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COHOMOGENEITY ONE MANIFOLDS OF EVEN DIMENSION WITH STRICTLY POSITIVE SECTIONAL CURVATURE

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Abstract

We show that the only compact positively curved Riemannian manifolds of even dimension acted on by a simple group with a codimension one orbit are the compact rank one symmetric spaces.

1. Introduction

In this paper, we deal with simply connected compact Riemannian manifolds with strictly positive sectional curvature. These manifolds have been studied by several authors and, in particular, the homogeneous ones have been classified by Wallach [**29**] and Berard-Bergery [**5**]. The only known non-homogeneous examples are biquotiens (i.e., quotients of a Lie group G with respect to the free action of a subgroup of $G \times G$ from the left and the right) and have been found by Eschenburg [**9**], [**10**] in dimensions 6 and 7, and then by Bazaikin [**4**] in dimension 13. When looking for new examples, it seems natural to consider the class of cohomogeneity one manifolds, that is Riemannian manifolds $(M, \langle\cdot,\cdot\rangle)$ which admit an action of a compact group G of isometries with a codimension one orbit. Our main result is the following.

Theorem 1.1. *Let $(M, \langle\cdot,\cdot\rangle)$ be an even dimensional compact, simply connected cohomogeneity one G-manifold with positive sectional curvature. If G is a compact Lie group, then M is equivariantly diffeomorphic to a compact rank one symmetric space.*

The statement is proved in [**23**] and [**28**] for the case of G non-semisimple and semisimple (but not simple), respectively. In this paper, we deal with the case of G simple. In some small dimensional case, the same result was obtained by Hamilton [**14**] (for $\dim M = 3$), Hsiang–Kleiner [**15**] (for $\dim M = 4$) and Searle [**25**] (for $\dim M = 5, 6$), see

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also [12]. We mention here that, in the odd dimensional case, examples other than the compact rank one symmetric spaces can occur. In [24], the authors prove that, when $\dim M = 7$, this is possible only if $G = SU(2) \times SU(2)$. In [13], the authors prove that indeed this group acts with cohomogeneity one on some of the Eschenburg spaces.

Recently, very strong results on the topology of compact positively curved manifolds M have been proved by Wilking [30], [31], [32] assuming the existence of totally geodesic submanifolds of small codimension. As a consequence, the author can prove that M is up to homotopy, a rank one symmetric spaces provided that its isometry group is large or contains a large torus. In most cases, these obstructions seem to indicate that the only 'large class' of positively curved manifolds is given by the rank one symmetric spaces.

The strategy of the proof of our main theorem is as follows. A standard result (see e.g. [23]) implies that there is a singular orbit G/H such that H has maximal rank in G. Maximal subalgebras $\mathfrak{h}$ of maximal rank of simple Lie algebras $\mathfrak{g}$ are classified (see e.g. [11]) and we list them in Table 1. Using iteration, we have an algorithm to get any maximal rank subalgebra of $\mathfrak{g}$. In general, at every step, we increase either the number of simple ideals or the dimension of the center of $\mathfrak{h}$. In Section 3, we give a bound on the number of simple ideals in $\mathfrak{h}$ and in Section 4, we bound the dimension of the center of $\mathfrak{h}$. Hence, the algorithm is applied only a 'few' times and reduces the problem to a short list of candidates. In Section 5, we examine all these cases.

2. Preliminary results

Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold and let G be a compact Lie group acting by isometries on $(M, \langle\cdot,\cdot\rangle)$ with a codimension one orbit. We say that $(M, \langle\cdot,\cdot\rangle)$ is a cohomogeneity one G-manifold. The geometry of cohomogeneity one manifolds is now well understood. After the pioneer work of Mostert [19] and Nagano [21], they have been studied by several authors. For the basic results on the structure of these manifolds and for the notations used throughout the paper, we refer to Alekseevski [1], [2] and Bredon [7].

We study the case of a compact cohomogeneity one G-manifold with positive sectional curvature acted (almost) effectively by a compact simple Lie group G. Since the manifold is compact and positively curved, the fundamental group of M is finite and we will suppose that M is simply connected. Then, the orbit space M/G is homeomorphic to a closed interval $[0, 1]$, and there are exactly two singular orbits for the G action on M which we denote by P and P'. Let $\gamma(t)$ be a normal geodesic in $(M, \langle\cdot,\cdot\rangle)$ and denote by H and H' the isotropy subgroups of $\gamma(0) \in P, \gamma(t_0) \in P'$ and by K the isotropy subgroup of the regular points $\gamma(t), 0 < t < t_0$ (in fact K will fix the entire normal geodesic). Then, $P = G/H, P' = G/H'$ and $H/K, H'/K$ are diffeomorphic to spheres of positive dimension; indeed, simply connected cohomogeneity one manifolds admit no exceptional orbits ([7], Theorem 3.12, p. 185).

The study of the singular orbits will be the key for the proof of the main result of this paper. First, we use the following fact (see e.g. [23], Lemma 3.2).

Lemma 2.1. *Let $(M, \langle\cdot,\cdot\rangle)$ be a compact cohomogeneity one G-manifold with positive sectional curvature. If $\dim M$ is even, then there exists a singular orbit with positive Euler characteristic.*

This lemma implies that there is a singular orbit G/H, with H maximal rank subgroup of G. We denote by gothic letters the corresponding Lie subalgebras. The pairs $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is maximal, have been classified by Borel and Siebenthal. We list them in Table 1 and refer to [11] (Section 8.4) for the details.

Table 1. Maximal subalgebras $\mathfrak{h}$ of maximal rank in $\mathfrak{g}$.

$\mathfrak{g}$	$\mathfrak{h}$	
$\mathfrak{a}_n$	$\mathbb{R} + \mathfrak{a}_{n-1}$, $\mathbb{R} + \mathfrak{a}_p + \mathfrak{a}_q$	$n = p + q + 1; p, q > 0$
$\mathfrak{b}_n$	$\mathfrak{d}_p + \mathfrak{b}_q$, $\mathbb{R} + \mathfrak{b}_{n-1}$, $\mathfrak{d}_n$, $2\mathfrak{a}_1 + \mathfrak{b}_{n-2}$	$n = p + q; p, q > 0$
$\mathfrak{c}_n$	$\mathfrak{c}_p + \mathfrak{c}_q$, $\mathfrak{a}_1 + \mathfrak{c}_{n-1}$, $\mathbb{R} + \mathfrak{a}_{n-1}$	$n = p + q$
$\mathfrak{d}_n$	$\mathfrak{d}_p + \mathfrak{d}_q$, $2\mathfrak{a}_1 + \mathfrak{d}_{n-2}$, $\mathbb{R} + \mathfrak{a}_{n-1}$, $\mathbb{R} + \mathfrak{d}_{n-1}$	$n = p + q$
$\mathfrak{g}_2$	$\mathfrak{a}_2$, $2\mathfrak{a}_1$	
$\mathfrak{f}_4$	$\mathfrak{b}_4$, $2\mathfrak{a}_2$, $\mathfrak{a}_1 + \mathfrak{c}_3$	
$\mathfrak{e}_6$	$\mathfrak{a}_1 + \mathfrak{a}_5$, $3\mathfrak{a}_2$, $\mathbb{R} + \mathfrak{d}_5$	
$\mathfrak{e}_7$	$\mathfrak{a}_1 + \mathfrak{d}_6$, $\mathfrak{a}_2 + \mathfrak{a}_5$, $\mathfrak{a}_7$, $\mathbb{R} + \mathfrak{e}_6$	
$\mathfrak{e}_8$	$\mathfrak{a}_2 + \mathfrak{e}_6$, $2\mathfrak{a}_4$, $\mathfrak{a}_8$, $\mathfrak{a}_1 + \mathfrak{e}_7$, $\mathfrak{d}_8$	

Note that maximal connected subgroups of maximal rank of G having isomorphic Lie algebras are conjugated through an element of G (cf. [11], Section 8.3). This allows us to choose the embedding of $\mathfrak{h}$ in $\mathfrak{g}$. When we 'iterate' the algorithm for finding maximal rank subalgebras of $\mathfrak{g}$, it may happen that the corresponding subgroups are not conjugated through an inner automorphism. In these cases, we will consider a different embedding of $\mathfrak{h}$ for any inner conjugation class of subgroups.

In many cases, the proofs are similar and we will sometimes just give an indication of the proof.

As a first consequence of the fact that $\mathfrak{h}$ has maximal rank in $\mathfrak{g}$, we have that $\mathfrak{h}$ is a regular subalgebra. In order to make the definition precise, we recall here some basic facts about the root space decomposition of a simple Lie algebra.

Let $\mathfrak{g}$ be a simple Lie algebra, denoted by $\mathfrak{g}^{\mathbb{C}} = \mathfrak{c}^{\mathbb{C}} + \sum_{\alpha\in\Delta} \mathfrak{g}^{\mathbb{C}}_{\alpha}$, the decomposition of the complexified algebra $\mathfrak{g}^{\mathbb{C}}$ associated to the choice of a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ and a simple root system Δ. Then, $\mathfrak{g}$ admits a decomposition of the form

$$\mathfrak{g} = \mathfrak{c} + \sum_{\alpha\in\Delta^+} \mathfrak{g}_{\alpha} \tag{1}$$

where $\mathfrak{g}_{\alpha} = (\mathfrak{g}^{\mathbb{C}}_{\alpha} \oplus \mathfrak{g}^{\mathbb{C}}_{-\alpha}) \cap \mathfrak{g}$ and Δ^+ is the set of the positive roots with respect to a Weyl chamber. This decomposition has the following properties:

(1) If $\alpha \neq \pm\beta$

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subset \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha-\beta}$$

where $\mathfrak{g}_{\gamma} = 0$ if $\gamma \notin \Delta$;

(2) There exists $H_{\alpha} \in \mathfrak{c}$ such that

$$[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\alpha}] = \mathbb{R}\, H_{\alpha}$$

and, for any $H \in \mathfrak{c}$, $\alpha(H) = B(H_{\alpha}, H)$, where B is the Killing form of $\mathfrak{g}$;

(3) There exist a B-orthonormal basis X_{α}, Y_{α} of $\mathfrak{g}_{\alpha}$ such that, for any $H \in \mathfrak{c}$,

$$[H, X_{\alpha}] = \alpha(H)\, Y_{\alpha}, \qquad [H, Y_{\alpha}] = -\alpha(H)\, X_{\alpha}.$$

Then, a regular subalgebra $\mathfrak{h}$ is a subalgebra which can be written as

$$\mathfrak{h} = \mathfrak{c} + \sum_{\alpha\in\Gamma} \mathfrak{g}_{\alpha}, \qquad \Gamma \subset \Delta^+$$

for a suitable choice of $\mathfrak{c}$ and of a Weyl chamber.

We now wish to prove the main tool for our classification. In order to fix the notations, we choose a singular point $p \in G/H$ such that $H \cdot p = p$ and a singular geodesic $\gamma(t)$ through $p = \gamma(0)$. We denote also by K the isotropy subgroup of the regular points of γ, and by $\mathfrak{k}$ its Lie algebra. We consider the $\mathrm{Ad}\,(H)$-invariant complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$, and an $\mathrm{Ad}\,(K)$-invariant complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{h}$. Both $\mathfrak{p}$ and $\mathfrak{m}$ are acted on by K by isotropy representation at the regular points of $\gamma(t)$, and $\mathfrak{g}$ admits a decomposition of the form

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_0 + \cdots + \mathfrak{p}_r + \mathfrak{m}_{10} + \cdots + \mathfrak{m}_{1,s_1} + \cdots + \mathfrak{m}_{q,s_q}, \tag{2}$$

Table 2. Wallach's list.

$\mathfrak{g}$	$\mathfrak{h}$	M
$\mathfrak{a}_n$	$\mathbb{R}+\mathfrak{a}_{n-1}$	$\mathbb{CP}^n$
$\mathfrak{a}_2$	$\mathbb{R}^2$	$SU(3)/T^2$
$\mathfrak{b}_n$	$\mathfrak{d}_n$	S^{2n}
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1}+\mathfrak{a}_1$	$\mathbb{HP}^n$
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1}+\mathbb{R}$	$\mathbb{CP}^{2n-1}$
$\mathfrak{c}_3$	$\mathfrak{a}_1+\mathfrak{a}_1+\mathfrak{a}_1$	$Sp(3)/SU(2)^3$
$\mathfrak{f}_4$	$\mathfrak{b}_4$	$\mathbb{C}a\mathbb{P}^2$
$\mathfrak{f}_4$	$\mathfrak{d}_4$	$F_4/\mathrm{Spin}\,(8)$
$\mathfrak{g}_2$	$\mathfrak{a}_2$	S^6

where $\mathfrak{p}_0$ and $\mathfrak{m}_{i,0}$ $(1\leq i\leq q)$ are trivial K-modules while $\mathfrak{p}_i$ $(1\leq i\leq r)$ and $\mathfrak{m}_{i,j}$ $(1\leq i\leq q, 1\leq j\leq s_q)$ are irreducible K-modules and $\mathfrak{m}_{i,j_1},\mathfrak{m}_{i,j_2}$ belong to the same irreducible H module $\mathfrak{m}_i$.

Remark 2.1. Since H/K is equivariantly diffeomorphic to a sphere, there exists an element σ in the normalizer of K in H such that $\sigma(\gamma'(0)) = -\gamma'(0)$. σ acts on $\mathfrak{m}$ through isotropy representation at $p=\gamma(0)$. In particular, if $X\in\mathfrak{m}$ and $\sigma(X)$ belongs to the K-orbit of X, then

$$\langle X,X\rangle_{|\gamma(t)} = \langle\sigma(X),\sigma(X)\rangle_{|\gamma(-t)} = \langle X,X\rangle_{|\gamma(-t)},$$

where, in the last equality, we have used the $\mathrm{Ad}\,(K)$-invariance of the metric along $\gamma(t)$. This implies that $\langle X,X\rangle_{|\gamma(t)}$ is an even function of t, hence

$$\langle\nabla_X X,\gamma'(t)\rangle_p = \gamma'(t)\langle X,X\rangle_p = 0.$$

As a particular case, if $\mathfrak{m}$ is the sum of irreducible inequivalent K-modules (as $\mathfrak{m}$ is $\mathrm{Ad}\,(H)$-invariant, this can be checked for a particular choice of a normal geodesic), and each of them is preserved by σ, then the singular orbit is totally geodesic. In fact, if $\nabla_X Y$ is not tangent to the singular orbit, there exists a normal geodesic $\gamma(t)$ such that $\gamma(0)=p$ and $\langle\nabla_X Y,\gamma'(t)\rangle_p = \gamma'(0)\langle X,Y\rangle_p\neq 0$. This is impossible since $\langle X,Y\rangle_{|\gamma(t)}$ is is an even function of t.

Then, G/H is a simply connected compact positively curved manifold of even dimension. These manifolds are classified in [**29**] (Proposition 6.1) and we list the pairs $(\mathfrak{g},\mathfrak{h})$ in Table 2.

Lemma 2.2. *Let G be a semisimple Lie group acting by isometries on a compact manifold $(M,\langle\,.,.\,\rangle)$ with a codimension one orbit. Let p be* *a singular point, and let* $\gamma(t)$ *be* any *normal geodesic through* $p = \gamma(0)$*. Denoted by* H *(resp.* K*), the isotropy subgroup of* p *(resp. any regular point in* $\gamma(t)$*). Assume that* H *has maximal rank in* G *and denoted by* $\Delta(resp.\Delta_{\mathfrak{h}} \subset \Delta)$*, a root system associated with the choice of a Cartan subalgebra* $\mathfrak{c}$ *of the Lie algebra* $\mathfrak{g}$ *of* G *(resp.* $\mathfrak{h} \subset \mathfrak{g}$*, which is a regular subalgebra). If* $\alpha, \beta \in \Delta$ *are such that*

1. $\alpha, \beta \notin \Delta_{\mathfrak{h}}$,
2. $\alpha \pm \beta \notin \Delta$,
3. $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_\beta$ *are orthogonal with respect to any* $\mathrm{Ad}\,(K)$*-invariant scalar product on* $\mathfrak{g}$*,*
4. $\mathfrak{g}_\alpha$ *and* $\sigma(\mathfrak{g}_\alpha)$ *are contained in the same non-trivial* K*-orbit.*

then $(M, \langle\, .,.\, \rangle)$ *is not positively curved.*

Proof. Let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. Then, by (1), X and Y belong to the $\mathrm{Ad}\,(H)$-invariant complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$. In particular, X and Y are non-zero at the singular point p. By (2), X and Y commute, hence

$$R_{XYXY} = \langle \nabla_X Y, \nabla_X Y \rangle - \langle \nabla_X X, \nabla_Y Y \rangle.$$

This condition implies also that $\langle \nabla_X Y, Z \rangle_p$ and $\langle \nabla_Y Y, Z \rangle_p$ are zero if Z is tangent at p to the singular orbit G/H. In fact,

$$2\,\langle \nabla_X Y, Z \rangle = \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle$$

and, if $Z \in \mathfrak{g}_\gamma$, $[X, Z] \in \mathfrak{g}_{\alpha+\gamma} \oplus \mathfrak{g}_{\alpha-\gamma}$, $[Y, Z] \in \mathfrak{g}_{\beta+\gamma} \oplus \mathfrak{g}_{\beta-\gamma}$. Since H has maximal rank, if $i \neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal with respect to any $\mathrm{Ad}\,(H)$-invariant scalar product. Since, by (2), $\alpha \pm \gamma \neq \beta$ and $\beta \pm \gamma \neq \alpha$, the assertion is proved. Then, at the point p

$$\nabla_X Y = \langle \nabla_X Y, N_1 \rangle\, N_1 = -\frac{1}{2} N_1 \langle X, Y \rangle\, N_1$$

and

$$\nabla_X X = -\frac{1}{2} N_2 \langle X, X \rangle\, N_2,$$

where N_1, N_2 are orthogonal to G/H at p. By (3), $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ along any normal geodesic through p, that is, $N_1 \langle X, Y \rangle_p = 0$. By (4), the norm of X is an even function of t along any normal geodesic $\gamma(t)$ through $p = \gamma(0)$, that is $N_2 \langle X, X \rangle_p = 0$. This implies $R_{XYXY} = 0$ at p. q.e.d.

Remark 2.2. In some cases, we will use the result of Lemma 2.2 under weaker assumptions:

1. If $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are not orthogonal with respect to a generic $\mathrm{Ad}\,(K)$-invariant scalar product, but $\langle \sigma(X), \sigma(Y) \rangle_{\gamma(-t)} = \langle X, Y \rangle_{\gamma(t)}$ along any normal geodesic $\gamma(t)$ through $p = \gamma(0)$, then $N \langle X, Y \rangle_p = 0$ since $\langle X, Y \rangle$ is an even function of t. Then, the result of Lemma

2.2 remains true. This happens, in particular, when σ acts as id (or $-id$) on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$.

(2) If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{m}$ are commuting vector fields such that for any $Z \in \mathfrak{m}$, $\langle [Z, X], Y\rangle_p = \langle [Z, Y], X\rangle_p = 0$ (this implies that $(\langle \nabla_X Y, Z\rangle)_p = 0$) and for any choice of a normal geodesic through p, $\mathfrak{g}_\alpha$ is preserved by σ and $\langle X, Y\rangle = 0$ along $\gamma(t)$, then the result of Lemma 2.2 remains true.

The proof of these facts is identical to the proof of Lemma 2.2.

In the cases when Lemma 2.2 fails, we can use the following tool from [**28**], which is substantially based on the fact that a strictly concave non-negative function must have a zero.

Lemma 2.3. *Let (M, g) be a compact positively curved manifold and let G be a compact Lie group acting on (M, g) by isometries. Let p be a singular point and let K be the principal isotropy subgroup at the points of a normal geodesic $\gamma(t)$ through p. Let $\mathfrak{n}$ be an irreducible K-module in $T_{\gamma(t)}G/K$ which is not equivalent to any other K module in $T_{\gamma(t)}G/K$. Then, the Killing vector fields in $\mathfrak{n}$ must vanish at some point of $\gamma(t)$.*

A result in the same direction which strongly improves Lemma 2.3 has recently been obtained by Wilking [**30**]. As this work is not easily accessible, we will not make use of it.

The proof of the main theorem can be made shorter using the following result from [**12**] (see Theorems 3.7, 3.9 and 3.11).

Theorem 2.1. *Let M be a simply connected closed manifold with positive sectional curvature, and let $\mathfrak{g}$ be the Lie algebra of a compact connected Lie group acting (almost) effectively by isometries on M. Then:*

(a) *If $\mathfrak{g} = \mathfrak{a}_n$ and $\dim(M) \leq 4n - 2$ for $n > 2$ and $\dim(M) \leq 7, 4$ for $n = 2, 1$, then M is diffeomorphic to one of either a sphere, a complex projective space, $SU(3)/T^2$, $SU(5)/(Sp(2) \times T^1)$.*

(b) *If $\mathfrak{g} = \mathfrak{b}_n$ and $\dim(M) \leq 4n$ for $n > 2$, then M is diffeomorphic to one of either a sphere, a complex projective space, the Cayley projective plane.*

(c) *If $\mathfrak{g} = \mathfrak{c}_n$ and $\dim(M) \leq 8n - 11$ for $n > 2$ and $\dim(M) \leq 8$ for $n = 2$, then M is diffeomorphic to one of either a sphere, a complex or quaternionic projective space, $Sp(3)/Sp(1)^3$, $Sp(2)/SU(2)$.*

(d) *If $\mathfrak{g} = \mathfrak{d}_n$ and $\dim(M) \leq 4n-2$ for $n > 3$, then M is diffeomorphic to one of either a sphere or a complex projective space.*

3. Structure of the semisimple part of $\mathfrak{g}$

We use the previous lemmata to analyze the structure of the semisimple part of $\mathfrak{h}$.

Proposition 3.1. *Using the notations of Lemma 2.2, if (M, g) is positively curved, then*

(1) $\mathfrak{h}$ *contains at most two simple ideals,*
(2) *the kernel of the slice representation ν at the singular orbit P contains at most one simple ideal of $\mathfrak{h}$,*
(3) *if $\mathfrak{h}$ contains two simple ideals $\mathfrak{h}_1, \mathfrak{h}_2$ and $\mathfrak{h}_1 \subset \ker(\nu)$ then $\mathfrak{h}_1 \simeq \mathfrak{a}_1$.*

Proof. The result is proven with a case by case check on the list of simple Lie algebras. We consider the decomposition $\mathfrak{h} = \mathbb{R}^s + \mathfrak{h}_1 + \cdots + \mathfrak{h}_r$, where $\mathfrak{h}_i$ are simple ideals of $\mathfrak{h}$. Denote by Δ_i the set of roots of $\mathfrak{h}$ with support in $\mathfrak{h}_i$ and let $\delta_i \in \Delta_i$. Since $\mathfrak{h}_i$ are distinct ideals of $\mathfrak{h}$, the roots δ_i are strongly orthogonal i.e., $\delta_i \pm \delta_j \notin \Delta$.

We list in Tables 3 and 4 the pairs of strongly orthogonal roots of a simple Lie algebra (using the notations of [22], Table 1, p. 224).

Table 3. Pairs of strongly orthogonal roots in classical Lie algebras.

$\mathfrak{g}$	δ_1	δ_2	
$\mathfrak{a}_n$	$\epsilon_i - \epsilon_j$	$\epsilon_r - \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
$\mathfrak{b}_n$	$\epsilon_i + \epsilon_j$	$\epsilon_i - \epsilon_j$	
		$\epsilon_r \pm \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
		ϵ_r	$r \notin \{i,j\}$
	$\epsilon_i - \epsilon_j$	$\epsilon_i + \epsilon_j$	
		$\epsilon_r \pm \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
		ϵ_r	$r \notin \{i,j\}$
	ϵ_i	$\epsilon_r \pm \epsilon_s$	$i \notin \{r,s\}$
$\mathfrak{c}_n$	$\epsilon_i \pm \epsilon_j$	$\epsilon_r \pm \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
		$2\epsilon_r$	$r \notin \{i,j\}$
	$2\epsilon_i$	$\epsilon_r \pm \epsilon_s$	$i \notin \{r,s\}$
		$2\epsilon_r$	$r \neq i$
$\mathfrak{d}_n$	$\epsilon_i + \epsilon_j$	$\epsilon_i - \epsilon_j$	
		$\epsilon_r \pm \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
	$\epsilon_i - \epsilon_j$	$\epsilon_i + \epsilon_j$	
		$\epsilon_r \pm \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$

Consider a normal geodesic $\gamma(t)$ through $p = \gamma(0)$ and denote by $\mathfrak{k}$ the Lie algebra of the regular isotropy subgroup of $\gamma(t)$. $\nu(\mathfrak{h})$ acts effectively and transitively on a sphere, and the pairs $(\nu(\mathfrak{h}), \nu(\mathfrak{k}))$ are classified [6]. We list them in Table 5 (extracted from [3]).

Table 4. Pairs of strongly orthogonal roots in exceptional Lie algebras.

$\mathfrak{g}$	δ_1	δ_2	
$\mathfrak{f}_4$	ϵ_i	$\epsilon_r \pm \epsilon_s$	$i \notin \{r,s\}$
	$\epsilon_i + \epsilon_j$	$\epsilon_i - \epsilon_j$	
		$(\epsilon_i - \epsilon_j \pm \epsilon_r \pm \epsilon_s)/2$	
		$\epsilon_r \pm \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
		ϵ_r	$r \notin \{i,j\}$
	$(\epsilon_i \pm \epsilon_j \pm \epsilon_r \pm \epsilon_s)/2$	$\epsilon_i \mp \epsilon_j$	
$\mathfrak{e}_6$	$\epsilon_i - \epsilon_j$	$\epsilon_r - \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
		2ϵ	
		$\epsilon_i + \epsilon_j + \epsilon_k \pm \epsilon$	
	2ϵ	$\epsilon_i - \epsilon_j$	
	$\epsilon_i + \epsilon_j + \epsilon_k \pm \epsilon$	$\epsilon_i - \epsilon_j$	
		$\epsilon_i + \epsilon_j + \epsilon_r \mp \epsilon$	$r \neq k$
$\mathfrak{e}_7$	$\epsilon_i - \epsilon_j$	$\epsilon_r - \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
		$\epsilon_i + \epsilon_j + \epsilon_r + \epsilon_s$	
	$\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l$	$\epsilon_i - \epsilon_j$	
		$\epsilon_r - \epsilon_s$	$\{i,j,k,l\} \cap \{r,s\} = \emptyset$
		$\epsilon_i + \epsilon_j + \epsilon_r + \epsilon_s$	$\{k,l\} \cap \{r,s\} = \emptyset$
$\mathfrak{e}_8$	$\epsilon_i - \epsilon_j$	$\epsilon_r - \epsilon_s$	$\{i,j\} \cap \{r,s\} = \emptyset$
		$\epsilon_i + \epsilon_j + \epsilon_r$	
		$\epsilon_p + \epsilon_q + \epsilon_r$	$\{i,j\} \cap \{p,q,r\} = \emptyset$
	$\epsilon_i + \epsilon_j + \epsilon_k$	$\epsilon_i - \epsilon_j$	
		$\epsilon_r - \epsilon_s$	$\{i,j,k\} \cap \{r,s\} = \emptyset$
		$\epsilon_i + \epsilon_r + \epsilon_s$	$\{j,k\} \cap \{r,s\} = \emptyset$

In particular, $\nu(\mathfrak{h})$ contains at most two simple ideals and this occurs only if $(\nu(\mathfrak{h}), \nu(\mathfrak{k})) = (\mathfrak{a}_1 + \mathfrak{c}_m, \mathfrak{a}_1^{\Delta} + \mathfrak{c}_{m-1})$, where $\mathfrak{a}_1^{\Delta}$ is diagonally embedded in $\nu(\mathfrak{h})$. One further peculiarity of this last case is that σ belongs to the center of H. In particular, every root space $\mathfrak{g}_\alpha$ is preserved by σ.

Expecting a contradiction, in order to prove (1), we may assume that $r \geq 3$ and that, if $\nu(\mathfrak{h})$ contains two simple ideals, $\mathfrak{h}_1 \subset \ker(\nu)$ and $\mathfrak{k}$ has a surjective projection over $\mathfrak{h}_2 \simeq \mathfrak{a}_1$. If $\nu(\mathfrak{h})$ contains at most one simple ideal, then $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \ker(\nu)$. In order to prove (2), we assume $r = 2$ and $\mathfrak{h}_1, \mathfrak{h}_2 \subset \ker(\nu)$. In order to prove that (3) holds, we assume $r = 2$, $\mathfrak{h}_1 \subset \ker(\nu)$ and $rank(\mathfrak{h}_i) > 1$ for $i = 1, 2$.

Table 5. Transitive actions of H on spheres $S^k = H/K$.

$\mathfrak{h}$	$\mathfrak{k}$	k	H/K	
$\mathfrak{a}_n$	$\mathfrak{a}_{n-1}$	$2n+1$	$SU(n+1)/SU(n)$	$n \geq 1$
$\mathfrak{b}_n$	$\mathfrak{d}_n$	$2n$	$SO(2n+1)/SO(2n)$	$n \geq 1$
$\mathfrak{c}_n$	$\mathfrak{c}_{n-1}$	$4n-1$	$Sp(n)/Sp(n-1)$	$n \geq 2$
$\mathfrak{d}_n$	$\mathfrak{b}_{n-1}$	$2n-1$	$SO(2n)/SO(2n-1)$	$n \geq 3$
$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R} + \mathfrak{a}_{n-2}$	$2n-1$	$U(n)/U(n-1)$	$n \geq 2$
$\mathbb{R} + \mathfrak{c}_{n-1}$	$\mathbb{R} + \mathfrak{c}_{n-2}$	$4n-5$	$U(1) \cdot Sp(n-1)/U(1) \cdot Sp(n-2)$	$n \geq 3$
$\mathfrak{a}_1 + \mathfrak{c}_{n-1}$	$\mathfrak{a}_1 + \mathfrak{c}_{n-2}$	$4n-5$	$Sp(1) \cdot Sp(n-1)/Sp(1) \cdot Sp(n-2)$	$n \geq 2$
$\mathfrak{g}_2$	$\mathfrak{a}_2$	6	$G_2/SU(3)$	
$\mathfrak{b}_3$	$\mathfrak{g}_2$	7	$\mathrm{Spin}\,(7)/G_2$	
$\mathfrak{b}_4$	$\mathfrak{b}_3$	15	$\mathrm{Spin}\,(9)/\mathrm{Spin}\,(7)$	
$\mathbb{R}$	0	1	$SO(2)/\{1\}$	
0	0	0	$\mathbb{Z}_2/\{1\}$	

Lemma 3.1. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{a}_n$.

Proof. (1) Let $\delta_1 = \epsilon_i - \epsilon_j$, $\delta_2 = \epsilon_r - \epsilon_s$, $\delta_3 = \epsilon_p - \epsilon_q$ and define $\alpha = \epsilon_i - \epsilon_r$ and $\beta = \epsilon_s - \epsilon_q$. We have $\alpha \pm \beta \notin \Delta$ and $\mathfrak{g}_\alpha \subset \mathfrak{m}$, $\mathfrak{g}_\beta \subset \mathfrak{m}$ since α and β are not orthogonal to some roots of distinct ideals of $\mathfrak{h}$. Moreover, $\mathfrak{g}_\alpha$ is contained in a non-trivial K-orbit since it is acted on irreducibly by $\mathbb{R}H_{\delta_1}$. These two root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product in $\mathfrak{g}$ since H_{δ_1} acts trivially on $\mathfrak{g}_\beta$. The K-orbit of $\mathfrak{g}_\alpha$ is preserved by σ. In fact, if $\nu(\mathfrak{h})$ contains two simple ideals, then σ belongs to the center of H and preserves $\mathfrak{g}_\alpha$. Otherwise, if σ has a projection on $\mathfrak{g}_\gamma$, $\mathfrak{g}_\gamma \nsubseteq \mathfrak{h}_1, \mathfrak{h}_2$, then γ is strongly orthogonal to α. This implies that σ can act on $\mathfrak{g}_\alpha$ at most like an element of the Cartan subalgebra of $\mathfrak{g}$. In any case, $\mathfrak{g}_\alpha$ is preserved by σ. We can then apply Lemma 2.2 to obtain a contradiction.

(2) If $n > 3$, let $\delta_1 = \epsilon_i - \epsilon_j$ and $\delta_2 = \epsilon_r - \epsilon_s$. We may define $\alpha = \epsilon_i - \epsilon_r$ and $\beta = \epsilon_j - \epsilon_t$, with ϵ_t orthogonal to the roots of $\mathfrak{h}_1$ (exchanging $\mathfrak{h}_1$ and $\mathfrak{h}_2$ if needed). Then, the corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product since H_{δ_2} acts trivially on $\mathfrak{g}_\beta$. The root space $\mathfrak{g}_\alpha$ is preserved by σ. This is trivially true if σ belongs to the Cartan subalgebra of $\mathfrak{h}$. If not, we may choose σ such that it has a projection on $\mathfrak{g}_\gamma$ only if γ is orthogonal to α since any root of $\Delta_{\mathfrak{h}} \setminus \Delta_{\mathfrak{k}}$ is orthogonal to α. We can conclude using Lemma 2.2.

If $n = 3$, we have $\mathfrak{h} = \mathfrak{a}_1 + \mathfrak{a}_1 + \mathbb{R}$, $\mathfrak{k} = \mathfrak{a}_1 + \mathfrak{a}_1$. We consider the one dimensional subgroup T corresponding to $\mathbb{R}(H_{\delta_1} + H_{\delta_2})$. One connected component of the fixed point set of T is a 6 dimensional totally geodesic submanifold M_t of M, which is acted on with cohomogeneity one by a subgroup of G with Lie algebra $\mathfrak{t} = \mathfrak{a}_1 + \mathfrak{a}_1 \subset \mathfrak{g}$. The triple associated to this action is $(\mathbb{R}+\mathbb{R} = \mathfrak{t}\cap\mathfrak{h}, \mathbb{R} = \mathfrak{t}\cap\mathfrak{k}, \mathfrak{t}\cap\mathfrak{h}')$. According to the main result of [25] and the list of cohomogeneity one action on spheres and complex projective spaces [16], [26], [27], M_t is equivariantly diffeomorphic to $\mathbb{CP}^3$. In this case, $\mathfrak{h}'\cap\mathfrak{t} = \mathfrak{b}_1$ and K has two connected components. This is possible only if $\mathfrak{h}' = \mathfrak{b}_2$. In this case, M is equivariantly diffeomorphic to $\mathbb{CP}^5$ as described in [26].

(3) We have a contradiction assuming $\mathfrak{h}_1 \subset \ker(\nu)$ and $rank(\mathfrak{h}_1) > 1$. In fact, consider the regular subalgebras $\mathfrak{h}_1 \simeq \mathfrak{a}_p$, $\mathfrak{h}_2 \simeq \mathfrak{a}_q$ for some $p > 1$ and $q \geq 1$. Here $\nu(\mathfrak{h}) = \mathbb{R} + \mathfrak{a}_q$ and $\nu(\mathfrak{k}) = \mathbb{R}^\Delta + \mathfrak{a}_{q-1}$ since the center of $\mathfrak{h}$ never lies in the kernel of the slice representation (see, for instance, [17] Section 5.6, p. 497). This implies that σ does not depend on the choice of the normal geodesic and preserves any root space. We may assume $\delta_{1a} = \epsilon_i - \epsilon_a$, $\delta_{1b} = \epsilon_i - \epsilon_b \in \Delta_{\mathfrak{h}_1}$ and $\delta_2 = \epsilon_r - \epsilon_s$. Define $\alpha = \epsilon_r - \epsilon_i$ and $\beta = \epsilon_b - \epsilon_s$. Then, we may apply Lemma 2.2. q.e.d.

Lemma 3.2. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{b}_n$.

Proof.
(1) If $\mathfrak{g} = \mathfrak{b}_n$, we have 8 possible choices for (δ_1, δ_2). We give the proof for the case $\delta_1 = \epsilon_i + \epsilon_j$, the same proof works, with minor changes, also for the case $\delta_1 = \epsilon_i - \epsilon_j$, which will be omitted.

If $\delta_2 = \epsilon_i - \epsilon_j$, then we define $\alpha = \epsilon_i$ and $\beta_\pm = \epsilon_j \pm \epsilon_r$, where ϵ_r is non-orthogonal to some root of $\Delta_{\mathfrak{h}_3}$. Then, $\alpha \pm \beta_\pm \notin \Delta$ and $\mathfrak{g}_\alpha \subset \mathfrak{m}$, $\mathfrak{g}_{\beta_\pm} \subset \mathfrak{m}$ since α and $\beta_\pm$ are not orthogonal to some roots of two distinct ideals of $\mathfrak{h}$. $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\beta_+}$ or $\mathfrak{g}_{\beta_-}$ are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product: if $\nu(\mathfrak{h})$ contains just one simple ideal, then the root vector H_{ϵ_i} belongs to K and acts trivially on $\mathfrak{g}_{\beta_\pm}$. If $\nu(\mathfrak{h})$ contains two simple ideals, then K has a surjective projection on $\mathfrak{h}_2$. Then, by a linear combination of an element of K which projects on H_{δ_2} and H_{δ_1}, it is possible to find an element of K which acts trivially on $\mathfrak{g}_\alpha$ and non-trivially on $\mathfrak{g}_{\beta_+}$ or $\mathfrak{g}_{\beta_-}$. Any root γ of $\Delta_{\mathfrak{h}}$ such that $\mathfrak{g}_\gamma$ does not belong to $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is orthogonal to ϵ_i. This implies that σ preserves $\mathfrak{g}_\alpha$, which is entirely contained in a K-orbit since $\mathfrak{k}$ has a surjective projection on $\mathbb{R}H_{\epsilon_i}$. We can then apply Lemma 2.2 to obtain a contradiction.

If $\delta_2 = \epsilon_r \pm \epsilon_s$, we divide the proof in two cases, according to $\mathfrak{g}_{\epsilon_i} \subset \mathfrak{h}$ or $\mathfrak{g}_{\epsilon_i} \not\subseteq \mathfrak{h}$. In the first case, define $\alpha = \epsilon_i + \epsilon_r$, $\beta = \epsilon_j + \epsilon_s$. Then, since $H_{\epsilon_i} \in \mathfrak{k}$, the corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. $\mathfrak{g}_\alpha$ is preserved by σ since any root $\gamma \in \delta_{\mathfrak{h}}$ which is not orthogonal to ϵ_i is such that $\mathfrak{g}_\gamma \subset \mathfrak{k}$. In the second case, define $\alpha = \epsilon_i$, $\beta = \epsilon_j + \epsilon_r$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. This is immediate if $\mathfrak{h}_2 \subset \mathfrak{k}$, if not, proceeding as in the case $\delta_2 = \epsilon_i - \epsilon_j$, it is possible to find an element of K which acts trivially on $\mathfrak{g}_\alpha$ and non-trivially on $\mathfrak{g}_\beta$. The only root of $\Delta_\mathfrak{h} \setminus \Delta_\mathfrak{k}$ which is not orthogonal to α is $\epsilon_i - \epsilon_j$. We may assume that $\mathfrak{g}_{\epsilon_i - \epsilon_j} \not\subseteq \mathfrak{h}$, then $\mathfrak{g}_\alpha$ is preserved by σ. Since $\mathfrak{g}_{\epsilon_j}$ is in the K-orbit of $\mathfrak{g}_\alpha$, we may conclude using Lemma 2.2.

If $\delta_2 = \epsilon_r$, we define $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_j + \epsilon_s$ (where ϵ_r is non-orthogonal to some root of $\Delta_{\mathfrak{h}_3}$) if $\epsilon_i \in \Delta_\mathfrak{h}$, and $\alpha = \epsilon_i$, $\beta = \epsilon_r + \epsilon_s$ if $\mathfrak{g}_{\epsilon_i} \not\subseteq \mathfrak{h}$. We omit the proof which is similar to the one of the previous case.

The only case left is $\delta_1 = \epsilon_i$ and $\delta_2 = \epsilon_r \pm \epsilon_s$. In this case, we may assume that $\mathfrak{h}_2$ never lies in the kernel of the slice representation unless we are in one of the previously considered cases. Then, σ preserves any root space. We define $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_s$. We get a contradiction using Lemma 2.2.

(2) Since rank $(\mathfrak{k}) <$ rank $(\mathfrak{h})$, the same proof given for the case (1) works also in this case except when $\delta_1 = \epsilon_i \pm \epsilon_j$ and $\delta_2 = \epsilon_r$. Here, we must define $\alpha = \epsilon_i$ and $\beta = \epsilon_j + \epsilon_r$. Then, we get a contradiction using Lemma 2.2.

(3) Here, we get a contradiction assuming $\mathfrak{h}_1 \subset \ker(\nu)$ and rank $(\mathfrak{h}_1) > 1$. Since the slice representation is at the singular orbit, P is even dimensional and $\mathfrak{h}$ is a maximal rank subalgebra of $\mathfrak{b}_n$. We may always assume that σ acts on $\mathfrak{m}$ as an element of the center of H. In particular, any root space is preserved by σ.

We consider, first, the case when $\delta_{1a} = \epsilon_i \pm \epsilon_j$, $\delta_{1b} = \epsilon_i \pm \epsilon_r \in \Delta_{\mathfrak{h}_1}$ with $r \neq j$ and $\delta_2 = \epsilon_p + \epsilon_q$. Then, we define $\alpha = \epsilon_j + \epsilon_p$ and $\beta = \epsilon_r + \epsilon_q$.

If $\delta_{1a} = \epsilon_i \pm \epsilon_j$, $\delta_{1b} = \epsilon_i \pm \epsilon_r$ and $\delta_2 = \epsilon_p$, then we define $\alpha = \epsilon_r + \epsilon_p$ and $\beta = \epsilon_i \mp \epsilon_j$ (note that in this case, we may assume that $\mathfrak{g}_{\epsilon_i} \not\subseteq \mathfrak{h}$).

If $\delta_{1a} = \epsilon_i \pm \epsilon_j$, $\delta_{1b} = \epsilon_i$ and $\delta_2 = \epsilon_p + \epsilon_q$, then we define $\alpha = \epsilon_i + \epsilon_p$ and $\beta = \epsilon_j + \epsilon_q$.

The corresponding root spaces are orthogonal with respect to any choice of Ad (K)-invariant scalar product, then we may apply Lemma 2.2. q.e.d.

Lemma 3.3. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{c}_n$.

Proof. In this case, as the slice representation at the first singular orbit is even dimensional and $\mathfrak{h}$ is a maximal rank subalgebra of $\mathfrak{c}_n$, we may always assume that σ belongs to the center of H. In particular, any root space is preserved by σ.

(1) We consider, first, the case when $\delta_1 = \epsilon_i + \epsilon_j$.

If $\delta_2 = \epsilon_r + \epsilon_s$, we define $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_j - \epsilon_s$. The corresponding root spaces are always orthogonal with respect to any Ad (K)-invariant scalar product. In fact, if we denote by H_δ an element of K which projects on H_{δ_2}, the action of H_δ and H_{δ_2} is not compatible with any equivariant equivalence between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$. $\mathfrak{g}_\alpha$ is preserved by σ. We have to prove it only in the case when $\mathfrak{h}_2 \subset \ker(\nu)$. But then, σ has always a trivial projection on the root spaces $\mathfrak{g}_\gamma \subset \mathfrak{h} \setminus \mathfrak{k}$ with γ non-orthogonal to α.

The case $\delta_2 = \epsilon_r - \epsilon_s$ can be treated in the same way.

If $\delta_2 = 2\epsilon_r$, we define $\alpha = 2\epsilon_i$ and $\beta = \epsilon_j + \epsilon_r$ if $2\epsilon_i \notin \Delta_{\mathfrak{h}}$ (the corresponding root spaces are orthogonal unless we are in one of the previous cases) and $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_j - \epsilon_s$ (where ϵ_s is non orthogonal to some root of $\Delta_{\mathfrak{h}_3}$) if $2\epsilon_i \in \Delta_{\mathfrak{h}}$. Then, the proof is similar to the proof of the previous cases.

The cases $\delta_1 = \epsilon_i - \epsilon_j$ and $\delta_1 = 2\epsilon_i$, $\delta_2 = \epsilon_r \pm \epsilon_s$ can be treated in the same way.

We are left with the case $\delta_1 = 2\epsilon_i$ and $\delta_2 = 2\epsilon_r$. If there exists an index s such that $2\epsilon_s \notin \Delta_{\mathfrak{h}}$, then we define $\alpha = \epsilon_i - \epsilon_r$ and $\beta = 2\epsilon_s$. If, for any index s, $2\epsilon_s \in \Delta_{\mathfrak{h}}$ and $n > 3$, we define $\alpha = \epsilon_i - \epsilon_r$ and $\beta = \epsilon_j - \epsilon_s$. In both these cases, the proof is similar to the one in the previous cases. If $\mathfrak{g} = \mathfrak{c}_3$, $\mathfrak{h} = \mathfrak{a}_1 + \mathfrak{a}_1 + \mathfrak{a}_1$ and $\mathfrak{k} = \mathfrak{a}_1 + \mathfrak{a}_1$, the complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ is the sum of irreducible K-modules, each preserved by σ. This implies that the singular orbit is totally geodesic. Up to an exchange of the role played by $\mathfrak{h}$ and $\mathfrak{h}'$, the only case we have to consider is $\mathfrak{h}' = \mathfrak{h}$. This implies that the singular orbits are both totally geodesic and we obtain a contradiction using Frankel's theorem.

(2) We omit the proof which is similar to the one of case (1).

(3) We prove that rank $(\mathfrak{h}_1) > 1$ implies that M is diffeomorphic to a compact rank one symmetric space.

Assume, first, $\delta_{1a} = \epsilon_i + \epsilon_j$ and $\delta_{1b} = \epsilon_i + \epsilon_r$ (with $2\epsilon_i \notin \Delta_{\mathfrak{h}}$). If $\delta_2 = \epsilon_p + \epsilon_q$, then let $\alpha = \epsilon_i + \epsilon_p$, $\beta = \epsilon_r + \epsilon_q$. If $\delta_2 = 2\epsilon_p$, let $\alpha = \epsilon_i - \epsilon_j$ and $\beta = \epsilon_r + \epsilon_p$. Then, we may apply Lemma 2.2.

If $\delta_{1a} = \epsilon_i + \epsilon_j$, $\delta_{1b} = 2\epsilon_i$ and $\delta_2 = \epsilon_r \pm \epsilon_s$, then let $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_j + \epsilon_s$. Then, apply Lemma 2.2.

If $\delta_{1a} = \epsilon_i + \epsilon_j$, $\delta_{1b} = 2\epsilon_i$ and $\delta_2 = 2\epsilon_2$, and we are not in one of the previous cases, then $\mathfrak{h} = \mathfrak{a}_1 + \mathfrak{c}_{n-1}$, $\mathfrak{k} = \mathfrak{c}_{n-1}$. Then, $\mathfrak{m}$ is irreducible as K-module, this implies $\mathfrak{h}' = \mathfrak{c}_n$ and M is equivariantly diffeomorphic to $\mathbb{HP}^n$ as described in [**17**] (Section 4.2, p. 487). q.e.d.

Lemma 3.4. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{d}_n$.

Proof. We assume $n > 3$. Note that in this case, the simple factors in a maximal rank subalgebra of $\mathfrak{d}_n$ are $\mathfrak{a}_p$ or $\mathfrak{d}_p$. Moreover, since the slice representation at the first singular orbit is even dimensional, the case

$\nu(\mathfrak{h}) = \mathfrak{a}_p$ never occurs when p is even since we have always $SU(p+1) \subset U(p+q)$ and the center of $U(p+1)$ never lies in the kernel of the slice representation. Hence, we may always assume that σ belongs to the center of H and any root space is preserved by σ. Note also that $\nu(\mathfrak{h})$ contains two simple ideals only in the case $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$.

(1) Consider, first, the case when $\delta_1 = \epsilon_i + \epsilon_j$ and $\delta_2 = \epsilon_i - \epsilon_j$. If $\mathfrak{g}_{\delta_2} \subset \ker(\nu)$, then define $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_j + \epsilon_s$ where $\delta_3 = \epsilon_r + \epsilon_s$. Otherwise, we may assume that $\epsilon_r - \epsilon_s \notin \Delta_{\mathfrak{h}_3}$ and let $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_r - \epsilon_s$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product and we may apply Lemma 2.2. The case corresponding to the choice $\delta_3 = \epsilon_r - \epsilon_s$ can be treated in the same way.

If $\delta_1 = \epsilon_i + \epsilon_j$ and $\delta_2 = \epsilon_r + \epsilon_s$, we define $\alpha = \epsilon_i + \epsilon_r$ and $\beta = \epsilon_j - \epsilon_s$. The corresponding root spaces are trivially inequivalent if $\mathfrak{g}_{\delta_2} \subset \ker(\nu)$ or $\mathfrak{g}_{\epsilon_i - \epsilon_j} \subset \ker(\nu)$. Otherwise, $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$ and $\ker(\nu)$ contains $H \in \mathfrak{c}$ such that $\delta_1(H) = \delta_2(H) = 0$, but $\epsilon_i(H) \neq 0$. Then again, we can prove that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product and we can conclude using Lemma 2.2. The proof for the cases left is similar and we omit it.

(2) The proof given in (1) also covers this case.

(3) We obtain a contradiction assuming that $\ker(\nu)$ contains simple ideals of rank greater than one isomorphic to either $\mathfrak{a}_p$ or $\mathfrak{d}_p$. Let $\delta_{1a} = \epsilon_i + \epsilon_j$, $\delta_{1b} = \epsilon_i + \epsilon_r$ and $\delta_2 = \epsilon_p + \epsilon_q$. Then, let $\alpha = \epsilon_i + \epsilon_q$ and $\beta = \epsilon_r + \epsilon_q$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product, and we can conclude using Lemma 2.2. A similar proof works for the remaining cases and we omit it. q.e.d.

Lemma 3.5. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{f}_4$*.*

Proof.
(1) We give the proof for the case $\delta_1 = (\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)/2$, $\delta_2 = \epsilon_1-\epsilon_2$, $\delta_3 = \epsilon_3 - \epsilon_4$. The proof in the omitted cases is either similar to this one or to one in Lemma 3.2. If $\delta = \epsilon_1 + \epsilon_2 \in \Delta_{\mathfrak{h}}$, then let $\alpha = \epsilon_1$ and $\beta = \epsilon_3$. Since δ is not orthogonal to δ_1, $H_\delta \in \mathfrak{k}$. Then, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. $\mathfrak{g}_\alpha$ is preserved by σ since either σ belongs to the center of H or σ acts trivially on $\mathfrak{g}_\alpha$.

If $\epsilon_1 + \epsilon_2 \notin \Delta_{\mathfrak{h}}$, let $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = \epsilon_p$. In both cases, we can conclude using Lemma 2.2.

(2) We omit the proof which is similar to the one of the previous case.

(3) There is only one case to consider, $\mathfrak{h} = \mathfrak{a}_2 + \overline{\mathfrak{a}_2}$, and we may assume $\overline{\mathfrak{a}_2} \subset \ker(\nu)$. Since the maximal subgroup corresponding to $\mathfrak{h}$ is

$SU(3) \times SU(3)/\Delta\mathbb{Z}_3$, $\nu(\mathfrak{h})$ cannot act transitively on a sphere, and this case does not occur. q.e.d.

Lemma 3.6. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{e}_6$.

Proof.

(1) If $\delta_1 = \epsilon_i - \epsilon_j$ and $\delta_2 = \epsilon_r - \epsilon_s$, the following are possible cases for δ_3: $\epsilon_p - \epsilon_q$, 2ϵ, $\epsilon_i + \epsilon_j + \epsilon_p \pm \epsilon$, $\epsilon_r + \epsilon_s + \epsilon_p \pm \epsilon$ where $p \neq q \notin \{i, j, r, s\}$. First, note that a maximal rank subalgebra of $\mathfrak{e}_6$ never contains a simple ideal $\simeq \mathfrak{c}_n$ for $n > 1$. Then, the only case when $\nu(\mathfrak{h})$ contains two simple ideals is $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$. In this case, σ preserves any root space, and we can choose $\alpha = \epsilon_i - \epsilon_r$ and $\beta = \epsilon_s - \epsilon_p$ or $\beta = \epsilon_s + \epsilon_p + \epsilon_q + \epsilon$. The corresponding root spaces are in $\mathfrak{m}$ and we may apply Lemma 2.2. We are left with the case $\mathfrak{h}_1, \mathfrak{h}_2 \subset \ker(\nu)$. If $\mathrm{rank}\,(\mathfrak{h}_3) = 1$, we let (α, β) be equal to $(\epsilon_i - \epsilon_r, \epsilon_s - \epsilon_p)$, $(\epsilon_i - \epsilon_r, \epsilon_s + \epsilon_p + \epsilon_q + \epsilon)$, $(\epsilon_i - \epsilon_p, \epsilon_s - \epsilon_q)$ or $(\epsilon_i - \epsilon_q, \epsilon_r - \epsilon_p)$ according to the different choices of δ_3. If $\mathrm{rank}\,(\mathfrak{h}_3) = 2$, then $\mathfrak{h}_3 \simeq \mathfrak{a}_2$. We may assume that $\mathfrak{a}_2$ corresponds to one of the following choices of root spaces $\{\epsilon_p - \epsilon_q, \epsilon_i + \epsilon_j + \epsilon_p + \epsilon, \epsilon_i + \epsilon_j + \epsilon_q + \epsilon)$ or $\{2\epsilon, \epsilon_i + \epsilon_j + \epsilon_p + \epsilon, \epsilon_i + \epsilon_j + \epsilon_p - \epsilon)$. In the first case, let $\alpha = \epsilon_j + \epsilon_r + \epsilon_s - \epsilon$, $\beta = \epsilon_i + \epsilon_r + \epsilon_p + \epsilon$, in the second, $\alpha = \epsilon_j - \epsilon_p$, $\beta = \epsilon_r - \epsilon_q$. If $\mathrm{rank}\,(\mathfrak{h}_3) = 3$, then $\mathfrak{h}_3 \simeq \mathfrak{a}_3 \simeq \mathfrak{d}_3$ and σ preserves any root space. We may then conclude as in the case of $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$.

If $\delta_1 = \epsilon_i - \epsilon_j$, $\delta_2 = 2\epsilon$, then $\delta_3 = \epsilon_r - \epsilon_s$ with $\{r, s\} \cap \{i, j\} = \emptyset$. If $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$, then let $\alpha = \epsilon_i + \epsilon_r + \epsilon_s + \epsilon$, $\beta = \epsilon_p - \epsilon_q$ with $\{p, q\} \cap \{r, s\} = \emptyset$ and apply Lemma 2.2. Now, we may assume that $\nu(\mathfrak{h})$ contains just one simple ideal. If this ideal has rank 3, then σ preserves any root space. Otherwise, we may assume $H_\gamma \subset \ker(\epsilon_p)$ for any $\gamma \in \mathfrak{h}_3$. In both cases, let $\alpha = \epsilon_i - \epsilon_p$ and $\beta = \epsilon_r + \epsilon_s + \epsilon_q + \epsilon$ and apply Lemma 2.2.

If $\delta_1 = \epsilon_i - \epsilon_j$, $\delta_2 = \epsilon_i + \epsilon_j + \epsilon_r + \epsilon$ and $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$, we have to consider the cases $\delta_3 = \epsilon_p - \epsilon_q$, $\delta_3 = \epsilon_r + \epsilon_p + \epsilon_q + \epsilon$, $\delta_3 = \epsilon_s + \epsilon_p + \epsilon_q + \epsilon$ where $\{p, q, s\} \cap \{i, j, r\} = \emptyset$. Then, let (α, β) be equal to $(\epsilon_j + \epsilon_r + \epsilon_p + \epsilon, \epsilon_r + \epsilon_p + \epsilon_s - \epsilon)$ in the first case, and $(\epsilon_i - \epsilon_p, \epsilon_r - \epsilon_q)$ in the last two cases.

If $\nu(\mathfrak{h})$ contains just one simple ideal and $\mathrm{rank}\,(\mathfrak{h}_3) = 3$, then σ preserves any root space and we let $\alpha = \epsilon_i - \epsilon_r$, $\beta = \epsilon_j - \epsilon_q$. If $\mathrm{rank}\,(\mathfrak{h}_3) < 3$, then $\mathfrak{h}_3 \simeq \mathfrak{a}_1$ or $\mathfrak{h}_3 \simeq \mathfrak{a}_2$. In both these cases (according to the different possible embeddings of $\mathfrak{h}_3$ in $\mathfrak{e}_6$), it is possible to find α and β such that all of the hypothesis of Lemma 2.2 are satisfied.

We omit the proof of the remaining cases since it is similar to the one in the cases already considered.

(2) The proof can be easily deduced from the one of case (1) and we omit it.

(3) The semisimple part of $\mathfrak{h}$ is either of the form $\mathfrak{a}_1+\mathfrak{a}_i$ with $2 \leq i \leq 5$ or $\mathfrak{a}_2+\mathfrak{a}_2$. In the first case, we assume $\mathfrak{a}_i \subset \ker(\nu)$. Then, since the slice representation is even dimensional, σ preserves any root space. If $i > 2$, then the semisimple part of $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{a}_1 + \mathfrak{a}_5$ and, assuming that $\mathfrak{a}_1$ corresponds to the root 2ϵ and $\mathfrak{a}_5$ corresponds to the roots (the proof in the other cases is similar) $\epsilon_i - \epsilon_j$, with $1 \leq i, j \leq 6$, we may apply Lemma 2.2 with $\alpha = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon$ $\beta = \epsilon_1 + \epsilon_2 + \epsilon_4 - \epsilon$. For $i = 2$, the proof divides in several cases according to the possible embeddings of $\mathfrak{h}$ in $\mathfrak{g}$, but it is similar to the previous one and we omit it.

In the second case, note that we cannot have $\nu(\mathfrak{h}) = \mathfrak{a}_2$. If $\mathfrak{h} \subset 3\mathfrak{a}_2$, this follows from the fact that corresponding maximal subgroup is $SU(3)^3/\Delta\mathbb{Z}_3$, and if $\mathfrak{h} \subset \mathfrak{a}_1 + \mathfrak{a}_5$, we already used this fact in the case $\mathfrak{g} = \mathfrak{a}_n$. Then, σ preserves any root space and we may conclude as in the previous cases. q.e.d.

Lemma 3.7. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{e}_7$*.*

Proof.

(1) If $\delta_1 = \epsilon_i - \epsilon_j$, $\delta_2 = \epsilon_r - \epsilon_s$, we have two possible choices for δ_3: $\delta_3 = \epsilon_i + \epsilon_j + \epsilon_r + \epsilon_s$ or $\delta_3 = \epsilon_p - \epsilon_q$ with $\{p, q\} \cap \{i, j, r, s\} = \emptyset$. Then, we may apply Lemma 2.2 with $\alpha = \epsilon_i - \epsilon_r$, $\beta = \epsilon_s - \epsilon_q$ where $q \notin \{i, j, r, s\}$ in the first case, and $\alpha = \epsilon_i - \epsilon_r$, $\beta = \epsilon_s - \epsilon_q$ in the second.

If $\delta_1 = \epsilon_i - \epsilon_j$, $\delta_2 = \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l$, we give the proof in the case $\delta_3 = \epsilon_k - \epsilon_l$ as the proof in the other cases is simpler. Note that a simple factor in $\nu(\mathfrak{h})$ is either of the form $\mathfrak{a}_m$ or $\mathfrak{d}_m$. We may assume $\nu(\mathfrak{h}) \neq \mathfrak{a}_4$ since there is no such factor in a maximal subalgebra of $\mathfrak{e}_7$ containing three simple ideals. Then, with the only exception of the case $\nu(\mathfrak{h}) = \mathfrak{a}_2$, we may assume that σ preserves any root space.Then, if $\nu(\mathfrak{h}) \neq \mathfrak{a}_2$, we may apply Lemma 2.2 with $\alpha = \epsilon_i - \epsilon_p$ and $\beta = \epsilon_k - \epsilon_q$ with $\{p, q\} \cap \{i, j, k, l\} = \emptyset$. If $\nu(\mathfrak{h}) = \mathfrak{a}_2$, we may assume $\operatorname{rank}(\mathfrak{h}_i) \leq 2$ for $i = 1, 2$ or $\mathfrak{h}_i = \mathfrak{a}_1$ for $i = 1$ or $i = 2$. In both cases, it is possible to find r, s such that the root spaces corresponding to $\alpha = \epsilon_i - \epsilon_r$ and $\beta = \epsilon_k - \epsilon_s$ are in $\mathfrak{m}$ and $\mathfrak{g}_\alpha$ is preserved by σ. Then, we may conclude using Lemma 2.2.

The proof in the remaining cases is similar and we omit it.

(2) We omit the proof which is similar to the one of case (1).

(3) We give the proof for the case $\delta_1 = \epsilon_i - \epsilon_j$, $\delta_2 = \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l$, the proof in the remaining cases is similar. The root system of $\mathfrak{h}_1$ must contain a root γ which is not orthogonal to δ_1 and it is orthogonal

to δ_2, we may assume it is of the form $\epsilon_i - \epsilon_k$. If $\nu(\mathfrak{h}) \neq \mathfrak{a}_2$, we define $\alpha = \epsilon_i - \epsilon_p$ and $\beta = \epsilon_j - \epsilon_q$ with $\{p, q\} \cap \{i, j, k, l\} = \emptyset$. The corresponding root spaces ore orthogonal because of the action of H_γ. If $\nu(\mathfrak{h}) = \mathfrak{a}_2$, the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ contains a simple ideal which is not in $\mathfrak{h}$ (this follows also from the form of the maximal subgroups of G corresponding to the maximal rank subalgebras of $\mathfrak{g}$). We may then assume the existence of a root $\delta = \epsilon_p - \epsilon_q$ which is strongly orthogonal to the roots of $\mathfrak{h}$. The center of $\ker(\nu)$ contains also an element which acts non-trivially on $\mathfrak{g}_\delta$. We may, then, define $\alpha = \epsilon_p - \epsilon_q$, $\beta = \epsilon_i - \epsilon_r$ with $r \notin \{i, j, k, l\}$ and conclude using Lemma 2.2. q.e.d.

Lemma 3.8. *Proposition 3.1 in the case of* $\mathfrak{g} = \mathfrak{e}_8$.

Proof. We omit the proof which is similar to the one of the case $\mathfrak{e}_7$. q.e.d.

This concludes the proof of Proposition 3.1. q.e.d.

4. Structure of the center of $\mathfrak{g}$

Now, we prove that the center of $\mathfrak{h}$ is at most one dimensional. First, we exclude the case when $\mathfrak{h}$ is abelian, then we consider the case when $\mathfrak{h}$ contains at least one simple factor.

Proposition 4.1. *Under the assumptions of Lemma 2.2, if* (M, g) *is positively curved, then* $\mathfrak{h}$ *is not abelian.*

Proof. Let $\mathfrak{h} = \mathbb{R}^n$, $\mathfrak{k} = \mathbb{R}^{n-1}$ and let α and β be two strongly orthogonal roots. Then, in order to apply Lemma 2.2, it is sufficient to prove that the corresponding root spaces are inequivalent as K-modules, in fact each root space is preserved by σ.

Suppose that $\phi : \mathfrak{g}_\alpha \to \mathfrak{g}_\beta$ is an equivalence between K-modules. Then, $\forall Z \in \mathfrak{k}, \forall X \in \mathfrak{g}_\alpha^{\mathbb{C}}$, $ad(Z)\phi(X) = \phi([Z, X])$. Hence, $\alpha(Z) = \pm\beta(Z)$. Since $\dim \mathfrak{k} = n-1$, this implies $\mathfrak{k} = \ker(\alpha+\beta)$ or $\mathfrak{k} = \ker(\alpha-\beta)$.

If $\mathfrak{g} = \mathfrak{a}_n$, since $n \geq 3$, we have at least 3 pairs of strongly orthogonal roots. If, for a given choice of $\mathfrak{k}$, α and β are strongly orthogonal and such that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are equivalent as K-modules, it is always possible to find another pair of strongly orthogonal roots such that the corresponding root spaces are inequivalent non-trivial K-modules. If, for example, $\mathfrak{k} = \ker(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$, then $\alpha = \epsilon_1 - \epsilon_4$ and $\beta = \epsilon_2 - \epsilon_3$ are inequivalent and non-trivial. We may then apply Lemma 2.2 to obtain a contradiction.

If $\mathfrak{g} = \mathfrak{b}_n$, let $\alpha = \epsilon_1 - \epsilon_2$ and $\beta = \epsilon_1 + \epsilon_2$, unless $\mathfrak{k} = \ker(\alpha \pm \beta)$ the corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. If K is connected, we may still apply a slightly modified version of Lemma 2.2 (see Remark 2.2) to obtain a contradiction since σ acts as $-id$ on both the root spaces. So, we will assume that K is not connected and $\mathfrak{k} = \ker(\epsilon_2)$. In this case, $\mathfrak{g}_{\epsilon_1}$ is a 2-dimensional irreducible K-module which is not equivalent to any other K-module. For a given $X \in \mathfrak{g}_{\epsilon_1}$, it is possible to find $Y \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ such that $[X,Y] = 0$ and such that for any $Z \in \mathfrak{m} + \mathfrak{p}$, we have $\langle [Z,X], Y\rangle_p = \langle [Z,Y], X\rangle_p = 0$. Since $\mathfrak{g}_{\epsilon_1}$ is preserved by σ, we get a contradiction using the proof of Lemma 2.2 (see Remark 2.2).

If $\mathfrak{g} = \mathfrak{c}_n$, as $\mathfrak{c}_2 \simeq \mathfrak{b}_2$, we may assume $n \geq 3$. If $\mathfrak{k} \neq \ker(2\epsilon_1 \pm 2\epsilon_2)$, then let $\alpha = 2\epsilon_1$ and $\beta = 2\epsilon_2$. At least one of these root spaces is non-trivial as K-module and we may apply Lemma 2.2. Otherwise, define $\alpha = 2\epsilon_i$ and $\beta = 2\epsilon_r$.

If $\mathfrak{g} = \mathfrak{d}_n$, we may assume $n > 3$. Let $\delta_1 = \epsilon_i + \epsilon_j$ and $\delta_2 = \epsilon_i - \epsilon_j$. The corresponding root spaces are equivalent as K-modules if $\mathfrak{k} = \ker(\epsilon_i)$ or $\mathfrak{k} = \ker(\epsilon_j)$. In any case, it is possible to find two strongly orthogonal roots $\alpha = \epsilon_r + \epsilon_s$ and $\beta = \epsilon_r - \epsilon_s$ such that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. Then, we can conclude using Lemma 2.2.

If $\mathfrak{g} = \mathfrak{f}_4$, let $\alpha = \epsilon_1 + \epsilon_2$, $\beta_1 = \epsilon_1 - \epsilon_2$, $\beta_2 = \epsilon_3 - \epsilon_4$. For any possible choice of $\ker(\nu)$, we may apply Lemma 2.2 to one of the pairs (α, β_1) or (α, β_2).

If $\mathfrak{g} = \mathfrak{e}_i, i = 6, 7, 8$, we can conclude using a proof similar to the one of the case $\mathfrak{a}_n$. q.e.d.

Proposition 4.2. *Under the assumptions of Lemma 2.2, if (M, g) is positively curved, then the center of $\mathfrak{h}$ is at most one dimensional.*

Proof. We prove the result by contradiction. By Proposition 4.1, we may suppose $\mathfrak{h} = \mathbb{R}^k + \mathfrak{h}_0$, where $k \geq 2$ and $\mathfrak{h}_0$ is a non-trivial semisimple factor.

Lemma 4.1. *Proposition 4.2 in the case of $\mathfrak{g} = \mathfrak{a}_n$.*

Proof. We have two possible choices for $\mathfrak{h}$: (1) $\mathfrak{h} = \mathbb{R}^k + \mathfrak{a}_{n-k}$ or (2) $\mathfrak{h} = \mathbb{R}^k + \mathfrak{a}_1 + \mathfrak{a}_{n-k-1}$.

(1) We may assume that $n \geq 3$ and that there exists $p_0 \geq 2$ such that $\mathfrak{a}_{n-k} \subset \ker(\epsilon_p)$ for $p \leq p_0$ and $\epsilon_p - \epsilon_q$ is a root of $\mathfrak{a}_{n-k}$ for $p, q > p_0$. The possible choices for $\mathfrak{k}$ are $\mathfrak{k} = \mathbb{R}^{k-1} + \mathfrak{a}_{n-k}$, $\mathfrak{k} = R^{k-1} + \mathbb{R}^\Delta + \mathfrak{a}_{n-k-1}$, $\mathfrak{k} = R^k + \mathfrak{a}_{n-k-1}$, where Δ denotes a diagonal embedding.

When $\mathfrak{k} = \mathbb{R}^{k-1} + \mathfrak{a}_{n-k}$, σ lies in the center of H. Hence, in order to apply Lemma 2.2, we must show that two root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product.

If $n - k \geq 2$, let $\alpha = \epsilon_i - \epsilon_p$, $\beta = \epsilon_j - \epsilon_q$, with $i < j \leq p_0$ and $q > p > p_0$. These roots are strongly orthogonal and the corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product, due to the action of the elements of the Cartan subalgebra of $\mathfrak{a}_{n-k}$.

If $k > 2$, let $i < j < l \leq p_0$, $p > p_0$. Let $\alpha = \epsilon_i - \epsilon_j$ and $\beta = \epsilon_l - \epsilon_p$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product since the simple factor in $\mathfrak{h}$ acts trivially on $\mathfrak{g}_\alpha$ and non-trivially on $\mathfrak{g}_\beta$.

If $n = 3$ and $\mathfrak{h} = \mathbb{R}^2 + \mathfrak{a}_1$, if $\alpha = \epsilon_1 - \epsilon_3$ and $\epsilon_2 - \epsilon_4$ $(p = 2)$, the corresponding root spaces are inequivalent as K-modules except when $diag(1, -1, 0, 0)$ lies in the kernel of the slice representation. But in this case, we have $\mathfrak{k} \subset \mathfrak{a}_1 + \mathfrak{a}_1$ and the same proof of Lemma 3.1, case 2) gives us the result.

If $\mathfrak{k} = \mathbb{R}^{k-1} + \mathbb{R}^{\Delta} + \mathfrak{a}_{n-k-1}$, and $k \geq 3$, we get a contradiction with the same proof of Proposition 4.1. Let us assume that $k = 2$ and $p_0 = 2$. The element $\nu(\sigma)$ does not depend on the choice of the normal geodesic $\gamma(t)$ through $p = \gamma(0)$ and belongs to the center of $\nu(H)$, i.e., we may assume that σ belongs to the orthogonal complement of the kernel of the slice representation in the center of $\mathfrak{h}$. The kernel of the slice representation is one dimensional and is generated by an element T which commutes with $\mathfrak{a}_{n-k}$. Define $\alpha = \epsilon_1 - \epsilon_3$ and $\beta = \epsilon_2 - \epsilon_4$. If the corresponding root spaces are equivalent as K-modules, then we may assume $T = (i, -i, 0, \ldots, 0)$ or $T = (i, i, ai, \ldots, ai)$, $a = -\frac{2}{(n-2)}$. In both cases, σ acts as $-id$ on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$. This implies that, even if these root spaces are non-orthogonal for the generic choice of an $\mathrm{Ad}\,(K)$-invariant scalar product in $\mathfrak{m}$, at p the scalar product $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle_{|\gamma(t)}$ is an even function of t. Since both the root spaces are preserved by σ and are entirely contained in some K-orbits, we may apply a slightly modified version of Lemma 2.2 to obtain a contradiction (see Remark 2.2). For any other choice of T, the two root spaces are inequivalent and we may apply Lemma 2.2.

If $\mathfrak{k} = \mathbb{R}^k + \mathfrak{a}_{n-k-1}$, we get a contradiction with the same proof of Proposition 4.1.

(2) We may assume that $n \geq 3$ and that there exists $p_0 > 2$ such that $\mathfrak{a}_{n-k-1} \subset \ker(\epsilon_p)$ for $p \leq p_0$, $\epsilon_p - \epsilon_q$ is a root of $\mathfrak{a}_{n-k-1}$ for $p, q > p_0$ and that $\epsilon_1 - \epsilon_2$ is a root for $\mathfrak{a}_1$.

The possible choices for $\nu(\mathfrak{h})$ are $\nu(\mathfrak{h}) = \mathbb{R} + \mathfrak{a}_{n-k-1}$, $\nu(\mathfrak{h}) = \mathfrak{a}_{n-k-1}$, $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$.

If $\nu(\mathfrak{h}) = \mathbb{R} + \mathfrak{a}_{n-k-1}$, then let $\alpha = \epsilon_1 - \epsilon_r$ and $\beta = \epsilon_{p_0} - \epsilon_s$ with $r, s > p_0$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product since the $\mathfrak{a}_1$ factor acts trivially on $\mathfrak{g}_\beta$. Both are preserved by σ which belongs to the center of H.

If $\nu(\mathfrak{h}) = \mathfrak{a}_{n-k-1}$, then let $\alpha = \epsilon_1 - \epsilon_{p_0}$ and $\beta = \epsilon_2 - \epsilon_r$ with $r > p_0$. $\mathfrak{k}$ contains an element of the form $diag(0, \ldots, 0, (n-k-1)i, -i, \ldots, -i)$. Hence, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are inequivalent as K-modules. The K-orbit containing $\mathfrak{g}_\alpha$ is preserved by σ.

If $\nu(\mathfrak{h}) = \mathfrak{a}_1 + \mathfrak{a}_1$, then let $\alpha = \epsilon_1 - \epsilon_{p_0}$ and $\beta = \epsilon_2 - \epsilon_r$ with $r > p_0$. As in the previous case, the corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. Both are preserved by σ which belongs to the center of H.

In all of these cases, we may then apply Lemma 2.2 to obtain a contradiction. q.e.d.

Lemma 4.2. *Proposition 4.2 in the case of* $\mathfrak{g} = \mathfrak{b}_n$.

Proof. Let $\mathfrak{c}_0 = \mathfrak{h}_0 \cap \mathfrak{c}$, we may assume that one of the following occurs: (1) $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\epsilon_2)$, (2) $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\epsilon_2 \pm \epsilon_3)$, (3) $\mathfrak{c}_0 \subset \ker(\epsilon_1 \pm \epsilon_2) \cap \ker(\epsilon_3 \pm \epsilon_4)$, (4) $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\sum_{i>1} \epsilon_i)$.

(1) Let $\alpha = \epsilon_1$ and $\beta_\pm = \epsilon_2 \pm \epsilon_r$ for some $r > 2$. Since the kernel of the slice representation contains an $\mathbb{R}^{k-1}$ factor, we may assume that $\mathfrak{g}_\alpha$ is non-trivial as K-module (otherwise, define $\alpha = \epsilon_2$ and $\beta_\pm = \epsilon_1 \pm \epsilon_r$). Then, $\mathfrak{g}_\alpha$ is preserved by σ and it is non equivalent, as K-module, to at least one between $\mathfrak{g}_{\beta_+}$ and $\mathfrak{g}_{\beta_-}$. We may then apply Lemma 2.2 to obtain a contradiction.

(2) We give the proof for the case $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\epsilon_2 + \epsilon_3)$, the remaining case can be treated in the same way. If the center of $\mathfrak{h}$ lies in the kernel of the slice representation, we let $\alpha = \epsilon_1$ and $\beta = \epsilon_2 + \epsilon_3$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product due to the action of $\mathbb{R}^k$. Since $\mathfrak{g}_\alpha$ is preserved by σ, we may apply Lemma 2.2. Otherwise, $\ker(\nu)$ contains $\mathbb{R}^{k-1}$. In this case, σ preserves any root space. If $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are not orthogonal, we may define $\alpha = \epsilon_r$ and $\beta_\pm = \epsilon_i \pm \epsilon_s$. For at least one choice of the sign, the corresponding root spaces are orthogonal and we may apply Lemma 2.2.

(3) we give the proof for the case $\mathfrak{c}_0 \subset \ker(\epsilon_1 + \epsilon_2) \cap \ker(\epsilon_3 + \epsilon_4)$ since the proof in the other cases is similar. Let $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = \epsilon_3 + \epsilon_4$. At least one of them (say $\mathfrak{g}_\alpha$) is preserved by σ. If the corresponding

root spaces are not orthogonal, then define $\beta = \epsilon_r$. We may then apply Lemma 2.2.

(4) Let $\alpha = \epsilon_1$ and $\beta = \epsilon_2 + \epsilon_3$. It is always possible to find an element in $\mathfrak{k}$ which acts trivially on $\mathfrak{g}_\alpha$ and non-trivially on $\mathfrak{g}_\beta$. At least one of these root spaces is preserved by σ and lies in a non-trivial K-module. q.e.d.

Lemma 4.3. *Proposition 4.2 in the case of* $\mathfrak{g} = \mathfrak{c}_n$.

Proof. Let $\mathfrak{c}_0 = \mathfrak{h}_0 \cap \mathfrak{c}$, we may assume that one of the following occurs: (1) $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\epsilon_2)$, (2) $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\epsilon_2 + \cdots + \epsilon_n)$, (3) $\mathfrak{c}_0 \subset \ker(\epsilon_1 + \cdots + \epsilon_p) \cap \ker(\epsilon_{p+1} + \cdots + \epsilon_n)$.

In case (1), let $\alpha = 2\epsilon_1$ and $\beta = 2\epsilon_2$. The corresponding root spaces are non-equivalent with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. At least one of them is non-trivial and it is preserved by σ. Then, we may apply Lemma 2.2.

In case (2), let $\alpha = 2\epsilon_1$ and $\beta_1 = \epsilon_2 + \epsilon_3$, $\beta_2 = 2\epsilon_2$. For at least one index i, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{\beta_i}$ are non-equivalent with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. At least one of these root spaces is non-trivial as K-module and it is preserved by σ.

In case (3), let $\alpha = 2\epsilon_1$ and $\beta_1 = 2\epsilon_2$, $\beta_2 = 2\epsilon_3$. Then, the proof is the same of the previous case. q.e.d.

Lemma 4.4. *Proposition 4.2 in the case of* $\mathfrak{g} = \mathfrak{d}_n$.

Proof. Let $\mathfrak{c}_0 = \mathfrak{h}_0 \cap \mathfrak{c}$, we assume that one of the following occurs: (1) $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\epsilon_2)$, (2) $\mathfrak{c}_0 \subset \ker(\epsilon_1) \cap \ker(\epsilon_2 + \cdots + \epsilon_n)$, (3) $\mathfrak{c}_0 \subset \ker(\epsilon_1 + \cdots + \epsilon_p) \cap \ker(\epsilon_{p+1} + \cdots + \epsilon_q)$.

The proof in the remaining cases is similar and we omit it.

In case (1), let $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = \epsilon_1 - \epsilon_2$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. This is trivial if $\mathbb{R}^k \subset \ker(\nu)$, otherwise $\mathfrak{k}$ has a surjective projection on $\mathbb{R}^k$ and acts on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ as $\mathbb{R}^k$. Then, we may apply Lemma 2.2.

In case (2), let $\delta_1 = \epsilon_1 + \epsilon_2$ and $\delta_2 = \epsilon_1 - \epsilon_2$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product in $\mathfrak{m}$ unless $\ker(\nu)$ contains H_{ϵ_1} or $H_{\epsilon_2} + \cdots + H_{\epsilon_n}$. In both cases, we may define $\alpha = \epsilon_1 + \epsilon_1$ and $\beta = \epsilon_3 + \epsilon_4$. The corresponding root spaces are orthogonal and we may apply Lemma 2.2.

In case (3), let $\alpha = \epsilon_1 + \epsilon_p$ and $\beta = \epsilon_2 - \epsilon_{p+1}$. If the corresponding root spaces are equivalent as $\ker(\nu)$-modules, then we may choose $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = \epsilon_p + \epsilon_{p+1}$. For at least one choice of α, β, we may apply Lemma 2.2. q.e.d.

Lemma 4.5. *Proposition 4.2 in the case of* $\mathfrak{g} = \mathfrak{f}_4$.

Proof. We have $\dim(\mathfrak{h}_0) \leq 2$. First, we consider the case when $\dim(\mathfrak{h}_0) = 1$. Then, we may assume that $\mathfrak{g}_\gamma \subset \mathfrak{h}_0$ for one of the following (1) $\gamma = \epsilon_1$, (2) $\gamma = \epsilon_1 - \epsilon_2$, (3) $\gamma = \frac{(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}{2}$. If $\mathfrak{z}(\mathfrak{h}) \subset \ker(\nu)$, then let $(\alpha, \beta) = (\epsilon_2 + \epsilon_3, \epsilon_4)$ in case (1), $(\epsilon_1 + \epsilon_2, \epsilon_3 + \epsilon_4)$ in case (2) and $(\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2)$ in case (3). In the remaining cases, we have that any root space is preserved by σ. If $\mathfrak{h}_0 \subset \ker(\nu)$, then let $(\alpha, \beta) = (\epsilon_1 + \epsilon_2, \epsilon_3 + \epsilon_4)$ in case (1), $(\epsilon_1, \epsilon_3 + \epsilon_4)$ in the other cases. If $\nu(\mathfrak{h}) = \mathbb{R} + \mathfrak{a}_1$, then $\mathfrak{k}$ has a surjective projection over the center of $\mathfrak{h}$ and the proof is similar to the previous ones since it is possible to choose α strongly orthogonal to the root corresponding to the simple part of $\mathfrak{h}$. In any of these cases, we may conclude using Lemma 2.2.

The proof in the case $\dim(\mathfrak{h}_0) = 2$ is similar and we omit it. q.e.d.

Lemma 4.6. *Proposition 4.2 in the case of* $\mathfrak{g} = \mathfrak{e}_6$.

Proof. If $\mathfrak{h} \subset \mathfrak{a}_1 + \mathfrak{a}_5$ and $k > 2$, then we may use the proof of the case $\mathfrak{a}_5$. Then, we have $\mathfrak{h} = \mathbb{R}^2 + \mathfrak{a}_4$ or $\mathfrak{h} = \mathbb{R}^2 + \mathfrak{a}_1 + \mathfrak{a}_3$. If the projection of the center of $\mathfrak{h}$ over the $\mathfrak{a}_5$ factor is two-dimensional, then we may again use the proof of the main result for the case $\mathfrak{a}_n$. Then, assuming that the $\mathfrak{a}_1$ factor is generated by the root 2ϵ while $\mathfrak{a}_5$ corresponds to the roots of the form $\epsilon_i - \epsilon_j$, we may find roots of the form $\alpha = \epsilon_i - \epsilon_j$ and $\beta = 2\epsilon$ such that the corresponding root spaces are in $\mathfrak{m}$. Since $\mathfrak{g}_\beta$ is preserved by σ, we may apply Lemma 2.2. We can treat in a similar way the case when $\mathfrak{h} \subset \mathbb{R} + \mathfrak{d}_5$ and we omit this proof. If $\mathfrak{h} \subset \mathfrak{a}_2^3$, the projection of the center of $\mathfrak{h}$ over one of the $\mathfrak{a}_2$ factors must be surjective and it must also be non-trivial over some other factor. Then, it is possible to choose roots $\alpha = \epsilon_i - \epsilon_j$, $\beta = \epsilon_i + \epsilon_j + \epsilon_k + \epsilon$ and we may apply Lemma 2.2 (up to an exchange of the role of α and β in certain cases). q.e.d.

Lemma 4.7. *Proposition 4.2 in the case of* $\mathfrak{g} = \mathfrak{e}_7$.

Proof. If the dimension of the center of $\mathfrak{h}$ is at least 3, its projection over at least one simple factor in a maximal subalgebra of maximal rank of $\mathfrak{g}$ has dimension at least 2 and we may adapt one of the previous proofs. We are then left with the cases (1) $\mathfrak{h} = \mathbb{R}^2 + \mathfrak{a}_5$, (2) $\mathfrak{h} = \mathbb{R}^2 + \mathfrak{a}_1 + \mathfrak{a}_4$ and (3) $\mathfrak{h} = \mathbb{R}^2 + \mathfrak{d}_5$.

In case (1), by the previous remark, we have $\mathfrak{h} \subset \mathfrak{a}_1 + \mathfrak{d}_6$ or $\mathfrak{h} \subset \mathbb{R} + \mathfrak{e}_6$. In the first case, we may assume that $\mathfrak{a}_5$ corresponds to the roots $\epsilon_i - \epsilon_j$ with $1 \leq i, j \leq 6$ while the center is spanned by $H_{\epsilon_7 - \epsilon_8}$ and $H_{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}$. If $\nu(\mathfrak{h}) = \mathbb{R}$, we let $\alpha = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_7$, $\beta = \epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_8$, if $\nu(\mathfrak{h}) = \mathbb{R} + \mathfrak{a}_5$, we let $\alpha = \epsilon_7 - \epsilon_8$, $\beta = \epsilon_1 + \epsilon_2 + \epsilon_7 + \epsilon_8$. The orthogonality between the corresponding root spaces is trivial in the first case and follows from the fact that $\mathfrak{k}$ has a surjective projection over the center of $\mathfrak{h}$. Since σ preserves all the root spaces, we may conclude using Lemma 2.2. In the second case, we have a different embedding for the center which can be assumed to be spanned by $H_{\epsilon_7-\epsilon_8}$ and $H_{\epsilon_7}+H_{\epsilon_8}$. Then, let $\alpha=\epsilon_1-\epsilon_7$, $\beta=\epsilon_2-\epsilon_8$ or $\alpha=\epsilon_7-\epsilon_8$, $\beta=\epsilon_1+\epsilon_2+\epsilon_7+\epsilon_8$ according to the two possibilities for $\nu(\mathfrak{h})$.

In case (2), we have to consider only the cases when $\mathfrak{h}\subset\mathfrak{a}_2+\mathfrak{a}_5$ or $\mathfrak{h}\subset\mathbb{R}+\mathfrak{e}_6$. There are two possible choices for $\nu(\mathfrak{h})$, $\nu(\mathfrak{h})=\mathbb{R}$ or $\mathbb{R}+\mathfrak{a}_4$. In any of these cases, σ preserves all the root spaces and we omit the proof as it is similar to the one of the previous case.

In case (3), we have to consider the cases when $\mathfrak{h}\subset\mathfrak{a}_1+\mathfrak{d}_6$ or $\mathfrak{h}\subset\mathbb{R}+\mathfrak{e}_6$. There are two possible choices for $\nu(\mathfrak{h})$, $\nu(\mathfrak{h})=\mathbb{R}$ or $\mathfrak{d}_5\rightarrow$. In any of these cases, σ preserves all the root spaces and the proof is similar to the one of the previous cases. q.e.d.

Lemma 4.8. *Proposition 4.2 in the case of* $\mathfrak{g}=\mathfrak{e}_8$.

Proof. We omit the proof which is similar to the one of the case $\mathfrak{e}_7$. q.e.d.

This concludes the proof of Proposition 4.2. q.e.d.

5. Proof of the main theorem

We are now ready to prove the main theorem of this paper. The results of the previous sections imply that we may restrict to the case when there is a singular orbit G/H such that the Lie algebra of the group H contains at most two simple factors and its center is at most one-dimensional. We analyze the possible cases left by a case by case check on the list of simple Lie algebras $\mathfrak{g}$.

5.1. Proof of the main Theorem in the case of $\mathfrak{g}=\mathfrak{a}_n$. Using the results of the previous sections, we are left with the cases $\mathfrak{h}=\mathbb{R}+\mathfrak{a}_1+\mathfrak{a}_{n-2}$ or $\mathfrak{h}=\mathbb{R}+\mathfrak{a}_{n-1}$.

The possible choices for $(\nu(\mathfrak{h}),\nu(\mathfrak{k}))$ are listed in Table 6.

(1) In this case, $\mathfrak{m}$ contains an irreducible 4-dimensional K module $\mathfrak{m}_1$. This K-module is not equivalent to any other K-module in $\mathfrak{m}+\mathfrak{p}$ and it is preserved by σ. For a fixed normal geodesic $\gamma(t)$ through $p=\gamma(0)$, let t' be the minimum $t>0$ such that $\gamma(t)$ is singular.

If $n>4$, any admissible choice for the singular isotropy subgroup H' of $\gamma(t')$ is such that $\mathfrak{m}_1$ is preserved by σ' (the element of H' such that

Table 6. $\mathfrak{g} = \mathfrak{a}_n$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$	
1	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{a}_{n-2}$	$\mathbb{R} + \mathfrak{a}_{n-2}$	$\mathbb{R}^\Delta + \mathfrak{a}_{n-3}$	$\mathfrak{a}_1$	$6n+8$	$n \geq 3$
2	"	$\mathfrak{a}_1 + \mathfrak{a}_1$	$\mathfrak{a}_1^\Delta$	$\mathbb{R}$	12	$n = 3$
3	"	$\mathfrak{a}_{n-2}$	$\mathfrak{a}_{n-3}$	$\mathbb{R} + \mathfrak{a}_1$	$6n+8$	$n \geq 3$
4	"	$\mathfrak{d}_3$	$\mathfrak{b}_2$	$\mathbb{R} + \mathfrak{a}_1$	22	$n = 5$
5	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R}^\Delta + \mathfrak{a}_{n-2}$	0	$4n$	$n \geq 3$
6	"	$\mathfrak{a}_{n-1}$	$\mathfrak{a}_{n-2}$	$\mathbb{R}$	$4n$	$n \geq 3$
7	"	$\mathbb{R}$	0	$\mathfrak{a}_{n-1}$	$2n+2$	$n \geq 3$
8	"	$\mathfrak{d}_3$	$\mathfrak{b}_2$	$\mathbb{R}$	14	$n = 4$

$\sigma'\gamma(t_0 - t) = \gamma(t_0 + t))$. This implies that any Killing vector field in $\mathfrak{m}_1$ never vanishes along $\gamma(t)$. This gives a contradiction using Lemma 2.3.

If $n = 4$, there is one more choice for H' due to the isomorphisms $\mathfrak{d}_2 \simeq \mathfrak{a}_1 + \mathfrak{a}_1$ and $\mathfrak{b}_2 \simeq \mathfrak{c}_2 \subset R + \mathfrak{a}_3$. In this case, $\mathfrak{m}_1$ is not preserved by σ' and we cannot apply the previous argument. But in this case, the manifold is equivariantly diffeomorphic to $\mathbb{CP}^9$ as described in [**26**] (see the case (ii) at p. 188 for more details).

If $n = 3$, $\mathfrak{m}$ is the sum of two irreducible K-modules. These modules are inequivalent since the $\mathbb{R}$ factor in $\mathfrak{k}$ centralizes the kernel of the slice representation and is diagonally embedded in $\mathbb{R} + \mathfrak{a}_1$, and they are not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$. Both the modules are sums of root spaces and therefore, are preserved by σ which lies in the center of H. This implies that the singular orbit is totally geodesic. Since the pair $(\mathfrak{a}_n, \mathbb{R} + \mathfrak{a}_1 + \mathfrak{a}_{n-2})$ do not appear in Wallach's list, this case can be excluded.

(2) $\mathfrak{m}$ is the sum of two irreducible and inequivalent K-modules, which are preserved by σ since they have different dimensions. Since they are not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$, this implies that the singular orbit is totally geodesic, giving a contradiction.

(3) If $n \neq 3$, $\mathfrak{m}$ is the sum of two irreducible and inequivalent K-modules, both preserved by σ since they have different dimensions and σ sends K-modules into K-modules. Since they are not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$, this implies that the singular orbit is totally geodesic and we can conclude as in the previous case.

If $n = 3$, $\mathfrak{m}$ is the sum of two equivalent 4-dimensional irreducible K-modules. Since σ, which does not depend on the choice of the normal geodesic, can be chosen to act as $-id$ on both of them, the singular orbit is once again totally geodesic, giving a contradiction.

(4) Assume $\mathfrak{a}_1 \subset \ker(\epsilon_i)$ for $i > 2$ and let $\alpha = \epsilon_1 - \epsilon_3$, $\beta = \epsilon_2 - \epsilon_4$. The corresponding root spaces are entirely contained in non-trivial K-orbits. They are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product, due to the action of the kernel of the slice representation. Since σ does not depend on the choice of the normal geodesic and belongs to the center of $\nu(H)$, $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are preserved by σ. We can exclude this case using Lemma 2.2.

(5) Let $\gamma(t)$ be a normal geodesic through $p = \gamma(0)$, denote by K the regular isotropy of the points of $\gamma(t)$ and let H' be defined as in case (a). There is a unique irreducible 2-dimensional K-module in $\mathfrak{m}$ which is not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$. This K-module is preserved by σ. Looking at the possible choices for H', there is only one case where one does not obtain a contradiction using Lemma 2.3, given by $\mathfrak{h}' = \mathfrak{a}_1 + \mathfrak{a}_{n-2}$. With this choice, M is equivariantly diffeomorphic to $\mathbb{HP}^n$ as described in [**17**] (Section 5.6, p. 497).

(6) This case cannot occur since the slice representation is not trivial on the center of $\mathfrak{h}$ (see [**17**], Section 5.6, p. 497).

(7) In this case, $\mathfrak{m}$ is irreducible as K-module hence the singular orbit is totally geodesic. Here, the only possible choices for H' are $H' = H$. In the first case, we have a contradiction using Frankel's theorem. In the second case, M is equivariantly diffeomorphic to a complex projective space as described in [**26**] (see Section 8.2, p. 172).

(8) This case cannot occur, if the center of $\mathfrak{h}$ lies in the kernel of the slice representation the subgroup corresponding to the $\mathfrak{d}_3$ factor cannot act transitively on S^5.

5.2. Proof of the main Theorem in the case of $\mathfrak{g} = \mathfrak{b}_n$. Using the results of the previous sections, we are left with the possible choices for $(\nu(\mathfrak{h}), \nu(\mathfrak{k}))$ listed in Table 7.

(1) and (2) These cases are ruled out with Theorem 2.1.

(3) In this case, since $n \geq 3$, $\mathfrak{m}$ is the sum of two irreducible K-modules of different dimensions. This implies that these modules are preserved by σ and the singular orbit is totally geodesic.

(4) This case can be ruled out with Theorem 2.1.

(5) In this case, $\mathfrak{b}_3 \subset \mathfrak{d}_4$ is non-standard. $\mathfrak{m}$ decomposes as the sum of two irreducible K-modules of dimension 16 and 2. Then, the singular orbit is totally geodesic.

(6) In this case, $\mathfrak{m}$ is irreducible as K-module and the singular orbit is totally geodesic.

(7) In this case, $\mathfrak{m}$ is the sum of three irreducible and inequivalent K-modules (two 4-dimensional and one 2-dimensional). Each of them

Table 7. $\mathfrak{g} = \mathfrak{b}_n$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$	
1	$\mathfrak{d}_n$	$\mathfrak{d}_n$	$\mathfrak{b}_{n-1}$	0	$4n$	$n \geq 2$
2	$\mathfrak{a}_1 + \mathfrak{a}_1$	$\mathfrak{a}_1$	0	$\mathfrak{a}_1$	8	$n = 2$
3	$\mathfrak{d}_{n-1} + \mathfrak{b}_1$	$\mathfrak{d}_{n-1}$	$\mathfrak{b}_{n-2}$	$\mathfrak{b}_1$	$8n - 8$	$n \geq 4$
4	$\mathbb{R} + \mathfrak{b}_{n-1}$	$\mathbb{R}$	0	$\mathfrak{b}_{n-1}$	$4n$	$n \geq 2$
5	$\mathbb{R} + \mathfrak{b}_4$	$\mathfrak{b}_4$	$\mathfrak{b}_3$	$\mathbb{R}$	34	$n = 5$
6	$\mathbb{R} + \mathfrak{b}_3$	$\mathfrak{b}_3$	$\mathfrak{g}_2$	$\mathbb{R}$	22	$n = 4$
7	$\mathbb{R} + \mathfrak{b}_2$	$\mathbb{R} + \mathfrak{c}_2$	$\mathbb{R}^\Delta + \mathfrak{c}_1$	0	18	$n = 3$
8	”	$\mathfrak{c}_2$	$\mathfrak{c}_1$	$\mathbb{R}$	18	$n = 3$
9	$\mathbb{R} + \mathfrak{d}_{n-1}$	$\mathbb{R}$	0	$\mathfrak{d}_{n-1}$	$6n - 2$	$n \geq 4$
10	”	$\mathfrak{d}_{n-1}$	$\mathfrak{b}_{n-2}$	$\mathbb{R}$	$8n - 6$	$n \geq 3$
11	$\mathbb{R} + \mathfrak{d}_2$	$\mathbb{R} + \mathfrak{a}_1$	$\mathbb{R}^\Delta$	$\mathfrak{a}_1$	18	$n = 3$
12	”	$\mathfrak{a}_1$	0	$\mathbb{R} + \mathfrak{a}_1$	18	$n = 3$
13	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R}^\Delta + \mathfrak{a}_{n-2}$	0	$n^2 + 3n$	$n \geq 2$
14	”	$\mathbb{R}$	0	$\mathfrak{a}_{n-1}$	$n^2 + n + 2$	$n \geq 2$
15	”	$\mathfrak{a}_{n-1}$	$\mathfrak{a}_{n-2}$	$\mathbb{R}$	$n^2 + 3n$	$n \geq 2$
16	$\mathbb{R} + \mathfrak{d}_{n-2} + \mathfrak{b}_1$	$\mathfrak{d}_{n-2}$	$\mathfrak{b}_{n-3}$	$\mathbb{R} + \mathfrak{b}_1$	$12n - 18$	$n \geq 5$
17	$\mathbb{R} + \mathfrak{a}_{n-2} + \mathfrak{b}_1$	$\mathbb{R} + \mathfrak{a}_{n-2}$	$\mathbb{R}^\Delta + \mathfrak{a}_{n-3}$	$\mathfrak{b}_1$	$n^2 + 5n - 6$	$n \geq 3$
18	”	$\mathfrak{a}_{n-2}$	$\mathfrak{a}_{n-3}$	$\mathbb{R} + \mathfrak{b}_1$	$n^2 + 5n + 6$	$n \geq 3$
19	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{b}_4$	$\mathfrak{b}_4$	$\mathfrak{b}_3$	$\mathbb{R} + \mathfrak{a}_1$	54	$n = 6$
20	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{b}_3$	$\mathfrak{b}_3$	$\mathfrak{g}_2$	$\mathbb{R} + \mathfrak{a}_1$	38	$n = 5$
21	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{b}_2$	$\mathbb{R} + \mathfrak{c}_2$	$\mathbb{R}^\Delta + \mathfrak{c}_1$	$\mathfrak{a}_1$	30	$n = 4$
22	”	$\mathfrak{c}_2$	$\mathfrak{c}_1$	$\mathbb{R} + \mathfrak{a}_1$	30	$n = 4$
23	”	$\mathfrak{a}_1 + \mathfrak{c}_2$	$\mathfrak{a}_1^\Delta + \mathfrak{c}_1$	$\mathbb{R}$	30	$n = 4$
24	$\mathbb{R} + \mathfrak{d}_3 + \mathfrak{a}_1$	$\mathbb{R} + \mathfrak{a}_3$	$\mathbb{R}^\Delta + \mathfrak{a}_2$	$\mathfrak{a}_1$	44	$n = 5$
25	”	$\mathfrak{a}_3$	$\mathfrak{a}_2$	$\mathbb{R} + \mathfrak{a}_1$	44	$n = 5$

is preserved by σ, and this implies that the singular orbit is totally geodesic (see Remark 2.1).

(8) In this case, $\mathfrak{m}$ is the sum of three 2-dimensional irreducible K-modules. Two of them are equivalent. Since σ preserves all these modules and we can choose it such that it acts as $-id$ on the sum of the two equivalent modules, the singular orbit is totally geodesic.

(9) Assume $\mathfrak{d}_{n-1} \cap \mathfrak{c} \subset \ker(\epsilon_1)$ and let $\alpha = \epsilon_2$, $\beta = \epsilon_1 + \epsilon_3$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product, both are preserved by σ. We may then apply Lemma 2.2 to obtain a contradiction.

(10) Assume $\mathfrak{d}_{n-1} \cap \mathfrak{c} \subset \ker(\epsilon_1)$ and let $\alpha = \epsilon_1 + \epsilon_2$, $\beta = \epsilon_3$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}(K)$-invariant scalar product (due to the action of the kernel of the slice representation), $\mathfrak{g}_\alpha$ is entirely contained in a K-orbit and it is preserved by σ. We may then apply Lemma 2.2 to obtain a contradiction.

(11) and (12) The proof given in case (10) also works in this case.

(13) We choose a normal geodesic $\gamma(t)$ through p. If $n > 4$, the complement of $\mathfrak{a}_{n-2}$ in $\mathfrak{d}_{n-1} \subset \mathfrak{d}_n$ is an irreducible K-module non-equivalent to any other K-module and it is preserved by σ. By Lemma 2.3, this module must be included in the Lie algebra of a singular isotropy group H' along $\gamma(t)$. Then, $\mathbb{R} + \mathfrak{d}_{n-1} \subset \mathfrak{h}'$ and we are back to one of the previous cases.

If $n = 4$, there is always at least one irreducible K-module $\mathfrak{n} \subset \mathfrak{m}$ (depending on the slice representation) which is preserved by σ and it is not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$ and we can show that the corresponding Killing vector fields must vanish at some singular point of $\gamma(t) \cap G/H'$. If $\mathfrak{n}$ has dimension 2, then we can conclude using Lemma 2.2 (using the fact there is an $\mathfrak{a}_2$ factor in the kernel of the slice representation at the second singular orbit). If $\mathfrak{n}$ has dimension 6, then we can prove that $\mathfrak{k} \subset \mathfrak{d}_3 \simeq \mathfrak{a}_3 \subset \mathfrak{h}'$ and we get a contradiction since H'/K is a sphere.

If $n = 3$, we may find again an irreducible K-module $\mathfrak{n}_1$ preserved by σ and non-equivalent to any other K-module. If $\mathfrak{n}$ has dimension 2, the corresponding Killing vector fields must vanish along γ at some point of the second singular orbit. But this is impossible for any admissible choice of $\mathfrak{h}'$. If $\mathfrak{n}$ has dimension 4 and $\mathfrak{n} \subset \mathfrak{h}'$, then $\mathfrak{h}' = \mathbb{R} + \mathfrak{a}_2$ and we obtain a contradiction since the center of $\mathfrak{h}'$ is then orthogonal to the center of $\mathfrak{k}$. If the Killing vector fields corresponding to elements of $\mathfrak{n}$ vanish at some point of the first singular orbit, then there is only one possible choice for $\mathfrak{h}'$ (since σ' never preserves all the K-modules in $\mathfrak{m}$), $\mathfrak{h}' = \mathfrak{a}_1 + \mathfrak{a}_1$ at some point $\gamma(t_0)$. This is possible only if the three 4-dimensional K-modules in $\mathfrak{m}+\mathfrak{p}$, $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$ are mutually orthogonal along $\gamma(t)$. This can be proved using the action of the cyclic group generated by σ' on the tangent space at the second singular orbit. This implies that σ' acts as an element of order 3 on $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ and as an element of order 2 on the slice. As a consequence, $\langle X, X\rangle_{\gamma(t-t_0)} = \langle X, X\rangle_{\gamma(t+t_0)}$ for any $X \in \mathfrak{n}$. Then, we may apply a slightly modified version of Lemma 2.2 to any pair $\mathfrak{g}_\alpha \subset \mathfrak{n}$, $\mathfrak{g}_\beta \subset \mathfrak{n}$ with α and β strongly orthogonal.

If $n = 2$ and $\mathfrak{a}_1$ is generated by a short root, then σ preserves any K-module and it is compatible with any equivalence between K-modules since it acts as $-id$ on the sum of the equivalent modules (if any). This follows from the fact that the slice representation is a 2-dimensional irreducible representation of $U(2)$. Then, the singular orbit is totally geodesic and we may apply a slightly modified version of Lemma 2.2 where $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are such that α and β are strongly orthogonal.

If $\mathfrak{a}_1$ is generated by a long root, for any possible choice of $\mathfrak{k}$, there is at least one short root α such that $\mathfrak{g}_\alpha$ is not equivalent to any other K-module in $\mathfrak{m}+\mathfrak{p}$. Then, there are two possible choices for $\mathfrak{h}'$, $\mathfrak{h}' = \mathbb{R}+\mathfrak{a}_1$ and $\mathfrak{h}' = \mathfrak{a}_1$. In the first case, we are back to the preceding step. In the second case, K fixes a 4 dimensional submanifold of M on which a group with Lie algebra $\mathfrak{a}_1$ acts with cohomogeneity one and no singular orbits, giving a contradiction.

(14) If $n > 3$, it is possible to find two strongly orthogonal roots α and β such that $\mathfrak{g}_\alpha \subset \mathfrak{b}_3 \setminus \mathfrak{d}_3$ and $\mathfrak{g}_\beta \subset \mathfrak{d}_3$. Then, the corresponding root spaces are non-equivalent with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. Then, we may apply Lemma 2.2.

If $n = 3$, we may consider the fixed point set of $\mathbb{R} \subset \mathfrak{k} \cap \mathfrak{c}$. This is a 6-dimensional totally geodesic submanifold acted with cohomogeneity one by a group with Lie algebra $\mathfrak{a}_1+\mathfrak{a}_1$. The admissible group diagrams for such manifold implies that the only possible choice for $\mathfrak{h}'$ is $\mathfrak{h}' = \mathfrak{g}_2$. Since $N_{\mathrm{Spin}\,(7)}(G_2) = \mathbb{Z}_2$, we have two possible choices for the regular isotropy K. If $K = SU(3)$, then the first singular orbit is totally geodesic and we get a contradiction. In the second case, M is equivariantly diffeomorphic to $\mathbb{CP}^7$ as described in [**26**] (Section 9.7, p. 188).

The case $n = 2$ can be ruled out using Theorem 2.1.

(15) If $n > 2$, there is a unique irreducible 2-dimensional K-module in $\mathfrak{m} + \mathfrak{p}$. This implies that the only possible choice for $\mathfrak{h}'$ is $\mathfrak{h}' = \mathfrak{a}_1 + \mathfrak{a}_{n-2}$. Then, it is possible to find two strongly orthogonal roots α and β such that the corresponding root spaces are in $\mathfrak{m}'$ and are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product along any normal geodesic through the singular point p' fixed by H' (due to the projection of $\mathfrak{k}$ on the $\mathfrak{a}_1$ factor). One of these root spaces is contained in a non-trivial K-module and it is preserved by σ. Then, we may conclude using Lemma 2.2.

(16) Assume that $\mathfrak{b}_1$ is generated by the root ϵ_1, and that the $\mathbb{R}$ factor is spanned by H_{ϵ_2}. Then, define $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = \epsilon_3$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product and $\mathfrak{g}_\alpha$ is preserved by σ. Then, we may apply Lemma 2.2.

(17) The proof is similar to the one of case (16). In this case, σ preserves any root space.

(18) We omit the proof for these cases since it is similar to the one of case (16).

(19) Assume that the $\mathfrak{a}_1$ factor is spanned by $\epsilon_1 - \epsilon_2$ and that the $\mathbb{R}$ factor is spanned by $H_{\epsilon_1+\epsilon_2}$. Then, let $\alpha = \epsilon_1$, $\beta = \epsilon_2 + \epsilon_3$. Since σ may be chosen in the center of H, we may apply Lemma 2.2.

(20) We omit the proof which is similar to the one of case (19).

(21) In this case, $\mathfrak{m}$ is the sum of irreducible, non-equivalent, K-modules. All these modules are preserved by σ. Then, the singular orbit is totally geodesic and we get a contradiction.

(22) Assume that $\mathfrak{a}_1$ is generated by $\epsilon_1 - \epsilon_2$ and $\mathbb{R}$ is generated by $H_{\epsilon_1+\epsilon_2}$. Let $\alpha = \epsilon_1 + \epsilon_3$, $\beta = \epsilon_2 + \epsilon_4$. Then, we may apply Lemma 2.2 since σ preserves any root space.

(23) $\mathfrak{m}$ is the sum of irreducible, inequivalent K-modules, preserved by σ, then the singular orbit is totally geodesic and we get a contradiction.

(24) Assume that $\mathfrak{a}_1$ is generated by ϵ_1 and $\mathbb{R}$ is generated by H_{ϵ_2}. Let $\alpha = \epsilon_1 + \epsilon_3$, $\beta = \epsilon_2 + \epsilon_4$. Then, we may apply Lemma 2.2 since σ preserves any root space.

(25) Assume that $\mathfrak{a}_1$ is generated by ϵ_1 and $\mathbb{R}$ is generated by H_{ϵ_2}. Let $\alpha = \epsilon_2$, $\beta = \epsilon_1 + \epsilon_3$. Then, we may apply Lemma 2.2 since σ preserves $\mathfrak{g}_\alpha$.

5.3. Proof of the main Theorem in the case of $\mathfrak{g} = \mathfrak{c}_n$. Using the results of the previous sections, we may assume that $n \geq 3$ and we are left with the possible choices for $(\nu(\mathfrak{h}), \nu(\mathfrak{k}))$ listed in Table 8.

(1) Let $\gamma(t)$ be a normal geodesic through p. There is a four-dimensional irreducible K-module $\mathfrak{n}$ which is not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$. By Lemma 2.3, it must be contained in some singular isotropy along $\gamma(t)$. This is possible only if $\mathfrak{n} \subset \mathfrak{h}'$. Then $\mathfrak{h}' = \mathfrak{c}_2 + \mathfrak{c}_{n-2}$ with $\nu'(\mathfrak{h}') = \mathfrak{c}_2$. But then, $\mathfrak{a}_1 \subset \mathfrak{c}_2$ is such that H'/K is not diffeomorphic to a sphere and this case cannot occur.

(2) In this case, $\mathfrak{m}$ is irreducible as K-module, this forces $\mathfrak{h}' = \mathfrak{c}_n$ and M is a rank one symmetric space.

(3) Let $\gamma(t)$ be a normal geodesic through p. In this case, $\mathfrak{m}$ is the sum of two irreducible K-modules which are not equivalent to any other K-module. By Lemma 2.3, the corresponding Killing vector fields must vanish at some singular point of $\gamma(t)$. Then, $\mathfrak{h}'$ has maximal rank and hence never contains two ideals of rank greater than one. This implies that the only possible case is when $n = 3$, $\mathfrak{h}' = \mathfrak{c}_2 \simeq \mathfrak{b}_2$. But then, M is equivariantly diffeomorphic to $\mathbb{C}a\mathbb{P}^2$ as described in [**18**] (cf. Proposition 3, p. 438).

Table 8. $\mathfrak{g} = \mathfrak{c}_n$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$	
1	$\mathfrak{a}_1 + \mathfrak{c}_{n-1}$	$\mathfrak{a}_1 + \mathfrak{c}_{n-1}$	$\mathfrak{a}_1^{\Delta} + \mathfrak{c}_{n-2}$	0	$8n-8$	$n \geq 3$
2	”	$\mathfrak{a}_1$	0	$\mathfrak{c}_{n-1}$	$4n$	”
3	”	$\mathfrak{c}_{n-1}$	$\mathfrak{c}_{n-2}$	$\mathfrak{a}_1$	$8n-8$	”
4	$\mathbb{R} + \mathfrak{c}_{n-1}$	$\mathbb{R} + \mathfrak{c}_{n-1}$	$\mathbb{R}^{\Delta} + \mathfrak{c}_{n-2}$	0	$8n-10$	”
5	”	$\mathbb{R}$	0	$\mathfrak{c}_{n-1}$	$4n$	”
6	”	$\mathfrak{c}_{n-1}$	$\mathfrak{c}_{n-2}$	$\mathbb{R}$	$8n-10$	”
7	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R}^{\Delta} + \mathfrak{a}_{n-2}$	0	n^2+3n	”
8	”	$\mathbb{R}$	0	$\mathfrak{a}_{n-1}$	n^2+n+2	”
9	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{c}_{n-2}$	$\mathbb{R} + \mathfrak{a}_1$	$\mathbb{R}^{\Delta}$	$\mathfrak{c}_{n-2}$	n^2+3n	”
10	”	$\mathbb{R} + \mathfrak{c}_{n-2}$	$\mathbb{R}^{\Delta} + \mathfrak{c}_{n-3}$	$\mathfrak{a}_1$	$12n-18$	”
11	”	$\mathfrak{a}_1 + \mathfrak{c}_{n-2}$	$\mathfrak{a}_1^{\Delta} + \mathfrak{c}_{n-3}$	$\mathbb{R}$	$12n-18$	”
12	”	$\mathbb{R}$	0	$\mathfrak{a}_1 + \mathfrak{c}_{n-2}$	$8n-8$	”
13	”	$\mathfrak{a}_1$	0	$\mathbb{R} + \mathfrak{c}_{n-2}$	$8n-10$	”
14	”	$\mathfrak{c}_{n-2}$	$\mathfrak{c}_{n-3}$	$\mathbb{R} + \mathfrak{a}_1$	$12n-18$	”
15	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{a}_{n-2}$	$\mathbb{R} + \mathfrak{a}_1$	$\mathbb{R}^{\Delta}$	$\mathfrak{a}_{n-2}$	n^2+4n	”
16	”	$\mathbb{R} + \mathfrak{a}_{n-2}$	$\mathbb{R}^{\Delta} + \mathfrak{a}_{n-3}$	$\mathfrak{a}_1$	n^2+5n-6	”
17	”	$\mathbb{R}$	0	$\mathfrak{a}_1 + \mathfrak{a}_{n-2}$	n^2+4n-2	”
18	”	$\mathfrak{a}_1$	0	$\mathbb{R} + \mathfrak{a}_{n-2}$	n^2+4n	”
19	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{a}_3$	$\mathfrak{d}_3$	$\mathfrak{b}_2$	$\mathbb{R} + \mathfrak{a}_1$	42	$n=5$
20	$\mathbb{R} + \mathfrak{a}_3$	$\mathfrak{d}_3$	$\mathfrak{b}_2$	$\mathbb{R}$	26	$n=4$

(4) In this case, M contains a totally geodesic submanifold acted with cohomogeneity one by a group with Lie algebra $\mathfrak{c}_2 \simeq \mathfrak{b}_2$ and group diagram $(\mathbb{R} + \mathfrak{a}_1, \mathbb{R}^{\Delta}, \mathfrak{h}' \cap \mathfrak{c}_2)$. Then, we may use the proof of case (13) in Section 5.2.

(5) The result follows from Theorem 2.1.

(6) In this case, M contains a totally geodesic submanifold acted with cohomogeneity one by a group with Lie algebra $\mathfrak{c}_2 \simeq \mathfrak{b}_2$ and group diagram $(\mathbb{R} + \mathfrak{a}_1, \mathbb{R}, \mathfrak{h}' \cap \mathfrak{c}_2)$. Then, we may use the proof of case (15) in Section 5.2.

(7) Let $\gamma(t)$ be a fixed normal geodesic through p. Then, $\mathfrak{m}$ contains one trivial K-module and two irreducible K-modules. One of them has dimension $n(n-1)$, and it is not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$ and it is preserved by σ. The corresponding Killing vector fields must vanish by Lemma 2.3 at some point of the second singular orbit. We may exclude this case since it is not possible to find a singular isotropy subalgebra which contains this module.

(8) We may assume that $\mathfrak{a}_{n-1}$ corresponds to the roots $\epsilon_i - \epsilon_j$ with $1 \leq i \neq j < n$. Then, let $\alpha = 2\epsilon_1$, $\beta = 2\epsilon_2$. Since $n \geq 3$, we may conclude using Lemma 2.2.

(9) We have two cases to consider, according to $\mathfrak{h} \subset \mathfrak{c}_2 + \mathfrak{c}_n - 2$ or $\mathfrak{h} \subset \mathfrak{a}_1 + \mathfrak{c}_{n-1}$. We assume that $\mathfrak{c}_{n-2}$ is generated by the roots $2\epsilon_i, \epsilon_i \pm \epsilon_j$ with $i, j \geq 3$ and that the $\mathfrak{a}_1$ factor corresponds to the long root $2\epsilon_2$ in the first case and to the short root $\epsilon_1 - \epsilon_2$ in the second. Then, let $\alpha = \epsilon_2 + \epsilon_3$, $\beta = 2\epsilon_1$. Since σ preserves any root space, we can conclude (in both cases) using Lemma 2.2.

(10) With the notations of case (9), we let $\alpha = 2\epsilon_1$, $\beta = \epsilon_2 + \epsilon_3$ in the first case (the corresponding root spaces are always orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product since the $\mathbb{R}$ factor has a surjective projection on $H_{2\epsilon_1}$) and $\alpha = 2\epsilon_1$, $\beta = 2\epsilon_2$ in the second. Then, we use Lemma 2.2.

(11) With the notations of case (9), we let $\alpha = 2\epsilon_1$, $\beta = \epsilon_2 + \epsilon_3$ in the first case and $\alpha = 2\epsilon_1$, $\beta = 2\epsilon_2$ in the second (the corresponding root spaces are always orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product since $\mathfrak{k}$ has a surjective projection on the $\mathfrak{a}_1$ factor). Then, we use Lemma 2.2.

(12) With the notations of case (9), we let $\alpha = \epsilon_2 + \epsilon_3$, $\beta = 2\epsilon_1$ in the first case and $\alpha = 2\epsilon_1$, $\beta = \epsilon_2 + \epsilon_3$ in the second. Then, we use Lemma 2.2.

(13) With the notations of case (9), we let $\alpha = 2\epsilon_1$, $\beta = \epsilon_2 + \epsilon_3$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product because of the action of $\ker(\nu)$. Since $\nu(H) = SU(2)$, σ preserves any root space and we can use Lemma 2.2.

(14) We omit the proof which is similar to the one of case (13).

(15) In this case, $\mathfrak{h} \subset \mathfrak{a}_1 + \mathfrak{c}_{n-1}$. It is sufficient to consider the case when $\mathfrak{a}_1$ is generated by $2\epsilon_1$ and $\mathfrak{a}_{n-1}$ by the roots $\epsilon_i - \epsilon_j$ with $2 \leq i \neq j \leq n$. The center of $\mathfrak{k}$ has a surjective projection on $H_{\epsilon_2+\epsilon_3}$, since $H_{\epsilon_2-\epsilon_3}$ belongs to $\mathfrak{k}$, if we let $\alpha = 2\epsilon_2$ and $\beta = 2\epsilon_3$, the corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. Since σ preserves any root space, we can apply Lemma 2.2.

(16) Using the notations of case (15), we let $\alpha = \epsilon_1 + \epsilon_3$ and $\beta = 2\epsilon_2$. The action of the kernel of the slice representation makes the corresponding root spaces orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product. Then, we use Lemma 2.2.

(17)–(19) We omit the proof which is similar to the one of case (16).

Table 9. $\mathfrak{g} = \mathfrak{d}_n$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$	
1	$\mathbb{R} + \mathfrak{d}_{n-1}$	$\mathbb{R}$	0	$\mathfrak{d}_{n-1}$	$4n - 2$	$n \geq 4$
2	”	$\mathfrak{d}_{n-1}$	$\mathfrak{b}_{n-2}$	$\mathbb{R}$	$6n - 6$	”
3	$\mathbb{R} + \mathfrak{d}_3$	$\mathbb{R} + \mathfrak{a}_3$	$\mathbb{R}^{\Delta} + \mathfrak{a}_2$	0	20	$n = 4$
4	”	$\mathfrak{a}_3$	$\mathfrak{a}_2$	$\mathbb{R}$	”	”
5	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R} + \mathfrak{a}_{n-1}$	$\mathbb{R}^{\Delta} + \mathfrak{a}_{n-2}$	0	$n^2 + n$	$n \geq 4$
6	”	$\mathbb{R}$	0	$\mathfrak{a}_{n-1}$	$n^2 - n + 2$	”
7	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{d}_{n-2}$	$\mathbb{R} + \mathfrak{a}_1$	$\mathbb{R}^{\Delta}$	$\mathfrak{d}_{n-2}$	$8n - 10$	$n \geq 5$
8	”	$\mathbb{R}$	0	$\mathfrak{a}_1 + \mathfrak{d}_{n-2}$	$8n - 12$	”
9	”	$\mathfrak{d}_{n-2}$	$\mathfrak{b}_{n-3}$	$\mathbb{R} + \mathfrak{a}_1$	$10n - 18$	”
10	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{d}_3$	$\mathbb{R} + \mathfrak{a}_3$	$\mathbb{R}^{\Delta} + \mathfrak{a}_2$	$\mathfrak{a}_1$	34	$n = 5$
11	”	$\mathfrak{a}_3$	$\mathfrak{a}_2$	$\mathbb{R} + \mathfrak{a}_1$	34	”

(20) In this case, $\mathfrak{m}$ is irreducible as H-module and it remains irreducible as K-module. This implies that the singular orbit is totally geodesic and we get a contradiction.

5.4. Proof of the main Theorem in the case of $\mathfrak{g} = \mathfrak{d}_n$. Using the results of the previous sections, we may assume that $n \geq 4$ and we are left with the possible choices for $(\nu(\mathfrak{h}), \nu(\mathfrak{k}))$ listed in Table 9.

(1) This case can be ruled out using Theorem 2.1.

(2) In this case, $\mathfrak{m}$ is the sum of two irreducible K-modules of different dimensions. Both are preserved by σ and the singular orbit is totally geodesic. This gives a contradiction.

(3) In this case, $\mathfrak{m}$ is the sum of two irreducible K-modules. These modules are not equivalent as K-modules and are preserved by σ. Then, the singular orbit is totally geodesic.

(4) The proof of (3) works also in this case.

(5) If $n > 5$, the singular orbit is totally geodesic since $\mathfrak{m}$ is the sum of two irreducible K-modules of different dimensions. If $n = 5$, then $\mathfrak{m}$ contains a 12-dimensional K-module $\mathfrak{n}_1$ which is irreducible except for one choice of $\mathbb{R}^{\Delta}$. When $\mathfrak{n}$ is irreducible, it is not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$ and should be contained in the singular isotropy subalgebra of the second singular orbit. But this is impossible since H'/K must be a sphere. Then, we may assume that $\mathfrak{n}$ is the sum of two irreducible six-dimensional K-modules $\mathfrak{n}_1$ and $\mathfrak{n}_2$. If these modules are equivalent as K-modules, then σ is compatible with this equivalence and the singular orbit is totally geodesic. If they are not equivalent, by Lemma 2.3, they must be contained in the singular isotropy subalgebra at the second singular orbit. This is possible only if $\mathfrak{h}' = \mathbb{R} + \mathfrak{b}_3$

and $\sigma(\mathfrak{n}_1) = \mathfrak{n}_2$. Then, M is equivariantly diffeomorphic to $\mathbb{CP}^{15}$ as described in [**26**] (see Section 9.7, p. 188 or [**8**]).

(6) Assume that $\mathbb{R}$ is spanned by $\sum H_{\epsilon_i}$. Then, let $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = \epsilon_3 + \epsilon_4$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product because of the action of $\mathfrak{c} \cap \ker(\nu)$. It is then possible to apply Lemma 2.2. The proof in the other cases is similar and we omit it.

(7) Assume that $\mathfrak{a}_1$ is generated by $\mathfrak{g}_{\epsilon_1-\epsilon_2}$. Then, let $\alpha = \epsilon_1 + \epsilon_3$ and $\beta = \epsilon_2 - \epsilon_4$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product because of the action of $\mathfrak{c} \cap \ker(\nu)$. It is then possible to apply Lemma 2.2. The proof in the other cases is similar and we omit it.

(8) The proof is identical to the proof of case (7).

(9) Assume that $\mathfrak{a}_1$ is generated by $\mathfrak{g}_{\epsilon_1-\epsilon_2}$. Then, let $\alpha = \epsilon_1 + \epsilon_3$ and $\beta = \epsilon_2 + \epsilon_4$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product because of the action of $\mathfrak{c} \cap \ker(\nu)$. It is then possible to apply Lemma 2.2. The proof in the other cases is similar and we omit it.

(10) Let $\gamma(t)$ be a normal geodesic through p. $\mathfrak{m}$ contains an irreducible two-dimensional K-module which is not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$. By Lemma 2.3, this module must be contained in the isotropy subalgebra at some singular point of $\gamma(t)$ in the second singular orbit. Then $\mathfrak{h}'$ must contains two ideals of rank greater then 1. By Proposition 3.1, this is possible only if $\mathfrak{h}'$ has rank 4. But this is not possible since $\mathfrak{h}'$ is a sum of K-modules and H'/K is a sphere.

(11) The proof is identical to the proof of case (9).

5.5. Proof of the main Theorem in the case of $\mathfrak{g} = \mathfrak{g}_2$. In order to fix the notations, we assume that the decomposition (1) has the form

$$\mathfrak{g} = \mathbb{R}^2 + \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{2\alpha+3\beta} + \mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_{\alpha+3\beta} + \mathfrak{g}_\beta,$$

where α is a long root and β is a short root.

The possibile choices for $\mathfrak{h}$ are listed in Table 10.

(1) In this case, we must have $\mathfrak{k} \simeq \mathbb{R}$. $\mathfrak{k}$ is contained in the kernel of the slice representation and does not depend on the choice of the normal geodesic through p. In this case, the decomposition (2) takes the form

$$\mathfrak{g} = \mathfrak{k} + \mathbb{R} + \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{2\alpha+3\beta} + \mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_{\alpha+3\beta} + \mathfrak{g}_\beta.$$

Note that $\mathfrak{g}_\gamma$ are also irreducible inequivalent H-modules. We consider the 3 pairs $(\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\gamma_2})$, where H_{γ_1} and H_{γ_2} are orthogonal. At most, two of these pairs are made by equivalent K-modules. In fact, if $\mathfrak{g}_{\gamma_1}$ and

Table 10. $\mathfrak{g} = \mathfrak{g}_2$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$
1	$\mathbb{R}^2$	$\mathbb{R}$	0	$\mathbb{R}$	14
2	$\mathfrak{a}_1 + \mathfrak{a}_1$	$\mathfrak{a}_1 + \mathfrak{a}_1$	$\mathfrak{a}_1^\Delta$	0	12
3	$\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_1$	0	12
4	$\mathbb{R} + \mathfrak{a}_1$	$\mathbb{R}$	0	$\mathfrak{a}_1$	12
5	”	$\mathfrak{a}_1$	0	$\mathbb{R}$	14

$\mathfrak{g}_{\gamma_2}$ are equivalent, then $\mathfrak{k}$ coincides with $\ker(\gamma_1 + \gamma_2)$ or $\ker(\gamma_1 - \gamma_2)$. Even in this case, it is possible to find a pair $(\mathfrak{g}_{\gamma_1}, \mathfrak{g}_{\gamma_2})$ such that $\mathfrak{g}_{\gamma_1}$ and $\mathfrak{g}_{\gamma_2}$ are non-equivalent as K-modules and at least one of them, say $\mathfrak{g}_{\gamma_1}$, is irreducible. We choose $X \in \mathfrak{g}_{\gamma_1}$ and $Y \in \mathfrak{g}_{\gamma_2}$. Then, for $Z \in \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$,

$$2\langle \nabla_X Y, Z\rangle_p = \langle [X, Z], Y\rangle_p + \langle [Y, Z], X\rangle_p$$

since $\mathfrak{g}_{\gamma_1} + \mathfrak{g}_{\gamma_2}$ and $\mathfrak{g}_{\gamma_1} - \mathfrak{g}_{\gamma_2}$ are not roots, and $\mathfrak{g}_\gamma$ are not equivalent as H-modules, we can conclude that $(\nabla_X Y)^T = 0$. For the same reason, $(\nabla_X X)^T = 0$. Since $\mathfrak{g}_{\gamma_1}$ and $\mathfrak{g}_{\gamma_2}$ are not equivalent as K-modules, we have $\langle X, Y\rangle_{\gamma(t)} = 0$, hence $\langle \nabla_X Y, N\rangle_p = -\frac{1}{2} N\langle X, Y\rangle_p = 0$. Since $\mathfrak{g}_{\gamma_1}$ is irreducible and it is preserved by σ, we have that $\langle X, X\rangle_{\gamma(t)}$ is an even function of t, hence $\langle \nabla_X X, N\rangle_p = -\frac{1}{2} N\langle X, X\rangle_p = 0$. Since this construction does not depend on the choice of the normal geodesic $\gamma(t)$, then $(\nabla_X Y)_p = (\nabla_X X)_p = 0$ and $(R_{XYXY})_p = 0$.

(2) We may assume $\mathfrak{h} = \mathfrak{a}_1 + \overline{\mathfrak{a}}_1$, where $\mathfrak{a}_1$ is generated by a short root and $\overline{\mathfrak{a}}_1$ is generated by a long root. The maximal connected subgroup of G_2 of type $\mathfrak{a}_1 + \mathfrak{a}_1$ is $SO(4)$. Since H/K is a sphere, we have the only choice $\mathfrak{k} = \mathfrak{a}_1^\Delta$, where Δ denotes a diagonal embedding. In this case, the decomposition (2) takes the form

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_1^\Delta + \mathfrak{m}_{11},$$

where $\mathfrak{m}_{11}$ is an irreducible eight-dimensional K-module. This module is preserved by σ and it is not equivalent to any other K-module. Then, for $X, Y \in \mathfrak{m}_{11}$, $\langle X, Y\rangle_{\gamma(t)}$ is an even function of t and, as before, we have $\langle \nabla_X Y, N\rangle_p = 0$. This fact does not depend on the choice of the normal geodesic through p. Hence, the singular orbit G/H is totally geodesic, but this is impossible since the pair $(\mathfrak{g}_2, \mathfrak{a}_1 + \mathfrak{a}_1)$ does not appear in Wallach's list 2.

(3) In this case, there is just one possibility for $\mathfrak{k}$, $\mathfrak{k} = \mathfrak{a}_1$. The decomposition (2) takes the form

$$\mathfrak{g} = \mathfrak{k} + \mathbb{R} + \mathfrak{p}_{11} + \mathbb{R}^2 + \mathfrak{m}_{11},$$

where $\mathfrak{p}_{11}$ and $\mathfrak{m}_{11}$ are 4-dimensional irreducible K-modules. The module $\mathfrak{m}_{11}$ is preserved by σ since it is the only 4-dimensional K-module in $\mathfrak{m}$. In order to study the action of σ on the trivial module $\mathbb{R}^2$, we consider the slice representation $\nu : H \to O(6)$. Since H acts transitively on S^5, this representation is uniquely determined up to equivalence, and we may assume that it coincides with the restriction to G_2 of the linear action of $SO(7)$ on $\mathbb{R}^7 = \text{span}\{e_1, \ldots, e_7\} \simeq \text{Im}(\mathbb{O})$. The connected subgroup $U \simeq SU(3)$ corresponding to $\mathfrak{a}_2$ may be viewed as the isotropy subgroup of e_3, while $\mathfrak{a}_1 \subset \mathbb{R} + \mathfrak{a}_1 \subset \mathfrak{a}_2$ fixes e_1, e_2, e_3 (note that, since U is a maximal connected subgroup of G_2 and $N_{G_2}(U)/U \simeq \mathbb{Z}_2$, there are just two possible choices for H, $H = U$ or $H = \mathbb{Z}_2 \cdot U$, in fact H/K must be a sphere). We may choose $\sigma = \text{diag}(-1, -1, 1, 1, -1, -1, 1) \in SO(7)$. Then, $\sigma = \exp(\Sigma_1)$, where $\Sigma \in \mathbb{R} + \mathfrak{a}_1 = \text{span}\{2E_{12} + E_{56} - E_{47}, E_{56} + E_{47}, E_{45} + E_{67}, E_{46} - E_{57}\}$ and E_{ij} is the standard basis for $\mathfrak{so}(7)$. The element σ acts as $-I_2$ on the trivial module $\mathbb{R}^2$, this implies that, for any $X, Y \in \mathfrak{m}$, $\langle X, Y\rangle_{\gamma(t)}$ is an even function of t, and one may conclude that the singular orbit is totally geodesic. By Frankel's theorem, we exclude the case $\mathfrak{h}_2 = \mathfrak{a}_2$. So, the only possibility for the second singular isotropy remains $\mathfrak{h}_2 = \mathbb{R} + \mathfrak{a}_1$. The corresponding decomposition is

$$\mathfrak{g}_2 = \mathfrak{k} + \mathbb{R} + \mathfrak{n}_{11} + \mathfrak{n}_{21} + \mathbb{R}^2.$$

First, we consider the case when K is connected. Then, σ' does not depend on the choice of the normal geodesic and preserves $\mathfrak{n}_{11}$ and $\mathfrak{n}_{21}$. We may assume that $\mathfrak{a}_1$ is generated by a long root (say $2\alpha + 3\beta$) and that $\mathfrak{n}_{11} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+3\beta}$. Then, $[\mathfrak{n}_{11}, \mathfrak{n}_{11}] \subset \mathfrak{k}$ and it is possible to find $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{\alpha+3\beta}$ such that $[X, Y] = 0$. This implies that $\langle \nabla_X Y, Z\rangle = \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle$ must vanish at the singular point p' for any $Z \in \mathfrak{m}$. X and Y belong to the same irreducible K module and are orthogonal along any normal geodesic through p', i.e., $\langle \nabla_X Y, N\rangle = 0$. Hence, $\nabla_X Y$ vanishes at p'. Similarly, we can prove that $\nabla_X X$ vanishes at p', then $R_{XYXY} = 0$ at p'.

If K is not connected, then $K = \mathbb{Z}_2 \cdot SU(2) \subset N_{G_2}(U)$ and $H' = N_{G_2}(K)^0 \not\subseteq SU(3)$. In this case, the modules $\mathfrak{m}_{11}$ and $\mathfrak{m}_{21}$, which are equivalent as $SU(2)$-modules, are not equivalent as K-modules. By Lemma 2.3, the Killing vector fields corresponding to elements of $\mathfrak{m}_{11}$ $\mathfrak{m}_{21}$ must have zero at some point of $\gamma(t)$. This is possible if and only if $\sigma'(\mathfrak{m}_{11}) = \mathfrak{m}_{21}$ (hence in this case, the proof we gave for K connected fails). In this case, $M \simeq \mathbb{CP}^6$, as described in [**26**] (Section 7.4).

(4) We have to consider only the case $\mathfrak{h}' = \mathbb{R} + \mathfrak{a}_1$.

If $\mathfrak{k}$ is generated by a short root, we have the decomposition

$$\mathfrak{g}_2 = \mathfrak{k} + \mathbb{R} + \mathfrak{m}_{11} + \mathbb{R}^2,$$

where $\mathfrak{m}_{11}$ is an irreducible eight-dimensional module. This K-module is preserved by both σ and σ'. Then, the corresponding Killing vector fields cannot have a zero along $\gamma(t)$ and we get a contradiction using Lemma 2.3.

If $\mathfrak{k}$ is generated by a long root, then using the same proof of the previous case (which used just a tubular neighborhood of the second singular orbit), we may assume that K is not connected. This implies that $\mathfrak{m}_{11}$ and $\mathfrak{m}_{21}$ are not equivalent as K-modules. $\mathfrak{m}_{11} \oplus \mathfrak{m}_{21}$ is preserved by the group generated by σ and σ'. Then, the vector fields corresponding to $\mathfrak{m}_{11}$ cannot have a zero along $\gamma(t)$ and we get a contradiction using Lemma 2.3.

(5) Again, we have to consider only the case $\mathfrak{h}' = \mathbb{R} + \mathfrak{a}_1$. If $\mathfrak{k} = \mathbb{R}$, then the decomposition (2) takes the form

$$\mathfrak{g} = \mathfrak{k} + \mathbb{R} + \mathbb{R}^2 + \mathfrak{m}_{11} + \mathfrak{m}_{21} + \mathfrak{m}_{31} + \mathfrak{m}_{41}.$$

The element σ belongs to the center of H and preserves the modules $\mathfrak{m}_{j1}$. Since at most one of them is contained in $\mathfrak{h}'$, we get a contradiction as in the previous cases.

5.6. Proof of the main Theorem in the case of $\mathfrak{g} = \mathfrak{f}_4$. Using the results of the previous sections, we are left with the possible choices for $(\nu(\mathfrak{h}), \nu(\mathfrak{k}))$ listed in Table 11. Due to the fact that the maximal connected subgroup corresponding to $\mathfrak{a}1+\mathfrak{c}_3$ is $Sp(1)\times Sp(3)/\mathbb{Z}_2$, some pairs $(\mathfrak{h}, \mathfrak{k})$ do not appear in the table since they do not lead to cohomogeneity one manifolds.

(1) Denote by $\mathfrak{m}$ the $\mathrm{Ad}\,(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then, $\mathfrak{m} = \mathbb{R} + \mathfrak{m}_1 + \mathfrak{m}_2$, where $\dim \mathfrak{m}_1 = 7$ and $\dim \mathfrak{m}_2 = 8$. By Lemma 2.3, the only possibility for $\mathfrak{h}'$ is $\mathfrak{h}' = \mathbb{R} + \mathfrak{b}_3$ (note that $\mathbb{R} + \mathfrak{b}_3$ is unique up to conjugation inside $\mathfrak{f}_4$). Since $\mathfrak{b}_3$ lies in the kernel of the slice representation at the second singular orbit, where $\nu'(\mathfrak{h}') = \mathbb{R}$, we may exclude this case using Lemma 2.2 with $\alpha = \epsilon_1 + \epsilon_4$ and $\beta = (\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)/2$.

(2) In this case, we have $\mathfrak{m} = \mathbb{R} + \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3$, where $\dim \mathfrak{m}_1 = 7$ and $\dim \mathfrak{m}_2 = \dim \mathfrak{m}_3 = 8$. We omit the proof which is similar to the one of the previous case.

(3) In this case, we have $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3$, where $\dim \mathfrak{m}_1 = 8$, $\dim \mathfrak{m}_2 = 5$ and $\dim \mathfrak{m}_3 = 15$. Then, we may exclude this case applying Lemma 2.2 to $\mathfrak{m}_3$.

(4) We give the proof for the case when $\mathfrak{a}_3$ is the regular subalgebra of $\mathfrak{f}_4$ corresponding to the roots $\epsilon_i - \epsilon_j$, $i \neq j = 1, \ldots, 4$, the proof in

Table 11. $\mathfrak{g} = \mathfrak{f}_4$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$
1	$\mathfrak{b}_4$	$\mathfrak{b}_4$	$\mathfrak{b}_3$	0	32
2	$\mathfrak{d}_4$	$\mathfrak{d}_4$	$\mathfrak{b}_3$	0	”
3	$\mathfrak{a}_1 + \mathfrak{c}_3$	$\mathfrak{a}_1 + \mathfrak{c}_3$	$\mathfrak{a}_1^\Delta + \mathfrak{c}_2$	0	40
4	$\mathbb{R} + \mathfrak{a}_3$	$\mathbb{R} + \mathfrak{a}_3$	$\mathbb{R}^\Delta + \mathfrak{a}_2$	0	44
5	”	$\mathbb{R}$	0	$\mathfrak{a}_3$	38
6	”	$\mathfrak{a}_3$	$\mathfrak{a}_2$	$\mathbb{R}$	44
7	”	$\mathfrak{d}_3$	$\mathfrak{b}_2$	$\mathbb{R}$	42
8	$\mathbb{R} + \mathfrak{b}_3$	$\mathbb{R}$	0	$\mathfrak{b}_3$	32
9	”	$\mathfrak{b}_3$	$\mathfrak{g}_2$	$\mathbb{R}$	38
10	$\mathbb{R} + \mathfrak{c}_3$	$\mathbb{R} + \mathfrak{c}_3$	$\mathbb{R}^\Delta + \mathfrak{c}_2$	0	42
11	”	$\mathbb{R}$	0	$\mathfrak{c}_3$	32
12	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{a}_2$	$\mathbb{R} + \mathfrak{a}_2$	$\mathbb{R}^\Delta + \mathfrak{a}_1$	$\mathfrak{a}_1$	46
13	”	$\mathfrak{a}_2$	$\mathfrak{a}_1$	$\mathbb{R} + \mathfrak{a}_1$	”
14	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{c}_2$	$\mathbb{R} + \mathfrak{a}_1$	$\mathbb{R}^\Delta$	$\mathfrak{c}_2$	42
15	”	$\mathbb{R} + \mathfrak{c}_2$	$\mathbb{R}^\Delta + \mathfrak{c}_1$	$\mathfrak{a}_1$	46
16	”	$\mathfrak{a}_1 + \mathfrak{c}_2$	$\mathfrak{a}_1^\Delta + \mathfrak{a}_1$	$\mathbb{R}$	”
17	”	$\mathfrak{a}_1$	0	$\mathbb{R} + \mathfrak{c}_2$	42
18	”	$\mathfrak{c}_2$	$\mathfrak{c}_1$	$\mathbb{R} + \mathfrak{a}_1$	46

any other case is similar. Then, let $\alpha = \epsilon_1 + \epsilon_2$, $\beta = \epsilon_3 + \epsilon_4$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product since $\mathbb{R}^\Delta \subset \mathfrak{k}$ has always a non-trivial component on the $\mathbb{R}$ factor in $\mathfrak{h}$. Since σ preserves any root space, we may exclude this case using Lemma 2.2.

(5) We give the proof for $\mathfrak{a}_3 \subset \mathfrak{f}_4$ as in (4). Then, let $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = (\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)/2$ and apply Lemma 2.2.

(6) Let $\mathfrak{a}_3 \subset \mathfrak{f}_4$ be as above. Since σ belongs to the center of H, it preserves any root space. Let $\alpha = \epsilon_1 + \epsilon_2$ and $\beta = (\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4)$, then $\alpha \pm \beta$ is not orthogonal to $\mathbb{R} = \ker(\nu)$. Hence, the corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product, and $\mathfrak{g}_\alpha$ is acted on non-trivially by K. We may then apply Lemma 2.2.

(7) The proof is similar to the one of case (6) and we omit it.

(8) We give the proof for the case when $\mathfrak{b}_3 \subset \mathfrak{b}_4$ is generated by ϵ_i, $i = 1, 2, 3$. In this case, σ preserves any root space, let $\alpha = \epsilon_1 + \epsilon_4$ and $\beta = (\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4)/2$. The corresponding root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product due to the action of $\ker(\nu)$ and we may conclude using Lemma 2.2.

(9) The proof is identical to the one of case (6) and we omit it.

(10) We give the proof for the case when $\mathfrak{c}_3 \subset \mathfrak{f}_4$ as in (3), the proof in any other case is similar. We consider a normal geodesic through p, $\mathfrak{m}$ contains an irreducible K-module $\mathfrak{m}_1$ of dimension 10. By Lemma 2.3, the corresponding Killing vector fields must vanish at some point of the normal geodesic, but this is not possible since we cannot find a singular isotropy subalgebra $\mathfrak{h}'$ such that $\mathfrak{m}_1 \subset \mathfrak{h}'$.

(11) Let $\mathfrak{c}_3 \subset \mathfrak{f}_4$ as above. In this case, σ preserves any root space and we let $\alpha = \epsilon_1 + \epsilon_3$ and $\beta = \epsilon_2 + \epsilon_4$. We may then exclude this case using Lemma 2.2.

(12) In this case, $\mathfrak{a}_1 + \mathbb{R} + \mathfrak{a}_2 \subset \mathfrak{a}_1 + \mathfrak{c}_3$ and we may assume $\mathfrak{a}_1 + \mathfrak{c}_3 \subset \mathfrak{f}_4$ as in (3). We give the proof for the case when $\mathfrak{a}_2$ corresponds to the roots $\epsilon_1, (\epsilon_1 \pm (\epsilon_2 + \epsilon_3 + \epsilon_4))/2$. Then, let $\alpha = \epsilon_1 - \epsilon_3$ and $\beta = \epsilon_2$. Since σ preserves any root space, we may conclude using Lemma 2.2.

(13) Let $\mathfrak{a}_1 + \mathbb{R} + \mathfrak{a}_2$ be as above. Then, let $\alpha = \epsilon_2 - \epsilon_3$ and $\beta = \epsilon_4$. The K-module containing the root space $\mathfrak{g}_\alpha$ is non-trivial and it is preserved by σ which acts on it through an element of K. The two root spaces are orthogonal with respect to any $\mathrm{Ad}\,(K)$-invariant scalar product since $\alpha \pm \beta$ is not orthogonal to some component of $\ker(\nu)$. Then, apply Lemma 2.2.

(14) Let $\mathfrak{a}_1$ and $\mathfrak{c}_3$ be as in (3). We prove the result for the case when $\mathfrak{c}_2$ corresponds to the roots $\epsilon_1, \epsilon_2, \epsilon_1 \pm \epsilon_2$. Then, let $\alpha = \epsilon_1 + \epsilon_3$, $\beta = \epsilon_2 + \epsilon_4$, all the conditions to apply Lemma 2.2 are satisfied.

(15) Assume the embeddings of the subalgebras are as above. Then, let $\alpha = \epsilon_1 + \epsilon_3$ and $\beta = (\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)/2$ and conclude using Lemma 2.2.

(16)–(18) With the embeddings described above, let $\alpha = \epsilon_1 + \epsilon_4$ and $\beta = (\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)/2$. We can conclude using Lemma 2.2.

5.7. Proof of the main Theorem in the case of $\mathfrak{g} = \mathfrak{e}_6$. Using the results of the previous sections, we are left with the possible choices for $(\nu(\mathfrak{h}), \nu(\mathfrak{k}))$ listed in Table 12. Due to the form of the maximal connected subgroup corresponding to maximal subalgebras of maximal rank in $\mathfrak{e}_6$, some pairs $(\mathfrak{h}, \mathfrak{k})$ which are admissible from the point of view of Table 5 do not appear in the table since they do not lead to cohomogeneity one manifolds.

(1) In this case, σ preserves all the root spaces. Assuming that $\mathfrak{a}_5$ corresponds to the roots of the form $\epsilon_i - \epsilon_j$, we may set $\alpha = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon$, $\beta = \epsilon_1 + \epsilon_2 + \epsilon_4 - \epsilon$. The corresponding root spaces are orthogonal because of the action of $\ker(\nu)$ and we may apply Lemma 2.2.

(2) We may assume that $\mathfrak{a}_4$ corresponds to the roots of the form $\epsilon_i - \epsilon_j$ with $1 \leq i, j \leq 5$ and $\mathfrak{a}_1$ corresponds to the root 2ϵ. Then, let $\alpha = \epsilon_1 + \epsilon_2 + \epsilon_6 + \epsilon$, $\beta = \epsilon_1 - \epsilon_6$ and apply Lemma 2.2.

Table 12. $\mathfrak{g} = \mathfrak{e}_6$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$
1	$\mathbb{R} + \mathfrak{a}_5$	$\mathbb{R}$	0	$\mathfrak{a}_5$	44
2	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{a}_4$	$\mathbb{R} + \mathfrak{a}_4$	$\mathbb{R}^\Delta + \mathfrak{a}_3$	$\mathfrak{a}_1$	60
3	$\mathbb{R} + \mathfrak{d}_5$	$\mathbb{R}$	0	$\mathfrak{d}_5$	34

Table 13. $\mathfrak{g} = \mathfrak{e}_7$.

	$\mathfrak{h}$	$\nu(\mathfrak{h})$	$\nu(\mathfrak{k})$	$\ker(\nu)$	$\dim(M)$
1	$\mathbb{R} + \mathfrak{d}_6$	$\mathbb{R}$	0	$\mathfrak{d}_6$	68
2	$\mathbb{R} + \mathfrak{a}_1 + \mathfrak{a}_5$	$\mathbb{R} + \mathfrak{a}_5$	$\mathbb{R}^\Delta + \mathfrak{a}_4$	$\mathfrak{a}_1$	106
3	$\mathbb{R} + \mathfrak{a}_6$	$\mathbb{R}$	0	$\mathfrak{a}_6$	86
4	"	$\mathbb{R} + \mathfrak{a}_6$	$\mathbb{R}^\Delta + \mathfrak{a}_5$	0	98

(3) We omit the proof which is similar to the ones of the previous cases.

5.8. Proof of the main Theorem in the case of $\mathfrak{g} = \mathfrak{e}_7$. Using the results of the previous sections, we are left with the possible choices for $(\nu(\mathfrak{h}), \nu(\mathfrak{k}))$ listed in Table 13. Again, due to the form of the maximal connected subgroup corresponding to maximal subalgebras of maximal rank in $\mathfrak{e}_7$, some pairs $(\mathfrak{h}, \mathfrak{k})$ which are admissible from the point of view of Table 5 do not appear in the table since they do not lead to cohomogeneity one manifolds. We recall here that these groups are $(\mathrm{Spin}\,(12)\times SU(2))/\Delta\mathbb{Z}_2$, $(SU(3)\times SU(6))/\Delta\mathbb{Z}_3$, $SU(8)/\mathbb{Z}_2$ and $(E(6)\times U(1))/\Delta\mathbb{Z}_3$.

(1) We may assume that $\mathfrak{d}_6$ corresponds to the roots $\epsilon_i - \epsilon_j$, $\epsilon_i + \epsilon_j + \epsilon_7 + \epsilon_8$ with $1 \leq i, j \leq 6$. Then, let $\alpha = \epsilon_1 - \epsilon_7$, $\beta = \epsilon_2 - \epsilon_8$. As σ preserves the root spaces, we may conclude using Lemma 2.2.

(2) If $\mathfrak{h} \subset \mathfrak{a}_1 + \mathfrak{d}_6$, then we may assume that $\mathfrak{a}_1$ corresponds to $\epsilon_7 - \epsilon_8$ and $\mathfrak{a}_5$ to the roots $\epsilon_i - \epsilon_j$ $1 \leq i, j \leq 6$. Then, let $\alpha = \epsilon_1 - \epsilon_7$, $\beta = \epsilon_1 + \epsilon_2 + \epsilon_7 + \epsilon_8$. As σ preserves all the root spaces, we may conclude using Lemma 2.2. If $\mathfrak{h} \subset \mathbb{R} + \mathfrak{e}_6$, the proof is similar. If $\mathfrak{h} \subset \mathfrak{a}_2 + \mathfrak{a}_5$, we assume that the $\mathfrak{a}_1$ factor corresponds to $\epsilon_7 - \epsilon_8$ and $\mathfrak{a}_5$ to the roots $\epsilon_i - \epsilon_j$, $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_i$, $4 \leq i, j \leq 8$. Then, let $\alpha = \epsilon_1 - \epsilon_4$, $\beta = \epsilon_3 - \epsilon_5$ and conclude using Lemma 2.2. If $\mathfrak{h} \subset \mathfrak{a}_7$, the proof is similar to the one of the previous cases and we omit it.

(3) In this case, we have $\mathfrak{h} \subset \mathfrak{a}_7$. We may assume that $\mathfrak{a}_6$ corresponds to the roots $\epsilon_i - \epsilon_j$ with $1 \leq i, j \leq 7$. Then, let $\alpha = \epsilon_1 - \epsilon_8$, $\beta =$

$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_8$. σ preserves all the root spaces and we may conclude using Lemma 2.2.

(4) In this case, $\mathfrak{m}$ contains an irreducible K-module $\mathfrak{n}$ of dimension 30 which is not equivalent to any other K-module in $\mathfrak{m} + \mathfrak{p}$. By Lemma 2.3, the corresponding Killing vector fields must vanish. This must happen at the points of the second singular orbit as $\mathfrak{p}$ never contains any such-module. But Table 5 shows that it is impossible to have $\mathfrak{n} \subset \mathfrak{h}'$.

We omit the proof of the case $\mathfrak{e}_8$ which is very similar to the one of the case $\mathfrak{e}_7$.

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