J. DIFFERENTIAL GEOMETRY
**67** (2004) 35-77

# THE COVERING SPECTRUM OF A COMPACT LENGTH SPACE

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## Abstract

We define a new spectrum for compact length spaces and Riemannian manifolds called the "covering spectrum" which roughly measures the size of the one dimensional holes in the space. More specifically, the covering spectrum is a set of real numbers $\delta > 0$ which identify the distinct $\delta$ covers of the space. We investigate the relationship between this covering spectrum, the length spectrum, the marked length spectrum and the Laplace spectrum. We analyze the behavior of the covering spectrum under Gromov–Hausdorff convergence and study its gap phenomenon.

## 1. Introduction

One of the most important subfields of Riemannian Geometry is the study of the Laplace spectrum of a compact Riemannian manifold. Recall that the Laplace spectrum is defined as the set of eigenvalues of the Laplace operator. The elements of the Laplace spectrum are assigned a multiplicity equal to the dimension of the corresponding eigenspace.

Another spectrum defined in an entirely different manner is the length spectrum of a manifold: the set of lengths of smoothly closed geodesics. There are various methods used to assign a multiplicity to each element of the length spectrum. The simplest notion is to count all geodesics of a given length. This becomes uninteresting when one has continua of geodesics of the same length as in a torus, so that all or some multiplicities become infinite. A common alternative definition of the multiplicity of a given length is the number of free homotopy classes of geodesics

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The first author was partially supported by NSF Grant # DMS-0102279 and a grant from the City University of New York PSC-CUNY Research Award Program and the second author was partially supported by NSF Grant # DMS-0204187.

Received 12/18/2003.

that contain a smoothly closed geodesic sharing that length (cf. [14]). We will use the latter definition.

It was proven by Colin de Verdiere [8] that the Laplace spectrum determines the length spectrum of a generic manifold. (See also Duistermaat–Guillemin's paper [10].) In particular, the Laplace spectrum determines the length spectrum on negatively curved manifolds of arbitrary dimension. However, there are pairs of isospectral manifolds first constructed by Carolyn Gordon [14] that have different length spectra when one takes multiplicity into account. These pairs are Heisenberg manifolds and Pesce has since shown that the length spectrum, not counting multiplicity, is determined by the Laplace spectrum on Heisenberg manifolds [23].

There is also a concept called the marked length spectrum which gives the lengths of smoothly closed geodesics freely homotopic to a representative of each element in the fundamental group. One has the remarkable result that compact surfaces of negative curvature with same marked length spectrum are isometric [22, 9, 12]. This is not true in general, as the sphere and the Zoll sphere have same marked length spectrum, but are not isometric [3]. Gornet has shown that Laplace isospectral nilmanifolds with the same marked length spectrum need not be isometric or have the same spectrum on one forms [17].

In this paper, we have defined a new spectrum for compact Riemannian manifolds which we call the *covering spectrum* (see Definition 3.1). In fact, this spectrum can be well defined on compact length spaces (Definition 2.1). Note that it isn't too difficult to extend the concept of a length spectrum to such spaces, but there is no natural Laplace spectrum unless one adds an appropriately defined measure on the metric space (see e.g., [7, Section 6]).

The authors first defined a special sequence of covering spaces for a given complete length space, $X$, called the delta covers of $X$ in [25] (see Definition 2.3). We used these delta covers to study the fundamental groups of these spaces and their universal covers. In particular, we proved that the universal cover of a compact length space $X$ is a $\delta$ cover for a sufficiently small real number $\delta$ [25, Proposition 3.2]. We can now show that a compact length space $X$ has a universal cover iff there is only a finite set of distinct delta covers (Theorem 3.4). We have named the corresponding finite list of distinct real numbers the "covering spectrum" of $X$ (Definition 3.1). Roughly, this covering spectrum lists the sizes of one dimensional holes in $X$. For example, the covering spectrum of a $1 \times 3$ flat torus is $\{1/2, 3/2\}$ and the covering spectrum of the standard $\mathbb{R}P^2$ is $\{\pi/2\}$. Recall that if $X$ is a compact Riemannian manifold, then it is a length space and has a universal cover, so its covering spectrum is well defined and finite.

Compact length spaces have grown in interest among Riemannian Geometers in recent years because they are the natural limits of Riemannian manifolds using Gromov's compactness theorem [**18**]. Gromov–Hausdorff limits of Cauchy sequences of Riemannian manifolds with a uniform upper bound on diameter are compact length spaces and, with appropriate curvature bounds on the manifolds, they are metric measure spaces [**18**], [**6**]. Cheeger and Colding have proven Fukaya's conjecture that the Laplace spectra of a sequence of manifolds with a uniform lower Ricci curvature bound converge to the Laplace spectrum of the metric measure limit space [**7**]. It is important to note that one needs metric measure convergence of the manifolds, not just Gromov–Hausdorff convergence to control the Laplace spectrum in this way [**13**].

On the other hand, Gromov–Hausdorff convergence does not interact well with length spectra in general. This is because closed geodesics can disappear and appear in the limit and the length spectrum (even the minimal length spectrum) of the sequence doesn't converge to the length spectrum of the limit (cf. Examples 8.1–8.3).

Here, we have shown that the covering spectrum interacts very nicely with Gromov–Hausdorff convergence (Theorem 8.4) and is fairly easy to define both on manifolds and limit spaces. This follows from the fact that the delta covering spaces are well controlled when the base spaces converge in the Gromov–Hausdorff sense (Theorem 7.3) and [**25**, Theorem 3.6]. Another interesting property is that the covering spectrum, when assigned an appropriate multiplicity, may be used to study fundamental groups (Definition 6.1, Proposition 6.4).

We prove that every element in the covering spectrum is $(1/2)$ of an element in the length spectrum (Theorem 4.7). We also prove that the marked length spectrum determines the covering spectrum on any compact length space with a universal cover (Theorem 5.7) . We also discuss the relationship between the covering and the Laplace spectra on compact Riemannian manifolds and give a number of examples as described below. In particular, we construct Laplace isospectral Heisenberg manifolds with different covering spectra (Example 10.3).

The paper is organized as follows.

In Section 2, we provide all the necessary background including the definition of delta covers and some key examples. In particular, we recall that a universal cover is a cover of all covers and that the Hawaii ring is a compact length space with no universal cover.

In Section 3, we define the covering spectrum, $\mathrm{CovSpec}\,(X)$, for an arbitrary compact length space, $X$, and prove that $\mathrm{CovSpec}\,(X)$ is discrete and $Cl(\mathrm{CovSpec}\,(X) \subset \mathbb{R}) \subset \mathrm{CovSpec}\,(X) \cup \{0\}$ (Proposition 3.2). We then prove that $\mathrm{CovSpec}\,(X)$ is finite iff $X$ has a universal cover (Theorem 3.4).

In Section 4, we restrict our attention to compact length spaces that have a universal cover. We extend the definition of length spectrum, (Definition 4.2) to these spaces and prove Theorem 4.7 that $\mathrm{CovSpec}\,(X) \subset (1/2)\mathrm{LengthSpec}\,(X)$. We then further restrict ourselves to compact length spaces with simply connected universal covers and extend the definition of the minimal length spectrum (Definition 4.10). We prove this spectrum is closed and discrete and that $\mathrm{CovSpec}\,(X) \subset (1/2)\,\mathrm{MinLengthSpec}\,(X)$ (Theorem 4.12) .

In Section 5, we extend the definition of the marked length spectrum to these compact length spaces with universal covers (Definition 5.1). Note that to extend the definition of the marked length spectrum which ordinarily depends on the fundamental group of the manifold, we use the "revised fundamental group" instead. This is the group of deck transforms of the universal cover (Definition 4.3).

We then prove Theorem 5.7 that the marked length spectrum determines the covering spectrum. In fact, Theorem 5.7 also relates the covering spectrum to a special sequence of subgroups of the revised fundamental group, which is then used to define multiplicity for the covering spectrum in Section 6.

As one would expect, the covering spectrum contains less information than the length spectrum. This can be seen in our example of a smooth one parameter family of non-isospectral tori with a common covering spectrum (Example 5.6). Note that flat tori are isospectral iff they share the same length spectrum, and are determined up to isometry by their marked length spectrum [**16**]. On the other hand, length spectrum alone does not determine the covering spectrum. We have examples of compact Riemannian manifolds with a common length spectrum, but a distinct covering spectrum (Example 5.10).

In Section 6, we define multiplicity (Definition 6.1) for the covering spectrum, and find a bound on $\#_m(\mathrm{CovSpec}\,(X) \cap [a, b])$ for $a > 0$, where

$\#_m$ is the cardinality of the set counting multiplicity (Lemma 6.2). We also define a special set of generators of the revised fundamental group (Definition 6.1) that we call the short basis. Roughly, these are the elements of the revised fundamental group represented by loops wrapped one time around a single hole in the space. We prove this set generates the revised fundamental group in Proposition 6.4, and show that the number of elements in this set is $\#_m(\mathrm{CovSpec}\,(X))$.

In Section 7, we focus on the relationship between the covering spectrum and Gromov–Hausdorff convergence. We begin by studying the Gromov–Hausdorff convergence of the delta covers, proving that if $X_i$ converge to $X$ in the GH sense, then a subsequence of the delta covers $\tilde{X}_i^\delta$ converges as well (Theorem 7.3). This involves reworking Gromov's precompactness theorem and carefully controlling the group of deck transforms of a delta cover. In Example 7.4, we show it is necessary to use a subsequence, and in Example 7.5, we show that universal covers need not have converging subsequences. An immediate application is that if we have a GH compact class of compact length spaces with universal covers, then for $b > a > 0$, $\#_m(\mathrm{CovSpec}\,(X) \cap [a, b])$ is uniformly bounded on this class (Corollary 7.7). One can also use the precompactness of the $\delta$-covers to show that the *revised* fundamental groups of such a compact class with an additional uniform lower bound on the first systole, have finitely many isomorphism classes extending Theorem 5 in [**24**].

In Section 8, we prove that if compact length spaces $X_i$ converge to a compact length space $Y$ in the GH sense, then the covering spectra converge (Theorem 8.4). In particular,

$$\lim_{i\to\infty} d_H(\mathrm{CovSpec}\,(X_i) \cup \{0\}, \mathrm{CovSpec}\,(Y) \cup \{0\}) \to 0, \tag{1.1}$$

where $d_H$ is the Hausdorff distance between subsets of the real line (Corollary 8.5). Note that it is easy to see that when $1/j \times 1$ tori converge to a circle, there are elements of the covering spectrum which converge to 0. Note also that if $M_i$ are compact Riemannian manifolds with $\mathrm{Ricci}\,(M_i) \geq -(n-1)H$ and $\mathrm{diam}\,(M_i) \leq D$, such that $M_i$ converge to $Y$, then $\#(\mathrm{CovSpec}\,(M_i)) \geq \#(\mathrm{CovSpec}\,(Y))$ for $i$ sufficiently large (not counting multiplicity) (Corollary 8.6).

We also prove that connected classes of compact length spaces with a common discrete length spectrum, have a common covering spectrum

(Theorem 8.7). In particular, a one parameter family of compact Riemannian manifolds with a common length spectrum must have a common covering spectrum (Corollary 8.8).

In Section 9, we study the gap phenomenon of the covering spectrum in certain classes of compact length spaces with universal covers (Propositions 9.2 and 9.6). We apply these results and [**25**, Theorem 1.1] to describe the gap and clumping properties of the covering spectra of Riemannian manifolds with $\mathrm{Ricci}\,(M_i) \geq -(n-1)H$ and $\mathrm{diam}\,(M_i) \leq D$ and their limit spaces (Corollary 9.5 and 9.7).

In Section 10, we relate the covering spectrum with the Laplace spectrum of a manifold. We first easily show that if we have a class of negatively curved compact Riemannian manifolds with a common Laplace spectrum, then there are only finitely many possible covering spectra in this class (Proposition 10.1). We conjecture that this is true without the negative sectional curvature condition, but with a uniform upper bound on diameter (Conjecture 10.2). In Example 10.3, we give a pair of Heisenberg manifolds which are Laplace isospectral and yet have distinct covering spectra. This example heavily uses the work of Carolyn Gordon in [**14**], but it should be noted that her famous pairs of Laplace isospectral Heisenberg manifolds with distinct length spectra, in fact, have the same covering spectrum. We close by demonstrating that special pairs of Sunada isospectral manifolds, the ones he attributes to Komatsu [**28**, Example 3], always share the same covering spectrum and, in fact, have only one element in that covering spectrum (Proposition 10.5).

## 2. Background

First, we recall some basic definitions.

**Definition 2.1.** A *complete length space* is a complete metric space such that every pair of points in the space is joined by a length minimizing rectifiable curve. The distance between the points is the length of that curve. A *compact length space* is a compact complete length space (cf. [**5**]).

**Definition 2.2.** We say $\bar{X}$ is a *covering space* of $X$ if there is a continuous map $\pi : \bar{X} \to X$ such that $\forall x \in X$ there is an open neighborhood $U$ such that $\pi^{-1}(U)$ is a disjoint union of open subsets of $\bar{X}$ each of which is mapped homeomorphically onto $U$ by $\pi$ (we say $U$ is evenly covered by $\pi$).

Let $\mathcal{U}$ be any open covering of $Y$. For any $p \in Y$, by [**27**, p. 81], there is a covering space, $\tilde{Y}_{\mathcal{U}}$, of $Y$ with covering group $\pi_1(Y,\mathcal{U},p)$, where $\pi_1(Y,\mathcal{U},p)$ is a normal subgroup of $\pi_1(Y,p)$, generated by homotopy classes of closed paths having a representative of the form $\alpha^{-1} \circ \beta \circ \alpha$, where $\beta$ is a closed path lying in some element of $\mathcal{U}$ and $\alpha$ is a path from $p$ to $\beta(0)$.

Now, let us recall the $\delta$-covers we introduced in [**25**].

**Definition 2.3.** Given $\delta > 0$, the $\delta$-cover, denoted $\tilde{Y}^{\delta}$, of a length space $Y$, is defined to be $\tilde{Y}_{\mathcal{U}_\delta}$ where $\mathcal{U}_\delta$ is the open covering of $Y$ consisting of all balls of radius $\delta$.

The covering group will be denoted $\pi_1(Y,\delta,p) \subset \pi_1(Y,p)$ and the group of deck transformations of $\tilde{Y}^{\delta}$ will be denoted $G(Y,\delta) = \pi_1(Y,p)/\pi_1(Y,\delta,p)$.

It is easy to see that a delta cover is a regular or Galois cover. That is, the lift of any closed loop in Y is either always closed or always open in its delta cover.

We now state some very simple lemmas.

**Lemma 2.4.** *If $\pi : \bar{Y} \to Y$ is a covering map between complete length spaces and $\forall y \in Y$, $\pi^{-1}(B_y(r))$ is a disjoint collection of balls of radius $r$ in $\bar{Y}$, then $\tilde{Y}^r$ covers $\bar{Y}$.*

*Proof.* Recall that in [**27**, Chapter 2, Section 5, Lemma 11], Spanier shows that if $\pi : \bar{Y} \to Y$ is a covering projection and $\mathcal{U}$ is an open covering of $Y$ such that each of its open sets is evenly covered by $\pi$, then $\tilde{Y}_{\mathcal{U}}$ covers $\bar{Y}$. Here $\mathcal{U}$ is the collection of balls of radius $r$, so we need only show that these balls are evenly covered by $\pi$.

Let $B_{\bar{y}}(r) \subset \pi^{-1}(B_y(r))$. We need only show $\pi : B_{\bar{y}}(r) \to B_y(r)$ is a homeomorphism. In fact, by the hypothesis, it is a covering map. Thus, if it is not 1:1, there are two preimages of $y$: $\bar{y}_1$ and $\bar{y}_2$. Note that $B_{\bar{y}_i}(r)$ is a connected subset of $\pi^{-1}(B_y(r))$, so it is a subset of $B_{\bar{y}}(r)$ in which case $\bar{y} = \bar{y}_i$ and $\pi$ is 1:1. q.e.d.

**Example 2.5.** Suppose $Y$ is a flat $3 \times 2$ torus: $S^1_3 \times S^1_2$, then it has the following delta covers:

$$\begin{aligned} \tilde{Y}^{\delta} &= Y \quad \text{for } \delta > 3/2, \\ \tilde{Y}^{\delta} &= S^1_2 \times \mathbb{R} \quad \text{for } \delta \in (1, 3/2], \\ \tilde{Y}^{\delta} &= \mathbb{R}^2 \quad \text{for } \delta \in (0, 1]. \end{aligned}$$

**Lemma 2.6.** *The $\delta$ covers of complete length spaces are monotone in the sense that if $r < t$, then $\tilde{X}^r$ covers $\tilde{X}^t$. In fact, $\tilde{X}^r$ is the $r$-cover of the complete length space $\tilde{X}^t$.*

*Proof.* Let $Y_t = \tilde{X}^t$. We need only show $\tilde{Y_t}^r = \tilde{X}^r$ for $r < t$. Since the balls of radius $r$ in $X$ lift to unions of disjoint balls of radius $r$ in $\tilde{Y_t}^r$, by applying Lemma 2.4 we have $\tilde{X}^r$ covers $\tilde{Y_t}^r$.

Recall from [**27**, Chapter 2, Section 5 and 8] that if $\mathcal{V}$ is an open covering of $X$ that refines $\mathcal{U}$, then $\pi_1(X, \mathcal{V}, p) \subset \pi_1(X, \mathcal{U}, p)$, or $\tilde{X}_{\mathcal{V}}$ covers $\tilde{X}_{\mathcal{U}}$.

Thus, clearly, $\tilde{X}^r$ covers $Y_t = \tilde{X}^t$.

Now, we apply Lemma 2.4 to balls of radius $r$ in $Y_t$. These must lift to unions of disjoint balls of radius $r$ in $\tilde{X}^r$, as can be seen by first projecting them down to $X$. Thus, $\tilde{Y_t}^r$ covers $\tilde{X}^r$ and we are done. q.e.d.

In the following lemma, we restrict ourselves to compact length spaces.

**Lemma 2.7.** *The $\delta$-covers of a compact length space $X$ are lower semi-continuous. In fact, for any $\delta > 0$, there exists $\epsilon \in (0, \delta)$ such that $\tilde{X}^\epsilon = \tilde{X}^\delta$.*

*Proof.* If not, there is a sequence of $\delta_i \to \delta$ increasingly, such that $\tilde{X}^{\delta_i} \neq \tilde{X}^\delta$ for each $i$. Namely there exist a sequence of closed curves $\gamma_i$ in $X$ with length $l(\gamma_i) \leq 2\,\mathrm{diam}\,(X) + 2\delta_i$, which lifts to an open curve in $\tilde{X}^{\delta_i}$, but a closed curve in $\tilde{X}^\delta$. Parametrize each curve by the unit interval $[0, 1]$ with constant speed. Since $X$ is compact, by Arzela–Ascoli theorem, there is a subsequence of $\gamma_i$ which converges to some closed curve $\gamma : [0, 1] \to X$ uniformly. So $d(\gamma_i(t), \gamma(t)) < \delta/2$ for all $i$ large and $t \in [0, 1]$. Hence, $\gamma_i, \gamma$ lift the same to the covering spaces $\tilde{X}^{\delta_i}$, $\tilde{X}^\delta$ for $i$ large. That is, $\gamma$ lifts to an open curve in $\tilde{X}^{\delta_i}$ for all $i$ large and a closed curved in $\tilde{X}^\delta$. From Definition 2.3, $\gamma$ lies in some finite union of open $\delta$-balls in $X$, so it must also lie in some union of open $\delta'$-balls for some $\delta' < \delta$, which contradicts to that $\gamma$ lifts to an open curve in $\tilde{X}^{\delta_i}$ for all $i$ large. q.e.d.

**Example 2.8.** The Hawaii ring (cf. [**27**]) is a compact length space which consists of an infinite set of rings of radii $r_i$ decreasing to 0, all joined at a common point. This space has an infinite sequence of distinct $\delta$ covers as $\delta$ converges to 0.

A modified complete non-compact Hawaii ring can be created by taking an infinite set of rings of radius $r_i$ increasing to $r_0 = 1$. This space also has an infinite sequence of distinct $\delta$ covers as $\delta$ approaches $r_0$ and demonstrates that compactness is a necessary condition for lower semi-continuity.

In both of these spaces, the $\tilde{X}^{\pi r_i}$ are all distinct covers.

This example demonstrates that the compactness hypothesis in Lemma 2.7 is necessary.

## 3. The covering spectrum

We now define the covering spectrum by singling out the deltas where the delta covering spaces change.

**Definition 3.1.** Given a complete length space $X$, the covering spectrum of $X$, denoted $\mathrm{CovSpec}\,(X)$ is the set of all $\delta > 0$ such that

$$\tilde{X}^{\delta} \neq \tilde{X}^{\delta'} \tag{3.1}$$

for all $\delta' > \delta$.

Since the $\delta$-covers are monotone, this is equivalent to, say, for any $\epsilon > 0$, there exists $\delta'$ with $0 < \delta' - \delta < \epsilon$ such that

$$\tilde{X}^{\delta} \neq \tilde{X}^{\delta'}.$$

In general, for a compact length space $X$, the $\mathrm{CovSpec}\,(X)$ lies in $(0, \mathrm{diam}\,(X))$.

In our above examples, the covering spectrum of the flat 3 x 2 torus is $\{1, 3/2\}$, and the traditional Hawaii ring with infinite circles of radii $r_i$ is $\{\pi r_i : i \in \mathbb{N}\}$.

We have the following property of the covering spectrum.

**Proposition 3.2.** *For a compact length space, $X$, its* $\mathrm{CovSpec}\,(X)$ *is discrete and*

$$Cl(\mathrm{CovSpec}\,(X) \subset \mathbb{R}) \subset \mathrm{CovSpec}\,(X) \cup \{0\}. \tag{3.2}$$

*Proof.* Since zero is not in $\mathrm{CovSpec}\,(X)$, if $\mathrm{CovSpec}\,(X)$ is not discrete, we can assume it has an accumulation at some $\delta > 0$. In fact, we can assume there is a strictly decreasing sequence of $\delta_i \in \mathrm{CovSpec}\,(X)$ converges to the $\delta > 0$ since $\delta$-covers are lower semi-continuous (Lemma 2.7). Let $\gamma_i$ be a loop at $p \in X$ such that $\gamma_i$ lifts trivially to $\tilde{X}^{\delta_i}$

but non-trivially to $\tilde{X}^{\delta_{i+1}}$ with length $l(\gamma_i) \leq 2\,\mathrm{diam}\,(X)+2\delta_i$. Parametrize each curve in the sequence by the unit interval $[0,1]$ with constant speed. Since $X$ is compact, by the Arzela–Ascoli theorem, there is an uniformly converging subsequence, which we will still call $\gamma_i$. So there is an $i$ sufficiently large that $d(\gamma_i(t), \gamma_{i+1}(t)) < \delta$ for all $t \in [0,1]$. Since $\delta_i > \delta$ the covering maps are isometric on $\delta$ balls for all $i$, so $\gamma_i, \gamma_{i+1}$ lift the same to the covering spaces $\tilde{X}^{\delta_i}$, $\tilde{X}^{\delta_{i+1}}$, contradicting that $\gamma_{i+1}$ lifts trivially to $\tilde{X}^{\delta_{i+1}}$ and $\gamma_i$ lifts non-trivially to $\tilde{X}^{\delta_{i+1}}$. Therefore, $\mathrm{CovSpec}\,(X)$ is discrete. q.e.d.

The example of the Hawaii ring shows that 0 could be in the closure of the covering spectrum of a compact length space. Proposition 3.2 is not true for a non-compact complete length space as another revised Hawaii ring, the union of the sequence of circles with a common point and radius $r_i$ decreasing to 1, shows.

We now turn to a discussion of the existence of universal covers. The original compact Hawaii ring with $r_i \to 0$ is a classic example of a compact length space with no universal cover. Recall the definition of a universal cover.

**Definition 3.3** ([**27**, pp. 62, 83])**.** We say $\tilde{X}$ is a *universal cover* of $X$ if $\tilde{X}$ is a cover of $X$ such that for any other cover $\bar{X}$ of $X$, there is a commutative triangle formed by a continuous map $f : \tilde{X} \to \bar{X}$ and the two covering projections.

In [**25**, Proposition 3.2], we proved that if a compact length space $Y$ has a universal cover $\tilde{Y}$, then $\tilde{Y}$ is a delta cover. In fact, $Y$ has a universal cover iff the delta covers stabilize: there exists a $\delta_0 > 0$ such that $\tilde{Y}^\delta = \tilde{Y}^{\delta_0}$ for all $\delta < \delta_0$ [**25**, Theorem 3.7]. Clearly, the delta covers of the Hawaii ring do not stabilize.

**Theorem 3.4.** *For a compact length space $X$, its universal cover $\tilde{X}$ exists iff its covering spectrum,* $\mathrm{CovSpec}\,(X)$*, is finite.*

*Proof.* If the $\mathrm{CovSpec}\,(X)$ is finite, then $\delta_0 = \min\{\mathrm{CovSpec}\,(X)\}$ is positive. So, the $\delta$-covers stabilize and by [**25**, Theorem 3.7], the universal cover of $X$ exists.

If the universal cover $\tilde{X}$ of $X$ exists, then $\mathrm{CovSpec}\,(X)$ lies in $[\delta_X, \mathrm{diam}\,(X)]$ for some $\delta_X > 0$. By Proposition 3.2, the $\mathrm{CovSpec}\,(X) \cap [\delta_X, \mathrm{diam}\,(X)]$ is closed and discrete. Thus, $\mathrm{CovSpec}\,(X)$ is finite. q.e.d.

Although the covering spectrum is defined using layers of covering spaces, $\#\{\mathrm{CovSpec}\,(X)\}$ does not count the total number of covering spaces of $X$. The $\delta$ covers are a very small selection of covering spaces. Clearly, the $3 \times 2$ torus has many covering spaces that are tori and cylinders which are not delta covers (i.e., $S^1_{3i} \times S^1_{2j}$, $\mathbb{R} \times S^1_{2j}$ and $S^1_{3j} \times \mathbb{R}$). Furthermore, the lens spaces, $S^3$ mod $\mathbb{Z}^k$ with the standard metric only have one $\delta$-cover, $S^3$, although they often have many covering spaces.

The covering spectrum can intuitively be thought of as capturing the size of holes in the length space. For the $2 \times 3$ torus, it captures information about both of the holes in the torus: both of the generators of the fundamental group. The fact that $\#\{\mathrm{CovSpec}\,(X)\} = 2$, in this example, is strongly related to the fact that there are two generators of the fundamental group.

On the other hand, the covering spectrum of a $1 \times 1$ torus has only one element because both holes in this torus have the same size. Later on, we will define multiplicity for the elements of the covering spectrum, which will better enable us to capture the fact that there are two "holes" in this torus as well.

## 4. The covering spectrum and length spectrum

In this section, we restrict our attention to complete length spaces $X$ which have a universal cover.

First, recall that a geodesic in a length space is a curve which is locally a distance minimizer in the following sense [**5**].

**Definition 4.1.** A curve $\gamma : I \to X$ is called a *geodesic* if for every $t \in I$, there exists an interval $J$ containing a neighborhood of $t$ in $I$ such that $\gamma|_J$ is a shortest path. A *closed geodesic* is a geodesic loop which is minimizing in a neighborhood of its end point.

It is easy to use the definition of a covering space to show that a length minimizing curve in a covering space projects to a geodesic, and that geodesics lift to geodesics.

Then, one can naturally extend the definition of length spectrum from manifolds to complete length spaces.

**Definition 4.2.** The *length spectrum*, $\mathrm{Length}(X)$, of a complete length space, $X$, is the set of lengths of closed geodesics. It is counted with multiplicity where the multiplicity refers to the number of distinct free homotopy classes that contain a closed geodesic of that length.

We recall the definition of the revised fundamental group from [25].

**Definition 4.3.** The *revised fundamental group*, $\bar{\pi}_1(X)$, of a complete length space, $X$, with a universal cover, $\tilde{X}$, is the group of deck transforms of the universal cover. Given an element, $g \in \bar{\pi}_1(X)$, and a base point $x \in X$ a *representative loop* of $g$ based at $x$ is a curve, $c$, such that $c(0) = x$ whose lift $\tilde{c}$ to the universal cover runs from a point $\tilde{c}(0)$ to $g\tilde{c}(0)$. If one does not mention the basepoint, a representative loop depends only on the conjugacy class of $g$ and can be based at any point in $X$.

For simplicity, the reader may wish to assume that $X$ has a simply connected universal cover, or equivalently, that $X$ is semi-locally simply connected. In that case, the fundamental group $\pi_1(X)$ of $X$ is isomorphic to $\bar{\pi}_1(X)$.

In general, however, the universal cover of a compact length space may not be simply connected. One example is the double suspension over the Hawaii Ring (cf. [27]) which is its own universal cover, but has an infinite fundamental group because the infinite alternation of loops in the Hawaii rings are not contractible. These loops are homotopic to loops in an arbitrarily small neighborhood, but not to a single point.

When the universal cover is not simply connected, the representive loops of the identity element are the projections of arbitrary loops in the universal cover, which are not necessarily contractible. Thus, the equivalence class of representative loops corresponding to an element $g \in \bar{\pi}_1(X)$ and a point $x \in X$ is not a homotopy equivalence class, but rather a collection of homotopy equivalence classes.

The following lemma is easy to prove using the fact that the universal cover is a $\delta$-cover and using the compactness of $X$.

**Lemma 4.4.** *Given a compact length space $X$ with a universal cover $\tilde{X}$, for all non-trivial elements $g \in \bar{\pi}_1(X)$, we have*

$$m(g) := \min_{\tilde{x} \in \tilde{X}} d_{\tilde{X}}(\tilde{x}, g\tilde{x}) \subset \text{Length}\,(X). \tag{4.1}$$

*If $\gamma_g$ is the projection of a minimizing curve joining a minimizing pair of points $\tilde{x}$ and $g\tilde{x}$, then $\gamma_g$ is a closed geodesic in $X$ of length $m(g)$ which is a shortest representative of $g$ and a shortest curve in its free homotopy class.*

*Proof.* There exists $\delta > 0$ such that the universal cover is a $\delta$ cover. Thus,

$$m(g) = \inf_{\tilde{x}\in\tilde{X}} d_{\tilde{X}}(\tilde{x}, g\tilde{x}) \geq 2\delta > 0. \tag{4.2}$$

Let $\tilde{x}_i \in \tilde{X}$ approach this infimum. Since $X$ is compact, a subsequence of $x_i = \pi(\tilde{x}_i)$ converges to some $x$ whose lift $\tilde{x}$ then achieves this infimum. Then, $\gamma_g$ defined above is the shortest representative of $g$ for any base point, it has length $m(g)$ and it is the projection of a geodesic to a loop. Extending the definition of $\gamma_g$ periodically, we can see that it is a representative of $g$ based at $\gamma_g(t)$ as well. So, it must be the projection of a length minimizing curve between $\tilde{\gamma}_g(t)$ and $g\tilde{\gamma}_g(t)$ which implies that it is a closed geodesic. q.e.d.

Thus, we have the following useful map.

**Definition 4.5.** The *minimum marked length map* of a compact length space $X$ with a universal cover is the function $m : \bar{\pi}_1(X) \to \mathrm{LengthSpec}\,(X) \cup \{0\}$ defined in Lemma 4.4.

**Remark.** The minimum marked length map is closely related to the translative delta length $l(g, \delta)$ we defined in [**25**, Definition 3.2]. Recall

$$l(g, \delta) = \min_{q\in\tilde{X}^\delta} d_{\tilde{Y}^\delta}(q, g(q)). \tag{4.3}$$

and $l(g, \delta) \geq 2\delta$ for all $g$ which act non-trivially on $\tilde{X}^\delta$. Note that, since covering maps are distance decreasing,

$$m(g) \geq l(g, \delta) \quad \text{for all } g \in \bar{\pi}_1(X). \tag{4.4}$$

So, $m(g) \geq 2\delta$ for the largest $\delta > 0$ such that a representative loop of $g$ lifts to a curve with distinct endpoints in $\tilde{X}^\delta$.

**Lemma 4.6.** *When $X$ is compact, the set* $\mathrm{Im}\,(m) = \{m(g) : g \in \bar{\pi}_1(X)\}$ *is closed and discrete. Furthermore, $m(g) = 0$ iff $g = e$.*

*Proof.* First note that, by the Arzela–Ascoli theorem, sequences of length minimizing curves have subsequences which converge to length minimizing curves, so if $m_i \in \mathrm{Im}(m)$ converge to $m_\infty$, then we have a subsequence of $x_i$ in the fundamental domain of $\tilde{X}$ converging to $x_\infty$, and $g_i x_i$ converging to some $y_\infty$ such that $d_{\tilde{X}}(x_\infty, y_\infty) = m_\infty$. Since the universal cover is a delta cover, if $g_j^{-1}g_i \neq e$, then it must move points at least a distance $\delta$. However,

$$\begin{aligned} d(g_j^{-1}g_i x_i, x_i) &\leq d(g_j^{-1}g_i x_i, x_j) + d(x_j, x_i) \\ &= d(g_i x_i, g_j x_j) + d(x_i, x_j) \\ &< \delta/2 + \delta/2 \quad \text{for } i, j \text{ sufficiently large.} \end{aligned}$$

So $g_j = g_i$ and $m_i = m_j$ for $i$ sufficiently large. Thus, $m_\infty = m_i$ for all $i$ large and $\mathrm{Im}(m)$ is closed and discrete.

We know $m(g) = 0$ iff $g = e$ because only a trivial deck transform fixes a point. q.e.d.

This leads to our first theorem.

**Theorem 4.7.** *When $X$ is a compact length space with a universal cover, then*

$$2\mathrm{CovSpec}\,(X) \subset \mathrm{Im}\,(m(\bar{\pi}_1(X)) \subset \mathrm{LengthSpec}\,(X) \cup \{0\}, \tag{4.5}$$

*where $m$ is the minimum marked length map defined in Definition 4.5.*

This theorem follows from the following definition and lemma.

**Definition 4.8.** If $X$ is a complete length space and $\delta > 0$, then we say a $\delta$-pair is a pair of points $\{x_1, x_2\}$ in $\tilde{X}^\delta$ which are not equal, but are projected to the same point in $\tilde{X}^{\delta'}$ for all $\delta' > \delta$.

**Lemma 4.9.** *Fix a compact length space with a universal cover $X$, and $\delta \in \mathrm{CovSpec}\,(X)$. Let $h_\delta = \inf d_{\tilde{X}^\delta}(x_1, x_2)$ over all $\delta$-pairs $x_1, x_2$. Then, this infimum is achieved, and there is an element $g \in \bar{\pi}_1$ such that $m(g) = h_\delta$ and $h_\delta = 2\delta$.*

*Proof.* By compactness, it is easy to show that there exists a $\delta$ pair $x_1, x_2$ which achieves this infimum. It is not necessarily a unique pair even up to deck transforms.

First, $h_\delta \geq 2\delta$, else a minimizing curve from $x_1$ to $x_2$ would have length $< 2\delta$ and its projection to $X$ would fit in $B_{\pi(x_1)}(\delta)$, so it would be lifted as a loop to $\tilde{X}^\delta$ making $x_1 = x_2$ by Definition 2.3.

Now, we will show $h = h_\delta \leq 2\delta$. By Theorem 3.4, the covering spectrum is finite, so there is $\epsilon > \delta$ such that for all $\delta' \in (\delta, \epsilon)$, $\tilde{X}^{\delta'} = \tilde{X}^\epsilon$. Naturally, $\tilde{X}^\delta$ is a non-trivial cover of $\tilde{X}^\epsilon$.

Note that $y_1$ and $y_2$ are a $\delta$ pair iff they are not equal, but project to the same point in $\tilde{X}^\epsilon$. So, we have for all $y_1 \neq y_2 \in \tilde{X}^\delta$ such that

$\pi_\epsilon(y_1) = \pi_\epsilon(y_2) \in \tilde{X}^\epsilon$, $B_{y_1}(h/2)$ and $B_{y_2}(h/2)$ are disjoint. So, for all $z \in \tilde{X}^\epsilon, B_z(h/2)$ lifts to a disjoint union of balls in $\tilde{X}^\delta$. Thus, by Lemma 2.4, applied to $\bar{Y} = \tilde{X}^\delta$ as a cover of $Y = \tilde{X}^\epsilon$, we get $\tilde{Y}^{h/2}$ covers $\bar{Y} = \tilde{X}^\delta$. If $h/2 > \epsilon$, then $\tilde{Y}^{h/2} = \tilde{X}^\epsilon$, which means $\tilde{X}^\delta = \tilde{X}^\epsilon$. This is a contradiction. So, we have $h/2 \leq \epsilon$, then $\tilde{Y}^{h/2} = \tilde{X}^{h/2}$, so, $\tilde{X}^{h/2}$ covers $\tilde{X}^\delta$. Therefore, $h \leq 2\delta$. So, $h_\delta = 2\delta$.

Now, let $C$ be a minimal geodesic connecting $x_1$ and $x_2$ and $g$ an element in $\bar{\pi}_1(X)$ which is represented by the projection of $C$. Then, $m(g) \leq 2\delta$. But $g$ acts non-trivially on $\tilde{X}^\delta$, so $l(g,\delta) \geq 2\delta$. Therefore, $m(g) \geq 2\delta$ by (4.4). Thus, $m(g) = 2\delta$. q.e.d.

Another standard length spectrum defined on manifolds is the minimal length spectrum.

**Definition 4.10.** The *minimal length spectrum* is the set of lengths of closed geodesics which are the shortest in their free homotopy class.

If a compact length space $X$ is semilocally simply connected, or equivalently has a simply connected universal cover, then the above definition makes sense and each homotopy class contains a curve of minimum length. In fact, the minimal length spectrum agrees with $im(m) \setminus \{0\}$ as can be seen in the following lemma combined with Lemma 4.4.

**Lemma 4.11.** *For a compact length space with a simply connected universal cover, the minimum marked length map $m$ maps surjectively onto the minimal length spectrum $\cup\{0\}$.*

*Proof.* Given any $L$ in the minimal length spectrum, there is a free homotopy class of loops whose minimum length is $L$. Let $c_1$ be the shortest such loop. It defines a deck transform $g$ and $m(g) \leq L(c) = L$. Suppose $m(g) < L$, then there exists $\tilde{x} \in \tilde{X}$ such that $d(g\tilde{x}, \tilde{x}) < L$. Join this pair of points by a length minimizing curve $\tilde{c}_2$.

If the universal cover is simply connected, then the projection $c_2$ is a loop freely homotopic to $c_1$ and we have a contradiction. q.e.d.

Theorem 4.7 and Lemmas 4.6 and 4.11 combine to give us the following theorem.

**Theorem 4.12.** *When $X$ is a compact length space with a simply connected universal cover, then the minimum length spectrum is closed and discrete and*

$$2\text{CovSpec}\,(X) \subset \text{MinLengthSpec}\,(X). \tag{4.6}$$

## 5. The marked length spectrum

A stronger concept than the length spectrum of a manifold is the marked length spectrum which includes information about the fundamental group itself. Here, we will study arbitrary compact length spaces with universal covers. The natural extension of the definition of marked length spectrum to such spaces involves the revised fundamental group $\bar{\pi}_1(X)$ instead of the fundamental group (Definition 4.3). For simplicity, the reader may wish to assume the universal cover is simply connected, in which case, the revised fundemantal group is just the fundamental group of the space.

**Definition 5.1.** Given a complete length space $X$, the *marked length spectrum* of $X$ is a function $MLS$ that associates to each element $g$ in $\bar{\pi}_1(X)$ the set of lengths, $MLS(g)$, of the closed geodesics freely homotopic to a representative loop of $g$. Clearly, this map only depends on the conjugacy class of $g$.

Two spaces $X_1$ and $X_2$ are said to have the same marked length spectrum iff there is an isomorphism between their revised fundamental groups which commutes with their marked length maps $MLS_1$ and $MLS_2$.

Recall the definition of the minimum marked length map, $m : \bar{\pi}_1(X) \to (0, \infty)$, in Definition 4.5 and Lemma 4.4. Since the image $MLS(g)$ includes the lengths of all geodesics representing $g$, we have $m(g) = \min(MLS(g))$.

**Definition 5.2.** We say two spaces with universal covers $X_1$ and $X_2$ have the same *minimum marked length spectrum* iff there is an isomorphism between their revised fundamental groups which commutes with their minimum marked length maps $m_1$ and $m_2$.

We can also mark the covering spectrum of a compact length space with a universal cover using the following simple map:

**Definition 5.3.** Given a complete length space $X$ with a universal cover, we define the *covering spectrum map*, $f : \bar{\pi}_1(X) \to \mathrm{CovSpec}\,(X) \cup \{0\}$ as follows: given $g \in \bar{\pi}_1(X)$, let $f(g)$ be the unique $\delta$ in CovSpec (X) such that $g$ acts non-trivially on $\tilde{X}^{\delta}$, but trivially on $\tilde{X}^{\delta'}$ for all $\delta' > \delta$. That is, there exists a loop $\gamma_g$ representing $g$ in $X$ lifts to a curve in $\tilde{X}^{\delta}$ that is not a loop, but lifts to a loop in $\tilde{X}^{\delta'}$ for all $\delta' > \delta$. Note that all loops freely homotopic to this one will then also share this property.

Equivalently, $f(g)$ is the largest $\delta$ such that the projections of $gx$ and $x$ from $\tilde{X}$ to $\tilde{X}^\delta$ are distinct points.

Note $f(g) = 0$ iff $g = e$ and $f(hgh^{-1}) = f(g)$ for all $h$ because the definition of $f$ is basepoint free.

**Lemma 5.4.** *Given a compact length space $X$ with a universal cover, the covering spectrum map $f : \bar{\pi}_1(X) \to \mathrm{CovSpec}\,(X) \cup \{0\}$ is surjective.*

*Proof.* If $\delta \in \mathrm{CovSpec}\,(X)$, then $\tilde{X}^\delta \neq \tilde{X}^{\delta'}$ for all $\delta' > \delta$. Since $X$ is compact, the covering spectrum is discrete away from 0 [Theorem 3.2], so there exists $\epsilon > \delta$ such that $\tilde{X}^\epsilon = \tilde{X}^{\delta'}$ for all $\delta' \in (\delta, \epsilon]$. Let $x_1$ and $x_2$ be a pair of distinct points in $\tilde{X}^\delta$ which are mapped to the same point in $\tilde{X}^\epsilon$. Let $C$ be a curve joining $x_1$ to $x_2$. Then, $C$ projects to a loop in $X$, which lifts as a loop to $\tilde{X}^{\delta'}$ for all $\delta' > \delta$ and lifts to a curve that is not a loop in $\tilde{X}^\delta$. Let $g \in \bar{\pi}_1(X)$ which is represented by the projection of $C$, then $f(g) = \delta$. q.e.d.

Note that the compactness in this lemma is necessary as the following example shows. Let $X$ be a revised Hawaii ring with circles of radius $1 + 1/n$ all attached at one point, then $X$ has a universal cover. Furthermore, $\pi$ is in $\mathrm{CovSpec}\,(X)$, but it does not lie in the image of $f$ since the circle of radius 1 is not in $X$.

**Lemma 5.5.** *When $X$ is a complete length space with a universal cover and $f$ is the covering spectrum map, then $f^{-1}([0, \delta])$ is a subgroup of $\bar{\pi}_1(X)$.*

*Proof.* If $g_1, g_2 \in f^{-1}([0, \delta])$, suppose $f(g_i) = \delta_i$, then there are loops $\gamma_i$ which lift as closed loops to $\tilde{M}^{\delta'}$ for $\delta' > \delta_i$ and as open curves to $\tilde{M}^{\delta_i}$. So, for any $\delta' > \max\{\delta_1, \delta_2\}$, both curves $\gamma_1, \gamma_2$ lift as closed loops to $\tilde{M}^{\delta'}$. Now, the element $g_1 g_2$ can be represented by the loop $\gamma_1$ following $\gamma_2$, so the lift of the combination is closed in $\tilde{M}^{\delta'}$. Thus,

$$f(g_1 g_2) \leq \max\{\delta_1, \delta_2\} = \max\{f(g_1), f(g_2)\}. \tag{5.1}$$

q.e.d.

A non-positively curved metric on a surface of genus $\geq 2$ with the set where the curvature is 0 has empty interior is determined up to isometry by its marked length spectrum [**22**, **9**, **12**]. The same is true for flat tori [**16**]. The following example demonstrates that even on flat tori, the covering spectrum does not determine the isometry class. In fact, it includes a smooth family of flat tori with a common covering spectra.

**Example 5.6.** Here, we examine a set of flat 2 dimensional tori, $T^2_\theta$, defined as rhombi with side length 1 and a variable angle $0 < \theta \leq \frac{\pi}{2}$ between the sides. Opposite sides are identified in the usual way and the universal cover of any of these examples is the Euclidean plane. Note that, in this case, the marked length spectrum has only one length per element of the abelian fundamental group, so we can denote it as $m(g)$.

If we locate the fundamental domain with corners at the points $(0,0)$, $(1,0)$, $(\cos(\theta),\sin(\theta))$ and $(1+\cos(\theta),\sin(\theta))$, then the group of deck transforms is generated by $g_1 : (x,y) \mapsto (x+1,y)$ and $g_2 : (x,y) \mapsto (x+\cos(\theta), y+\sin(\theta))$.

Now, for $\theta \in [\pi/3, \pi/2]$, it is easy to see that

$$m(g_1^a g_2^b) = \sqrt{a^2+b^2+2ab\cos(\theta)},$$

So, the length spectrum is

$$\{\sqrt{a^2+b^2+2ab\cos(\theta)} : a,b \in \mathbb{Z} \setminus \{0\}\}. \tag{5.2}$$

Furthermore, $f(g_1^a g_2^b) = 1/2$, unless $a = b = 0$, so the covering spectrum is just $\{1/2\}$. This provides us with a one parameter family of flat tori with a common covering spectrum.

Now for $\theta \in (0, \pi/3)$, we get the same formula

$$m(g_1^a g_2^b) = \sqrt{a^2+b^2+2ab\cos(\theta)},$$

So, the length spectrum is

$$\{\sqrt{a^2+b^2+2ab\cos(\theta)} : a,b \in \mathbb{Z} \setminus \{0\}\}. \tag{5.3}$$

However, now

$$f(g_1^a g_2^b) = \begin{cases} (1/2)\sqrt{1+1-2\cos(\theta)} < 1/2 & \text{if } a = -b, \\ 1/2 & \text{otherwise.} \end{cases} \tag{5.4}$$

So, the covering spectrum has two distinct elements.

Since these two families together form a single one parameter family, we have also shown that the number of elements of the covering spectrum may change. It is nice to see that here the covering spectra do vary continuously in Hausdorff sense.

In fact, the covering spectrum is determined by the minimum marked length spectrum (Definition 5.2).

**Theorem 5.7.** *Let $X_1$ and $X_2$ be compact length spaces with universal covers. If they have the same minimum marked length spectrum, then they have the same covering spectrum.*

*Moreover, if* $\mathrm{CovSpec}\,(X) = \{\delta_1 < \delta_2 < \cdots < \delta_k\}$, *then there exists a special sequence of subgroups* $\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots G_k = \bar{\pi}_1(X)$ *such that each $G_i$ is generated by*

$$S_i = \{h \in \bar{\pi}_1(X) : m(h) = 2\delta_i\} \tag{5.5}$$

*combined with the elements of $G_{i-1}$. Furthermore, the covering map $f$ has the following property: $f(g) = \delta_i$ implies $g \in G_i$ while $g \in G_i \setminus G_{i-1}$ has $f(g) = \delta_i$.*

First, we state a simple lemma which we will need.

**Lemma 5.8.** *Suppose $C : [0, L] \to B_q(\delta) \subset X$ where $X$ is a complete length space, then $C$ is freely homotopic to a product of curves of length $< 2\delta$ based at $q$.*

*Proof.* We assume $C$ is parametrized by arclength. Since its image is closed, it is in fact contained in $B_q(\delta - \epsilon)$ for some $\epsilon > 0$. Partition $[0, L]$ into pieces of length $< \epsilon$: $t_1 = 0 < t_2 < t_3 < \cdots < t_k = L$. Let $\sigma_i$ run minimally from $q$ to $C(t_i)$ so it has length $< \delta - \epsilon$. Set $\sigma_k = \sigma_1$. So, $C_i$ starting at $q$ running along $\sigma_i$ to $C(t_i)$ running along $C$ to $C(t_{i+1})$ and running backwards along $\sigma_{i+1}$ to $q$ is a closed curve of length $< 2\delta$.

The product of these $C_i$ is a curve which is freely homotopic to $C$ (where the homotopy runs along $\sigma_1 = \sigma_k$). q.e.d.

**Corollary 5.9.** *If $C : [0, L] \to B_q(\delta) \subset X$ parametrized by arclength is the shortest non-contractible curve in $X$, then $L < 2\delta$.*

*Proof of Theorem 5.7.* We will derive the marked covering map $f \colon \bar{\pi}_1(X) \to \mathrm{CovSpec}\,(X) \cup \{0\}$ from the marked shortest length spectrum $m$.

We first claim that

$$f(g) \le (1/2)m(g) \quad \text{for all } g \in \bar{\pi}_1(X). \tag{5.6}$$

If $(1/2)m(g) < f(g)$, then there is a loop representing $g$ of length $< 2f(g)$. Such a curve must be contained in a ball of radius $f(g)$, so it would lift as a closed curve to the $f(g)$ cover of $X$, but it cannot by the definition of $f(g)$ (Definition 5.3).

Let $G_0 = \{e\}$ and $\delta_0 = 0$. We will construct the covering spectrum and the map $f$ by induction.

**5.1. Induction hypotheses. (a)** We have defined distinct subgroups $\emptyset = G_{-1} \subset G_0 \subset G_1 \subset \cdots G_k \subset \bar{\pi}_1(X)$ and defined $\delta_0 < \delta_1 < \cdots < \delta_k$ such that each

$$\delta_i = \min\{(1/2)m(g) : g \in \bar{\pi}_1(X) \setminus G_{i-1}\} \subset \{0\} \cup \mathrm{CovSpec}\,(X). \tag{5.7}$$

**(b)** Each $G_i$ is generated by all the elements $h \in \bar{\pi}_1(X)$ such that $m(h) = 2\delta_i$ combined with the elements of $G_{i-1}$.

**(c)** Each $G_i$ contains every element $g \in \bar{\pi}_1(X)$ such that $f(g) = \delta_i$ and

$$\{\delta_0, \delta_1, \ldots, \delta_k\} = \mathrm{CovSpec}\,(X) \cap [0, \delta_k] \cup \{0\}. \tag{5.8}$$

Note that for all $g \in G_i \setminus G_{i-1}$, we have $f(g) = \delta_i$ as a consequence of the above hypotheses (b) and (c), and of (5.6) and (5.1).

First, the hypotheses (a)–(c) are trivially true for $k = 0$.

Next, we prove the induction step assuming (a)–(c) for $k$:
If $G_k \neq \bar{\pi}_1(x)$, let

$$\delta_{k+1} = \min\{(1/2)m(g) : g \in \bar{\pi}_1(X) \setminus G_k\}. \tag{5.9}$$

Since $\{m(g)\}$ is closed and discrete (Lemma 4.6), the minimum exists and is achieved.

We will now show $\delta_{k+1} \in \mathrm{CovSpec}\,(X)$ thus completing the proof of (a) for $k + 1$. To do so, we show $f(h) = \delta_{k+1}$ for the $h$ achieving the minimum in (5.9).

We have $h \in \bar{\pi}_1(X) \setminus G_k$ such that $m(h) = 2\delta_{k+1}$. By (5.6), $f(h) \leq \delta_{k+1}$. Suppose, we assume that $f(h) < \delta_{k+1}$. Then, for any $f(h) < \delta' < \delta_{k+1}$ $h$ lifts trivially to $\tilde{X}^{\delta'}$. So $h$ is a product of elements $g_1 g_2 ... g_l$ where each $g_i$ has a representative curve based at $p$ of the form $\alpha_i^{-1}\beta_i\alpha_i$ where $\alpha_i$ runs from $p$ to some $p_i$ and $\beta_i$ is in a ball $B_{q_i}(\delta')$. By Lemma 5.8, each $\beta_i$ is freely homotopic to a product of curves of length $< 2\delta' < 2\delta_{k+1}$. Thus, each $g_i$ is a product of $g_{i,j} \in \bar{\pi}_1(X)$ which have representative curves freely homotopic to curves of length $< 2\delta_{k+1}$. So, $m(g_{i,j}) < 2\delta_{k+1}$. We also know by the definition of $\delta_{k+1}$ in (5.9) that for any $g \in \bar{\pi}_1(X) \setminus G_k$, we have $m(g) \geq \delta_{k+1}$. Thus, $g_{i,j} \in G_k$ and so are their products $g_i$ and $h = g_1 g_2 \cdots g_l$. This is a contradiction since $h$ was chosen to be in $\bar{\pi}_1(X) \setminus G_k$. So, $f(h) = \delta_{k+1}$.

For (b), we just define $G_{k+1}$ to be the group generated by $G_k$ and elements $h \in \bar{\pi}_1$ such that $m(h) = 2\delta_{k+1}$.

For (c), we first show (5.8) for $k + 1$. So, we must show that if $\delta \in (\delta_k, \delta_{k+1})$, then $\delta$ is not in $\mathrm{CovSpec}\,(X)$. Suppose $\delta \in (\delta_k, \delta_{k+1})$

is in CovSpec $(X)$. By Lemma 4.9, there exists an element $g \in \bar{\pi}_1(X)$ with $f(g) = \delta, m(g) = 2\delta$. So, $\frac{1}{2}m(g) < \delta_{k+1}$ and $g$ must be in $G_k$. But then, $f(g) \leq \delta_k$ which is a contradiction.

To finish (c), we need only show $G_{k+1}$ includes all $g$ such that $f(g) = \delta_{k+1}$. Let $h \in \bar{\pi}_1(X)$ be an element such that $f(h) = \delta_{k+1}$. Then, for any $\delta' > \delta_{k+1}$, $h$ lifts trivially to $\tilde{X}^{\delta'}$. By Lemma 4.6, we can choose

$$\delta_{k+1} < \delta' < (1/2)\min(\{m(g) : g \in \bar{\pi}_1(X)\} \cap (2\delta_{k+1}, \infty)) \tag{5.10}$$

so that if $m(g) < 2\delta'$ then $m(g) \leq 2\delta_{k+1}$.

Now, $h$ is a product of elements $g_1 g_2 \cdots g_l$ where each $g_i$ has a representative curve based at $p$ of the form $\alpha_i^{-1}\beta_i\alpha_i$ where $\alpha_i$ runs from $p$ to some $p_i$ and $\beta_i$ is in a ball $B_{q_i}(\delta')$. By Lemma 5.8, each $\beta_i$ is freely homotopic to a product of curves of length $< 2\delta'$ and, by the choice of $\delta'$, to a product of curves of length $\leq 2\delta_{k+1}$. So, $h$ is a product of elements of $\bar{\pi}_1$ which have representative curves freely homotopic to curves of length $\leq 2\delta_{k+1}$ so $m$ of these elements is $\leq 2\delta_{k+1}$. Thus, $h$ is a product of elements in $G_{k+1}$ and $h$ itself is in $G_{k+1}$.

This completes the proof of the induction hypothesis.

Finally, using the finiteness of the covering spectrum [Lemma 3.4], we know that this process must terminate. Thus, by (c), eventually $G_k$ must equal $\bar{\pi}_1(X)$. So, we have determined the value of $f$ for every element of $\bar{\pi}_1(X)$ and determined the marked covering spectrum of $X$.
q.e.d.

The following examples demonstrate that the length spectrum alone does not determine the covering spectrum. We have many more examples in Section 10 which have the same Laplace spectra and length spectra, but different covering spectra.

**Example 5.10.** Let $M_1 = S^2_{\pi/2}$ be the standard sphere of diameter $\pi/2$, $M_2 = \mathbb{R}P^2_\pi = S^2_\pi/\mathbb{Z}_2$. Then, the length spectra of both $M_1$ and $M_2$ are $\{l\pi : l \in \mathbb{N}\}$, while the covering spectrum of $M_1$ is empty and the covering spectrum of $M_2$ is $\{\pi/2\}$. Here, $M_1, M_2$ have different fundamental groups.

There are also examples with the same fundamental group. The product spaces $M_1 = \mathbb{R}P^2_\pi \times S^2_\pi$ and $M_2 = \mathbb{R}P^2_{2\pi} \times S^2_{\pi/2}$ are diffeomorphic and share the following length spectrum:

$$\left\{ \sqrt{(k\pi)^2 + (2l\pi)^2} : k, l \in \mathbb{N} \cup \{0\} \right\} \setminus \{0\}. \tag{5.11}$$

The covering spectrum of $M_1$ is $\{\pi/2\}$ while the covering spectrum of $M_2$ is $\{\pi\}$.

## 6. Counting generators of fundamental groups

In this section, we restrict ourselves to compact length spaces $X$ which have universal covers.

The sequence of groups $G_i$ and sets $S_i$ in Theorem 5.7 give us a way to construct a short basis of $\bar{\pi}_1(X)$ and to define the multiplicity of the covering spectrum.

**Definition 6.1.** For each $\delta_j \in \mathrm{CovSpec}\,(X)$, the *basis multiplicity* of $\delta_j$ is the minimum number of $g \in S_j$ required to generate $G_j$.

Let $\bar{S}_j \subset S_j$ be a list of such generators. Let a *short basis* of $\bar{\pi}_1(X)$ be $S = \bigcup \bar{S}_j$.

Note that the covering space of a compact length space with lifted metric is a locally compact complete length space, therefore by the Hopf-Rinow theorem for metric spaces (see [**18**] or [**5**, Theorem 2.5.28]), each bounded closed domain is compact. Hence, we show below that the multiplicity in the above is always finite for compact length spaces. In fact, we have the following lemma:

**Lemma 6.2.** *Let $X$ be a compact length space with a universal cover $\tilde{X}$. For $b \geq a > 0$ and $D = \mathrm{diam}\,(X)$,*

$$\#_m\{\mathrm{CovSpec}\,(X) \cap [a,b]\} = \sum_{j:\delta_j \in [a,b]} \#\bar{S}_j \leq \tilde{N}(a, 2b + 2D + a), \tag{6.1}$$

*where $\tilde{N}(a,b)$ is the maximum number of disjoint balls of radius $a$ that fit in a ball of radius $b$ in $\tilde{X}$.*

This estimate, in particular, gives an estimate on the multiplicity of a fixed element $\delta$. Lemma 6.2 will be improved later, see Corollary 7.7.

*Proof of Lemma 6.2.* Let $\{\lambda_1, \ldots, \lambda_k\} = \mathrm{CovSpec}\,(X) \cap [a,b]$ counted with multiplicity. By Lemma 4.9 for each $\lambda_i$, there is a $g_i$ in $\bar{\pi}_1(X)$ such that $a \leq f(g_i) = \frac{1}{2}m(g_i) \leq b$. By the proof of Theorem 5.7, $\frac{1}{2}m(g_i^{-1}g_j) \geq f(g_i^{-1}g_j) \geq a$ for all $i \neq j$. Fix $\tilde{p} \in \tilde{X}$, we have $d(\tilde{p}, g\tilde{p})) \leq m(g) + 2D$ for any $g \in \bar{\pi}_1(X)$. Therefore, each ball $B(g_i\tilde{p}, a)$ is disjoint from each other for $i = 1, \ldots, k$ and all are isometric and lie in the ball $B(\tilde{p}, 2b + 2D + a)$. This gives (6.1). q.e.d.

The following example shows that the multiplicities of short elements of the covering spectrum can grow to infinity, while elements in the covering spectrum converge to 0.

**Example 6.3.** Let $M_j^2$ be a handlebody with $j$ handles which looks like a standard 2 sphere with many small handles on the scale of $1/j$. The multiplicity of $1/j$ goes to infinity as $j$ goes to infinity.

Note that the multiplicity in Definition 6.1 does not agree with the multiplicity of the length spectrum. We have deliberately related it to the revised fundamental group rather than to free homotopy classes of loops. This way, Theorem 5.7 immediately gives us the following proposition.

**Proposition 6.4.** *For a compact length space $X$ with a universal cover, $\bar{\pi}_1(X)$ can be generated by the short basis $S$ of Definition 6.1 and* $\#S = \#_m\{\mathrm{CovSpec}\,(X)\}$.

Note that the number of generators of a fundamental group $\pi_1(X,p)$ may not be finite for a compact length space, $X$, with a non-simply connected universal cover. The double cone over the Hawaiian earring is its own universal cover, so $\#\{\mathrm{CovSpec}\,X\} = 0$, but its fundamental group is uncountable and, in particular, not finitely generated.

## 7. Gromov–Hausdorff convergence and $\delta$ covers

Here, we first prove a convergence property of $\delta$-covering spaces which does not hold for universal covers. Then, we show that, unlike the length spectrum, the covering spectrum behaves nicely under Gromov–Hausdorff convergence. We begin with the definition of the Gromov–Hausdorff distance between compact length spaces.

**Definition 7.1** ([**18**, Definition 3.4])**.** Given two metric spaces $X$ and $Y$ the *Gromov Hausdorff distance* between them is defined,

(7.1)
$$d_{GH}(X,Y) = \inf\left\{ d_H^Z(f(X),f(Y)) : \begin{array}{l} \text{all metric spaces, } Z\text{, and} \\ \text{all isometric embeddings:} \\ f: X \to Z, g: Y \to Z \end{array} \right\},$$

where, $d_H^Z$ is the Hausdorff distance between subsets of $Z$,

$$d_H^Z(A,B) = \inf\{\epsilon > 0 : B \subset T_\epsilon(A) \text{ and } A \subset T_\epsilon(B)\}. \tag{7.2}$$

Here, $T_\epsilon(A) = \{x \in Z : d_Z(x,A) < \epsilon\}$.

It is then clear what we mean by the Gromov–Hausdorff convergence of compact metric spaces. However, for non-compact metric spaces, the following looser definition of convergence was defined by Gromov.

**Definition 7.2** ([**18**, Definition 3.14])**.** We say that non-compact length spaces $(X_n, x_n)$ converge in the pointed Gromov–Hausdorff sense to $(Y, y)$ if for any $R > 0$, there exists a sequence $\epsilon_n \to 0$ such that $B_{x_n}(R + \epsilon_n)$ converges to $B_y(R)$ in the Gromov–Hausdorff sense.

It is easy to see that neither the topology of a metric space nor the dimension is conserved under Gromov–Hausdorff convergence. Two compact spaces are close in the GH sense if they look almost the same with "blurry vision" so that "small holes" cannot be seen. A sequence of $1 \times 1/j$ tori collapses to a circle losing both dimension and topology. The sequence of handlebodies, $M_j^2$, of Example 6.3 converges to a standard sphere, thus losing topology without collapsing to a lower dimension. One also can lose regularity, as can be seen, when taking a sequence of one-sheeted hyperboloids converging to a cone.

**Proposition 7.3.** *If a sequence of compact length spaces $X_i$ converges to a compact length space $X$ in the Gromov–Hausdorff topology, then for any $\delta > 0$ there is a subsequence of $X_i$ such that their $\delta$-covers also converges in the pointed Gromov–Hausdorff topology.*

This answers a question in [**26**]. Compare Proposition 3.1 in there.

By Theorem 3.6 in [**25**], the limit of the $\delta$-covers (if it exists) is always a cover of $X$, but note that two different subsequences could have different limits as the next example shows.

**Example 7.4.** Let $X_i$ be tori of side lengths 1 by $(n+1)/(2n)$ alternating with the tori of length 1 by $(n-1)/(2n)$. Then $X_i$ converges to the 1 by $1/2$ torus. For $\delta = 1/2$, we get two limits of the $\delta$-covers: one is the cylinder and the other is Euclidean space.

In the following examples, we demonstrate that universal covers may not have any converging subsequences. Recall that Gromov's Precompactness Theorem [**18**] states that a set, $S$, of compact length spaces is precompact iff there is a uniform upper bound, $N(r, R)$, on the number of disjoint $r$ balls contained in a ball of radius $R$, $N_X(r, R)$:

(7.3)

$$\forall r, R > 0\ \exists N(r, R) \in \mathbb{N} \text{ s.t. we have } N_X(r, R) \le N(r, R) \quad \forall X \in S.$$

**Example 7.5.** Let $M_j^j$ be a flat $j$ dimensional $1 \times (1/j) \times (1/j) \times \cdots \times (1/j)$ torus. Then the Gromov–Hausdorff limit of $M_j^j$ is a circle. The universal covers of the $M_j^j$ are Euclidean $j$ dimensional spaces, so $N_{M_j}(1,5) \geq 2j$ and the $M_j$ do not have a converging subsequence.

Other examples include the sequence of spheres with small handles, $M_j^2$ in Example 6.3, which converges in the Gromov–Hausdorff sense to the standard two sphere and a sequence of finite sets of circles joined at a common point which converges to the standard Hawaii ring. In both cases, the sequences of universal covers do not having any converging subsequences.

Gromov proved that if $M_j$ are closed manifolds with Ricci curvature uniformly bounded from below and dimension bounded above, then by the Bishop Gromov Volume Comparison Theorem, $N_{M_j}(r,R)$ is uniformly bounded [**18**]. Since the universal covers of the $M_j$ share these curvature and dimension bounds, they do have converging subsequences. However, even in this case, the limits of universal covers are not necessarily covers of the limit space. An example is a sequence of flat $1 \times 1/j$ tori which collapse to a circle. The limit of the universal covers is the Euclidean plane which is not a cover of a circle.

*Proof of Proposition* 7.3. It is enough to show that the set of $\delta$-covers of $X_i$ is precompact by finding a uniform bound on $N_{\tilde{X}_i^\delta}(r,R)$. Since $X_i$ converge to a limit space $X$ in the GH sense, they also converge in the pointed GH sense, so there exists $x_i \in X_i$ and $x \in X$ such that $(X_i, x_i)$ converges to $(X, x)$. So, we need only prove that for all $\epsilon, R > 0$, the number of disjoint balls of radius $\epsilon$ centered in $B_{\tilde{x}_i}(R)$ is uniformly bounded. In fact, we can fix $\epsilon < \delta < R$ since bounds for these $\epsilon$ and $R$ will control the others.

Let $\tilde{x}_i \in \tilde{X}_i^\delta$ be a lift of $x_i$. Let $\mathrm{FD}_i$ be a (closed) fundamental domain of $X_i$ based at $\tilde{x}_i$. Let the $\epsilon$ almost adjacent generators

$$F_{\epsilon,i} = \{g \in G(X_i,\delta) :\, gT_\epsilon(\mathrm{FD}_i) \cap T_\epsilon(\mathrm{FD}_i) \neq \emptyset\}, \tag{7.4}$$

and, let the adjacent generators be the set

$$F_i = \{g \in G(X_i,\delta) :\, g(\mathrm{FD}_i) \cap (\mathrm{FD}_i) \neq \emptyset\} \subset F_{\epsilon,i}. \tag{7.5}$$

Now, examine $B(\tilde{x}_i, R) \subset \tilde{X}_i^\delta$. By Milnor's lemma [**21**, Lemma 2], if $d(\tilde{x}_i, g\tilde{x}_i) < \delta k$ for some positive integer $k$, then $g$ can be expressed as a $k$-fold product, $g = h_1 h_2 \cdots h_k$, with $h_1, \ldots, h_k \in F_i$. Let $k = [R/\delta] + 1$, where $[R/\delta]$ is the integer part of $R/\delta$. Thus, the number of fundamental domains $g\mathrm{FD}_i$ intersecting $B(\tilde{x}_i, R)$ is bounded by $(\#F_i)^{[R/\delta]+1}$.

On the other hand, if $N_i(\epsilon, D)$ is the number of maximal disjoint $\epsilon$-balls in $X_i$, then if $\epsilon < \delta$, we claim the maximal number of disjoint $\epsilon$-balls centered in each fundamental domain $\mathrm{FD}_i$ is bounded by $N = N_i(\epsilon, D) \cdot \#F_{\epsilon,i}$. If not, then let $\tilde{y}_1, \ldots, \tilde{y}_{N+1}$ be the centers of these balls and $y_1, \ldots, y_{N+1}$ be their projections to $X_i$. Since the covering map $\tilde{X}_i^\delta$ is isometric on $\delta$ balls,

$$B_{y_j}(\epsilon) \cap B_{y_k}(\epsilon) = \emptyset \text{ iff } B_{\tilde{y}_j}(\epsilon) \cap B_{g\tilde{y}_k}(\epsilon) = \emptyset \ \forall g \in G(X_i, \delta). \tag{7.6}$$

which is equivalent to checking that

$$B_{\tilde{y}_j}(\epsilon) \cap B_{g\tilde{y}_k}(\epsilon) = \emptyset \ \forall g \in F_{i,\epsilon}. \tag{7.7}$$

So, we can select $N_i(\epsilon, D)$ disjoint $\epsilon$ balls in $X_i$ by first choosing $y_1$ and eliminating the at most $\#F_{\epsilon,i}$ $y_k$ that fail to satisfy (7.7) for $y_1$, then choosing the next remaining $y_j$ and eliminating the at most $\#F_{\epsilon,i}$ $y_k$ that fail to satisfy (7.7) for that $y_j$, and so on. This is a contradiction.

So, the total number of balls of radius $\epsilon$ in $B_{\tilde{x}_i}(R)$ is bounded by $(\#F_i)^{[R/\delta]+1} \cdot N_i(\epsilon, D) \cdot \#F_{\epsilon,i}$. Since $F_i \subset F_{\epsilon,i}$, we need only bound $\#F_{\epsilon,i}$ uniformly in $i$.

Note that by Theorem 3.4 in [**25**], we have surjective homomorphisms

$$\Phi_i : G(X, \delta/2) \to G(X_i, \delta) \tag{7.8}$$

for all $i$ large. We can assume that $\operatorname{diam} X_i \leq D$. If $\bar{\alpha} \in F_{\epsilon,i} \subset G(X_i, \delta)$, then it can be represented by a closed curve $\bar{\sigma}$ passing through $x_i = \pi(\tilde{x}_i)$ of length $\leq 2(D + \epsilon)$. From the proof of surjectivity in [**25**, Theorem 3.4 ], we can take an $\epsilon$ partition of $\bar{\sigma}$ and get a curve $\sigma$ passing through $\pi(\tilde{x}) \in X$ such that $\Phi_i(\sigma) = \bar{\alpha}$ and the length of $\sigma$ is at most 5 times as long as $\bar{\sigma}$. Thus, each element $g \in F_{\epsilon,i} \subset G(X_i, \delta)$ is mapped to by $\Phi_i$ of some element $h \in G(X, \delta/2)$ such that $d_{\tilde{X}^{\delta/2}}(h\tilde{x}, \tilde{x}) < 10(D + \epsilon)$.

Now, if $g_1$ and $g_2$ are two distinct elements in $G(X_i, \delta)$ and $\Phi_i(h_1) = g_1$ and $\Phi_i(h_2) = g_2$, then $h_1 h_2^{-1} \in G(X, \delta/2)$ is non-trivial. Any non-trivial element $h \in G(X, \delta/2)$ has $d_{\tilde{X}^{\delta/2}}(hx, x) \geq \delta$. So, $d_{\tilde{X}^{\delta/2}}(h_1 x, h_2 x) \geq \delta$.

Hence, for all $i$ large

$$\#\{F_{\epsilon,i}\} \leq \tilde{N}(\delta/2, 10(D + \epsilon) + \delta/2), \tag{7.9}$$

where $\tilde{N}_i(\delta/2, R')$ is the maximal number of disjoint balls of radius $\delta/2$ that fit in a ball, $B(\tilde{x}, R') \subset \tilde{X}^{\delta/2}$, in the limit spaces $\delta$ cover. q.e.d.

An immediate corollary of this is the following.

**Corollary 7.6.** *Let $\mathcal{M}$ be a GH compact set of length spaces and $\mathcal{M}^\delta$ be the set consisting of their delta covers. Then, $\mathcal{M}^\delta$ is precompact and $\tilde{N}(\epsilon, R)$ is uniformly bounded on $\mathcal{M}^\delta$.*

Note that $\mathcal{M}^\delta$ need not be compact since a limit of $\delta$ covers need not be a $\delta$ cover. See the example in [**25**] immediately above Theorem 3.6.

Corollary 7.6 enables us to give an improvement of Lemma 6.2.

**Corollary 7.7.** *For all $X$ in a Gromov–Hausdorff compact set $\mathcal{M}$ of compact length spaces with universal covers and $b > a > 0$, $\#_m$ $(\mathrm{CovSpec}\,(X) \cap [a, b])$ is uniformly bounded.*

As another nice application of Corollary 7.6 we have the following.

**Proposition 7.8.** *The revised fundamental groups of a Gromov–Hausdorff compact set of complete length spaces with a uniform lower bound on their first systole have finitely many isomorphism classes.*

Here, the first systole of $(X) = \inf \mathrm{CovSpec}\,(X)$, which is a natural way of extending the definition of first systole to length spaces that are not semilocally simply connected.

This result generalizes Theorem 5 in [**24**] on manifolds. The same proof in [**24**] carries over once we have a uniform bound for $\tilde{N}(\epsilon, R)$ on the universal covers.

## 8. Convergence of the covering spectrum

Note that the length spectrum can change dramatically under Gromov–Hausdorff convergence as the following examples show. First, we see that lengths can disappear in the limit.

**Example 8.1.** Let $X_n$ be the boundary of the $1/n$-neighborhood of the closed planar unit disk in $\mathbb{R}^3$ with the induced length metric, then $X_n$ converges to the double disk (identification of two closed unit disks along the boundary circles). The circle $x^2 + y^2 = (1 + 1/n)^2$ is a closed geodesic in $X_n$, but the limit curve $x^2 + y^2 = 1$ is not a geodesic in the limit space. In fact, $2\pi(1 + 1/n) \in$Length Spectrum of $X_n$, but its limit $2\pi$ is not. $X_n$ can be easily approximated by 2-dimensional smooth manifolds with same properties.

The following example shows that even minimal length spectrum may not converge.

**Example 8.2.** Let $M$ be a smooth manifold which is $S^2$ with two small handles: one near the pole and one near the equator, and $M_i$ is a sequence of $M$ as the handles getting smaller and smaller and converges to $Y = S^2$ in Gromov–Hausdorff sense (see Figure 1).

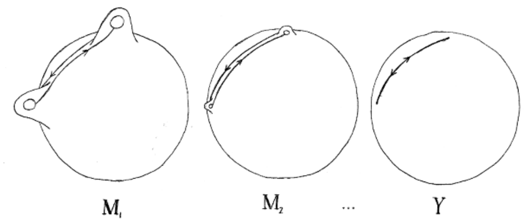


**Figure 1.**

Let $\gamma_i$ be the shortest curve in the free homotopy class representing the non-trivial loop passing through the two holes. The $\gamma_i$'s are closed geodesics with length almost $\pi$. The limit of $\gamma_i$ is a back and forth curve on a geodesic segment. So, we have minimal lengths $\lambda_i \in \text{MinLengthSpec}\,(M_i)$ with $\lambda_i \to \pi \notin \text{LengthSpec}\,(S^2)$.

Next is an example with the sudden appearance of elements in the limit's length spectrum far from elements in the sequences' spectra. Note this occurs even with smooth convergence.

**Example 8.3.** Let $M_i$ be a sequence of rotationally symmetric manifolds diffeomorphic to $S^2$, which have annular regions that are annuli in flat cones with the shorter end capped off smoothly with positive curvature and the wide end capped with a region of negative curvature glued to a large sphere (see Figure 2). As $i$ increases, the cones converge to a cylinder and the $M_i$ converge to a space $Y$ which is a capped off cylinder attached smoothly to a large sphere. A new geodesic appears on the cylinder with a length not approximated by lengths in the spectrum of the $M_i$.

Using the result [**25**, Theorem 3.6] that the Gromov–Hausdorff limit of the $\delta$-covers of a sequence is almost the $\delta$-cover of their limit space, we can show that the covering spectrum of the sequence and the covering spectrum of the limit space are very closely related. Note that the counting here is without multiplicity.

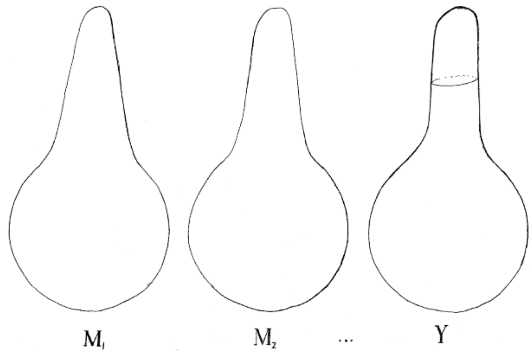


**Figure 2.**

**Theorem 8.4.** *If $X_i$ is a sequence of compact length spaces converging to a compact length space $Y$, then for each $\delta \in \mathrm{CovSpec}\, Y$, there is $\delta_i \in \mathrm{CovSpec}\, X_i$ such that $\delta_i \to \delta$. Conversely, if $\delta_i \in \mathrm{CovSpec}\, X_i$ and $\delta_i \to \delta > 0$, then $\delta \in \mathrm{CovSpec}\, Y$. Moreover, if the universal cover of the sequence and $Y$ exist, then $\#\{\mathrm{CovSpec}\, X_i\} \geq \#\{\mathrm{CovSpec}\, Y\}$ for all $i$ large.*

*Proof.* Let us prove the first statement. If it is not true, there is a $\delta \in \mathrm{CovSpec}\, Y$ such that no subsequence of $\mathrm{CovSpec}\,(X_i)$ converges to $\delta$, namely there exists an $\epsilon > 0$ such that

$$\mathrm{CovSpec}\,(X_i) \cap [\delta - \epsilon, \delta + \epsilon] = \emptyset \tag{8.1}$$

for all except finitely many $X_i$. So $\tilde{X}_i^{\delta-\epsilon} = \tilde{X}_i^{\delta+\epsilon}$ for all except finite many $i$. Now, by Proposition 7.3, a subsequence of the covers converges (for both $\delta - \epsilon$ and for $\delta + \epsilon$). Therefore, their limits $Y^{\delta-\epsilon} = Y^{\delta+\epsilon}$. By [**25**, Theorem 3.6] $\tilde{Y}^{\delta} = \tilde{Y}^{\delta+\epsilon/2}$, contradicting to $\delta \in \mathrm{CovSpec}\, Y$.

To prove the second statement, note that $\tilde{X}_i^{\delta_i} \to \tilde{X}_i^{\delta'}$ is non-trivial for all $\delta' > \delta_i$ and $\delta_i$ converges to $\delta > 0$. So, for all $\delta' > \delta > 0$ and $\epsilon \in (0, \delta)$, we have $\delta - \epsilon < \delta_i < \delta'$ for $i$ sufficiently large and $\tilde{X}_i^{\delta-\epsilon} \to \tilde{X}_i^{\delta'}$ is non-trivial. Now, take the limit as $i \to \infty$ and we get $Y^{\delta-\epsilon} \to Y^{\delta'}$ is non-trivial. This is true for all $\epsilon \in (0, \delta)$ and $\delta' > \delta$. Now, by the properties of limit covers [**25**, Theorem 3.6], we have for all $\epsilon \in (0, \delta)$ and $\delta' > \delta$, $\tilde{Y}^{\delta-\epsilon} \to \tilde{Y}^{\delta'}$ is non-trivial. So, $\mathrm{CovSpec}\,(Y) \cap [\delta - \epsilon, \delta']$ is non-empty. But $\mathrm{CovSpec}\,(Y)$ is discrete at $\delta$, so this forces $\mathrm{CovSpec}\,(Y)$ to include $\delta$. q.e.d.

An immediate corollary of this is as follows.

**Corollary 8.5.** *If $X_i$ is a sequence of compact length spaces converging to a compact length space $Y$, then the covering spectra converge in the Hasudorff sense as subsets of $\mathbb{R}$:*

$$\lim_{i\to\infty} d_H(\mathrm{CovSpec}\,(X_i)\cup\{0\}, \mathrm{CovSpec}\,Y\cup\{0\}) = 0. \tag{8.2}$$

*Proof.* By the definition of Hausdorff convergence (see inside Definition 7.1), we need only show that for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for all $i \geq N_\epsilon$,

$$\mathrm{CovSpec}\,(X_i) \subset T_\epsilon(\mathrm{CovSpec}\,(Y)\cup\{0\}), \tag{8.3}$$

$$\mathrm{CovSpec}\,(Y) \subset T_\epsilon(\mathrm{CovSpec}\,(X_i)\cup\{0\}). \tag{8.4}$$

If (8.3) is not true, then there is an $\epsilon > 0$ and a subsequence of the $i$ such that there exists

$$\lambda_i \in \mathrm{CovSpec}\,(X_i) \setminus T_\epsilon(\mathrm{CovSpec}\,(Y)\cup\{0\}). \tag{8.5}$$

Since the $X_i$ converge to $Y$, they have a uniform upper bound on diameter, $D$, and the $\lambda_i \in (0, D]$, so a subsequence converges to some

$$\lambda \in [0, D] \setminus T_\epsilon(\mathrm{CovSpec}\,(Y)\cup\{0\}). \tag{8.6}$$

Thus $\lambda \notin \mathrm{CovSpec}\,(Y)\cup\{0\}$ contradicting Theorem 8.4.

If (8.4) is not true, then there is an $\epsilon > 0$ and a subsequence of the $i$ such that there exists

$$\lambda_i \in \mathrm{CovSpec}\,(Y) \setminus T_\epsilon(\mathrm{CovSpec}\,(X_i)\cup\{0\}). \tag{8.7}$$

Since $\lambda_i \in [\epsilon, D]$, where $D = \mathrm{diam}\,(Y)$, and $(\mathrm{CovSpec}\,(Y)) \cap [\epsilon, D]$ is closed by Proposition 3.2, a subsequence of $\lambda_i$ converges to some $\lambda \in \mathrm{CovSpec}\,(Y)\cap[\epsilon, D]$. In particular, for $i$ sufficiently large $|\lambda_i - \lambda| < \epsilon/2$ and

$$\lambda \in \mathrm{CovSpec}\,(Y) \setminus T_{\epsilon/2}(\mathrm{CovSpec}\,(X_i)\cup\{0\}). \tag{8.8}$$

Then, by Theorem 8.4, we know there exists $\delta_i \in \mathrm{CovSpec}\,(X_i)$ converging to $\lambda$ which is a contradiction. q.e.d.

Applying Theorem 8.4 to manifolds with Ricci curvature lower bound and combining Theorem 1.1 in [**25**], we have the following.

**Corollary 8.6.** *If $M_i^n$ is a sequence of manifolds with* $\mathrm{Ric} \geq (n-1)H$ *converges to a compact length space $Y$, then* $\#\{\mathrm{CovSpec}\, M_i\} \geq \#\{\mathrm{CovSpec}\, Y\}$ *for all $i$ large.*

Another useful application of Theorem 8.4 concerns the covering spectra of classes of isolengthspectral manifolds:

**Theorem 8.7.** *If $\mathcal{M}$ is a Gromov–Hausdorff connected class of compact length spaces with a common discrete length spectrum, then all compact length spaces in $Cl(\mathcal{M})$ have the same covering spectrum as well.*

*Proof.* We need only show that all the spaces in $\mathcal{M}$ have the same covering spectrum. Then, the same holds true for all compact $Y$ in the closure by Theorem 8.4 since there will be $X_i$ with uniform covering spectra converging to $Y$.

Suppose there are at least two distinct covering spectra $C_1$ and $C_2$ for spaces in $\mathcal{M}$. Let $\mathcal{M}_i$ be the subset of $\mathcal{M}$ of spaces with covering spectra $C_i$. Clearly, these are disjoint sets. Each is closed as a subset of $\mathcal{M}$ by Theorem 8.4. Thus, we need only show each is relatively open to get a contradiction.

Suppose $\mathcal{M}_1$ is not relatively open. Then, there is a space $Y \in \mathcal{M}_1$ which can be approximated by $X_j \in \mathcal{M}$ such that $\mathrm{CovSpec}\,(X_j) \neq C_1$.

Thus, for each $j$ either there exists $c_j \in C_1 \setminus \mathrm{CovSpec}\, X_j$ or there exists $c_j \in \mathrm{CovSpec}\, X_j \setminus C_1$.

Since $Y$ is compact, there is a $D$ such that $\mathrm{diam}\, X_j \leq D$. Furthermore, all the spaces share the same discrete length spectrum, $L$, and length spectra are closed sets. Thus $\frac{1}{2}L \cap (0, D]$ is finite and, by Theorem 4.7, $c_j \subset \frac{1}{2}L \cap (0, D]$. Thus, by the pigeonhole principle, there exists $c > 0$ and a subsequence of the $j$ such that $c_j = c$.

If $c \in C_1 \setminus \mathrm{CovSpec}\,(X_j)$ for this subsequence, then by Theorem 8.4, there exists $\delta_j \in \mathrm{CovSpec}\,(X_j)$ such that $\delta_j$ converges to $c$. But these $\delta_j \subset (1/2)L \cap [0, D]$, so eventually they must repeat and we have a contradiction.

Thus, $c \in \mathrm{CovSpec}\,(X_j) \setminus C_1$ for this subsequence. By Theorem 8.4 again, $c \in \mathrm{CovSpec}\,(Y) = C_1$ (since $c > 0$) which is also a contradiction. q.e.d.

This leads immediately to the following corollary.

**Corollary 8.8.** *If $M_t$ is a one parameter family of compact Riemannian manifolds with a common discrete length spectrum (not counting multiplicity), then they have the same covering spectrum.*

## 9. Gaps in the covering spectrum

In this section, we discuss gap and clumping phenomenon in the covering spectra of compact length spaces.

Theorem 8.4 immediately gives us the following gap phenomenon near 0.

**Proposition 9.1.** *Given a sequence of compact length spaces $X_i$ which converges to a compact length $X$ that has a universal cover, there is $\lambda_X > 0$ such that for all $0 < \epsilon < \lambda_X$, $\exists N_\epsilon \in \mathbb{N}$ such that the Covering Spectum of $X_i$ has a gap at $(\epsilon, \lambda_X)$ for all $i \geq N_\epsilon$:*

$$\mathrm{CovSpec}\,(X_i) \cap (\epsilon, \lambda_X) = \emptyset. \tag{9.1}$$

Note that the gap here depends on the limit space. The simplest example which illustrates the restrictions on this gap, is a sequence of tori collapsing to a circle. The size of the limit circle determines $\lambda_X$ and the speed of collapse determines the relationship between $\epsilon$ and $N_\epsilon$.

In the following, we show there are many gaps in the covering spectrum which are uniform in size for a compact class of length spaces. Note that a Gromov–Hausdorff compact set of compact length spaces have a uniform upper diameter bound.

**Proposition 9.2.** *Let $\mathcal{M}$ be a Gromov–Hausdorff compact set of compact length spaces with universal covers and* $\mathrm{diam} \leq D$*, and let $S \subset [L_1, L_2] \subset [0, D]$ be a discrete set which includes the end points $L_1$ and $L_2$, then if*

$$\mathrm{gap}_N(X, S) = N\text{th largest element in } \{\lambda_i - \lambda_{i-1}\} \tag{9.2}$$

*among all $\lambda_i \in (\mathrm{CovSpec}\,(X) \cap [L_1, L_2]) \cup S$ in increasing order, then $\mathrm{gap}_{\#S-1}(X, S)$ has a uniform lower bound for all $X \in \mathcal{M}$. This lower bound depends on $S$. In particular,*

$$\mathrm{gap}_1(X, S) = \max\{\lambda_i - \lambda_{i-1} : \lambda_i \in (\mathrm{CovSpec}\,(X) \cap [L_1, L_2]) \cup S\} \tag{9.3}$$

*is uniformly bounded below depending on $S$.*

Note that the importance of this result is that the length of the gap interval of the covering spectrum is uniform for all $X \in \mathcal{M}$. On the other hand, the exact location of the gap can not be uniform as one can see if we take $\mathcal{M}$ to be the set including all flat 2-dimensional tori, circles and the one point space.

Note that if $S = \{0, D\}$, then for simply connected length spaces, $\mathrm{gap}_{\#S-1}(X, S) = D$, for $X$ with $\mathrm{CovSpec}\,(X) = \{\lambda_1\}$, then $\mathrm{gap}_{\#S-1}(X, S) = \max\{\lambda_1, D - \lambda_1\} \geq D/2$. So, it is not a strong bound for these length spaces. But then, as we progress to length spaces with large numbers of elements in the covering spectrum, this will force at least one gap which will be significantly larger than the average distance between elements.

By taking $S = \{0, D/N, 2D/N \ldots D\}$, we only start getting interesting controls over spaces with more than $N$ elements in the covering spectrum.

Proposition 9.2 implies that there are sequences of gaps approaching 0. That is, for any $L > 0$, there exists a $\delta_{\mathcal{M},L} > 0$ such that for any $X \in \mathcal{M}$, $\mathrm{CovSpec}\,(X)$ has a gap of size $\delta_{\mathcal{M},L}$ between 0 and $L$.

We now prove the gap theorem. Note that when $S = \{L_1, L_2\}$ with $L_1 > 0$, the lower bound for $\mathrm{gap}_1(X, S)$ also follows from Corollary 7.7.

*Proof of Proposition* 9.2. We already know that when $X_i$ converge to $X$ in the GH sense, then the $\mathrm{CovSpec}\,(X_i) \cup \{0\}$ converges to $\mathrm{CovSpec}\, X \cup \{0\}$ in the Hausdorff sense [Corollary 8.5]. So, $(\mathrm{CovSpec}\,(X_i) \cap [L_1, L_2]) \cup \{L_1, L_2\}$ converges to $(\mathrm{CovSpec}\,(X) \cap [L_1, L_2]) \cup \{L_1, L_2\}$ for any $[L_1, L_2] \subset [0, D]$.

Since $S$ includes the endpoints, $L_1$ and $L_2$, the set $(\mathrm{CovSpec}\,(X_i) \cap [L_1, L_2]) \cup S$ converges to the set $(\mathrm{CovSpec}\,(X) \cap [L_1, L_2]) \cup S$ in the Hausdorff sense.

When two discrete sets of numbers in $[L_1, L_2]$ are close in the Hausdorff sense, then the gaps are close as well. That is, the largest gaps are close, and the second largest and so on. Eventually, many of the gaps will be close to 0 or non-existent.

So $\mathrm{gap}_N(X_i, S)$ converges to $\mathrm{gap}_N(X, S)$ as long as $N \leq \#S$, and in fact, converges for all $N$, if we set the gap to 0 when there are not enough elements in the set.

On the other hand, all the covspecs in $\mathcal{M}$ are closed and discrete and so is S, so for each $X$, $\mathrm{gap}_{\#S-1}(X, S) > 0$. Since a positive continuous function defined on a compact set has a uniform positive lower bound, we are done. q.e.d.

**Example 9.3.** If we look at the following compact set of metric spaces: $X_i$ Hawaii ring with rings of radius $1/i^2, 2/i^2 \dots (i-1)/i^2, 1/i, 1$, and $X$, a circle of radius 1, then this space is compact and all elements have discrete covering spectra. There is no uniform bound on the number of elements in the covering spectra.

The largest gap in $\mathrm{CovSpec}\,(X_i)$ is $\pi(1-1/i)$ and the rest of the gaps are the same size, $\pi/i^2$.

Taking $S = \{0, \pi\}$, our uniform lower bound on the largest gap exists and is $\pi(1/2)$.

Taking $S = \{0, \frac{\pi}{2}, \pi\}$, our uniform bound on the largest gap is $\pi(1/2)$ and on the second largest gap is $\pi(1/4)$. But this second largest gap just records the fact that the covering spectra are below $\pi/2$.

Taking $S_j = \{0, \pi/j\}$ says more, since we know there is a uniform lower bound on the largest gap between 0 and $\pi/j$. But, in fact, this gap is basically above the majority of the spectra for all but finitely many of the $X_i$.

This theorem can also be used to show sets of complete metric spaces are not compact in the Gromov–Hausdorff sense.

**Example 9.4.** Let $X_j$ be a compact length space formed by $2^j$ circles of radii

$$\{1/2^j, 2/2^j, 3/2^j, \dots, (2^j-1)/2^j, 1\} \tag{9.4}$$

joined at a common point. Then,

$$\mathrm{CovSpec}\,(X_j) = \{\pi/2^j, 2\pi/2^j, 3\pi/2^j, \dots, (2^j-1)\pi/2^j, \pi\} \tag{9.5}$$

and $\mathrm{gap}_1(X, \{0, \pi\}) = \pi/2^j$ is not uniformly bounded below. Sure enough, this sequence has no converging subsequence in the Gromov–Hausdorff sense.

Applying Proposition 9.2 to the compact class of manifolds with a uniform lower bound on Ricci curvature, we have the following.

**Corollary 9.5.** *For all $H \in \mathbb{R}$, $D > 0$, $n \in \mathbb{N}$, $L > 0$, there exists a $\delta(H, D, L, n) > 0$ such that for any compact manifolds $M^n$ with* $\mathrm{diam}\,(M) \le D$, $\mathrm{Ric}\,(M^n) \ge (n-1)H$, *$\exists \lambda_M < L$ such that*

$$\mathrm{CovSpec}\,(M) \cap [\lambda_M, \lambda_M + \delta(H, D, L, n)] = \emptyset. \tag{9.6}$$

In addition to showing the existence of gaps of a certain size, one can study the location of elements in the covering spectrum. We call the following theorem a clumping theorem, since it shows that elements in the covering spectra have tendencies to clump around certain locations.

**Proposition 9.6.** *If $\mathcal{M}$ is a Gromov–Hausdorff compact set of compact length spaces with universal covers and diam $\leq D$, then for all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ and subsets $S_1, S_2, \ldots, S_{N_\epsilon} \subset [0, D]$ such that $m(S_i) < \epsilon$ and each $S_i$ is a finite set of intervals of the form:*

$$S_i = [0, \epsilon_i) \cup \bigcup_{j=1}^{N_i} (d_j^i - \epsilon_i, d_j^i + \epsilon_i)$$

*and for all $X \in \mathcal{M}, \exists i \in 1, \ldots, N_\epsilon$ s.t.* $\operatorname{CovSpec} X \subset S_i$.

*Proof.* Let $\mathcal{D}$ be the set of discrete subsets of $[0, D]$ which include $\{0\}$. Let $F : \mathcal{M} \to \mathcal{D}$ be defined as $F(X) = \operatorname{CovSpec}(X) \cup \{0\}$.

By Corollary 8.5, $F$ is a continuous map when the metric on $\mathcal{D}$ is the Hausdorff metric. Now, the continuous image of compact set is compact, so $F(\mathcal{M})$ is compact. In particular, any open cover of $F(\mathcal{M})$ has a finite subcover.

Fix $X \in \mathcal{M}$, denote $\operatorname{CovSpec}(X) = \{d_1, d_2, \ldots, d_N\}$. Define $U_X \subset \mathcal{D} = B_{F(X)}(r_{h,X})$, where $r_{h,X} = h \min\{d_1, d_2 - d_1, \ldots, d_N - d_{N-1}\}$. Note that $W_h = [0, \epsilon) \cup \bigcup_{j=1}^{N} (d_j - \epsilon, d_j + \epsilon)$ is an open subset of $[0, D]$, and for $h < 1/2$, this is a disjoint collection of intervals. For fixed $\epsilon > 0$, choose $h > 0$ very small so that $m(W_h) < \epsilon$. This determines $r_{h,X}$ for each $X$.

Now, $U_X$ form an open cover of $F(\mathcal{M})$, so there is a finite subcover. Let $U_{X_1}, \ldots, U_{X_{N_\epsilon}}$ be that finite subcover and let $\epsilon_i = r_{h,X_i}$ and $d_j^i$ the $j^{th}$ element in $\operatorname{CovSpec}(X_i)$. Then,

$$S_i = B_{F(X_i)}(\epsilon_i) = U_{X_i}. \tag{9.7}$$

So for every $X \in \mathcal{M}$, there is an $i \in 1, \ldots, N_\epsilon$ such that $\operatorname{CovSpec}(X) \subset U_{X_i} = S_i$. q.e.d.

One can easily see that Example 9.4 also fails to satisfy this clumping phenomenon.

**Corollary 9.7.** *For all $H \in \mathbb{R}$, $D > 0$, $n \in \mathbb{N}, \epsilon > 0$, there exists $N = N(\epsilon, H, D, n) \in \mathbb{N}$ and subsets $S_1, S_2, \ldots, S_N \subset [0, D]$ depending on $\epsilon, H, D$ and $n$ such that $m(S_i) < \epsilon$ and each $S_i$ is a finite set of* *intervals of the form:*

$$S_i = [0, \epsilon_i) \cup \bigcup_{j=1}^{N_i} (d_j^i - \epsilon_i, d_j^i + \epsilon_i),$$

*and such that for any compact manifold* $M^n$ *with* $\mathrm{diam}\,(M) \le D$, $\mathrm{Ric}\,(M^n) \ge (n-1)H$

$$\exists i \in 1, \dots, N_\epsilon \ \ s.t. \ \mathrm{CovSpec}\, M^n \subset S_i. \tag{9.8}$$

## 10. The Laplace spectrum

In this section, we discuss the relationship between the Laplace spectrum and the covering spectrum of a compact Riemannian manifold. Recall that the Laplace spectrum is defined as the set of eigenvalues of the Laplace operator. The elements of the Laplace spectrum are assigned a multiplicity equal to the dimension of the corresponding eigenspace.

It was proven by Colin de Verdiere that the Laplace spectrum determines the length spectrum of a generic manifold [**8**]. A generic manifold is one with a "bumpy metric" in the sense of Abraham and, given any Riemannian manifold, there is a nearby generic manifold which is close in the $C^5$ sense [**1**]. In particular, negatively curved manifolds are generic in this sense [**2**]. The generic manifolds are known to have discrete length spectra [**2**]. Thus, the Laplace spectrum determines the length spectrum on negatively curved manifolds of arbitrary dimension.

On Riemann surfaces, Huber proved the length and the Laplace spectrums determine each other completely [**20**]. Eberlein has shown that on two step nilmanifolds, the marked length spectrum determines the Laplace spectrum [**11**].

However, there are pairs of Laplace isospectral manifolds first constructed by Carolyn Gordon that have different length spectra when one takes multiplicity into account [**14**].

The simplest result we can get from the above is the following.

**Proposition 10.1.** *If* $\mathcal{M}$ *is a set of Laplace isospectral manifolds which are negatively curved, then there are only finitely many distinct covering spectra for the manifolds in this class.*

By Proposition 6.4, this implies that there is a uniform bound on the number of generators of the fundamental groups of these manifolds. However, this last fact was already known, since this class of manifolds is known to have only finitely many homeomorphism classes [**4**].

Another application is that complete length spaces in the GH closure of $\mathcal{M}$ have universal covers. Furthermore, $Cl(\mathcal{M})$ also has only finitely many distinct covering spectra and there is a uniform bound on the number of generators of the revised fundamental groups of these spaces.

*Proof.* Since $M \in \mathcal{M}$ are negatively curved and Laplace isospectral, they share the same length spectrum and this length spectrum is closed and discrete. They also have a uniform upper bound on diameter by [**4**]. The covering spectra are contained in one half times the length spectrum $\cap[0, D]$ by Theorem 4.7. Thus, there are only finitely many possible covering spectra. q.e.d.

Since, as yet, all known examples of Laplace isospectral sets of manifolds share the same length spectrum not counting multiplicity, we make the following conjecture.

**Conjecture 10.2.** If $\mathcal{M}$ is a set of Laplace isospectral manifolds which are with a uniform upper bound on diameter, then there are only finitely many distinct covering spectra for the manifolds in this class.

In the following example, we show that the Laplace spectrum does not determine the covering spectrum. In particular, we find a pair of Laplace isospectral Riemannian Heisenberg manifolds which do not share the same covering spectrum. Note that Pesce has proven that all Laplace isospectral Riemannian Heisenberg manifolds have the same length spectrum not counting multiplicities [**23**].

**Example 10.3.** In [**14**], Gordon studied the Heisenberg manifolds which are of the following form: $H_n(\Gamma, g) = (\Gamma \setminus H_n, g)$ where

$$\Gamma = \Gamma_{\bar{r},\bar{s},c} = \Big\{(\bar{x}, \bar{y}, u) \in H_n : \bar{x} \in r_1\mathbb{Z} \times \cdots \times r_n\mathbb{Z}, \bar{y} \in s_1\mathbb{Z} \times \cdots \times s_n\mathbb{Z}, u \in c\mathbb{Z}\Big\} \tag{10.1}$$

where $H_n$ is the $(2n+1)$ dimensional Heisenberg group with multiplication

$$(\bar{x}, \bar{y}, u)(\bar{x}', \bar{y}', u') = (\bar{x} + \bar{x}', \bar{y} + \bar{y}', u + u' + \bar{x}\bar{y}') \tag{10.2}$$

and the metric $g$ at $T_eH_n$ is a diagonal matrix with diagonal $\{a_1, a_2, \ldots, a_n, a_1, \ldots, a_n, 1\}$ where we have $0 < a_1 \le a_2 \le \ldots a_n$. Note that one needs $\Gamma$ is a subgroup of $H_n$ which is true iff $r_is_i \in c\mathbb{Z}$.

Then, the elements of the fundamental group of $H_n(\Gamma, g)$ are elements of $H_n$ of the form $(r_1x_1, \ldots, r_nx_n, s_1y_1, \ldots, s_ny_n, cu)$ where $x_i, y_i, u \in \mathbb{Z}$. By [14] Corollary 2.9, if $x_i$ or $y_i$ is not zero, we have the simple formula:

$$m((r_1x_1, \ldots, r_nx_n, s_1y_1, \ldots, s_ny_n, cu)) = \sqrt{\sum_{i=1}^{n} a_i(r_i^2x_i^2 + s_i^2y_i^2)}. \tag{10.3}$$

Otherwise,

$$m((0, \ldots, 0, cz)) = \min\Big\{|cz|, (4j\pi a_i(|cz| - j\pi a_i))^{1/2} : j \in \mathbb{Z}, \\ i = 1, \ldots, n, 2j\pi a_i < |cz|\Big\}. \tag{10.4}$$

For a proof of (10.4), see [11]. Since $(4j\pi a_i(|cz|-j\pi a_i))^{1/2}$ is increasing in $j$ for $0 < j < (|cz|)/(2\pi a_i)$ we have

$$m((0, \ldots, 0, 0, \ldots, 0, cz)) \\ = \min\{|cz|, (4\pi a_i(|cz| - \pi a_i))^{1/2}, i = 1, \ldots, n\} \tag{10.5}$$

and

$$m(0, 0, cz) \geq m(0, 0, c) \quad \text{for all integers } z. \tag{10.6}$$

Note that the elements $(r_ie_i, 0, 0)$ and $(0, s_ie_i, 0)$ generate all the elements of $\Gamma$ of the form $(\bar{x}, \bar{y}, 0)$, so the covering spectrum map is determined on these elements:

$$f((r_1x_1, \ldots r_nx_n, s_1y_1, \ldots, s_ny_n, 0)) \\ \in \left\{\frac{1}{2}\sqrt{a_i}r_i, \frac{1}{2}\sqrt{a_i}s_i : i = 1, \ldots, n\right\}, \tag{10.7}$$

Note also that $(r_ie_i, 0, 0)(0, s_ie_i, 0) = (r_ie_i, s_ie_i, r_is_i)$, so these elements also generate elements in the center of the form $(0, 0, kr_is_i)$.

If $c \neq kr_is_i$ for all $i$ and integer $k$ and $m(0, 0, c) \notin \{\sqrt{a_i}r_i, \sqrt{a_i}s_i : i = 1, \ldots, n\}$, by (10.6) $f(0, 0, cz) = m(0, 0, c)$ and the covering spectrum is

$$\left\{\frac{1}{2}\sqrt{a_i}r_i, \frac{1}{2}\sqrt{a_i}s_i, \frac{1}{2}m(0, 0, c) : i = 1, \ldots, n\right\}. \tag{10.8}$$

When there is an $i$ and an integer $k$ such that $c = kr_is_i$ and $m(0, 0, c) \geq \max\{\sqrt{a_i}r_i, \sqrt{a_i}s_i\}$ for that particular $i$, then $f(0, 0, cz) = \max\{\sqrt{a_i}r_i, \sqrt{a_i}s_i\}$ for that particular $i$, and then the covering spectrum is only $\{\frac{1}{2}\sqrt{a_i}r_i, \frac{1}{2}\sqrt{a_i}s_i : i = 1, \ldots, n\}$.

This is particularly interesting because Gordon states that two Heisenberg manifolds $H_n(\Gamma',\, g')$ and $H_n(\Gamma,\, g)$ are Laplace isospectral iff $a_i = a_i',\ c = c'$ and $\{a_1 r_1^2, \ldots, a_n r_n^2,\ a_1 s_1^2, \ldots, a_n s_n^2\}$ is a permutation of $\{a_1 (r_1')^2, \ldots, a_n (r_n')^2, a_1 (s_1')^2, \ldots, a_n (s_n')^2\}$. Thus, the only way to get an isospectral pair with different covering spectra is to have one which includes $m(0,0,c)$ and one which does not.

Let $a_1 = 1/8,\ a_2 = 1/2$ and $c = 1$, by (10.5) $m(0,0,1) = (\pi/2(1 - \pi/4))^{1/2} \sim .9767$.

If we take $r_1 = 20,\ r_2 = 1,\ s_1 = 10$ and $s_2 = 1$, then $c = r_2 s_2$ and

$$m(0,0,c) \geq \max\{\sqrt{a_2} r_2, \sqrt{a_2} s_2\} = 1/\sqrt{2}, \tag{10.9}$$

so

$$\operatorname{CovSpec}(H_2(\Gamma, c)) = \Big\{ 20/(2\sqrt{8}) = 5\sqrt{2}/2,\ 1/(2\sqrt{2}) = \sqrt{2}/4,\ 10/(2\sqrt{8}) = 5\sqrt{2}/4 \Big\}. \tag{10.10}$$

If we take $r_1' = 2,\ r_2' = 10,\ s_1' = 10$, and $s_2' = 1$, then $c \neq r_i' s_i'$ for any $i$ and

$$\operatorname{CovSpec}(H_2(\Gamma', c)) = \{\sqrt{2}/4, 5\sqrt{2}/2, 5\sqrt{2}/4, 1/2(\pi/2(1 - \pi/4))^{1/2}\}. \tag{10.11}$$

It is easy to see that one can construct quite a number pairs of isospectral Heisenberg manifolds with different covering spectra in this manner. Interestingly, Gordon's particular pair of isospectral Heisenberg manifolds with different length spectrum (counting multiplicity) [**14**](Example 2.4 a) do share the same covering spectra: $\{1/2, 1\}$. So, there are distinct pairs of Laplace isopectral manifolds that share the same covering spectrum.

Next, one questions what happens to the covering spectra in a continuous family of Laplace isospectral manifolds. Note that since Pesce has shown Laplace isospectral Heisenberg manifolds share the same discrete length spectrum, by Theorem 8.7, we know that a one parameter family of Laplace isospectral Heisenberg manifolds must share the same covering spectrum. Thus, the two manifolds constructed in Example 10.3 are not joined by such a one parameter family.

The most explored method of constructing Laplace isospectral pairs of Riemannian manifolds is using Sunada's method [**28**]. Such isospectral manifolds are called Sunada isospectal pairs:

**Definition 10.4.** Sunada isospectral pairs of manifolds are pairs of manifolds $M_1 = M/H_1$ and $M_2 = M/H_2$ with $\pi : M \to M/G$ is a finite normal covering and $H_i$ are subgroups of $G$ such that for any conjugacy class $G_j \subset G$,

$$\#(G_j \cap H_1) = \#(G_j \cap H_2). \tag{10.12}$$

Sunada proved that these spaces are Laplace isospectral and length isospectral.

A special case of Sunada isospectral pairs of manifolds are the examples Sunada attributes to Komatsu [**28**, Example 3].

**Example 10.5.** A Sunada isospectral pair of manifolds is a Komatsu pair if $H_1$ and $H_2$ are any pair of finite groups of the same order with exponents of the same odd prime $p$. Both are identified with a set $S$ and they are embedded into the symmetric group on $S$ using the left actions of the $H_i$ on $S$.

Now, two permutations of the symmetric group are conjugate iff they have the same cycle decomposition (cf. [**19**]). So a conjugacy class $G_i$ corresponds to a partition $\#S = p_1 + \cdots + p_k$, where $p_i \geq 1$. Since $H_i$ have exponents of order $p$, they only contain $p$ cycles. And since they act on the left on $S$, their non-trivial elements must move every point in $S$, and thus they are complete sets of $p$ cycles and they are all in the same conjugacy class: $G_1$ corresponding to $\#S = p + p + \cdots + p$. So,

$$\#(H_1 \cap G_1) = \#S - 1 = \#(H_2 \cap G_1) \tag{10.13}$$

and $\#(H_i \cap G_j) = 0$ otherwise.

Since every symmetric group can be shown to act by isometries on some Riemannian manifold, this creates a Sunada isospectral pair. In particular, they can be constructed as a Sunada isospectral pair whose common finite cover, $M$, is a simply connected compact manifold.

**Proposition 10.6.** *Komatsu pairs of Sunada isospectral manifolds share the same covering spectrum which in fact consists of a single element.*

*Proof.* Let $M_1 = M/H_1$ and $M_2/H_2$ be the Komatsu pair with a common simply connected finite cover $M$. Let $M_0 = M/G$ where $G$ is the symmetric group.

Now, let $m_i : H_i \to \mathbb{R}$ be the minimum marked length map for $M_i$ and $m : G \to \mathbb{R}$ be the minimum marked length map for $M/G$. Note that $m_i(h) = \inf_{x\in M} d_M(x, hx) = m(h)$. Furthermore, $m(g_1) = m(g_2)$

whenever $g_1$ and $g_2$ are conjugate because this is the minimum length of a loop freely homotopic to a loop representing $g_i$.

However, every non-trivial element in either of the $H_i$ is a member of the same conjugacy class corresponding to $\#S = p + p + \cdots + p$. So, $m_1(h_1) = m(h_1) = m(h_2) = m_2(h_2)$ for all non-trivial $h_i \in H_i$. Thus, the covering maps are equal as well, and the only element in the covering spectrum is this $m(h_i)$. q.e.d.

Note that Komatsu pairs of Sunada isospectral manifolds do not necessarily have the same covering spectrum counting multiplicity. In [**28**, Example 3], $H_1 = (\mathbb{Z}/p\mathbb{Z})^3$ has three generators and thus the only element in its covering spectrum must have multiplicity 3 while

$$H_2 = \Big\langle\, a, b \mid a^p = b^p = (aba^{-1}b^{-1})^p = e, \\ a(aba^{-1}b^{-1}) = (aba^{-1}b^{-1})a,\ b(aba^{-1}b^{-1}) = (aba^{-1}b^{-1})b \,\Big\rangle$$

has two generators and thus the only element in its covering spectrum must have multiplicity 2.

## 11. Acknowledgements

The authors would like to thank Carolyn Gordon for conversations and references regarding the length and Laplace spectra. We would also like to thank Jeff Cheeger for suggesting that we investigate the possible existence of a gap phenomenon related to Theorem 1.1 of [**25**].

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