one defines the corresponding Möbius structure via pullback.

Example 2.3. Let $G \subset \text{Möb}(S^4)$ be a subgroup acting properly discontinuously on an open subset $\Omega \subset S^4$. Then the quotient space Ω/G has a natural Möbius orbifold structure. The local charts ϕ_α appear in this case as restrictions of the projection $p : \Omega \to \Omega/G$ to open subsets with finite stabilizers.

Example 2.4. In particular, suppose that G is a discrete subgroup of $\text{Möb}(S^4)$ generated by reflections in faces of a spherical polyhedron $D \subset S^4$. Let Ω denote a G-invariant component of the domain of discontinuity of G. Then the quotient *reflection orbifold* $Q := \Omega/G$ can be identified with the intersection $D \cap \Omega$.

Example 2.5. Another example is obtained by taking a manifold M and a local homeomorphism $h : M \to S^4$, so that $Q \subset h(M)$. Then we can pull back the Möbius orbifold structure on Q to an appropriate subset X of M, to get a 4-dimensional Möbius orbifold. As another example of a pullback construction, let O be a Möbius orbifold and $M \to O$ be an orbifold cover such that M is a manifold. Then one can pull back the Möbius orbifold structure from O to an ordinary Möbius structure on M.

Although the Möbius structures on 4-dimensional manifolds and orbifolds constructed in this paper definitely do not arise as reflection orbifolds and pullbacks, the above Examples 2.4 and 2.5 will appear as "building blocks" in our construction.

Idea of the construction. We first produce a class of compact spherical polyhedra $D \subset S^4$ which have *prescribed combinatorics*, i.e., the nerve of the associated family of round balls in $\mathbb{R}^4$ is the 1-st barycentric subdivision of the prescribed 2-dimensional finite subcomplex of the standard cubulation of $\mathbb{R}^4$; this is done in Section 3. We then construct a certain open 4-dimensional manifold W and a local diffeomorphism $h : W \to S^4$. Taking the pullback of the Möbius orbifold structure from $D = \Omega/G$ to W via h we obtain a Möbius orbifold O whose underlying set X_O is a compact submanifold with boundary in W. This is done in Section 4. The open manifold W (and the manifold with boundary X_O) are constructed in such a way that the given smooth compact spin-manifold M appears as a connected summand of W and of X_O. Therefore the orbifold O is the connected sum of the manifold M with a certain closed orbifold. This proves an orbifold version of Theorem 1.1. To prove Theorem 1.1 we construct a finite manifold cover $\widetilde{O}$ over O to which the submanifold $M \setminus B^4 \subset O$ lifts homeomorphically. Then $\widetilde{O}$ is a conformally-flat closed 4-manifold which contains M as a connected summand.

3. Reflection groups in S^4 with prescribed combinatorics of the fundamental domains

Consider the standard cubulation $\mathcal{Q}$ of $\mathbb{R}^4$ by the Euclidean cubes with the edges of length 2 and let X denote the 2-skeleton of this cubulation. Given a collection of round balls $\{B_i, i \in I\}$ in $\mathbb{R}^4$, with the nerve $\mathcal{N}$, we define the canonical simplicial mapping $f : \mathcal{N} \to \mathbb{R}^4$ by sending each vertex of $\mathcal{N}$ to the center of the corresponding ball and extending f linearly to the simplices of $\mathcal{N}$. For a subcomplex $K \subset X \subset \mathcal{Q}$ define its *barycentric subdivision* $\beta(K)$ to be the following simplicial complex: subdivide each edge of K by its midpoint. Then subdivide each 2-cube Q in K by coning off the barycentric subdivision of ∂Q from the center of Q, see Figure 1 for the barycentric subdivision of the 2-cube.

Proposition 3.1. *Suppose that $K \subset X$ is a 2-dimensional compact subcomplex such that each vertex belongs to a 2-cell. Then there exists a*

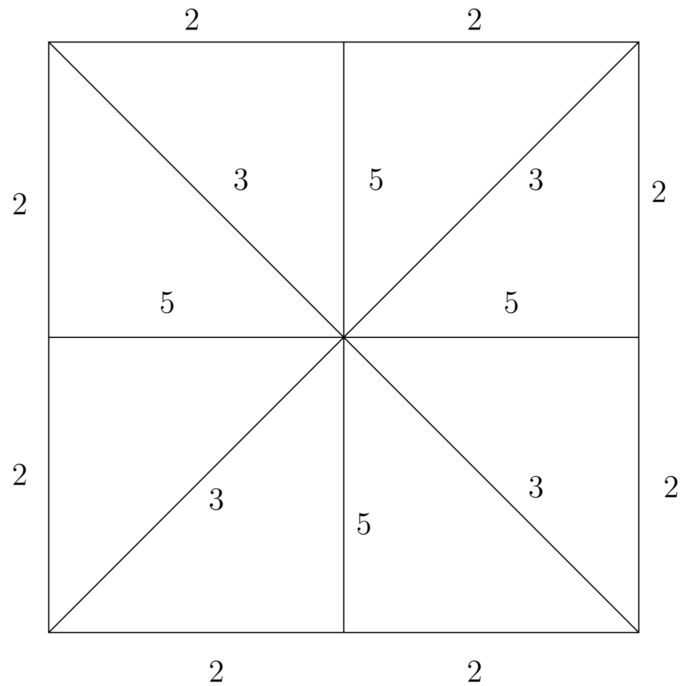

Figure 1. Barycentric subdivision of a square.

collection of open round 4*-balls* B_i, $i = 1, \ldots, k$, *centered at the vertices of* $\beta(K)$, *so that:*

(1) *The Möbius inversions* R_i *in the round spheres* $S_i = \partial B_i$ *generate a discrete subgroup* $G \subset \text{Möb}(S^4)$.
(2) *The complement* $S^4 \setminus \cup_{i=1}^k B_i$ *is a fundamental domain* Φ *of* G.
(3) *The canonical mapping from the nerve of* $\{B_i, i = 1, \ldots, k\}$ *to* $\mathbb{R}^4$ *is a simplicial isomorphism onto* $\beta(K)$.

Proof. We begin by constructing the family of spheres $S_i, i \in \mathbb{N}$ centered at certain points of X. For each square Q in X we pick 9 points $x_1, \ldots, x_9$: $x_5, \ldots, x_8$ are the vertices of Q, $x_1, \ldots, x_4$ are midpoints of the edges of Q and x_9 is the center of Q. Pick radii r, R, ρ so that:

(a) The spheres $S_1 = S(x_1, r), S_5 = S(x_5, R)$ are orthogonal.
(b) The (exterior) angle of intersection between the spheres $S(x_1, r)$ and $S_9 = S(x_9, \rho)$ equals $\pi/3$.

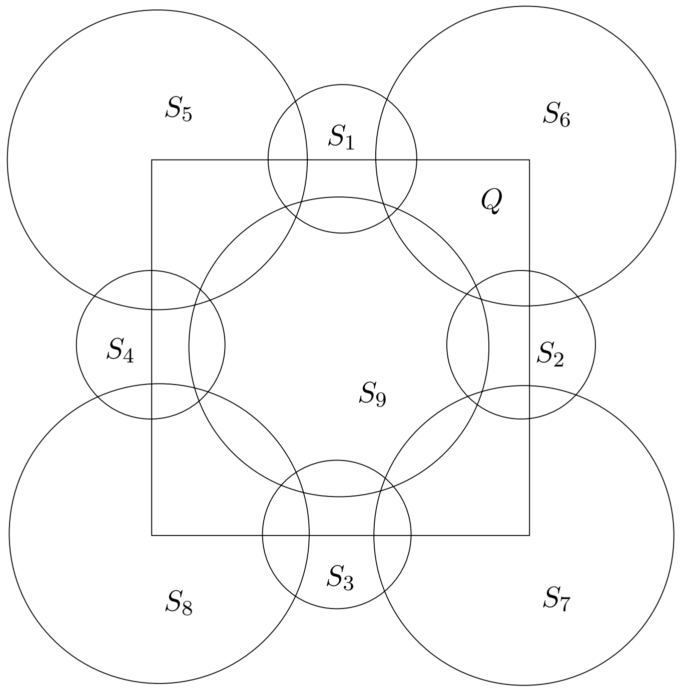

Figure 2.

(c) The (exterior) angle of intersection between the spheres $S(x_5, R)$ and $S(x_9, \rho)$ equals $\pi/5$. (Actually the latter angle can be taken $\pi/4$ as well, but $\pi/3$ would not suffice.)

Then for the radii r, R, ρ we get:

$$R \approx 0.8534646790, r \approx 0.5211506901, \rho \approx 0.6317819089.$$

In particular, r and ρ are both less than $1/\sqrt{2} \approx 0.7071067810$.

We then consider the collection of round balls $B(x_9, \rho)$, $B(x_i, r)$, $i = 1, \dots, 4$ and $B(x_i, R)$, $i = 4, \dots, 8$; see Figure 2. The condition $r < \sqrt{2}/2$ and our choice of the angles of intersection between the spheres imply that the nerve of the above collection of balls is the barycentric subdivision of Q. See Figure 1 for the Coxeter graph of the Coxeter group generated by inversions in the spheres $S_1, \dots, S_9$.

Suppose now that Q^4 is a 4-cube in the cubulation $\mathcal{Q}$, apply the above construction to each 2-face of Q^4. Then the condition $\rho, r < \sqrt{2}/2$ implies that the the nerve $\mathcal{N}_{Q^4}$ of the resulting collection of balls $\{B_i\}$

is such that the canonical mapping $\mathcal{N}_{Q^4} \to \beta((Q^4)^{(2)})$ is a simplicial isomorphism.

Now we are ready to construct the covering $\{B_i : i = 1, \ldots, k\}$ of the 2-complex K. For each 2-face Q of K introduce the family of nine round spheres S_i constructed above, consider the inversions R_i is these spheres; the spheres S_i bound balls $\{B_i : i = 1, \ldots, k\}$. The fact that for each 4-cube Q^4 the mapping $\mathcal{N}_{Q^4} \to \beta((Q^4)^{(2)})$ is a simplicial isomorphism, implies that the mapping from the nerve of the covering $\{B_i : i = 1, \ldots, k\}$ to $\beta(K)$ is a simplicial isomorphism as well. Thus the exterior angles of intersections between the spheres equal $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{5}$, thus we can apply Poincare's fundamental polyhedron theorem [10] to ensure that the intersection of the complements to the balls B_i is a fundamental domain for the Möbius group G generated by the above reflections.

q.e.d.

Remark 3.2. Instead of collections of round balls based on a cubulation of $\mathbb{R}^4$ one could use a periodic triangulation of $\mathbb{R}^4$, however in this case the construction of a collection of balls covering the 2-skeleton of a 4-simplex would be more complicated.

4. Proof of Theorem 1.1

Recall that the manifold M is almost parallelizable, i.e $M^\circ = M \setminus \{p\}$ is parallelizable; hence, by [6], there exists an immersion $f : M^\circ \to \mathbb{R}^4$. Let B denote a small open round ball centered at p and let M' denote the complement $M \setminus B$. We retain the notation f for the restriction $f|M'$. We next convert to the piecewise-cubical setting: consider the pullback of the standard cubulation $\mathcal{Q}$ of $\mathbb{R}^4$ to M' via f. The image $f(M')$ is compact; hence, after replacing if necessary $\mathcal{Q}$ by its sufficiently fine cubical subdivision, we may assume that for each point $x \in M'$ there exists a cube $Q_x \in \mathcal{Q}$ and a component $\widetilde{Q}_x \subset M^\circ$ of $f^{-1}(Q)$ so that $f|\widetilde{Q}_x : \widetilde{Q}_x \to Q_x$ is a homeomorphism. In particular, if $C \subset M'$ denotes the union of the cubes $\widetilde{Q}_x, x \in M'$, then C is a compact cubical complex. After replacing B by a slightly larger open ball B' we get: $\partial B' \subset \mathrm{int}(C)$. Let $M^\bullet$ denote the complement $M \setminus B'$. Now consider the 2-nd cubical subdivision C'' of C and the regular neighborhood $N := N(\mathrm{Fr}_M(C))$ in C'' of the frontier $\mathrm{Fr}_M(C)$: the frontier of N in C is a 3-dimensional submanifold $Y \subset B'$ which is contained in the 3-skeleton of C''. Thus $f(Y)$ is also contained in the 3-skeleton of $\mathcal{Q}''$. Let M'' denote the closure of the component of $M \setminus Y$ which is disjoint from

p. Clearly, M'' is a compact cubulated manifold with the boundary Y. We retain the notation f for the restriction $f|M''$. We now double M'' across its boundary Y, the result is a closed cubulated manifold DM, let $\tau : DM \to DM$ denote the involution fixing Y pointwise; the mapping f extends to $DM \setminus M''$ by $f \circ \tau$. Thus we get a globally defined piecewise-linear map $F : DM \to \mathbb{R}^4$, which is a homeomorphism on each 4-cube in DM and is a local diffeomorphism on $M^\bullet$. By cutting DM along the sphere $\partial M^\bullet$ we get a connected sum decomposition $DM = M\#W$. We orient DM so that F preserves the orientation on $M^\bullet$.

We now borrow the standard arguments from the proof of Alexander's theorem which states that each closed n-dimensional PL manifold is a branched cover over the n-sphere, see e.g., [**4**]. For each cube $Q' \subset DM$ such that $F|Q'$ is orientation-reversing we replace $F|Q'$ with the composition $J \circ F|Q'$, where J is the PL *inversion* in the boundary of the unit cube $F(\partial Q')$ (see Definition 2.1). The resulting mapping $h : DM \to S^4$ has the property that it is a local PL homeomorphism away from a 2-dimensional subcomplex $L \subset DM \setminus M''$, which is therefore disjoint from $M^\bullet$. (Note that L has dimension 2 near every point: each vertex in L belongs to a 2-cube in L.) Thus the mapping h is a branched covering over S^4 with the singular locus L contained in $DM \setminus M^\bullet$, the branch-locus of h is the compact subcomplex $K = h(L) \subset \mathbb{R}^4$. The branched covering h has the property that for each point $x \in K$ there exists a neighborhood $U(x) \subset \mathbb{R}^4$ such that $h^{-1}(U(x))$ is a disjoint union of balls $V(y), y \in h^{-1}(x) \in DM \setminus M''$, (whose interiors contain y), so that for each $y \in h^{-1}(x)$, the restriction $h|V(y)$ is a branched covering onto $U(x)$. Moreover, each branched covering $h|V(y)$ is obtained by coning off a branched covering from the 3-sphere $\partial V(y)$ to the 3-sphere $U(x)$.

Question 4.1. Given a local diffeomorphism $f : M^\circ \to \mathbb{R}^4$, is it possible to modify f within the ball B to make it a branched cover $M \to S^4$ which is ramified over a smooth surface in S^4? This can be easily arranged in the case of 3-manifolds. In dimension 4 this would be a relative version of a recent theorem of Iori and Piergallini [**8**].

Let T denote a regular neighborhood of K in $\mathbb{R}^4$, so that $U(x) \subset T$ for each $x \in K$. Next, subdivide the cubulation of $\mathbb{R}^4$ and scale the subdivision up to the standard unit cubulation, so that the discrete group G and the collection of balls $\{B_j, j = 1, \dots, k\}$ associated with the subcomplex K in Section 3 have the properties:

1. $T \subset \cup_{i=1}^k B_k$.

2. Each ball B_j, $j = 1, \dots, k$, (centered at $x_j \in K$) is contained in the neighborhood $U(x_j)$.

We set $W := DM \setminus L$; this is an open submanifold of DM which admits a local diffeomorphism $h : W \to S^4$. Observe that M appears as a connected summand of W. By pullback via h we of course get a Möbius structure on W, our goal however is to produce a Möbius structure on a compact orbifold. To do so we consider a compact submanifold with boundary $X_O \subset W$ which is the complement $DM \setminus \mathcal{N}^0(L)$, where $\mathcal{N}^0(L)$ is an open tubular neighborhood of the subcomplex $L \subset DM$.

We now use the branched covering h to introduce a Möbius orbifold structure O on the space X_O as follows:

For each ball $B_j \subset U(x_j)$ centered at $x_j \in K$ and for each $y_j \in h^{-1}(x_j) \cap L$, such that the restriction $h|V(y_j)$ is *not a homeomorphism onto its image*, we let $\widetilde{B}(y_j)$ denote the inverse image $h^{-1}(B_j) \cap V(y_j)$. It follows that each $\widetilde{B}(y_j)$ is a polyhedral 4-ball in M and the union of these balls is a tubular neighborhood $\mathcal{N}(L)$ of L. The boundary of $\mathcal{N}(L)$ has a natural partition into subcomplexes: "vertices", "edges", "2-faces" and "3-faces":

- The "vertices" are the points of triple intersections of the 3-spheres $\partial\widetilde{B}(y_j)$, $\partial\widetilde{B}(y_i)$, $\partial\widetilde{B}(y_l)$.
- The "2-faces" are the connected components of the double intersections of the 3-spheres $\partial\widetilde{B}(y_j)$, $\partial\widetilde{B}(y_i)$.
- The "3-faces" are the connected components of the complements

$$\partial\widetilde{B}(y_j) \setminus \cup_{i \neq j} \widetilde{B}(y_i).$$

We declare each "3-face" a boundary reflector of the orbifold O. The dihedral angles between the balls B_j define the dihedral angles between the boundary reflectors in O. Since the restriction $h|M \setminus L$ is a local homeomorphism, this construction defines a Möbius orbifold O. The mapping $h|X_O$ is the projection of the developing mapping $\widetilde{h} : \widetilde{O} \to S^4$ of this Möbius orbifold. Let $O^\bullet$ denote the orbifold with boundary $O \setminus M^\bullet$; let O' be the closed orbifold obtained by attaching 4-disk D^4 to $O^\bullet$ along the boundary sphere S^3.

We next convert back to the smooth category. It is clear from the construction that the orbifold O is obtained by (smooth) gluing of the manifold with boundary $M \setminus B$ and the orbifold with boundary $O^\bullet$. Hence O is diffeomorphic to the connected sum of the manifold M with the orbifold O'.

It remains to construct a finite manifold covering $\hat{M}$ over the orbifold O, so that $M \setminus B$ lifts homeomorphically to $\hat{M}$; the construction is analogous to the one used by M. Davis in [**3**]. The universal cover $\widetilde{O}$ is a manifold since it admits a (locally homeomorphic) developing mapping to S^4. The fundamental group $\pi_1(O)$ is the free product $\pi_1(M) * \pi_1(Q)$. We have holonomy homomorphism

$$\phi : \pi_1(O) \to G,$$

the subgroup $\pi_1(M)$ is contained in the kernel of this homomorphism; by construction, the kernel of ϕ acts freely on $\widetilde{O}$. The Coxeter group G is virtually torsion-free, let $\theta : G \to A$ be a homomorphism onto a finite group A, so that $\mathrm{Ker}(\theta)$ is torsion-free and orientation-preserving. Then the kernel of the homomorphism $\psi = \theta \circ \phi : \pi_1(O) \to A$ is a finite index subgroup of $\pi_1(O)$, which contains $\pi_1(M)$ and still acts freely on $\widetilde{O}$. Let $\hat{M} \to O$ denote the finite orbifold cover corresponding to the subgroup $\mathrm{Ker}(\psi)$. Then $\hat{M}$ is a smooth oriented conformally flat manifold, the submanifold $M^\bullet$ lifts diffeomorphically to $M^\bullet \subset \hat{M}$. Thus the connected sum decomposition $O = M \# O'$ also lifts to $\hat{M}$, so that the latter manifold is diffeomorphic to the connected sum of M and a 4-manifold N. q.e.d.

We observe that the proof of Theorem 1.1 can be modified to prove the following:

Theorem 4.2. *Suppose that M is a closed smooth 4-manifold whose orientable 2-fold cover is Spin. Then there exists a closed smooth 4-manifold N so that $\hat{M} = M \# N$ admits a conformally-Euclidean Riemannian metric.*

Proof. The difference with Theorem 1.1 is that M can be nonorientable. Let $\widetilde{M} \to M$ be the orientable double cover with the deck-transformation group $D \cong \mathbb{Z}/2$. Then all Stiefel–Whitney classes of $\widetilde{M}$ are trivial. As before, let $p \in M$, $\{p_1, p_2\}$ be the preimage of $\{p\}$ in M. Consider a Euclidean reflection τ in $\mathbb{R}^4$ and an epimorphism $\theta : D \to \langle \tau \rangle$. Then, arguing as in the proof of Hirsch's theorem [**6**, Theorem 4.7], one gets a θ-equivariant immersion $\widetilde{f} : \widetilde{M} \setminus \{p_1, p_2\} \to \mathbb{R}^4$. This yields a D-invariant flat conformal structure on $\widetilde{M} \setminus \{p_1, p_2\}$ via pullback of the flat conformal structure from $\mathbb{R}^4$. Let $B_1 \sqcup B_2$ be a D-invariant disjoint union of open balls around the points p_1, p_2. Then the rest of the proof of Theorem 1.1 goes through: replace the ball B_1 with a manifold with boundary N_1 so that the flat conformal structure on

$\widetilde{M} \setminus (B_1 \cup B_2)$ extends over N_1. Then glue a copy of N_1 along the boundary of B_2 in D-invariant fashion. Note that the quotient of the manifold $P := (\widetilde{M} \setminus (B_1 \cup B_2)) \cup (N_1 \cup N_2)$ by the group D is diffeomorphic to a closed manifold $M \# N$, where N is obtained from N_1 by attaching the 4-ball along the boundary. Finally, project the D-invariant Möbius structure on P to a Möbius structure on the manifold $M \# N$. q.e.d.

As a corollary of Theorem 1.1 we get:

Corollary 4.3. *Let* Γ *be a finitely-presented group. Then there exists a* 3*-dimensional complex manifold* Z *which admits a complex-projective structure, so that the fundamental group of* Z *splits as* $\Gamma * \Gamma'$.

Proof. Our argument is similar to the one used to construct 3-dimensional complex manifolds with the prescribed finitely-presented fundamental group. We first construct a smooth closed oriented 4-dimensional spin manifold M with the fundamental group Γ. This can be done for instance as follows: let $\langle x_1, \dots, x_n | R_1, \dots, R_\ell \rangle$ be a presentation of Γ. Consider a 4-manifold X which is the connected sum of n copies of $S^3 \times S^1$. This manifold is clearly spin. Pick a collection of disjoint embedded smooth loops $\gamma_1, \dots, \gamma_\ell$ in X, which represent the conjugacy classes of the words $R_1, \dots, R_\ell$ in the free group $\pi_1(X)$. Consider the pair (S^4, γ), where γ is an embedded smooth loop in S^4. For each i pick a diffeomorphism f_i between a tubular neighborhood $T(\gamma)$ of γ in S^4 and a tubular neighborhood $T(\gamma_i)$ of γ_i in X. We can choose f_i so that it matches the spin structures of $T(\gamma)$ and $T(\gamma_i)$. Now, attach n copies of $S^4 \setminus T(\gamma)$ to $X \setminus \cup_i T(\gamma_i)$ via the diffeomorphisms f_i. The result is a smooth spin 4-manifold M with the fundamental group Γ.

Next, by Theorem 1.1 there exists a smooth 4-manifold N (with the fundamental group Γ') such that $\hat{M} = M \# N$ admits a conformally-Euclidean Riemannian metric. Applying the twistor construction to the manifold $\hat{M}$ we get a complex 3-manifold Z which is an S^2-bundle over $\hat{M}$ and the flat conformal structure on $\hat{M}$ lifts to a complex-projective structure on Z, see [**1**]. Clearly, $\pi_1(Z) \cong \pi_1(\hat{M}) = \Gamma * \Gamma'$. q.e.d.

Remark 4.4. We refer the reader to the papers [**12**], [**9**], [**14**] for further discussion of complex-projective structures on higher-dimensional manifolds.

References

[1] M.F. Atiyah, N.J. Hitchin & I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362(1711)** (1978) 425–461, MR 0506229 (80d:53023), Zbl 0389.53011.

[2] R. Charney & M. Davis, *Strict hyperbolization*, Topology **34** (1995) 329–350, MR 1318879 (95m:57034), Zbl 0826.53040.

[3] M. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117** (1983) 293–324, MR 0690848 (86d:57025), Zbl 0531.57041.

[4] M. Feighn, *Branched covers according to J.W. Alexander*, Collect. Math. **37**(**1**) (1986) 55–60, MR 0882791 (88d:57002), Zbl 0615.57006.

[5] W. Goldman, *Conformally flat manifolds with nilpotent holonomy*, Transactions of AMS **278** (1983) 573–583, MR 0701512 (84i:53043), Zbl 0518.53041.

[6] M. Hirsch, *The imbedding of bounding manifolds in Euclidean space*, Ann. of Math. (2) **74** (1961) 494–497, MR 0133136 (24 #A2970), Zbl 0102.38604.

[7] S. Hwang, *Möbius structures on* 3*-manifolds*, Ph. D. Thesis, University of Utah, 2001.

[8] M. Iori & R. Piergallini, 4*-manifolds as covers of the* 4*-sphere branched over non-singular surfaces*, Geom. Topol. **6** (2002) 393–401, MR 1914574 (2003f:57007), Zbl 1021.57003.

[9] F. Lárusson, *Compact quotients of large domains in complex projective space*, Ann. Inst. Fourier (Grenoble) **48**(**1**) (1998) 223–246, MR 1614953 (99d:32035), Zbl 0912.32020.

[10] B. Maskit, *Kleinian groups*, Grundlehren der Mathematischen Wissenschaften, **287**, Springer Verlag, 1988, MR 0959135 (90a:30132), Zbl 0627.30039.

[11] Y. Matsumoto & J. Montesinos, *A proof of Thurston's uniformization theorem of geometric orbifolds*, Tokyo J. Math. **14** (1991) 181–196, MR 1108165 (92g:57005), Zbl 0732.57015.

[12] M. Nori, *The Schottky groups in higher dimensions*, in 'The Lefschetz centennial conference, Part I' (Mexico City, 1984), Contemp. Math., **58**, 195–197, MR 0860414 (88c:32017), Zbl 0597.32026.

[13] A. Phillips, *Submersions of open manifolds*, Topology **6** (1967) 171–206, MR 0208611 (34 #8420), Zbl 0204.23701.

[14] J. Seade & A. Verjovsky, *Higher-dimensional complex Kleinian groups*, Math. Ann. **322**(**2**) (2002) 279–300, MR 1893917 (2003c:32022), Zbl 1022.37032.

[15] C. Taubes, *The existence of anti-self-dual conformal structures*, J. Differential Geom. **36** (1992) 163–253, MR 1168984 (93j:53063), Zbl 0822.53006.

Department of Mathematics
University of California
Davis, CA 95616

E-mail address: kapovich@math.ucdavis.edu