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HOMOGENEOUS EINSTEIN METRICS AND SIMPLICIAL COMPLEXES

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Abstract

On compact homogeneous spaces, we investigate the Hilbert action restricted to the space of homogeneous metrics of volume one. Based on a detailed understanding of the high energy levels of this action, we assign to a compact homogeneous space a simplicial complex, whose non-contractibility is a sufficient condition for the existence of homogeneous Einstein metrics.

Einstein metrics of volume one on a closed manifold can be characterized variationally as the critical points of the Hilbert action [**30**], which associates to each Riemannian metric of volume one the integral of its scalar curvature, with respect to its volume element. Even though there exist many interesting classes of Einstein metrics, e.g., Kähler–Einstein metrics [**59**], [**48**], metrics with small holonomy group [**33**], Sasakian–Einstein metrics [**11**] and homogeneous Einstein metrics [**28**], [**10**], general existence and non-existence results are hard to obtain. For instance, in dimensions greater than or equal to five, no obstructions to the existence of Einstein metrics are known (cf. [**36**] for the 4-dimensional case).

On a compact homogeneous space G/H, the scalar curvature of a homogeneous Einstein metric is non-negative by a theorem of Bochner [**6**]. It is zero if and only if the metric is flat [**2**]. As a consequence, we are only interested in homogeneous Einstein metrics with positive scalar curvature; by the theorem of Bonnet–Myers, we may assume that G/H has finite fundamental group. For such homogeneous spaces G/H, the moduli space of G-invariant Einstein metrics is compact and has at most finitely many path components [**10**].

In the following, we assume that G and H are connected, referring to Section 1 for the general case. Let T be a maximal torus of a compact

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complement of H in $N_G(H)$. Then, there exist only finitely many connected Lie subgroups K of G with $TH \subsetneq K \subsetneq G$. We consider all flags $K_{\alpha_1} < \cdots < K_{\alpha_r}$ of such subgroups with $K_{\alpha_i} \subsetneq K_{\alpha_{i+1}}$. The extended simplicial complex $\hat{\Delta}_{G/H}$ of G/H is the associated flag complex; that is, flags K_α of length one correspond to vertices, flags $K_\alpha < K_{\alpha'}$ of length two correspond to edges and so on. The simplicial complex $\Delta_{G/H}$ of G/H is a subcomplex of the extended simplicial complex, which is homotopy equivalent to $\hat{\Delta}_{G/H}$.

Theorem A. *Let G/H be a compact homogeneous space with both G and H connected. If the simplicial complex of G/H is not contractible, then G/H admits a G-invariant Einstein metric.*

Geometrically, the simplicial complex and the extended simplicial complex can be thought of as the space of certain non-toral G-invariant foliations on G/H.

The starting point for the proof of Theorem A is the well known fact that G-invariant Einstein metrics on a compact homogeneous space G/H are the critical points of the Hilbert action, restricted to the space $\mathcal{M}_1^G$ of G-invariant metrics of volume one. This characterization of homogeneous Einstein metrics was first used by Jensen [**32**]. Later on, global variational methods have been described [**54**], [**10**]. Wang and Ziller attached to a compact homogeneous space G/H a graph $\Gamma_{G/H}$, defined by certain intermediate subgroups. By the Graph Theorem [**10**], G/H admits a G-invariant Einstein metric if $\Gamma_{G/H}$ has at least two non-toral components. To prove Theorem A, we apply variational methods similar to those described in [**54**], [**10**], which, however, rely on new scalar curvature estimates.

In the (generic) case that each G-invariant metric on G/H is even $(G\times T)$-invariant, the simplicial complex $\Delta_{G/H}$ is a much finer invariant for describing the topology of high energy levels of the Hilbert action restricted to $\mathcal{M}_1^G$ than the graph $\Gamma_{G/H}$. In fact, we conjecture that on these homogeneous spaces, the simplicial complex and high energy levels of the Hilbert action restricted to $\mathcal{M}_1^G$ are homotopy equivalent.

We turn to a monoid structure on the set of connected simply connected compact homogeneous spaces. As is well known, any such homogeneous space has a presentation G/H with G connected simply connected and semisimple and H connected (cf. [**44**]), and in the following, we will restrict our attention to such presentations.

From the homotopy sequence of the fibration $H \to G \to G/H$, it follows that the second homotopy group of G/H is finite if and only if the isotropy group H is semisimple. For such a homogeneous space G/H, we have either $\operatorname{rk} N_G(H) = \operatorname{rk} H$ or $\operatorname{rk} N_G(H) > \operatorname{rk} H$, where the rank of a compact Lie group is the dimension of a maximal torus. In the latter case, let us choose a maximal torus T in a compact complement of H in $N_G(H)$. Then, G/H is the total space of the principal torus bundle $TH/H \to G/H \to G/TH$. We note that the base G/TH does not depend on the choice of T and that $\operatorname{rk} N_G(TH) = \operatorname{rk} TH$.

Definition. Let G/H be a compact simply connected non-product homogeneous space with G connected, simply connected and semisimple. Then, we call G/H a prime homogeneous space, if $N_G(H)$ and H have the same rank.

A homogeneous space G/H is called a product homogeneous space if $G = G_1 \times G_2$ and $H = H_1 \times H_2$ with $H_i \subsetneq G_i$.

A simply connected homogeneous space G/H, G as above, is either a product of prime homogeneous spaces or the total space of a principal torus bundle over such a product. In both cases, we call the factors of this product the prime factors of G/H.

Theorem B. *Let G/H be a compact simply connected homogeneous space with G connected, simply connected and semisimple. If there exists a field $\mathbb{F}$ such that the reduced homology with coefficients in $\mathbb{F}$ of the simplicial complexes of all prime factors of G/H does not vanish, then G/H admits a G-invariant Einstein metric.*

Since in each dimension there exist at most finitely many equivariant diffeomorphism types of prime homogeneous spaces, the computation of the simplicial complexes $\Delta_{G/H}$ for all prime homogeneous spaces G/H up to a fixed dimension is a finite enumeration problem in Lie theory.

Even though we did not pursue this non-trivial task in this paper, we will indicate below by a number of examples, that Theorem B provides an extremely powerful result for proving existence of homogeneous and sometimes even inhomogeneous Einstein metrics (see [**8**]).

Let us turn to the classification problem of simply connected compact homogeneous Einstein manifolds. There exists a wealth of examples of homogeneous Einstein manifolds and also partial classification results have been obtained [**37**], [**38**], [**39**], [**58**], [**56**], [**60**], [**53**], [**34**], [**42**] (cf. [5], [52] for many more examples and details). With the help of Theorem B, the following result could be proved.

Theorem ([9]). *Any compact simply connected homogeneous space of dimension less or equal than eleven admits a homogeneous Einstein metric.*

This theorem is optimal, since there exist 12-dimensional homogeneous spaces which do not admit homogeneous Einstein metrics (cf. [54], [9]). More generally, in [7] we could describe large classes of simply connected non-product homogeneous spaces not carrying homogeneous Einstein metrics (cf. [54], [51], [45]).

We turn to important properties of the simplicial complex $\Delta_{G/H}$ of a compact homogeneous space G/H. The homological dimension of $\Delta_{G/H}$ can be arbitrary large, e.g. $\dim \tilde{H}_{n-3}(\Delta_{\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1)^n)}, \mathbb{Z}_2) \geq n-1$. Maximal simplices of $\Delta_{G/H}$ may also have distinct dimension. In such a case, the simplicial complex $\Delta_{G/H}$ is of course not a Tits building [49] even though the known non-contractible simplicial complexes $\Delta_{G/H}$ are likely to have the homotopy type of a bouquet of spheres.

By Quillen's work [46], the simplicial complex of a product homogeneous space $G/H = G_1/H_1 \times G_2/H_2$ can be computed: We have $\Delta_{G/H} = \Delta_{G_1/H_1} * \Delta_{G_2/H_2} * S^0$. Here, $\Delta * \tilde{\Delta}$ denotes the join of simplicial complexes Δ and $\tilde{\Delta}$ and $\Delta * S^0$ is the suspension of Δ. Theorem B is now an immediate consequence of Theorem A and this product formula together with Milnor's computation of homology groups of joins [41].

Finally, let us present examples of prime homogeneous spaces with non-contractible simplicial complex $\Delta_{G/H}$. We begin with the homogeneous space $G/H = \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1)^3)$. There are three connected intermediate Lie subgroups, all isomorphic to $\mathrm{S}(\mathrm{U}(2)\mathrm{U}(1))$, hence the reduced homology group $\tilde{H}_0(\Delta_{G/H}, \mathbb{Z}_2)$ with coefficients in the field $\mathbb{F} = \mathbb{Z}_2$ equals $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The homogeneous spaces with prime factor decomposition G/H are G/H itself, $\mathrm{SU}(3)/\{\mathrm{e}\}$ and the Aloff-Wallach spaces $G/H_{k,q} = \mathrm{SU}(3)/\mathrm{U}_{k,q}(1)$, where k, q are coprime integers and $\mathrm{U}_{k,q}(1)$ is embedded in $\mathrm{S}(\mathrm{U}(1)^3)$ with slope determined by (k, q). We recover the fact that the Aloff-Wallach spaces admit homogeneous Einstein metrics [50].

By Theorem B, the homogeneous space $P = \mathrm{SU}(3) \times \cdots \times \mathrm{SU}(3)/\Delta T$ admits a homogeneous Einstein metric too for any diagonally embedded subtorus ΔT of $\mathrm{SU}(3) \times \cdots \times \mathrm{SU}(3)$. In the very special case that the characteristic classes of P are integral linear combinations of the indivisible classes $\alpha_i \in H^2(M_i, \mathbb{Z})$ of the first Chern classes $c_1(M_i)$ of the homogeneous Kähler–Einstein manifolds $M_i = \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1)^3)$, $i = 1, \ldots, p$, existence of a homogeneous Einstein metric on P follows also from [**55**].

As a consequence, we can deduce from the existence of a single non-contractible simplicial complex the existence of infinitely many homotopy types of simply connected homogeneous Einstein n-manifolds, for infinitely many dimensions $n \geq 7$ (cf. [**50**], [**55**], [**13**], [**12**]).

More elaborate examples of prime homogeneous spaces with non-contractible simplicial complex $\Delta_{G/H}$ are given by G/H, where G is a connected simple classical Lie group and H a maximal torus of G, $G/H = \mathrm{SU}(n)/K$, where K is a connected subgroup of maximal rank, and $G/H = \mathrm{Spin}(p_1 + \cdots + p_r)/\mathrm{Spin}(p_1) \times \cdots \times \mathrm{Spin}(p_r)$, where $r \geq 2$ and $p_1, \ldots, p_r \geq 2$.

Since for all these prime homogeneous spaces the reduced homology with coefficients in $\mathbb{Z}_2$ of the simplicial complex does not vanish (by definition, we have $\tilde{H}_*(\emptyset, \mathbb{F}) \neq 0$ for any field $\mathbb{F}$), Theorem B implies the existence of large classes of homogeneous Einstein manifolds. For instance, we obtain the following new classification result:

Proposition. *Let $G = \mathrm{SU}(n_1) \times \cdots \times \mathrm{SU}(n_p)$, $n_1, \ldots, n_p \geq 2$, and let H be a compact, connected subgroup of G. If $\mathfrak{h}$, the Lie algebra of H, is a regular subalgebra of $\mathfrak{g}$, the Lie algebra of G, then G/H admits a G-invariant Einstein metric.*

A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called regular, if the normalizer $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$ and $\mathfrak{g}$ have the same rank (cf. [**23**]). Let us note that in general it seems to be impossible to solve the Einstein equations explicitly in these cases.

Our paper contains eight sections. In the first section, we sketch the proof of Theorem A and provide further general existence results on homogeneous Einstein metrics. In Section 2, we give many examples of homogeneous spaces with non-contractible simplicial complex and in Section 3, we describe the simplicial complex of product homogeneous spaces. Basic results on the space of homogeneous metrics and the scalar curvature of homogeneous metrics are described in Section 4. In Section 5, we examine the asymptotic behavior of the Hilbert action, restricted to the space of homogeneous metrics of volume one. Homogeneous spaces of finite type are defined in Section 6. In Section 7, we assign to arbitrary homogeneous spaces simplicial complexes. The proof of Theorem A is given in Section 8.

1. On the proof of Theorem A

We will give a brief outline of the main steps of the proof of Theorem A in this section. We consider arbitrary compact, connected, almost effective homogeneous spaces G/H, where G and H are compact Lie groups (not necessarily connected).

In Section 4.1, we investigate the space of G-invariant, unit volume metrics on G/H, denoted by $\mathcal{M}_1^G$. When endowed with the L^2-metric, $\mathcal{M}_1^G$ becomes a finite-dimensional, non-compact symmetric space. By homogeneity, the scalar curvature $\mathrm{sc}(g)$ is a constant function on G/H for $g \in \mathcal{M}_1^G$. Therefore, the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ is a differentiable, real-valued function. Let $Q \in \mathcal{M}_1^G$ be a suitable base point (a normal homogeneous metric) and let γ_v denote a unit speed geodesic in $\mathcal{M}_1^G$ emanating from Q. For $v \in T_Q\mathcal{M}_1^G$, let α_v be defined by $v(\cdot\,,\cdot) = Q(\alpha_v\,\cdot\,,\cdot)$ and let $\mathfrak{m}_{I_1^v(f)}$ denote the eigenspace of α_v associated to the smallest eigenvalue.

We call a subalgebra, which lies properly in between $\mathfrak{h}$ and $\mathfrak{g}$, H-subalgebra, if it is invariant under the adjoint action of H on $\mathfrak{g}$. If H is connected, then every subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ with $\dim \mathfrak{h} < \dim \mathfrak{k} < \dim \mathfrak{g}$ is an H-subalgebra. Hence, in this case H-subalgebras are in one-to-one correspondence with connected Lie subgroups K of G with $H < K < G$. An H-subalgebra is called toral if its semisimple part is contained in $\mathfrak{h}$, otherwise non-toral. If K is connected and $\mathfrak{k}$ is a compact, toral H-subalgebra, then K/H is a torus.

Lemma 1.1. *If the scalar curvature functional is bounded from below along a geodesic ray γ_v, that is $\mathrm{sc}(\gamma_v(t)) \geq C$ for all $t \geq 0$, then $\mathfrak{h}\oplus\mathfrak{m}_{I_1^v(f)}$ is an H-subalgebra.*

This observation gives rise to the definition of the compact, semi-algebraic set of non-negative directions $W^\Sigma \subset \Sigma$ (Definition 5.11), where Σ denotes the unit sphere in $T_Q\mathcal{M}_1^G$. The "eigenvalues" of $v \in W^\Sigma$ satisfy linear inequalities which are coupled with Lie bracket relations of the associated eigenspaces. In particular, the eigenspace associated to the smallest eigenvalue corresponds to an H-subalgebra (Lemma 5.16). If this H-subalgebra is toral, then we call the direction $v \in W^\Sigma$ toral, otherwise non-toral. We prove that for $v \in \Sigma \backslash W^\Sigma$ we have

$\lim_{t\to+\infty} \mathrm{sc}(\gamma_v(t)) = -\infty$. More generally, Theorem 5.18 provides uniform scalar curvature estimates.

As a first consequence, the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ is bounded from above for homogeneous spaces G/H, which do not admit H-subalgebras. Since the existence of a global maximum point follows immediately, we obtain a new proof of Theorem 2.2 in [**54**].

In the next step, we investigate, which non-negative directions are positive, that is for which $v \in W^\Sigma \subset \Sigma$ the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ tends to $+\infty$ along the geodesic ray γ_v. To this end, we define in Section 5.6 the extended non-toral directions $X_{\mathrm{ent}}^\Sigma \subset W^\Sigma$. The extended non-toral directions X_{ent}^Σ contain the non-toral directions X_{nt}^Σ. If G/H does admit toral H-subalgebras, then $X_{nt}^\Sigma \subset X_{\mathrm{ent}}^\Sigma \subsetneq W^\Sigma$, otherwise $X_{nt}^\Sigma = W^\Sigma$. By Theorem 5.48, the non-toral directions and the extended non-toral directions are homotopy equivalent.

In Section 5.7, we show $\lim_{t\to+\infty} \mathrm{sc}(\gamma_v(t)) \leq 0$ for $v \in W^\Sigma \backslash X_{\mathrm{ent}}^\Sigma$. More generally, Theorem 5.52 provides uniform scalar curvature estimates. Notice that such scalar curvature estimates are definitively wrong for arbitrary directions $v \in X_{\mathrm{ent}}^\Sigma \backslash X_{nt}^\Sigma$. We conclude that $W^\Sigma \backslash X_{\mathrm{ent}}^\Sigma$ also does not contain positive directions.

As a consequence, we obtain the following structure result:

Theorem 1.2. *Let G/H be a compact homogeneous space. Then, the scalar curvature functional* $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ *is bounded from above if and only if there exist no non-toral H-subalgebras.*

For homogeneous spaces G/H for which G and H are connected, this result is well known (cf. [**54**, Theorem 2.2, 2.4]).

In the next step, we investigate the homotopy type of the set of positive directions in the non-toral directions $X_{nt}^\Sigma \subset W^\Sigma \subset \Sigma$. To this end, we define the nerve $X_{G/H}^\Sigma \subset X_{nt}^\Sigma$ of G/H. The nerve of G/H does not only consist of positive directions (Proposition 5.6), but also provides a link between the asymptotic behavior of the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ and certain flags of non-toral intermediate H-subalgebras.

For an H-subalgebra $\mathfrak{k}$, the canonical direction $v_{\mathrm{can}}(\mathfrak{k}) \in W^\Sigma \subset \Sigma$ has two distinct eigenvalues $v_1(\mathfrak{k}) < 0 < v_2(\mathfrak{k})$, such that the eigenspace associated to $v_1(\mathfrak{k})$ is given by the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$ (cf. Definition 5.1). Next, denoting by $\mathrm{conv}^E(x_1, \ldots, x_p)$ the convex hull of $x_1, \ldots, x_p \in T_Q\mathcal{M}_1^G$, for a flag $e = \mathfrak{k}_1 < \cdots < \mathfrak{k}_p$ of H-subalgebras with $\mathfrak{k}_i \subsetneq \mathfrak{k}_{i+1}$ let

$$\Delta_e = \Delta_{\mathfrak{k}_1,\dots,\mathfrak{k}_p} = \mathrm{conv}^E(v_{\mathrm{can}}(\mathfrak{k}_1),\dots,v_{\mathrm{can}}(\mathfrak{k}_p)) \subset T_Q\mathcal{M}_1^G\backslash\{0\}.$$

We call a non-toral H-subalgebra $\mathfrak{k}$ minimal non-toral, if an H-subalgebra $\mathfrak{k}'$ with $\mathfrak{k}' < \mathfrak{k}$ must be toral. Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}$. If $\mathfrak{k}$ is the smallest subalgebra of $\mathfrak{g}$ with $\mathfrak{k}_1, \mathfrak{k}_2 \leq \mathfrak{k}$, then we say that $\mathfrak{k}$ is generated by $\mathfrak{k}_1$ and $\mathfrak{k}_2$. We consider all minimal non-toral H-subalgebras of $\mathfrak{g}$, all H-subalgebras generated by them and finally, all flags of such H-subalgebras. We call such flags H-flags and we denote the (possibly infinite) set of maximal H-flags by $F_{G/H}$. Notice that not every flag of H-subalgebras is an H-flag.

Definition 1.3 (The nerve of a homogeneous space)**.** The nerve $X^{\Sigma}_{G/H}$ of a compact homogeneous space G/H is defined as follows:

$$X^{\Sigma}_{G/H} := \bigcup_{e\in F_{G/H}} \left\{ v = \frac{\bar{v}}{\|\bar{v}\|} \,\Big|\, \bar{v} \in \Delta_e \right\} \subset \Sigma.$$

The nerve is a compact, semialgebraic variety. Since there is a one-to-one correspondence between the set of canonical directions, the set of H-subalgebras of $\mathfrak{g}$ and the set of (non-trivial) G-invariant foliations on G/H, the nerve can be considered the space of (non-trivial) G-invariant foliations on G/H, generated by minimal non-toral foliations.

We propose the following:

Conjecture. Let G/H be a compact homogeneous space. Then, the high energy levels of the Hilbert action restricted to $\mathcal{M}_1^G$ and the nerve of G/H are homotopy equivalent.

We proved one part of this conjecture in the following case: In Section 6, we consider compact homogeneous spaces of finite type, which by definition admit at most finitely many minimal non-toral H-subalgebras. Consequently, the set of H-flags is finite. As in the introduction, a simplicial complex can be associated to this set of flags, which we denote by $\Delta^{\min}_{G/H}$. Notice that a homogeneous space of finite type may admit infinitely many non-toral H-subalgebras.

For homogeneous spaces of finite type, we show that an open semialgebraic neighborhood of the semialgebraic, compact set of extended non-toral directions $X^{\Sigma}_{\mathrm{ent}}$ in Σ is homotopy equivalent to the nerve (Theorem 6.10) and that the nerve is homeomorphic to the simplicial complex $\Delta^{\min}_{G/H}$ (Proposition 6.5). Hence, if $\Delta^{\min}_{G/H}$ is not contractible (and not empty), then high energy levels of the scalar curvature functional restricted to $\mathcal{M}_1^G$ carry topology.

In Section 8, we obtain by variational methods a Palais–Smale sequence of G-invariant metrics of volume one with scalar curvature bounded from below by a positive constant. Recall that a sequence (g_i) in $\mathcal{M}_1^G$ is called a Palais–Smale sequence if $\mathrm{sc}(g_i)$ is bounded and $\|(\mathrm{grad}\,\mathrm{sc})_{g_i}\|$ converges to 0 in an appropriate norm. Since we choose for $\|\ \|$ the L^2-norm, which by homogeneity reduces to $\|w\|_g$ for a tangent vector w at $g \in \mathcal{M}_1^G$, the existence of a critical point follows from [**10**].

Theorem 1.4. *Let G/H be a compact homogeneous space of finite type. If the simplicial complex $\Delta_{G/H}^{\min}$ is not contractible, then G/H admits a G-invariant Einstein metric.*

Next, we apply these methods to arbitrary homogeneous spaces G/H. Notice that if $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$, then G/H is of finite type. In this case, we set $\Delta_{G/H}^T := \Delta_{G/H}^{\min}$ where $T = \{e\}$ is the trivial group. If $\dim \mathfrak{n}(\mathfrak{h}) > \dim \mathfrak{h}$, then $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{m}_0$, where $\mathfrak{m}_0$ is a compact, non-trivial $\mathrm{Ad}(H)$-invariant subalgebra of $\mathfrak{g}$ (Lemma 4.27). Notice that $\mathfrak{n}(\mathfrak{h})$ is the Lie algebra of $N_{G_0}(H_0)$, where K_0 denotes the identity component of a Lie group K. Let T denote a maximal torus of the connected, compact subgroup of G with Lie algebra $\mathfrak{m}_0$. We consider non-toral H-subalgebras which are invariant under the adjoint action of T. Such H-subalgebras are called T-adapted. A non-toral T-adapted H-subalgebra $\mathfrak{k}$ is called T-minimal non-toral, if a T-adapted H-subalgebra $\mathfrak{k}'$ with $\mathfrak{k}' < \mathfrak{k}$ must be toral. Since there are at most finitely many T-minimal non-toral H-subalgebras (Corollary 7.2), we can assign to G/H and T a simplicial complex $\Delta_{G/H}^T$ as described above. For different choices of T, these simplicial complexes may be different for H disconnected.

In the next step, we observe that the space $(\mathcal{M}_1^G)^T$ of unit volume G-invariant metrics fixed by the "adjoint action" of T on $\mathcal{M}_1^G$ is invariant under the Ricci flow (Lemma 4.28). Hamilton's Ricci flow [**27**] restricted to $\mathcal{M}_1^G$ is just the gradient flow of the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$. As a consequence, we can apply the variational methods described above to $\mathrm{sc} : (\mathcal{M}_1^G)^T \to \mathbb{R}$.

Theorem 1.5. *Let G/H be a compact homogeneous space and let T be a maximal torus of a compact complement of H_0 in $N_{G_0}(H_0)$. If the simplicial complex $\Delta_{G/H}^T$ is not contractible, then G/H admits a G-invariant Einstein metric.*

In the special case that there exist at most finitely many T-adapted non-toral H-subalgebras, the extended simplicial complex $\hat{\Delta}^T_{G/H}$ of G/H is defined as $\Delta^T_{G/H}$, but employs all T-adapted non-toral H-subalgebras of $\mathfrak{g}$. Even though $\Delta^T_{G/H}$ is in general a proper subcomplex of $\hat{\Delta}^T_{G/H}$, both complexes are homotopy equivalent (Corollary 6.12).

Finally, let us assume that both G and H are connected. In this case, the simplicial complex $\Delta^T_{G/H}$ does not depend on the choice of T.

Definition 1.6. Let G/H be a compact homogeneous space with both G and H connected. Then, the simplicial complex $\Delta_{G/H}$ of G/H is defined by $\Delta_{G/H} := \Delta^T_{G/H}$.

If, in addition, G/H has finite fundamental group, by Proposition 7.3 we have $\Delta_{G/H} = \Delta_{G/TH}$. Theorem A follows now from Theorem 1.5, since the simplicial complex $\Delta_{G/TH} = \Delta^{\{e\}}_{G/TH}$ and the extended simplicial complex $\hat{\Delta}_{G/H} = \hat{\Delta}^{\{e\}}_{G/TH}$ of G/H are homotopy equivalent.

2. Examples

In this section, we will describe examples of compact homogeneous spaces G/H with non-contractible simplicial complex $\Delta_{G/H}$. For instance, we examine the prime homogeneous spaces G/T, where G is a simple Lie group and T a maximal torus of G, and show that the simplicial complex is non-contractible if G is one of the classical Lie groups.

From Corollary 3.3 and Milnor's computation of the reduced homology groups of joins, the following theorem can be deduced:

Theorem 2.1. *Let G/H be a compact homogeneous space with G and H connected and with finite fundamental group. If there exists a connected subgroup K of G, such that $K/H = T^r$, $G = G_1 \times \cdots \times G_p$ and $K = K_1 \times \cdots \times K_p$, $K_i \subsetneq G_i$, and a field $\mathbb{F}$, such that for each $i \in \{1, \ldots, p\}$ the reduced homology $\tilde{H}_*(\Delta_{G_i/K_i}, \mathbb{F})$ does not vanish, then $\Delta_{G/H}$ is not contractible.*

In the following lemma, we present a simple combinatorial criterion, which guarantees that the simplicial complex $\Delta_{G/H}$, respectively the extended simplicial complex $\hat{\Delta}_{G/H}$, is not contractible. Due to [**47**], a simplicial complex Z^r is called *pure*, if all the maximal simplices of Z^r

have dimension r. Pure simplicial complexes, which have the additional property that every $(r-1)$-dimensional simplex of Z^r is contained in precisely two r-dimensional simplices, are called pseudo-manifolds (a zero-dimensional simplicial complex Z^0 is a pseudo-manifold if $Z^0 = S^0$). Notice that there exist pure, contractible simplicial complexes, such that each hyper-face is contained in at least two top-dimensional simplices (cf. Remark 3.5).

Lemma 2.2. *Let G/H be a compact homogeneous space with both G and H connected and with finite fundamental group. Let Z^r be a pure subcomplex of $\hat{\Delta}_{G/H}$ of dimension r. Then, $\tilde{H}_r(\hat{\Delta}_{G/H}, \mathbb{Z}_2) \neq 0$, if Z^r is a pseudo-manifold and if one of its defining flags is maximal.*

Proof. Since Z^r is a pseudo-manifold, the subcomplex Z^r is a cycle modulo $\mathbb{Z}_2$. But Z^r is not a boundary, since one of its defining flags is maximal. q.e.d.

Now, let G be a connected, compact simple Lie group and let T be a maximal torus of G. Then, $\mathfrak{n}(\mathfrak{t}) = \mathfrak{t}$. Therefore, in order to determine the extended simplicial complex $\hat{\Delta}_{G/T}$, we have to compute all flags $\mathfrak{k}_1 < \cdots < \mathfrak{k}_p$ of subalgebras of $\mathfrak{g}$ with $\mathfrak{t} < \mathfrak{k}_i < \mathfrak{g}$.

In what follows, we adopt notation to [**15**]. Let R denote the set of real roots of $\mathfrak{g}$ (see [**15**], p. 185). For $\alpha \in R$, let $L_\alpha \subset \mathfrak{g} \otimes \mathbb{C}$ denote the corresponding weight space. Then, $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{t} \otimes \mathbb{C} \oplus \bigoplus_{\alpha \in R} L_\alpha$. Next, for $\alpha \in R$ let $\mathfrak{m}_\alpha := (L_\alpha \oplus L_{-\alpha}) \cap \mathfrak{g}$. The summands $\mathfrak{m}_\alpha$ are the non-trivial, real iso-typical summands of the real $\mathrm{Ad}(T)$-module $\mathfrak{g}$. For $\alpha, \beta \in R$, $\alpha \neq \pm\beta$ we have

$$\operatorname{span}_{\mathbb{R}}\{\,[\mathfrak{m}_\alpha, \mathfrak{m}_\beta]\,\} = \begin{cases} \mathfrak{m}_{\alpha+\beta} \oplus \mathfrak{m}_{\alpha-\beta} \\ \mathfrak{m}_{\alpha+\beta} \\ \mathfrak{m}_{\alpha-\beta} \\ \{0\} \end{cases} \Longleftrightarrow \begin{cases} \alpha+\beta,\ \alpha-\beta \in R \\ \alpha+\beta \in R,\ \alpha-\beta \notin R \\ \alpha+\beta \notin R,\ \alpha-\beta \in R \\ \alpha-\beta,\ \alpha+\beta \notin R \end{cases}$$

(cf. [**29**, III, Theorem 4.3, (iv), Theorem 6.3]).

Every T-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ can be written as the direct sum of $\mathfrak{t}$ with a direct sum of root spaces, that is $\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in I} \mathfrak{m}_\alpha$, where $I \subsetneq R$. Hence, flags of T-subalgebras are in one-to-one correspondence with the corresponding flags of direct sums of root spaces. A flag of direct sums of root spaces is given by a flag of unions of the corresponding roots. The latter flags will be called flags of roots, if they come from flags of T-subalgebras. Up to sign, these roots are determined uniquely and we will consider two flags of roots equal if they are equal up to signs. Furthermore, we will call a flag of roots maximal if the corresponding flag of T-subalgebras is maximal.

Example 2.3. Let G be a simple classical Lie group, and let H denote a maximal torus of G. Then, $\tilde{H}_*(\Delta_{G/H}, \mathbb{Z}_2) \neq 0$.

Proof.

Case A_{n-1}: Let $G/H = \mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1)^n)$, $n \geq 2$. Then $\hat{\Delta}_{\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1)^n)} = \hat{\Delta}_{\mathrm{U}(n)/(\mathrm{U}(1)^n)}$. If $n = 2$, then H is maximal in G, hence $\hat{\Delta}_{G/H} = \emptyset$, and we obtain the claim in this case ($\tilde{H}_*(\emptyset, \mathbb{Z}_2) \neq 0$ by definition). If $n = 3$, then there exist three maximal flags $(\vartheta_1 - \vartheta_2)$, $(\vartheta_1 - \vartheta_3)$ and $(\vartheta_2 - \vartheta_3)$, hence $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

For $n \geq 4$, let $\pi \in \mathrm{S}_{n-1}$, the symmetric group acting on the set $\{2, 3, \ldots, n\}$, be any permutation and consider the maximal flag

$$\begin{aligned}\{\vartheta_1 - \vartheta_{\pi(2)}\} &\subset \{\vartheta_1 - \vartheta_{\pi(2)}, \vartheta_1 - \vartheta_{\pi(3)}, \vartheta_{\pi(2)} - \vartheta_{\pi(3)}\} \\ &\subset \cdots \\ &\subset \{\vartheta_1 - \vartheta_{\pi(2)}, \ldots, \vartheta_1 - \vartheta_{\pi(n-1)}, \ldots, \vartheta_{\pi(n-2)} - \vartheta_{\pi(n-1)}\}.\end{aligned}$$

Let Δ' be the subcomplex of $\hat{\Delta}_{G/H}$ defined by all these maximal flags. The subcomplex Δ' is isomorphic to a flag complex of subsets of $\{2, \ldots, n\}$ by identifying the above flag of roots with the flag $\{\pi(2)\} \subset \{\pi(2), \pi(3)\} \subset \cdots \subset \{\pi(2), \ldots, \pi(n-1)\}$. Obviously, Δ' is a pseudo-manifold (in fact we have $\Delta' = S^{n-3}$), hence Lemma 2.2 can be applied, and we obtain the claim.

Case D_n: Let $G/H = \mathrm{SO}(2n)/\mathrm{SO}(2)^n$, $n \geq 2$. If $n = 2$, then the only maximal flags are $(\vartheta_1 - \vartheta_2)$ and $(\vartheta_1 + \vartheta_2)$ and we conclude $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Z}_2) = \mathbb{Z}_2 \neq 0$. If $n = 3$, then $\mathfrak{so}(6) = \mathfrak{su}(4)$ (cf. [**1**, p. 31]). For $n \geq 4$, let $\pi \in \mathrm{S}_{n-1}$ (acting on $\{2, \ldots, n\}$), $\epsilon_{\pi(i)} \in \{+1, -1\}$, $2 \leq i \leq n$, and

$$\begin{aligned}&\{\vartheta_1 + \epsilon_{\pi(2)}\vartheta_{\pi(2)}\} \\ &\quad\subset \{\vartheta_1 + \epsilon_{\pi(2)}\vartheta_{\pi(2)}, \vartheta_1 + \epsilon_{\pi(3)}\vartheta_{\pi(3)}, \vartheta_{\pi(2)} - \epsilon_{\pi(2)}\epsilon_{\pi(3)}\vartheta_{\pi(3)}\} \\ &\quad\subset \cdots \\ &\quad\subset \{\vartheta_1 + \epsilon_{\pi(2)}\vartheta_{\pi(2)}, \ldots, \vartheta_1 + \epsilon_{\pi(n)}\vartheta_{\pi(n)}, \\ &\qquad \vartheta_{\pi(2)} - \epsilon_{\pi(2)}\epsilon_{\pi(3)}\vartheta_{\pi(3)}, \ldots, \vartheta_{\pi(n-1)} - \epsilon_{\pi(n-1)}\epsilon_{\pi(n)}\vartheta_{\pi(n)}\}\end{aligned}$$

be a flag of roots. It is easy to check that this flag corresponds to a maximal flag of H-subalgebras.

Let Δ' be the subcomplex of $\hat{\Delta}_{G/H}$ defined by all these maximal flags. The subcomplex Δ' is isomorphic to a flag complex of subsets of $\{\pm 2, \pm 3, \dots, \pm n\}$ by identifying the above flag of roots with the flag $\{\epsilon_{\pi(2)}\pi(2)\} \subset \{\epsilon_{\pi(2)}\pi(2), \epsilon_{\pi(3)}\pi(3)\} \subset \cdots \subset \{\epsilon_{\pi(2)}\pi(2), \dots, \epsilon_{\pi(n)}\pi(n)\}$. Obviously, Δ' is a pseudo-manifold. Hence, Lemma 2.2 can be applied and we obtain the claim.

Case C_n: Let $G/H = \mathrm{Sp}(n)/\mathrm{U}(1)^n$, $n \geq 1$. If $n = 1$, then $\hat{\Delta}_{G/H} = \emptyset$. If $n = 2$, then $\mathfrak{sp}(2) = \mathfrak{so}(5)$ (cf. [**1**, p. 31]). For $n \geq 3$, let us define $\Delta' \subsetneq \hat{\Delta}_{G/H}$ as in case D_n. Then, the claim follows.

Case B_n: Let $G/H = \mathrm{SO}(2n+1)/\mathrm{SO}(2)^n$, $n \geq 1$. If $n = 1$, then $\hat{\Delta}_{G/H} = \emptyset$. If $n = 2$, then $\hat{\Delta}_{G/H}$ is disconnected. For $n \geq 3$ let $\pi \in \mathrm{S}_n$. We consider the flag

$$\begin{aligned}
\{\vartheta_{\pi(1)}\} &\subset \{\vartheta_{\pi(1)}, \vartheta_{\pi(2)}, \vartheta_{\pi(1)} \pm \vartheta_{\pi(2)}\} \\
&\subset \cdots \\
&\subset \{\vartheta_{\pi(1)}, \dots, \vartheta_{\pi(n-1)}, \vartheta_{\pi(1)} \pm \vartheta_{\pi(2)}, \dots, \vartheta_{\pi(n-2)} \pm \vartheta_{\pi(n-1)}\}
\end{aligned}$$

and the subcomplex Δ' of $\hat{\Delta}_{G/H}$ defined by all these flags. Since Δ' is a pseudo-manifold and all its defining flags are maximal, we obtain the claim by Lemma 2.2. q.e.d.

Notice that the graph $\Gamma_{G/T'}$ is connected for a generic subtorus T' of H and that it seems to be impossible to solve the Einstein equations explicitly in such a case.

Next, we describe homogeneous spaces, which have the same extended simplicial complex as $\mathrm{U}(r)/(\mathrm{U}(1))^r$: The first example is

$$G/H = \mathrm{SO}(p_1 + \cdots + p_r)/\mathrm{SO}(p_1) \times \cdots \times \mathrm{SO}(p_r)\,,$$

where $r \geq 2$ and $p_1, \dots, p_r \geq 3$, a second one

$$G/H = \mathrm{Sp}(p_1 + \cdots + p_r)/\mathrm{Sp}(p_1) \times \cdots \times \mathrm{Sp}(p_r)\,,$$

where $r \geq 2$ and $p_1, \dots, p_r \geq 1$. These homogeneous spaces do not admit homogeneous Kähler–Einstein metrics (cf. [**5**, 8.113, 8.115, 8.116]) nor do they admit a normal homogeneous Einstein metric for generic choices of $(p_1, \dots, p_r)$ (see [**53**]).

For the unitary group, there exist corresponding examples. They are contained in the much larger class of homogeneous spaces described below.

Example 2.4. Let $G = \mathrm{SU}(n_1)\times\cdots\times\mathrm{SU}(n_p)$, $n_1,\ldots,n_p \geq 2$, and let H be a compact, connected subgroup of G. If $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{g}$, then G/H admits a G-invariant Einstein metric.

Proof. Since $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{g}$, $\mathfrak{n}(\mathfrak{h})$ contains a maximal torus of $\mathfrak{g}$ (cf. [**43**, p. 150], [**23**, p. 142]). Consequently, there exists an H-subalgebra $\mathfrak{k}$ with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{a}$, $\mathfrak{a}$ abelian or $\mathfrak{a} = 0$, and $\operatorname{rk}\mathfrak{k} = \operatorname{rk}\mathfrak{g}$. Let K denote the compact, connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. Since $\operatorname{rk} K = \operatorname{rk} G$, we get $K = K_1 \times \cdots \times K_p$, with $\operatorname{rk} K_i = \operatorname{rk}\mathrm{SU}(n_i)$. As is well known, $K_i = \mathrm{S}(\mathrm{U}(p^i_1)\times\cdots\times\mathrm{U}(p^i_{r_i}))$ with $\sum_{k=1}^{r_i} p^i_k = n_i$. Since $\Delta_{\mathrm{SU}(n_i)/K_i} = \Delta_{\mathrm{U}(r_i)/(\mathrm{U}(1))^{r_i}}$, we have $\tilde{H}_*(\Delta_{\mathrm{SU}(n_i)/K_i},\mathbb{Z}_2) \neq 0$. The claim follows now from Theorem 2.1. q.e.d.

This class of homogeneous spaces covers the following very intriguing example: On $S^2 \times S^{2m+1}$, the moduli space of Einstein metrics has infinitely many connected components [**55**]. The corresponding Einstein metrics are homogeneous with respect to one of the infinitely many inequivalent, transitive actions of $\mathrm{SU}(2)\times\mathrm{SU}(m+1)$ on $S^2 \times S^{2m+1}$.

Example 2.5. Let $G/H = \mathrm{SO}(p_1+\cdots+p_r)/\mathrm{SO}(p_1)\times\cdots\times\mathrm{SO}(p_r)$, with $r \geq 2$ and $p_1,\ldots,p_r \geq 2$. Then, $\tilde{H}_*(\Delta_{G/H},\mathbb{Z}_2) \neq 0$.

Proof. The case $r = 2$ is trivial. For $r \geq 3$: By Example 2.3 we may assume $p_1 > 2$. If also $p_2,\ldots,p_r > 2$, then $\hat{\Delta}_{G/H} = \hat{\Delta}_{\mathrm{U}(r)/\mathrm{U}(1)^r}$. Hence, we are left with the case $p_1 \geq \cdots \geq p_s > p_{s+1} = \cdots = p_r = 2$. Let $\pi \in S_{r-1}$, acting on $\{2,\ldots,r\}$, and consider the flag of H-subalgebras $\mathfrak{so}(p_1 + p_{\pi(2)}) + \mathfrak{h} < \cdots < \mathfrak{so}(p_1 + p_{\pi(2)} + \cdots + p_{\pi(r-1)}) + \mathfrak{h}$. Let the subcomplex Δ' of $\hat{\Delta}_{G/H}$ be defined by all these flags. As in Example 2.3, the claim follows. q.e.d.

For most of these homogeneous spaces, neither $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{g}$ nor G/H is a homogeneous Kähler–Einstein manifold with positive first Chern class.

The following prime homogeneous spaces G/H have the common feature, that G is an exceptional Lie group and H a non-semisimple subgroup of the same rank.

Example 2.6. Let $G/H = \mathrm{G}_2/\mathrm{U}(2)$, $\mathrm{F}_4/\mathrm{U}(3)\mathrm{Sp}(1)$ or let $G/H = \mathrm{E}_6/\mathrm{Spin}(10)\mathrm{SO}(2)$, $\mathrm{E}_7/\mathrm{E}_6\mathrm{SO}(2)$, $\mathrm{E}_8/\mathrm{E}_6\mathrm{U}(2)$. Then, $\tilde{H}_0(\Delta_{G/H},\mathbb{Z}_2) \neq 0$.

Proof. Since $\mathfrak{so}(10)\oplus\mathfrak{so}(2)$ is maximal in $\mathfrak{e}_6$ and $\mathfrak{e}_6\oplus\mathfrak{so}(2)$ is maximal in $\mathfrak{e}_7$, the simplicial complex of the first two symmetric spaces is empty. For the other three homogeneous spaces G/H, we show that their simplicial complex is not connected. This will be established by proving that there exists a proper subalgebra $\mathfrak{k}_1$ containing $\mathfrak{h}$ properly, such that $\mathfrak{h}$ is maximal in $\mathfrak{k}_1$ and $\mathfrak{k}_1$ is maximal in $\mathfrak{g}$, and a second intermediate subalgebra $\mathfrak{k}_2$ not isomorphic to $\mathfrak{k}_1$ (see [10]). In order to fix the embedding of the isotropy group H in G, recall that $\mathfrak{u}(2) < \mathfrak{su}(3) < \mathfrak{g}_2$, $\mathfrak{u}(3)\oplus\mathfrak{sp}(1) < \mathfrak{sp}(3)\oplus\mathfrak{sp}(1) < \mathfrak{f}_4$ and $\mathfrak{e}_6\oplus\mathfrak{u}(2) < \mathfrak{e}_7\oplus\mathfrak{su}(2) < \mathfrak{e}_8$. In the first case, we set $\mathfrak{k}_1 = \mathfrak{so}(4)$ and $\mathfrak{k}_2 = \mathfrak{su}(3)$, in the second case, $\mathfrak{k}_1 = \mathfrak{sp}(3)\oplus\mathfrak{sp}(1)$ and $\mathfrak{k}_2 = \mathfrak{su}(3)\oplus\mathfrak{su}(3)$ and in the last case, $\mathfrak{k}_1 = \mathfrak{e}_7\oplus\mathfrak{su}(2)$ and $\mathfrak{k}_2 = \mathfrak{e}_6\oplus\mathfrak{su}(3)$ (cf. [1, p. 55-57]). q.e.d.

For all the prime homogeneous spaces G/H described above, the reduced homology with coefficients in $\mathbb{Z}_2$ of the simplicial complex $\Delta_{G/H}$ does not vanish. By Theorem A, the existence of a G-invariant Einstein metric follows. Even though this was known before in many cases, Theorem B implies also the existence of a homogeneous Einstein metric on each homogeneous torus bundle (with finite fundamental group) over an arbitrary product of such prime homogeneous spaces. In many cases, one obtains this way infinitely many homotopy types of simply connected homogeneous Einstein manifolds in a fixed dimension. The Einstein equations for these torus bundles are (in general) very complicated and it seems that none of the known methods for proving existence of homogeneous Einstein metrics can be applied.

Let us now turn to compact homogeneous spaces G/H with contractible simplicial complex. If G/H does not admit G-invariant Einstein metrics, then $\Delta_{G/H}$ must be contractible. For instance, we have the following:

Example 2.7 ([7])**.** Let G/H be a compact simply connected homogeneous space with G connected simply connected and semisimple. Then, G/H does not admit a G-invariant Einstein metric, if

$$\mathrm{Spin}(n)\times\mathrm{Spin}(n)/\Delta\mathrm{Spin}(n-2)\cdot(\mathrm{Spin}(2)\times\mathrm{Spin}(2)),\ n>8$$

is a prime factor of G/H and if G/H is generic.

Here, $\mathrm{Spin}(n)$ denotes the double covering space of $\mathrm{SO}(n)$ and the group $\mathrm{Spin}(n-2)\times\mathrm{Spin}(2)$ is a maximal subgroup of $\mathrm{Spin}(n)$ of maximal rank. The group $\Delta\mathrm{Spin}(n-2)$ denotes the diagonal embedding of

$\mathrm{Spin}(n-2)$ in $\mathrm{Spin}(n-2) \times \mathrm{Spin}(n-2)$. The homogeneous spaces

$$G/H = \mathrm{Spin}(n) \times \mathrm{Spin}(n)/\Delta\mathrm{Spin}(n-2) \cdot \Delta_{k,q}\mathrm{Spin}(2)$$

provide the simplest examples of such spaces, where k, q are coprime integers and $\Delta_{k,q}\mathrm{Spin}(2)$ is embedded in $\mathrm{Spin}(2) \times \mathrm{Spin}(2)$ with slope determined by (k, q). In this case, G/H is generic if $(k, q) \neq \pm(1, 1)$, $(0, \pm 1)$, $(\pm 1, 0)$. If $(k, q) = \pm(1, 1)$, then G/H admits a G-invariant Einstein metric by the Graph Theorem (but $\Delta_{G/H}$ is contractible of course).

This example illustrates one serious difficulty with respect to the classification problem of compact homogeneous Einstein manifolds. There exist torus bundles G/H over products of prime homogeneous spaces such that not all G-invariant metrics on G/H are $(G \times T)$-invariant. As a consequence, such torus bundles may admit G-invariant Einstein metrics, even in case that generic torus bundles do not.

Theorem B overcomes this problem: the simplicial complexes of the prime factors of a torus bundle are the crucial data in this theorem and not the bundle G/H itself or its space of G-invariant metrics.

Finally, let us also mention that Theorem B can only detect homogeneous Einstein metrics which have to exist for global reasons. There exist homogeneous Einstein spaces with contractible simplicial complex (cf. [**54**], [**10**]). The sphere $S^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$ is one example where we have $\Delta_{\mathrm{Sp}(n+1)/\mathrm{Sp}(n)} = \{\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\}$, another example is $G/H = \mathrm{G}_2/T^2$.

3. The simplicial complex of product homogeneous spaces

Let $\hat{G}/\hat{H}$ be a compact product homogeneous space, that is $\hat{G} = G \times \tilde{G}$, $\hat{H} = H \times \tilde{H}$, $H \subsetneq G$ and $\tilde{H} \subsetneq \tilde{G}$, where H, $\tilde{H}$, G and $\tilde{G}$ are compact Lie groups. We have $\hat{G}/\hat{H} = G/H \times \tilde{G}/\tilde{H}$ and

$$\mathfrak{n}(\hat{\mathfrak{h}}) = \hat{\mathfrak{h}} \oplus \hat{\mathfrak{m}}_0 = \mathfrak{n}(\mathfrak{h}) \oplus \mathfrak{n}(\tilde{\mathfrak{h}}) = \mathfrak{h} \oplus \mathfrak{m}_0 \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{m}}_0\,,$$

where $\mathfrak{m}_0$, $\tilde{\mathfrak{m}}_0$ and $\hat{\mathfrak{m}}_0 = \mathfrak{m}_0 \oplus \tilde{\mathfrak{m}}_0$ are compact subalgebras of $\mathfrak{g}$. Let T and $\tilde{T}$ be maximal tori of the compact, connected Lie groups with Lie algebra $\mathfrak{m}_0$ and $\tilde{\mathfrak{m}}_0$, respectively. Then, $\hat{T} = T \times \tilde{T}$ is a maximal torus of the compact, connected Lie group with Lie algebra $\hat{\mathfrak{m}}_0$.

Next, let Δ and $\tilde{\Delta}$ be abstract simplicial complexes with realizations $|\Delta| \subset \mathbb{R}^p$ and $|\tilde{\Delta}| \subset \mathbb{R}^q$. The join of Δ and $\tilde{\Delta}$, denoted by $\Delta * \tilde{\Delta}$, is the abstract simplicial complex which has the realization

$$|\Delta * \tilde{\Delta}| = \Big\{((1-\lambda)x, \lambda y, \lambda) \mid 0 \le \lambda \le 1,\ (x,y) \in |\Delta| \times |\tilde{\Delta}|\Big\}$$

in $\mathbb{R}^{p+q+1}$ (cf. [40, 2.3.18]). We have $\emptyset * \Delta = \Delta$, $\Delta * \tilde{\Delta} = \tilde{\Delta} * \Delta$ and $(\Delta * \tilde{\Delta}) * \Delta' = \Delta * (\tilde{\Delta} * \Delta')$. Notice $\dim |\Delta * \tilde{\Delta}| = \dim |\Delta| + \dim |\tilde{\Delta}| + 1$. If $\tilde{\Delta} = S^0$, then $\Delta * \tilde{\Delta}$ is the suspension of Δ.

Let $C\Delta$ denote the cone over a simplicial complex Δ, that is the simplicial complex, whose realization $|C\Delta|$ is the cone on $|\Delta|$. Furthermore, let 0_Δ denote the vertex of $C\Delta$ not being contained in Δ.

Proposition 3.1 ([46]). *Let Δ and $\tilde{\Delta}$ be simplicial complexes. Then, there is a canonical homeomorphism $|\Delta * \tilde{\Delta}| = |(C\Delta \times C\tilde{\Delta}) \backslash \{0_\Delta \times 0_{\tilde{\Delta}}\}|$.*

Here, $(C\Delta \times C\tilde{\Delta}) \backslash \{0_\Delta \times 0_{\tilde{\Delta}}\}$ denotes the maximal subcomplex of $(C\Delta \times C\tilde{\Delta})$ which does not contain the simplicial neighborhood of the vertex $0_\Delta \times 0_{\tilde{\Delta}}$.

Theorem 3.2. *Let $\hat{G}/\hat{H}$ be a compact product homogeneous space, that is $\hat{G} = G \times \tilde{G}$ and $\hat{H} = H \times \tilde{H}$, $H \subsetneq G$ and $\tilde{H} \subsetneq \tilde{G}$. Then, the simplicial complex $\Delta^{\hat{T}}_{\hat{G}/\hat{H}}$ is homeomorphic to $\Delta^T_{G/H} * \Delta^{\tilde{T}}_{\tilde{G}/\tilde{H}} * S^0$.*

Proof. Since our proof works for arbitrary flag complexes, we will assume that $\mathfrak{n}(\hat{\mathfrak{h}}) = \hat{\mathfrak{h}}$ and $\Delta^T_{G/H} = \hat{\Delta}^T_{G/H}$, $\Delta^{\tilde{T}}_{\tilde{G}/\tilde{H}} = \hat{\Delta}^{\tilde{T}}_{\tilde{G}/\tilde{H}}$. We set $\Delta := \Delta^T_{G/H}$ and $\tilde{\Delta} := \Delta^{\tilde{T}}_{\tilde{G}/\tilde{H}}$.

We consider the flag complex defined by all flags of $\mathrm{Ad}(H \times \tilde{H})$-invariant subalgebras $\mathfrak{k}$ of $\mathfrak{g} \times \tilde{\mathfrak{g}}$ with $\mathfrak{h} \times \tilde{\mathfrak{h}} \le \mathfrak{k} \le \mathfrak{g} \times \tilde{\mathfrak{g}}$. By [46], this simplicial complex is homeomorphic to $C_{\mathfrak{h}}(C_{\mathfrak{g}}\Delta) \times C_{\tilde{\mathfrak{h}}}(C_{\tilde{\mathfrak{g}}}\tilde{\Delta})$ and by the above proposition, we obtain

$$|(C_{\mathfrak{h}}(C_{\mathfrak{g}}\Delta) \times C_{\tilde{\mathfrak{h}}}(C_{\tilde{\mathfrak{g}}}\tilde{\Delta})) \backslash \{0_{\mathfrak{h}} \times 0_{\tilde{\mathfrak{h}}}\}| = |C_{\mathfrak{g}}\Delta * C_{\tilde{\mathfrak{g}}}\tilde{\Delta}|.$$

From the simplicial complex $C_{\mathfrak{g}}\Delta * C_{\tilde{\mathfrak{g}}}\tilde{\Delta}$, we still need to remove (a simplicial neighborhood of) the vertex $0_{\mathfrak{g}} \times 0_{\tilde{\mathfrak{g}}}$. We may assume

$$\begin{aligned} |C_{\mathfrak{g}}\Delta| &= \{(t, t\bar{x}) \in \mathbb{R} \times \mathbb{R}^p \mid 0 \le t \le 1,\ \bar{x} \in |\Delta|\}, \\ |C_{\tilde{\mathfrak{g}}}\tilde{\Delta}| &= \{(s, s\bar{y}) \in \mathbb{R} \times \mathbb{R}^q \mid 0 \le s \le 1,\ \bar{y} \in |\tilde{\Delta}|\}, \end{aligned}$$

where $|\Delta| \subset \mathbb{R}^p$ and $|\tilde{\Delta}| \subset \mathbb{R}^q$ are realizations of Δ and $\tilde{\Delta}$, respectively. We deduce

$$\begin{aligned}
&|(C_{\mathfrak{g}}\Delta * C_{\tilde{\mathfrak{g}}}\tilde{\Delta})\backslash\{0_{\mathfrak{g}} \times 0_{\tilde{\mathfrak{g}}}\}| \\
&\quad = \Big\{((1-\lambda)(1,\bar{x}), \lambda(s, s\bar{y}), \lambda) \mid 0 \le \lambda, s \le 1,\ (\bar{x},\bar{y}) \in |\Delta| \times |\tilde{\Delta}|\Big\} \\
&\quad \cup \Big\{((1-\lambda)(t, t\bar{x}), \lambda(1,\bar{y}), \lambda) \mid 0 \le \lambda, t \le 1,\ (\bar{x},\bar{y}) \in |\Delta| \times |\tilde{\Delta}|\Big\} \\
&\quad = \Big\{((1-\lambda)\alpha(2\tau)(1,\bar{x}), \lambda\beta(2\tau)(1,\bar{y}), \lambda) \mid \\
&\qquad\qquad 0 \le \lambda, \tau \le 1,\ (\bar{x},\bar{y}) \in |\Delta| \times |\tilde{\Delta}|\Big\},
\end{aligned}$$

where $\alpha(\tau) = 1$ for $0 \le \tau \le 1$, $\alpha(\tau) = 2 - \tau$ for $1 \le \tau \le 2$ and $\beta(\tau) = \tau$ for $0 \le \tau \le 1$, $\beta(\tau) = 1$ for $1 \le \tau \le 2$.

Notice that $(\lambda, \tau) = (0, 1)$ and $(\lambda, \tau) = (1, 0)$ parameterize only one point, namely $(0, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 1)$, respectively. By contrast, the diagonal $\{\lambda = \tau\}$ of $[0, 1] \times [0, 1]$ parameterizes $\Delta * \tilde{\Delta}$, since the function $f(\lambda) = (1 - \lambda)\alpha(2\lambda)$ is continuous and strong monotonously decreasing with $f(0) = 1$ and $f(1) = 0$ and the function $g(\lambda) = \lambda\beta(2\lambda)$ is continuous and strong monotonously increasing with $f(0) = 0$ and $f(1) = 1$. By the same reasoning, the sets $\{\lambda \le \tau\}$ and $\{\lambda \ge \tau\}$ parameterize a simplicial complex homeomorphic to the cone over $\Delta * \tilde{\Delta}$. q.e.d.

Corollary 3.3. *Let $\hat{G}/\hat{H}$ be a compact homogeneous space. If $\hat{G} = G_1 \times \cdots \times G_p$ and $\hat{H} = H_1 \times \cdots \times H_p$, $H_i \subsetneq G_i$, then $\Delta^{\hat{T}}_{\hat{G}/\hat{H}}$ is homeomorphic to $\Delta^{T_1}_{G_1/H_1} * \cdots * \Delta^{T_p}_{G_p/H_p} * S^{p-2}$, where $\hat{T} = T_1 \times \cdots \times T_p$.*

Even if $\Delta^{T_i}_{G_i/H_i} = \emptyset$ for certain i, this gives interesting results. For instance, in case $\Delta^{T_i}_{G_i/H_i} = \emptyset$, $1 \le i \le p$, we get $\Delta^{\hat{T}}_{\hat{G}/\hat{H}} = S^{p-2}$.

Corollary 3.4. *Let $\hat{G}/\hat{H} = G/H \times \tilde{G}/\tilde{H}$ be a compact product homogeneous space. Then, if either $\Delta^{T}_{G/H}$ or $\Delta^{\tilde{T}}_{\tilde{G}/\tilde{H}}$ is contractible $\Delta^{\hat{T}}_{\hat{G}/\hat{H}}$ is contractible too.*

Vice versa, we expect that if $\Delta^{T}_{G/H}$ and $\Delta^{\tilde{T}}_{\tilde{G}/\tilde{H}}$ are not contractible, so is $\Delta^{\hat{T}}_{\hat{G}/\hat{H}}$. To some extent, this is known. The reduced homology groups of joins have been computed by Milnor [41]: If $\tilde{H}_q(\Delta, \mathbb{F})$ and $\tilde{H}_{\tilde{q}}(\tilde{\Delta}, \mathbb{F})$ are non-trivial for a field $\mathbb{F}$, then $\tilde{H}_{q+\tilde{q}+1}(\Delta * \tilde{\Delta}, \mathbb{F})$ is also non-trivial.

Note, however, that in general this might be wrong. There exists a 2-dimensional, non-contractible simplicial complex Δ such that $\Delta * S^0$ is contractible (see below). Therefore, if $\Delta^T_{G/H} = \Delta$ and $\Delta^{\tilde{T}}_{\tilde{G}/\tilde{H}} = \emptyset$, then $\Delta^{\hat{T}}_{\hat{G}/\hat{H}}$ would be contractible.

Remark 3.5. By the work of Whitehead, a simply connected CW-complex Δ which $\tilde{H}_n(\Delta, \mathbb{Z}) = 0$ for all $n \in \mathbb{N}_0$ is contractible (see Corollary 8.3.11 in [**40**]). However, there exist non-contractible CW-complexes with $\tilde{H}_n(\Delta, \mathbb{Z}) = 0$ for all $n \in \mathbb{N}_0$: The space Δ obtained from five points a_0, a_1, a_2, a_3, a_4, ten 1-cells (joining each pair of points) and six 2-cells given by the edge-loops $a_0a_1a_2a_3a_4a_0$, $a_0a_4a_1a_3a_2a_0$, $a_0a_2a_4a_3a_1a_0$, $a_0a_4a_2a_1a_3a_0$, $a_0a_3a_2a_4a_1a_0$, $a_0a_2a_1a_4a_3a_0$ is such an example (cf. Example 3.3.22 and Remark 8.3.12 in [**40**]). This space can be triangulated by adding six points (the barycenter of the 2-cells), thirty edges and by replacing the six 2-cells by thirty faces. This implies that there exist pure, non-contractible simplicial complexes Δ whose suspension $\Delta * S^0$ is contractible.

Note that for the 2-dimensional simplicial complex Δ given above, every edge is contained in two or three faces. Hence, every 2-dimensional simplex of $\Delta * S^0$ is contained in two or three 3-dimensional simplices of $\Delta * S^0$.

4. Preliminaries

In this section the space of G-invariant metrics is discussed in detail, and how the gauge group $N_G(H)$ of G/H is acting on it. We will also state a well known formula for the scalar curvature of a homogeneous metric on a compact homogeneous space (cf. [**54**]). Finally, basic properties of toral and non-toral H-subalgebras will be investigated.

4.1. The space of G-invariant metrics. The space of G-invariant metrics on G/H, denoted by $\mathcal{M}^G$, endowed with the L^2-metric becomes a symmetric space. For a suitable base point Q, we will "parameterize" $T_Q\mathcal{M}^G$ with the help of a polynomial map, which "unfolds" the tangent spaces of maximal flats in $(\mathcal{M}^G, L^2)$, which contain Q. We also derive a canonical splitting of $\mathcal{M}^G$ related to the isotypical summands of the isotropy representation of H.

Let G/H be a connected, almost effective, n-dimensional homogeneous space, where G and H are compact Lie groups (not necessarily connected). By [14, Chapter 0, Theorem 5.1], we may assume $G \subset \mathrm{O}(6N)$. The negative of the Killing form on $\mathrm{O}(6N)$ induces a biinvariant metric Q on G. The metric Q induces a normal homogeneous metric on G/H, again denoted by Q, which we may assume to have volume one after rescaling. Let us fix Q once and for all.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the $\mathrm{Ad}(H)$-invariant decomposition of $\mathfrak{g}$ with $Q(\mathfrak{h}, \mathfrak{m}) = 0$. The set of G-invariant metrics on G/H can be identified with the set of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$. Furthermore, for every $g \in \mathcal{M}^G$ we have $g(\,\cdot\,,\,\cdot\,) = Q(\alpha_g\,\cdot\,,\,\cdot\,)$, where α_g is an $\mathrm{Ad}(H)$-equivariant, Q-selfadjoint and positive definite endomorphism of $\mathfrak{m}$. If $\langle\,\cdot\,,\cdot\,\rangle$ denotes the L^2-metric on $\mathcal{M}^G$, then we have

$$\langle \phi, \psi \rangle_{\alpha_g} = \operatorname{tr}\, \alpha_g^{-1} \phi\, \alpha_g^{-1} \psi \tag{4.1}$$

for $\alpha_g \in \mathcal{M}^G$ and Q-selfadjoint tangent vectors $\phi, \psi \in T_{\alpha_g}\mathcal{M}^G$ (see [28, Section 3.1]). Having fixed a Q-orthonormal basis of $\mathfrak{m}$, we consider the endomorphism α_g a symmetric, positive definite and $\mathrm{Ad}(H)$-equivariant $(n \times n)$-matrix. Similarly, for $h \in H$ the endomorphism $\mathrm{Ad}(h)|_{\mathfrak{m}}$ will be considered an orthogonal $(n \times n)$-matrix.

The space of symmetric, positive definite $(n \times n)$-matrices with real entries, denoted by $P(n)$, endowed with the following Riemannian metric is a symmetric space with non-positive sectional curvatures (cf. [24]): $\langle V, W \rangle_P = \operatorname{tr} P^{-1} V P^{-1} W$ where $P \in P(n)$ and $V, W \in T_P P(n)$. The group $\mathrm{Gl}(n)$ acts on $P(n)$ by $\mathrm{Gl}(n) \times P(n) \to P(n)$; $(A, P) \mapsto APA^t$. This action is transitive and isometric, and $(\mathrm{Gl}(n), \mathrm{O}(n))$ is the corresponding symmetric pair. As is well known, $P(n)$ is isometric to $\mathbb{R} \times \mathrm{Sl}(n)/\mathrm{SO}(n)$. We conclude that $\mathcal{M}^G$, considered as the set of symmetric, positive definite and $\mathrm{Ad}(H)$-equivariant $(n \times n)$-matrices, is the fixed point set of the isometric action of H on $P(n)$ given by

$$(h, P) \mapsto \mathrm{Ad}(h)|_{\mathfrak{m}} \cdot P \cdot \mathrm{Ad}(h)^t|_{\mathfrak{m}}\,.$$

In particular, $\mathcal{M}^G$ is a totally geodesic subspace of $\mathrm{Gl}(n)/\mathrm{O}(n)$. Hence, again, a symmetric space with non-positive sectional curvatures. Note that the L^2-metric is the restriction of the above defined Riemannian metric to $\mathcal{M}^G$.

Let H_0 denote the identity component of the Lie group H.

Definition 4.2 (Trivial and almost trivial summands)**.** Let $\tilde{\mathfrak{m}} \leq \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant submodule of $\mathfrak{m}$. We call $\tilde{\mathfrak{m}}$ trivial (almost trivial) if $\mathrm{Ad}(H)$ $(\mathrm{Ad}(H_0))$ acts trivially on $\tilde{\mathfrak{m}}$.

We fix an $\mathrm{Ad}(H)$-isotypical splitting

$$\mathfrak{m} = \bigoplus_{i=1}^{\ell_{\mathrm{iso}}} \mathfrak{p}_i$$

of $\mathfrak{m}$ once and for all (cf. [**15**, II, Proposition 6.9]). We may assume that $\mathfrak{p}_i$ is almost trivial if and only if $1 \le i \le \ell_0$ where $0 \le \ell_0 \le \ell_{\mathrm{iso}}$. In case $\ell_0 > 0$, we set

$$\mathfrak{m}_0 = \bigoplus_{i=1}^{\ell_0} \mathfrak{p}_i.$$

The subspace $\mathfrak{m}_0$ of $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subalgebra of $\mathfrak{g}$, namely the Q-orthogonal complement of $\mathfrak{h}$ in the normalizer $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ (see Lemma 4.26).

By Schur's Lemma, every $\mathrm{Ad}(H)$-invariant inner product $Q(\alpha_g \cdot\,, \cdot\,)$ on $\mathfrak{m}$ respects this splitting. In other words

$$Q(\alpha_g \cdot\,, \cdot\,) = Q((\alpha_g)_1 \cdot\,, \cdot\,)|_{\mathfrak{p}_1} \perp \cdots \perp Q((\alpha_g)_{\ell_{\mathrm{iso}}} \cdot\,, \cdot\,)|_{\mathfrak{p}_{\ell_{\mathrm{iso}}}} \tag{4.3}$$

where $(\alpha_g)_i$ is an $\mathrm{Ad}(H)$-equivariant, selfadjoint and positive definite endomorphism of $\mathfrak{p}_i$. As above, by choosing the Q-orthonormal basis of $\mathfrak{m}$ appropriately, we can consider $(\alpha_g)_i$ a symmetric, positive definite and $\mathrm{Ad}(H)$-equivariant $(\dim \mathfrak{p}_i \times \dim \mathfrak{p}_i)$-matrix. Let

$$\mathfrak{p}_i = \bigoplus_{j=1}^{\ell_i} \mathfrak{m}_j^i \tag{4.4}$$

denote a Q-orthogonal splitting of $\mathfrak{p}_i$ into irreducible $\mathrm{Ad}(H)$-invariant summands. Let

$$D(\mathfrak{m}_1^i) := \mathrm{Hom}_{\mathrm{Ad}(H)}(\mathfrak{m}_1^i, \mathfrak{m}_1^i)$$

denote the division algebra of $\mathrm{Ad}(H)$-equivariant endomorphisms of $\mathfrak{m}_1^i$. Then, $D(\mathfrak{m}_1^i)$ equals $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ if and only if $\mathfrak{m}_1^i$ is a real representation of real, complex or quaternionic type, respectively (cf. [**15**, II, 6.2, II, Theorem 6.7]). Let

$$S(\ell_i, D(\mathfrak{m}_1^i)) := \begin{cases} \mathrm{O}(\ell_i) & \text{if } D(\mathfrak{m}_1^i) = \mathbb{R}, \\ \mathrm{U}(\ell_i) & \text{if } D(\mathfrak{m}_1^i) = \mathbb{C}, \\ \mathrm{Sp}(\ell_i) & \text{if } D(\mathfrak{m}_1^i) = \mathbb{H}. \end{cases}$$

Since an isotypical summand $\mathfrak{p}_i$ sums up equivalent irreducible summands of $\mathfrak{m}$, there exist $\mathrm{Ad}(H)$-equivariant Q-isometries $A_{1j}^i : \mathfrak{m}_1^i \to \mathfrak{m}_j^i$,

$1 \leq j \leq \ell_i$. We obtain an $\mathrm{Ad}(H)$-equivariant isometry from $(\mathfrak{p}_i, Q|_{\mathfrak{p}_i})$ to $(\mathfrak{m}_1^i, Q|_{\mathfrak{m}_1^i})^{\ell_i}$ endowed with the obvious $\mathrm{Ad}(H)$-action.

Proposition 4.5. *The symmetric space $\mathcal{M}^G$ of G-invariant metrics on G/H is* (*up to scaling the factors*) *isometric to the following direct product of symmetric spaces:*

$$\mathcal{M}^G = \prod_{i=1}^{\ell_{\mathrm{iso}}} \mathrm{Gl}(\ell_i, D(\mathfrak{m}_1^i))/S(\ell_i, D(\mathfrak{m}_1^i)).$$

Proof. By (4.3), we have $\mathcal{M}^G = \prod_{i=1}^{\ell_{\mathrm{iso}}} (\mathcal{M}^G)_i$, where $(\mathcal{M}^G)_i$ is a totally geodesic subspace of $\mathrm{Gl}(\dim \mathfrak{p}_i, \mathbb{R})/\mathrm{O}(\dim \mathfrak{p}_i)$. As remarked above, we may assume $\mathfrak{p}_i = (\mathfrak{m}_1^i)^{\ell_i}$. The space of $\mathrm{Ad}(H)$-invariant, not necessarily symmetric bilinear forms on $(\mathfrak{m}_1^i)^{\ell_i}$ can be identified with $\mathrm{Mat}(\ell_i, D(\mathfrak{m}_1^i))$ embedded into $\mathrm{Mat}(\dim \mathfrak{m}_1^i \cdot \ell_i, \mathbb{R})$ in the following way:

If $D(\mathfrak{m}_1^i) = \mathbb{R}$, then let $(\hat{e}_1, \ldots, \hat{e}_{d_i^{\mathbb{R}}})$ denote a Q-orthonormal basis of $\mathfrak{m}_1^i$. The group elements $r = a \cdot \mathrm{id}_{\mathfrak{m}_1^i} \in D(\mathfrak{m}_1^i)$ are embedded into $\mathrm{Mat}(d_i^{\mathbb{R}}, \mathbb{R})$ as follows:

$$r = a \cdot I_{d_i^{\mathbb{R}}}.$$

Consequently, $A_i = (a_{kj}^i)_{1 \leq k,j \leq \ell_i} \in \mathrm{Mat}(\ell_i, D(\mathfrak{m}_1^i))$ is embedded into $\mathrm{Mat}(\dim \mathfrak{m}_1^i \cdot \ell_i, \mathbb{R})$ as follows:

$$A_i = \begin{pmatrix} a_{11}^i I_{d_i^{\mathbb{R}}} & a_{12}^i I_{d_i^{\mathbb{R}}} & \ldots & a_{1\ell_i}^i I_{d_i^{\mathbb{R}}} \\ a_{21}^i I_{d_i^{\mathbb{R}}} & a_{22}^i I_{d_i^{\mathbb{R}}} & \ldots & a_{2\ell_i}^i I_{d_i^{\mathbb{R}}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell_i 1}^i I_{d_i^{\mathbb{R}}} & a_{\ell_i 2}^i I_{d_i^{\mathbb{R}}} & \ldots & a_{\ell_i \ell_i}^i I_{d_i^{\mathbb{R}}} \end{pmatrix}. \tag{4.6}$$

If we have $D(\mathfrak{m}_1^i) = \mathbb{C}$, then $\mathfrak{m}_1^i$ admits a Q-isometry $J \in D(\mathfrak{m}_1^i)$ with $J^2 = -\mathrm{id}_{\mathfrak{m}_1^i}$, thus $\dim \mathfrak{m}_1^i = 2d_i^{\mathbb{C}}$, $d_i^{\mathbb{C}} \in \mathbb{N}$. Let $(\hat{e}_1, \ldots, \hat{e}_{d_i^{\mathbb{C}}}, J\hat{e}_1, \ldots, J\hat{e}_{d_i^{\mathbb{C}}})$ denote a Q-orthonormal basis of $\mathfrak{m}_1^i$. Then, the group elements $z = a \cdot \mathrm{id}_{\mathfrak{m}_1^i} + b \cdot J \in D(\mathfrak{m}_1^i)$ are embedded into $\mathrm{Mat}(2d_i^{\mathbb{C}}, \mathbb{R})$ as follows:

$$z = \begin{pmatrix} aI_{d_i^{\mathbb{C}}} & -bI_{d_i^{\mathbb{C}}} \\ bI_{d_i^{\mathbb{C}}} & aI_{d_i^{\mathbb{C}}} \end{pmatrix}.$$

As a consequence, $A_i = (z_{kj}^i)_{1 \leq k,j \leq \ell_i} \in \mathrm{Mat}(\ell_i, \mathbb{C})$ is embedded into $\mathrm{Mat}(\dim \mathfrak{m}_1^i \cdot \ell_i, \mathbb{R})$ as in (4.6).

In case $D(\mathfrak{m}_1^i)=\mathbb{H}$, the summand $\mathfrak{m}_1^i$ admits Q-isometries $I,J,K\in D(\mathfrak{m}_1^i)$ with $I^2=J^2=K^2=-\mathrm{id}_{\mathfrak{m}_1^i}$ and $IJ=K$. Hence, $\dim\,\mathfrak{m}_1^i=4d_i^{\mathbb{H}}$, $d_i^{\mathbb{H}}\in\mathbb{N}$. Let $(\hat e_1,\dots,\hat e_{d_i^{\mathbb{H}}},I\hat e_1,\dots,I\hat e_{d_i^{\mathbb{H}}},-J\hat e_1,\dots,-J\hat e_{d_i^{\mathbb{H}}},K\hat e_1,\dots,K\hat e_{d_i^{\mathbb{H}}})$ denote a Q-orthonormal basis of $\mathfrak{m}_1^i$. Then, the group elements $q=a\cdot\mathrm{id}_{\mathfrak{m}_1^i}+b\cdot I+c\cdot J+d\cdot K\in D(\mathfrak{m}_1^i)$ are embedded into $\mathrm{Mat}(4d_i^{\mathbb{H}},\mathbb{R})$ as follows:

$$q=\begin{pmatrix} aI_{d_i^{\mathbb{H}}} & -bI_{d_i^{\mathbb{H}}} & cI_{d_i^{\mathbb{H}}} & -dI_{d_i^{\mathbb{H}}}\\ bI_{d_i^{\mathbb{H}}} & aI_{d_i^{\mathbb{H}}} & dI_{d_i^{\mathbb{H}}} & cI_{d_i^{\mathbb{H}}}\\ -cI_{d_i^{\mathbb{H}}} & -dI_{d_i^{\mathbb{H}}} & aI_{d_i^{\mathbb{H}}} & bI_{d_i^{\mathbb{C}}}\\ dI_{d_i^{\mathbb{H}}} & -cI_{d_i^{\mathbb{H}}} & -bI_{d_i^{\mathbb{H}}} & aI_{d_i^{\mathbb{C}}} \end{pmatrix}.$$

It follows as above that $A_i=(q_{kj}^i)_{1\le k,j\le \ell_i}\in\mathrm{Mat}(\ell_i,\mathbb{H})$ is embedded into $\mathrm{Mat}(\dim\,\mathfrak{m}_1^i\cdot\ell_i,\mathbb{R})$ as in (4.6).

The symmetric space $(\mathcal{M}^G)_i$ can, therefore, be thought of as the symmetric and positive definite matrices in $\mathrm{Mat}(\ell_i,D(\mathfrak{m}_1^i))$, embedded into $\mathrm{Mat}(\dim\,\mathfrak{m}_1^i\cdot\ell_i,\mathbb{R})$ as above. The group $\mathrm{Gl}(\ell_i,D(\mathfrak{m}_1^i))$ acts isometrically on $((\mathcal{M}^G)_i,L^2|_{(\mathcal{M}^G)_i})$ by $(A_i,P_i)\mapsto A_i^tP_iA_i$ and the isotropy group of $\mathrm{I}_{\dim\,\mathfrak{p}_i}$ is given by $S(\ell_i,D(\mathfrak{m}_1^i))$. Since the orbit of $\mathrm{I}_{\dim\,\mathfrak{p}_i}$ is open in $(\mathcal{M}^G)_i$, $\mathrm{Gl}(\ell_i,D(\mathfrak{m}_1^i))$ acts transitively. q.e.d.

Before we will define a "parameterization" of $T_Q\mathcal{M}^G$, we will make a brief digression in semialgebraic geometry. A set $X\subset\mathbb{R}^m$ is called semialgebraic if X is defined by finitely many polynomial equations and inequalities (cf. [**4**, Definition 2.1.1]). Let $X\subset\mathbb{R}^m$ and $Y\subset\mathbb{R}^n$ be semialgebraic sets. We call a map $f:X\to Y$ semialgebraic if the graph of f is a semialgebraic subset of $\mathbb{R}^{m+n}$ (cf. [**4**, Definition 2.3.2]). Semialgebraic maps do not have to be continuous (cf. [**4**, Example 2.7.3]), even though some authors require a semialgebraic map to be continuous by definition (cf. [**31**], [**20**, Theorem 7.6], [**22**, I, Proposition 3.13]). The most fundamental examples of semialgebraic maps are polynomials.

Let

$$\hat{\mathcal{M}}^G:=\widehat{T_Q\mathcal{M}^G}:=\prod_{i=1}^{\ell_{\mathrm{iso}}}\left(S(\ell_i,D(\mathfrak{m}_1^i))\times\mathbb{R}^{\ell_i}\right).$$

Then, the subset $\hat{\mathcal{M}}^G$ of the Euclidean space $\prod_{i=1}^{\ell_{\rm iso}}(\mathrm{Mat}(\ell_i, D(\mathfrak{m}_1^i))\times\mathbb{R}^{\ell_i})$, is semialgebraic. Next, let

$$T_Q\mathcal{M}^G = \prod_{i=1}^{\ell_{\rm iso}} T_{\mathrm{I}_{\dim\,\mathfrak{p}_i}}\mathrm{Gl}(\ell_i, D(\mathfrak{m}_1^i))/S(\ell_i, D(\mathfrak{m}_1^i))$$

and consider the cubic polynomial

$$\begin{aligned} F: \prod_{i=1}^{\ell_{\rm iso}}\Big(\mathrm{Mat}(\ell_i, D(\mathfrak{m}_1^i))\times\mathbb{R}^{\ell_i}\Big) \quad &\to \quad T_Q\mathcal{M}^G \ ; \\ ((A_1,D_1),\dots,(A_{\ell_{\rm iso}},D_{\ell_{\rm iso}})) \quad &\mapsto \quad \big(A_1D_1A_1^t,\dots,A_{\ell_{\rm iso}}D_{\ell_{\rm iso}}A_{\ell_{\rm iso}}^t\big)\,, \end{aligned}$$

where $D_i\in\mathbb{R}^{\ell_i}$ is considered a diagonal $(\ell_i\times\ell_i)$-matrix with real entries.

Proposition 4.7. *The polynomial map $F|_{\hat{\mathcal{M}}^G} : \hat{\mathcal{M}}^G \to T_Q\mathcal{M}^G$ is surjective.*

We call the map $F|_{\hat{\mathcal{M}}^G}$ a parameterization, since in case that $\mathfrak{m}$ is of real type it is a local diffeomorphism at generic points.

4.2. The scalar curvature. Recall that in Section 4.1 we have fixed an isotypical splitting $\mathfrak{m}=\oplus_{i=1}^{\ell_{\rm iso}}\mathfrak{p}_i$ of $\mathfrak{m}$. We call an ordered, Q-orthogonal decomposition

$$f=\bigoplus_{i=1}^{\ell}\mathfrak{m}_i$$

of $\mathfrak{m}$ into irreducible $\mathrm{Ad}(H)$-invariant summands $\mathfrak{m}_i$ iso-ordered if $\mathfrak{p}_1=\oplus_{j=1}^{\ell_1}\mathfrak{m}_j$, $\mathfrak{p}_2=\oplus_{j=1}^{\ell_2}\mathfrak{m}_{\ell_1+j}$ and so on. In what follows, we restrict our attention to such decompositions of $\mathfrak{m}$. Note that the maximal *flats* in the symmetric space $\mathcal{M}^G$, which contain Q, are in one-to-one correspondence with the finite sets $[f]$ of decompositions of $\mathfrak{m}$ obtained by reordering f.

Since $(\mathcal{M}^G, L^2)$ is a symmetric space with non-positive sectional curvatures, the geodesic exponential map from $T_Q\mathcal{M}^G$ to $\mathcal{M}^G$ is surjective by the Theorem of Hadamard-Cartan. As is well known, for each geodesic γ_v emanating from Q we have

$$\gamma_v(t)=e^{tv_1}Q|_{\mathfrak{m}_1}\perp\cdots\perp e^{tv_\ell}Q|_{\mathfrak{m}_\ell}, \tag{4.8}$$

where $f=\oplus_{i=1}^{\ell}\mathfrak{m}_i$ is a decomposition of $\mathfrak{m}$ (cf. [**29**, p. 226]). Here, $(v_1,\dots,v_\ell)\in\mathbb{R}^\ell$ and the tangent vector $v=\gamma_v'(0)\in T_Q\mathcal{M}^G$ is given by

$v = v_1 Q|_{\mathfrak{m}_1} \perp \cdots \perp v_\ell Q|_{\mathfrak{m}_\ell}$. Applying (4.1) yields

$$\langle \gamma_v'(0), \gamma_v'(0)\rangle_Q = \langle v, v\rangle_Q = \sum_{i=1}^{\ell} \underbrace{\dim \mathfrak{m}_i}_{=:d_i} \cdot v_i^2. \tag{4.9}$$

We will call such a decomposition f of $\mathfrak{m}$ a good decomposition of γ_v. For a fixed geodesic γ_v, a good decomposition might not be unique.

Now, let

$$\mathcal{M}_1^G \quad = \quad \left\{\gamma_v(1) \mid v \in T_Q\mathcal{M}^G \text{ and } \sum_{i=1}^{\ell} d_i \cdot v_i = 0\right\} \tag{4.10}$$

denote the set of G-invariant metrics of volume one. Then, $\mathcal{M}_1^G$ is a totally geodesic subspace of $\mathcal{M}^G$, hence, again, a symmetric space with non-positive sectional curvatures.

Let $v \in T_Q\mathcal{M}_1^G$ and let f be a good decomposition of γ_v. Next, let $\hat{v}_1 < \cdots < \hat{v}_{\ell_v}$ denote the distinct eigenvalues of α_v ordered by size, $\ell_v \geq 1$. Let the index sets $I_1^v(f), \ldots, I_{\ell_v}^v(f) \subset \{1, \ldots, l\}$ be defined by

$$v_i = \hat{v}_m \quad \Longleftrightarrow \quad i \in I_m^v(f)$$

for $m \in \{1, \ldots, \ell_v\}$ and $i \in \{1, \ldots, \ell\}$ and let $v_{\min} = \hat{v}_1$ and $v_{\max} = \hat{v}_{\ell_v}$ denote the smallest and the largest eigenvalue of α_v, respectively.

Finally, let

$$\Sigma = \left\{v \in T_Q\mathcal{M}_1^G \mid \sum_{i=1}^{\ell} d_i \cdot v_i^2 = 1\right\} \tag{4.11}$$

denote the unit sphere of $T_Q\mathcal{M}_1^G$ with respect to $L^2|_{T_Q\mathcal{M}_1^G}$. Note that in case $\dim \mathcal{M}^G = 1$ the space $\mathcal{M}_1^G$ is just a single point. This happens if and only if G/H is isotropy irreducible. In this case, we set $\Sigma = \emptyset$.

Lemma 4.12. *For $v \in \Sigma$, the number of distinct eigenvalues of α_v is bigger or equal than 2. Furthermore, there exists a constant $c_{G/H} < 0$, only depending on G/H, such that*

$$v_{\min} \leq c_{G/H} \quad \textit{and} \quad v_{\max} \geq -c_{G/H}.$$

Let $f = \oplus_{i=1}^{\ell} \mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$. For a non-empty subset I of $\{1, \ldots, \ell\}$, we set

$$\mathfrak{m}_I := \oplus_{i \in I} \mathfrak{m}_i \quad \text{and} \quad d_I := \dim \mathfrak{m}_I.$$

Since the spaces $\mathfrak{m}_{I^v_m(f)}$ are the eigenspaces of α_v, they do not depend on the choice of the good decomposition f of γ_v for $1 \le m \le \ell_v$. As a consequence,

$$\gamma_v(t) = e^{t\hat{v}_1} Q|_{\mathfrak{m}_{I^v_1(f)}} \perp \cdots \perp e^{t\hat{v}_{\ell_v}} Q|_{\mathfrak{m}_{I^v_{\ell_v}(f)}}$$

for any good decomposition f of γ_v.

We turn to the formula for the scalar curvature of homogeneous metrics, described in [**54**]. Let f be a fixed decomposition of $\mathfrak{m}$ and let $\{\hat{e}_1, \dots, \hat{e}_n\}$ be a Q-orthonormal basis of $\mathfrak{m}$ adapted to f. Let I, J, K be non-empty subsets of $\{1, \dots, \ell\}$. Following [**54**], we define

$$[IJK]_f := \sum Q([\hat{e}_\alpha, \hat{e}_\beta], \hat{e}_\gamma)^2,$$

where we sum over all indices $\alpha, \beta, \gamma \in \{1, \dots, n\}$ with $\hat{e}_\alpha \in \mathfrak{m}_I$, $\hat{e}_\beta \in \mathfrak{m}_J$ and $\hat{e}_\gamma \in \mathfrak{m}_K$. Notice $[IJK]_f$ is symmetric in all three entries and does not depend on the Q-orthonormal bases chosen for $\mathfrak{m}_I$, $\mathfrak{m}_J$ and $\mathfrak{m}_K$. In the special case $I = \{i\}$, $J = \{j\}$ and $K = \{k\}$, we simply write $[ijk]_f$. We have $[ijk]_f \ge 0$ with $[ijk]_f = 0$ if and only if $Q([\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k) = 0$. Furthermore, $f \mapsto [ijk]_f$ is a continuous function on the space of all ordered, Q-orthogonal decompositions of $\mathfrak{m}$ (cf. Section 4.3).

Lemma 4.13. *Let $v \in T_Q\mathcal{M}^G_1$ and let f be a good decomposition of γ_v. Then, for $1 \le i, j, k \le \ell_v$ the real number $[I^v_i(f) I^v_j(f) I^v_k(f)]_f$ does not depend on the choice of the good decomposition f of γ_v.*

Let $v \in T_Q\mathcal{M}^G$ and let f be a good decomposition of γ_v. Then, by [**54**] the scalar curvature of $\gamma_v(t)$ is given by

$$\operatorname{sc}(\gamma_v(t)) = \frac{1}{2}\sum_{i=1}^{\ell} d_i \cdot b_i \cdot e^{t(-v_i)} - \frac{1}{4}\sum_{i,j,k=1}^{\ell} [ijk]_f \cdot e^{t(v_i - v_j - v_k)} \tag{4.14}$$

$$= \frac{1}{2}\sum_{i=1}^{\ell_v} \left(\sum_{j \in I^v_i(f)} d_j \cdot b_j \right) \cdot e^{t(-\hat{v}_i)} \tag{4.15}$$

$$- \frac{1}{4}\sum_{i,j,k=1}^{\ell_v} [I^v_i(f) I^v_j(f) I^v_k(f)]_f \cdot e^{t(\hat{v}_i - \hat{v}_j - \hat{v}_k)},$$

where $b_i \ge 0$ is defined by $B(\,\cdot\,,\,\cdot\,)|_{\mathfrak{m}_i} = -\operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(\cdot) \circ \operatorname{ad}(\cdot))|_{\mathfrak{m}_i} = b_i \cdot Q|_{\mathfrak{m}_i}$. Recall that B is the negative of the Killing form. We have $B(X, X) \ge 0$ with equality if and only if $X \in \mathfrak{z}(\mathfrak{g})$.

The real numbers $\sum_{j\in I_i^v(f)} \frac{d_j}{2}\cdot b_j$ and $[I_i^v(f)I_j^v(f)I_k^v(f)]_f$ in (4.15) do not depend on the particular choice of the good decomposition f. However, the "constant" b_i does depend on the choice of a good decomposition f.

Let $(h_i)_{1\le i\le \dim H}$ denote a Q-orthonormal basis of $\mathfrak{h}$. The Casimir operator

$$C_{\mathfrak{m}_i,Q|_{\mathfrak{h}}} = -\sum_i \mathrm{ad}(h_i)\circ \mathrm{ad}(h_i)$$

of the $\mathrm{Ad}(H)$-irreducible summand $\mathfrak{m}_i$ satisfies

$$C_{\mathfrak{m}_i,Q|_{\mathfrak{h}}} = c_i\cdot \mathrm{id}_{\mathfrak{m}_i}\,.$$

We have $c_i\ge 0$ and $c_i=0$ if and only if $\mathfrak{m}_i$ is almost trivial (cf. Lemma 4.26).

Lemma 4.16 ([**54**])**.** *Let f be a decomposition of $\mathfrak{m}$. Then, we have*

$$d_ib_i - \frac{1}{2}\sum_{j,k=1}^{\ell}[ijk]_f = 2d_ic_i + \frac{1}{2}\sum_{j,k=1}^{\ell}[ijk]_f \ge 0.$$

Corollary 4.17. *Let G/H be a compact homogeneous space with finite fundamental group and let $f=\oplus_{i=1}^{\ell}\mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$. Then, $d_ib_i - \frac{1}{2}\sum_{j,k=1}^{\ell}[ijk]_f > 0$ for $1\le i\le \ell$.*

Proof. If $2d_ic_i + \frac{1}{2}\sum_{j,k=1}^{\ell}[ijk]_f = 0$, then $\mathfrak{m}_i$ is an abelian subalgebra which commutes with the subalgebra $\mathfrak{h}\oplus(\bigoplus_{j=1,j\ne i}^{\ell}\mathfrak{m}_j)$. We conclude that G_0/H_0 has infinite fundamental group. On the other hand, we have $G/H = G_0/(G_0\cap H)$ as manifolds. Hence, G_0/H_0 is a finite covering space of G/H. Contradiction. q.e.d.

Therefore, if G/H has finite fundamental group, then

$$b_{G/H} = \sum_{i=1}^{\ell} d_ib_i > 0. \tag{4.18}$$

4.3. The structure constants. We determine the space $\mathcal{F}^G$ of ordered (iso-ordered) decompositions of $\mathfrak{m}$. Furthermore, we show that there exist semialgebraic lifts $\widehat{[ijk]} : \hat{\mathcal{M}}^G \to \mathbb{R}$ of the structure constants $[ijk]:\mathcal{F}^G\to\mathbb{R}$.

Let

$$S_{f_0}(\ell_i, D(\mathfrak{m}_1^i)) = \begin{cases} (\mathbb{Z}_2)^{\ell_i} & \text{if } D(\mathfrak{m}_1^i) = \mathbb{R}, \\ (\mathrm{U}(1))^{\ell_i} & \text{if } D(\mathfrak{m}_1^i) = \mathbb{C}, \\ (\mathrm{Sp}(1))^{\ell_i} & \text{if } D(\mathfrak{m}_1^i) = \mathbb{H} \end{cases}$$

and let

$$\mathcal{F}^G = \prod_{i=1}^{\ell_{\text{iso}}} S(\ell_i, D(\mathfrak{m}_1^i))/S_{f_0}(\ell_i, D(\mathfrak{m}_1^i)).$$

Lemma 4.19. *The compact homogeneous space $\mathcal{F}^G$ is diffeomorphic to the space of ordered (iso-ordered) decompositions of $\mathfrak{m}$.*

Next, let us identify $\prod_{i=1}^{\ell_{\text{iso}}}(S(\ell_i, D(\mathfrak{m}_1^i))/S_{f_0}(\ell_i, D(\mathfrak{m}_1^i))) \times \mathbb{R}^{\ell_i}$ with $\mathcal{F}^G \times \mathbb{R}^\ell$. Then, the following diagram commutes:

$$\begin{tikzcd} \widehat{T_Q\mathcal{M}^G} = \hat{\mathcal{M}}^G \arrow[dd, "F|_{\hat{\mathcal{M}}^G}"'] \arrow[dr, "\pi_1"] & \\ & \mathcal{F}^G \times \mathbb{R}^l \arrow[dl, "\pi_2"] \\ T_Q\mathcal{M}^G & \end{tikzcd} \qquad \begin{tikzcd} (A, D) \arrow[dd] \arrow[dr] & \\ & ([A], D) \arrow[dl] \\ ADA^t & \end{tikzcd}$$

where $[A] = (A_1 S_{f_0}(\ell_1, D(\mathfrak{m}_1^1)), \ldots, A_{\ell_{\text{iso}}} S_{f_0}(\ell_{\ell_{\text{iso}}}, D(\mathfrak{m}_1^{\ell_{\text{iso}}})))$. Let

$$\pi_{\mathcal{F}^G} = \mathrm{pr}_1 \circ \pi_1 : \hat{\mathcal{M}}^G \to \mathcal{F}^G \ ; \ \ (A, D) \mapsto [A].$$

Next, for $1 \le i \le \ell_{\text{iso}}$, let $({}^i_1\hat{e}_1, \ldots, {}^i_1\hat{e}_{\dim\, \mathfrak{m}_1^i})$ denote the Q-orthonormal basis of $\mathfrak{m}_1^i$ chosen in Proposition 4.5. For $1 \le k \le \ell_i$, $1 \le j \le \dim\, \mathfrak{m}_1^i$ let ${}^i_k\hat{e}_j := A^i_{1k} \cdot {}^i_1\hat{e}_j$. Let $A_i = (a^i_{kj})_{1 \le j,k \le \ell_i}$ with $a^i_{kj} \in \mathrm{Mat}(\dim\, \mathfrak{m}_1^i, \mathbb{R})$. If $A_i \in \mathrm{O}(\dim\, \mathfrak{p}_i)$, then a Q-orthonormal basis of $\tilde{\mathfrak{m}}^i_j = A_i(\mathfrak{m}^i_j)$ is given by $(\sum_{k=1}^{\ell_i} a^i_{kj} \cdot {}^i_k\hat{e}_1, \ldots, \sum_{k=1}^{\ell_i} a^i_{kj} \cdot {}^i_k\hat{e}_{\dim\, \mathfrak{m}_1^i})$, where we have dropped some obvious isomorphisms.

Let $i = (\sum_{m=1}^{i_1 - 1} \ell_m) + i'$, $j = (\sum_{m=1}^{j_1 - 1} \ell_m) + j'$ and $k = (\sum_{m=1}^{k_1 - 1} \ell_m) + k'$, where $i_1, j_1, k_1 \in \{1, \ldots, \ell_{\text{iso}}\}$, $i' \in \{1, \ldots, \ell_{i_1}\}$, $j' \in \{1, \ldots, \ell_{j_1}\}$ and $k' \in \{1, \ldots, \ell_{k_1}\}$. For

$$A = (A_1, \ldots, A_{\ell_{\text{iso}}}) \in \prod_{i=1}^{\ell_{\text{iso}}} \mathrm{Mat}(\dim\, \mathfrak{p}_i, \mathbb{R})$$

let

$$\widehat{[ijk]}_A := \sum_{\alpha,\beta,\gamma} Q\left(\left[\sum_{\delta_\alpha=1}^{\ell_{i_1}} a^{i_1}_{\delta_\alpha i'} \cdot {}^{i_1}_{\delta_\alpha}\hat{e}_\alpha, \sum_{\delta_\beta=1}^{\ell_{j_1}} a^{j_1}_{\delta_\beta j'} \cdot {}^{j_1}_{\delta_\beta}\hat{e}_\beta \right], \sum_{\delta_\gamma=1}^{\ell_{k_1}} a^{k_1}_{\delta_\gamma k'} \cdot {}^{k_1}_{\delta_\gamma}\hat{e}_\gamma \right)^2,$$

where the sum is taken over $\alpha = 1, \ldots, \dim \mathfrak{m}_1^{i_1}$, $\beta = 1, \ldots, \dim \mathfrak{m}_1^{j_1}$ and $\gamma = 1, \ldots, \dim \mathfrak{m}_1^{k_1}$

The map $\widehat{[ijk]} : \prod_{i=1}^{\ell_{\mathrm{iso}}} \mathrm{Mat}(\dim \mathfrak{p}_i, \mathbb{R}) \to \mathbb{R}$ is a polynomial of degree 6, which will also be considered as a map on $\prod_{i=1}^{\ell_{\mathrm{iso}}} \big(\mathrm{Mat}(\dim \mathfrak{p}_i, \mathbb{R}) \times \mathbb{R}^{\ell_i}\big)$.

Proposition 4.20. *The function $\widehat{[ijk]} : \hat{\mathcal{M}}^G \to \mathbb{R}$ is semialgebraic, continuous and satisfies $\widehat{[ijk]}_{(A,D)} = [ijk]_{\pi_{\mathcal{F}^G}(A,D)}$.*

In later sections, it will be convenient to denote the diagonal matrix $D = (D_1, \ldots, D_{\ell_{\mathrm{iso}}})$ by $v = (v^1, \ldots, v^{\ell_{\mathrm{iso}}})$. For $i \in \{1, \ldots, \ell_{\mathrm{iso}}\}$, let $v^i \in \mathbb{R}^{\ell_i}$ be given by $v^i = (v^i_1, \ldots, v^i_{\ell_i})$. Let

$$\hat{\mathcal{M}}_1^G := \widehat{T_Q\mathcal{M}_1^G} := \Big\{ (A, v) \in \hat{\mathcal{M}}^G \mid \sum_{i=1}^{\ell_{\mathrm{iso}}} \sum_{j=1}^{\ell_i} \dim \mathfrak{m}_1^i \cdot v^i_j = 0 \Big\}.$$

Obviously, $\hat{\mathcal{M}}_1^G$ is a semialgebraic subset of $\hat{\mathcal{M}}^G$. Note that we will write $v = (v^1, \ldots, v^{\ell_{\mathrm{iso}}}) \in \prod_{i=1}^{\ell_{\mathrm{iso}}} \mathbb{R}^{\ell_i} = \mathbb{R}^\ell$ also as $v = (v_1, \ldots, v_\ell) \in \mathbb{R}^\ell$.

In what follows, we will also denote the restrictions $\widehat{[ijk]}|_{\hat{\mathcal{M}}_1^G}$, $\pi_1|_{\hat{\mathcal{M}}_1^G}$, $\pi_{\mathcal{F}^G}|_{\hat{\mathcal{M}}_1^G}$ and $\pi_2|_{\pi_1(\hat{\mathcal{M}}_1^G)}$ by $\widehat{[ijk]}$, π_1, π_2 and $\pi_{\mathcal{F}^G}$, respectively.

Corollary 4.21. *The polynomial map $F|_{\hat{\mathcal{M}}_1^G} : \hat{\mathcal{M}}_1^G \to T_Q\mathcal{M}_1^G$ is surjective and the function $\widehat{[ijk]} : \hat{\mathcal{M}}_1^G \to \mathbb{R}$ is semialgebraic, continuous and satisfies $\widehat{[ijk]}_{(A,v)} = [ijk]_{\pi_{\mathcal{F}^G}(A,v)}$.*

4.4. Toral and non-toral subalgebras. In this section, we will state basic properties of intermediate subalgebras of $\mathfrak{h}$ and $\mathfrak{g}$ and Lie groups associated to them.

Definition 4.22. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \leq \mathfrak{k}$. We call $\mathfrak{k}$ an H-subalgebra, if $\dim \mathfrak{h} < \dim \mathfrak{k} < \dim \mathfrak{g}$ and if $\mathfrak{k}$ is $\mathrm{Ad}(H)$-invariant. Let $\mathfrak{m}_\mathfrak{k}$ denote the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$. An H-subalgebra $\mathfrak{k}$ is called toral if $\mathfrak{m}_\mathfrak{k}$ is an abelian subalgebra of $\mathfrak{g}$, otherwise non-toral.

Note that H-subalgebras are not required to be compact. Here and in the following, we call a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ compact if it is the Lie algebra of a compact subgroup K of G. If H is connected, then every subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ with $\dim \mathfrak{h} < \dim \mathfrak{k} < \dim \mathfrak{g}$ is an H-subalgebra. We have $\mathfrak{m}_{\mathfrak{k}} \subset \mathfrak{m}_0$ for toral H-subalgebras $\mathfrak{k}$ (see Lemma 4.26), but $\mathfrak{m}_{\mathfrak{k}} \subset \mathfrak{m}_0$ does in general not imply, that $\mathfrak{k}$ is toral.

The Stiefel manifolds $\mathrm{SO}(n+2)/\mathrm{SO}(n)$, $n \geq 3$, admit one toral H-subalgebra, given by $\mathfrak{so}(2) \oplus \mathfrak{so}(n)$, and non-toral H-subalgebras, isomorphic to $\mathfrak{so}(n+1)$. In this case, the set of non-toral H-subalgebras is diffeomorphic to S^1.

Definition 4.23. For an H-subalgebra $\mathfrak{k}$ let $H(\mathfrak{k})$ denote the subgroup of G generated by H and the connected Lie subgroup K_0 of G with Lie algebra $\mathfrak{k}$.

If H is connected, then $H(\mathfrak{k}) = K_0$. But if H is disconnected, then H might not be a subgroup of K_0. In any event, the subgroup $H(\mathfrak{k})$ is a Lie subgroup of G with $T_1 H(\mathfrak{k}) = \mathfrak{k}$. Notice that the subgroup $H(\mathfrak{k})$ does not have to be compact. In this case, we would like to understand how $H(\mathfrak{k})$ and $\overline{H(\mathfrak{k})}$ (the closure of $H(\mathfrak{k})$ in G) are related.

Lemma 4.24. *Let $\mathfrak{k}$ be an H-subalgebra, such that $H(\mathfrak{k})$ is a non-compact Lie subgroup of G. Then, there exist $\mathrm{Ad}(H)$-invariant abelian subalgebras $\mathfrak{a}', \mathfrak{a} \subset \mathfrak{m}$ of $\mathfrak{g}$ with $\mathfrak{m}_{\mathfrak{k}} = \mathfrak{m}_{\mathfrak{k}'} \oplus \mathfrak{a}'$ and $T_1 \overline{H(\mathfrak{k})} = \mathfrak{k} \oplus \mathfrak{a}$.*

Proof. If $H(\mathfrak{k})$ is non-compact, then the projection of $\mathfrak{z}(\mathfrak{k})$ onto $\mathfrak{m}_{\mathfrak{k}}$ is non-trivial. Hence, the existence of $\mathfrak{a}'$ follows.

A subspace of $\mathfrak{g}$ is $\mathrm{Ad}(H(\mathfrak{k}))$-invariant if and only if it is $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-invariant. Hence, the Q-orthogonal complement of $\mathfrak{k}$ in $T_1 \overline{H(\mathfrak{k})}$, denoted by $\mathfrak{a}$, is $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-invariant. Thus, $\mathfrak{a}$ is an ideal in $T_1 \overline{H(\mathfrak{k})}$ and it is well known, that $\mathfrak{a}$ is abelian. q.e.d.

The following lemma will be needed later on.

Lemma 4.25. *Let $\tilde{\mathfrak{m}}$ be an $\mathrm{Ad}(H)$-invariant submodule of $\mathfrak{m}$, and let $\mathfrak{k}$ be an H-subalgebra. Then, $\tilde{\mathfrak{m}}$ is $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-invariant if and only if $[\mathfrak{k}, \tilde{\mathfrak{m}}] \subset \tilde{\mathfrak{m}}$.*

Proof. If $\tilde{\mathfrak{m}}$ is $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-invariant, then $\tilde{\mathfrak{m}}$ is $\mathrm{Ad}(H(\mathfrak{k}))$-invariant as well and we obtain $[\mathfrak{k}, \tilde{\mathfrak{m}}] \subset \tilde{\mathfrak{m}}$. Now, suppose $[\mathfrak{k}, \tilde{\mathfrak{m}}] \subset \tilde{\mathfrak{m}}$ and let K_0 denote the connected Lie subgroup of G with $T_1 K_0 = \mathfrak{k}$. Let $w \in \mathfrak{k}$.

Then $\mathrm{Ad}(\exp(w)) = \exp(\mathrm{ad}(w))$. Hence, $\tilde{\mathfrak{m}}$ is $\mathrm{Ad}(K_0)$-invariant. Since by assumption $\tilde{\mathfrak{m}}$ is $\mathrm{Ad}(H)$-invariant, the claim follows. q.e.d.

Recall that $N_G(H)$ denotes the normalizer of H in G and that $\mathfrak{n}(\mathfrak{h})$ denotes the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Lemma 4.26. *Let $\tilde{\mathfrak{m}}$ be an $\mathrm{Ad}(H)$-invariant submodule of $\mathfrak{m}$. Then, $\tilde{\mathfrak{m}}$ is almost trivial if and only if $[\mathfrak{h}, \tilde{\mathfrak{m}}] = 0$ if and only if $\tilde{\mathfrak{m}} \subset \mathfrak{n}(\mathfrak{h})$. In particular, $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{m}_0$.*

As is well known, the connected Lie subgroup of G with Lie algebra $\mathfrak{h} \oplus \mathfrak{m}_0$ equals $(N_G(H_0))_0 = (N_{G_0}(H_0))_0$. Note that, in general, we can have $\dim N_G(H) < \dim N_G(H_0)$.

Lemma 4.27. *The Lie algebra $\mathfrak{m}_0$ is compact.*

Proof. Let us decompose the compact Lie algebra $\mathfrak{h} \oplus \mathfrak{m}_0$ into the Q-orthogonal sum of its semisimple part $(\mathfrak{h} \oplus \mathfrak{m}_0)_s = \mathfrak{h}_s \oplus (\mathfrak{m}_0)_s$ and its center $\mathfrak{z}(\mathfrak{h} \oplus \mathfrak{m}_0)$. The Lie algebras $\mathfrak{h}_s$, $(\mathfrak{m}_0)_s$ and $\mathfrak{z}(\mathfrak{h} \oplus \mathfrak{m}_0)$ are compact. Also, $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{h} \oplus \mathfrak{m}_0)$ is compact. It remains to show that the Q-orthogonal complement of $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{h} \oplus \mathfrak{m}_0)$ in $\mathfrak{z}(\mathfrak{h} \oplus \mathfrak{m}_0)$ is compact. Since by the very definition of Q (cf. Section 4.1) the compact, abelian subalgebras $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{h} \oplus \mathfrak{m}_0)$ and $\mathfrak{z}(\mathfrak{h} \oplus \mathfrak{m}_0)$ can be considered subalgebras of $\mathfrak{so}(6N)$, the claim follows. q.e.d.

4.5. The gauge group. The group of G-equivariant diffeomorphisms of G/H acts on the symmetric space $(\mathcal{M}_1^G, L^2)$ by pulling back G-invariant metrics. Therefore, this action preserves the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ and its L^2-gradient $(\nabla_{L^2}\mathrm{sc})_g = -(\mathrm{ric}_g - (\mathrm{sc}(g)/n)g)$. This observation will enable us to define totally geodesic subspaces of $(\mathcal{M}_1^G, L^2)$ invariant under the Ricci flow.

Due to [**14**, I, Corollary 4.3] a G-equivariant diffeomorphism of G/H is a right translation $R_n : G/H \to G/H;\ gH \mapsto gnH$ for some $n \in N_G(H)$. Since $N_G(H)$ normalizes H, we can restrict the adjoint action of $N_G(H)$ to $\mathfrak{m}$. Thus, $\mathrm{Ad}(n)|_{\mathfrak{m}} \in \mathrm{O}(\mathfrak{m}, Q|_{\mathfrak{m}})$ for $n \in N_G(H)$. This yields an isometric action of $N_G(H)$ on $(\mathcal{M}^G, L^2)$ given by

$$g \mapsto (R_n)_*(g) = \mathrm{Ad}(n)|_{\mathfrak{m}} \circ g \circ \mathrm{Ad}(n)^{-1}|_{\mathfrak{m}}$$

where g is considered an $\mathrm{Ad}(H)$-equivariant endomorphism of $\mathfrak{m}$. Since $N_G(H)$ acts on $\mathcal{M}_1^G$ by pulling back G-invariant metrics, integral curves of the Ricci flow on $\mathcal{M}_1^G$ are mapped onto integral curves.

Now, for a subgroup $L \leq N_G(H)$, let

$$(\mathcal{M}_1^G)^L = \{g \in \mathcal{M}_1^G \mid g = (R_l)_*(g) \text{ for all } l \in L\}.$$

Note that $(\mathcal{M}_1^G)^L$ consists of metrics on G/H, not only invariant under the left-action of G, but under the right-action of L as well, and that $Q \in (\mathcal{M}_1^G)^L$. It follows that $(\mathcal{M}_1^G)^L$ is a totally geodesic subspace of $\mathcal{M}_1^G$ invariant under the Ricci flow.

As mentioned earlier, we have $G/H = G_0/(G_0 \cap H)$ as manifolds. Hence, G_0/H_0 is a finite covering space of G/H. Therefore, the space $\mathcal{M}_1^G$ of G-invariant, unit volume metrics on G/H can be considered a totally geodesic subspace of the space $\mathcal{M}_1^{G_0}$ of G_0-invariant, unit volume metrics on G_0/H_0. Certainly, $\mathcal{M}_1^G$ is invariant under the Ricci flow on $\mathcal{M}_1^{G_0}$.

Lemma 4.28. *Let G/H be a compact homogeneous space. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}_0$ and let T denote the corresponding maximal torus. Then,*

$$(\mathcal{M}_1^G)^T := \mathcal{M}_1^G \cap (\mathcal{M}_1^{G_0})^T$$

is a totally geodesic subspace of $\mathcal{M}_1^G$ invariant under the Ricci flow.

We have $T \leq N_G(H_0)$. However, if H is disconnected, then we can have $T \not\leq N_G(H)$. For instance, there are isotropy irreducible homogeneous spaces G/Γ, where G_0 is a simple compact Lie group and Γ is a finite subgroup of G (see [**56**]). This implies that in general T does not act on G/H.

5. The asymptotic behavior of the scalar curvature functional

We turn now to the investigation of the asymptotic behavior of the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$. We will obtain the following rough picture: "Generically", we expect that along a geodesic ray γ_v, emanating from the base point Q, the scalar curvature functional tends to $-\infty$. If the scalar curvature functional is bounded from below along a geodesic ray γ_v, then the direct sum of the most shrinking directions of γ_v, that is, the eigenspace associated to the smallest eigenvalue of α_v, with $\mathfrak{h}$ is an H-subalgebra. This gives rise to the definition of the non-negative directions. We develop a structure theory for such directions, which finally links analytic and Lie theoretical data.

In Section 5.6, we define the extended non-toral directions. Based on this definition, the scalar curvature estimates provided in Section 5.7

yield a very detailed understanding of the asymptotic behavior of the scalar curvature functional. For instance, we are able to characterize compact homogeneous spaces whose scalar curvature functional is bounded from above or from below (the case of connected isotropy groups has been settled in [54]).

5.1. Canonical directions. We assign to every H-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ a canonical direction $v_{\mathrm{can}}(\mathfrak{k})$ in the unit sphere Σ of $T_Q\mathcal{M}_1^G$ and we investigate the scalar curvature functional along the geodesic $\gamma_{v_{\mathrm{can}}(\mathfrak{k})}$ emanating from Q. If $\mathfrak{k}$ is a compact subalgebra, then the canonical direction $v_{\mathrm{can}}(\mathfrak{k})$ comes from the canonical variation of Q with respect to the fibration $H(\mathfrak{k})/H \to G/H \to G/H(\mathfrak{k})$ [5, 9.67].

In the second part of this section, we investigate flags of H-subalgebras and directions which are closely related to such flags. We provide scalar curvature estimates along the corresponding geodesic rays γ_v emanating from Q.

Let $\mathfrak{k}$ be an H-subalgebra and let $\mathfrak{m}_{\mathfrak{k}}^{\perp}$ denote the Q-orthogonal complement of $\mathfrak{m}_{\mathfrak{k}}$ in $\mathfrak{m}$ (recall $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_{\mathfrak{k}}$).

Definition 5.1 (Canonical directions)**.** Let $\mathfrak{k}$ be an H-subalgebra. The canonical direction $v_{\mathrm{can}}(\mathfrak{k}) \in \Sigma$ associated to $\mathfrak{k}$ is defined by

$$v_{\mathrm{can}}(\mathfrak{k}) := v_1(\mathfrak{k})Q|_{\mathfrak{m}_{\mathfrak{k}}} \perp v_2(\mathfrak{k})Q|_{\mathfrak{m}_{\mathfrak{k}}^{\perp}},$$

where $\dim \mathfrak{m}_{\mathfrak{k}} \cdot v_1(\mathfrak{k}) + \dim \mathfrak{m}_{\mathfrak{k}}^{\perp} \cdot v_2(\mathfrak{k}) = 0$, $\dim \mathfrak{m}_{\mathfrak{k}} \cdot v_1^2(\mathfrak{k}) + \dim \mathfrak{m}_{\mathfrak{k}}^{\perp} \cdot v_2^2(\mathfrak{k}) = 1$ and $v_1(\mathfrak{k}) < v_2(\mathfrak{k})$.

Let $f = \oplus_{i=1}^{\ell} \mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$. We call an H-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ f-adapted if there exists an index set $I_1^{\mathfrak{k}}(f) \subset \{1, \ldots, \ell\}$ with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_{I_1^{\mathfrak{k}}(f)}$. For every H-subalgebra $\mathfrak{k}$, there exists a decomposition f of $\mathfrak{m}$, such that $\mathfrak{k}$ is f-adapted. Let $I_1 = I_1^{\mathfrak{k}}(f)$, $I_2 = \{1, \ldots, \ell\} \backslash I_1$ and let

$$\begin{aligned}
a_{\mathfrak{k}} &:= \frac{1}{2}\left(\sum_{j \in I_1} d_j \cdot b_j - \frac{1}{2}[I_1 I_1 I_1]_f - [I_1 I_2 I_2]_f\right),\\
b_{\mathfrak{k}} &:= \frac{1}{2}\left(\sum_{j \in I_2} d_j \cdot b_j - \frac{1}{2}[I_2 I_2 I_2]_f\right),\\
c_{\mathfrak{k}} &:= \frac{1}{4}[I_1 I_2 I_2]_f.
\end{aligned}$$

Then $a_{\mathfrak{k}}$, $b_{\mathfrak{k}}$, $c_{\mathfrak{k}}$ do not depend on the choice of f. By (4.15), we obtain

$$\mathrm{sc}(\gamma_{v_{\mathrm{can}}(\mathfrak{k})}(t)) = e^{t(-v_1(\mathfrak{k}))}\big(a_{\mathfrak{k}} + b_{\mathfrak{k}}x(t) - c_{\mathfrak{k}}x^2(t)\big), \tag{5.2}$$

where $x(t) := e^{t(v_1(\mathfrak{k})-v_2(\mathfrak{k}))} \in [1,0)$ for $t \in [0,+\infty)$.

Lemma 5.3. *Let $\mathfrak{k}$ be an H-subalgebra, f a decomposition of $\mathfrak{m}$, such that $\mathfrak{k}$ is f-adapted, and let $I_1 = I_1^{\mathfrak{k}}(f)$ and $I_2 = \{1,\dots,\ell\}\backslash I_1$. Then, $a_{\mathfrak{k}} = \sum_{j\in I_1} d_j \cdot c_j + \frac{1}{4}[I_1I_1I_1]_f$. Furthermore, $\mathfrak{k}$ is toral if and only if $a_{\mathfrak{k}} = 0$.*

Proof. By Lemma 4.16, we have $a_{\mathfrak{k}} = \sum_{j\in I_1} d_j \cdot c_j + \frac{1}{4}[I_1I_1I_1]_f \geq 0$. Suppose $a_{\mathfrak{k}} = 0$. Since $c_j \geq 0$, we have $c_j = [I_1I_1I_1]_f = 0$. Lemma 4.26 implies $[\mathfrak{h},\mathfrak{m}_{\mathfrak{k}}] = 0$, thus $\mathfrak{k}$ is toral.

Conversely, suppose that $\mathfrak{k}$ is toral, that is $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_{\mathfrak{k}}$ for an abelian algebra $\mathfrak{m}_{\mathfrak{k}}$. We have $[\mathfrak{m}_{\mathfrak{k}},\mathfrak{h}] = 0$. Hence, $c_j = 0$ for $j \in I_1$. Since $\mathfrak{m}_{\mathfrak{k}}$ is abelian, the claim follows. q.e.d.

Now, we can state a first result towards the asymptotic behavior of the scalar curvature functional.

Lemma 5.4 ([**10**])**.** *Let $\mathfrak{k}$ be an H-subalgebra. Then,*

$$\lim_{t\to+\infty} \mathrm{sc}(\gamma_{v_{\mathrm{can}}(\mathfrak{k})}(t)) = \begin{cases} +\infty \\ 0 \end{cases} \iff \begin{cases} \mathfrak{k} \text{ non-toral,} \\ \mathfrak{k} \text{ toral.} \end{cases}$$

If, in addition, G/H is not a torus, then $\mathrm{sc}(\gamma_{v_{\mathrm{can}}(\mathfrak{k})}(t)) > 0$ for all $t \geq 0$.

Proof. The first statement follows from (5.2). If G/H is not a torus, then $\mathrm{sc}(\gamma_{v_{\mathrm{can}}(\mathfrak{k})}(0)) = \mathrm{sc}(Q) = a_{\mathfrak{k}} + b_{\mathfrak{k}} - c_{\mathfrak{k}} > 0$. By means of $a_{\mathfrak{k}} \geq 0$, the parabola $p(x) = a_{\mathfrak{k}} + b_{\mathfrak{k}}x - c_{\mathfrak{k}}x^2$ satisfies $p(0) \geq 0$ and $p(1) > 0$. Since $c_{\mathfrak{k}} \geq 0$, we conclude $p(x) > 0$ for all $x \in (0,1]$. q.e.d.

Let $f = \oplus_{i=1}^{\ell}\mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$ and let

$$\mathcal{M}^G(f) = \{e^{v_1}Q|_{\mathfrak{m}_1} \perp \cdots \perp e^{v_\ell}Q|_{\mathfrak{m}_\ell} \mid (v_1,\dots,v_\ell) \in \mathbb{R}^\ell\}.$$

$\mathcal{M}^G(f)$ consists of all G-invariant metrics which can be "diagonalized" with respect to the decomposition f of $\mathfrak{m}$. Geometrically, $\mathcal{M}^G(f)$ is a maximal flat in $(\mathcal{M}^G, L^2)$. Hence, $\mathcal{M}_1^G(f) = \mathcal{M}_1^G \cap \mathcal{M}^G(f)$ is a maximal flat in $\mathcal{M}_1^G$, thus isometric to $\mathbb{R}^{\ell-1}$. In the following, we will identify very often

$$v = v_1Q|_{\mathfrak{m}_1} \perp \cdots \perp v_\ell Q|_{\mathfrak{m}_\ell} \in T_Q\mathcal{M}_1^G(f)$$

with $(v_1,\dots,v_\ell) \in \mathbb{R}^\ell$ satisfying $\sum_{i=1}^{\ell} d_i \cdot v_i = 0$.

Lemma 5.5. *Let $\mathfrak{k}_1, \dots, \mathfrak{k}_p$ be H-subalgebras with $\mathfrak{k}_1 < \cdots < \mathfrak{k}_p$. Then, the canonical directions $(v_{\mathrm{can}}(\mathfrak{k}_1), \dots, v_{\mathrm{can}}(\mathfrak{k}_p))$ are linearly independent.*

Proof. Let f be a decomposition of $\mathfrak{m}$, such that the H-subalgebras $\mathfrak{k}_1, \dots, \mathfrak{k}_p$ are f-adapted. In order to make the notation as simple as possible, let us assume $\mathfrak{k}_i = \mathfrak{h} \oplus \bigoplus_{m=1}^{j_i} \mathfrak{m}_m$ with $1 \leq j_1 < \cdots < j_p < \ell$. Hence,

$$v_{\mathrm{can}}(\mathfrak{k}_i) = (\underbrace{v_1(\mathfrak{k}_i), \dots, v_1(\mathfrak{k}_i)}_{j_i}, v_2(\mathfrak{k}_i), \dots, v_2(\mathfrak{k}_i)) \in \mathbb{R}^\ell$$

with $v_1(\mathfrak{k}_i) < 0$ and $v_2(\mathfrak{k}_i) > 0$. The claim follows now by induction. q.e.d.

For $X, Y \subset T_Q \mathcal{M}_1^G$, let $\mathrm{conv}^E(X, Y)$ denote the convex hull of X and Y in $T_Q \mathcal{M}_1^G$. The following scalar curvature estimates are important ingredients for the variational methods described in Section 8.

Proposition 5.6. *Let $\mathfrak{k}_1, \dots, \mathfrak{k}_p$ be H-subalgebras with $\mathfrak{k}_1 < \cdots < \mathfrak{k}_p$ and let*

$$\Delta_{\mathfrak{k}_1, \dots, \mathfrak{k}_p} = \mathrm{conv}^E(v_{\mathrm{can}}(\mathfrak{k}_1), \dots, v_{\mathrm{can}}(\mathfrak{k}_p)).$$

If $\mathfrak{k}_1$ is a non-toral H-subalgebra, then for all $v \in \Delta_{\mathfrak{k}_1, \dots, \mathfrak{k}_p}$ we have $\lim_{t \to +\infty} \mathrm{sc}(\gamma_v(t)) = +\infty$ and $\mathrm{sc}(\gamma_v(t)) > 0$ for $t \geq 0$.

Proof. As above, let f be a decomposition of $\mathfrak{m}$, such that $\mathfrak{k}_1, \dots, \mathfrak{k}_p$ are f-adapted. Let $0 \leq \lambda_1, \dots, \lambda_p \leq 1$ be given with $\sum_{i=1}^p \lambda_i = 1$. By the special shape of $v_{\mathrm{can}}(\mathfrak{k}_i)$, we can consider $v_{\mathrm{can}}(\mathfrak{k}_i)$ elements in $\mathbb{R}^{p+1}$ (rather than elements in $\mathbb{R}^\ell$), that is

$$v_{\mathrm{can}}(\mathfrak{k}_i) = (\underbrace{v_1(\mathfrak{k}_i), \dots, v_1(\mathfrak{k}_i)}_{i}, v_2(\mathfrak{k}_i), \dots, v_2(\mathfrak{k}_i)).$$

Hence

(5.7)

$$\sum_{i=1}^{p} \lambda_i \cdot v_{\mathrm{can}}(\mathfrak{k}_i) = \begin{pmatrix} \lambda_1 v_1(\mathfrak{k}_1) & + & \lambda_2 v_1(\mathfrak{k}_2) & + & \cdots & + & \lambda_p v_1(\mathfrak{k}_p) \\ \lambda_1 v_2(\mathfrak{k}_1) & + & \lambda_2 v_1(\mathfrak{k}_2) & + & \cdots & + & \lambda_p v_1(\mathfrak{k}_p) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \lambda_1 v_2(\mathfrak{k}_1) & + & \lambda_2 v_2(\mathfrak{k}_2) & + & \cdots & + & \lambda_p v_2(\mathfrak{k}_p) \end{pmatrix} = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_{p+1} \end{pmatrix}.$$

Let $j_1, \dots, j_p$ be defined as in Lemma 5.5 and let $I_1 := \{1, \dots, j_1\}, \dots, I_{p+1} := \{j_p + 1, \dots, \ell\}$. Since $\mathfrak{k}_i, \mathfrak{k}_j, \mathfrak{k}_k$ are subalgebras contained in each other, we have $[I_i I_j I_k]_f = 0$, whenever all three indices i, j, k are pairwise distinct, and $[I_i I_i I_k]_f = 0$ for $k > i$.

With the help of Lemma 4.16 we conclude

$$\sum_{j \in I_m} d_j \cdot b_j = [I_m I_m I_m]_f + \sum_{s=m+1}^{p+1} [I_m I_s I_s]_f + 2 \sum_{s=1}^{m-1} [I_s I_m I_m]_f + \sum_{j \in I_m} 2 d_j c_j$$

for $m \in \{1, \dots, p+1\}$. Now let

$$a_m := \frac{1}{2} \left(\sum_{j \in I_m} d_j \cdot b_j - \frac{1}{2} [I_m I_m I_m]_f - \sum_{s=m+1}^{p+1} [I_m I_s I_s]_f \right).$$

By (4.15) we obtain

$$\mathrm{sc}(\gamma_v(t)) = \sum_{m=1}^{p+1} a_m e^{t(-\hat{v}_m)} - \frac{1}{4} \sum_{m=1}^{p+1} \sum_{s=1}^{m-1} [I_s I_m I_m]_f \, e^{t(\hat{v}_s - 2\hat{v}_m)}.$$

Since $\lambda_i \geq 0$ and $v_1(\mathfrak{k}_i) < v_2(\mathfrak{k}_i)$, (5.7) yields $\hat{v}_s - 2\hat{v}_m \leq -\hat{v}_m$ for $s \leq m - 1$. Thus

$$\mathrm{sc}(\gamma_v(t)) \geq \sum_{m=1}^{p+1} e^{t(-\hat{v}_m)} \left(\sum_{j \in I_m} d_j c_j + \frac{1}{4} [I_m I_m I_m]_f + \frac{3}{4} \sum_{s=1}^{m-1} [I_s I_m I_m]_f \right).$$

Since by assumption $\mathfrak{k}_1$ is not toral, we obtain the claim by Lemma 5.3. q.e.d.

Corollary 5.8. *Let $\mathfrak{k}_1, \dots, \mathfrak{k}_p$ be H-subalgebras with $\mathfrak{k}_1 < \cdots < \mathfrak{k}_p$ and $p \geq 2$. Let $v \in \Delta_{\mathfrak{k}_1,\dots,\mathfrak{k}_p}$ and let f be a decomposition of $\mathfrak{m}$, such that $\mathfrak{k}_1, \dots, \mathfrak{k}_p$ are f-adapted. Then, we have $I_1^v(f) = I_1^{\mathfrak{k}_1}(f)$ if and only if $v \in \Delta_{\mathfrak{k}_1,\dots,\mathfrak{k}_p} \backslash \Delta_{\mathfrak{k}_2,\dots,\mathfrak{k}_p}$.*

Proof. This follows from (5.7). q.e.d.

Later on, it will be convenient to work with directions $v \in T_Q\mathcal{M}_1^G$ of unit length. Therefore, let us consider the round sphere Σ of radius 1 in $(T_Q\mathcal{M}_1^G, L^2|_{T_Q\mathcal{M}_1^G})$, and let

$$\pi_\Sigma : T_Q\mathcal{M}_1^G \backslash \{0\} \to \Sigma;\ v \mapsto v/||v||.$$

Note that from the theorem of Tarski–Seidenberg, it easily follows that π_Σ is semialgebraic (cf. [**4**, Remark 2.3.8]).

We call a set $\Delta \subset T_Q\mathcal{M}_1^G$ a convex polyhedron if it is given as the solution set of finitely many affine linear inequalities. These inequalities are allowed to be strict or non-strict. A set $X \subset \Sigma$ is called a convex spherical polyhedron if $(\pi_\Sigma)^{-1}(X)$ or $(\pi_\Sigma)^{-1}(X) \cup \{0\}$ is a convex polyhedron in $T_Q\mathcal{M}_1^G$. If X is a convex spherical polyhedron, so is $\bar{X}$ (the closure of X in Σ).

Definition 5.9. A bounded convex polyhedron Δ_X in $T_Q\mathcal{M}_1^G \backslash \{0\}$ is called a slice for a convex spherical polyhedron $X \subset \Sigma$ if π_Σ maps Δ_X homeomorphicly onto X.

If a convex spherical polyhedron X admits a slice, then, since $0 \notin \bar{\Delta}_X$, $\bar{X}$ lies in the interior of a hemisphere. In particular, the diameter of $\bar{X}$ is strictly less than π. Vice versa, if X is a convex spherical polyhedron with $\operatorname{diam}(\bar{X}) < \pi$, then $\bar{X}$ lies in the interior of a hemisphere and a slice can be found.

Proposition 5.10. *Let $\Delta_{\mathfrak{k}_1,\dots,\mathfrak{k}_p}$ be defined as in Proposition 5.6. Then, $\Delta_{\mathfrak{k}_1,\dots,\mathfrak{k}_p}$ is a slice for the compact, convex spherical polyhedron $\pi_\Sigma(\Delta_{\mathfrak{k}_1,\dots,\mathfrak{k}_p})$.*

Proof. This follows from Lemma 5.5. q.e.d.

5.2. Non-negative directions. We will give the definition of the set of non-negative directions $W^\Sigma \subset \Sigma$ and prove that W^Σ is a compact and semialgebraic subset of $T_Q\mathcal{M}_1^G$. Furthermore, for $v \in W^\Sigma$, we will provide scalar curvature estimates along γ_v which yield a first rough understanding of the asymptotic behavior of the scalar curvature functional.

A direction $v \in \Sigma$ is called negative if the scalar curvature functional tends to $-\infty$ along γ_v, that is $\lim_{t\to+\infty} \mathrm{sc}(\gamma_v(t)) = -\infty$. This gives rise to the following definition:

Definition 5.11 (Non-negative directions)**.** Let W^Σ denote the set of all $v \in \Sigma$ with the following property: If f is any good decomposition of γ_v, then for all $(i,j,k) \in \{1,\dots,\ell\}^3$ we have

$$\{\,[ijk]_f > 0 \Rightarrow v_i - v_j - v_k + v_{\min} \leq 0\}. \tag{5.12}$$

Note that if (5.12) is true for a single good decomposition f of γ_v, then it is true for all good decompositions f of γ_v.

If G/H has finite fundamental group, then by Corollary 4.17, we have $b_i > 0$. Therefore, the set W^Σ gathers roughly the non-negative directions of the scalar curvature functional (cf. (4.15)). For instance, all $v \in \Sigma$ with $\lim_{t\to+\infty} \mathrm{sc}(\gamma_v(t)) \neq -\infty$ belong to W^Σ by Theorem 5.18.

However, W^Σ might still contain negative directions mainly due to the existence of toral subalgebras. This is an essential problem which will complicate the investigation of the asymptotic behavior of the scalar curvature functional considerably (cf. Sections 5.6 and 5.7).

Remark 5.13. In the definition of W^Σ, it would have been enough to demand (5.12) to be true for $(i,j,k) \in \{1,\dots,\ell\}^3$ with $(i,j,k) \notin (I_1^v(f), I_m^v(f), I_m^v(f))^S$, $1 \leq m \leq \ell_v$, where ^{S} denotes cyclic permutation.

For a set $X \subset T_Q\mathcal{M}_1^G$, let

$$\hat{X} := (F|_{\hat{\mathcal{M}}_1^G})^{-1}(X) \tag{5.14}$$

denote the pre-image of $F|_{\hat{\mathcal{M}}_1^G} : \hat{\mathcal{M}}_1^G = \widehat{T_Q\mathcal{M}_1^G} \to T_Q\mathcal{M}_1^G$ (cf. Corollary 4.21). The cubic polynomial F was defined by $F(A,v) = AvA^t$, where $v = (v_1,\dots,v_\ell) \in \mathbb{R}^\ell$ is considered a diagonal matrix. Abusing notation slightly, we write $F(A,v) = v \in T_Q\mathcal{M}^G$. If $(A,v) \in \hat{\mathcal{M}}_1^G$, then $\sum_{i=1}^\ell \dim \mathfrak{m}_i \cdot v_i = 0$, where $f = \oplus_{i=1}^\ell \mathfrak{m}_i$ is the ordered (iso-ordered) decomposition of $\mathfrak{m}$, which corresponds to $[A] \in \mathcal{F}^G$. Recall that $f = [A]$

is a good decomposition of γ_v, that is $\mathfrak{m}_1,\dots,\mathfrak{m}_\ell$ are eigenspaces of α_v associated to the eigenvalues $v_1,\dots,v_\ell$ (cf. Section 4.3).

Proposition 5.15. *The set $W^\Sigma \subset \Sigma$ is a semialgebraic, compact subset of $T_Q\mathcal{M}_1^G$.*

Proof. We have

$$\hat{W}^\Sigma \;=\; \hat{\Sigma} \;\cap \bigcap_{\substack{(i,j,k)\in\\ \{1,\dots,\ell\}^3}} \{\widehat{[ijk]}=0\} \;\cup\; \{\cup_{\tilde{i}=1}^{\ell}\{v_i - v_j - v_k + v_{\tilde{i}} \le 0\}\},$$

thus $\hat{W}^\Sigma$ is a compact, semialgebraic subset of $\hat{\mathcal{M}}_1^G$. By the Theorem of Tarski–Seidenberg, W^Σ is a semialgebraic subset of Σ too. Since $F|_{\hat{\mathcal{M}}_1^G}$ is continuous, the claim follows. q.e.d.

The following lemma is to some extent the starting point for the entire theory developed in this article.

Lemma 5.16. *Let $v \in W^\Sigma$ and let f be a good decomposition of γ_v. Then,*

$$[I_1^v(f) I_{m_1}^v(f) I_{m_2}^v(f)]_f = 0 \tag{5.17}$$

for $m_1 \neq m_2$ and $1 \le m_1, m_2 \le \ell_v$. In particular, $\mathfrak{h} \oplus \mathfrak{m}_{I_1^v(f)}$ is an f-adapted H-subalgebra.

Proof. This follows from (5.12). q.e.d.

Note that condition (5.17) does not depend on the chosen good decomposition f of γ_v (cf. Lemma 4.13). Let $W^\Sigma(\delta)$ denote the open δ-neighborhood of W^Σ in Σ.

Theorem 5.18. *Let G/H be a compact homogeneous space. Then, for each $\delta > 0$ there exists $t_0(\delta) > 0$, such that for $t \ge t_0(\delta)$ and $v \in \Sigma \backslash W^\Sigma(\delta)$ we have $\mathrm{sc}(\gamma_v(t)) \le 0$ and $\lim_{t\to+\infty} \mathrm{sc}(\gamma_v(t)) = -\infty$.*

Proof. We claim that for $\delta > 0$ there exists $\epsilon(\delta) > 0$, such that for all $v \in \Sigma \backslash W^\Sigma(\delta)$ there exists a good decomposition f of γ_v with the following property: There exists $(i,j,k) \in \{1,\dots,\ell\}^3$ with $[ijk]_f \ge \epsilon(\delta)$ and $v_i - v_j - v_k + v_{\min} \ge \epsilon(\delta)$. This is seen as follows: Suppose no such $\epsilon(\delta) > 0$ exists. Then, for all $\alpha \in \mathbb{N}$ there exists $v(\alpha) \in \Sigma \backslash W^\Sigma(\delta)$, such that for all good decompositions f_α of $\gamma_{v(\alpha)}$ the following holds: For all $(i,j,k) \in \{1,\dots,\ell\}^3$, we have either $[ijk]_{f_\alpha} < \frac{1}{\alpha}$ or $v_i(\alpha) - v_j(\alpha) - v_k(\alpha) + v_{\min}(\alpha) < \frac{1}{\alpha}$. By passing to a subsequence of $(v(\alpha))_{\alpha\in\mathbb{N}}$, we may assume $v(\alpha) \to v_\infty \in \Sigma\backslash W^\Sigma(\delta)$ and $f_\alpha \to f_\infty$ where f_∞ is a good decomposition of γ_{v_∞}. We obtain either $[ijk]_{f_\infty} = 0$ or $v_i^\infty - v_j^\infty - v_k^\infty + v_{\min}^\infty \leq 0$, hence $v_\infty \in W^\Sigma$. This is a contradiction and the above claim is proved. The proof of the theorem follows now from (4.15) and $b_i \leq b_{G/H}$ (cf. (4.18)). q.e.d.

Corollary 5.19 ([**54**]). *Let G/H be a compact homogeneous space. If $W^\Sigma = \emptyset$, then the scalar curvature functional* $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ *is bounded from above and attains its global maximum.*

A homogeneous space G/H is primitive, that is the only G-invariant foliations are the foliation by points and the foliation by the whole space (cf. [**26**], [**57**]), if and only if $W^\Sigma = \emptyset$.

Lemma 5.20 ([**54**]). *Let G/H be a compact homogeneous space. If $W^\Sigma = \Sigma$, then the scalar curvature functional* $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ *is bounded from below by zero. If, in addition, G/H has finite fundamental group or if G/H is a torus, then* $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ *attains its global minimum.*

Notice that also the isotropy irreducible homogeneous spaces G/H are covered by Corollary 5.19 and Lemma 5.20 since here $W^\Sigma = \Sigma = \emptyset$.

5.3. Scalar curvature functional bounded from below or above. Corollary 5.19 and Lemma 5.20 raise the question for which compact homogeneous spaces G/H the scalar curvature functional is bounded from above or below. For homogeneous spaces G/H, for which G and H are connected, thanks to [**54**, Theorem 2.1, 2.2, 2.4], this question has been completely clarified.

Theorem 5.21 ([**54**]). *Let G/H be a compact homogeneous space. Then, the scalar curvature functional* $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ *is bounded from below if and only if the following holds: There exists a unique decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \bigoplus_{i=1}^r \mathfrak{g}_i$ of $\mathfrak{g}$, where $\mathfrak{g}_i$ are non-abelian* $\mathrm{Ad}(H)$*-invariant subalgebras of $\mathfrak{g}$, and a unique decomposition $\mathfrak{h} = \bigoplus_{i=1}^r \mathfrak{h}_i$ of $\mathfrak{h}$, where $\mathfrak{h}_i < \mathfrak{g}_i$, such that if $\mathfrak{p}_i$ denotes the Q-orthogonal complement of $\mathfrak{h}_i$ in $\mathfrak{g}_i$, then $\mathfrak{p}_i$ is an irreducible, isotypical summand of $\mathfrak{m} = \mathfrak{z}(\mathfrak{g}) \oplus \bigoplus_{i=1}^r \mathfrak{p}_i$.*

If, in addition, G/H has finite fundamental group, then $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ attains its global positive minimum at a uniquely determined Einstein metric on G/H (cf. [**54**]). If H is not connected, then $\dim \mathfrak{h}_i = 0$ is possible, since there exist isotropy irreducible homogeneous spaces which are not strongly isotropy irreducible (see [**56**]).

The proof of the next theorem relies on new scalar curvature estimates.

Theorem 5.22. *Let G/H be a compact homogeneous space. Then, the scalar curvature functional* $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ *is bounded from above if and only if there exist no non-toral H-subalgebras. In this case, the functional* $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ *attains its global maximum.*

Proof. If $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ is bounded from above, then by Lemma 5.4 there exist no non-toral H-subalgebras. Vice versa, if there exist no non-toral H-subalgebras, then by Corollaries 5.19 and 5.53 $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ is bounded from above. Since by [**10**], a maximizing sequence has a convergent subsequence (if G/H is not the torus), it follows that $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ attains its global maximum. q.e.d.

5.4. A stratification of the non-negative directions. In this section, we will investigate the non-negative directions W^Σ in greater detail. We will obtain a "stratification" of W^Σ into semialgebraic, contractible subsets $W^\Sigma(\mathfrak{k})$ of W^Σ where $\mathfrak{k}$ runs through the set of all H-subalgebras. These strata obey the inclusion relation for H-subalgebras in the following manner:

$$\mathfrak{k} \leq \mathfrak{l} \quad \Longleftrightarrow \quad \overline{W^\Sigma(\mathfrak{l})} \subset \overline{W^\Sigma(\mathfrak{k})}.$$

Definition 5.23 ($\mathfrak{k}$-Non-negative directions)**.** For an H-subalgebra $\mathfrak{k}$, let the $\mathfrak{k}$-non-negative directions be defined by

$$W^\Sigma(\mathfrak{k}) = \{v \in W^\Sigma \mid \mathfrak{m}_{I_1^v(f)} = \mathfrak{m}_\mathfrak{k}\},$$

where f is any good decomposition of γ_v.

The set $W^\Sigma(\mathfrak{k})$ consists of all non-negative directions $v \in W^\Sigma$ whose eigenspace associated to the smallest eigenvalue equals $\mathfrak{m}_\mathfrak{k}$.

Next, let $\mathcal{M}_{G/H(\mathfrak{k})}$ denote the set of $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-invariant inner products on $\mathfrak{m}_\mathfrak{k}^\perp$ and let

$$\mathcal{M}^G(\mathfrak{k}) = \{a \cdot Q|_{\mathfrak{m}_\mathfrak{k}} \perp \bar{g} \mid a > 0,\ \bar{g} \in \mathcal{M}_{G/H(\mathfrak{k})}\}.$$

Then, $\mathcal{M}^G(\mathfrak{k})$ is the fixed point set of the isometric action of $\mathrm{O}(\mathfrak{m}_\mathfrak{k}, Q|_{\mathfrak{m}_\mathfrak{k}}) \times \mathrm{Ad}(\overline{H(\mathfrak{k})})$ on the symmetric space $\mathrm{Gl}(\mathfrak{m}_\mathfrak{k} \oplus \mathfrak{m}_\mathfrak{k}^\perp)/\mathrm{O}(\mathfrak{m}_\mathfrak{k} \oplus \mathfrak{m}_\mathfrak{k}^\perp, Q|_{\mathfrak{m}_\mathfrak{k} \oplus \mathfrak{m}_\mathfrak{k}^\perp})$, hence a totally geodesic, symmetric subspace of $\mathcal{M}^G$. Notice that we have $\gamma_{v_{\mathrm{can}}(\mathfrak{k})}(t) \in \mathcal{M}^G(\mathfrak{k}) \cap \mathcal{M}_1^G$ for $t \in \mathbb{R}$.

Remark 5.24. If the group $H(\mathfrak{k})$ is compact, then all metrics in the symmetric space $\mathcal{M}^G(\mathfrak{k})$ are submersion metrics with respect to the fibration $H(\mathfrak{k})/H \to G/H \to G/H(\mathfrak{k})$. In this case, $\mathcal{M}_{G/H(\mathfrak{k})}$ can be identified with the set of G-invariant metrics on $G/H(\mathfrak{k})$.

Let $\mathcal{M}^G_1(\mathfrak{k}) = \mathcal{M}^G(\mathfrak{k}) \cap \mathcal{M}^G_1$. We claim

$$W^\Sigma(\mathfrak{k}) \subset W^\Sigma \cap T_Q\mathcal{M}^G_1(\mathfrak{k}) \subset \Sigma^{\mathfrak{k}} := \Sigma \cap T_Q\mathcal{M}^G_1(\mathfrak{k}).$$

This is seen as follows: Let $v \in W^\Sigma(\mathfrak{k})$ and let f be a good decomposition of γ_v. Then, $\mathfrak{m}_{I^v_1(f)} = \mathfrak{m}_{\mathfrak{k}}$. Since $\mathfrak{k}$ is a subalgebra, we have $Q([\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{I^v_i(f)}], \mathfrak{k}) = 0$ for $2 \leq i \leq \ell_v$. By Lemma 5.16, we conclude $[\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{I^v_i(f)}] \subset \mathfrak{m}_{I^v_i(f)}$, hence $\mathfrak{m}_{I^v_i(f)}$ is $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-invariant (cf. Lemma 4.25). Note that in general $W^\Sigma(\mathfrak{k}) \neq W^\Sigma \cap T_Q\mathcal{M}^G_1(\mathfrak{k})$.

For $\epsilon > 0$ and $v \in \Sigma$, let $B^\Sigma_\epsilon(v)$ denote the open ϵ-ball in Σ, centered at v, with respect to $(\Sigma, L^2|_\Sigma)$. Let

$$B^{\Sigma^{\mathfrak{k}}}_\epsilon(v) = B^\Sigma_\epsilon(v) \cap \Sigma^{\mathfrak{k}}.$$

For notational reasons, we set

$$X^\Sigma(\mathfrak{k}) = \overline{W^\Sigma(\mathfrak{k})}.$$

For the definition of the negative constant $c_{G/H}$, we refer to Lemma 4.12. We set

$$\epsilon_{G/H} := -\frac{1}{2} c_{G/H}.$$

Now, we can state a first structure result on the set of $\mathfrak{k}$-non-negative directions $W^\Sigma(\mathfrak{k})$.

Theorem 5.25. *Let $\mathfrak{k}$ be an H-subalgebra. Then, we have:*

1. $X^\Sigma(\mathfrak{k})$ *is a compact, semialgebraic and star-shaped subset of* $\Sigma^{\mathfrak{k}}$.
2. $B^{\Sigma^{\mathfrak{k}}}_{\epsilon_{G/H}}(v_{\mathrm{can}}(\mathfrak{k})) \subset X^\Sigma(\mathfrak{k})$.
3. $B^{\Sigma^{\mathfrak{k}}}_{\epsilon_{G/H}}(-v_{\mathrm{can}}(\mathfrak{k})) \cap X^\Sigma(\mathfrak{k}) = \emptyset$.

Proof. By Proposition 5.26, the set $W^\Sigma(\mathfrak{k})$ is semialgebraic, and hence its closure $X^\Sigma(\mathfrak{k})$ as well (cf. [**4**, Proposition 2.3.7]). Next, $B^{\Sigma^{\mathfrak{k}}}_{\epsilon_{G/H}}(v_{\mathrm{can}}(\mathfrak{k})) \subset X^\Sigma(\mathfrak{k})$ by Corollary 5.30. With the help of Lemma 5.29, we conclude that $B^{\Sigma^{\mathfrak{k}}}_{\epsilon_{G/H}}(-v_{\mathrm{can}}(\mathfrak{k})) \cap X^\Sigma(\mathfrak{k}) = \emptyset$ and that $X^\Sigma(\mathfrak{k})$ is star-shaped. q.e.d.

We call a decomposition f of $\mathfrak{m}$ a $\mathfrak{k}$-decomposition of $\mathfrak{m}$, if there exists a partition

$$I_1^{\mathfrak{k}}(f) \cup \cdots \cup I_{\ell_{\mathfrak{k}}}^{\mathfrak{k}}(f) = \{1, \ldots, \ell\}$$

of $\{1, \ldots, \ell\}$, such that $\mathfrak{m}_{I_1^{\mathfrak{k}}(f)} = \mathfrak{m}_{\mathfrak{k}}$, $\mathfrak{m}_{I_i^{\mathfrak{k}}(f)}$ is $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-invariant and $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-irreducible for $i \in \{2, \ldots, \ell_{\mathfrak{k}}\}$.

Proposition 5.26. *Let $\mathfrak{k}$ be an H-subalgebra. Then, $W^{\Sigma}(\mathfrak{k})$ is a semialgebraic subset of $T_Q\mathcal{M}_1^G(\mathfrak{k})$.*

Proof. Let $\mathfrak{m}_{\mathfrak{k}}^{\perp} = \oplus_{i=2}^{\ell_{\mathrm{iso}}^{\mathfrak{k}}} \mathfrak{p}_i^{\mathfrak{k}}$ denote an $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-isotypical splitting of $\mathfrak{m}_{\mathfrak{k}}^{\perp}$. Setting $\mathfrak{p}_1^{\mathfrak{k}} = \mathfrak{m}_{\mathfrak{k}}$, we obtain an "$\mathrm{O}(\mathfrak{m}_{\mathfrak{k}}, Q|_{\mathfrak{m}_{\mathfrak{k}}}) \times \mathrm{Ad}(\overline{H(\mathfrak{k})})$-isotypical splitting" of $\mathfrak{m}$. As in Section 4.1, we can define a parameterization $F^{\mathfrak{k}}|_{\hat{\mathcal{M}}_1^G(\mathfrak{k})}$ of $T_Q\mathcal{M}_1^G(\mathfrak{k})$ (cf. Corollary 4.21). Next, for any $\mathfrak{k}$-decomposition $f_{\mathfrak{k}} = \mathfrak{m}_{\mathfrak{k}} \oplus \bigoplus_{i=2}^{\ell_{\mathfrak{k}}} \mathfrak{m}_i^{\mathfrak{k}}$ of $\mathfrak{m}$ into $\mathrm{O}(\mathfrak{m}_{\mathfrak{k}}, Q|_{\mathfrak{m}_{\mathfrak{k}}}) \times \mathrm{Ad}(\overline{H(\mathfrak{k})})$-irreducible summands, let us define structure constants $[ijk]_{f_{\mathfrak{k}}}$ as in Section 4.2. As in Section 4.3, we can lift these structure constants. The corresponding functions $\widehat{[ijk]}^{\mathfrak{k}} : \hat{\mathcal{M}}_1^G(\mathfrak{k}) \to \mathbb{R}$ are semialgebraic and compatible with $[ijk]_{f_{\mathfrak{k}}}$ as described in Corollary 4.21. As in Proposition 5.15, we conclude that $W^{\Sigma} \cap T_Q\mathcal{M}_1^G(\mathfrak{k})$ is semialgebraic. Since $\mathfrak{m}_{\mathfrak{k}}$ is an iso-typical summand of $\mathfrak{m} = \mathfrak{m}_{\mathfrak{k}} \oplus \mathfrak{m}_{\mathfrak{k}}^{\perp}$, the constraint $\mathfrak{m}_{I_1^v(f)} = \mathfrak{m}_{\mathfrak{k}}$ comes from a linear inequality between the first eigenvalue and the other ones. q.e.d.

Let f be a decomposition of $\mathfrak{m}$, such that $\mathfrak{k}$ is f-adapted. Recall that we have already defined the index set $I_1^{\mathfrak{k}}(f) \subset \{1, \ldots, \ell\}$ by $\mathfrak{m}_{I_1^{\mathfrak{k}}(f)} = \mathfrak{m}_{\mathfrak{k}}$. Since the intersection of the totally geodesic subspaces $\mathcal{M}^G(f)$ and $\mathcal{M}^G(\mathfrak{k})$ of $\mathcal{M}^G$, denoted by $\mathcal{M}^G(f, \mathfrak{k})$, is a totally geodesic subspace of the maximal flat $\mathcal{M}^G(f)$, we have:

Lemma 5.27. *Let $\mathfrak{k}$ be an H-subalgebra and let f be a decomposition of $\mathfrak{m}$, such that $\mathfrak{k}$ is f-adapted. Then, there exist index sets $I_2^{\mathfrak{k}}(f), \ldots, I_{\ell_{\mathfrak{k}}(f)}^{\mathfrak{k}}(f) \subset \{1, \ldots, \ell\} \backslash I_1^{\mathfrak{k}}(f)$ with $[I_1^{\mathfrak{k}}(f) I_j^{\mathfrak{k}}(f) I_k^{\mathfrak{k}}(f)]_f = 0$ for $1 \leq j, k \leq \ell_{\mathfrak{k}}(f)$ and $j \neq k$, such that we have:*

$$\mathcal{M}^G(f, \mathfrak{k}) = \{\gamma_{v_1, \ldots, v_{\ell_{\mathfrak{k}}(f)}}(1) \mid (v_1, \ldots, v_{\ell_{\mathfrak{k}}(f)}) \in \mathbb{R}^{\ell_{\mathfrak{k}}(f)}\}$$

where

$$\gamma_{v_1, \ldots, v_{\ell_{\mathfrak{k}}(f)}}(t) = e^{tv_1} Q|_{\mathfrak{m}_{I_1^{\mathfrak{k}}(f)}} \perp \cdots \perp e^{tv_{\ell_{\mathfrak{k}}(f)}} Q|_{\mathfrak{m}_{I_{\ell_{\mathfrak{k}}(f)}^{\mathfrak{k}}(f)}}. \tag{5.28}$$

If we require in Equation (5.28), in addition, $\sum_{i=1}^{\ell_{\mathfrak{k}}(f)} d_{I_i^{\mathfrak{k}}(f)} \cdot v_i = 0$, then we obtain an $(\ell_{\mathfrak{k}}(f)-1)$-dimensional family of G-invariant metrics of volume one, the flat $\mathcal{M}_1^G(f,\mathfrak{k}) = \mathcal{M}_1^G(f) \cap \mathcal{M}_1^G(\mathfrak{k})$ of $\mathcal{M}_1^G(\mathfrak{k})$.

We set $W^{\Sigma}(f,\mathfrak{k}) = T_Q\mathcal{M}_1^G(f) \cap W^{\Sigma}(\mathfrak{k})$ and again for notational reasons

$$X^{\Sigma}(f,\mathfrak{k}) = \overline{W^{\Sigma}(f,\mathfrak{k})}$$

(that is replacing W by X means taking the closure).

Lemma 5.29. *Let $\mathfrak{k}$ be an H-subalgebra, let f be a decomposition of $\mathfrak{m}$, such that $\mathfrak{k}$ is f-adapted, and let $C(f,\mathfrak{k}) = \{v \in T_Q\mathcal{M}^G(f,\mathfrak{k}) \mid v = (\gamma_{v_1,\dots,v_{\ell_{\mathfrak{k}}(f)}})'(0)$, such that (1) and (2) hold$\}$, where*

(1) $v_1 < v_i$ *for* $2 \le i \le \ell_{\mathfrak{k}}(f)$.

(2) $[I_i^{\mathfrak{k}}(f) I_j^{\mathfrak{k}}(f) I_k^{\mathfrak{k}}(f)]_f > 0 \Rightarrow v_i - v_j - v_k + v_1 \le 0$ *for* $(i,j,k) \in \{1,\dots,\ell_{\mathfrak{k}}(f)\}^3$.

Then, $W^{\Sigma}(f,\mathfrak{k}) = C(f,\mathfrak{k}) \cap \Sigma$. Furthermore, $W^{\Sigma}(f,\mathfrak{k})$ and $X^{\Sigma}(f,\mathfrak{k})$ are convex spherical polyhedra, and $X^{\Sigma}(f,\mathfrak{k})$ does not contain antipodal points.

Proof. We show $W^{\Sigma}(f,\mathfrak{k}) = C(f,\mathfrak{k}) \cap \Sigma$. For "$\supset$": This is obvious. For "$\subset$": Let $v \in T_Q\mathcal{M}_1^G(f) \cap W^{\Sigma}(\mathfrak{k}) \subset T_Q\mathcal{M}_1^G(f) \cap T_Q\mathcal{M}^G(\mathfrak{k}) \cap \Sigma$. By Lemma 5.27, γ_v can be written as in (5.28). Since $\mathfrak{m}_{\mathfrak{k}} = \mathfrak{m}_{I_1^{\mathfrak{k}}(f)}$, condition (1) in the definition of $C(f,\mathfrak{k})$ holds true. With the help of (5.12), we conclude that condition (2) is satisfied as well.

Since $C(f,\mathfrak{k})$ is a cone containing the geodesic ray $\{tv_{\mathrm{can}}(\mathfrak{k}) \mid t > 0\}$, so is $T_Q\mathcal{M}_1^G(f,\mathfrak{k}) \cap C(f,\mathfrak{k})$. Hence, $W^{\Sigma}(f,\mathfrak{k})$ and $X^{\Sigma}(f,\mathfrak{k})$ are convex spherical polyhedra (cf. end of Section 5.1).

Finally, suppose $w, -w \in X^{\Sigma}(f,\mathfrak{k})$. Then $I_1^{\mathfrak{k}}(f) \subset I_1^{w}(f) \cap I_1^{-w}(f)$. On the other hand, $I_1^{w}(f) \cap I_1^{-w}(f) = \emptyset$. q.e.d.

Corollary 5.30. *We have $B_{\epsilon_{G/H}}^{\Sigma^{\mathfrak{k}}}(v_{\mathrm{can}}(\mathfrak{k})) \subset W^{\Sigma}(\mathfrak{k})$ for every H-subalgebra $\mathfrak{k}$.*

Proof. If $\ell_{\mathfrak{k}} = 2$, then $\mathfrak{m}_{\mathfrak{k}}^{\perp}$ is $\mathrm{Ad}(\overline{H(\mathfrak{k})})$-irreducible. It follows that $\mathcal{M}_1^G(\mathfrak{k}) = \{\gamma_{v_{\mathrm{can}}(\mathfrak{k})}(t) \mid t \in \mathbb{R}\}$ and we obtain the claim. Suppose now $\ell_{\mathfrak{k}} > 2$. Let $v \in \Sigma^{\mathfrak{k}}$ and let f be a $\mathfrak{k}$-decomposition with $v \in T_Q\mathcal{M}_1^G(f)$. Since $v \in T_Q\mathcal{M}_1^G(f,\mathfrak{k})$, γ_v can be written as in (5.28) and we have $1 = \langle v, v\rangle = \sum_{i=1}^{\ell_{\mathfrak{k}}} \dim \mathfrak{m}_{I_i^{\mathfrak{k}}(f)} \cdot v_i^2$. Let $v^{\epsilon} = v - v_{\mathrm{can}}(\mathfrak{k})$ and suppose $\langle v^{\epsilon}, v^{\epsilon}\rangle^{\frac{1}{2}} < \epsilon_{G/H} = -\frac{1}{2}c_{G/H}$. Then, $|v_i^{\epsilon}| < -\frac{1}{2}c_{G/H}$. We will prove that all such $v \in \Sigma^{\mathfrak{k}}$ satisfy condition (1) and (2) in the definition of the cone $C(f, \mathfrak{k})$. Let $i \in \{2, \dots, \ell_{\mathfrak{k}}\}$. Then, $v_1 - v_i < 0$ is equivalent to $v_1^\epsilon - v_i^\epsilon < v_2(\mathfrak{k}) - v_1(\mathfrak{k})$. Since $v_2(\mathfrak{k}) - v_1(\mathfrak{k}) \geq -2c_{G/H}$, condition (1) is fulfilled. In case $i, j, k \geq 2$, it follows that $v_i - v_j - v_k + v_1 \leq 0$ is equivalent to $v_i^\epsilon - v_j^\epsilon - v_k^\epsilon + v_1^\epsilon \leq v_2(\mathfrak{k}) - v_1(\mathfrak{k})$, hence condition (2) is satisfied in this case. Since by Lemma 5.27 $[I_1^{\mathfrak{k}}(f) I_j^{\mathfrak{k}}(f) I_k^{\mathfrak{k}}(f)]_f = 0$, for $j \neq k$, the claim follows from Remark 5.13. q.e.d.

Note that $W^\Sigma(\mathfrak{k}) = \{v_{\mathrm{can}}(\mathfrak{k})\} \subset \{\pm v_{\mathrm{can}}(\mathfrak{k})\} = \Sigma^{\mathfrak{k}} \subset \mathcal{M}_1^G(\mathfrak{k})$ for $\ell_{\mathfrak{k}} = 2$.

After having finished the proof of Theorem 5.25, let us now investigate how $W^\Sigma(\mathfrak{k})$ and $W^\Sigma(\mathfrak{k}')$ are related for distinct H-subalgebras $\mathfrak{k}$ and $\mathfrak{k}'$. For H-subalgebras $\mathfrak{k}_1, \dots, \mathfrak{k}_p$, let

$$\langle \mathfrak{k}_1, \dots, \mathfrak{k}_p \rangle = \bigcap \{\mathfrak{k} \mid \mathfrak{k} \leq \mathfrak{g} \text{ and } \mathfrak{k}_i \leq \mathfrak{k} \text{ for all } 1 \leq i \leq p\}$$

denote the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{k}_1, \dots, \mathfrak{k}_p$. If $\langle \mathfrak{k}_1, \dots, \mathfrak{k}_p \rangle < \mathfrak{g}$, then $\langle \mathfrak{k}_1, \dots, \mathfrak{k}_p \rangle$ is an H-subalgebra. If $\langle \mathfrak{k}_1, \dots, \mathfrak{k}_p \rangle = \mathfrak{g}$, let us define $W^\Sigma(\mathfrak{g}) = \emptyset$.

Theorem 5.31. *Let $\mathfrak{k}$ and $\mathfrak{k}'$ be H-subalgebras. Then, we have:*

1. *If $\mathfrak{k} \neq \mathfrak{k}'$, then $W^\Sigma(\mathfrak{k}) \cap W^\Sigma(\mathfrak{k}') = \emptyset$.*
2. *If $\mathfrak{k} < \mathfrak{k}'$, then $X^\Sigma(\mathfrak{k}') \subset X^\Sigma(\mathfrak{k}) \backslash W^\Sigma(\mathfrak{k})$.*
3. *$X^\Sigma(\mathfrak{k}) \cap X^\Sigma(\mathfrak{k}') = X^\Sigma(\langle \mathfrak{k}, \mathfrak{k}' \rangle)$.*

Proof.

For 1: This follows from Definition 5.23.

For 2: We have $X^\Sigma(\mathfrak{k}') \cap W^\Sigma(\mathfrak{k}) = \emptyset$ by $\mathfrak{k} < \mathfrak{k}'$ and the definition of $W^\Sigma(\mathfrak{k}')$. Vice versa, let $v \in W^\Sigma(\mathfrak{k}')$. Let f be a good $\mathfrak{k}'$-decomposition of γ_v. Since $\mathfrak{k} < \mathfrak{k}'$, we may assume, that $\mathfrak{k}$ is an f-adapted H-subalgebra. Set $I_0 := I_1^{\mathfrak{k}}(f)$, $I_1 := I_1^{\mathfrak{k}'}(f) \backslash I_1^{\mathfrak{k}}(f)$ and $I_i := I_i^{\mathfrak{k}'}(f)$ for $2 \leq i \leq \ell_{\mathfrak{k}'}$. By means of these index sets, we can consider v as $v = (v_0, v_1, v_2, \dots, v_{\ell_{\mathfrak{k}'}})$ satisfying the following conditions: $v_0 = v_1$, $v_1 < v_j$ for $j = 2, \dots, \ell_{\mathfrak{k}'}$, $\sum_{i=0}^{\ell_{\mathfrak{k}'}} d_{I_i} \cdot v_i = 0$ and $\sum_{i=0}^{\ell_{\mathfrak{k}'}} d_{I_i} \cdot v_i^2 = \|v\|^2 = 1$.

Finally, let $v_* := \min\{v_j \mid j = 2, \dots, \ell_{\mathfrak{k}'}\}$. For $\epsilon \in (0, (v_* - v_1)d_{I_1})$, set $\tilde{v}^\epsilon := (v_0 - \frac{\epsilon}{d_{I_0}}, v_1 + \frac{\epsilon}{d_{I_1}}, v_2, \dots, v_{\ell_{\mathfrak{k}'}})$ and $v^\epsilon := \tilde{v}^\epsilon / ||\tilde{v}^\epsilon||$. We are left proving $v^\epsilon \in W^\Sigma(\mathfrak{k})$; that is for $(i, j, k) \in \{0, 1, \dots, \ell_{\mathfrak{k}'}\}^3$, we need to show, that

$$\{\, [I_i I_j I_k]_f > 0 \Rightarrow v_i^\epsilon - v_j^\epsilon - v_k^\epsilon + v_0^\epsilon \leq 0 \,\} \tag{5.32}$$

holds true. Suppose that $0 \in \{i,j,k\}$, say $i=0$. If $j=0$ and $k>0$, then $[I_0 I_0 I_k]_f = 0$, since $\mathfrak{k}$ is a subalgebra. If $j=1$ and $k>1$, then $[I_0 I_1 I_k]_f = 0$, since $\mathfrak{k} < \mathfrak{k}'$ and $\mathfrak{k}'$ is a subalgebra. If $j,k \geq 2$ with $j \neq k$, then by Lemma 5.27 $[I_0 I_j I_k]_f = 0$. Hence, if $i=0$, then the only structure constants which possibly are not zero are $[I_0 I_m I_m]_f$ for $0 \leq m \leq \ell_{\mathfrak{k}'}$. Therefore, (5.32) holds true in this case.

Suppose, now, that $1 \in \{i,j,k\}$, but $0 \notin \{i,j,k\}$. Again, the only structure constants which possibly are not zero are $[I_1 I_m I_m]_f$ where $1 \leq m \leq \ell_{\mathfrak{k}'}$. Since $\epsilon < (v_* - v_1) d_{I_1}$, (5.32) is fulfilled. Finally, suppose $i,j,k \geq 2$. By means of $v \in W^{\Sigma}$, (5.32) holds true again. We conclude $W^{\Sigma}(\mathfrak{k}') \subset X^{\Sigma}(\mathfrak{k})$ and the claim follows.

For 3: The inclusion "$\supset$" follows from 2. For "$\subset$": Let $v_\infty \in X^{\Sigma}(\mathfrak{k}) \cap X^{\Sigma}(\mathfrak{k}') \subset W^{\Sigma}$ and let f be a good decomposition of γ_{v_∞}. By Lemma 5.16, $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}_{I_1^{v_\infty}(f)}$ is an H-subalgebra. Thus, $v_\infty \in W^{\Sigma}(\mathfrak{l})$. Since $v_\infty \in X^{\Sigma}(\mathfrak{k}) \cap X^{\Sigma}(\mathfrak{k}')$, we have $\mathfrak{k}, \mathfrak{k}' \leq \mathfrak{l}$, hence $\langle \mathfrak{k}, \mathfrak{k}' \rangle \leq \mathfrak{l}$. With the help of 2, we conclude $v_\infty \in X^{\Sigma}(\mathfrak{l}) \subset X^{\Sigma}(\langle \mathfrak{k}, \mathfrak{k}' \rangle)$. q.e.d.

In Section 5.7, the following extension of Theorem 5.31 will be crucial.

Corollary 5.33. *Let $\mathfrak{k}$ be an H-subalgebra and let f be a decomposition of $\mathfrak{m}$, such that $\mathfrak{k}$ is f-adapted. Then, for each face F of $X^{\Sigma}(f,\mathfrak{k})$ one of the following two possibilities occurs:*

1. *There exists an f-adapted H-subalgebra $\mathfrak{l}$ with $\mathfrak{k} < \mathfrak{l}$, such that $F \subset X^{\Sigma}(f,\mathfrak{l})$.*
2. *For $v = (\gamma_{v_1,\ldots,v_{\ell_{\mathfrak{k}}(f)}})'(0) \in \operatorname{int} F$ we have $v_1 < v_i$, $i \in \{2,\ldots,\ell_{\mathfrak{k}}(f)\}$, and for $v \in F$, there exists $(i_0,j_0,k_0) \in \{2,\ldots,\ell_{\mathfrak{k}}(f)\}^3$ with*

$$[I_{i_0}^{\mathfrak{k}}(f) I_{j_0}^{\mathfrak{k}}(f) I_{k_0}^{\mathfrak{k}}(f)]_f > 0 \quad \textit{and} \quad v_{i_0} - v_{j_0} - v_{k_0} + v_1 = 0.$$

Proof. Let F be a face of $X^{\Sigma}(f,\mathfrak{k})$. If $\dim F = 0$, then $F = \{v\}$. If $I_1^{\mathfrak{k}}(f) \subsetneq I_1^{v}(f)$, then $v \in X^{\Sigma}(f,\mathfrak{l})$ for an f-adapted H-subalgebra $\mathfrak{l}$ with $\mathfrak{k} < \mathfrak{l}$. If $I_1^{\mathfrak{k}}(f) = I_1^{v}(f)$, then at least one of the inequalities in condition (2) in the definition of $C(f,\mathfrak{k})$ gets sharp. By Remark 5.13, there exists $(i_0,j_0,k_0) \in \{1,\ldots,\ell_{\mathfrak{k}}(f)\}^3 \backslash \{(m,m,1)^S \mid 1 \leq m \leq \ell_{\mathfrak{k}}(f)\}$ with $[I_{i_0}^{\mathfrak{k}}(f) I_{j_0}^{\mathfrak{k}}(f) I_{k_0}^{\mathfrak{k}}(f)]_f > 0$ and $v_{i_0} - v_{j_0} - v_{k_0} + v_1 = 0$. With the help of Lemma 5.27, we obtain $i_0, j_0, k_0 \geq 2$.

If $\dim F > 0$, then let $v \in \operatorname{int} F$. Suppose $v \in X^{\Sigma}(f,\mathfrak{l})$ for an f-adapted H-subalgebra $\mathfrak{l}$ with $\mathfrak{k} < \mathfrak{l}$. Then, for each $v \in F$, we have $I_1^{\mathfrak{l}}(f) \subset I_1^{v}(f)$ since the corresponding inequalities in condition (1) of the definition of $C(f,\mathfrak{k})$ get equalities on F. Thus, $F \subset X^{\Sigma}(f,\mathfrak{l})$.

If for all $v \in \operatorname{int} F$ we have $I_1^{\mathfrak{k}}(f) = I_1^v(f)$, then the claim follows as above. q.e.d.

5.5. The essential part of $\mathfrak{k}$-non-negative directions. The essential part $X_+^\Sigma(\mathfrak{k})$ of the closure $X^\Sigma(\mathfrak{k})$ of the $\mathfrak{k}$-non-negative directions $W^\Sigma(\mathfrak{k})$ is by definition the spherical cone of $\cup_{\mathfrak{k}<\mathfrak{l}} X^\Sigma(\mathfrak{l})$ over $v_{\mathrm{can}}(\mathfrak{k})$ in Σ. We will construct a strong deformation retraction from $X^\Sigma(\mathfrak{k})$ to $X_+^\Sigma(\mathfrak{k})$ using the fact, that both $X^\Sigma(\mathfrak{k})$ and $X_+^\Sigma(\mathfrak{k})$ are compact and semialgebraic subsets of Σ. In Section 6.2, this result will be used to determine the homotopy type of the non-toral directions for homogeneous spaces of finite type.

By Theorem 5.25, $v_{\mathrm{can}}(\mathfrak{k})$ and $v \in X^\Sigma(\mathfrak{k})$ can be joined by a unique unit speed geodesic $c_v^{\mathfrak{k}}$ in $\Sigma^{\mathfrak{k}}$ with $c_v^{\mathfrak{k}}(0) = v_{\mathrm{can}}(\mathfrak{k})$. Let

$$s_v = d^{\Sigma^{\mathfrak{k}}}(v_{\mathrm{can}}(\mathfrak{k}), v) \tag{5.34}$$

denote the spherical distance from v to $v_{\mathrm{can}}(\mathfrak{k})$. We have $c_v^{\mathfrak{k}}(s_v) = v$. Furthermore, there exists

$$s_v^\partial \in [\epsilon_{G/H}, \pi - \epsilon_{G/H}] \tag{5.35}$$

with $c_v^{\mathfrak{k}}(s) \in X^\Sigma(\mathfrak{k})$ for $0 \le s \le s_v^\partial$ and $c_v^{\mathfrak{k}}(s) \notin X^\Sigma(\mathfrak{k})$ for $s_v^\partial < s \le \pi$. Let

$$\pi_{\mathfrak{k}} : X^\Sigma(\mathfrak{k}) \backslash \{v_{\mathrm{can}}(\mathfrak{k})\} \to X^\Sigma(\mathfrak{k}) \backslash \{v_{\mathrm{can}}(\mathfrak{k})\};\ \ v \mapsto c_v^{\mathfrak{k}}(s_v^\partial) \tag{5.36}$$

denote the radial projection of $X^\Sigma(\mathfrak{k})$ onto the radial boundary

$$\partial_{\mathrm{rad}} X^\Sigma(\mathfrak{k}) = \pi_{\mathfrak{k}}(X^\Sigma(\mathfrak{k}) \backslash \{v_{\mathrm{can}}(\mathfrak{k})\})$$

of $X^\Sigma(\mathfrak{k})$. If $\ell_{\mathfrak{k}} = 2$, then $X^\Sigma(\mathfrak{k}) = \{v_{\mathrm{can}}(\mathfrak{k})\}$ and we set $\partial_{\mathrm{rad}} X^\Sigma(\mathfrak{k}) = \emptyset$.

Now, let $\mathfrak{k}$, $\mathfrak{l}$ be H-subalgebras with $\mathfrak{k} < \mathfrak{l}$. Then, $X^\Sigma(\mathfrak{l}) \subset \partial_{\mathrm{rad}} X^\Sigma(\mathfrak{k})$ (cf. Corollary 5.33). Note that in general $\cup_{\mathfrak{k}<\mathfrak{l}} X^\Sigma(\mathfrak{l}) \neq \partial_{\mathrm{rad}} X^\Sigma(\mathfrak{k})$. For $R, S \subset \Sigma$, let $\operatorname{conv}^\Sigma(R, S)$ denote the convex hull of R and S in the round sphere $(\Sigma, L^2|_\Sigma)$. If $S = \emptyset$, then we set $\operatorname{conv}^\Sigma(R, \emptyset) = R$ (for R convex).

Definition 5.37. Let $\mathfrak{k}$ be an H-subalgebra. For a subset S of $\partial_{\mathrm{rad}} X^\Sigma(\mathfrak{k})$, let

$$C^\Sigma(\mathfrak{k}, S) = \operatorname{conv}^\Sigma(v_{\mathrm{can}}(\mathfrak{k}), S).$$

Furthermore, let

$$X_+^\Sigma(\mathfrak{k}) = C^\Sigma(\mathfrak{k}, \cup_{\mathfrak{k}<\mathfrak{l}} X^\Sigma(\mathfrak{l})) \quad \text{and} \quad W_-^\Sigma(\mathfrak{k}) = X^\Sigma(\mathfrak{k}) \backslash X_+^\Sigma(\mathfrak{k}).$$

We have

$$W_-^{\Sigma}(\mathfrak{k}) \subset W^{\Sigma}(\mathfrak{k}) = X^{\Sigma}(\mathfrak{k}) \setminus \cup_{\mathfrak{k}<\mathfrak{l}} X^{\Sigma}(\mathfrak{l}).$$

Note that $W_-^{\Sigma}(\mathfrak{k}) = \emptyset$ is possible even if $\ell_{\mathfrak{k}} > 2$. The essential part $X_+^{\Sigma}(\mathfrak{k})$ of $X^{\Sigma}(\mathfrak{k})$ is non-empty and star-shaped. We have $X_+^{\Sigma}(\mathfrak{k}) = \{v_{\mathrm{can}}(\mathfrak{k})\}$ if and only if $\mathfrak{k}$ not contained properly in an H-subalgebra $\mathfrak{l}$.

Lemma 5.38. *Let $\mathfrak{k}$ be an H-subalgebra. Then, the essential part $X_+^{\Sigma}(\mathfrak{k})$ of $X^{\Sigma}(\mathfrak{k})$ is a compact, semialgebraic subset of $X^{\Sigma}(\mathfrak{k})$.*

Proof. If $\cup_{\mathfrak{k}<\mathfrak{l}} X^{\Sigma}(\mathfrak{l}) = \emptyset$, then $X_+^{\Sigma}(\mathfrak{k}) = \{v_{\mathrm{can}}(\mathfrak{k})\}$ and the claim follows. If $\emptyset \neq \cup_{\mathfrak{k}<\mathfrak{l}} X^{\Sigma}(\mathfrak{l})$, then $\cup_{\mathfrak{k}<\mathfrak{l}} X^{\Sigma}(\mathfrak{l})$ is a compact, semialgebraic subset of $\Sigma^{\mathfrak{k}}$ (see below) having positive distance to $-v_{\mathrm{can}}(\mathfrak{k})$ (see Theorem 5.25). Hence, $\mathrm{conv}^E(v_{\mathrm{can}}(\mathfrak{k}), \cup_{\mathfrak{k}<\mathfrak{l}} X^{\Sigma}(\mathfrak{l}))$ is a compact, semialgebraic subset of $T_Q\mathcal{M}_1^G(\mathfrak{k})$ not containing the origin (cf. [**31**, Example 1.5]). Consequently, $\pi_{\Sigma}(\mathrm{conv}^E(v_{\mathrm{can}}(\mathfrak{k}), \cup_{\mathfrak{k}<\mathfrak{l}} X^{\Sigma}(\mathfrak{l}))) = X_+^{\Sigma}(\mathfrak{k})$ is compact and semialgebraic, since the map π_{Σ} is continuous and semialgebraic.

Recall that in Proposition 5.26, we have defined the parametrization $F^{\mathfrak{k}}|_{\hat{\mathcal{M}}_1^G(\mathfrak{k})} : \hat{\mathcal{M}}_1^G(\mathfrak{k}) \to T_Q\mathcal{M}_1^G(\mathfrak{k})$, the structure constants $[ijk]_{f_{\mathfrak{k}}}$ and their lifts $\widehat{[ijk]}^{\mathfrak{k}} : \hat{\mathcal{M}}_1^G(\mathfrak{k}) \to \mathbb{R}$. Let $I \subset \{1, \dots, \ell_{\mathfrak{k}}\}$ with $1 \in I$ and $2 \leq |I| < \ell_{\mathfrak{k}}$ and let $P_{\mathfrak{k}}$ denote the set of all such subsets. Furthermore, for $I \in P_{\mathfrak{k}}$ let

$$\hat{G}_I^{\mathfrak{k}}(A, v) = \sum_{i,j \in I} \sum_{k \in I^C} \widehat{[ijk]}^{\mathfrak{k}}_{(A,v)},$$

where $I^C = \{1, \dots, \ell_{\mathfrak{k}}\} \setminus I$. We claim

$$\begin{aligned} &(F^{\mathfrak{k}}|_{\hat{\mathcal{M}}_1^G(\mathfrak{k})})^{-1}(\cup_{\mathfrak{k}<\mathfrak{l}} X^{\Sigma}(\mathfrak{l})) = (F^{\mathfrak{k}}|_{\hat{\mathcal{M}}_1^G(\mathfrak{k})})^{-1}(X^{\Sigma}(\mathfrak{k})) \ \cap \\ &\qquad \bigcup_{I \in P_{\mathfrak{k}}} \Big\{ (\hat{G}_I^{\mathfrak{k}})^{-1}(0) \ \cap \bigcap_{m \in I, n \in I^C} \{v_m \leq v_n\} \ \cap \bigcap_{m,m' \in I} \{v_m = v_{m'}\} \Big\}. \end{aligned}$$

For "$\subset$": This is obvious. For "$\supset$": Let $(A, v) \in (F^{\mathfrak{k}}|_{\hat{\mathcal{M}}_1^G(\mathfrak{k})})^{-1}(X^{\Sigma}(\mathfrak{k}))$. By Theorem 5.31, we have $F^{\mathfrak{k}}|_{\hat{\mathcal{M}}_1^G(\mathfrak{k})}(A, v) \in W^{\Sigma}(\mathfrak{l})$ for an H-subalgebra $\mathfrak{l}$ with $\mathfrak{k} \leq \mathfrak{l}$. Since there exists $I \in P_{\mathfrak{k}}$ with

$$(A, v) \ \in \bigcap_{m \in I, n \in I^C} \{v_m \leq v_n\} \ \cap \bigcap_{m,m' \in I} \{v_m = v_{m'}\},$$

we obtain $\mathfrak{k} < \mathfrak{l}$. q.e.d.

Now, we turn to the main result of this section.

Theorem 5.39. *For an H-subalgebra $\mathfrak{k}$, there exists a strong deformation retraction from the closure $X^\Sigma(\mathfrak{k})$ of the $\mathfrak{k}$-non-negative directions $W^\Sigma(\mathfrak{k})$ to the essential part $X^\Sigma_+(\mathfrak{k})$ of $X^\Sigma(\mathfrak{k})$.*

Proof. If $X^\Sigma(\mathfrak{k}) = X^\Sigma_+(\mathfrak{k})$, then there is nothing to show. This covers, in particular, the case $\ell_\mathfrak{k} = 2$. If $\ell_\mathfrak{k} > 2$ and $X^\Sigma_+(\mathfrak{k}) = \{v_{\mathrm{can}}(\mathfrak{k})\}$, then the claim follows from Theorem 5.25. Hence, we may assume that $\ell_\mathfrak{k} > 2$ and $\emptyset \subsetneq \cup_{\mathfrak{k}<\mathfrak{l}} X^\Sigma(\mathfrak{l}) \subsetneq \partial_{\mathrm{rad}} X^\Sigma(\mathfrak{k})$.

It follows from Theorem 1 of [**21**] (see also [**22**, III, Theorem 1.1]) that there exists an open semialgebraic neighborhood U of $X^\Sigma_+(\mathfrak{k})$ in $X^\Sigma(\mathfrak{k})$ and a semialgebraic, continuous map $G : [0,1] \times \overline{U} \to \overline{U}$, such that the restriction $G|_{[0,1]\times U}$ yields a strong deformation retraction from U to $X^\Sigma_+(\mathfrak{k})$.

Let $A = \{0\} \times X^\Sigma(\mathfrak{k}) \cup (0,1] \times \overline{U}$. Then, A is a closed, semialgebraic subset of $[0,1] \times X^\Sigma(\mathfrak{k})$. We extend G to a map from A to $X^\Sigma(\mathfrak{k})$ by $G(0,v) = v$ for all $v \in X^\Sigma(\mathfrak{k})$. Denote this map again by G. Then, G is continuous and the graph of G is a semialgebraic subset of $A \times X^\Sigma(\mathfrak{k})$. Thus, G is a continuous and semialgebraic map, and hence by [**22**, I, Proposition 3.13], semialgebraic in their terminology. Since $X^\Sigma(\mathfrak{k})$ is contractible, by Theorem 3 of [**21**], the map G can be extended to a continuous, semialgebraic map from $[0,1] \times X^\Sigma(\mathfrak{k})$ to $X^\Sigma(\mathfrak{k})$, again denoted by G.

Since both $X^\Sigma(\mathfrak{k})$ and $X^\Sigma_+(\mathfrak{k})$ are star-shaped with respect to the base point $v_{\mathrm{can}}(\mathfrak{k})$, there exists a strong deformation retraction F from $X^\Sigma(\mathfrak{k})$ to a compact neighborhood K of $X^\Sigma_+(\mathfrak{k})$ with $K \subset U$. Then, $H : [0,1]\times X^\Sigma(\mathfrak{k}) \to X^\Sigma(\mathfrak{k})\,;\ (\mu, v) \mapsto G_\mu(F_\mu(v))$ is a strong deformation retraction from $X^\Sigma(\mathfrak{k})$ to $X^\Sigma_+(\mathfrak{k})$. q.e.d.

5.6. Extended non-toral directions. In this section, we define the set of toral directions $X^\Sigma_t \subset W^\Sigma$, the set of non-toral directions $X^\Sigma_{nt} \subset W^\Sigma$ and the set of extended non-toral directions $X^\Sigma_{\mathrm{ent}} \supset X^\Sigma_{nt}$. All these sets are compact and semialgebraic. We will prove that there exists a strong deformation retraction from X^Σ_{ent} to X^Σ_{nt}.

The extended non-toral directions X^Σ_{ent} are obtained from the non-negative directions W^Σ by removing (inductively) certain toral directions. For the toral directions removed, that is for $v \in W^\Sigma \backslash X^\Sigma_{\mathrm{ent}}$, we will provide then in Section 5.7 scalar curvature estimates along γ_v, which are crucial ingredients for the variational methods described in Section 8.

Before we define the above mentioned sets, let us consider the following explicit example.

Example 5.40 (Aloff–Wallach spaces)**.** Let $G/H = \mathrm{SU}(3)/\mathrm{U}_{k,q}(1)$ for generic $k, q \in \mathbb{Z}$. We have dim $\Sigma = 2$. Let $\mathfrak{t} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1))$. Then, $W^\Sigma = X^\Sigma(\mathfrak{t}) = \overline{W^\Sigma(\mathfrak{t})}$.

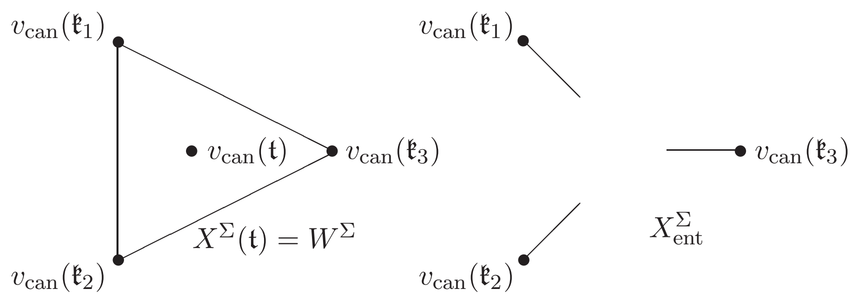

Figure 1. Non-negative and extended non-toral directions for the Aloff–Wallach spaces.

In fact, it is easy to check that W^Σ is a spherical triangle with barycentre $v_{\text{can}}(\mathfrak{t})$. The vertices of this triangle are precisely the three canonical directions, associated to the three non-toral H-subalgebras $\mathfrak{k}_1$, $\mathfrak{k}_2$, $\mathfrak{k}_3$ of type $\mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1))$. They constitute the non-toral directions X^Σ_{nt}.

The extended non-toral directions X^Σ_{ent} consist of a compact neighborhood of X^Σ_{nt} in the essential part $X^\Sigma_+(\mathfrak{t})$ of $X^\Sigma(\mathfrak{t})$. Notice that the extended non-toral directions (up to homotopy) can be obtained from the non-negative directions W^Σ by removing a propeller from W^Σ with center $v_{\text{can}}(\mathfrak{t})$.

We call a non-toral H-subalgebra $\mathfrak{k}$ minimal non-toral if an H-subalgebra $\mathfrak{l}$ with $\mathfrak{l} < \mathfrak{k}$ must be toral. We call a toral H-subalgebra $\mathfrak{t}$ minimal toral if $\mathfrak{l} \leq \mathfrak{t}$ implies $\mathfrak{l} = \mathfrak{t}$ for all H-subalgebras $\mathfrak{l}$. Let

$$W^\Sigma_{nt} = \bigcup_{\mathfrak{k}\ \text{minimal non-toral}} W^\Sigma(\mathfrak{k}) \quad \text{and} \quad W^\Sigma_t = \bigcup_{\mathfrak{k}\ \text{minimal toral}} W^\Sigma(\mathfrak{k}).$$

Definition 5.41 (Non-toral and toral directions)**.** The non-toral directions X^Σ_{nt} and the toral directions X^Σ_t are defined by $X^\Sigma_{nt} = \overline{W^\Sigma_{nt}}$ and $X^\Sigma_t = \overline{W^\Sigma_t}$, respectively.

Notice that the set of toral directions X^Σ_t will, in general, contain also non-toral directions $v \in W^\Sigma$ as defined in Section 1.

Lemma 5.42. *We have*

$$X^{\Sigma}_{nt} = \bigcup_{\mathfrak{k} \text{ minimal non-toral}} X^{\Sigma}(\mathfrak{k}) \quad \text{and} \quad X^{\Sigma}_{t} = \bigcup_{\mathfrak{k} \text{ minimal toral}} X^{\Sigma}(\mathfrak{k}).$$

Proof. Let $v \in X^{\Sigma}_{nt} \subset W^{\Sigma}$ and let $(v(\alpha))_{\alpha\in\mathbb{N}}$ be a sequence in W^{Σ}_{nt} converging to v with $v(\alpha) \in W^{\Sigma}(\mathfrak{k}(\alpha))$. By passing to a subsequence, we may assume that $\mathfrak{k}(\alpha)$ converges to an H-subalgebra $\mathfrak{l}$. Since $\mathfrak{l}$ is not toral (cf. [**10**]), there exists a minimal non-toral H-subalgebra $\mathfrak{k}$ with $\mathfrak{k} \leq \mathfrak{l}$. By Theorem 5.31, we conclude $v \in X^{\Sigma}(\mathfrak{l}) \subset X^{\Sigma}(\mathfrak{k})$ and obtain the first claim.

The proof of the second claim follows as above since a convergent sequence $(\mathfrak{k}(\alpha))_{\alpha\in\mathbb{N}}$ of toral H-subalgebras has to converge to a toral H-subalgebra. q.e.d.

Let us turn to the definition of the extended non-toral directions. For the definition of the semialgebraic parameterization $F : \hat{\mathcal{M}}^G_1 \to T_Q\mathcal{M}^G_1$ of $T_Q\mathcal{M}^G_1$, we refer to (5.14). Recall that the pre-image of W^{Σ}, Σ has been denoted by $\hat{W}^{\Sigma}$, $\hat{\Sigma}$, respectively. For the definition of the structure constants $\widehat{[ijk]}$, see Section 4.3.

For a non-empty, proper subset $I \subset \{1, \ldots, \ell\}$ let

$$\hat{G}_I : \hat{\mathcal{M}}^G_1 \to \mathbb{R};\ \ (A, v) \ \ \mapsto \ \sum_{i,j\in I}\sum_{k\in I^C} \widehat{[ijk]}_{(A,v)} + \sum_{i,j,k\in I} \widehat{[ijk]}_{(A,v)}.$$

The functions G_I are semialgebraic and continuous. For $r \in \{1,\ldots,\ell-1\}$, let

$$\begin{aligned} \hat{X}^{\Sigma}_t(r) \ \ = \ \bigcup_{I\subset\{1,\ldots,\ell\},\,|I|=r} \Big\{ &\hat{W}^{\Sigma} \ \cap \ (\hat{G}_I)^{-1}(0) \ \cap \bigcap_{m\in I,n\in I^C} \{v_m \leq v_n\} \\ &\cap \bigcap_{m,m'\in I} \{v_m = v_{m'}\} \Big\}. \end{aligned}$$

The semialgebraic and compact set $\hat{X}^{\Sigma}_t(r)$ parameterizes the set

$$X^{\Sigma}_t(r) := F(\hat{X}^{\Sigma}_t(r)) \subset W^{\Sigma},$$

the (possibly infinite) union of $X^{\Sigma}(\mathfrak{t})$'s where $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ is a toral subalgebra and the $\mathfrak{m}_i$'s are irreducible for all i.

Let the semialgebraic set $\hat{W}^{\Sigma}_t(r)$ be defined as $\hat{X}^{\Sigma}_t(r)$, by replacing the term $\{v_m \leq v_n\}$ by $\{v_m < v_n\}$. Then, $\hat{W}^{\Sigma}_t(r)$ parameterizes $W^{\Sigma}_t(r)$, the union of $W^\Sigma(\mathfrak{t})$'s with $\mathfrak{t}$ as above. Below, we will show

$$\overline{\hat{W}_t^\Sigma(r)} = \hat{X}_t^\Sigma(r).$$

In order to do so, note that $\hat{W}_t^\Sigma(r) = \hat{X}_t^\Sigma(r)$ is possible. However, if $\hat{W}_t^\Sigma(r) \subsetneq \hat{X}_t^\Sigma(r)$, then let

$$\partial_+ \hat{X}_t^\Sigma(r) = \hat{X}_t^\Sigma(r) \backslash \hat{W}_t^\Sigma(r)$$

denote the essential radial boundary of $\hat{X}_t^\Sigma(r)$ (the set $\partial_+ \hat{X}_t^\Sigma(r)$ is in general a proper subset of the boundary of $\hat{X}_t^\Sigma(r)$). The set $\partial_+ \hat{X}_t^\Sigma(r)$ parameterizes the essential-radial boundary

$$\partial_+ X_t^\Sigma(r) = F(\partial_+ \hat{X}_t^\Sigma(r))$$

of $X_t^\Sigma(r)$.

We claim that $\hat{W}_t^\Sigma(r)$ is relatively open in $\hat{X}_t^\Sigma(r)$ in order to show that $\partial_+ \hat{X}_t^\Sigma(r)$ is a compact and semialgebraic set. To this end, let $(A, v) \in \hat{W}_t^\Sigma(r)$. Then, $F(A, v) \in W^\Sigma(\mathfrak{k})$, where $\mathfrak{k}$ is a toral H-subalgebra. Note that $\mathfrak{m}_\mathfrak{k} = \mathfrak{m}_{i_1} \oplus \cdots \oplus \mathfrak{m}_{i_r}$, where $f = \oplus_{i=1}^\ell \mathfrak{m}_i$ is the decomposition of $\mathfrak{m}$ which corresponds to $[A]$. Now let $(\tilde{A}, \tilde{v}) \in \hat{X}_t^\Sigma(r) \backslash \hat{W}_t^\Sigma(r)$. Then, $F(\tilde{A}, \tilde{v}) \in W^\Sigma(\mathfrak{l})$ for an H-subalgebra $\mathfrak{l}$. We have $\mathfrak{m}_\mathfrak{l} = \tilde{\mathfrak{m}}_{i_1} \oplus \cdots \oplus \tilde{\mathfrak{m}}_{i_{\tilde{r}}}$, with $\tilde{r} > r$, where $\tilde{f} = \oplus_{i=1}^\ell \tilde{\mathfrak{m}}_i$ is the decomposition of $\mathfrak{m}$, which corresponds to $[\tilde{A}]$. Since the summands $\mathfrak{m}_i$ and $\tilde{\mathfrak{m}}_i$ are irreducible and since $\tilde{r} > r$, $\hat{W}_t^\Sigma(r)$ is relatively open in $\hat{X}_t^\Sigma(r)$.

Next, we claim

$$\hat{W}_t^\Sigma(r+1) \subset \partial_+ \hat{X}_t^\Sigma(r).$$

To this end, let $(A, v) \in \hat{W}_t^\Sigma(r+1)$. Then, $F(A, v) \in W^\Sigma(\mathfrak{k})$, where $\mathfrak{k}$ is a toral H-subalgebra. Note that $\mathfrak{m}_\mathfrak{k} = \mathfrak{m}_{i_1} \oplus \cdots \oplus \mathfrak{m}_{i_{r+1}}$, where $f = \oplus_{i=1}^\ell \mathfrak{m}_i$ is the decomposition of $\mathfrak{m}$ which corresponds to $[A]$. Since $\mathfrak{k}$ is toral, $\mathfrak{m}_\mathfrak{k}$ is an abelian subalgebra of $\mathfrak{m}_0$, hence $\mathfrak{m}_{i_1} \oplus \cdots \oplus \mathfrak{m}_{i_r} = \mathfrak{m}_\mathfrak{l}$ as well. Thus, $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}_\mathfrak{l}$ is a toral H-subalgebra with $\mathfrak{l} < \mathfrak{k}$. We conclude $F(A, v) \in X^\Sigma(\mathfrak{l})$ by Theorem 5.31, hence $(A, v) \in \hat{X}_t^\Sigma(r)$.

Another consequence of Theorem 5.31 is that $\overline{\hat{W}_t^\Sigma(r)} = \hat{X}_t^\Sigma(r)$. This implies

$$\hat{X}_t^\Sigma(r+1) \subset \hat{X}_t^\Sigma(r) \subset \cdots \subset \hat{X}_t^\Sigma(1).$$

Therefore, $\cup_{r=1}^{\ell-1} \hat{X}_t^\Sigma(r) = \hat{X}_t^\Sigma(1) = F^{-1}(X_t^\Sigma)$. Consequently, X_t^Σ is a compact and semialgebraic subset of the non-negative directions W^Σ.

Remark 5.43. The elements $(A,v) \in \hat{W}^{\Sigma}_t(r) \subset \hat{W}^{\Sigma}$ are characterized as follows: If $F(A,v) \in W^{\Sigma}(\mathfrak{k})$, then $\mathfrak{k}$ is toral and there exists a flag $\mathfrak{t}_1 < \cdots < \mathfrak{t}_r = \mathfrak{k}$ of toral H-subalgebras of length r, such that $v_{\mathrm{can}}(\mathfrak{t}_i) \in F(\hat{W}^{\Sigma}_t(i))$ for $i = 1, \dots, r$, but there exists no flag $\mathfrak{t}'_1 < \cdots < \mathfrak{t}'_{r+1} = \mathfrak{k}$ of toral H-subalgebras.

The elements $(A', v') \in \partial_+ \hat{X}^{\Sigma}_t(r)$ are characterized as follows: There exists $(A,v) \in \hat{W}^{\Sigma}_t(r)$, such that $F(A,v) = v_{\mathrm{can}}(\mathfrak{k})$, where $\mathfrak{k}$ is a toral H-subalgebra, and an H-subalgebra $\mathfrak{l}$ with $\mathfrak{k} < \mathfrak{l}$, such that $F(A',v') \in W^{\Sigma}(\mathfrak{l}) \subset X^{\Sigma}(\mathfrak{k})$. The H-subalgebra $\mathfrak{l}$ can be either toral or non-toral.

Notice that for the Aloff–Wallach spaces (discussed above), we have $X^{\Sigma}_t(1) = X^{\Sigma}(\mathfrak{t})$, $W^{\Sigma}_t(1) = W^{\Sigma}(\mathfrak{t}) = X^{\Sigma}(\mathfrak{t}) \backslash \{v_{\mathrm{can}}(\mathfrak{k}_1), v_{\mathrm{can}}(\mathfrak{k}_2), v_{\mathrm{can}}(\mathfrak{k}_3)\}$ and $\partial_+ X^{\Sigma}_t(1) = \{v_{\mathrm{can}}(\mathfrak{k}_1), v_{\mathrm{can}}(\mathfrak{k}_2), v_{\mathrm{can}}(\mathfrak{k}_3)\}$.

Before we define the extended non-toral directions, we have to introduce further subsets of $\hat{W}^{\Sigma}$. For a non-empty, proper subset $I \subset \{1, \dots, \ell\}$ and $t_1, t_2 \in \mathbb{R}$ let $v_I(t_1,t_2) \in \mathbb{R}^l$ be defined by $v_I(t_1,t_2)_i = t_1$ for $i \in I$ and $v_I(t_1,t_2)_j = t_2$ for $j \in I^C$. Let

$$\hat{P}^{\Sigma}_t(r) \;=\; \hat{W}^{\Sigma}_t(r) \;\cap \bigcup_{I \subset \{1,\dots,\ell\}} \{(A,v) \in \hat{\Sigma} \mid \exists (t_1,t_2) \in \mathbb{R}^2 : v = v_I(t_1,t_2)\}$$

denote the soul of $\hat{W}^{\Sigma}_t(r)$. By the Theorem of Tarski–Seidenberg $\hat{P}^{\Sigma}_t(r)$ is a semialgebraic subset of $\hat{\Sigma}$ (cf. [**4**, p. 60-62], [**17**, p. 268–269]). Notice that

$$P^{\Sigma}_t(r) := F(\hat{P}^{\Sigma}_t(r))$$

equals the set of canonical directions in $W^{\Sigma}_t(r)$.

In the next step, we show that the soul $\hat{P}^{\Sigma}_t(r)$ of $\hat{W}^{\Sigma}_t(r)$ is compact. Since $\hat{P}^{\Sigma}_t(r) \subset \hat{X}^{\Sigma}_t(r)$, we are left proving that $\hat{P}^{\Sigma}_t(r)$ is closed. Let $(A(\alpha), v(\alpha))_{\alpha \in \mathbb{N}}$ be a sequence in $\hat{P}^{\Sigma}_t(r)$ with $\lim_{\alpha \to \infty}(A(\alpha), v(\alpha)) = (A(\infty), v(\infty)) \in \hat{X}^{\Sigma}_t(r)$. With the help of Lemma 4.12, we conclude that the two eigenvalues of canonical directions are bounded away from zero, thus $F((A(\infty), v(\infty)) = v_{\mathrm{can}}(\mathfrak{k})$, where $\mathfrak{k}$ is a toral H-subalgebra. We obtain the claim by Remark 5.43.

Now, consider the map $v_{\min} : \mathbb{R}^l \to \mathbb{R}\,;\ v \mapsto \min\{v_1, \dots, v_\ell\}$. Let $i \in \{1, \dots, \ell\}$. Then $\{v \in \mathbb{R}^\ell \mid v_i - v_{\min}(v) = 0\}$ is a semialgebraic subset of $\mathbb{R}^\ell$. For $r \in \{1, \dots, \ell-1\}$ with $\partial_+ \hat{X}_t^\Sigma(r) \neq \emptyset$, let

$$\begin{aligned}
&\hat{J}_{t,+}^\Sigma(r) \\
&= \Big\{((A,v),(A',v')) \in \hat{P}_t^\Sigma(r) \times \partial_+ \hat{X}_t^\Sigma(r) \;\mid\; \big\{[F(A,v),F(A',v')] = 0\big\} \\
&\quad \wedge\; \big\{\forall i \in \{1,\dots,\ell\} : \{v_i - v_{\min}(v) = 0\} \Rightarrow \{v_i' - v_{\min}(v') = 0\}\big\}\Big\}.
\end{aligned}$$

Then, $\hat{J}_{t,+}^\Sigma(r) \subset \hat{\Sigma} \times \hat{\Sigma}$ is semialgebraic. For $((A,v),(A',v')) \in \hat{J}_{t,+}^\Sigma(r)$, we have $F(A,v) = v_{\mathrm{can}}(\mathfrak{k})$ for a toral subalgebra $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ and $F(A',v') \in X^\Sigma(\mathfrak{k}) \backslash W^\Sigma(\mathfrak{k})$; that is, $F(A',v')$ is contained in the radial boundary of the essential part $X_+^\Sigma(\mathfrak{k})$ of $X^\Sigma(\mathfrak{k})$. We claim that $\hat{J}_{t,+}^\Sigma(r)$ is compact. Let $((A(\alpha),v(\alpha)),(A'(\alpha),v'(\alpha)))_{\alpha\in\mathbb{N}}$ be a sequence in $\hat{J}_{t,+}^\Sigma(r)$ converging to $((A(\infty),v(\infty)),(A'(\infty),v'(\infty))) \in \hat{P}_t^\Sigma(r) \times \partial_+ \hat{X}_t^\Sigma(r)$. By the very definition of $\hat{J}_{t,+}^\Sigma(r)$, the endomorphisms $F((A(\alpha),v(\alpha)))$ and $F((A'(\alpha),v'(\alpha)))$ commute. Thus, the same is true for $F((A(\infty),v(\infty)))$ and $F((A'(\infty),v'(\infty)))$. Since the two eigenvalues of canonical directions are bounded away from zero, the claim follows.

With the help of the above defined sets we now define subsets of the non-negative directions W^Σ in order to give the inductive definition of the extended non-toral directions X_{ent}^Σ. For $\mu \in [0,1]$, we denote by

$$\begin{aligned}
X_{t,+}^\Sigma(r)_\mu \;=\; \pi_\Sigma\Big(&\{(1-\kappa)F(A,v) + \kappa F(A',v') \mid \\
&\kappa \in [\mu,1],\, ((A,v),(A',v')) \in \hat{J}_{t,+}^\Sigma(r)\}\Big)
\end{aligned}$$

the μ-essential part of $X_t^\Sigma(r)$, where $\pi_\Sigma : T_Q\mathcal{M}_1^G \backslash \{0\} \to \Sigma;\; v \mapsto v/||v||$ is the radial projection in $T_Q\mathcal{M}_1^G$ onto the unit sphere Σ. It follows that the set $X_{t,+}^\Sigma(r)_\mu$ is compact and semialgebraic and that we have $X_{t,+}^\Sigma(r)_\mu \subset X_t^\Sigma \subset W^\Sigma$. Notice

$$X_{t,+}^\Sigma(r)_0 = \{X_+^\Sigma(\mathfrak{t}) \mid v_{\mathrm{can}}(\mathfrak{t}) \in P_t^\Sigma(r)\}.$$

The elements $\pi_\Sigma((1-\kappa)F(A,v) + \kappa F(A',v'))$ in $X_{t,+}^\Sigma(r)_0$ determine uniquely κ, (A,v) and (A',v') if $\kappa \neq 0,1$.

To see this, let $((A,v),(A',v')),((\tilde{A},\tilde{v}),(\tilde{A}',\tilde{v}')) \in \hat{J}_{t,+}^\Sigma(r)$ and let $\kappa, \tilde{\kappa} \in [0,1]$. We claim that if

$$\pi_\Sigma\Big((1-\kappa)F(A,v) + \kappa F(A',v')\Big) = \pi_\Sigma\Big((1-\tilde{\kappa})F(\tilde{A},\tilde{v}) + \tilde{\kappa} F(\tilde{A}',\tilde{v}')\Big)$$

then $\kappa = \tilde{\kappa}$, $(1-\kappa)F(A,v) = (1-\tilde{\kappa})F(\tilde{A},\tilde{v})$ and $\kappa F(A',v') = \tilde{\kappa} F(\tilde{A}',\tilde{v}')$.

We have $F(A,v) = v_{\mathrm{can}}(\mathfrak{t})$, $F(\tilde{A},\tilde{v}) = v_{\mathrm{can}}(\tilde{\mathfrak{t}})$ for toral H-subalgebras $\mathfrak{t}$, $\tilde{\mathfrak{t}}$. Furthermore, $F(A',v') \in X^{\Sigma}(\mathfrak{l})$ and $F(\tilde{A}',\tilde{v}') \in X^{\Sigma}(\tilde{\mathfrak{l}})$ for H-subalgebras $\mathfrak{l}$, $\tilde{\mathfrak{l}}$ with $\mathfrak{t} < \mathfrak{l}$ and $\tilde{\mathfrak{t}} < \tilde{\mathfrak{l}}$. If the intersection of the two spherical geodesic segments $\overline{F(A,v)F(A',v')}$ and $\overline{F(\tilde{A},\tilde{v})F(\tilde{A}',\tilde{v}')}$ contains an interior point (of one of the segments), then $\mathfrak{t} = \tilde{\mathfrak{t}}$ and $F(A',v') = F(\tilde{A}',\tilde{v}')$. If this intersection does not contain interior points, then $F(A,v) = F(\tilde{A},\tilde{v})$ or $F(A',v') = F(\tilde{A}',\tilde{v}')$ and the above claim follows.

Hence, if $(\pi_\Sigma((1-\kappa(\alpha))F(A(\alpha),v(\alpha)) + \kappa(\alpha)F(A'(\alpha),v'(\alpha))))_{\alpha\in\mathbb{N}}$ is a sequence in $X^{\Sigma}_{t,+}(r)_0$ converging to $\pi_\Sigma((1-\kappa(\infty))F(A(\infty),v(\infty)) + \kappa(\infty)F(A'(\infty),v'(\infty)))$, then we have

$$\kappa(\alpha) \to \kappa(\infty), \tag{5.44}$$

$$(1-\kappa(\alpha))F(A(\alpha),v(\alpha)) \to (1-\kappa(\infty))F(A(\infty),v(\infty)), \tag{5.45}$$

$$\kappa(\alpha)F(A'(\alpha),v'(\alpha)) \to \kappa(\infty)F(A'(\infty),v'(\infty)). \tag{5.46}$$

The next lemma guarantees that we can remove from the 0-essential part $X^{\Sigma}_{t,+}(r)_0$ of $X^{\Sigma}_t(r)$ a small neighborhood of the soul $P^{\Sigma}_t(r)$ of $W^{\Sigma}_t(r) \subset X^{\Sigma}_t(r)$ in $X^{\Sigma}_{t,+}(r)_0$ controlling both the homotopy type of the complement and its distance to $P^{\Sigma}_t(r)$. Recall that $\epsilon_{G/H} = -\frac{1}{2}c_{G/H}$ with $c_{G/H}$ as in Lemma 4.12.

Lemma 5.47. *For $r \in \{1,\dots,\ell-1\}$, there exists $\epsilon_r > 0$, such that the following holds true: If $v \in X^{\Sigma}_{t,+}(r)_0 \backslash X^{\Sigma}_{t,+}(r)_{\epsilon_r}$, then there exist a toral H-subalgebra $\mathfrak{t}$, such that $v \in B^{\Sigma\mathfrak{t}}_{\epsilon_{G/H}}(v_{\mathrm{can}}(\mathfrak{t}))$.*

Proof. Suppose that such an $\epsilon_r > 0$ does not exist. Then, we obtain a sequence $(v(\alpha))_{\alpha\in\mathbb{N}}$ in $X^{\Sigma}_{t,+}(r)_0$ with $v(\alpha) \in X^{\Sigma}_{t,+}(r)_0 \backslash X^{\Sigma}_{t,+}(r)_{\frac{1}{\alpha}} \cap W^{\Sigma}(\mathfrak{t}(\alpha)) \backslash B^{\Sigma\mathfrak{t}(\alpha)}_{\epsilon_{G/H}}(v_{\mathrm{can}}(\mathfrak{t}(\alpha)))$, where $\mathfrak{t}(\alpha)$ is a toral H-subalgebra. Since $X^{\Sigma}_{t,+}(r)_0$ is compact, we may assume $v(\alpha) \to v_\infty \in X^{\Sigma}_{t,+}(r)_0$. We conclude $v_\infty \in P^{\Sigma}_t(r)$, that is $v_\infty = v_{\mathrm{can}}(\mathfrak{t})$ for a toral H-subalgebra $\mathfrak{t}$. Contradiction. q.e.d.

Finally, we define the extended non-toral directions $X^{\Sigma}_{\mathrm{ent}}$ and the strong deformation retraction from $X^{\Sigma}_{\mathrm{ent}}$ to the non-toral directions X^{Σ}_{nt}. Let $\epsilon := \min\{\epsilon_1,\dots,\epsilon_{\ell-1},\frac{1}{2}\}$.

Step 1: If there exists no toral H-subalgebra, then we set $X^{\Sigma}_{\mathrm{ent}} = X^{\Sigma}_{nt} = W^{\Sigma}$.

If $X^{\Sigma}_t(1) \neq \emptyset$, but $X^{\Sigma}_t(1) \cap X^{\Sigma}_{nt} = \emptyset$, then $X^{\Sigma}_t(1)$ is a connected component of W^{Σ}. In this case, we set $X^{\Sigma}_{\mathrm{ent}} = W^{\Sigma} \backslash X^{\Sigma}_t(1) = X^{\Sigma}_{nt}$. If

$X_t^\Sigma(1) \cap X_{nt}^\Sigma \neq \emptyset$, then $\partial_+ X_t^\Sigma(1) \cap X_{nt}^\Sigma \neq \emptyset$ by Theorem 5.31. Let

$$\begin{aligned} H(1) : [0,1] \times X_{t,+}^\Sigma(1)_\epsilon &\to X_{t,+}^\Sigma(1)_\epsilon \; ; \\ \big(s, \pi_\Sigma \left((1-\kappa)F(A,v) + \kappa F(A',v')\right)\big) \\ \mapsto \pi_\Sigma \left((1-h(s,\kappa))F(A,v) + h(s,\kappa)F(A',v')\right), \end{aligned}$$

where $h(s,\kappa) = s+\kappa-s\kappa$. The map $H(1)$ is well-defined and continuous, since (5.44), (5.45) and (5.46) hold true. We obtain a continuous, strong deformation retraction from $X_{t,+}^\Sigma(1)_\epsilon$ to $X_{t,+}^\Sigma(1)_1 = \partial_+ X_t^\Sigma(1)$. Now, let

$$X_{\text{ent}}^\Sigma(1) = X_{t,+}^\Sigma(1)_\epsilon \cup X_{nt}^\Sigma.$$

Since $H(1)|_{[0,1]\times(X_{t,+}^\Sigma(1)_\epsilon \cap X_{nt}^\Sigma)} = \text{id}$, the map $H(1)$ can be extended to a continuous, semialgebraic map

$$H(1) : [0,1] \times X_{\text{ent}}^\Sigma(1) \to X_{\text{ent}}^\Sigma(1),$$

with $H(1)|_{[0,1]\times X_{nt}^\Sigma} = \text{id}$. Let $\pi(1) = H(1)_1$ denote the corresponding strong deformation retraction from $X_{\text{ent}}^\Sigma(1)$ to $\partial_+ X_t^\Sigma(1) \cup X_{nt}^\Sigma$. Since $H(1)$ is semialgebraic, so is $\pi(1)$.

Step 2: In case $X_t^\Sigma(2) = \emptyset$, we set $X_{\text{ent}}^\Sigma = X_{\text{ent}}^\Sigma(1)$. If $X_t^\Sigma(2) \neq \emptyset$, but $X_t^\Sigma(2) \cap X_{nt}^\Sigma = \emptyset$, then $(\pi(1))^{-1}(X_t^\Sigma(2))$ is a connected component of $X_{\text{ent}}^\Sigma(1)$. In this case, we set $X_{\text{ent}}^\Sigma = X_{\text{ent}}^\Sigma(1) \backslash (\pi(1))^{-1}(X_t^\Sigma(2))$. If $X_t^\Sigma(2) \cap X_{nt}^\Sigma \neq \emptyset$, then $\partial_+ X_t^\Sigma(2) \cap X_{nt}^\Sigma \neq \emptyset$.

Next, let the map $H(2,2) : [0,1] \times X_{t,+}^\Sigma(2)_\epsilon \to X_{t,+}^\Sigma(2)_\epsilon$ be defined as above. We obtain a strong deformation retraction from $X_{t,+}^\Sigma(2)_\epsilon$ to $X_{t,+}^\Sigma(2)_1 = \partial_+ X_t^\Sigma(2)$. Let

$$X_{\text{ent}}^\Sigma(2,2) = X_{t,+}^\Sigma(2)_\epsilon \cup X_{nt}^\Sigma \,.$$

As above, the map $H(2,2)$ can be extended to a continuous, semialgebraic map

$$H(2,2) : [0,1] \times X_{\text{ent}}^\Sigma(2,2) \to X_{\text{ent}}^\Sigma(2,2)\,.$$

Let $\pi(2) = H(2,2)_1$ denote the corresponding strong deformation retraction from $X_{\text{ent}}^\Sigma(2,2)$ to $\partial_+ X_t^\Sigma(2) \cup X_{nt}^\Sigma$. Then $\pi(2)$ is semialgebraic. Let

$$X_{\text{ent}}^\Sigma(2) = \pi(1)^{-1}(X_{\text{ent}}^\Sigma(2,2))\,.$$

Since $\pi(1)$ is continuous and semialgebraic, $X_{\text{ent}}^\Sigma(2)$ is compact and semialgebraic (see [**4**, Exercise 2.3.9]). With the help of $H(1)$ and $H(2,2)$, we obtain a continuous map

$$H(2) : [0,1] \times X^{\Sigma}_{\text{ent}}(2) \to X^{\Sigma}_{\text{ent}}(2),$$

which yields a strong deformation retraction from $X^{\Sigma}_{\text{ent}}(2)$ to $\partial_+ X^{\Sigma}_{t}(2) \cup X^{\Sigma}_{nt}$.

By induction, we obtain both the definition of the extended non-toral directions X^{Σ}_{ent} and a strong deformation retraction from X^{Σ}_{ent} to the non-toral directions X^{Σ}_{nt}.

Theorem 5.48. *The set of non-toral directions X^{Σ}_{nt} is compact, semi-algebraic and a strong deformation retract of the set of extended non-toral directions X^{Σ}_{ent}.*

Now, let us give a characterization of non-negative directions $v \in W^{\Sigma}$ which are not extended non-toral directions. Recall that we write $F(A, v) = v \in T_Q \mathcal{M}^G_1$. Let $v \in W^{\Sigma} \backslash X^{\Sigma}_{\text{ent}}$. Then, $v \subset W^{\Sigma}(\mathfrak{t}_1)$ for a toral H-subalgebra $\mathfrak{t}_1$. Either v is contained in $B^{\Sigma^{\mathfrak{t}_1}}_{\epsilon_{G/H}}(v_{\text{can}}(\mathfrak{t}_1)) \cup W^{\Sigma}_{-}(\mathfrak{t}_1)$ or it is not (where $W^{\Sigma}_{-}(\mathfrak{t}_1) = X^{\Sigma}(\mathfrak{t}_1) \backslash X^{\Sigma}_{+}(\mathfrak{t}_1)$). In the latter case, there exists an H-subalgebra $\mathfrak{k}_2$ with $\mathfrak{k}_2 > \mathfrak{t}_1$, such that $\pi_{\mathfrak{t}_1}(v) \in X^{\Sigma}(\mathfrak{k}_2)$ where $\pi_{\mathfrak{t}_1}$ denotes the radial projection defined in (5.36). Proceeding inductively, we obtain toral H-subalgebras $\mathfrak{t}_1, \dots, \mathfrak{t}_m$, such that either

$$w = (\pi_{\mathfrak{t}_{m-1}} \circ \cdots \circ \pi_{\mathfrak{t}_1})(v) \in B^{\Sigma^{\mathfrak{t}_m}}_{\epsilon_{G/H}}(v_{\text{can}}(\mathfrak{t}_m)) \cup W^{\Sigma}_{-}(\mathfrak{t}_m),$$

or there exists a non-toral H-subalgebra $\mathfrak{k}_{m+1}$, such that

$$(\pi_{\mathfrak{t}_m} \circ \cdots \circ \pi_{\mathfrak{t}_1})(v) \in X^{\Sigma}(\mathfrak{k}_{m+1}).$$

Corollary 5.49. *For every direction $v \in W^{\Sigma} \backslash X^{\Sigma}_{\text{ent}}$, there exists a flag $\mathfrak{t}_1 < \cdots < \mathfrak{t}_m$ of toral H-subalgebras, such that either*

$$v \in \operatorname{conv}^{\Sigma}(v_{\text{can}}(\mathfrak{t}_m) + x, v_{\text{can}}(\mathfrak{t}_{m-1}), \dots, v_{\text{can}}(\mathfrak{t}_1)), \tag{5.50}$$

where $v_{\text{can}}(\mathfrak{t}_m) + x \in B^{\Sigma^{\mathfrak{t}_m}}_{\epsilon_{G/H}}(v_{\text{can}}(\mathfrak{t}_m))$, $x \in T_Q \mathcal{M}^G_1$ and $\|x\| < \epsilon_{G/H} = -\frac{1}{2} c_{G/H}$, or

$$v \in \operatorname{conv}^{\Sigma}(\tilde{x}, v_{\text{can}}(\mathfrak{t}_{m-1}), \dots, v_{\text{can}}(\mathfrak{t}_1)), \tag{5.51}$$

where $\tilde{x} \in W^{\Sigma}_{-}(\mathfrak{t}_m) = X^{\Sigma}(\mathfrak{t}_m) \backslash X^{\Sigma}_{+}(\mathfrak{t}_m)$.

Proof. From Lemma 5.47 and the definition of X^{Σ}_{ent}, it follows that if $(\pi_{\mathfrak{t}_m} \circ \cdots \circ \pi_{\mathfrak{t}_1})(v) \in X^{\Sigma}(\mathfrak{k}_{m+1})$, then there exists $1 \le i \le m-1$ with $(\pi_{\mathfrak{t}_i} \circ \cdots \circ \pi_{\mathfrak{t}_1})(v) \in B^{\Sigma^{\mathfrak{t}_{i+1}}}_{\epsilon_{G/H}}(v_{\text{can}}(\mathfrak{t}_{i+1}))$. q.e.d.

5.7. Scalar curvature estimates. In this section, we investigate, which directions in W^{Σ} are not positive, that is, we examine for which $v \in W^{\Sigma}$ $\mathrm{sc}(\gamma_v(t))$ does not tend to $+\infty$ for $t \to +\infty$. The scalar curvature estimates provided below are quite involved. The key result is Proposition 5.58. There, we are able to relate a priori independent structure constants to each other.

We begin with the main results of this section. Recall that the non-positive number $-b_{G/H}$ is nothing, but the trace of the Killing form. Let $U^{\Sigma}_{\text{ent}}(\delta)$ denote the open δ-neighborhood of the extended non-toral directions X^{Σ}_{ent} in Σ.

Theorem 5.52. *For $\delta > 0$ there exists $t_0(\delta) > 0$, such that for all $v \in \Sigma\backslash U^{\Sigma}_{\text{ent}}(\delta)$ we have $\mathrm{sc}(\gamma_v(t)) \leq \frac{1}{2} b_{G/H}$ for all $t \geq t_0(\delta)$. Furthermore, $\lim_{t\to+\infty} \mathrm{sc}(\gamma_v(t)) \leq 0$.*

Proof. Suppose, there exists $\delta > 0$ and a sequence $(v(\alpha), t(\alpha))_{\alpha\in\mathbb{N}}$ with $v(\alpha) \in \Sigma\backslash U^{\Sigma}_{\text{ent}}(\delta)$, $t(\alpha) \geq \alpha$ and $\mathrm{sc}(\gamma_{v(\alpha)}(t(\alpha))) > \frac{1}{2} b_{G/H}$. Let f_α be a good decomposition of $\gamma_{v(\alpha)}$. We may assume $v(\alpha) \to v \in \Sigma$ and $f_\alpha \to f$ where f is a good decomposition of γ_v. From Theorem 5.18, it follows $v \in W^{\Sigma}$. Hence, $v \in W^{\Sigma}\backslash U^{\Sigma}_{\text{ent}}(\delta)$. By Corollary 5.49, we are in position to apply Theorem 5.54. This yields a contradiction. q.e.d.

Corollary 5.53. *Let G/H be a compact homogeneous space. If G/H admits no non-toral H-subalgebras, then the scalar curvature functional $\mathrm{sc} : \mathcal{M}^G_1 \to \mathbb{R}$ is bounded from above.*

The following theorem is our main result towards scalar curvature estimates for homogeneous metrics on compact homogeneous spaces.

Theorem 5.54. *Let G/H be a compact homogeneous space. Let $\mathfrak{t}_1 < \cdots < \mathfrak{t}_m$ be a flag of toral H-subalgebras. Let $v \in W^{\Sigma}$ and suppose, that either (5.50) or (5.51) hold true. Furthermore, let $(v(\alpha))_{\alpha\in\mathbb{N}}$ be a sequence in Σ converging to v. Let f_α be a good decomposition of $\gamma_{v(\alpha)}$. Suppose that $(f_\alpha)_{\alpha\in\mathbb{N}}$ converges to a good decomposition f of γ_v. Then, there exists $t_0 > 0$, such that $\mathrm{sc}(\gamma_{v(\alpha)}(t)) \leq \frac{1}{2} b_{G/H}$ for all $t \geq t_0$ and all but finitely many α. Furthermore, $\lim_{t\to+\infty} \mathrm{sc}(\gamma_{v(\alpha)}(t)) \leq 0$ for α large.*

Proof.
Case 1: Let $v \in \mathrm{conv}^{\Sigma}(v_{\text{can}}(\mathfrak{t}_m) + x, v_{\text{can}}(\mathfrak{t}_{m-1}), \ldots, v_{\text{can}}(\mathfrak{t}_1))$ where $x \in T_Q\mathcal{M}^G_1$ with $\|x\| < \epsilon_{G/H}$ (cf. (5.50)). As in the proof of Lemma 5.5, it follows that $(v_{\text{can}}(\mathfrak{t}_m) + x, v_{\text{can}}(\mathfrak{t}_{m-1}), \ldots, v_{\text{can}}(\mathfrak{t}_1))$ are linearly independent. It follows that the convex hull $\mathrm{conv}^E(v_{\text{can}}(\mathfrak{t}_m)+x, \ldots, v_{\text{can}}(\mathfrak{t}_1))$

of the vectors $v_{\text{can}}(\mathfrak{t}_m) + x, \ldots, v_{\text{can}}(\mathfrak{t}_1) \in T_Q\mathcal{M}_1^G$ is a slice for $\operatorname{conv}^{\Sigma}(v_{\text{can}}(\mathfrak{t}_m) + x, \ldots, v_{\text{can}}(\mathfrak{t}_1))$ (cf. Proposition 5.10). Hence, $v = \pi_{\Sigma}(v^E)$ where

$$v^E = \lambda_m(v_{\text{can}}(\mathfrak{t}_m) + x) + \underbrace{\sum_{j=1}^{m-1} \lambda_j v_{\text{can}}(\mathfrak{t}_j)}_{=:y} \in T_Q\mathcal{M}_1^G \backslash \{0\}$$

with $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^{m} \lambda_j = 1$. By induction over the length of the toral flags $\mathfrak{t}_1 < \cdots < \mathfrak{t}_m$, we may assume $\lambda_m > 0$. Another induction shows that $\mathfrak{m}_{\mathfrak{t}_m}$ and $\mathfrak{m}_{\mathfrak{t}_m}^{\perp}$ equal direct sums of eigenspaces of v. Since $\lambda_m > 0$ and $v_{\text{can}}(\mathfrak{t}_m) + x \in W^{\Sigma}(\mathfrak{t}_m)$, we conclude that the eigenvalues of $v|_{\mathfrak{m}_{\mathfrak{t}_m}}$ are strictly less than the eigenvalues of $v|_{\mathfrak{m}_{\mathfrak{t}_m}^{\perp}}$ (cf. Proposition 5.6). Thus, $\mathfrak{t}_m$ is f-adapted for the good decomposition f of γ_v given above. Let $I = I_1^{\mathfrak{t}_m}(f)$ and $I^C = \{1, \ldots, \ell\} \backslash I$. We obtain $v_k^E < v_j^E$ for $k \in I$ and $j \in I^C$.

Let $v^E(\alpha) = v^E + w(\alpha)$ with $w(\alpha) \in T_Q\mathcal{M}_1^G$ and $\pi_{\Sigma}(v^E(\alpha)) = v(\alpha)$. In particular, $v^E(\alpha) \neq 0$. Since $v(\alpha) \to v$, we may assume $w(\alpha) \to 0$. By $v_k^E < v_j^E$, $k \in I$, $j \in I^C$, we are in position to apply Lemma 5.55. This yields

$$\operatorname{sc}(\gamma_{v^E(\alpha)}(t)) \leq \frac{1}{2} \sum_{i \in I^C} d_i b_i e^{t(-\lambda_m(v_2(\mathfrak{t}_m)+x_i) - \sum_{j=1}^{m-1} \lambda_j v_2(\mathfrak{t}_j) + \tilde{w}_i(\alpha))}$$

for $t \geq t_0$ and α large, where $\tilde{w}_i(\alpha) \to 0$. Since $v_2(\mathfrak{t}_m) + x_i \geq \epsilon_{G/H}$, we get

$$\operatorname{sc}(\gamma_{v_\alpha}(t)) \leq \frac{1}{2} \sum_{i \in I^C} d_i b_i e^{t(-\frac{1}{2}\epsilon_{G/H})} \leq \frac{1}{2} b_{G/H}$$

for $t \geq t_0$ and α large.

Case 2: Let $v \in \operatorname{conv}^{\Sigma}(\tilde{x}, v_{\text{can}}(\mathfrak{t}_{m-1}), \ldots, v_{\text{can}}(\mathfrak{t}_1))$ with $\tilde{x} \in W_-^{\Sigma}(\mathfrak{t}_m)$ (cf. (5.51)). We may assume that $\tilde{x} \notin B_{\epsilon_{G/H}}^{\Sigma \mathfrak{t}_m}(v_{\text{can}}(\mathfrak{t}_m))$, since otherwise we are covered by case 1. Let

$$x = \pi_{\mathfrak{t}_m}(\tilde{x}) \in \partial_{\text{rad}} X^{\Sigma}(\mathfrak{t}_m) \cap W_-^{\Sigma}(\mathfrak{t}_m)$$

(see Section 5.5). Consequently, $v \in \operatorname{conv}^{\Sigma}(x, v_{\text{can}}(\mathfrak{t}_m), \ldots, v_{\text{can}}(\mathfrak{t}_1))$ with $x \in W_-^{\Sigma}(\mathfrak{t}_m)$. As in the proof of Lemma 5.5, it follows that $(x, v_{\text{can}}(\mathfrak{t}_m), v_{\text{can}}(\mathfrak{t}_{m-1}), \ldots, v_{\text{can}}(\mathfrak{t}_1))$ are linearly independent. We get

$v = \pi_\Sigma(v^E)$ where

$$v^E = \lambda_{m+1} x + \sum_{j=1}^{m} \lambda_j v_{\rm can}(\mathfrak{t}_j) \in T_Q \mathcal{M}_1^G \backslash \{0\}$$

with $0 \le \lambda_j \le 1$, $\sum_{j=1}^{m+1} \lambda_j = 1$ and $\lambda_{m+1} > 0$. As in case 1, it follows that $\mathfrak{t}_m$ is f-adapted for the good decomposition f of γ_v given above. Hence, f is a good decomposition of $\gamma_{v_{\rm can}(\mathfrak{t}_m)}$. This implies that f is a good decomposition of γ_x as well. Let I and I^C be defined as above. We get $x_k < x_j$ for $k \in I$ and $j \in I^C$, and hence, $v_k^E < v_j^E$ for $k \in I$ and $j \in I^C$.

Let $v^E(\alpha) = v^E + w(\alpha)$ with $w(\alpha) \in T_Q\mathcal{M}_1^G$ and $\pi_\Sigma(v^E(\alpha)) = v(\alpha)$. Since $v(\alpha) \to v$, we may assume $w(\alpha) \to 0$. Suppose $1 \in I_1^x(f) = I$ and $i_m + 1 \in I_2^x(f) \subset I^C$. Since $x \in \partial_{\rm rad} X^\Sigma(\mathfrak{t}_m)$ and $x_k < x_j$, $k \in I$, $j \in I^C$ holds true, by Corollary 5.33, there exists $(i_0, j_0, k_0) \in \{I^C\}^3$ with

$$[i_0 j_0 k_0]_f > 0 \quad \text{and} \quad x_{i_0} - x_{j_0} - x_{k_0} + x_{i_m+1} = x_{i_m+1} - x_1 > 0\,.$$

By $v_k^E < v_j^E$, $k \in I$, $j \in I^C$, we can apply Lemma 5.55 and obtain

$$\begin{aligned}
&4 \cdot \mathrm{sc}(\gamma_{v^E_\alpha}(t)) \\
&\le \sum_{i \in I^C} 2 d_i b_i e^{t(-\sum_{j=1}^m \lambda_j v_2(\mathfrak{t}_j) - \lambda_{m+1} x_i - \tilde{w}_i(\alpha))} \\
&\quad - \sum_{i,j,k \in I^C} [ijk]_{f_\alpha} e^{t(-\sum_{j=1}^m \lambda_j v_2(\mathfrak{t}_j) + \lambda_{m+1}(x_i - x_j - x_k) + \tilde{w}_i(\alpha) - \tilde{w}_j(\alpha) - \tilde{w}_k(\alpha))} \\
&\le e^{-t(\sum_{j=1}^m \lambda_j v_2(\mathfrak{t}_j) + \lambda_{m+1} x_{i_m+1})} \\
&\qquad \cdot \Big(\sum_{i \in I^C} 2 d_i b_i e^{-t(\lambda_{m+1}(\overbrace{x_i - x_{i_m+1}}^{\ge 0}) + \tilde{w}_i(\alpha))} \\
&\qquad\quad - [i_0 j_0 k_0]_{f_\alpha} e^{t(\lambda_{m+1}(x_{i_m+1} - x_1) + \tilde{w}_{i_0}(\alpha) - \tilde{w}_{j_0}(\alpha) - \tilde{w}_{k_0}(\alpha))} \Big) \\
&\le e^{t(-\sum_{j=1}^m \lambda_j v_2(\mathfrak{t}_j) - \lambda_{m+1} x_{i_m+1})} \cdot \Big(\sum_{i \in I^C} 2 d_i b_i e^{t(-\tilde{w}_i(\alpha))} \\
&\qquad\quad - [i_0 j_0 k_0]_{f_\alpha} e^{t(\lambda_{m+1}(x_{i_m+1} - x_1) + \tilde{w}_{i_0}(\alpha) - \tilde{w}_{j_0}(\alpha) - \tilde{w}_{k_0}(\alpha))} \Big)
\end{aligned}$$

for $t \geq t_0$ and α large, where $\tilde{w}_i(\alpha) \to 0$. Since the structure constants $[ijk]_f$ depend continuously on f and since $f_\alpha \to f$, we get $\mathrm{sc}(\gamma_{v_\alpha}(t)) < 0$ for $t \geq t_0$ and α large. q.e.d.

Lemma 5.55. *Let $\mathfrak{t}$ be a toral H-subalgebra, $v \in \Sigma$ and let f be a good decomposition of γ_v. Suppose $\mathfrak{t}$ is f-adapted. Let $I = I_1^{\mathfrak{t}}(f)$, $I^C := \{1, \dots, \ell\} \backslash I$ and $(v(\alpha))_{\alpha \in \mathbb{N}}$ be a sequence in Σ with $v(\alpha) \to v$. Let f_α be a good decomposition of $\gamma_{v(\alpha)}$ and suppose $f_\alpha \to f$. If for all $k \in I$ and $j \in I^C$ $v_k < v_j$ holds true, then there exists $t_0 > 0$, such that for all $t \geq t_0$ and for all, but finitely many, $\alpha \in \mathbb{N}$, we have*

$$\begin{aligned} &\mathrm{sc}(\gamma_{v_\alpha}(t)) \\ &\leq \frac{1}{2} \sum_{i \in I^C} d_i b_i e^{t(-v_i(\alpha))} - \frac{1}{4} \sum_{i,j,k \in I^C} [ijk]_{f_\alpha} e^{t(v_i(\alpha) - v_j(\alpha) - v_k(\alpha))}. \end{aligned} \tag{5.56}$$

Proof. By Lemma 4.16, we have $d_i b_i = 2 d_i c_i + \sum_{j,k \in I \cup I^C} [ijk]_{f_\alpha}$ for $1 \leq i \leq \ell$. Since $\mathfrak{t}$ is a toral H-subalgebra, we have $\mathfrak{m}_{\mathfrak{t}} \leq \mathfrak{m}_0$, hence $\mathfrak{m}_{\mathfrak{t}}$ is almost trivial. For $i \in I$, we conclude

$$d_i b_i = \sum_{j,k \in I} [ijk]_{f_\alpha} + 2 \sum_{j \in I, k \in I^C} [ijk]_{f_\alpha} + \sum_{j,k \in I^C} [ijk]_{f_\alpha}.$$

With the help of (4.14) we obtain

$$\begin{aligned} &\mathrm{sc}(\gamma_{v(\alpha)}(t)) \\ &= \frac{1}{2} \sum_{i \in I \cup I^C} d_i b_i e^{t(-v_i(\alpha))} - \frac{1}{4} \sum_{i,j,k \in I \cup I^C} [ijk]_{f_\alpha} e^{t(v_i(\alpha) - v_j(\alpha) - v_k(\alpha))} \\ &= \frac{1}{2} \sum_{i \in I} e^{t(-v_i(\alpha))} \cdot \Bigg\{ \sum_{j,k \in I} [ijk]_{f_\alpha} \left(1 - \frac{1}{2} e^{t(v_j(\alpha) - v_k(\alpha))} \right) \\ &\quad + \sum_{j \in I, k \in I^C} [ijk]_{f_\alpha} \left(2 - \frac{1}{2} e^{t(v_k(\alpha) - v_j(\alpha))} \right) \\ &\quad + \sum_{j,k \in I^C} [ijk]_{f_\alpha} \left(1 - \frac{1}{2} e^{t(v_j(\alpha) - v_k(\alpha))} - \frac{1}{2} e^{t(v_k(\alpha) - v_j(\alpha))} \right) \\ &\quad - \sum_{j \in I, k \in I^C} [ijk]_{f_\alpha} e^{t(v_j(\alpha) - v_k(\alpha))} \end{aligned}$$

$$- \frac{1}{2} \sum_{j,k \in I^C} [ijk]_{f_\alpha} e^{t(2v_i(\alpha) - v_j(\alpha) - v_k(\alpha))} \Bigg\}$$
$$+ \frac{1}{2} \sum_{i \in I^C} d_i b_i e^{t(-v_i(\alpha))} - \frac{1}{4} \sum_{i,j,k \in I^C} [ijk]_{f_\alpha} e^{t(v_i(\alpha) - v_j(\alpha) - v_k(\alpha))}.$$

Note that it is enough to control the first term by the second. By assumption, there exists $\epsilon > 0$, such that $v_k(\alpha) - v_j(\alpha) > \epsilon$ for α large and $j \in I, k \in I^C$. Now, suppose that there exists a positive constant C with

$$\sum_{j,k \in I} [ijk]_{f_\alpha} \leq C \cdot \sum_{j \in I, k \in I^C} [ijk]_{f_\alpha} \tag{5.57}$$

for $i \in I$ and $\alpha \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{j,k \in I} [ijk]_{f_\alpha} \left(1 - \frac{1}{2} e^{t(v_j(\alpha) - v_k(\alpha))}\right) &+ \sum_{j \in I, k \in I^C} [ijk]_{f_\alpha} \left(2 - \frac{1}{2} e^{t(v_k(\alpha) - v_j(\alpha))}\right) \\ &\leq \sum_{j,k \in I} [ijk]_{f_\alpha} + \sum_{j \in I, k \in I^C} [ijk]_{f_\alpha} \left(2 - \frac{1}{2} e^{t\epsilon}\right) \\ &\leq \sum_{j \in I, k \in I^C} [ijk]_{f_\alpha} \left(C + 2 - \frac{1}{2} e^{t\epsilon}\right) \end{aligned}$$

and the desired estimate follows.

It remains to prove the estimate (5.57). Recall $\mathfrak{m}_{\mathfrak{t}} \leq \mathfrak{m}_0$. Since $\mathfrak{m}_0$ is a compact subalgebra of $\mathfrak{g}$ (Lemma 4.27) and since by definition $\mathfrak{m}_0$ sums up almost trivial, isotypical summands of $\mathfrak{m}$, (5.57) can be considered an estimate for a sequence of $\mathrm{Ad}(H)$-invariant decompositions $(f_\alpha)_{\alpha \in \mathbb{N}}$ of $\mathfrak{m}_0$. By the very definition of $[ijk]_{f_\alpha}$, (5.57) is nothing but an estimate for a sequence $(\hat{e}_1(\alpha), \ldots, \hat{e}_{\dim\, \mathfrak{m}_0}(\alpha))_{\alpha \in \mathbb{N}}$ of Q-orthonormal bases of $\mathfrak{m}_0$ converging to a Q-orthonormal basis $(\hat{e}_1, \ldots, \hat{e}_{\dim\, \mathfrak{m}_0})$ of $\mathfrak{m}_0$. Consequently, we obtain the claim by Proposition 5.58. q.e.d.

The following Łojasiewicz inequality is the key estimate for the proof of Theorem 5.54. Note that standard estimates cannot be applied to the above situation (cf. [**4**, Proposition 2.3.11]).

Proposition 5.58. *Let G' be a compact, connected Lie group. Let Q' be a biinvariant metric on G' and let $(\hat{e}_1, \ldots, \hat{e}_{\dim\, \mathfrak{g}'})$ denote a Q'-orthonormal basis of $\mathfrak{g}'$. Furthermore, let $I \subset \{1, \ldots, \dim\, \mathfrak{g}'\}$ and $I^C = \{1, \ldots, \dim\, \mathfrak{g}'\} \backslash I$ with $1 \in I$. If $\oplus_{i \in I} \langle \hat{e}_i \rangle$ is an abelian subalgebra of $\mathfrak{g}'$*

and if $(\hat{e}_1(\alpha), \dots, \hat{e}_{\dim\, \mathfrak{g}'}(\alpha))_{\alpha\in\mathbb{N}}$ *is a sequence of* Q'*-orthonormal bases of* $\mathfrak{g}'$ *converging to* $(\hat{e}_1, \dots, \hat{e}_{\dim\, \mathfrak{g}'})$*, then there exists* $C > 0$*, such that*

$$\sum_{j,k\in I} Q'([\hat{e}_1(\alpha), \hat{e}_j(\alpha)], \hat{e}_k(\alpha))^2 \leq C \cdot \sum_{j\in I, k\in I^C} Q'([\hat{e}_1(\alpha), \hat{e}_j(\alpha)], \hat{e}_k(\alpha))^2$$

for all but finitely many α*.*

Proof. First, note that if $I^C = \emptyset$, then $\mathfrak{g}'$ is abelian and the above claim is obviously true. Hence, we may assume that $I^C \neq \emptyset$ and that $\mathfrak{g}'$ is not abelian. Now, suppose that such a constant $C > 0$ does not exist. (In what follows, we will not distinguish between the original sequence and its subsequences in order to make notation as simple as possible. Furthermore, we will pass, whenever convenient, to a subsequence, without mentioning this explicitly.) We get

$$\begin{aligned} &\sum_{j,k\in I} Q'([\hat{e}_1(\alpha), \hat{e}_j(\alpha)], \hat{e}_k(\alpha))^2 \\ &\qquad > \sum_{j\in I, k\in I^C} Q'([\hat{e}_1(\alpha), \hat{e}_j(\alpha)], \hat{e}_k(\alpha))^2 \cdot g(\alpha) \end{aligned} \tag{5.59}$$

where $g : \mathbb{R} \to \mathbb{R}$ with $\lim_{\alpha\to\infty} g(\alpha) = +\infty$.

Suppose $\hat{e}_1(\alpha) \in \mathfrak{t}(\alpha)$, where $\mathfrak{t}(\alpha)$ denotes a maximal abelian subalgebra of $\mathfrak{g}'$. The subalgebras $\mathfrak{t}(\alpha)$ converge to a maximal abelian subalgebra $\mathfrak{t}(\infty)$ of $\mathfrak{g}'$. Since maximal abelian subalgebras of $\mathfrak{g}'$ are conjugate, there exists a sequence $(g'_\alpha)_{\alpha\in\mathbb{N}}$ of group elements in G' with $\mathrm{Ad}(g'_\alpha)(\mathfrak{t}(\alpha)) = \mathfrak{t}(\infty)$ and $\lim_{\alpha\to\infty} g'_\alpha \to e$. We gain a sequence of Q'-orthonormal bases of $\mathfrak{g}'$ converging to $(\hat{e}_1, \dots, \hat{e}_{\dim\, \mathfrak{g}'})$ with $\hat{e}_1(\alpha) \in \mathfrak{t}(\infty)$ for $\alpha \in \mathbb{N}$, again denoted by $(\hat{e}_1(\alpha), \dots, \hat{e}_{\dim\, \mathfrak{g}'}(\alpha))_{\alpha\in\mathbb{N}}$. Since (5.59) is $\mathrm{Ad}(G')$-invariant, this sequence satisfies (5.59) as well.

Step 1: Let $j \in I$. Since $\hat{e}_1(\alpha) \in \mathfrak{t}(\infty)$, we have

$$[\hat{e}_1(\alpha), \hat{e}_j(\alpha)] \;=\; \sum_{k\in I\setminus\{1,j\}} a_{jk}(\alpha)\hat{e}_k(\alpha) + Z_j(\alpha) \;\in\; \mathfrak{t}(\infty)^\perp,$$

where $Z_j(\alpha) \in \oplus_{m\in I^C}\langle \hat{e}_m(\alpha)\rangle = (\oplus_{m\in I}\langle \hat{e}_m(\alpha)\rangle)^\perp$. From (5.59), it follows $|I| \geq 3$. Now, suppose $\{2, 3\} \subset I$. Then, we may assume $|a_{23}(\alpha)| \geq |a_{jk}(\alpha)|$ for $j, k \in I$ and $\alpha \in \mathbb{N}$. From (5.59), we get $|a_{23}(\alpha)| > 0$. Thus, $\lim_{\alpha\to\infty} \frac{|a_{jk}(\alpha)|}{|a_{23}(\alpha)|} \in [0, 1]$. Note that $|a_{23}(\alpha)|$ is bounded and that $\lim_{\alpha\to\infty} a_{23}(\alpha) = 0$, since $\oplus_{i=1}^{|I|}\langle \hat{e}_i\rangle$ is abelian. Again, by (5.59), we may assume that for all $j \in I\backslash\{1\}$, there exists $k \in I$ with $\lim_{\alpha\to\infty} \frac{|a_{jk}(\alpha)|}{|a_{23}(\alpha)|} > 0$.

Step 2: For $j \in I\backslash\{1\}$, there exists $k(j) \in I\backslash\{1, j\}$ with $|a_{jk(j)}(\alpha)| \geq |a_{jk}(\alpha)|$ for $k \in I$ and $\alpha \in \mathbb{N}$. Since $\lim_{\alpha\to\infty} \frac{|a_{jk(j)}(\alpha)|}{|a_{23}(\alpha)|} > 0$, we have $|a_{jk(j)}(\alpha)| > 0$ for $\alpha \in \mathbb{N}$. We obtain

$$[\hat{e}_1(\alpha), \hat{e}_j(\alpha)] = a_{jk(j)}(\alpha) \cdot (E_j(\alpha) + X_j(\alpha)) \in \mathfrak{t}(\infty)^{\perp}, \tag{5.60}$$

with

$$E_j(\alpha) = \sum_{k\in I\backslash\{1,j\}} \frac{a_{jk}(\alpha)}{a_{jk(j)}(\alpha)} \hat{e}_k(\alpha) \quad \text{ and } \quad X_j(\alpha) = \frac{Z_j(\alpha)}{a_{jk(j)}(\alpha)}.$$

From (5.59), we deduce $\lim_{\alpha\to\infty} X_j(\alpha) = 0$ and since $\oplus_{i=2}^{|I|}\langle \hat{e}_i\rangle$ is abelian, we get $\lim_{\alpha\to\infty} a_{jk(j)}(\alpha) = 0$. Let $\mathfrak{c}(\hat{e}_1) \leq \mathfrak{g}'$ denote the centralizer of $\hat{e}_1$. Then, we obtain $\mathfrak{t}(\infty) \subset \mathfrak{t}(\infty) \oplus \langle \hat{e}_2, \dots, \hat{e}_{|I|}\rangle \subset \mathfrak{c}(\hat{e}_1)$. Since

$$\lim_{\alpha\to\infty} E_j(\alpha) =: \underbrace{E_j(\infty)}_{\neq 0} \in \mathfrak{c}(\hat{e}_1) \cap \mathfrak{t}(\infty)^{\perp}, \tag{5.61}$$

we conclude

$$\mathfrak{t}(\infty) \subsetneq \mathfrak{t}(\infty) \oplus \langle \hat{e}_2, \dots, \hat{e}_{|I|}\rangle \subset \mathfrak{c}(\hat{e}_1). \tag{5.62}$$

Step 3: We will show $\mathfrak{t}(\infty) \subsetneq \mathfrak{c}(\mathfrak{t}(\infty))$, which is clearly impossible. To see this, let us decompose $\hat{e}_1(\alpha)$ and $\hat{e}_j(\alpha)$, $j \in I\backslash\{1\}$, in the following manner: $\hat{e}_1(\alpha) = d(\alpha) \cdot \hat{e}_1 + r_1^2(\alpha)$, where $r_1^2(\alpha) \in \mathfrak{t}(\infty)$, $r_1^2(\alpha) \perp \hat{e}_1$, and $\hat{e}_j(\alpha) = t_j(\alpha) + u_j^2(\alpha) + Y_j^2(\alpha)$ where $t_j(\alpha) \in \mathfrak{t}(\infty)$, $u_j^2(\alpha) \in \mathfrak{c}(\hat{e}_1) \cap \mathfrak{t}(\infty)^{\perp}$ and $Y_j^2(\alpha) \in \mathfrak{c}(\hat{e}_1)^{\perp}$. We obtain

$$[\hat{e}_1(\alpha), \hat{e}_j(\alpha)] = \underbrace{[r_1^2(\alpha), u_j^2(\alpha)]}_{\in \mathfrak{c}(\hat{e}_1)\cap\mathfrak{t}(\infty)^{\perp}} + \underbrace{[\hat{e}_1(\alpha), Y_j^2(\alpha)]}_{\in \mathfrak{c}(\hat{e}_1)^{\perp}}$$

and from (5.60), we deduce

$$[r_1^2(\alpha), u_j^2(\alpha)] = a_{jk(j)}(\alpha) \cdot \underbrace{\mathrm{pr}_{\mathfrak{c}(\hat{e}_1)}(E_j(\alpha) + X_j(\alpha))}_{=: \tilde{E}_j^2(\alpha)} \in \mathfrak{c}(\hat{e}_1) \cap \mathfrak{t}(\infty)^{\perp}.$$

Equation (5.61) implies $\lim_{\alpha\to\infty} \tilde{E}_j^2(\alpha) = E_j(\infty) \neq 0$ and we deduce $r_1^2(\alpha) \neq 0$ for $\alpha \in \mathbb{N}$.

Set $\hat{e}_1^2(\alpha) := \frac{r_1^2(\alpha)}{\|r_1^2(\alpha)\|}$. Then, $\lim_{\alpha\to\infty} \hat{e}_1^2(\alpha) = \hat{e}_1^2 \in \mathfrak{t}(\infty)$ with $\hat{e}_1^2 \perp \hat{e}_1$ and $\|\hat{e}_1^2\| = 1$. Next, for $j \in I\backslash\{1\}$ let $\hat{e}_j^2(\alpha) := t_j(\alpha) + u_j^2(\alpha) \in \mathfrak{c}(\hat{e}_1)$. Note that by (5.62), it follows $\lim_{\alpha\to\infty} \hat{e}_j^2(\alpha) = \hat{e}_j$. We obtain

$$[\hat{e}_1^2(\alpha), \hat{e}_j^2(\alpha)] = a_{jk(j)}^2(\alpha) \cdot (E_j^2(\alpha) + X_j^2(\alpha)) \in \mathfrak{c}(\hat{e}_1) \cap \mathfrak{t}(\infty)^\perp\,, \tag{5.63}$$

where

$$E_j^2(\alpha) := \mathrm{pr}_{\oplus_{i=2}^{|I|}\langle \hat{e}_i^2(\alpha)\rangle}(\tilde{E}_j^2(\alpha)) \in \mathfrak{c}(\hat{e}_1),$$
$$X_j^2(\alpha) := \mathrm{pr}_{(\oplus_{i=2}^{|I|}\langle \hat{e}_i^2(\alpha)\rangle)^\perp}(\tilde{E}_j^2(\alpha)) \in \mathfrak{c}(\hat{e}_1)$$

and $a_{jk(j)}^2(\alpha) := \frac{a_{jk(j)}(\alpha)}{\|r_1^2(\alpha)\|}$. Since $\lim_{\alpha\to\infty} \tilde{E}_j^2(\alpha) = E_j(\infty) \in \oplus_{i=2}^{|I|}\langle \hat{e}_i\rangle$, we get $\lim_{\alpha\to\infty} X_j^2(\alpha) = 0$, and since $\oplus_{i=2}^{|I|}\langle \hat{e}_i\rangle$ is abelian, it follows $\lim_{\alpha\to\infty} a_{jk(j)}^2(\alpha) = 0$. This yields $\mathfrak{t}(\infty) \subset \mathfrak{t}(\infty) \oplus \langle \hat{e}_2, \ldots, \hat{e}_{|I|}\rangle \subset \mathfrak{c}(\hat{e}_1^2)$. Since

$$\lim_{\alpha\to\infty} E_j^2(\alpha) = \underbrace{E_j(\infty)}_{\neq 0} \in \mathfrak{c}(\hat{e}_1) \cap \mathfrak{t}(\infty)^\perp, \tag{5.64}$$

we conclude with the help of (5.62)

$$\mathfrak{t}(\infty) \subsetneq \mathfrak{t}(\infty) \oplus \langle \hat{e}_2, \ldots, \hat{e}_{|I|}\rangle \subset \mathfrak{c}(\hat{e}_1) \cap \mathfrak{c}(\hat{e}_1^2). \tag{5.65}$$

By induction, we obtain the claim. q.e.d.

6. Homogeneous spaces of finite type

The scalar curvature estimates provided in Theorem 5.52 show that $W^\Sigma \backslash X_{\mathrm{ent}}^\Sigma$ does not contain positive directions. Since the extended non-toral directions X_{ent}^Σ and the non-toral directions X_{nt}^Σ are homotopy equivalent, we restrict our attention to positive directions in X_{nt}^Σ. The homotopy type of such directions and a Lie theoretical description of it is of great interest, since this yields further Lie theoretically defined invariants of compact homogeneous spaces guaranteeing existence of homogeneous Einstein metrics.

We are not pursuing this here in full generality, but turn to homogeneous spaces G/H of finite type to whom we associate a simplicial complex $\Delta_{G/H}^{\min}$. This simplicial complex is homeomorphic to the nerve $X_{G/H}^\Sigma$ of G/H, defined in Section 1, and moreover, homotopy equivalent to the non-toral directions X_{nt}^Σ (Theorem 6.10).

For homogeneous spaces admitting at most finitely many non-toral H-subalgebras, we will also introduce the extended simplicial complex $\hat{\Delta}^{\min}_{G/H}$ of G/H. The simplicial complexes $\Delta^{\min}_{G/H}$ and $\hat{\Delta}^{\min}_{G/H}$ are homotopy equivalent, but in general not homeomorphic.

Definition 6.1 (Homogeneous spaces of finite type)**.** We call a compact homogeneous space G/H a homogeneous space of finite type, if there exist at most finitely many minimal non-toral H-subalgebras.

The most elementary examples of homogeneous spaces of finite type are homogeneous spaces G/H admitting only one (iso-ordered) decomposition f of $\mathfrak{m}$. Obviously, these spaces admit only finitely many H-subalgebras. Examples are homogeneous spaces G/H of equal rank, that is $\operatorname{rk} G = \operatorname{rk} H$. But there exist also many homogeneous spaces of finite type which admit infinitely many decompositions f of $\mathfrak{m}$.

Lemma 6.2. *Let G/H be a compact homogeneous space. If the normalizer $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$ is contained in $\mathfrak{h}$, then G/H is of finite type.*

Proof. Recall that G_0/H_0 is a finite covering space of G/H. By [**10**], there exist at most finitely many H_0-subalgebras, hence at most finitely many H-subalgebras. q.e.d.

Homogeneous spaces of finite type, e.g., triple products of Aloff–Wallach spaces, may admit infinitely many non-toral H-subalgebras. Furthermore, note that the product of two homogeneous spaces of finite type does not have to be of finite type anymore, for instance $\mathrm{SU}(2) \times \mathrm{SU}(2)/\{e\}$ is not of finite type.

6.1. The simplicial complex of homogeneous spaces of finite type. To every compact homogeneous spaces G/H of finite type, we will associate a simplicial complex $\Delta^{\min}_{G/H}$. We show that there exists a realization $X_{G/H} \subset T_Q \mathcal{M}^G_1$ of $\Delta^{\min}_{G/H}$, homeomorphic to the nerve $X^{\Sigma}_{G/H}$ of G/H.

Let $\{\mathfrak{k}^*_1, \dots, \mathfrak{k}^*_m\}$ denote the minimal non-toral H-subalgebras of $\mathfrak{g}$ and consider the finite set

$$V := \{\mathfrak{k}_1, \dots, \mathfrak{k}_N\}$$

of H-subalgebras, generated by these subalgebras, i.e. $\mathfrak{k}_i = \langle \mathfrak{k}^*_{i_1}, \dots, \mathfrak{k}^*_{i_p} \rangle$, where $1 \leq i_1 < \cdots < i_p \leq m$. The set V is partially ordered by the inclusion relation. We will call flags $\mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_m}$ with $\mathfrak{k}_i \subsetneq \mathfrak{k}_{i+1}$ of such

H-subalgebras H-flags. Notice that not every flag of H-subalgebras is an H-flag.

The set of H-flags itself is ordered in the following manner:

$$\mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_m} \quad \subset \quad \mathfrak{k}_{i'_1} < \cdots < \mathfrak{k}_{i'_{m'}}$$

if and only if for all $j \in \{1, \ldots, m\}$ there exists an index $j' \in \{1, \ldots, m'\}$ with $\mathfrak{k}_{i_j} = \mathfrak{k}_{i'_{j'}}$. The simplicial complex $\Delta^{\min}_{G/H}$ of a compact homogeneous space G/H of finite type is the corresponding flag complex (cf. [**16**, p. 28], [**46**]).

Definition 6.3. For an H-flag $e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_m}$ let

$$\begin{aligned} \Delta_e &= \Delta_{\mathfrak{k}_{i_1},\ldots,\mathfrak{k}_{i_m}} = \operatorname{conv}^E(v_{\text{can}}(\mathfrak{k}_{i_1}), \ldots, v_{\text{can}}(\mathfrak{k}_{i_m})), \\ \Delta^{\Sigma}_e &= \pi_{\Sigma}(\Delta_e). \end{aligned}$$

Recall that $\pi_{\Sigma}(v) = v/\|v\|$. Furthermore, let

$$F_{G/H} = \{e \mid e \text{ maximal } H\text{-flag}\}$$

denote the set of maximal H-flags and let

$$X_{G/H} = \bigcup_{e \in F_{G/H}} \Delta_e \quad \text{and} \quad X^{\Sigma}_{G/H} = \pi_{\Sigma}\left(X_{G/H}\right).$$

The nerve $X^{\Sigma}_{G/H}$ of G/H is a subset of the non-toral directions X^{Σ}_{nt}.

Proposition 6.4. *Let G/H be a compact homogeneous space of finite type. Then, $X_{G/H}$ is a simplicial complex isomorphic to $\Delta^{\min}_{G/H}$.*

Proof. Let $e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_m}$ be an H-flag. Since, by Lemma 5.5, the vertices of Δ_e are linearly independent, e and Δ_e are isometric as simplices. Let $e' = \mathfrak{k}_{i'_1} < \cdots < \mathfrak{k}_{i'_{m'}}$ be a second H-flag. We have to show $\Delta_e \cap \Delta_{e'} = \Delta_{e \cap e'}$.

Let $v \in \Delta_e$ and let f be a good decomposition of γ_v. Then, by Corollary 5.8, we have $\mathfrak{m}_{I^v_1(f)} = \mathfrak{k}_{i_j}$ for $j \in \{1, \ldots, m\}$. As a consequence, $\Delta_e \cap \Delta_{e'} = \emptyset$ if and only if $e \cap e' = \emptyset$. Furthermore, $\Delta_{e \cap e'} \subset \Delta_e \cap \Delta_{e'}$. We claim that the other inclusion holds as well. To see this, let $w_1 \in \Delta_e \cap \Delta_{e'}$. Since $w_1 \in \Delta_e$, there exists a good decomposition f of γ_{w_1}, such that $\mathfrak{h} \oplus \mathfrak{m}_{I^{w_1}_1(f)} = \mathfrak{l}_1$ is an H-subalgebra. As explained above, we have $\mathfrak{l}_1 = \mathfrak{k}_{i_{k_1}} = \mathfrak{k}_{i'_{k'}}$ for $k_1 \in \{1, \ldots, m\}$ and $k' \in \{1, \ldots, m'\}$. Next, let $e^1 = \mathfrak{k}_{i_{k_1}} < \mathfrak{k}_{i_{k_1+1}} < \cdots < \mathfrak{k}_{i_m}$ and $e'^1 = \mathfrak{k}_{i'_{k'}} < \mathfrak{k}_{i'_{k'+1}} < \cdots < \mathfrak{k}_{i'_{m'}}$. Then, we have $w_1 \in \Delta_{e^1} \cap \Delta_{e'^1}$ by Corollary 5.8. If $w_1 = v_{\text{can}}(\mathfrak{l}_1)$, then $w_1 \in \Delta_{e \cap e'}$ follows. If $w_1 \neq v_{\text{can}}(\mathfrak{l}_1)$, then $\tilde{e}^1 = \mathfrak{k}_{i_{k_1+1}} < \cdots < \mathfrak{k}_{i_m}$ and $\tilde{e}'^1 = \mathfrak{k}_{i'_{k'+1}} < \cdots < \mathfrak{k}_{i'_{m'}}$ are non-empty H-flags. The half ray $\{v_{\text{can}}(\mathfrak{l}_1) + s(w_1 - v_{\text{can}}(\mathfrak{l}_1)) \mid s \geq 0\}$ intersects $\Delta_{\tilde{e}^1}$ and $\Delta_{\tilde{e}'^1}$, say in w_2 and w_2', respectively. If $w_2 \neq w_2'$, then we may assume, that w_2 lies in the interior of the segment $\operatorname{conv}^E(v_{\text{can}}(\mathfrak{l}), w_2')$. But this is impossible by Corollary 5.8. Thus $w_2 = w_2'$. Inductively, we obtain an H-flag $e'' = \mathfrak{k}_{i_{k_1}} < \cdots < \mathfrak{k}_{i_{k_*}} \subset e \cap e'$ with $w_1 \in \Delta_{e''}$. q.e.d.

Proposition 6.5. *Let G/H be a compact homogeneous space of finite type. Then, the simplicial complex $X_{G/H}$ and the nerve $X^{\Sigma}_{G/H}$ of G/H are homeomorphic.*

Proof. Recall that by Proposition 5.10, for a flag e of H-subalgebras, the simplex Δ_e is a slice for $\pi_\Sigma(\Delta_e)$. In particular, $\pi_\Sigma : \Delta_e \to \pi_\Sigma(\Delta_e)$ is injective. In order to show that $\pi_\Sigma : X_{G/H} \to X^{\Sigma}_{G/H}$ is injective as well, suppose that there exist two maximal H-flags $e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_m}$ and $e' = \mathfrak{k}_{i'_1} < \cdots < \mathfrak{k}_{i'_{m'}}$ with $w_1 \in \Delta_e$ and $\alpha_1 w_1 \in \Delta_{e'}$ for $\alpha_1 \neq 1$. Since $w_1 \in \Delta_e$, there exists a good decomposition f of γ_{w_1}, such that $\mathfrak{h} \oplus \mathfrak{m}_{I_1^{w_1}(f)} = \mathfrak{h} \oplus \mathfrak{m}_{I_1^{\alpha_1 w_1}(f)} = \mathfrak{l}_1$ is an H-subalgebra. By Corollary 5.8, we have $\mathfrak{l}_1 = \mathfrak{k}_{i_{k_1}} = \mathfrak{k}_{i'_{k'}}$. If $w_1 = v_{\text{can}}(\mathfrak{l}_1)$ or $\alpha_1 w_1 = v_{\text{can}}(\mathfrak{l}_1)$, then $\alpha_1 = 1$ would follow. Contradiction.

Now, let the H-flags $\tilde{e}^1$ and $\tilde{e}'^1$ be defined as in Proposition 6.4. Then, we have $w_1 \in \operatorname{conv}^E(v_{\text{can}}(\mathfrak{l}_1), \Delta_{\tilde{e}^1})$ and $\alpha_1 w_1 \in \operatorname{conv}^E(v_{\text{can}}(\mathfrak{l}_1), \Delta_{\tilde{e}'^1})$. The half ray $\{v_{\text{can}}(\mathfrak{l}_1) + s(w_1 - v_{\text{can}}(\mathfrak{l}_1)) \mid s \geq 0\}$ and the half ray $\{v_{\text{can}}(\mathfrak{l}_1) + s(\alpha_1 w_1 - v_{\text{can}}(\mathfrak{l}_1)) \mid s \geq 0\}$ intersect $\Delta_{\tilde{e}^1}$ and $\Delta_{\tilde{e}'^1}$, say in w_2 and w_2', respectively. If w_2 and w_2' are linearly independent, then we may assume that there exists $\alpha_2 \neq 1$, such that $\alpha_2 w_2$ lies in the interior of the segment $\operatorname{conv}^E(v_{\text{can}}(\mathfrak{l}_1), w_2')$. By Corollary 5.8, we obtain a contradiction. Hence, $w_2' = \alpha_2 w_2$ for $\alpha_2 \neq 1$ and $w_2 \in \Delta_{\tilde{e}^1}$, $\alpha_2 w_2 \in \Delta_{\tilde{e}'^1}$. By induction, we obtain the claim. q.e.d.

6.2. The strong deformation retraction. We turn to the main result of this section. For compact homogeneous spaces of finite type, we will construct an explicit strong deformation retraction from the non-toral directions X^{Σ}_{nt} to the nerve $X^{\Sigma}_{G/H}$ of G/H.

Recall that the partially ordered set V of vertices of the simplicial complex $\Delta^{\min}_{G/H}$ consists of H-flags of length one. It is convenient to define the set of s-minimal vertices of V, $s \geq 1$: Let V_1 denote the set of minimal elements in V, that is the set of minimal non-toral H-subalgebras, let V_2 denote the minimal elements in $V \backslash V_1$ and so on.

Remark 6.6. Let $\mathfrak{k} \in V_s$ be an s-minimal vertex of $\Delta^{\min}_{G/H}$. Then, there exists $m \geq s$ and a maximal H-flag $e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_m} \in F_{G/H}$ with $\mathfrak{k} = \mathfrak{k}_{i_s}$. Furthermore, there exist no $e \in F_{G/H}$ with $\mathfrak{k} = \mathfrak{k}_{i_{\tilde{s}}}$ for $\tilde{s} \geq s+1$. Note, however, that there might exist $e \in F_{G/H}$ with $\mathfrak{k} = \mathfrak{k}_{i_{\tilde{s}}}$ for $\tilde{s} \leq s-1$. For instance, if $G/H = \mathrm{SO}(7)/\mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$, then the H-flag $\mathfrak{so}(2) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(3) < \mathfrak{so}(2) \oplus \mathfrak{so}(5)$ and the H-flag $\mathfrak{so}(2) \oplus \mathfrak{u}(2) < \mathfrak{so}(2) \oplus \mathfrak{so}(4) < \mathfrak{so}(2) \oplus \mathfrak{so}(5)$ are maximal.

Let us now write the set V_s of s-minimal vertices in V as a disjoint union:

$$V_s = V_s(\max) \cup V_s(ext),$$

where $V_s(\max)$ denotes the set of those vertices in V_s which are maximal in V. For $\mathfrak{k} \in V$, let

$$\mathrm{st}(\mathfrak{k}) = \big\{ e \in F_{G/H} \mid e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k} < \cdots < \mathfrak{k}_{i_m} \big\}$$

denote the set of H-flags containing the vertex $\mathfrak{k}$, that is $\mathrm{st}(\mathfrak{k})$ is the simplicial neighborhood of the vertex $\mathfrak{k}$ (the star of the vertex $\mathfrak{k}$). Let $M_{G/H}(s) = \cup_{\mathfrak{k} \in V_s(\max)} \mathrm{st}(\mathfrak{k})$, $X_{G/H}(s) = \cup_{e \in M_{G/H}(s)} \Delta_e$ and

$$X^{\Sigma}_{G/H}(s) = \pi_{\Sigma}(X_{G/H}(s)).$$

Then, we have

$$X^{\Sigma}_{G/H} = \bigcup_{s=1}^{N} X^{\Sigma}_{G/H}(s). \tag{6.7}$$

Next, for $\mathfrak{k} \in V_s(ext)$ let

$$\begin{aligned} F^{\subsetneq}(\mathfrak{k}) \;\; = \;\; & \big\{ e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_m} \in F_{G/H} \mid \\ & \quad \exists\, j \in \{2, \ldots, m\} : \mathfrak{k} \subsetneq \mathfrak{k}_{i_j} \text{ and } \mathfrak{k}_{i_{j-1}} \in \cup_{r=1}^{s} V_r \big\}. \end{aligned}$$

For fixed $e \in F^{\subsetneq}(\mathfrak{k})$, the index j is uniquely determined, but j may vary as e varies. We introduce the following notation for $\mathfrak{k} \in V_s(ext)$ and $e \in F^{\subsetneq}(\mathfrak{k})$:

$$\mathfrak{k}(e) := \mathfrak{k}_{i_j}. \tag{6.8}$$

There are three possibilities: $\mathfrak{k}(e) \in V_{s+1}(\max)$, $\mathfrak{k}(e) \in V_{s+1}(ext)$ or $\mathfrak{k}(e) \in \cup_{r=s+2}^{N} V_r$. Let $F^{\subsetneq}_{\max}(\mathfrak{k})$, $F^{\subsetneq}_{ext}(\mathfrak{k})$ and $F^{\subsetneq}_{\geq s+2}(\mathfrak{k})$ denote the corresponding subsets of $F^{\subsetneq}(\mathfrak{k})$. Then,

$$F^{\subsetneq}(\mathfrak{k}) = F^{\subsetneq}_{\max}(\mathfrak{k}) \cup F^{\subsetneq}_{ext}(\mathfrak{k}) \cup F^{\subsetneq}_{\geq s+2}(\mathfrak{k}).$$

Furthermore, for $\mathfrak{k} \in V_s(ext)$, let

$$\begin{aligned} V_{\geq s+1}(\mathfrak{k}) &= \bigcup_{j \geq s+1} \{\mathfrak{k}' \in V_j \mid \mathfrak{k} \subsetneq_{\max} \mathfrak{k}'\}, \\ C^{\Sigma}(V_{\geq s+1}(\mathfrak{k})) &= C^{\Sigma}\left(v_{\mathrm{can}}(\mathfrak{k}), \cup_{\mathfrak{k}' \in V_{\geq s+1}(\mathfrak{k})} X^{\Sigma}(\mathfrak{k}')\right) \end{aligned}$$

(cf. Definition 5.37). Here, $\mathfrak{k} \subsetneq_{\max} \mathfrak{k}'$ means, that $\mathfrak{k} \subsetneq \mathfrak{k}'$ and that there exists no vertex $\tilde{\mathfrak{k}} \in \Delta^{\min}_{G/H}$ with $\mathfrak{k} \subsetneq \tilde{\mathfrak{k}} \subsetneq \mathfrak{k}'$.

More generally, for an H-flag $e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_j} < \cdots < \mathfrak{k}_{i_m}$ and for $j \geq 2$, let

$$C^{\Sigma}(e, \mathfrak{k}_{i_j}) = C^{\Sigma}(v_{\mathrm{can}}(\mathfrak{k}_{i_1}), C^{\Sigma}(\ldots, C^{\Sigma}(v_{\mathrm{can}}(\mathfrak{k}_{i_{j-1}}), X^{\Sigma}(\mathfrak{k}_{i_j}))\ldots)).$$

For $j = 1$, we set $C^{\Sigma}(e, \mathfrak{k}_{i_1}) = X^{\Sigma}(\mathfrak{k}_{i_1})$. If we consider $v \in C^{\Sigma}(e, \mathfrak{k}_{i_j})$ as an endomorphism α_v, then the eigenspace corresponding to the smallest eigenvalue of α_v is either $\mathfrak{m}_{\mathfrak{l}}$ for an arbitrary H-subalgebra $\mathfrak{l}$ with $\mathfrak{l} \geq \mathfrak{k}_{i_j}$ (see Theorem 5.31) or given by $\mathfrak{m}_{\mathfrak{k}_{i_m}}$ for $m \in \{1, \ldots, j-1\}$.

Lemma 6.9. *Let $e_* = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_{j-1}}$, $e = e_* < \mathfrak{k}_{i_j}$ and $e'_* = \mathfrak{k}_{i'_1} < \cdots < \mathfrak{k}_{i'_{j'-1}}$, $e' = e'_* < \mathfrak{k}_{i'_{j'}}$ be H-flags. Then, we have:*

1. *If $\langle \mathfrak{k}_{i_j}, \mathfrak{k}_{i'_{j'}} \rangle < \mathfrak{g}$, $\mathfrak{k}_{i_j}, \mathfrak{k}_{i'_{j'}} \in \cup_{r=s}^{N} V_r$ and $\mathfrak{k}_{i_{j-1}}, \mathfrak{k}_{i'_{j'-1}} \in \cup_{r=1}^{s-1} V_r$, then $C^{\Sigma}(e, \mathfrak{k}_{i_j}) \cap C^{\Sigma}(e', \mathfrak{k}_{i'_{j'}}) = C^{\Sigma}(e_* \cap e'_* < \langle \mathfrak{k}_{i_j}, \mathfrak{k}_{i'_{j'}} \rangle, \langle \mathfrak{k}_{i_j}, \mathfrak{k}_{i'_{j'}} \rangle)$;*
2. *If $\langle \mathfrak{k}_{i_j}, \mathfrak{k}_{i'_{j'}} \rangle = \mathfrak{g}$, then $C^{\Sigma}(e, \mathfrak{k}_{i_j}) \cap C^{\Sigma}(e', \mathfrak{k}_{i'_{j'}}) = \Delta^{\Sigma}_{e_* \cap e'_*}$.*

Proof.

For 1: The inclusion "$\supset$" is obvious. For "$\subset$": Let $v \in C^{\Sigma}(e, \mathfrak{k}_{i_j}) \cap C^{\Sigma}(e', \mathfrak{k}_{i'_{j'}})$. Then, there exists a good decomposition f of γ_v, such that $\mathfrak{h} \oplus \mathfrak{m}_{I_1^v(f)} = \mathfrak{l}_1$ is an H-subalgebra. If $\mathfrak{k}_{i'_{j'}} \leq \mathfrak{l}_1$, then $v \in X^{\Sigma}(\mathfrak{k}_{i'_{j'}})$ by Theorem 5.31. Since $\mathfrak{k}_{i'_{j'}} \in \cup_{r=s}^{N} V_r$ and $\mathfrak{k}_{i_{j-1}} \in \cup_{r=1}^{s-1} V_r$, we cannot have $\mathfrak{l}_1 = \mathfrak{k}_{i_m}$ for $m \in \{1, \ldots, j-1\}$. Hence, $\mathfrak{l}_1 \geq \mathfrak{k}_{i_j}$ and we conclude $v \in X^{\Sigma}(\mathfrak{k}_{i_j})$. By Theorem 5.31, we get $v \in X^{\Sigma}(\langle \mathfrak{k}_{i_j}, \mathfrak{k}_{i'_{j'}} \rangle)$.

We are left with the case $\mathfrak{l}_1 = \mathfrak{k}_{i_m} = \mathfrak{k}_{i'_{m'}}$ for $m \in \{1, \ldots, j-1\}$ and $m' \in \{1, \ldots, j'-1\}$. If $v = v_{\mathrm{can}}(\mathfrak{l}_1)$, then we are done as well. If

$v \notin v_{\mathrm{can}}(\mathfrak{l}_1)$, then by the very definition of $C^{\Sigma}(e, \mathfrak{k}_{i_j})$ and $C^{\Sigma}(e', \mathfrak{k}_{i'_{j'}})$, we have $\pi_{\mathfrak{l}_1}(v) \in C^{\Sigma}(e, \mathfrak{k}_{i_j}) \cap C^{\Sigma}(e', \mathfrak{k}_{i'_{j'}})$ where $\pi_{\mathfrak{l}_1}$ denotes the radial projection as defined in (5.36). By induction over the dimension of $\mathfrak{l}_1$, we obtain the claim.

For 2: The inclusion "$\supset$" is again obvious. For "$\subset$": Let $v \in C^{\Sigma}(e, \mathfrak{k}_{i_j}) \cap C^{\Sigma}(e', \mathfrak{k}_{i'_{j'}})$. Then, there exists a good decomposition f of γ_v, such that $\mathfrak{h} \oplus \mathfrak{m}_{I_1^v(f)} = \mathfrak{l}_1$ is an H-subalgebra. It is easy to see that we cannot have $\mathfrak{k}_{i_j} \leq \mathfrak{l}_1$. Thus, we are left with the case $\mathfrak{l}_1 = \mathfrak{k}_{i_m} = \mathfrak{k}_{i'_{m'}}$ for $m \in \{1, \ldots, j-1\}$ and $m' \in \{1, \ldots, j'-1\}$ and the claim follows as above. q.e.d.

Before we give the proof of the main theorem in this section, let us explain why canonical directions $v_{\mathrm{can}}(\mathfrak{k})$, $\mathfrak{k}$ not being contained in V, are (first of all) not important for determining the homotopy type of the non-toral directions X^{Σ}_{nt}.

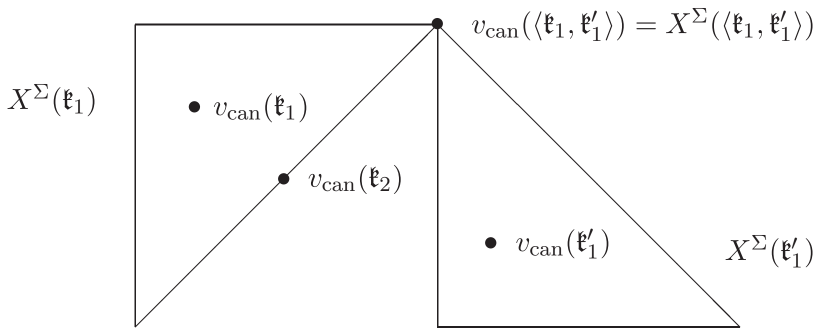

Figure 2.

In Figure 2, we assume that only four H-subalgebras exist all being non-toral: $\mathfrak{k}_1, \mathfrak{k}_1'$ are minimal non-toral H-subalgebras, $\mathfrak{k}_1 \subsetneq \mathfrak{k}_2$ and $\langle \mathfrak{k}_1, \mathfrak{k}_1' \rangle$ is a maximal H-subalgebra of $\mathfrak{g}$ with $X^{\Sigma}(\langle \mathfrak{k}_1, \mathfrak{k}_1' \rangle) = v_{\mathrm{can}}(\langle \mathfrak{k}_1, \mathfrak{k}_1' \rangle)$. We have $W^{\Sigma} = X^{\Sigma}_{nt} = X^{\Sigma}(\mathfrak{k}_1) \cup X^{\Sigma}(\mathfrak{k}_1')$ since there are no toral H-subalgebras. Obviously, the homotopy type of the non-toral directions X^{Σ}_{nt} is already determined by considering the flag complex of all H-flags, that is, we do not have to take care of the H-subalgebra $\mathfrak{k}_2$. But we also see that in this special case, we might consider all flags of H-subalgebras in order to describe the homotopy type of X^{Σ}_{nt}.

Theorem 6.10. *Let G/H be a compact homogeneous space of finite type. Then, the nerve $X^{\Sigma}_{G/H}$ of G/H is a strong deformation retract of the non-toral directions X^{Σ}_{nt}.*

Proof. We will proceed by induction over $s \geq 1$ in order to show, that there exists a strong deformation retraction from X^{Σ}_{nt} to

$$\tilde{X}^{\Sigma}_{G/H}(s) \;=\; \bigcup_{j=1}^{s} X^{\Sigma}_{G/H}(j) \;\cup \bigcup_{\mathfrak{k} \in V_s(\mathrm{ext})} \Big\{ C^{\Sigma}(e, \mathfrak{k}(e)) \mid e \in F^{\subsetneq}(\mathfrak{k}) \Big\}.$$

Then, by (6.7), the claim follows.

Step 1: Let $s = 1$ and $\mathfrak{k}_{i_1} \in V_1(\max)$. By Theorem 5.31, $X^{\Sigma}(\mathfrak{k}_{i_1})$ is a connected component of X^{Σ}_{nt} and by Theorem 5.25, $X^{\Sigma}(\mathfrak{k}_{i_1})$ is contractible. Hence, there exists a strong deformation retraction from X^{Σ}_{nt} to $X^{\Sigma}_{G/H}(1) \cup \{X^{\Sigma}(\mathfrak{k}_{i_1}) \mid \mathfrak{k}_{i_1} \in V_1(\mathrm{ext})\}$.

Next, let $\mathfrak{k}_{i_1} \in V_1(\mathrm{ext})$. Since $C^{\Sigma}(V_{\geq 2}(\mathfrak{k}_{i_1}))$ is the spherical cone of the compact semialgebraic set $\cup_{\mathfrak{k}' \in V_{\geq s+1}(\mathfrak{k}_{i_1})} X^{\Sigma}(\mathfrak{k}')$ over the vertex $v_{\mathrm{can}}(\mathfrak{k}_{i_1})$ it follows, as in the proof of Theorem 5.39, that there exists a strong deformation retraction from $X^{\Sigma}(\mathfrak{k}_{i_1})$ to $C^{\Sigma}(V_{\geq 2}(\mathfrak{k}_{i_1}))$. For any $\mathfrak{k}_{i'_1} \in V_1(\mathrm{ext}) \backslash \{\mathfrak{k}_{i_1}\}$, we obtain in the same manner a strong deformation retraction from $X^{\Sigma}(\mathfrak{k}_{i'_1})$ to $C^{\Sigma}(V_{\geq 2}(\mathfrak{k}_{i'_1}))$. Since by Theorem 5.31 $X^{\Sigma}(\mathfrak{k}_{i_1}) \cap X^{\Sigma}(\mathfrak{k}_{i'_1}) = X^{\Sigma}(\langle \mathfrak{k}_{i_1}, \mathfrak{k}_{i'_1} \rangle)$, both these deformation retractions can be extended to a continuous, strong deformation retraction from $X^{\Sigma}(\mathfrak{k}_{i_1}) \cup X^{\Sigma}(\mathfrak{k}_{i'_1})$ to $C^{\Sigma}(V_{\geq 2}(\mathfrak{k}_{i_1})) \cup C^{\Sigma}(V_{\geq 2}(\mathfrak{k}_{i'_1}))$. Proceeding inductively, we obtain a continuous, strong deformation retraction from X^{Σ}_{nt} to $\tilde{X}^{\Sigma}_{G/H}(1)$.

Now, let $s \geq 2$ and suppose that we have already constructed a strong deformation retraction from X^{Σ}_{nt} to $\tilde{X}^{\Sigma}_{G/H}(s-1)$.

Step 2: Let $\mathfrak{k}_{i_s} \in V_s(\max)$. By Theorem 5.25, there exists a strong deformation retraction from $X^{\Sigma}(\mathfrak{k}_{i_s})$ to $v_{\mathrm{can}}(\mathfrak{k}_{i_s})$. Let $e \in \mathrm{st}(\mathfrak{k}_{i_s})$ be a maximal H-flag. We obtain in a canonical manner a strong deformation retraction from $C^{\Sigma}(e, \mathfrak{k}_{i_s})$ to $\Delta^{\Sigma}_e \in X^{\Sigma}_{G/H}(s)$.

Note that for two H-subalgebras $\mathfrak{k}_1, \mathfrak{k}_2$ with $\mathfrak{k}_1 < \mathfrak{k}_2$ the radial projection $\pi_{\mathfrak{k}_1}$ from $C^{\Sigma}(\mathfrak{k}_1, X^{\Sigma}(\mathfrak{k}_2)) \backslash \{v_{\mathrm{can}}(\mathfrak{k}_1)\}$ to $X^{\Sigma}(\mathfrak{k}_2) \subset \Sigma^{\mathfrak{k}_2}$, given by $\pi_{\mathfrak{k}_1}(v) = c^{\mathfrak{k}_1}_v(s^{\partial}_v)$, is continuous (for the definition of s_v and s^{∂}_v see (5.34) and (5.35)). Hence, any continuous map $H(2) : [0,1] \times X^{\Sigma}(\mathfrak{k}_2) \to X^{\Sigma}(\mathfrak{k}_2)$

can be extended in a canonical manner to a map

$$H(1):[0,1]\times C^{\Sigma}(\mathfrak{k}_1,X^{\Sigma}(\mathfrak{k}_2))\to C^{\Sigma}(\mathfrak{k}_1,X^{\Sigma}(\mathfrak{k}_2))\ ;$$
$$(\mu,v)\mapsto\begin{cases} c^{\mathfrak{k}_1}_{H(2)(\mu,\pi_{\mathfrak{k}_1}(v))}\Big(\frac{s_v}{s^{\partial}_v}\cdot s^{\partial}_{H(2)(\mu,\pi_{\mathfrak{k}_1}(v))}\Big), & v\neq v_{\mathrm{can}}(\mathfrak{k}_1)\\ v_{\mathrm{can}}(\mathfrak{k}_1), & v=v_{\mathrm{can}}(\mathfrak{k}_1)\end{cases}.$$

Since the maps $s, s^{\partial}, H(2)$ and $c^{\mathfrak{k}_1}$ are continuous, so is $H(1)$. We define the above mentioned deformation retraction and the corresponding homotopy now inductively.

For any $e'\in \mathrm{st}(\mathfrak{k}_{i_s})\backslash\{e\}$, we obtain in the same manner a strong deformation retraction from $C^{\Sigma}(e',\mathfrak{k}_{i_s})$ to $\Delta^{\Sigma}_{e'}$. By Lemma 6.9, 1. this yields a strong deformation retraction from $\cup_{e\in\mathrm{st}(\mathfrak{k}_{i_s})}C^{\Sigma}(e,\mathfrak{k}_{i_s})$ to $\cup_{e\in\mathrm{st}(\mathfrak{k}_{i_s})}\Delta^{\Sigma}_{e}$. As a consequence, there exists a strong deformation retraction from $\cup_{e\in\mathrm{st}(\mathfrak{k}_{i'_s})}C^{\Sigma}(e,\mathfrak{k}_{i'_s})$ to $\cup_{e\in\mathrm{st}(\mathfrak{k}_{i'_s})}\Delta_e$ for any $\mathfrak{k}_{i'_s}\in V_s(\max)$. Since we have $\langle\mathfrak{k}_{i_s},\mathfrak{k}_{i'_s}\rangle=\mathfrak{g}$ for $\mathfrak{k}_{i_s}\neq\mathfrak{k}_{i'_s}$, by Lemma 6.9, 2. this yields a strong deformation retraction from

$$\Big\{C^{\Sigma}(e,\mathfrak{k})\mid \mathfrak{k}\in V_s(\max),\, e\in\mathrm{st}(\mathfrak{k})\Big\}$$
$$=\Big\{C^{\Sigma}(e,\mathfrak{k}(e))\mid \mathfrak{k}\in V_{s-1}(\mathrm{ext}),\, e\in F^{\subsetneq}_{\max}(\mathfrak{k})\Big\}$$

to $X^{\Sigma}_{G/H}(s)$ (see (6.8) for the definition of $\mathfrak{k}(e)$), and from $\tilde X^{\Sigma}_{G/H}(s-1)$ to

$$\bigcup_{j=1}^{s}X^{\Sigma}_{G/H}(j)\cup\Big\{C^{\Sigma}(e,\mathfrak{k}(e))\mid \mathfrak{k}\in V_{s-1}(\mathrm{ext}),\, e\in F^{\subsetneq}_{\mathrm{ext}}(\mathfrak{k})\cup F^{\subsetneq}_{\geq s+1}(\mathfrak{k})\Big\}$$

Step 3: Let $\mathfrak{k}_{i_s}\in V_s(\mathrm{ext})$. As above, we obtain a strong deformation retraction from $X^{\Sigma}(\mathfrak{k}_{i_s})$ to $C^{\Sigma}(V_{\geq s+1}(\mathfrak{k}_{i_s}))$ and from $\cup_{e\in\mathrm{st}(\mathfrak{k}_{i_s})}C^{\Sigma}(e,\mathfrak{k}_{i_s})$ to

$$\big\{C^{\Sigma}(e,\mathfrak{k})\mid \mathfrak{k}\in V_{\geq s+1}(\mathfrak{k}_{i_s}),\ e\in\mathrm{st}(\mathfrak{k})\cap\mathrm{st}(\mathfrak{k}_{i_s})\big\}.$$

For any $\mathfrak{k}_{i'_s}\in V_s(\mathrm{ext})\backslash\{\mathfrak{k}_{i_s}\}$, we obtain in the same manner a corresponding strong deformation retraction. We have either $\langle\mathfrak{k}_{i_s},\mathfrak{k}_{i'_s}\rangle=\mathfrak{g}$ or $\langle\mathfrak{k}_{i_s},\mathfrak{k}_{i'_s}\rangle\in\cup_{r=s+1}^{N}V_r$. In the latter case, there exist $\mathfrak{k}\in V_{\geq s+1}(\mathfrak{k}_{i_s})$ and $\mathfrak{k}'\in V_{\geq s+1}(\mathfrak{k}_{i'_s})$ with $\langle\mathfrak{k}_{i_s},\mathfrak{k}_{i'_s}\rangle\geq\mathfrak{k}$ and $\langle\mathfrak{k}_{i_s},\mathfrak{k}_{i'_s}\rangle\geq\mathfrak{k}'$. By Lemma 6.9, we obtain a strong deformation retraction from

$$\Big\{C^{\Sigma}(e,\mathfrak{k}(e))\mid \mathfrak{k}\in V_{s-1}(\mathrm{ext}),\, e\in F^{\subsetneq}_{\mathrm{ext}}(\mathfrak{k})\Big\}$$
$$=\Big\{C^{\Sigma}(e,\mathfrak{k})\mid \mathfrak{k}\in V_s(\mathrm{ext}),\, e\in\mathrm{st}(\mathfrak{k})\Big\}$$

to $B = \cup_{\mathfrak{k}_{i_s} \in V_s(\text{ext})} \{C^\Sigma(e, \mathfrak{k}) \mid \mathfrak{k} \in V_{\geq s+1}(\mathfrak{k}_{i_s}),\ e \in \mathrm{st}(\mathfrak{k}) \cap \mathrm{st}(\mathfrak{k}_{i_s})\}$ and as above from $\tilde{X}^\Sigma_{G/H}(s-1)$ to

$$\cup_{j=1}^s X^\Sigma_{G/H}(j) \cup B \cup C,$$

where $C = \{C^\Sigma(e, \mathfrak{k}(e)) \mid \mathfrak{k} \in V_{s-1}(\text{ext}),\, e \in F^{\subsetneq}_{\geq s+1}(\mathfrak{k})\}$. It remains to show that

$$A := \bigcup_{\mathfrak{k}_{i_s} \in V_s(\text{ext})} \left\{ C^\Sigma(e, \mathfrak{k}_{i_s}(e)) \mid e \in F^{\subsetneq}(\mathfrak{k}_{i_s}) \right\} = B \cup C.$$

We clearly have $B \cup C \subset A$. For "$A \subset B \cup C$": Let $e = \mathfrak{k}_{i_1} < \cdots < \mathfrak{k}_{i_{j-1}} < \mathfrak{k}_{i_s}(e) < \cdots < \mathfrak{k}_{i_m} \in F^{\subsetneq}(\mathfrak{k}_{i_s})$ for $\mathfrak{k}_{i_s} \in V_s(\text{ext})$. By definition, $\mathfrak{k}_{i_s} \subsetneq \mathfrak{k}_{i_s}(e)$. If $\mathfrak{k}_{i_{j-1}} \in V_s$, then we have $C^\Sigma(e, \mathfrak{k}_{i_s}(e)) \subset B$, since $\mathfrak{k}_{i_{j-1}} \subsetneq_{\max} \mathfrak{k}_{i_s}(e)$. If $\mathfrak{k}_{i_{j-1}} \in \cup_{r=1}^{s-1} V_r$, then we have $C^\Sigma(e, \mathfrak{k}_{i_s}(e)) \subset C$. q.e.d.

6.3. The extended simplicial complex. Let G/H be a compact homogeneous space, such that there exist at most finitely many non-toral H-subalgebras. Then G/H is a compact homogeneous space of finite type. Recall that the converse in not true. The set of non-toral H-subalgebras is partially ordered by the inclusion relation. The extended simplicial complex $\hat{\Delta}^{\min}_{G/H}$ of G/H is the corresponding flag complex. Certainly, we have $\Delta^{\min}_{G/H} \subset \hat{\Delta}^{\min}_{G/H}$ but equality does not hold in general: For instance, let $G = \mathrm{SO}(9)$ and $H = (\mathrm{SO}(3) \cdot \mathrm{U}(1)) \times \mathrm{SO}(3)$, where $\mathrm{SO}(3) \cdot \mathrm{U}(1) < \mathrm{U}(3) < \mathrm{SO}(6)$. Then, $\mathfrak{k}_1 = \mathfrak{u}(3) \oplus \mathfrak{so}(3)$ and $\mathfrak{k}_2 = \mathfrak{so}(6) \oplus \mathfrak{so}(3)$ are the only H-subalgebras of $\mathfrak{g}$ (cf. [**45**]).

As in Definition 6.3, we can now define a simplicial complex $\hat{X}_{G/H}$. It follows as in Section 6.1, that $\hat{\Delta}^{\min}_{G/H}$ and $\hat{X}_{G/H}$ are isometric and that $\hat{X}_{G/H}$ and $\hat{X}^\Sigma_{G/H} = \pi_\Sigma(\hat{X}_{G/H})$ are homeomorphic.

Proposition 6.11. *Let G/H be a compact homogeneous space. If there exist at most finitely many non-toral H-subalgebras, then $\hat{X}^\Sigma_{G/H}$ is a strong deformation retract of X^Σ_{nt}.*

Proof. The proof is the same as in Theorem 6.10. The only difference is that the set V_s of s-minimal vertices is now a subset of the set of all non-toral H-subalgebras of $\mathfrak{g}$. q.e.d.

Corollary 6.12. *Let G/H be a compact homogeneous space, such that there exist at most finitely many non-toral H-subalgebras. Then, $\Delta^{\min}_{G/H}$ and $\hat{\Delta}^{\min}_{G/H}$ are homotopy equivalent.*

This result is a purely Lie theoretical fact about H-subalgebras. Note that the extended simplicial complex is sometimes more convenient to work with, since we do not have to check, whether or not a non-toral H-subalgebra is generated by minimal non-toral H-subalgebras.

7. The simplicial complex of arbitrary homogeneous spaces

If G/H is a compact homogeneous space, such that the normalizer $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ equals $\mathfrak{h}$, then G/H is of finite type (Lemma 6.2) and in Section 6.1 we have assigned to G/H a simplicial complex $\Delta^{\min}_{G/H}$.

If $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{n}(\mathfrak{h})$, then there exists a compact complement $\mathfrak{m}_0$ of $\mathfrak{h}$ in $\mathfrak{n}(\mathfrak{h})$. Let T denote a maximal torus of the compact, connected Lie group with Lie algebra $\mathfrak{m}_0$. We call an H-subalgebra $\mathfrak{k}$ T-adapted if $\mathfrak{k}$ is invariant under the adjoint action of T. A non-toral T-adapted H-subalgebra $\mathfrak{k}$ is called T-minimal non-toral if a T-adapted H-subalgebra $\mathfrak{k}'$ with $\mathfrak{k}' < \mathfrak{k}$ must be toral. The set of T-adapted non-toral H-subalgebras, generated by T-minimal non-toral H-subalgebras, is finite and partially ordered by the inclusion relation. The simplicial complex $\Delta^T_{G/H}$ attached to G/H and T is the corresponding flag complex (cf. Section 6.1 or Section 1). Vertices correspond to flags of length one, edges to flags of length two and so on. Notice that if $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$, the above definitions make also sense for $T = \{e\}$. In this case, we have, of course $\Delta^T_{G/H} := \Delta^{\min}_{G/H}$.

Let G/H be a compact homogeneous space. Since in case $\mathfrak{h} = \mathfrak{n}(\mathfrak{h})$ all the results stated subsequently are trivial, we may assume $\mathfrak{h} < \mathfrak{n}(\mathfrak{h})$. In this case, there exists a compact subalgebra $\mathfrak{m}_0$ of $\mathfrak{g}$ (cf. Lemma 4.27), such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{m}_0$ (cf. Lemma 4.26). Let T be a maximal torus of the compact, connected Lie group with Lie algebra $\mathfrak{m}_0$.

Suppose that H is connected. Let $\mathfrak{t} \leq \mathfrak{m}_0$ denote the Lie algebra of T. Then, we have $\mathfrak{m} = \mathfrak{t} \oplus (\mathfrak{t}^\perp \cap \mathfrak{m}_0) \oplus \mathfrak{m}_0^\perp$. The Lie groups H and T act via the adjoint action on these three summands of $\mathfrak{m}$ in the following manner: H and T act trivially on $\mathfrak{t}$, H acts trivially on $\mathfrak{t}^\perp \cap \mathfrak{m}_0$, but T does not and H does not act trivially on $\mathfrak{m}_0^\perp$. We obtain

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1^{\mathfrak{k}} \oplus \mathfrak{m}_2^{\mathfrak{k}} \oplus \mathfrak{m}_3^{\mathfrak{k}}$$

for each T-adapted H-subalgebra $\mathfrak{k}$, where $\mathfrak{m}^1_{\mathfrak{k}} \subset \mathfrak{t}$, $\mathfrak{m}^2_{\mathfrak{k}} \subset \mathfrak{t}^{\perp} \cap \mathfrak{m}_0$ and $\mathfrak{m}^3_{\mathfrak{k}} \subset \mathfrak{m}_0^{\perp}$.

Lemma 7.1. *Let G/H be a compact homogeneous space with both G and H connected. Suppose that G/H has finite fundamental group. Then, T-minimal non-toral H-subalgebras are in one-to-one correspondence with minimal non-toral TH-subalgebras of G/TH.*

Proof. Since $\mathfrak{n}(\mathfrak{h} \oplus \mathfrak{t}) = \mathfrak{h} \oplus \mathfrak{t}$, there exist at most finitely many TH-subalgebras of $\mathfrak{g}$, all being non-toral (cf. Lemma 6.2). Let $\mathfrak{h} \oplus \mathfrak{t} \oplus \mathfrak{m}^{\mathfrak{k}}_2 \oplus \mathfrak{m}^{\mathfrak{k}}_3$ be a minimal non-toral TH-subalgebra and let $\mathfrak{m}^{\mathfrak{k}}_1 = [\mathfrak{m}^{\mathfrak{k}}_2, \mathfrak{m}^{\mathfrak{k}}_2]_{\mathfrak{t}} \oplus [\mathfrak{m}^{\mathfrak{k}}_3, \mathfrak{m}^{\mathfrak{k}}_3]_{\mathfrak{t}}$. Then, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}^{\mathfrak{k}}_1 \oplus \mathfrak{m}^{\mathfrak{k}}_2 \oplus \mathfrak{m}^{\mathfrak{k}}_3$ is a T-minimal non-toral H-subalgebra.

Vice versa, let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}^{\mathfrak{k}}_1 \oplus \mathfrak{m}^{\mathfrak{k}}_2 \oplus \mathfrak{m}^{\mathfrak{k}}_3 < \mathfrak{g}$ be a T-minimal non-toral H-subalgebra. Then, $\mathfrak{h} \oplus \mathfrak{t} \oplus \mathfrak{m}^{\mathfrak{k}}_2 \oplus \mathfrak{m}^{\mathfrak{k}}_3$ is a subalgebra of $\mathfrak{g}$ invariant under the adjoint action of both T and H. Since G/H has finite fundamental group, this subalgebra is a proper subalgebra of $\mathfrak{g}$, and hence, a minimal non-toral TH-subalgebra (cf. proof of Proposition 7.5). q.e.d.

It is necessary to assume that G/H has finite fundamental group, as the homogeneous space $S^1 \times S^3/\{e\}$ shows.

Corollary 7.2. *Let G/H be a compact homogeneous space. Then, there exist at most finitely many T-minimal non-toral H-subalgebras.*

The set of H-subalgebras, generated by the T-minimal non-toral H-subalgebras is finite and partially ordered by the inclusion relation. We will call flags of such H-subalgebras T-adapted H-flags. The simplicial complex $\Delta^T_{G/H}$ attached to G/H and T is the corresponding flag complex.

If H is connected, then the simplicial complex $\Delta^T_{G/H}$ does not depend on the choice of T, since the maximal tori of the compact, connected subgroup of $N_G(H)$ with Lie algebra $\mathfrak{m}_0$ are all conjugate to each other. In Section 1, we have defined the simplicial complex $\Delta_{G/H} := \Delta^T_{G/H}$ of G/H and the simplicial complex $\Delta_{G/TH} := \Delta^{\{e\}}_{G/TH} = \Delta^{\min}_{G/TH}$ of G/TH.

Proposition 7.3. *Let G/H be a compact homogeneous space with both G and H connected. Suppose that G/H has finite fundamental group. Then $\Delta_{G/H} = \Delta_{G/TH}$.*

Proof. We have to show $\Delta^T_{G/H} = \Delta^{\min}_{G/TH}$. By Lemma 7.1, the T-minimal non-toral H-subalgebras and the minimal non-toral TH-subalgebras of G/TH are in one-to-one correspondence. More generally, to any TH-subalgebra $\mathfrak{k}^{\mathfrak{t}} = \mathfrak{h} \oplus \mathfrak{t} \oplus \mathfrak{m}_2^{\mathfrak{k}} \oplus \mathfrak{m}_3^{\mathfrak{k}}$, we can assign a T-adapted H-subalgebra $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1^{\mathfrak{k}} \oplus \mathfrak{m}_2^{\mathfrak{k}} \oplus \mathfrak{m}_3^{\mathfrak{k}}$, where $\mathfrak{m}_1^{\mathfrak{k}} = [\mathfrak{m}_2^{\mathfrak{k}}, \mathfrak{m}_2^{\mathfrak{k}}]_{\mathfrak{t}} \oplus [\mathfrak{m}_3^{\mathfrak{k}}, \mathfrak{m}_3^{\mathfrak{k}}]_{\mathfrak{t}}$.

Next, let $\mathfrak{l}^{\mathfrak{t}} = \mathfrak{h} \oplus \mathfrak{t} \oplus \mathfrak{m}_2^{\mathfrak{l}} \oplus \mathfrak{m}_3^{\mathfrak{l}}$ be a further TH-subalgebra and let $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}_1^{\mathfrak{l}} \oplus \mathfrak{m}_2^{\mathfrak{l}} \oplus \mathfrak{m}_3^{\mathfrak{l}}$ denote the corresponding T-adapted H-subalgebra. Let $\mathfrak{k}^{\mathfrak{t}} = \mathfrak{k} \oplus \mathfrak{t}_{\mathfrak{k}}$ and $\mathfrak{l}^{\mathfrak{t}} = \mathfrak{l} \oplus \mathfrak{t}_{\mathfrak{l}}$. Since $\mathfrak{k}$ and $\mathfrak{l}$ are invariant under the adjoint action of TH, we obtain $[\mathfrak{k}^{\mathfrak{t}}, \mathfrak{l}^{\mathfrak{t}}] \subset \mathfrak{k} + \mathfrak{l} + [\mathfrak{k}, \mathfrak{l}] \subset \langle \mathfrak{k}, \mathfrak{l} \rangle$. Proceeding inductively yields $\langle \mathfrak{k}, \mathfrak{l} \rangle \oplus \mathfrak{t}' = \langle \mathfrak{k}^{\mathfrak{t}}, \mathfrak{l}^{\mathfrak{t}} \rangle$. Therefore, $\langle \mathfrak{k}^{\mathfrak{t}}, \mathfrak{l}^{\mathfrak{t}} \rangle$ corresponds to $\langle \mathfrak{k}, \mathfrak{l} \rangle$ via the above assignment.

We conclude that $\Delta_{G/TH}^{\min}$ can be considered a subcomplex of $\Delta_{G/H}^{T}$. Since G/H has finite fundamental group, there exists no T-adapted H-subalgebra, generated by T-minimal non-toral H-subalgebras, which corresponds to $\mathfrak{g}$, thus $\Delta_{G/TH}^{\min} = \Delta_{G/H}^{T}$. q.e.d.

For compact homogeneous spaces $M^n = G/H$ with finite fundamental group the connected component of the semisimple part of G, which contains the identity, acts transitively on M^n. If M^n is simply connected, then the isotropy group of this action is connected. Hence, $M^n = \tilde{G}/\tilde{H}$, where $\tilde{G}$ is semisimple and $\tilde{H}$ is connected. By the very definition of the simplicial complex of such homogeneous spaces, we have $\Delta_{\tilde{G}/\tilde{H}} = \Delta_{\tilde{G}/\tilde{H}_s}$, where $\tilde{H}_s$ denotes the semisimple part of $\tilde{H}$.

Remark 7.4. The dimension of a compact, connected, simply connected, semisimple Lie group G acting almost effectively on M^n is bounded between n and $\frac{1}{2}n(n+1)$. Hence, there exist only finitely many such groups acting transitively on M^n. For each of these Lie groups, there exist (up to conjugation) at most finitely many connected, semisimple Lie subgroups. Therefore, it is a finite task to compute these simplicial complexes up to a fixed dimension.

By contrast, recall that in dimension seven and each dimension greater than or equal to nine, there exist infinitely many homotopy types of simply connected compact homogeneous spaces (cf. [**35**]).

Finally, we turn to simply criteria, which imply that the simplicial complex $\Delta_{G/H}^{T}$ of a compact homogeneous space is contractible.

Proposition 7.5. *Let G/H be a compact homogeneous space. If G/H has infinite fundamental group, then either $\Delta_{G/H}^{T}$ is contractible or $\Delta_{G/H}^{T} = \emptyset$ and G/H is a torus.*

Proof. If $\mathfrak{g}$ is abelian, then $\Delta^T_{G/H} = \emptyset$ and $G/H = T^n$. So, suppose that $\mathfrak{g}$ is not abelian. Let T be a maximal torus of the compact, connected Lie group with Lie algebra $\mathfrak{m}_0$: Since by assumption G/H has infinite fundamental group, $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}'$, where $\mathfrak{a}$ is an abelian, $\mathrm{Ad}(TH)$-invariant subalgebra of $\mathfrak{g}$, and $\mathfrak{g}'$ is a non-toral H-subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, which is invariant under the adjoint action of T. We conclude $\Delta^T_{G/H} \neq \emptyset$. Furthermore, all T-minimal non-toral H-subalgebras of $\mathfrak{g}$ are contained in $\mathfrak{g}'$. It follows that the subalgebra $\mathfrak{g}''$ generated by all T-minimal non-toral H-subalgebras is T-adapted and a proper subalgebra of $\mathfrak{g}$, hence an H-subalgebra. This implies that $\Delta^T_{G/H}$ is a cone over $\mathfrak{g}''$. q.e.d.

Corollary 7.6. *Let G/H be a compact homogeneous space. If there exists an H-subalgebra $\mathfrak{g}'$ containing all T-minimal non-toral H-subalgebras of $\mathfrak{g}$, then $\Delta^T_{G/H}$ is contractible.*

8. Variational methods

In this section, we apply variational methods to the scalar curvature functional $\mathrm{sc} : \mathcal{M}^G_1 \to \mathbb{R}$ of compact homogeneous spaces G/H, admitting a non-contractible simplicial complex $\Delta^T_{G/H}$, in order to prove the existence of a critical point. More precisely, we show that there exists a critical point g, such that the augmented coindex of the scalar curvature functional at g is bounded from below by the homological dimension of $\Delta^T_{G/H}$. Finally, we indicate how Lyusternik–Schnirelmann-theory can be applied.

The following theorem is our main existence result on compact homogeneous Einstein manifolds. Recall that the empty simplicial complex $\Delta^T_{G/H}$ is non-contractible.

Theorem 8.1. *Let G/H be a compact homogeneous space. If a simplicial complex $\Delta^T_{G/H}$ of G/H is not contractible, then G/H admits a G-invariant Einstein metric.*

Proof. First, we consider the case $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$. Then, G/H is a homogeneous space of finite type and $\Delta^T_{G/H} = \Delta^{\min}_{G/H}$. If $\Delta^T_{G/H} = \emptyset$, then the claim follows from Theorem 5.22. Now, suppose $\Delta^T_{G/H} \neq \emptyset$. With the help of Proposition 7.5, we conclude that G/H has finite fundamental group. By Lemma 5.20, we may assume that the set of extended non-toral directions X^{Σ}_{ent} is a proper subset of the unit sphere Σ in $T_Q\mathcal{M}^G_1$. Since X^{Σ}_{ent} is a compact, semialgebraic subset of Σ (Theorem 5.48), by Theorem 1 in [**21**], there exists an open neighborhood U^{Σ} of X^{Σ}_{ent} in Σ, such that X^{Σ}_{ent} is a strong deformation retraction of U^{Σ}. Let $\delta > 0$, such that the open δ-neighborhood $U^{\Sigma}_{\text{ent}}(\delta)$ of X^{Σ}_{ent} in Σ is contained in U^{Σ}. We define the open truncated cone

$$C(U^{\Sigma}, t_0(\delta)) := \{\gamma_v(t) \mid v \in U^{\Sigma},\ t > t_0(\delta)\} \subset T_Q\mathcal{M}^G_1$$

over U^{Σ} with $t_0(\delta) > 0$ as in Theorem 5.52. Hence, for all directions $v \in \Sigma \backslash U^{\Sigma}_{\text{ent}}(\delta) \supset \Sigma \backslash U^{\Sigma}$, we have $\text{sc}(\gamma_v(t)) \leq (1/2) b_{G/H}$ for all $t \geq t_0(\delta)$. This implies that there exists a positive constant $\text{sc}_+ > 2\text{sc}(Q) > 0$, such that $\{\text{sc} \geq \text{sc}_+\} \subset C(U^{\Sigma}, t_0(\delta))$.

Since the nerve $X^{\Sigma}_{G/H}$ of G/H is compact, by Proposition 5.6, there exists $T_0 > t_0(\delta)$, such that for all $t \geq T_0$ and all $v \in X^{\Sigma}_{G/H}$ we have $\text{sc}(\gamma_v(t)) > \text{sc}_+$. We define the cycle

$$B := \{\gamma_v(T_0) \mid v \in X^{\Sigma}_{G/H}\} \subset \{\text{sc} > \text{sc}_+\} \subset C(U^{\Sigma}, t_0(\delta)) \tag{8.2}$$

and the contractible cone

$$A := \{\gamma_v(t) \mid v \in X^{\Sigma}_{G/H},\ 0 \leq t \leq T_0\}$$

over B. By Proposition 5.6, we have $\text{sc}(p) > 0$ for all $p \in A$. Since A is compact, there exists $\epsilon > 0$ with $A \subset \{\text{sc} \geq \epsilon\}$.

We claim that B is not contractible in $C(U^{\Sigma}, t_0(\delta))$. To this end, recall that $\Delta^T_{G/H}$ and $X^{\Sigma}_{G/H}$ are homeomorphic (Proposition 6.5), and that we have the following inclusion chain:

$$X^{\Sigma}_{G/H} \subset X^{\Sigma}_{nt} \subset X^{\Sigma}_{\text{ent}} \subset U^{\Sigma}_{\text{ent}}(\delta) \subset U^{\Sigma}.$$

There exist strong deformation retractions from U^{Σ} to X^{Σ}_{ent} (by choice of U^{Σ}) from X^{Σ}_{ent} to X^{Σ}_{nt} (Theorem 5.48) and from X^{Σ}_{nt} to $X^{\Sigma}_{G/H}$ (Theorem 6.10). If B would be contractible in $C(U^{\Sigma}, t_0(\delta))$, then $X^{\Sigma}_{G/H}$ would be contractible in U^{Σ}. This would imply that $X^{\Sigma}_{G/H}$ is contractible and we would obtain a contradiction.

By [**10**], the set $Z = \{g \in \mathcal{M}^G_1 \mid (\nabla_{L^2}\text{sc})_g = 0\}$ of critical points is compact. Hence, there exists a smooth positive function Ψ on $\mathcal{M}^G_1$ with $\Psi \equiv 1$ close to Z, with $\Psi(g) = \frac{1}{\|(\nabla_{L^2}\text{sc})_g\|_g}$ far away from Z and with $\Psi \leq \frac{1}{\|(\nabla_{L^2}\text{sc})\|}$. Here, $\|\cdot\|_g$ denotes the norm on $T_g\mathcal{M}^G_1$ induced by the

L^2-metric. We obtain a smooth vector field $X(g) = \Psi(g) \cdot (\nabla_{L^2}\mathrm{sc})_g$ on $\mathcal{M}_1^G$ with $\|X\| \leq 1$.

Let $g(t)$ be an integral curve of X. Since $(\mathcal{M}_1^G, L^2)$ is complete and $\|X\|$ is bounded, $g(t)$ is defined for all $t \geq 0$. We have

$$\frac{d}{dt}\mathrm{sc}(g(t)) = \Psi(g(t)) \cdot \|(\nabla_{L^2}\mathrm{sc})_{g(t)}\|^2_{g(t)} \geq 0.$$

Let Φ denote the flow of X. Since $\Phi_u(B) \subset \{\mathrm{sc} \geq \mathrm{sc}_+\} \subset C(U^\Sigma, t_0(\delta))$ for all $u \geq 0$, there exists $p_u \in A$ with $\mathrm{sc}(\Phi_u(p_u)) < \mathrm{sc}_+$ for all $u \geq 0$; otherwise, there would exist $u_0 > 0$, such that $\Phi_{u_0}(A) \subset \{\mathrm{sc} \geq \mathrm{sc}_+\} \subset C(U^\Sigma, t_0(\delta))$ and we would obtain an explicit contraction of $\Phi_{u_0}(B)$ in $C(U^\Sigma, t_0(\delta))$.

Next, let $(p_j)_{j\in\mathbb{N}}$ subconverge to $p_\infty \in A$. We have $\mathrm{sc}(\Phi_j(p_\infty)) \leq \mathrm{sc}_+$ for all $j \in \mathbb{N}$ and it follows

$$\int_0^j \Psi(\Phi_s(p_\infty)) \cdot \|(\nabla_{L^2}\mathrm{sc})_{\Phi_s(p_\infty)}\|^2_{\Phi_s(p_\infty)}\, ds \leq \mathrm{sc}_+ - \mathrm{sc}(\Phi_0(p_\infty)).$$

This yields a Palais–Smale sequence of $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$, hence by [**10**] the existence of a G-invariant Einstein metric on G/H.

Now, suppose $\mathfrak{h} < \mathfrak{n}(\mathfrak{h})$, where $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{m}_0$. Let T be a maximal torus of the compact, connected Lie group with Lie algebra $\mathfrak{m}_0$. We consider the restriction of the scalar curvature functional to the subspace $(\mathcal{M}_1^G)^T \subset \mathcal{M}_1^G$ of metrics which are invariant under the adjoint action of both H and T (see Lemma 4.28). Note that an H-subalgebra $\mathfrak{k}$ is T-adapted if and only if the canonical direction $v_{\mathrm{can}}(\mathfrak{k})$ of $\mathfrak{k}$ is contained in $T_Q(\mathcal{M}_1^G)^T$. All results provided in Sections 5 and 6 can be applied to $\mathrm{sc} : (\mathcal{M}_1^G)^T \to \mathbb{R}$, since there exist only finitely many T-minimal non-toral H-subalgebras (Corollary 7.2). Therefore, the simplicial complex $\Delta^T_{G/H}$ associated to G/H and T provides information about high energy levels of the scalar curvature functional restricted to $(\mathcal{M}_1^G)^T$. The claim follows now as above, since $(\mathcal{M}_1^G)^T$ is invariant under the Ricci flow. q.e.d.

For the compact homogeneous space $G/H = \mathrm{SU}(4)/\mathrm{U}(2)$, where $\mathrm{U}(2) \subset \mathrm{SO}(4) \cap \mathrm{Sp}(2)$, Figure 3 shows the graph of the scalar curvature functional $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$.

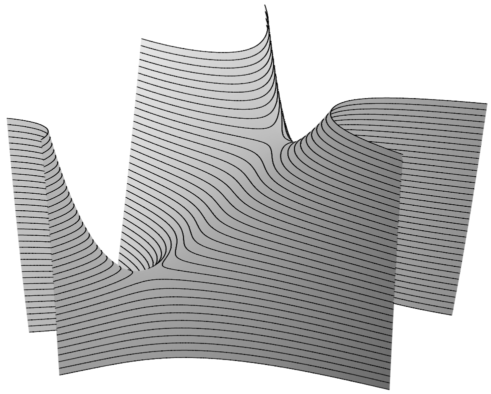

Figure 3. Scalar curvature functional on SU(4)/U(2).

In this case, the set B consists of three points g_1, g_2, g_3, such that $(\mathrm{sc}(g_1), g_1)$, $(\mathrm{sc}(g_2), g_2)$ and $(\mathrm{sc}(g_3), g_3)$ are located on the three connected components of the highest energy level visible. The base point $(\mathrm{sc}(Q), Q)$ is a point on a lower energy level (say below the lower saddle point) and the cone A would correspond to a union of three curves joining $(\mathrm{sc}(Q), Q)$ with $(\mathrm{sc}(g_i), g_i)$, $i = 1, 2, 3$. Then, applying the gradient flow Φ to the cone A, the points $p_u \in \Phi_u(A)$ having the lowest energy among all $p \in \Phi_u(A)$, will converge for $u \to +\infty$ to the lower saddle point.

The coindex (augmented coindex) of a critical point $g \in \mathcal{M}_1^G$ is by definition the dimension of the maximal subspace of $T_g\mathcal{M}_1^G$ on which the Hessian of $\mathrm{sc} : \mathcal{M}_1^G \to \mathbb{R}$ is positive definite (semi-definite).

Lemma 8.3. *Let G/H be a compact homogeneous space. Suppose that for a field $\mathbb{F}$ and for $q \in \mathbb{N}_0$, we have $\tilde{H}_q(\Delta^T_{G/H}, \mathbb{F}) \neq 0$. Then, G/H carries an Einstein metric $g \in \mathcal{M}_1^G$ whose augmented coindex is greater or equal than $q + 1$.*

Proof. Suppose $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$. We will invoke Theorem (5) in [**25**]. Let B be as in the proof of Theorem 8.1 and let α' be a non-trivial class in $\tilde{H}_q(B) \cong \tilde{H}_q(\Delta^T_{G/H})$. Since $\mathcal{M}_1^G$ is contractible, there exists an isomorphism $j_* : \tilde{H}_{q+1}(\mathcal{M}_1^G, B) \to \tilde{H}_q(B)$. Consequently, $\alpha := (j_*)^{-1}(\alpha')$ is a non-trivial class in $\tilde{H}_{q+1}(\mathcal{M}_1^G, B)$. For $A \subset \mathcal{M}_1^G$ with $B \subset A$, let $i : A \to \mathcal{M}_1^G$ denote the inclusion map and let $i_* : \tilde{H}_{q+1}(A, B) \to \tilde{H}_{q+1}(\mathcal{M}_1^G, B)$ denote the corresponding map in homology. Following [**25**], we define the homological family

$$\mathcal{F}(\alpha) = \{A \subset \mathcal{M}_1^G \mid A \text{ compact}, \ B \subset A \text{ and } \alpha \in \operatorname{Im}(i_*)\}$$

of dimension $q+1$ with boundary B. We have $\mathcal{F}(\alpha) \neq \emptyset$, since $A \in \mathcal{F}(\alpha)$, for A defined as in the proof of Theorem 8.1. Next, let

$$c = \sup_{A \in \mathcal{F}(\alpha)} \inf \{\operatorname{sc}(x) \mid x \in A\}.$$

From the proof of Theorem 8.1, we deduce $c > 0$. Moreover, $c \leq \operatorname{sc}_+$. Otherwise, there exists $A \subset \mathcal{F}(\alpha)$ with $A \subset \{\operatorname{sc} \geq \operatorname{sc}_+\} \subset C(U^\Sigma, t_0(\delta))$ such that $\alpha = i_*(\hat{\alpha})$ for $\hat{\alpha} \in \tilde{H}_{q+1}(A, B)$. Let $i^1 : A \to C(U^\Sigma, t_0(\delta))$ and $i^2 : C(U^\Sigma, t_0(\delta)) \to \mathcal{M}_1^G$ denote inclusion maps. Then, we have $i_* = i_*^2 \circ i_*^1$. But since $C(U^\Sigma, t_0(\delta))$ and B are homotopy equivalent, we conclude $\tilde{H}_{q+1}(C(U^\Sigma, t_0(\delta)), B) = 0$. Contradiction.

Now, let $F := \{\operatorname{sc} \leq c\}$. Then, F is closed and $F \cap B = \emptyset$ (see (8.2)). By the very definition of F and c, the closed subsets F and B are linked via $\mathcal{F}(\alpha)$, that is $\alpha \notin \operatorname{Im}(l_*)$ where $l_* : \tilde{H}_{q+1}(\mathcal{M}_1^G \backslash F, B) \to \tilde{H}_{q+1}(\mathcal{M}_1^G, B)$ is induced by the inclusion map $l : \mathcal{M}_1^G \backslash F \to \mathcal{M}_1^G$. By Theorem (5) in [**25**], we obtain the claim. In case $\mathfrak{n}(\mathfrak{h}) > \mathfrak{h}$, the claim follows in the same manner. q.e.d.

We turn now to the question, when existence of more than one critical point can be guaranteed. Since the corresponding Lyusternik–Schnirelmann-theory is well known, we present this part quite briefly essentially following [**3**].

Lemma 8.4. *Let G/H be a compact homogeneous space. Then, the scalar curvature functional has at least* $\operatorname{cuplength}(\Delta_{G/H}^T)$ *critical points.*

Proof. We repeat the proof described in [**3**] on p. 255–256 up to one minor modification. Consider

$$F : (\mathcal{M}_1^G, L^2) \to \mathbb{R}; \ g \mapsto -\operatorname{sc}(g)$$

(to adjust notation) and the continuous deformation $\Phi : \mathcal{M}_1^G \times \mathbb{R} \to \mathcal{M}_1^G$ with Φ as in the proof of Theorem 8.1. Then, we have $F(\Phi_t(p)) \leq F(p)$ for $p \in \mathcal{M}_1^G$ and $t \geq 0$. By [**10**], we obtain conclusion $(\star)$ in [**3**] (the Palais-Smale condition C) for $\kappa < 0$. We define the subsets $Y = \mathcal{M}_1^G$ and $Z = \{F \leq -\operatorname{sc}_+ - 1\}$ of $\mathcal{M}_1^G$. Both Y and Z are invariant under the deformation Φ.

Next, let $h \in \tilde{H}_*(\Delta_{G/H}^T)$ be non-zero. We think of h as a non-trivial class in $H_*(Z)$. Therefore, h corresponds to a non-trivial relative class in $H_*(Y,Z)$, again denoted by h. Conclusion (1.2) on page 255 follows as in [**3**], since there exists a cycle z in h, such that $|z| \subset \{F < 0\}$ (cf. proof of Theorem 8.1). Now, let $h_1, h_2 \in H_*(Y,Z)$ be non-zero, and suppose that there exists a cohomology class $\omega \in H^{\geq 1}(Y,Z)$, such that $\omega \cap h_2 = h_1$. We say, that h_1 is subordinate to h_2. (Here, we have to consider relative cohomology classes, since Y is contractible.) The claim follows now as in [**3**]. q.e.d.

Note that, in general, the above lemma does not imply that these distinct critical points are non-isometric.

Corollary 8.5. *Let G/H be a compact homogeneous space. Suppose that there exist at most finitely many G-invariant metrics of volume one in $(\mathcal{M}_1^G)^T$, which are solutions to the Einstein equation. Then, G/H admits at least* $\mathrm{cuplength}(\Delta^T_{G/H})$ *non-isometric Einstein metrics.*

There exist homogeneous spaces G/H, such that infinitely many G-invariant metrics of volume one are solutions to the Einstein equation. However, for homogeneous spaces G/H with $\mathrm{rk}\, G = \mathrm{rk}\, H$, it has been conjectured in [**10**] that this is impossible. Notice, furthermore, that there exist compact homogeneous spaces G/H admitting many non-isometric, G-invariant Einstein metrics, for instance $\mathrm{SO}(n)$, $n \geq 12$, admits at least n non-isometric Einstein metrics.

Let us conclude this section with the following important question:

Question 8.6. Does there exist an "algebraic" proof of Palais–Smale condition C for the Hilbert action, restricted to the space of G-invariant metrics of volume one with scalar curvature bounded from below by a positive constant?

The proof given in [**10**] is based on the deep theory on spaces with Ricci curvature bounded from below, developed by J. Cheeger and T.H. Colding [**18**], [**19**].

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