J. DIFFERENTIAL GEOMETRY
**66** (2004) 211-232

# CONVERGENCE OF THE KÄHLER–RICCI FLOW ON NONCOMPACT KÄHLER MANIFOLDS

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## Abstract

We study the Kähler–Ricci flow on noncompact Kähler manifolds and provide conditions under which the flow has a long time solution converging to a complete negative Kähler–Einstein metric. We also study the complex parabolic Monge–Ampère equation.


## 1. Introduction

In [15] Yau provided necessary and sufficient conditions for a compact Kähler manifold to admit a Kähler–Einstein metric with either zero or negative scalar curvature. Under these conditions, Cao ([2]) was able to prove that a Kähler metric converges to a Kähler–Einstein metric under the normalized Kähler–Ricci flow. It would be interesting to know the extent to which this result can be generalized to complete noncompact Kähler manifolds. The main results generalizing Yau's work in this direction appear in ([4],[5], [12]) in the case of negative scalar curvature, and in ([13],[14]) in the case of zero scalar curvature. In this paper we determine sufficient conditions under which a complete noncompact Kähler manifold $(M, g_{i\bar{j}})$ converges to a complete Kähler–Einstein manifold with negative scalar curvature under the Kähler–Ricci flow:

$$\begin{aligned} \frac{d\widetilde{g}_{i\bar{j}}}{dt} &= -\widetilde{R}_{i\bar{j}} + \widetilde{g}_{i\bar{j}}, \\ \widetilde{g}_{i\bar{j}}(x,0) &= g_{i\bar{j}}. \end{aligned} \tag{1.1}$$

Received 09/19/2002.

We also study the following type of complex parabolic Monge–Ampère equations on $(Mg_{i\bar{j}})$:

$$\frac{du}{dt} = \log \frac{\det(g_{k\bar{l}} + u_{k\bar{l}})}{\det(g_{k\bar{l}})} - u - f, \qquad u(x,0) = 0, \tag{1.2}$$

for appropriate choices of the function $f$. Our main result is:

**Theorem 1.1.** *Assume $(M, g_{i\bar{j}})$ is a complete noncompact Kähler manifold with bounded curvature and that $R_{i\bar{j}}+g_{i\bar{j}} = f_{i\bar{j}}$ for some smooth bounded function $f$ on $M$. Then (1.2) has a long time smooth solution $u(x,t)$ which converges, as $t \to \infty$, on every compact subset of $M$, to a smooth limit $u(x,\infty)$. Moreover, $\widetilde{g}_{i\bar{j}}(x,t) := g_{i\bar{j}}(x) + u_{i\bar{j}}(x,t)$ provides a long time smooth solution to (1.1) and converges, as $t \to \infty$, on every compact subset of $M$, to a complete Kähler–Einstein metric $\widetilde{g}_{i\bar{j}}(x,\infty)$ where $\widetilde{g}_{i\bar{j}}(x,\infty)$ has negative scalar curvature, is equivalent to $g_{i\bar{j}}$, and has all covariant derivatives of its curvature tensor bounded.*

Analogous analytic results are obtained in ([**4**], [**5**], [**12**]) for the corresponding complex elliptic Monge–Ampère equation, and examples of complete noncompact Kähler–Einstein manifolds of negative scalar curvature are provided there. Their examples are either pseudocovex domains in $\mathbb{C}^n$, or the complement of a divisor in a compact projective variety.

Using Shi's basic theory for (1.1), established in ([**9**], [**11**]), we will derive short time existence and basic theory for (1.2). In doing this we will see how the smoothing effect of the Ricci flow allows us to avoid various Hölder type conditions typically required when dealing with the complex elliptic Monge–Ampère equation. We will then follow the approach in [**2**]; to adapt the a priori estimates for the complex elliptic Monge–Ampère equation to the parabolic case. We will also formulate higher-order a priori estimates for (1.2) which are necessary in the noncompact setting. These higher-order estimates correspond to the curvature estimates formulated by Shi [**11**]. Most of our estimates will be proved by the parabolic maximum principles of Shi ([**9**], [**11**]) for noncompact manifolds.

The paper is organized as follows: in Section 2 we establish some analytic preliminaries as well as state Shi's short time existence result for (1.1). In Section 3 we derive (1.2) and show its equivalence to (1.1).

Short time existence for (1.2) is proved in Section 4 and a priori estimates for (1.2) are proved in Section 5. Finally, we prove Theorem 1.1 in Section 6.

**Acknowledgments.** The author would like to thank his Ph.D. thesis advisor M. Kuranishi for his advice and his enduring support and encouragement, R. Hamilton and J. Loftin for many helpful discussions and S.T. Yau for overall support in preparing this paper.

## 2. Preliminaries

**2.1. Parabolic Hölder spaces.** The natural setting for the Monge–Ampère equation on a general Kähler manifold $(M, g_{i\bar{j}})$ is the elliptic Hölder space defined relative to the metric (see [**4**], [**12**], [**15**]). We will use these Hölder spaces, as well as their parabolic versions, to apply standard Schauder estimates and to provide appropriate norms to state our convergence results. In this section we define the parabolic and elliptic Hölder spaces of a noncompact Kähler manifold.

**Definition 2.1.** Let $m$ be a positive integer and $\alpha \in (0, 1)$. A complex $n$-dimensional Kähler manifold $(M, g_{i\bar{j}})$ is said to have bounded geometry of order $m + \alpha$ if there are numbers $r_1, r_2, k_1, k_2, C > 0$ such that for every $p \in M$:

1) There is a neighborhood $U_p$ of $p$ and a holomorphic covering map $\xi_p : V_p \to U_p$ where $V_p \subset \mathbb{C}^n$, $B_{r_1}(0) \subseteq V_p \subseteq B_{r_2}(0)$ and $\xi_p(0) = p$.
2) $k_1\delta_{a\bar{b}} \le \xi_p^* g_{a\bar{b}} \le k_2\delta_{a\bar{b}}$ on $V_p$.
3) For all $a, b$ we have $\|\xi_p^* g_{a\bar{b}}\|_{p,m+\alpha} \le C$ where $\|\cdot\|_{p,m+\alpha}$ is the standard $C^{m+\alpha}$ Hölder norm on $V_p \in \mathbb{C}^n$.

$(M, g_{i\bar{j}})$ is said to have bounded geometry of order $\infty$ if it has bounded geometry of order $m + \alpha$ for every $m$. Let $(M, g_{i\bar{j}})$ have bounded geometry of order $m + \alpha$ and let $[0, T)$ be an arbitrary time interval. For some choice of maps $\xi_p$ as in Definition 2.1, consider the following norm for any smooth function $u$ on $M \times [0, T)$:

$$\|u\|_{m+\alpha,m/2+\alpha/2} := \sup_{p\in M}\{\|\xi_p^* u\|_{p,m+\alpha,m/2+\alpha/2}\}, \tag{2.1}$$

where $\xi_p^* u$ is the pull back of $u$ to $V_p$ and $\|\cdot\|_{p,m+\alpha,m/2+\alpha/2}$ is the standard parabolic Hölder norm on $V_p \times [0, T)$. The following definition is independent of the choice of $\xi_p' s$.

**Definition 2.2.** : Let $(M, g_{i\bar{j}})$ be a complete Kähler manifold with bounded geometry of order $m + \alpha$. With respect to (2.1), we define the parabolic Hölder spaces $C^{m+\alpha,m/2+\alpha/2}(M \times [0,T))$ to be the closure of the set of all smooth functions $u(x,t) : M \times [0,T) \longmapsto \mathbb{R}$ for which (2.1) is finite. Also, given $(M, g_{i\bar{j}})$ above, one can define the elliptic Hölder spaces $C^{m+\alpha}(M)$ in an obvious way.

$C^{m+\alpha,m/2+\alpha/2}(M \times [0,T))$ with the norm (2.1) for some choice of maps $\xi_p$, and $C^{m+\alpha}(M)$ with the analogous elliptic norm, are easily checked to be Banach spaces.

**2.2. Short time existence for Kähler–Ricci flow.** The Ricci flow was first introduced by Hamilton [**6**] to evolve a real Riemannian metric with positive Ricci curvature to one of constant positive curvature on a compact three manifold. In [**6**] Hamilton showed that on an arbitrary smooth compact Riemanian manifold, the Ricci flow always has a short time solution. Under the hypothesis of bounded sectional curvature, Shi [**9**] extended this result to complete noncompact Riemannian manifolds and later applied this result to the case of Kähler manifolds ([**10**] [**11**]). We will use the following version of Shi's short time existence theorem for the Kähler–Ricci flow.

**Theorem 2.3.** *Let $(M, g_{i\bar{j}})$ be an $n$-dimensional complete noncompact Kähler manifold with Riemannian curvature tensor* Rm *satisfying*

$$\sup_{x\in M} \|\mathrm{Rm}(x)\|_g \leq K,$$

*where $K$ is a positive constant. Then for some constant $T > 0$ depending only on $n$ and $K$, there is a smooth short time solution $\widetilde{g}_{i\bar{j}}(x,t)$ to the following Kähler–Ricci flow equation on $M$:*

$$\begin{aligned} \frac{d\widetilde{g}_{i\bar{j}}}{dt} &= -\widetilde{R}_{i\bar{j}} - \widetilde{g}_{i\bar{j}}, \\ \widetilde{g}_{i\bar{j}}(x,0) &= g_{i\bar{j}}, \end{aligned} \tag{2.2}$$

*for $t \in [0,T)$. Moreover, for all $t \in [0,T)$, $\widetilde{g}_{i\bar{j}}(x,t)$ is a Kähler metric on $M$, equivalent to $g_{i\bar{j}}$, and satisfies the following estimates for the covariant derivatives of its curvature tensor:*

$$\sup_{x\in M} \|\widetilde{\nabla}^m \widetilde{\mathrm{Rm}}(x,t)\|^2_{\widetilde{g}} \leq C_{n,m,k}(1/t)^m, \tag{2.3}$$

*where the covariant differentiation and the norm is with respect to the metric $\widetilde{g}_{i\bar{j}}(x,t)$ and $C_{n,m,k}$ is a constant depending only on $n, m$ and $k$.*

**Remark 2.4.** (2.2) is a normalization of the Kähler–Ricci flow equation treated in [**9**] and [**11**].

**2.3. Maximum principles.** We now state some maximum principles for parabolic equations on a noncompact Riemannian manifold.

**Lemma 2.5.** *For $t \in [0, T)$, Let $g_{i\bar{j}}(x,t)$ be a family of complete Riemannian metrics on a noncompact manifold $M$ such that:*

1) *$g_{i\bar{j}}(x,t)$ varies smoothly in $t$.*
2) *For all $t$, $g_{i\bar{j}}(x,t)$ is equivalent to $g_{i\bar{j}}(x,0)$.*
3) *For all $t$, $g_{i\bar{j}}(x,t)$ has bounded curvature.*

*Suppose $f(x,t)$ is a smooth bounded function on $M \times [0,T)$ such that:*

1) *$f(x,0) \geq 0$,*
2) *$df/dt = \Delta_t f + Q(f,x,t)$,*
3) *$Q(f,x,t) \geq 0$ whenever $f \leq 0$,*

*where $\Delta_t$ denotes the Laplacian of the metric $g_{i\bar{j}}(x,t)$. Then $f(x,t) \geq 0$ on $M \times [0,T)$.*

*Proof.* See [**9**] and [**11**]. q.e.d.

**Lemma 2.6.** *Let $(M, g_{i\bar{j}}(x,t))$ satisfy the hypothesis of Lemma 2.5. Suppose $f(x,t)$ is a smooth bounded function on $M \times [0,T)$ such that:*

1) *$df/dt = \Delta_t f + Q(f,x,t)$,*
2) *$Q(f,x,t) \leq -f^\alpha + K$,*

*where $K$ and $\alpha$ are positive constants. Then $f(x,t) \leq C_{K,\alpha}$ on $M \times [0,T)$ for some constant $C_{K,\alpha}$ depending only on $K$, $\alpha$ and $\sup_{x\in M} |f(x,0)|$.*

*Proof.* Let $C = \max\{\sup_{x\in M} |f(x,0)|, K^{1/\alpha}\}$, and consider the function $\psi := C - f$. Then $\psi$ satisfies the first and second condition of the Lemma 2.5. Moreover, we have

$$\begin{aligned} d\psi/dt &= -df/dt \\ &= -\Delta_t f - Q(f,x,t) \\ &= \Delta_t \psi - Q(f,x,t) \\ &\geq \Delta_t \psi + f^\alpha - K. \end{aligned} \tag{2.4}$$

It follows that when $\psi \leq 0$ we have $f \geq C$ and thus $-Q(f,x,t) \geq f^\alpha - K \geq C^\alpha - K \geq 0$. Thus the evolution of $\psi$ satisfies the third condition of Lemma 2.5 and $\psi$ remains positive for all $t$. From this, and the the definition of $\psi$, the lemma is readily seen to be true. q.e.d.

At times, we will use versions of the maximum principle which are slightly different from Lemma 2.5 and Lemma 2.6, but which can also be derived from Lemma 2.5 by an argument similar to that above. We will omit these derivations.

## 3. The parabolic Monge–Ampère equation

Let $(M, g_{i\bar{j}})$ be a complete noncompact Kähler manifold satisfying the following condition:

$$g_{i\bar{j}} + R_{i\bar{j}} = f_{i\bar{j}}, \tag{3.1}$$

for some smooth function $f$ on $M$. We would then like to find a smooth function $u$ on $M$ such that the tensor

$$\widetilde{g}_{i\bar{j}} := g_{i\bar{j}} + u_{i\bar{j}}, \tag{3.2}$$

is a complete negative Kähler–Einstein metric on $M$. We will refer to such a function $u$ as a negative Kähler–Einstein potential for $(M, g_{i\bar{j}})$. We now derive the parabolic equation (1.2) on $M$ and show that its stationary solutions are precisely given by negative Kähler–Einstein potentials for $(M, g_{i\bar{j}})$.

Recall that $\widetilde{g}_{i\bar{j}}$ is Kähler–Einstein if its Ricci tensor satisfies

$$\widetilde{R}_{i\bar{j}} = \rho \widetilde{g}_{i\bar{j}}, \tag{3.3}$$

for some constant $\rho$, which in our case, we assume to be $-1$. Also recall that on any Kähler manifold $(M, g_{i\bar{j}})$ the Ricci tensor $R_{i\bar{j}}$ is given locally by

$$R_{i\bar{j}} = -[\log(\det g_{k\bar{l}})]_{i\bar{j}}. \tag{3.4}$$

It follows from (3.1), (3.3) and (3.4) that $u$ is a negative Kähler–Einstein potential for $(M, g_{i\bar{j}})$ iff $u$ satisfies

$$0 = \left[\log \frac{\det(g_{k\bar{l}} + u_{k\bar{l}})}{\det(g_{k\bar{l}})} - u - f\right]_{i\bar{j}} \tag{3.5}$$

on $M$, and thus, a sufficient condition for $u$ to be a negative Kähler–Einstein potential for $(M, g_{i\bar{j}})$ is that

$$0 = \log \frac{\det(g_{k\bar{l}} + u_{k\bar{l}})}{\det(g_{k\bar{l}})} - u - f. \tag{3.6}$$

on $M$. We point out that necessity in (3.6) holds (up to a constant) in the the case that $M$ is compact, while in the case $M$ is noncompact it does not. This results from the possible existence of nontrivial pluriharmonic functions on a noncompact Kähler manifold. (3.6) is a special case of the complex Monge–Ampère equation and has been solved on compact Kähler manifolds in [**15**] and [**1**] and on noncompact Kähler manifolds in [**4**] and [**12**] under special conditions.

It follows, from (3.3)–(3.6), that the stationary solutions to (1.2) are negative Kähler–Einstein potentials, and if $u$ is a smooth solution to (1.2), then $\widetilde{g}_{i\bar{j}} := g_{i\bar{j}} + u_{i\bar{j}}$ defines a family of Kähler matrics on $M$ evolving by (1.1). We refer to [**2**] for a treatment of (1.2) in the case of compact manifolds.

## 4. Short time existence

We now prove short time existence for (1.2).

**Lemma 4.1.** *Let $(M, g_{i\bar{j}})$ be a complete noncompact Kähler manifold such that:*

1) $\sup_{x\in M} \|\mathrm{Rm}(x)\|_g \leq K$ *for some constant* $K > 0$.
2) $R_{i\bar{j}} + g_{i\bar{j}} = f_{i\bar{j}}$ *for some smooth bounded function $f$ on $M$.*

*Then for some constant $T > 0$, depending only on $n$ and $K$, (1.2) has a smooth solution $u(x,t)$ for $t \in [0,T)$. Moreover, for every $t \in [0,T)$ we have:*

1) *The Kähler metric $\hat{g}_{i\bar{j}} := g_{i\bar{j}}(x) + u_{i\bar{j}}(x,t)$ is equivalent to $g_{i\bar{j}}$ and has bounded geometry of infinite order.*
2) $\hat{R}_{i\bar{j}} + \hat{g}_{i\bar{j}} = \hat{f}_{i\bar{j}}$ *where $\hat{R}_{i\bar{j}}$ is the Ricci curvature tensor of $\hat{g}_{i\bar{j}}$ and $\hat{f}$ is a smooth bounded function on M.*

*Proof.* Starting with $(M, g_{i\bar{j}})$, let $\widetilde{g}_{i\bar{j}}(x,t)$ and $T$ be as in Theorem 2.3. We will show that $\widetilde{g}_{i\bar{j}}(x,t)$ gives us a smooth solution $u(x,t)$ to (1.2) on $[0,T)$. To show this we note first that $\forall x \in M$ the following ODE in $t$ has a smooth solution $u(x,t)$ for $t \in [0,T)$:

$$\frac{d}{dt}u(x,t) = \log \frac{\det(\widetilde{g}_{k\bar{l}})(x,t)}{\det(g_{k\bar{l}})(x)} - u(x,t) - f(x), \tag{4.1}$$

$$u(x,0) = 0.$$

By the equivalence of the metrics $\widetilde{g}_{i\bar{j}}(x,t)$ and $g_{i\bar{j}}$ for all $t \in [0,T)$, the logarithmic term in (4.1) is seen to be a smooth bounded function on $M \times [0,T)$ thus making (4.1) an ODE with smooth bounded terms. Consider now the difference tensor $S_{i\bar{j}} := (\widetilde{g}_{i\bar{j}}(x,t) - g_{i\bar{j}}(x)) - u_{i\bar{j}}(x,t)$. It is easily checked that $S_{i\bar{j}}$ satisfies the following evolution equation:

$$\frac{dS_{i\bar{j}}}{dt} = -S_{i\bar{j}}, \tag{4.2}$$

$$S_{i\bar{j}}(x,0) := 0,$$

on $M \times [0,T)$. Since the solution to (4.2) is unique, the zero solution, we conclude that $\widetilde{g}_{i\bar{j}}(x,t) = g_{i\bar{j}}(x) + u_{i\bar{j}}(x,t)$. Finally, by substituting

$g_{i\bar{j}}(x) + u_{i\bar{j}}(x,t)$ for $\widetilde{g}_{i\bar{j}}(x,t)$ in (4.1), we see that $u(x,t)$ is in fact a smooth solution to (1.2) on $M \times [0,T)$.

Now fix an arbitrary $T' \in [0,T)$ and let $\hat{g}_{i\bar{j}} := \widetilde{g}_{i\bar{j}}(x,T')$. We want to show now that $\hat{R}_{i\bar{j}} + \hat{g}_{i\bar{j}} = \hat{f}_{i\bar{j}}$ where $\hat{R}_{i\bar{j}}$ is the Ricci tensor of $\hat{g}_{i\bar{j}}$ and $\hat{f}$ is some smooth bounded function on $M$. By taking applying the $\partial\bar{\partial}$ operator to (4.1) we get

$$(u_t(x,T'))_{i\bar{j}} = -\widetilde{R}_{i\bar{j}}(x,T') - \widetilde{g}_{i\bar{j}}(x,T'). \tag{4.3}$$

Also, by differentiating (1.2) in $t$ we get

$$\frac{du_t}{dt} = \widetilde{\Delta} u_t - u_t, \tag{4.4}$$
$$u_t(x,0) = -f(x).$$

where $\widetilde{\Delta}$ is the Laplace operator for the metric $\widetilde{g}_{i\bar{j}}(x,t)$. Applying the maximum principle to (4.4), we see that $u_t(x,t)$ begins as a bounded function on $M$ and remains uniformly bounded on M $\forall t \in [0,T)$. By (4.3) we see that $u_t(x,T')$ is precisely the function $\hat{f}$ we are looking for.

To complete the proof of Lemma 4.1 we need to show that $\hat{g}_{i\bar{j}}$ has bounded geometry of infinite order. In [**13**] the authors prove that on a noncompact Kähler manifold, one has bounded geometry of order $2+\alpha$ provided one has bounded curvature and gradient of scalar curvature. Their proof can be extended to show that one has bounded geometry of infinite order provided one has all covariant derivatives of curvature bounded. Thus since $\hat{g}_{i\bar{j}}$ has all covariant derivatives of its curvature bounded by Theorem (2.3), we see that $\hat{g}_{i\bar{j}}$ in fact has bounded geometry of order infinity. This completes the proof of Lemma 4.1. q.e.d.

## 5. A priori estimates

Assume $(M, g_{i\bar{j}})$ satisfies the hypothesis of Theorem 1.1 with the additional assumption of bounded geometry of order $\infty$. By the previous sections we know that on some time interval $[0,T)$, (1.2) has a smooth solution $u(x,t)$, and $\widetilde{g}_{i\bar{j}} := g_{i\bar{j}} + u_{i\bar{j}}$ solves (1.1). We will show that $u$ stays bounded in every Hölder norm on $(M, g_{i\bar{j}})$ independent of $T$. This will be done by establishing several a priori estimates.

### 5.1. Estimates for $u$ and $u_t$.

Differentiating (1.2) in $t$ we get

$$\frac{du_t}{dt} = \widetilde{\Delta} u_t - u_t, \tag{5.1}$$
$$u_t(x,0) = -f.$$

Applying the maximum principle to (5.1), we get that $\sup_{x\in M}|u_t(x,t)|$ starts off bounded and continues to decay exponentially:

$$\sup_{x\in M}|u_t(x,t)| \le Ke^{-t}, \tag{5.2}$$

for some constant $K$ independent of $T$. From this we may bound $\sup_{(x,t)\in M\times[0,T)}|u(x,t)|$, independent of $T$, by the following estimate:

$$|u(x,t)| = \left|\int_0^t u_s(x,s)ds\right| \le K\int_0^t e^{-s}ds \le C, \tag{5.3}$$

where $C$ is some constant independent of $T$.

**5.2. Estimates for $\boldsymbol{\Delta u}$.** Consider the quantity

$$A = \log(n+\Delta u) - ku, \tag{5.4}$$

defined on $M\times[0,T)$, where we choose the constant $k$ later. Clearly a bound on $|A|$ implies a bound on $|\Delta u|$. We will bound $A$ from above using the maximum principle. The bound from below will follow directly from some simple inequalities derived from (1.2). We begin by computing $\frac{dA}{dt}$ and $\widetilde{\Delta}A$ separately as

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{(n+\Delta u)}\frac{d\Delta u}{dt} - k\frac{du}{dt}, \\ -\widetilde{\Delta}A &\le -(k-C)\widetilde{g}^{\lambda\bar{\mu}}g_{\lambda\bar{\mu}} - \frac{1}{(n+\Delta u)}\left(\Delta u + \Delta f + \Delta\frac{du}{dt}\right) + nk, \end{aligned} \tag{5.5}$$

where $C$ depends only on the curvature of the initial metric $g_{i\bar{j}}$. We estimate some of the terms on the right side of (5.5).

$$\begin{aligned} (n+\Delta u) &\ge \left[\frac{\det \widetilde{g}_{i\bar{j}}}{\det g_{i\bar{j}}}\right]^{\frac{1}{n}} = e^{\frac{1}{n}(u+f+\frac{du}{dt})}, \\ \widetilde{g}^{\lambda\bar{\mu}}g_{\lambda\bar{\mu}} &\ge \frac{1}{n-1}\left[(n+\Delta u)e^{\frac{1}{n}(-u-f-\frac{du}{dt})}\left(\frac{1}{n-1}\right)^{1-n}\right]^{\frac{1}{n-1}}. \end{aligned} \tag{5.6}$$

The details of (5.5) and (5.6) can be found in [**1**]. Note that by (5.2), (5.3) and (1.2), (5.6) already gives a bound on $\inf_{(x,t)\in M\times[0,T)}A(x,t)$ independent of $T$. Substituting estimates (5.6) back into (5.5) gives the following:

$$
\begin{aligned}
(5.7)\quad \frac{dA}{dt} &\leq \widetilde{\Delta} A + \frac{1}{(n+\Delta u)}\frac{d\Delta u}{dt} - k\frac{du}{dt} - (k-C)\widetilde{g}^{\lambda\bar{\mu}}g_{\lambda\bar{\mu}} \\
&\quad - \frac{1}{(n+\Delta u)}\left(\Delta u + \Delta f + \Delta\frac{du}{dt}\right) + nk \\
&\leq \widetilde{\Delta} A - (k-C)\widetilde{g}^{\lambda\bar{\mu}}g_{\lambda\bar{\mu}} - \frac{1}{(n+\Delta u)}(\Delta u + \Delta f) + nk - k\frac{du}{dt} \\
&\leq \widetilde{\Delta} A - (k-C)\widetilde{g}^{\lambda\bar{\mu}}g_{\lambda\bar{\mu}} - \frac{1}{(n+\Delta u)}(-n + \Delta f) + nk - k\frac{du}{dt} \\
&\leq \widetilde{\Delta} A + e^{-\frac{1}{n}(u+f+\frac{du}{dt})}|n+\Delta f| + nk - k\frac{du}{dt} \\
&\quad - \frac{k-C}{n-1}\left[e^{\frac{1}{n}\left(-u-f-\frac{du}{dt}\right)}\left(\frac{1}{n-1}\right)^{1-n}\right]^{\frac{1}{n-1}}(n+\Delta u)^{\frac{1}{n-1}}.
\end{aligned}
$$

Choosing $k$ sufficiently large, we have

$$
\begin{aligned}
(5.8)\qquad \frac{dA}{dt} &\leq \widetilde{\Delta} A - N_1(n+\Delta u)^{\frac{1}{n-1}} + N_2 \\
&\leq \widetilde{\Delta} A - N_1 e^{\frac{A+ku}{n-1}} + N_2 \\
&\leq \widetilde{\Delta} A - N_3 e^{\frac{A}{n-1}} + N_2,
\end{aligned}
$$

where the constants $N_i$ are positive and independent of $T$. The bound on $A$ from above now follows from the maximum principle. From this and our previous bound on A from below we obtain a bound on $sup_{(x,t)\in M\times[0,T)}|\Delta u(x,t)|$ independent of $T$. We now show that these bounds imply the equivalence of the metrics $g_{i\bar{j}}$ and $\widetilde{g}_{i\bar{j}}$ where the factor of equivalence is independent of $T$.

Consider any point $q \in M$ and any time $t_0 \in [0,T)$. Then at this point in time consider orthonormal holomorphic local coordinates $z_i$ at $q$ such that:

1) $g_{i\bar{j}} = \delta_{ij}$,
2) $\widetilde{g}_{i\bar{j}}(q,t) = g_{i\bar{j}}(q) + u_{i\bar{j}}(q,t) = 0$; for $i \neq j$.

By our estimates above we have

$$
\begin{aligned}
(5.9)\qquad & \widetilde{g}_{i\bar{i}}(q,t) \geq 0; \quad \forall i, \\
& \sum_i \widetilde{g}_{i\bar{i}}(q,t) = (n+\Delta u)(q,t) \leq C_1, \\
& \prod_i \widetilde{g}_{i\bar{i}}(q,t) = \frac{\det(\widetilde{g}_{k\bar{l}}(q,t))}{\det(g_{k\bar{l}}(q))} = e^{(u+f+\frac{du}{dt})} \geq C_2,
\end{aligned}
$$

where the constants $C_i$ are independent of $q$ and $T$. We see that these inequalities provide us with the following equivalences:

$$K_1 \leq \widetilde{g}_{i\bar{i}}(q,t) \leq K_2; \quad \forall i, \tag{5.10}$$

where the constants $K_i$ are independent of $q$ and $T$. This provides, for all $t \in [0,T)$, the uniform equivalence of the metrics $g_{i\bar{j}}$ and $\widetilde{g}_{i\bar{j}} := g_{i\bar{j}} + u_{i\bar{j}}$, where the uniformity is independent of $T$.

**5.3. Estimates for third derivatives of mixed type.** Consider the quantity

$$Q = \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\gamma}} u_{,\alpha\bar{\mu}\nu} u_{,\bar{\beta}\lambda\bar{\gamma}}, \tag{5.11}$$

defined on $M \times [0,T)$, where the covariant differentiation is in the original metric $g_{i\bar{j}}$ and $\widetilde{g}^{a\bar{b}}$ represents the inverse of the time dependent metric $\widetilde{g}_{\alpha\bar{\beta}}$. By the previous subsection, this norm is equivalent to that using the original metric $g^{\alpha\bar{\beta}}$. We will apply the maximum principle to the evolution of $Q$. We begin by noting the following expansions, the details of which can be found in [1]:

(5.12)

$$\begin{aligned}
\frac{dQ}{dt} &= -\widetilde{g}^{c\bar{d}}(2\widetilde{g}^{\alpha\bar{\delta}}\widetilde{g}^{\gamma\bar{\beta}}\widetilde{g}^{a\bar{b}} + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{a\bar{\delta}}\widetilde{g}^{\gamma\bar{b}}) u_{,c\bar{b}\alpha} u_{,\bar{d}a\bar{\beta}} (u_t)_{\gamma\bar{\delta}} \\
&\quad + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{a\bar{b}}\widetilde{g}^{c\bar{d}} \left[ u_{,\bar{d}a\bar{\beta}}(u_t)_{,c\bar{b}\alpha} + u_{,c\bar{b}\alpha}(u_t)_{,\bar{d}a\bar{\beta}} \right], \\
\widetilde{\Delta} Q &= \widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{a\bar{b}}\widetilde{g}^{c\bar{d}} [(u_{,c\bar{b}\alpha\bar{\mu}} - u_{,\bar{b}\gamma\bar{\mu}} u_{,c\bar{\delta}\alpha}\widetilde{g}^{\gamma\bar{\delta}})(u_{,\bar{d}a\bar{\beta}\lambda} - u_{,\bar{d}\bar{\delta}\bar{\beta}} u_{,a\gamma\lambda}\widetilde{g}^{\gamma\bar{\delta}}) \\
&\quad + (u_{,c\bar{b}\alpha\lambda} - u_{,\rho\bar{b}\lambda} u_{,c\bar{\nu}\alpha}\widetilde{g}^{\rho\bar{\nu}} - u_{,c\bar{\nu}\lambda} u_{,\alpha\bar{b}\rho}\widetilde{g}^{\rho\bar{\nu}}) \times \\
&\qquad \cdot (u_{,\bar{d}a\bar{\beta}\bar{\mu}} - u_{,\rho a\bar{\mu}} u_{,\bar{d}\bar{\nu}\bar{\beta}}\widetilde{g}^{\rho\bar{\nu}} - u_{,\bar{d}\bar{\nu}\bar{\mu}} u_{,\bar{\beta}a\rho}\widetilde{g}^{\rho\bar{\nu}})] \\
&\quad - \widetilde{g}^{c\bar{d}}(2\widetilde{g}^{\alpha\bar{\delta}}\widetilde{g}^{\gamma\bar{\beta}}\widetilde{g}^{a\bar{b}} + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{a\bar{\delta}}\widetilde{g}^{\gamma\bar{b}}) u_{,c\bar{b}\alpha} u_{\bar{d}a\bar{\beta}} [(u_t + u + f)_{\gamma\bar{\delta}} - R_{\gamma\bar{\delta}}] \\
&\quad + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{a\bar{b}}\widetilde{g}^{c\bar{d}} [u_{,\bar{d}a\bar{\beta}}(u_t + u + f)_{,c\bar{b}\alpha} + u_{,c\bar{b}\alpha}(u_t + u + f)_{,\bar{d}a\bar{\beta}}] \\
&\quad + \widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{a\bar{b}}\widetilde{g}^{c\bar{d}} [u_{,\bar{d}a\bar{\beta}}(R^{\nu}_{c\lambda\bar{b}} u_{,\nu\bar{\mu}\alpha} + R^{\bar{\rho}}_{\bar{b}\bar{\mu}\alpha} u_{,c\bar{\rho}\lambda} + R^{\nu}_{c\bar{\mu}\alpha} u_{,\nu\bar{b}\lambda}) \\
&\quad + u_{,c\bar{b}\alpha}(R^{\nu}_{\bar{d}\bar{\mu}c} u_{,\nu\lambda\bar{\beta}} + R^{\bar{\rho}}_{a\lambda\bar{\beta}} u_{,\bar{d}\bar{\rho}\bar{\mu}} + R^{\nu}_{\bar{d}\lambda\bar{\beta}} u_{,\nu c\bar{\mu}})] \\
&\quad + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{c\bar{d}} [u_{,\bar{d}a\bar{\beta}}(\widetilde{g}^{\lambda\bar{\mu}} R^{a}_{c\bar{\mu}\alpha,\lambda} - \widetilde{g}^{a\bar{b}} R_{c\bar{b},\alpha}) \\
&\qquad \cdot u_{,c\bar{b}\alpha}(\widetilde{g}^{\lambda\bar{\mu}} R^{a}_{\bar{d}\lambda\bar{\beta},\bar{\mu}} - \widetilde{g}^{a\bar{b}} R_{\bar{d}c,\bar{\beta}})].
\end{aligned}$$

We note that in $\frac{dQ}{dt} - \widetilde{\Delta} Q$, all terms involving $\frac{du}{dt}$ cancel, and all terms containing fourth derivatives of $u$ can be collected as above to give negative terms, while all other terms are contractions involving second and/or third mixed derivatives of $u$ and $f$, the metric $\widetilde{g}_{i\bar{j}}$ and the curvature tensor of the initial metric $g_{i\bar{j}}$. Thus by our previous estimates on the second derivatives of $u$ we have

$$\frac{dQ}{dt} \leq \widetilde{\Delta} Q + C_1 Q + C_2 \sqrt{Q}, \tag{5.13}$$

where the constants $C_i$ are positive and independent of $T$. The evolution of $Q$ does not admit to the standard maximum principle argument since $C_1$ is positive. To fix this we consider the modified quantity

$$Q' = Q + h\Delta u, \tag{5.14}$$

where we determine the constant $h$ later. Certainly, a bound on $Q'$ will provide a bounding on $Q$. We proceed by computing the evolution of $\Delta u$. The following estimate can be found in [1]:

$$\begin{aligned} \widetilde{\Delta}\Delta u &= \widetilde{g}^{\alpha\bar{\mu}}\widetilde{g}^{\gamma\bar{\beta}} u_{,\gamma\bar{\mu}}^{,\nu} u_{,\alpha\bar{\beta}\nu} + \Delta \frac{du}{dt} + \Delta u + \Delta f + E \\ &\geq BQ + \Delta \frac{du}{dt} + \Delta u + \Delta f + E, \end{aligned} \tag{5.15}$$

where the constant $B$ and the term $E$ is bounded on $M \times [0, T)$ independent $T$. Equivalently, we write this as an evolution for $\Delta u$ as

$$\frac{d\Delta u}{dt} \leq \widetilde{\Delta}(\Delta u) - BQ - \Delta u - \Delta f - E. \tag{5.16}$$

This provides the following estimate for the evolution of $Q'$:

$$\begin{aligned} \frac{dQ'}{dt} &\leq \widetilde{\Delta} Q' + C(Q + \sqrt{Q}) - BhQ - h\Delta u - h\delta f - hE \\ &\leq \widetilde{\Delta} Q' + (C - hB)Q + C\sqrt{Q} - h\Delta u - h\delta f - hE. \end{aligned} \tag{5.17}$$

Choosing the constant $h$ such that $(C - hB)$ is negative, we can apply the maximum principle to the above equation to conclude that $\sup_{(x,\,t) \in M \times [0,\,T)} Q'(x, t)$, and thus also $\sup_{(x,t)\in M\times[0,T)} Q(x, t)$, is bounded independent of $T$.

**5.4. Estimates for fourth derivatives of mixed type.** Consider the quantity

$$Q = \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}} u_{,\alpha\bar{\sigma}\lambda\bar{\rho}} u_{,\bar{\beta}\delta\bar{\mu}\nu}, \tag{5.18}$$

defined on $M \times [0, T)$, where the covariant differentiation is in the original metric $g_{i\bar{j}}$ and $\widetilde{g}^{a\bar{b}}$ represents the inverse of the time dependent metric $\widetilde{g}_{\alpha\bar{\beta}}$. We begin by computing the evolution of $Q$:

(5.19)
$$\begin{aligned}\frac{dQ}{dt} &= \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}(u_t)_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu} + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}(u_t)_{,\bar{\beta}\delta\bar{\mu}\nu}\\ &+ \widetilde{g}_t^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu} + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}_t^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu}\\ &+ \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}_t^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu} + \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}_t^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu}.\end{aligned}$$

Expanding the term $\widetilde{g}_t^{\alpha\bar{\beta}}$ gives

$$\begin{aligned}\widetilde{g}_t^{\alpha\bar{\beta}} &= -\widetilde{g}^{\alpha\bar{l}}\widetilde{g}^{k\bar{\beta}}u_{t\bar{l}k}\\ &= -\widetilde{g}^{\alpha\bar{l}}\widetilde{g}^{k\bar{\beta}}\left[\log\frac{\det(g_{i\bar{j}}+u_{i\bar{j}})}{\det(g_{i\bar{j}})} - u - f\right]_{,\bar{l}k}\\ &= -\widetilde{g}^{\alpha\bar{l}}\widetilde{g}^{k\bar{\beta}}[\widetilde{g}^{i\bar{j}}u_{,i\bar{j}\bar{l}} - u_{,\bar{l}} - f_{,\bar{l}}]_{,k}\\ &= -\widetilde{g}^{\alpha\bar{l}}\widetilde{g}^{k\bar{\beta}}[-\widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}sk}u_{,i\bar{j}\bar{l}} + \widetilde{g}^{i\bar{j}}u_{,i\bar{j}\bar{l}k} - u_{,\bar{l}k} - f_{,\bar{l}k}],\end{aligned}\tag{5.20}$$

while the term $(u_t)_{,\alpha\bar{\sigma}\lambda\bar{\rho}}$ gives

(5.21)
$$\begin{aligned}(u_t)_{,\alpha\bar{\sigma}\lambda\bar{\rho}} &= \left[\log\frac{\det(g_{i\bar{j}}+u_{i\bar{j}})}{\det(g_{i\bar{j}})} - u - f\right]_{,\alpha\bar{\sigma}\lambda\bar{\rho}}\\ &= [\widetilde{g}^{i\bar{j}}u_{,i\bar{j}\alpha} - u_{,\alpha} - f_{,\alpha}]_{,\bar{\sigma}\lambda\bar{\rho}}\\ &= [-\widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha} + \widetilde{g}^{i\bar{j}}u_{,i\bar{j}\alpha\bar{\sigma}} - u_{,\alpha\bar{\sigma}} - f_{,\alpha\bar{\sigma}}]_{,\lambda\bar{\rho}}\\ &= [\widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha} + \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha}\\ &\quad - \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{l}k\bar{\sigma}\lambda}u_{,i\bar{j}\alpha} - \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha\lambda}\\ &\quad - \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}s\lambda}u_{,i\bar{j}\alpha\bar{\sigma}} + \widetilde{g}^{i\bar{j}}u_{,i\bar{j}\alpha\bar{\sigma}\lambda} - u_{,\alpha\bar{\sigma}\lambda} - f_{,\alpha\bar{\sigma}\lambda}]_{\bar{\rho}}\\ &= -\widetilde{g}^{i\bar{p}}\widetilde{g}^{q\bar{r}}\widetilde{g}^{s\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha}\\ &\quad - \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{p}}\widetilde{g}^{q\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha}\\ &\quad - \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{l}}\widetilde{g}^{k\bar{p}}\widetilde{g}^{q\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha} + \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{r}s\lambda\bar{\rho}}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha}\\ &\quad + \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}\bar{\rho}}u_{,i\bar{j}\alpha} + \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha\bar{\rho}}\\ &\quad - \widetilde{g}^{i\bar{p}}\widetilde{g}^{q\bar{l}}\widetilde{g}^{k\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha}\\ &\quad - \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{p}}\widetilde{g}^{q\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha}\\ &\quad - \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{r}}\widetilde{g}^{s\bar{p}}\widetilde{g}^{q\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha} + \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}s\lambda\bar{\rho}}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha}\end{aligned}$$

$$
\begin{aligned}
&+ \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}\bar{\rho}}u_{,i\bar{j}\alpha} + \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha\bar{\rho}} \\
&+ \widetilde{g}^{i\bar{p}}\widetilde{g}^{q\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{\bar{r}s\lambda}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha} + \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{p}}\widetilde{g}^{q\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{l}k\bar{\sigma}\lambda}u_{,i\bar{j}\alpha} \\
&- \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{l}k\bar{\sigma}\lambda\bar{\rho}}u_{,i\bar{j}\alpha} - \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{l}k\bar{\sigma}\lambda}u_{,i\bar{j}\alpha\bar{\rho}} \\
&+ \widetilde{g}^{i\bar{p}}\widetilde{g}^{q\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha\lambda} + \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{p}}\widetilde{g}^{q\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha\lambda} \\
&- \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{l}k\bar{\sigma}}u_{,i\bar{j}\alpha\lambda\bar{\rho}} - \widetilde{g}^{i\bar{l}}\widetilde{g}^{k\bar{j}}u_{,\bar{l}k\bar{\sigma}\bar{\rho}}u_{,i\bar{j}\alpha\lambda} \\
&+ \widetilde{g}^{i\bar{p}}\widetilde{g}^{q\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,i\bar{j}\alpha\bar{\sigma}} + \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{p}}\widetilde{g}^{q\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,\bar{r}s\lambda}u_{,i\bar{j}\alpha\bar{\sigma}} \\
&- \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}s\lambda\bar{\rho}}u_{,i\bar{j}\alpha\bar{\sigma}} - \widetilde{g}^{i\bar{r}}\widetilde{g}^{s\bar{j}}u_{,\bar{r}s\lambda}u_{,i\bar{j}\alpha\bar{\sigma}\bar{\rho}} \\
&- \widetilde{g}^{i\bar{p}}\widetilde{g}^{q\bar{j}}u_{,\bar{p}q\bar{\rho}}u_{,i\bar{j}\alpha\bar{\sigma}\lambda} + \widetilde{g}^{i\bar{j}}u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}} - u_{,\alpha\bar{\sigma}\lambda\bar{\rho}} - f_{,\alpha\bar{\sigma}\lambda\bar{\rho}}.
\end{aligned}
$$

Upon substituting this and its conjugate expression back into (5.19), we get two terms involving sixth derivatives of $u$ which we examine below. Our goal of course is to recover $\widetilde{\Delta}Q = \widetilde{g}^{i\bar{j}}(\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu})_{,i\bar{j}}$ out of these two terms. In doing this, we show how to permute indices in any covariant derivative of $u$ occurring above. The two sixth derivative terms are

$$
\begin{aligned}
&\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu}, \\
&\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,i\bar{j}\bar{\beta}\delta\bar{\mu}\nu}.
\end{aligned}
\tag{5.22}
$$

Consider the term $u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}}$ in (5.22). We would like to express this in terms of $u_{,\alpha\bar{\sigma}\lambda\bar{\rho}i\bar{j}}$ as this is a term occurring in the expansion of $\widetilde{\Delta}Q$. We have,

$$
\begin{aligned}
u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}} &= u_{,\bar{j}i\alpha\bar{\sigma}\lambda\bar{\rho}} \\
&= u_{,\bar{j}\alpha i\bar{\sigma}\lambda\bar{\rho}} \\
&= (u_{,\bar{j}\alpha\bar{\sigma}i} + R^{\bar{a}}_{\bar{j}\bar{\sigma}i}u_{\bar{a}\alpha} + R^{a}_{\alpha\bar{\sigma}i}u_{\bar{j}a})_{,\lambda\bar{\rho}} \\
&= u_{,\bar{j}\alpha\bar{\sigma}i\lambda\bar{\rho}} + (R^{\bar{a}}_{\bar{j}\bar{\sigma}i}u_{\bar{a}\alpha} + R^{a}_{\alpha\bar{\sigma}i}u_{\bar{j}a})_{,\lambda\bar{\rho}} \\
&= u_{,\bar{j}\alpha\bar{\sigma}\lambda i\bar{\rho}} + (R^{\bar{a}}_{\bar{j}\bar{\sigma}i}u_{\bar{a}\alpha} + R^{a}_{\alpha\bar{\sigma}i}u_{\bar{j}a})_{,\lambda\bar{\rho}} \\
&= u_{,\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}i} + R^{\bar{a}}_{\bar{j}\bar{\rho}i}u_{\bar{a}\alpha\bar{\sigma}\lambda} + R^{a}_{\alpha\bar{\rho}i}u_{\bar{j}a\bar{\sigma}\lambda} \\
&\quad + R^{\bar{a}}_{\bar{\sigma}\bar{\rho}i}u_{\bar{j}\alpha\bar{a}\lambda} + R^{a}_{\lambda\bar{\rho}i}u_{\bar{j}\alpha\bar{\sigma}a} + (R^{\bar{a}}_{\bar{j}\bar{\sigma}i}u_{\bar{a}\alpha} + R^{a}_{\alpha\bar{\sigma}i}u_{\bar{j}a})_{,\lambda\bar{\rho}}.
\end{aligned}
\tag{5.23}
$$

Note that so far we have expressed $u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}}$ in terms of $u_{,\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}i}$. Repeating the above process allows us to express $u_{,\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}i}$ in terms of $u_{,\alpha\bar{\sigma}\lambda\bar{\rho}i\bar{j}}$. Doing the same for the term $u_{,i\bar{j}\bar{\beta}\delta\bar{\mu}\nu}$ in (5.22) and examining all the terms involving the curvature tensor $R$ above, we have,

$$u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}} = u_{,\alpha\bar{\sigma}\lambda\bar{\rho}i\bar{j}} + (\text{terms}),$$
$$u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}} = u_{,\alpha\bar{\sigma}\lambda\bar{\rho}i\bar{j}} + (\text{terms}), \tag{5.24}$$

where by (terms) we mean terms involving a contraction of the curvature tensor $R$ (or up to three covariant derivatives of $R$) with derivatives of $u$ such that the contraction is linear in $u$, and the derivatives of $u$ are of degree at most four. This now gives

$$\begin{aligned} &\widetilde{g}^{i\bar{j}}\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,i\bar{j}\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu} + \widetilde{g}^{i\bar{j}}\widetilde{g}^{i\bar{j}\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,i\bar{j}\bar{\beta}\delta\bar{\mu}\nu} \\ &= \widetilde{g}^{i\bar{j}}(\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu})_{i\bar{j}} - 2\widetilde{g}^{i\bar{j}}\widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}i}u_{,\bar{\beta}\delta\bar{\mu}\nu\bar{j}} \\ &\quad + (\text{terms}) \\ &= \widetilde{\Delta}Q - 2|\nabla^5_m u|^2 + (\text{terms}). \end{aligned} \tag{5.25}$$

where now, by (terms) we mean terms involving a contraction of the curvature tensor $R$ (or up to three covariant derivatives of $R$) with derivatives of $u$ where the contraction is now quadratic in $u$, and the derivatives of $u$ are of degree at most four (the notation $|\nabla^5_m u|^2$ will be explained below). Expanding (5.19) and using the above technique to permute covariant derivatives of $u$, we have the following:

**Remark 5.1.** In (5.19), by adding terms involving contractions of the curvature tensor $R$ (or up to three covariant derivatives of $R$) with either third mixed derivatives or second derivatives of $u$, we may assume all fifth derivatives occur in the form $u_{,i\bar{j}k\bar{l}m}$ or $u_{,\bar{i}j\bar{k}l\bar{m}}$ and all fourth derivatives of $u$ occur in the form $u_{,i\bar{j}k\bar{l}}$, $u_{,\bar{i}j\bar{k}l}$, $u_{,i\bar{j}lk}$ or $u_{,\bar{i}j\bar{l}\bar{k}}$.

We distinguish between the different types of fourth and fifth derivatives above by establishing the following notation:

$$\begin{aligned} |\nabla^3_m u|^2 &= \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\gamma}}u_{,\alpha\bar{\mu}\nu}u_{,\bar{\beta}\lambda\bar{\gamma}} \\ |\nabla^4_m u|^2 &= \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}}u_{,\bar{\beta}\delta\bar{\mu}\nu} \\ |\nabla^4_{m'} u|^2 &= \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\rho\bar{\nu}}u_{,\alpha\bar{\sigma}\lambda\rho}u_{,\bar{\beta}\delta\bar{\mu}\bar{\nu}} \\ |\nabla^5_m u|^2 &= \widetilde{g}^{\alpha\bar{\beta}}\widetilde{g}^{\delta\bar{\sigma}}\widetilde{g}^{\lambda\bar{\mu}}\widetilde{g}^{\nu\bar{\rho}}\widetilde{g}^{a\bar{b}}u_{,\alpha\bar{\sigma}\lambda\bar{\rho}a}u_{,\bar{\beta}\delta\bar{\mu}\nu\bar{b}}. \end{aligned} \tag{5.26}$$

By the remark above, and noting our previous bounds on the mixed third derivatives and all second derivatives of $u$ we can now estimate the evolution of $Q$ as follows:

$$\begin{aligned}
\frac{d|\nabla_m^4 u|^2}{dt} &\le \widetilde{\Delta}|\nabla_m^4 u|^2 - 2|\nabla_m^5 u|^2 + K|\nabla_m^4 u||\nabla_m^5 u| + K|\nabla_m^4 u|^3 \\
&\quad + K|\nabla_{m'}^4 u|^2 + K|\nabla_m^4 u|^2 + K|\nabla_m^4 u||\nabla_{m'}^4 u|^2 \\
&\quad + K|\nabla_{m'}^4 u||\nabla_m^4 u| + K \\
&\le \widetilde{\Delta}|\nabla_m^4 u|^2 - 2|\nabla_m^5 u|^2 + K|\nabla_m^4 u||\nabla_m^5 u| + K|\nabla_m^4 u|^3 \\
&\quad + K|\nabla_{m'}^4 u|^2 + K|\nabla_m^4 u||\nabla_{m'}^4 u|^2 + K,
\end{aligned} \tag{5.27}$$

where in (5.27) and in all that follows, we will denote any strictly positive constants, not necessarily the same, by $C$ and any other constants, not necessarily the same, by $K$. Of course, all constants $C$ and $K$ will be independent of $T$.

We now combine the evolution of $|\nabla_m^3 u|^2$ and $Q$. Remembering that $|\nabla_m^3 u|^2$ has already been estimated, using (5.12), we can estimate its evolution by

$$\frac{d|\nabla_m^3 u|^2}{dt} \le \widetilde{\Delta}|\nabla_m^3 u|^2 - C|\nabla_m^4 u|^2 - C|\nabla_{m'}^4 u|^2 + K. \tag{5.28}$$

Consider the quantity

$$S = (|\nabla_m^3 u|^2 + A)(|\nabla_m^4 u|^2 + B), \tag{5.29}$$

where $A$ and $B$ are positive constants to be chosen later. Clearly, a bound on $S$ implies a bound on $Q$. We begin by computing the evolution of $S$ as

(5.30)
$$\begin{aligned}
\frac{dS}{dt} &= \frac{d|\nabla_m^3 u|^2}{dt}(|\nabla_m^4 u|^2 + B) + (|\nabla_m^3 u|^2 + A)\frac{d|\nabla_m^4 u|^2}{dt} \\
&\le (\widetilde{\Delta}|\nabla_m^3 u|^2 - C|\nabla_m^4 u|^2 - C|\nabla_{m'}^4 u|^2 + K)(|\nabla_m^4 u|^2 + B) \\
&\quad + (|\nabla_m^3 u|^2 + A)(\widetilde{\Delta}|\nabla_m^4 u|^2 - 2|\nabla_m^5 u|^2 + K|\nabla_m^4 u||\nabla_m^5 u| + K \\
&\quad + K|\nabla_{m'}^4 u|^2 + K|\nabla_m^4 u||\nabla_{m'}^4 u|^2 + K) \\
&\le \widetilde{\Delta}|\nabla_m^3 u|^2(|\nabla_m^4 u|^2 + B) - C|\nabla_m^4 u|^4 - C|\nabla_m^4 u|^2|\nabla_{m'}^4 u|^2 \\
&\quad - CB|\nabla_{m'}^4 u|^2 + (|\nabla_m^4 u|^2 m2 + A)\widetilde{\Delta}|\nabla_m^4 u|^2 - 2A|\nabla_m^5 u|^2 \\
&\quad + AK|\nabla_m^4 u||\nabla_m^5 u| + AK|\nabla_m^4 u|^3 + AK|\nabla_{m'}^4 u|^2 \\
&\quad + AK|\nabla_m^4 u||\nabla_{m'}^4 u|^2 + K
\end{aligned}$$

$$\begin{aligned}
&\leq \widetilde{\Delta} S - 2A|\nabla^5_m u|^2 - C|\nabla^4_m u|^4 - C|\nabla^4_m u|^2|\nabla^4_{m'} u|^2 - CB|\nabla^4_{m'} u|^2 \\
&\quad + K|\nabla^4_m u|^2|\nabla^5_m u| + AK|\nabla^4_m u||\nabla^5_m u| + K|\nabla^4_{m'} u||\nabla^4_m u||\nabla^5_m u| \\
&\quad + AK|\nabla^4_m u|^3 + AK|\nabla^4_{m'} u|^2 + K|\nabla^4_m u|^2|\nabla^4_{m'} u| \\
&\quad + AK|\nabla^4_m u||\nabla^4_{m'} u|^2 + K.
\end{aligned}$$

The last inequality follows from the identity

$$\begin{aligned}
\widetilde{\Delta} S &= \widetilde{\Delta}|\nabla^3_m u|^2(|\nabla^4_m u|^2 + B) + (|\nabla^3_m u|^2 + A)\widetilde{\Delta}|\nabla^4_m u|^2 \\
&\quad + \widetilde{g}^{\alpha\bar{\beta}}|\nabla^3_m u|^2_{\alpha}|\nabla^4_m u|^2_{\bar{\beta}},
\end{aligned} \tag{5.31}$$

and by noting that the last term above is bounded by a linear combination of the terms $|\nabla^4_m u|^3$ , $|\nabla^4_{m'} u||\nabla^4_m u|^2$ , $|\nabla^4_{m'} u||\nabla^4_m u||\nabla^5_m u|$ and $|\nabla^4_m u|^2|\nabla^5_m u|$.

We now state two elementary propositions which we will use to estimate the last expression in (5.30).

**Proposition 5.2.** *Let $C_1$ and $C_2$ be given constants with $C_1 \geq 0$. Then there exists a constant $C_4$, depending only on $C_1$ and $C_2$, such that for all $x, y \geq 0$ the following inequality holds:*

$$-C_1 x^2 y + C_2 xy \leq -(C_1/2)x^2 y + C_4 y.$$

*Proof.* Choose $C_4$ such that for all $x \geq 0$,

$$x^2 - (C_2/C_1)x \geq (1/2)x^2 - (C_4/C_1). \qquad \text{q.e.d.}$$

**Proposition 5.3.** *Let $C_1$ and $C_2$ be given constants with $C_1 \geq 0$. Then there exists a constant $C_3 \geq 0$, depending only on $C_1$ and $C_2$, such that for all $x, y \geq 0$ the following inequality holds:*

$$-C_3 x^2 + C_2 xy - C_1 y^2 \leq -(C_3/2)x^2 - (C_1/2)y^2.$$

*Proof.* Begin by writing

$$\left(\sqrt{(C_3/2)}x - \sqrt{(C_1/2)}y\right)^2 = (C_3/2)x^2 - \sqrt{C_1 C_3}xy + (C_1/2)y^2 \geq 0.$$

Then from the above inequality and for a sufficiently large choice of $C_3$ we have:

$$-(C_3/2)x^2 - (C_1/2)y^2 \leq -\sqrt{C_1 C_3}xy. \leq -C_2 xy.$$

This gives

$$-(C_3/2)x^2 + C_2 xy - (C_1/2)y^2 \leq 0,$$

and thus

$$-C_3 x^2 + C_2 xy - C_1 y^2 \leq -(C_3/2)x^2 - (C_1/2)y^2.$$

q.e.d.

Using the two propositions above we now continue our estimate on the evolution of $S$. $K_A$ will denoted any constant depending only on $A$.

(5.32)

$$\begin{aligned}
\frac{dS}{dt} &\leq \widetilde{\Delta} S - 2A|\nabla^5_m u|^2 - C|\nabla^4_m u|^4 - C|\nabla^4_m u|^2|\nabla^4_{m'} u|^2 - CB|\nabla^4_{m'} u|^2 \\
&\quad + K|\nabla^4_m u|^2|\nabla^5_m u| + AK|\nabla^4_m u||\nabla^5_m u| + K|\nabla^4_{m'} u||\nabla^4_m u||\nabla^5_m u| \\
&\quad + AK|\nabla^4_m u|^3 + AK|\nabla^4_{m'} u|^2 + K|\nabla^4_m u|^2|\nabla^4_{m'} u| \\
&\quad + AK|\nabla^4_m u||\nabla^4_{m'} u|^2 + K \\
&\leq \widetilde{\Delta} S - 2A|\nabla^5_m u|^2 - C|\nabla^4_m u|^4 - C|\nabla^4_m u|^2|\nabla^4_{m'} u|^2 - CB|\nabla^4_{m'} u|^2 \\
&\quad + C|\nabla^4_m u|^2|\nabla^5_m u| + K_A|\nabla^5_m u| + K|\nabla^4_{m'} u||\nabla^4_m u||\nabla^5_m u| \\
&\quad + AK|\nabla^4_m u|^3 + AK|\nabla^4_{m'} u|^2 + K|\nabla^4_m u|^2 + K_A|\nabla^4_{m'} u|^2 + K \\
&\leq \widetilde{\Delta} S - A|\nabla^5_m u|^2 - (C/2)|\nabla^4_m u|^4 - (C/2)|\nabla^4_m u|^2|\nabla^4_{m'} u|^2 \\
&\quad - CB|\nabla^4_{m'} u|^2 + K_A|\nabla^5_m u| + AK|\nabla^4_m u|^3 + AK|\nabla^4_{m'} u|^2 \\
&\quad + K|\nabla^4_m u|^2 + K_A|\nabla^4_{m'} u|^2 + K.
\end{aligned}$$

The second inequality follows by applying Proposition 5.2 to the expressions

$$-(C/2)|\nabla^4_m u|^2|\nabla^4_{m'} u|^2 + K|\nabla^4_m u|^2|\nabla^4_{m'} u|,$$

and

$$-(C/2)|\nabla^4_m u|^2|\nabla^4_{m'} u|^2 + AK|\nabla^4_m u||\nabla^4_{m'} u|^2,$$

formed by grouping terms from the right-hand side of the first inequality, and also by estimating

$$K|\nabla^4_m u|^2|\nabla^5_m u| + AK|\nabla^4_m u||\nabla^5_m u| \leq C|\nabla^4_m u|^2|\nabla^5_m u| + K_A|\nabla^5_m u|,$$

for $C$ large enough. The third inequality follows by applying the Proposition 5.3 to the terms

$$-A|\nabla^5_m u|^2 + C|\nabla^4_m u|^2|\nabla^5_m u| - C|\nabla^4_m u|^4,$$

and

$$-A|\nabla^5_m u|^2 + K|\nabla^4_{m'} u||\nabla^4_m u||\nabla^5_m u| - C|\nabla^4_m u|^2|\nabla^4_{m'} u|^2,$$

appearing in the right-hand side of the second inequality.

By the last inequality in (5.32), we can choose the constants $A$ and $B$ large enough to give the following estimate:

$$
\frac{dS}{dt} \leq \widetilde{\Delta} S - A|\nabla_m^5 u|^2 - (C/2)|\nabla_m^4 u|^4 - (C/2)|\nabla_m^4 u|^2|\nabla_{m'}^4 u|^2 - CB|\nabla_{m'}^4 u|^2 + K. \tag{5.33}
$$

By applying the maximum principle to (5.33), we can bound $\sup_{(x,t)\in M\times[0,T)} S(x,t)$, and thus also $\sup_{(x,t)\in M\times[0,T)} Q(x,t)$, independent of $T$.

**5.5. Hölder norm estimates.** Recall that $(M, g_{i\bar{j}})$ has bounded geometry of order $\infty$. In Definition 2.1, choose some large $m$ and consider a holomorphic chart $\xi_p : V_p \subseteq \mathbb{C}^n \to U_i$ for arbitrary $p \in M$. Throughout this subsection, all local expressions will be with respect to the standard coordinates on $V_p \in \mathbb{C}^n$; $C^{k+\alpha}$ and $C^{k+\alpha,k/2+\alpha/2}$ will denote the standard Hölder spaces on $V_p$ and $V_p \times [0,T)$ (or subsets of these) respectively; $C^{k+\alpha}(M)$ and $C^{k+\alpha,k/2+\alpha/2}(M\times[0,T))$ will denote Hölder spaces relative to $g_{i\bar{j}}$; all estimates derived will be independent of $p$ and $T$ and in general we will refer to any quantity independent of $p$ and $T$ as being a uniform quantity.

At any time $t \in [0,T)$, in $V_p$, we have an estimate for $|u|$, $\Delta u$ and the Hölder derivative in space of $\Delta u$ (this last estimate follows from our estimate on the third mixed derivatives of $u$). We apply the standard elliptic Schauder interior estimates to $u$ in $V_p$ to get an estimate on the $C^{2+\alpha}$ norm of $u$ in some uniformly large subset of $V_p$. Also, by looking at (1.2) we see that our third and fourth derivative estimates in space are in fact equivalent to estimates on $u_{it}$, $u_{\bar{j}t}$, $u_{i\bar{j}t}$ and $u_{tt}$ in $V_p$. Using all these we obtain an estimate on the $C^{2+\alpha,1+\alpha/2}$ norm of $u$ in $U\times[0,T)$ for some uniformly large subset $U \subset V_p$. To see that we can extend this to an estimate on the $C^{4+\alpha,2+\alpha/2}$ norm, we differentiate (1.2) in the natural coordinates on $V_p$ giving

$$
\begin{aligned}
\frac{du_k}{dt} &= \widetilde{g}^{\alpha\bar{\beta}} u_{,\alpha\bar{\beta}k} - u_k - f_k \\
&= \widetilde{g}^{\alpha\bar{\beta}} u_{,k\alpha\bar{\beta}} + \widetilde{g}^{\alpha\bar{\beta}} R^a_{\alpha k\bar{\beta}} u_a - u_k - f_k \\
&= \widetilde{\Delta} u_k + \widetilde{g}^{\alpha\bar{\beta}} R^a_{\alpha k\bar{\beta}} u_a - u_k - f_k \\
&= \widetilde{\Delta} u_k - u_k + (\widetilde{g}^{\alpha\bar{\beta}} R^a_{\alpha k\bar{\beta}} u_a - f_k).
\end{aligned} \tag{5.34}
$$

We view (5.34) as a linear parabolic equation for $u_k$ in $V_p$ with the last term viewed as a single forcing term. It is clear that our previous estimate on the $C^{2+\alpha,1+\alpha/2}$ norm of $u$ provides an estimate on the $C^{\alpha,\alpha/2}$ norm of the coefficients and forcing term in the linear equation under consideration. By standard parabolic Schauder estimates, we get an estimate on the $C^{2+\alpha,1+\alpha/2}$ norm of $u_k$ in $\widetilde{U} \times [0,T)$ for a uniformly large subset $\widetilde{U} \subset U$ . This then gives an estimate on the $C^{\alpha,\alpha/2}$ norm of the first space derivatives of the coefficients and forcing term in $\widetilde{U}$. We again apply the Schauder estimates to obtain an estimate on the $C^{2+\alpha,1+\alpha/2}$ norm of the first space derivatives of $u_k$ in $U' \times [0,T)$ for a uniformly large subset $U' \subset \widetilde{U}$. Noting that the index $k$ is arbitrary, and repeating the above process for a barred index $\bar{k}$ it is easily checked that this is in fact equivalent to an estimate on the $C^{4+\alpha,2+\alpha/2}$ norm of $u$ in $U' \times [0,T)$. Noting that $U'$ in $V_p$ can be made uniform and the choice of $V_p$ was arbitrary, we see that this gives an estimate for the $C^{4+\alpha,2+\alpha/2}(M \times [0,T))$ norm of $u$ relative to the maps $\xi_p$. In particular, we see that the curvature of the corresponding evolving metrics stays uniformly bounded. Finally, we note that the above process can be repeated successively giving an estimate for the $C^{k+\alpha,k/2+\alpha/2}(M \times [0,T))$ norm of $u$ relative to the maps $\xi_p$ for every $k \leq m$, and that $m$ was chosen arbitrarily large. In particular, $u(x,t)$ is a smooth function of space and time.

## 6. Long time existence and convergence

Lemma 4.1 together with our a priori estimates allow us to prove the following:

**Lemma 6.1.** *Assume $(M, g_{i\bar{j}})$ is a complete noncompact Kähler manifold with bounded geometry of infinite order and that $R_{i\bar{j}} + g_{i\bar{j}} = f_{i\bar{j}}$ for some smooth bounded function $f$ on $M$. Let $C^{n+\alpha}$ denote the Hölder spaces on $M$ relative to the metric $g_{i\bar{j}}$. Then (1.2) has a smooth long time solution $u(x,t)$ which converges, as $t \to \infty$, in every $C^{n+\alpha}$ norm, to a smooth limit $u(x,\infty)$. Moreover, $\widetilde{g}_{i\bar{j}}(x,t) := g_{i\bar{j}}(x) + u_{i\bar{j}}(x,t)$ provides a smooth long time solution to (1.1) and converges, as $t \to \infty$, in every $C^{n+\alpha}$ norm, to a complete Kähler–Einstein metric $\widetilde{g}_{i\bar{j}}(x,\infty)$ where $\widetilde{g}_{i\bar{j}}(x,\infty)$ has negative scalar curvature on $M$, is equivalent to $g_{i\bar{j}}$, and has all covariant derivatives of its curvature tensor bounded.*

*Proof.* By Lemma 4.1 we know $\exists T > 0$ such that there is a smooth short time solution $u(x,t)$ to (1.2) on $M \times [0,T)$. Moreover, we may assume that $[0, T)$ is the maximal time interval on which we have such a solution. Our estimates from Section 5 show that the $C^{2+\alpha,1+\frac{\alpha}{2}}$ norm of $u$ is bounded on $M \times [0, T)$ independent of $T$. Combining this with the arguments from Section 5.5 and the assumption of bounded geometry of order $\infty$ implies that for every $n$ the $C^{n+\alpha}$ norm of $u(x, t)$ is bounded independent of $t$. This allows us to take a limit of $u(x, t)$, in $C^{n+\alpha}$ for any $n$, as $t \to T$, yielding a limit metric $\widetilde{g}_{i\bar{j}}(x, T) := g_{i\bar{j}}(x) + u_{i\bar{j}}(x, T)$ on $M$ equivalent to $g_{i\bar{j}}$ and with bounded geometry of order $\infty$. Now if $T < \infty$, by Lemma 4.1, with $\widetilde{g}_{i\bar{j}}(x, T)$ as initial metric, we see that we may continue our solution $u(x, t)$ past $T$ for some short time thus contradicting our assumption that $T$ is maximal and thus proving long time existence of $u(x, t)$.

It remains to be shown that for any $n$, $u(x, t)$ converges in $C^{n+\alpha}$ as $t \to \infty$. By subsection 5.1 $u(x, t)$ converges in the $C^0$ norm to a unique limit $u(x, \infty)$. Moreover, we have seen that for any $n$ the $C^{n+\alpha}$ norm of $u(x, t)$ remains uniformly bounded in time. Combining these two facts, it is easily seen that $u(x, \infty) \in C^{n+\alpha}$ $\forall n$, and $u(x, t)$ must converge in every $C^{n+\alpha}$ norm to $u(x, \infty)$. Taking the limit of (1.2) as $t \to \infty$, and noting the estimates of the previous section, we see that the limit metric $\widetilde{g}_{i\bar{j}}(x, \infty)$ is a Kähler–Einstein metric as stated in the lemma. q.e.d.

Our main Theorem 1.1 now follows immediately from Lemma 4.1 and Lemma 6.1.

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