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THE EXISTENCE OF HYPERSURFACES OF CONSTANT GAUSS CURVATURE WITH PRESCRIBED BOUNDARY

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Abstract

We are concerned with the problem of finding hypersurfaces of constant Gauss curvature (K-hypersurfaces) with prescribed boundary Γ in R^{n+1}, using the theory of Monge-Ampère equations. We prove that if Γ bounds a suitable locally convex hypersurface Σ, then Γ bounds a locally convex K-hypersurface. The major difficulty lies in the lack of a global coordinate system to reduce the problem to solving a fixed Dirichlet problem of Monge-Ampère type. In order to overcome this difficulty we introduced a Perron method to deform (lift) Σ to a solution. The success of this method is due to some important properties of locally convex hypersurfaces, which are of independent interest. The regularity of the resulting hypersurfaces is also studied and some interesting applications are given.

1. Introduction

In this paper we are concerned with the problem of finding hypersurfaces of constant Gauss-Kronecker curvature (K-hypersurfaces) in $\mathbb{R}^{n+1}$ ($n \geq 2$) with prescribed boundary: given a disjoint collection $\Gamma = \{\Gamma_1, \dots, \Gamma_m\}$ of closed smooth embedded $(n-1)$ dimensional submanifolds of $\mathbb{R}^{n+1}$, decide whether there exist (immersed) K-hypersurfaces M in $\mathbb{R}^{n+1}$ with $\partial M = \Gamma$. Locally this problem reduces to questions concerning Monge-Ampère type equations and we seek solutions for which the resulting equation is elliptic. This means that we must confine ourselves to the class of *locally strictly convex* hypersurfaces, i.e., those whose principal curvatures are all positive. Such

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hypersurfaces locally lie on one side of their tangent planes at any point but need not do so globally as they have nonempty boundary.

Finding hypersurfaces with prescribed curvature and boundary has been a major challenge in geometric analysis because of the highly nonlinear nature of the problem and the lack of variational methods. Beginning around 1980, some success was achieved due to breakthroughs in the theory of Monge-Ampère equations and general fully nonlinear equations, but only for hypersurfaces which are globally graphs of functions over domains with geometric restrictions (e.g., strictly convex domains). This was the case even for K-hypersurfaces where the only general existence results were consequences of the existence theory for Monge-Ampère equations (see [6], [20], [23]), and was restricted to strictly convex domains. This means that the resulting surfaces must be simply connected graphs, a very strong restriction geometrically.

The first idea that a more general result was possible came in the paper of Hoffman-Rosenberg-Spruck [18] and subsequently such a general result was developed in [12] and [11]. In these papers, the authors proved an essentially optimal existence theorem for Monge-Ampère equations in domains of arbitrary geometry and thus the limit of our understanding of K-hypersurfaces with boundary was reached, as far as global graphs (including multi-sheeted radial graphs) are concerned. This theory already led to striking geometric applications [12], [28], [8].

To solve the problem in its full parametric generality seemed to require substantial new techniques. A necessary condition for Γ to bound a locally strictly convex hypersurface is that its second fundamental form (as a submanifold of $\mathbb{R}^{n+1}$) is nondegenerate everywhere. This however, is not a sufficient condition; Rosenberg [27] (see also [10]) shows there are topological obstructions. It is natural to seek geometric conditions that guarantee the existence of locally strictly convex K-hypersurfaces spanning a given Γ. Based on the results in [12], the second author [29] made the following conjecture: Γ must bound an immersed K-hypersurface if it bounds a *locally* strictly convex immersed hypersurface. The first main result of the present paper settles this conjecture affirmatively. More precisely, we will prove:

Theorem 1.1. *Assume that there exists a locally convex immersed hypersurface Σ in $\mathbb{R}^{n+1}$ with $\partial\Sigma = \Gamma$ and $K_\Sigma \geq K$ everywhere, where K is a positive constant. Suppose, in addition, that Σ is C^2 and locally strictly convex along its boundary. Then there exists a smooth* (*up to the boundary*) *locally strictly convex immersed hypersurface M with $\partial M = \Gamma$ such that $K_M \equiv K$. Moreover, M is homeomorphic to Σ.*

We note that this is a huge jump in generality from our previous results in [12] as it deals with general immersed K-hypersurfaces and not just graphs (or radial graphs). Because of the presence of boundary, locally convex surfaces can be very complicated. In particular, in Theorem 1.1 M need not be embedded even if Σ is embedded.

It is also important to understand hypersurfaces of vanishing Gauss curvature. These hypersurfaces are clearly related to convex hulls of codimension 2 submanifolds in space. Our second main result in this article is the following

Theorem 1.2. *Suppose Γ bounds a locally convex hypersurface which is C^2 and locally strictly convex along its boundary. Then there exists a locally convex hypersurface M of Gauss curvature $K_M \equiv 0$ with $\partial M = \Gamma$, and M is of class $C^{1,1}$ up to the boundary. Moreover, for any interior point $p \in M$, all the extreme points of the (intrinsic) component of $M \cap T_pM$ containing p lie on ∂M, where T_pM denotes the tangent plane of M at p. In particular, if Γ is extreme, i.e., Γ lies on the boundary of its convex hull, then M coincides with part of the boundary of the convex hull of Γ and, therefore, is globally convex.*

The $C^{1,1}$ regularity in Theorem 1.2 is optimal for hypersurfaces of vanishing Gauss curvature, as shown by counterexamples (see [7]). We also remark that Theorem 1.2 does not hold without the assumption that Γ bounds a locally convex hypersurface which is locally strictly convex near its boundary. Ghomi [8] has constructed a smooth extreme Jordan curve γ in $\mathbb{R}^3$ with the properties that:

(a) γ bounds a convex surface of vanishing Gauss curvature which is not $C^{1,1}$,

(b) γ does not bound any locally strictly convex surface, and

(c) γ does not bound any locally convex surface of class $C^{1,1}$ with vanishing Gauss curvature.

As a consequence of Theorem 1.2 we have

Corollary 1.3. *Suppose Γ is extreme and let Σ be a locally convex hypersurface with $\partial\Sigma = \Gamma$. If Σ is C^2 up to the boundary and locally strictly convex in a neighborhood of its boundary, then the interior of Σ lies strictly outside the convex hull of Γ.*

We remark that such a hypersurface need not be globally convex, nor embedded. A somewhat stronger version of Corollary 1.3 has been proved by Alexander-Ghomi [1]. In [8], Ghomi made the following conjecture: every compact connected hypersurface of positive curvature with connected extreme boundary is embedded and its interior lies outside the convex hull of its boundary. We see that Corollary 1.3 settles affirmatively part of this conjecture. On the other hand, we will construct an example which shows such a hypersurface may fail to be embedded. Furthermore, using the bridge principle of Hauswirth [16], we will show there exist smooth K-surfaces in $\mathbb{R}^3$, with connected extreme boundary, which are not embedded.

Suppose Γ is extreme and let H_Γ be the boundary of its convex hull. Theorem 1.2 indicates that if Γ bounds a locally convex hypersurface which is C^2 (up to the boundary) and locally strictly convex in a neighborhood of its boundary, then one of the components of $H_\Gamma \setminus \Gamma$ must be $C^{1,1}$ up to the boundary. However, as we will show by an example, the other components may have interior singularities. A result of Ghomi [8] states that every component of $H_\Gamma \setminus \Gamma$ is $C^{1,1}$ up to the boundary if Γ is strictly convex, i.e., through every point of Γ there passes a (global) supporting hyperplane with first order contact.

Hypersurfaces of vanishing Gauss curvature are closely related to the homogeneous degenerate Monge-Ampère equation

$$\det D^2 u = 0. \tag{1.1}$$

In general, the Dirichlet problem for (1.1), even with smooth boundary data, does not have C^2 solutions, as shown by an example of Urbas (see [7]). Under suitable regularity assumptions on the boundary data, the interior and global $C^{1,1}$ regularity was established by Trudinger-Urbas [30] and Caffarelli-Nirenberg-Spruck [7], respectively, for strictly convex domains. Later the first author [11] extended the global regularity result of [7] to non-convex domains. These regularity results will play important role in our proof of Theorem 1.2. For more general (nonhomogeneous) degenerate Monge-Ampère equations, the $C^{1,1}$ regularity has been studied by Caffarelli-Kohn-Nirenberg-Spruck [5], Hong [19], Krylov [24], P.-F. Guan [13] and Guan-Trudinger-Wang [14], etc.

A major difficulty in proving Theorems 1.1 and 1.2 lies in the lack of global coordinate systems to reduce the problem to solving certain boundary value problem for Monge-Ampère type equations. To overcome this difficulty, we adopt a Perron method to deform (lift) Σ into a K-hypersurface by solving the Dirichlet problem for the Gauss curvature equation (2.1) locally. This approach, while classical for PDE's, requires substantial technical work as we are dealing with general locally convex hypersurfaces in space. A key ingredient, among others, is an *a priori* estimate for the local Lipschitz constants ($C^{0,1}$ norms) of locally convex hypersurfaces spanning Γ. This is established in Section 3 where we also derive *a priori* estimates for the lower and upper bounds of principal curvatures of locally strictly convex K-hypersurfaces spanning Γ. The Perron method is carried out in Section 4 where we define the deformation space $\mathcal{L}$ of liftings of Σ and construct M as the limit of a suitable sequence of hypersurfaces in $\mathcal{L}$. In Section 5, we study the regularity of the resulting hypersurface constructed in Section 4, to complete the proofs of Theorems 1.1 and 1.2. Finally, in Section 6 we prove Corollary 1.3 and construct an extreme curve in $\mathbb{R}^3$ which bounds a locally strictly convex K-surface with self-intersection and for which the boundary of its convex hull has interior singularities.

For general Monge-Ampère equations, there is a vast literature, with fundamental contributions from Pogorelov, Cheng-Yau, Lions, Ivochkina, Krylov, Caffarelli-Nirenberg-Spruck, Trudinger, Urbas and others in the 1970-1980's, and that of Caffarelli [2], [3] on the regularity theory. For further references the reader is referred to [9], [15] and the expository article [25].

An earlier version of this article was circulated as a preprint starting March 2001. At about the same time Trudinger-Wang [31] independently proved Theorem 1.1.

2. Notation and preliminaries

Let $\Phi : \Sigma_0^n \to \mathbb{R}^{n+1}$ be an immersion where Σ_0 is a manifold of dimension $n \geq 2$ with boundary $\partial\Sigma_0$ which may be empty. We will often identify Φ with its image $\Sigma := \Phi(\Sigma_0)$ and call Σ a hypersurface of $\mathbb{R}^{n+1}$. Similarly, the boundary of Σ, $\partial\Sigma$, means the immersion $\Phi : \partial\Sigma_0 \to \mathbb{R}^{n+1}$. When we consider a point $p \in \Sigma$, it should be understood as one of its preimages in Σ_0. For a subset U of $\mathbb{R}^{n+1}$, $\Sigma \cap_p U$ will denote the component of $\Sigma \cap U$ that contains p, that is, $\Sigma \cap_p U = \Phi(U_0)$ where U_0 is the component of $\Phi^{-1}(\Sigma \cap U)$ that contains the point identified to p in $\Phi^{-1}(p) \subset \Sigma_0$. In this paper, all hypersurfaces in $\mathbb{R}^{n+1}$ we consider are assumed to be connected, orientable and compact with or without boundary. Unless otherwise indicated, if two hypersurfaces have the same boundary, they are assumed to be oriented in such a way that they induce the same orientation on the boundary.

Let Σ be a C^2 hypersurface in $\mathbb{R}^{n+1}$. We will use K_Σ, ν_Σ and d_Σ to denote the Gauss curvature, the unit normal vector field, and the extrinsic diameter of Σ, respectively. The orientation of Σ is assumed to be consistent with ν_Σ which is continuously defined on entire Σ. At a point on Σ the Gauss curvature K_Σ is the product of the principal curvatures which are the eigenvalues of the second fundamental form of Σ computed with respect to ν_Σ. We denote by $\kappa_{\min}[\Sigma]$ and $\kappa_{\max}[\Sigma]$ the minimum and maximum, respectively, of all principal curvatures of Σ. We say Σ is *locally convex* (*locally strictly convex*) if $\kappa_{\min}[\Sigma] \geq 0$ ($\kappa_{\min}[\Sigma] > 0$, respectively).

We will also need to consider hypersurfaces with less regularity. In general, a hypersurface Σ in $\mathbb{R}^{n+1}$ is said to be locally convex if at every point $p \in \Sigma$ there exists a neighborhood which is the graph of a convex function $x_{n+1} = u(x)$, $x \in \mathbb{R}^n$, for a suitable coordinate system in $\mathbb{R}^{n+1}$, such that locally the region $x_{n+1} \geq u(x)$ always lies on a fixed side of Σ. (Note that Σ is assumed to be orientable so it has two sides; for convenience we will refer to the *inner* side as the one facing $x_{n+1} \geq u(x)$.) The latter requirement that the region $x_{n+1} \geq u(x)$ lie on one fixed side of Σ is to ensure that the local convexity at each point is consistent with a fixed orientation; see [1] for a detailed discussion. Note that a locally convex hypersurface is necessarily of class $C^{0,1}$ in the interior.

For a locally convex hypersurface Σ which is not necessarily C^1, ν_Σ is understood as the Gauss map from Σ to the subsets of $\mathbb{S}^n$: for a point $p \in \Sigma$, $\nu_\Sigma(p)$ is the set of all unit normal vectors of local supporting hyperplanes of Σ at p. For convenience, we will say ν_Σ has a certain property of a vector if every element of ν_Σ has that property. For the definition in weak sense of Gauss curvature we refer to [26]. According to Caffarelli [2], if Σ is the graph of a locally convex function $x_{n+1} = u(x)$ over a domain Ω in $\mathbb{R}^n$ then $K_\Sigma = K$ if and only if u is a viscosity solution of the Gauss curvature equation

$$\det(u_{ij}) = K(1 + |\nabla u|^2)^{\frac{n+2}{2}} \text{ in } \Omega. \tag{2.1}$$

One can similarly interpret the meanings of $K_\Sigma \leq K$ and $K_\Sigma \geq K$. We will need the following existence result which follows from, for example, Theorem 1.1 of [11] by approximation.

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{0,1}$. Suppose there exists a locally convex viscosity subsolution $\underline{u} \in C^{0,1}(\overline{\Omega})$ of* (2.1), *i.e.,*

$$\det(\underline{u}_{ij}) \geq K(1 + |\nabla \underline{u}|^2)^{\frac{n+2}{2}} \quad \text{in } \Omega,$$

where $K \geq 0$ is a constant. Then there exists a unique locally convex viscosity solution $u \in C^{0,1}(\overline{\Omega})$ of (2.1) *satisfying $u = \underline{u}$ on $\partial\Omega$.*

3. A priori estimates and compactness

In this section we prove some important local properties of locally convex hypersurfaces with boundary. Throughout the section, let Σ and M be locally convex hypersurfaces in $\mathbb{R}^{n+1}$ with $\partial\Sigma = \partial M$ and assume that there exists a fixed constant $\delta > 0$ such that the hypersurface

$$\Sigma_\delta := \{x \in \Sigma : \mathrm{dist}_\Sigma(x, \partial\Sigma) < \delta\}$$

is C^2 up to the boundary and locally strictly convex, where dist_Σ denotes the intrinsic distance on Σ. We furthermore assume that M locally lies on the *inner* side of Σ along the boundary and any neighborhood of ∂M in M does not intersect Σ_δ in the interior. By this we mean that $\nu_\Sigma(p) \cdot (q - p) > 0$ for all $p \in \Sigma_\delta$ and $q \in M$ near ∂M. In particular, both Σ and M locally lie on the same side of the tangent plane to Σ at any point of $\partial\Sigma$. Let $\mathit{\Pi}$ denote the second fundamental form of $\partial\Sigma$ (as a submanifold of $\mathbb{R}^{n+1}$). The main result of this section, which plays a key role in our proof of Theorems 1.1 and 1.2 and is of independent interest, may be stated as follows.

Theorem 3.1. *At every point on M, locally M can be represented as the graph of a convex function u defined in a domain $\Omega \subset \mathbb{R}^n$ of a fixed lower bound in size* (*depending only on δ, $\kappa_{\min}[\Sigma_\delta]$, $\kappa_{\max}[\Sigma_\delta]$, $\max_{\partial\Sigma} |\mathit{\Pi}|$ and d_M*) *such that*

$$\|u\|_{C^{0,1}(\overline{\Omega})} \leq C_1 \tag{3.1}$$

where C_1 depends on δ, $\kappa_{\min}[\Sigma_\delta]$, $\kappa_{\max}[\Sigma_\delta]$, $\max_{\partial\Sigma} |\mathit{\Pi}|$ and d_M.

More precisely, that Ω is of a lower bound in size means that there exists some constant $\delta_0 > 0$ such that Ω contains a ball of radius δ_0 or a portion of a ball of radius δ_0 separated by a smooth hypersurface (in

$\mathbb{R}^n$) with controlled geometric quantities, with the center of the ball in $\overline{\Omega}$.

Proof.

Step 1. We first note the following simple fact. Let p be an arbitrary point on $\partial\Sigma$ and X a unit tangent vector to $\partial\Sigma$ at p . Since Σ is locally strictly convex near $\partial\Sigma$, we have

$$\nu_\Sigma(p) \cdot \mathit{\Pi}(X,X) \geq \kappa_{\min}[\Sigma_\delta] > 0. \tag{3.2}$$

Throughout this proof let $\beta = \frac{1}{2}\sin^{-1}(\kappa_{\min}[\Sigma_\delta]/\max_{\partial\Sigma}|\mathit{\Pi}|)$. Then

$$\frac{\nu_\Sigma(p) \cdot \mathit{\Pi}(X,X)}{|\mathit{\Pi}(X,X)|} \geq \frac{\kappa_{\min}[\Sigma_\delta]}{\max_{\partial\Sigma}|\mathit{\Pi}|} = \sin 2\beta > 0. \tag{3.3}$$

Thus the angle between $\nu_\Sigma(p)$ and $\mathit{\Pi}(X,X)$ does not exceed $\frac{\pi}{2} - 2\beta$.

Now, for a fixed point $p \in \partial\Sigma$, we take p to be the origin and choose a coordinate system of $\mathbb{R}^{n+1}$ such that e_n and e_{n+1} are normal to $\partial\Sigma$ at p and

$$\nu_\Sigma(p) = e_n \cos\beta + e_{n+1}\sin\beta. \tag{3.4}$$

Here e_k is the unit vector in the positive x_k-axis direction ($1 \leq k \leq n+1$). For later reference we will call this the *special coordinate system* at p. It follows that Σ (locally at p) can be represented as the graph of a strictly convex function $x_{n+1} = \underline{u}(x)$ over a domain Ω' with a lower bound in size which depends on δ, β and $\kappa_{\max}[\Sigma_\delta]$. In particular, $\partial\Sigma$ is locally a graph over a portion, which we denote as Γ', of $\partial\Omega'$. By (3.3) and (3.4), the angle between e_n and $\mathit{\Pi}(X,X)$ does not excess $\frac{\pi}{2} - \beta$, that is

$$e_n \cdot \mathit{\Pi}(X,X) \geq |\mathit{\Pi}(X,X)|\sin\beta \geq \kappa_{\min}[\Sigma_\delta]\sin\beta \tag{3.5}$$

for any unit tangent vector X to $\partial\Sigma$ at p. Consequently, (possibly after a rotation of the $(x_1, \ldots, x_{n-1})$ coordinates) we may represent Γ' as a graph

$$x_n = \varphi(x') \equiv \sum_{i=1}^{n-1} a_i x_i^2 + o(|x'|^2), \quad x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \tag{3.6}$$

for some constants a_i, $1 \leq i \leq n-1$, satisfying

$$0 < \kappa_{\min}[\Sigma_\delta]\sin\beta \leq a_i \leq \max_{\partial\Sigma}|\mathit{\Pi}|, \quad 1 \leq i \leq n-1. \tag{3.7}$$

By shrinking the size of Ω' as necessary, we may assume $\Omega' = \{\varphi < x_n < 2r\}$ for some uniform constant $r > 0$.

Let v be the convex function defined on $\overline{\Omega'}$ by

$$v(x) = \sup\{L(x) : L \text{ is an affine function, } L \leq \underline{u} \text{ on } \Gamma'\}. \tag{3.8}$$

We have

$$\underline{u} \leq v \leq \max_{\Gamma'} \underline{u} \text{ in } \overline{\Omega'}, \quad v = \underline{u} \text{ on } \Gamma'$$

and

$$\mathrm{Lip}_{\Omega'}(v) \leq \max_{\Gamma'} |\nabla \underline{u}| + C$$

where $\mathrm{Lip}_{\Omega'}(v)$ denotes the Lipschitz coefficient of v on Ω'.

By the local convexity of M we have $\nu_M \cdot \mathit{\Pi}(X, X) \geq 0$ for any tangent vector X to $\partial\Sigma$. From (3.3) we see that the angle between ν_M and ν_Σ at any point on $\partial\Sigma$ does not exceed $\pi - 2\beta$. Therefore,

$$\nu_M(p) \cdot e_{n+1} \geq \sin\beta.$$

That is, the angle between $\nu_M(p)$ and e_{n+1} does not exceed $\frac{\pi}{2} - \beta$. Consequently, M locally (near p) can be represented as the graph of a convex function $x_{n+1} = u(x)$. Since M is locally convex, we see that u is defined on a smooth strictly convex domain Ω_p satisfying

$$\{\varphi < x_n < r\} \subset \Omega_p \subset \{\varphi < x_n < 2r\} \tag{3.9}$$

with

$$\underline{u} \leq u \leq v \text{ in } \overline{\Omega_p} \text{ and } \mathrm{Lip}_{\Omega_p}(u) \leq C \tag{3.10}$$

where C depends on r and $\|\underline{u}\|_{C^1(\Omega')}$. For later reference we set

$$\partial'\Omega_p := \{x_n = \varphi | 0 \leq x_n \leq r\} \subset \partial\Omega_p$$

and by $\Gamma(p)$ the graph of u over $\partial'\Omega_p$. Note that $\Gamma(p) \subset \partial M$.

Step 2. Next, let q be an interior point of M. We will consider two different cases. We first assume that there exists a hyperplane P through q, which either is a local supporting hyperplane or is transversal to M at q, such that

$$U_t \cap \partial M = \emptyset \text{ for all } t > 0 \text{ sufficiently small} \tag{3.11}$$

where $U_t = M \cap_q \{z \in \mathbb{R}^{n+1} : (z - q) \cdot \nu_P \leq t\}$. We first note the following fact which will often be used in the sequel without being explicitly referred to.

Lemma 3.2. *Suppose $s > 0$ such that* (3.11) *holds for all nonnegative $t \leq s$. Then U_s is transversal to $P_s := \{z \in \mathbb{R}^n : (z-q) \cdot \nu_P = s\}$.*

Proof. We note that U_t is transversal to P_t for all $t > 0$ sufficiently small. Suppose $s > 0$ is the first value such that (3.11) holds for all nonnegative $t \leq s$ while U_s is not transversal to P_s at a point $p \in \partial U_s$. By the local convexity of M, P_s is a local supporting hyperplane to M at p where M locally lies in the half space $(z-q)\cdot\nu_P \leq s$. For $\epsilon > 0$ small enough,

$$V_\epsilon := M \cap_p \{(z-p)\cdot(-\nu_P) \leq \epsilon\}$$

is transversal to $P_{s-\epsilon}$ and is a convex disk. Moreover, $\partial V_\epsilon = \partial U_{s-\epsilon}$, for $\partial U_{s-\epsilon}$ is a globally convex disk as its boundary is contained in a hyperplane (see [17]). This implies $M = V_\epsilon \cup U_{s-\epsilon}$ and therefore is a closed convex sphere without boundary, which is a contradiction. q.e.d.

We now return to the proof of Theorem 3.1. Let $t_0 > 0$ be the smallest value such that $U_{t_0} \cap \partial M \neq \emptyset$ and choose a point $p \in U_{t_0} \cap \partial M$. Note that $U := U_{t_0}$ is globally convex. We consider the special coordinate system at p which satisfies (3.4). Under this coordinate system, q lies in the region $|x_{n+1}| \leq x_n \cot\beta$. In particular, $x_n(q) > 0$. We also note that

$$\nu_P = e_n \cos\theta + e_{n+1}\sin\theta$$

for some $\theta \in [\beta, \pi - \beta]$. Moreover, M locally (near p) is given as the graph of a function u on a domain Ω_p as in (3.9) satisfying (3.1).

Let $r > 0$ be as in (3.9). We see from above that if $x_n(q) < r/2$ then q is on the graph of u over Ω_p and we are done. So we next consider the case that $x_n(q) \geq r/2$. Let $C_q = C_q(\partial U)$ be the convex cone generated by ∂U with vertex q. We will show that C_q contains a nondegenerate cone of fixed size that contains p. This means there exists a point $q_0 \in \mathbb{R}^{n+1}$, $|q_0 - q| = 1$, and a uniform constant $\delta_0 > 0$ such that $p \in C_q(B_{\delta_0}(q_0)) \subset C_q$ where $C_q(B_{\delta_0}(q_0))$ is the cone generated by $B_{\delta_0}(q_0)$ with vertex q. Since $|q-p| \geq r/2$, this will complete our proof under assumption (3.11).

Choose new coordinates $(y_1, \ldots, y_{n+1})$ in $\mathbb{R}^{n+1}$ with origin at p such that $y_i = x_i$ $(1 \leq i \leq n-1)$, $y_n(q) > 0$, $y_{n+1}(q) = 0$ and let τ_i denote the unit vector in the positive y_i direction $(0 \leq i \leq n+1)$. We have

$y_n(q) \geq x_n(q) \geq \frac{r}{2}$ and hence

$$\frac{(\tau_k \cdot (q-p))^2}{|q-p|^2} = \frac{(e_k \cdot (q-p))^2}{|q-p|^2} \leq 1 - \Big(\frac{r}{2d_M}\Big)^2, \ \ \forall\, 1 \leq k \leq n-1. \tag{3.12}$$

From the convexity of U we see that C_q contains the cone generated by $\Gamma(p)$ with vertex q since $\Gamma(p)$ and q are separated by the hyperplane containing ∂U. By (3.6), (3.7) and (3.12) the projection of C_q to the hyperplane $\mathbb{R}^n \equiv \{y_{n+1} = 0\}$ contains an n-ball $B_\rho(0)$ in $\mathbb{R}^n$ where $\rho \geq c_0$ for a uniform constant $c_0 > 0$. To complete the proof, therefore, we only have to find a point p_0 with

$$y_{n+1}(p_0) > 0 \ \text{ and } \ \frac{y_n(q) - y_n(p_0)}{y_{n+1}(p_0)} \leq C_0 \tag{3.13}$$

for some uniform constant $C_0 > 0$ such that the cone generated by the convex hull of $\Gamma(p) \cup \{p_0\}$ with vertex q is contained in C_q. (We note that it is always possible to find such p_0 on $\Gamma(p)$ with $C_0 = C_0(t_0)$ depending on t_0; $C_0(t_0)$ may, however, tend to infinity as $t_0 \to 0$.)

For $0 \leq t \leq y_n(q)$, let $W_t = \{y_n \geq t\} \cap_q M$. If $W_{y_n(q)} \cap \partial U \neq \emptyset$ then we are done since ∂U lies in the upper half space $\{y_{n+1} \geq 0\}$. We thus may assume $W_{y_n(q)} \cap \partial U = \emptyset$. Note that then $W_{y_n(q)} \subset U$ and is therefore a convex cap. We may find $t_1 \in [0, y_n(q))$ such that $W_t \cap \partial M = \emptyset$ for all $t_1 < t \leq y_n(q)$ and $W_{t_1} \cap \partial M \neq \emptyset$. Note that W_{t_1} is also a convex cap and $W_{t_1} \setminus U \subset C_p$ by convexity.

If $t_1 = 0$ then

$$\tau_n \cdot \mathit{\Pi}(X, X) \leq 0, \ \ \forall\, X \in T_p\partial M \tag{3.14}$$

since $\Gamma(p)$ lies in the half space $y_n \leq 0$. (This implies that $\Gamma(p)$ is contained in the half space $x_{n+1} \geq 0$.) It follows from (3.5) that $\tau_{n+1} \cdot e_n \geq \sin\beta$, that is, the angle between τ_{n+1} and e_n does not exceed $\frac{\pi}{2} - \beta$. Consequently,

$$y_{n+1}(z) = z \cdot \tau_{n+1} \geq x_n(z) e_n \cdot \tau_{n+1} \geq x_n(z) \sin\beta, \ \ \forall\, z \in \Gamma(p)$$

since $x_{n+1}(z) \geq 0$. We see any point p_0 on $\Gamma(p)$ with $x_n(p_0) \geq \frac{r}{2}$ must satisfy (3.13).

We now assume $t_1 > 0$ and take an arbitrary point $p_1 \in W_{t_1} \cap \partial M$. We have $p_1 \in (W_{t_1} \setminus U) \cup \partial U \subset C_q$ and, similarly to (3.14),

$$\tau_n \cdot X = 0 \ \text{ and } \ \tau_n \cdot \mathit{\Pi}(X, X) \leq 0, \ \ \forall\, X \in T_p\partial M. \tag{3.15}$$

Moreover, since $W_{t_1} \cap \{y_{n+1} \le 0\} \subset U$,

$$y_{n+1}(p_1) > 0 \text{ and } 0 < y_n(p_1) = t_1 < y_n(q). \tag{3.16}$$

We may further assume that there exists a uniform constant $\varepsilon_0 > 0$ such that

$$|X \cdot \tau_{n+1}| \le \varepsilon_0 \tag{3.17}$$

and

$$|\mathit{\Pi}(X, X) \cdot \tau_{n+1}| \le \varepsilon_0 \tag{3.18}$$

for all unit tangent vector X to ∂M at p_1. This can be seen as follows. Suppose there is a unit vector $X \in T_{p_1}\partial M$ which does not satisfy (3.17) or (3.18) and let γ_X be the geodesic on $\Gamma(p_1)$ tangential to X at p_0. We can then find a point $p_0 \in \gamma_X$ near p_1 such that, if (3.17) is violated then (3.13) holds for $C_0 = C_0(\varepsilon_0)$, while if (3.18) fails,

$$|Y \cdot \tau_{n+1}| \ge \varepsilon_1$$

for some unit tangent vector Y to ∂M at p_0 and some uniform constant $\varepsilon_1 > 0$.

Note that (3.17) and (3.18) imply

$$\nu_\Sigma(p_1) \cdot \tau_n \le 0 \tag{3.19}$$

when ε_0 is sufficiently small, since the angle between $\mathit{\Pi}(X, X)$ and $-\tau_n$ is sufficiently small while that between $\mathit{\Pi}(X, X)$ and $\nu_\Sigma(p_1)$ does not exceed $\frac{\pi}{2} - 2\beta$. By (3.16) and (3.19) we obtain

$$\nu_\Sigma(p_1) \cdot \tau_{n+1} \le 0, \tag{3.20}$$

since the segment joining p_1 and q locally lies on the inner side of Σ near p_1. Finally, by (3.19), (3.20) and the local strict convexity of Σ near boundary there exists a point $z \in \Sigma_\delta \cap V$ with

$$y_{n+1}(p_1) - y_{n+1}(z) \ge c_0$$

for some uniform constant $c_0 > 0$ depending on δ and $\kappa_{\min}[\Sigma_\delta]$, where V is the vertical 2-plane (in y-coordinates) through p_1 and q. Since z must lie above the line through q and p_1 by the convexity of M and the assumption that Σ_δ does not intersect M in interior, we have

$y_{n+1}(z) \geq y_{n+1}(q) = 0$. Thus $y_{n+1}(p_1) \geq c_0$ and $p_0 := p_1$ satisfies (3.13) where $C_0 > 0$ depends on δ and $\kappa_{\min}[\Sigma_\delta]$.

Step 3. We now assume there is no hyperplane through q satisfying assumption (3.11). We will first prove that M has a unique local supporting hyperplane (thus a tangent hyperplane) at q.

Let P be a local supporting hyperplane at q to M and let E denote the set of points on ∂M that (intrinsically) belong to $P \cap_q M$. Clearly $E \neq \emptyset$. We claim that q is contained in the convex hull of E. Indeed, if this is not the case, that is, q and E are separated by a hyperplane, we may assume $P = \{x_{n+1} = 0\}$ and q lies in the region $x_n > \varepsilon$ while E in $x_n < -\varepsilon$ for some $\varepsilon > 0$. Then $M \cap_q \{x_{n+1} \leq a x_n\}$ does not intersect ∂M where $a > 0$ is sufficiently small, which is a contradiction.

By Caratheodory's theorem (cf. [22]) q is contained in an l-dimensional simplex S with vertices in E for some $1 \leq l \leq n$. We have $S \subseteq P \cap_q M$ by the local convexity of M.

Let p be a vertex of S and consider the special coordinate system at p. We note that, by the local convexity of M, P is a local supporting hyperplane to M at every point on the segment $\overline{pq}$ joining p and q. It follows that

$$\nu_P = e_n \cos\theta + e_{n+1} \sin\theta \tag{3.21}$$

for some $\theta \in [\beta, \pi - \beta]$. Recall M locally near p is the graph of a convex function over a domain Ω_p in $\{x_{n+1} = 0\}$. Since Ω_p is strictly convex the segment $\overline{pq}$ is transversal to ∂M at p. For otherwise $\overline{pq}$ would be contained in $\{x_{n+1} = 0\}$ and tangential to $\partial\Omega_p$ at p, resulting in a contradiction as Ω_p would contain points on $\overline{pq}$ other than p. Consequently, P is the tangent hyperplane to M at p as P contains $\overline{pq}$ and is tangential to ∂M at p.

Next, assume furthermore that $\overline{pq} \subset \overline{pp_1} \subseteq S$ for some $p_1 \neq q$. Let Q be a local supporting hyperplane to M at q. Then $\overline{pp_1} \subset Q$ and therefore Q is a local supporting hyperplane to M at every point on $\overline{pp_1}$. We have $Q = P$ since both are the tangent hyperplane to M at p. This also shows that P is the tangent hyperplane of M at every point on $\overline{pp_1}$ (except possibly p_1). Consequently, u extends along $\overline{pp_1}$.

As we can always find a point $p \in E$ such that the segment $\overline{pq}$ extends in S, we have proved the uniqueness of the local supporting plane to M at q. Using induction on l we will next prove the assertion in the Theorem at point q.

Let us first consider the case $l = 1$, that is, $S = \overline{pp_1}$ where $p, p_1 \in \partial M$. Suppose $|p - q| \leq |p_1 - q|$ and let $\widetilde{\Omega}$ be the convex hull (in $\mathbb{R}^n = \{x_{n+1} = 0\}$) of $\{p_1'\} \cup \Omega_p$ where $p_1' \in \mathbb{R}^n$ with $p_1 = (p_1', x_{n+1}(p_1))$. (Similar meaning for q' below.) As in Step 2 we may assume $x_n(q) \geq \frac{r}{2}$ where r as in (3.9). This implies (3.12), that is the angle between $\overline{pq}$ and e_k has a uniform positive lower bound for all $1 \leq k \leq n-1$.

Let

$$\text{(3.22)} \quad v(x) = \sup\{L(x) : L \text{ is an affine function}, \; L \leq u \text{ at } p_1' \text{ and on } \Omega_p\}, \;\; x \in \widetilde{\Omega}.$$

Then v is a convex function and $\det D^2 v = 0$ in $\widetilde{\Omega}$. We have $u \leq v$ where u is defined in $\widetilde{\Omega}$. Since $|p_1 - q| \geq |p - q| \geq \frac{r}{2}$, by (3.12) there exists a uniform constant $\lambda > 0$ depending on r and $\max_{\partial M} |\mathit{\Pi}|$, such that the n-ball $B_\lambda(q')$ is contained in $\widetilde{\Omega}$. By the local convexity of M we see u is defined on $B_{\lambda/2}(q') \subset \widetilde{\Omega}$ with a uniform bound on $\|\widetilde{u}\|_{C^{0,1}(B_{\lambda/2}(q'))}$. This completes the proof for $l = 1$.

Assume now $l > 1$ and suppose we have proved the assertion for any point in a simplex of dimension less than l with vertices in E. Choose $p \in E$ and p_1 on an $(l-1)$ dimensional face of S such that $q \in \overline{pp_1}$. If $|p - q| \leq |p_1 - q|$ then the proof follows as exactly in case $l = 1$. Let us therefore assume $|p - q| \geq |p_1 - q|$. By induction, in a suitable coordinate system (y, y_{n+1}), $y \in \mathbb{R}^n$ with origin at p_1, M locally near p_1 is the graph of a convex function $y_{n+1} = u(y)$ with a uniform $C^{0,1}$ bound in an n-ball $B_R(0)$ where R is a uniform constant. Since P is the tangent hyperplane to M at any point on $\overline{pp_1}$ (except possibly p_1), we have $\nu_P \cdot (0, \ldots, 0, 1) \geq c_0$ for some uniform constant $c_0 > 0$. Thus u extends along $\overline{pp_1}$. Replacing the convex function v in (3.22) by

$$\text{(3.23)} \quad v(y) = \sup\{L(y) : L \text{ is an affine function}, \; L \leq u \text{ at } p \text{ and on } B_R(0)\},$$

defined in the convex hull of $\{p\} \cup B_R(0)$, the rest of proof follows that of case $l = 1$. This, finally, completes our proof. q.e.d.

An important consequence of Theorem 3.1 is a compactness result (Theorem 3.4) which we will need in the next section. First, it follows immediately from Theorem 3.1 that:

Corollary 3.3. *There exist uniform constants $R, r > 0$ depending on δ, $\kappa_{\min}[\Sigma_\delta]$, $\kappa_{\max}[\Sigma_\delta]$, $\max_{\partial\Sigma}|\mathit{\Pi}|$ and d_M such that for any $p \in M$, $M \cap_p B_R(p)$ is embedded and the convex body in $B_R(p)$ bounded by $M \cap_p B_R(p)$ contains a ball of radius r.*

According to a convergence theorem of Alexander-Ghomi [1] we thus have:

Theorem 3.4. *Let $\{M_k\}$ be a sequence of locally convex hypersurfaces contained in a bounded region in $\mathbb{R}^{n+1}$ with $\partial M_k = \partial\Sigma$ for all k. Suppose each M_k lies on the inner side of Σ and does not intersect Σ_δ. Then there exists a subsequence $\{M_{k_i}\}$ converging in Hausdorff metric to a locally convex hypersurface M with $\partial M = \partial\Sigma$. Moreover, for each i there exists a homeomorphism from M_{k_i} on to M with boundary fixed.*

Proof. We refer to [1] (Theorem 7.1) for the major part of the proof. Here we only point out that by Corollary 3.3 the conditions of Theorem 7.1 in [1] are satisfied, and give a brief proof of the fact $\partial M = \partial\Sigma$ and that M is $C^{0,1}$ up to the boundary. Given a point $p \in \partial\Sigma$, we consider the special coordinates at p satisfying (3.4). Then each Σ_k locally near p can be represented as a convex graph $x_{n+1} = u_k(x)$ over a domain Ω_p of form (3.9) with a uniform $C^{0,1}$ norm bound. By compactness there exists a subsequence of $\{u_k\}$ converging to a convex function $u \in C^{0,1}(\overline{\Omega_p})$. Moreover, we have $u = \underline{u}$ on $\{x \in \partial\Omega_p : x_n = \varphi\}$ since $\underline{u} \le u_k \le v$ in $\overline{\Omega_p}$ where v is as in (3.8), φ as in (3.6) and the graph of $\underline{u}$ represents Σ. Note that M must coincide with the graph of u near p. Consequently, M is a locally convex hypersurface of class $C^{0,1}$ up to the boundary and $\partial M = \partial\Sigma$. q.e.d.

We next derive *a priori* bounds for all principal curvatures for smooth locally strictly convex K-hypersurfaces.

Theorem 3.5. *Assume in addition that M is a smooth locally strictly convex hypersurface of constant Gauss curvature $K > 0$. Then*

$$\frac{1}{C_2} \le \kappa_{\min}[M] \le \kappa_{\max}[M] \le C_2$$

where $C_2 > 0$ depends on K, K^{-1}, δ, $\kappa_{\min}[\Sigma_\delta]$, $\kappa_{\max}[\Sigma_\delta]$, $\max_{\partial\Sigma}|\mathit{\Pi}|$, and d_M.

Proof. We first establish the estimates on the boundary. Given any point $p \in \partial M$, by Theorem 3.1 we may write M locally (near p) as a graph $x_{n+1} = u(x)$ with an *a priori* gradient bound over a smooth strictly convex domain Ω_p where u satisfies the Gauss curvature equation (2.1). As $\partial\Omega_p$ is strictly convex we may appeal to the boundary estimates for $|\nabla^2 u|$ due to Caffarelli-Nirenberg-Spruck [6] (which is local in nature) to obtain

$$|u_{ij}(0)| \leq C \tag{3.24}$$

where C depends on $\|u\|_{C^1(\overline{\Omega_p})}$ and geometric quantities of Σ_δ and $\partial\Sigma$. Since the principal curvatures of M at p are the eigenvalues of the matrix

$$\left\{(1+|\nabla u|^2)^{-\frac{1}{2}} u_{ij}\right\}$$

(with respect to $\{\delta_{ij} + u_i u_j\}$, the metric of M), the desired estimates follow from (3.24) and the fact that the Gauss curvature is the product of all principal curvatures.

Turning to the global estimates, consider $\Lambda := \max \kappa e^\rho$ where

$$\rho(\mathbf{x}) = |\mathbf{x} - \mathbf{x}^0|^2, \quad \mathbf{x} \in \mathbb{R}^{n+1}$$

($\mathbf{x}^0$ is a fixed point in $\mathbb{R}^{n+1}$), and the maximum is taken for all normal curvatures κ over M. As we already have estimates for principal curvatures on ∂M, we may assume Λ is attained at an interior point $p \in M$. Choose coordinates in $\mathbb{R}^{n+1}$ with origin at p such that the tangent hyperplane of M at p is given by $x_{n+1} = 0$ and M locally is written as a strictly convex graph $x_{n+1} = u(x)$ where $x = (x_1, \dots x_n) \in \mathbb{R}^n$. We may also assume the Hessian matrix $\{u_{ij}\}$ to be diagonal at 0 with $u_{11}(0) \geq u_{ii}(0) > 0$ for all $1 < i \leq n$. Note that, since $Du(0) = 0$, $u_{ii}(0)$ $(1 \leq i \leq n)$ are the principal curvatures of M at p. Thus Λ is achieved at p with respect to the normal curvature in x_1 direction which is locally given by

$$\kappa = \frac{u_{11}}{(1+u_1^2)w}, \quad w = (1+|\nabla u|^2)^{\frac{1}{2}}.$$

Since the function $\log u_{11} - \log(1+u_1^2) - \log w + \rho$ then has a maximum at the origin where $Du = 0$, $w = 1$, $Dw = 0$ and $w_{ii} = u_{ii}^2$ for all $1 \leq i \leq n$, we have at 0,

$$\frac{u_{11i}}{u_{11}} - \frac{w_i}{w} - \frac{2u_1 u_{1i}}{1+u_1^2} + \rho_i = 0, \quad 1 \leq i \leq n \tag{3.25}$$

and

$$\frac{u_{11ii}}{u_{11}} - \left(\frac{u_{11i}}{u_{11}}\right)^2 - u_{ii}^2 - 2u_{1i}^2 + \rho_{ii} \leq 0, \quad 1 \leq i \leq n. \tag{3.26}$$

Multiplying (3.26) by u_{11}/u_{ii} and taking sum over i from 1 to n, one obtains

$$\sum \frac{u_{11ii}}{u_{ii}} - \sum \frac{(u_{11i})^2}{u_{11}u_{ii}} - u_{11}\Delta u - 2u_{11}^2 + \sum \frac{\rho_{ii}}{u_{ii}} \leq 0. \tag{3.27}$$

Differentiating Equation (2.1) we have for $1 \leq k \leq n$,

$$u^{ij}u_{ijk} = (n+2)\frac{w_k}{w}$$

and

$$u^{ij}u_{ijkk} - u^{il}u^{jm}u_{ijk}u_{lmk} = (n+2)\frac{w_{kk}}{w} - (n+2)\frac{w_k^2}{w^2},$$

where $\{u^{ij}\}$ is the inverse matrix of $\{u_{ij}\}$. Combining these and (3.27) we obtain

$$nu_{11}^2 - u_{11}\Delta u + \sum \frac{\rho_{ii}}{u_{ii}} \leq 0. \tag{3.28}$$

Next,

$$\rho(\mathbf{x}) = |x - x^0|^2 + (u(x) - x_{n+1}^0)^2, \ \ x \in \mathbb{R}^n$$

where $\mathbf{x}^0 = (x^0, x_{n+1}^0)$, and therefore,

$$\rho_{ii} = 2 + 2(u(x) - x_{n+1}^0)u_{ii} + u_i^2.$$

Since $\Delta u \leq nu_{11}$, by (3.28) one sees that at 0,

$$\begin{aligned} 0 &\geq \sum_{i=1}^{n} \frac{\rho_{ii}}{u_{ii}} \geq \sum_{i=2}^{n} \frac{2}{u_{ii}} - 2nx_{n+1}^0 \\ &\geq \frac{2}{(u_{22}\dots u_{nn})^{\frac{1}{n-1}}} - 2nx_{n+1}^0 \\ &\geq 2\Big(\frac{u_{11}}{\det u_{ij}}\Big)^{\frac{1}{n-1}} - C \\ &= 2\Big(\frac{u_{11}}{K}\Big)^{\frac{1}{n-1}} - C. \end{aligned}$$

It follows that

$$u_{11}(0) \leq CK.$$

This proves an upper bound for $\kappa_{\max}[M]$, from which a lower bound for $\kappa_{\min}[M]$ can be derived in terms of K^{-1}. The proof is complete. q.e.d.

Remark 3.6. Using an estimate of Guan-Trudinger-Wang [14] in place of that of [6], it is possible to obtain an upper bound for the principal curvatures which does not depend on the lower bound of Gauss curvature.

4. Deformation to K-hypersurfaces

The primary purpose of this section is to prove the existence part in Theorems 1.1 and 1.2. Throughout the section, let Σ be a locally convex immersed hypersurface in $\mathbb{R}^{n+1}$ with embedded boundary $\partial\Sigma$ and Gauss curvature $K_\Sigma \geq K$ everywhere on Σ, where K is a fixed nonnegative constant. Our idea is to deform Σ to a locally convex immersed hypersurface M with $K_M \equiv K$ and $\partial M = \partial\Sigma$.

Let $D \subseteq \Sigma$ be a disk on Σ which, as a hypersurface in $\mathbb{R}^{n+1}$, may be represented as the graph of a convex function $\underline{u}$ defined in a domain Ω (in some hyperplane) with Lipschitz boundary. By Theorem 2.1, there is a unique function $u \in C^{0,1}(\overline{\Omega})$ whose graph is a convex hypersurface $\widetilde{D}$ of constant Gauss curvature K with $\partial\widetilde{D} = \partial D$. By the maximum principle, we have $u \geq \underline{u}$ in $\overline{\Omega}$. Thus $\widetilde{D}$ lies on the inner side of D.

This naturally induces a $C^{0,1}$-diffeomorphism $\Psi_D : \Sigma \to \widetilde{\Sigma} := \widetilde{D} \cup (\Sigma \setminus D)$ which is fixed on $\Sigma \setminus D$. The hypersurface $\widetilde{\Sigma}$ is locally convex with $K_{\widetilde{\Sigma}} \geq K$ and $\partial\widetilde{\Sigma} = \partial\Sigma$. We call $\widetilde{\Sigma}$ a *basic lifting* of Σ (by $\widetilde{D}$ over D). A *lifting* of Σ is a hypersurface which is obtained by a finite number of basic liftings starting from Σ. We introduce a partial order $\preceq$ between liftings of Σ: $\Sigma_1 \preceq \Sigma_2$ if and only if Σ_2 is a lifting of Σ_1 or $\Sigma_2 = \Sigma_1$.

Lemma 4.1. *Let Σ_1 and Σ_2 be any two liftings of Σ. Then there exists a unique lifting, which we denote as $\Sigma_1 \vee \Sigma_2$, of Σ such that $\Sigma_1 \preceq \Sigma_1 \vee \Sigma_2$, $\Sigma_2 \preceq \Sigma_1 \vee \Sigma_2$, and $\Sigma_1 \vee \Sigma_2 \preceq N$ for any lifting N with $\Sigma_1 \preceq N$ and $\Sigma_2 \preceq N$.*

Proof. We first assume Σ_1 is a basic lifting of Σ by $\widetilde{D_1}$ over a disk $D_1 \subseteq \Sigma$ and let A be the open region in $\mathbb{R}^{n+1}$ bounded by $D_1 \cup \widetilde{D_1}$. Assume Σ_2 to be a lifting of Σ over a region D_2. (D_2 is not necessarily a disk.) Intuitively, if Σ, Σ_1 and Σ_2 are all embedded, then it is obvious that the hypersurface

$$\Sigma_1 \vee \Sigma_2 := (\Sigma_2 \setminus (\Sigma_2 \cap A)) \cup (\widetilde{D}_1 \setminus (\widetilde{D}_1 \cap B))$$

where B is the open regions in $\mathbb{R}^{n+1}$ bounded by $\Sigma_2 \cup \Sigma$, is a lifting of

Σ with the desired properties. In the general case when some of these hypersurfaces may be immersed, we view Σ as an immersion

$$\Phi_0 : \Sigma_0 \to \Sigma \subset \mathbb{R}^{n+1} \tag{4.1}$$

of a differentiable manifold Σ_0 and let

$$\Phi_i : \Sigma_0 \to \Sigma_i \subset \mathbb{R}^{n+1}, \quad i = 1, 2 \tag{4.2}$$

be the immersions induced from the liftings. (Note that $\Phi_i = \Phi_0$ on $\Sigma_0 \setminus \Phi_0^{-1}(D_i)$.) The lifting $\Sigma_1 \vee \Sigma_2$ is then given by the immersion

$$\Phi : \Sigma_0 \to \Sigma_1 \vee \Sigma_2 := \Phi(\Sigma_0) \subset \mathbb{R}^{n+1}$$

defined as

$$\Phi(p) := \begin{cases} \Phi_1(p), & \text{if } p \in \Phi_0^{-1}(D_1) \setminus \Phi_0^{-1}(D_2), \\ \Phi_1(p), & \text{if } p \in \Phi_0^{-1}(D_1) \cap \Phi_0^{-1}(D_2) \text{ and } \Phi_2(p) \in A, \\ \Phi_2(p), & \text{otherwise,} \end{cases} \tag{4.3}$$

for $p \in \Sigma_0$. The general case now can be proved by induction. q.e.d.

The next lemma, which states that volume decreases under lifting, is well-known; for completeness we include a proof.

Lemma 4.2. *Let Σ_1 and Σ_2 be liftings of Σ. If $\Sigma_1 \preceq \Sigma_2$ then* $\mathrm{Vol}(\Sigma_1) \geq \mathrm{Vol}(\Sigma_2)$. *Moreover, the equality holds if and only if $\Sigma_1 = \Sigma_2$.*

Proof. Obviously we may assume Σ_2 is a basic lifting of Σ_1 over a disk $D_1 \subset \Sigma_1$. Suppose D_1 and its lifting $D_2 \subset \Sigma_2$ are the graphs of convex functions u_1 and u_2 over a domain $\Omega \subset \mathbb{R}^n$, respectively. We have $u_1 \leq u_2$ on $\overline{\Omega}$ and $u_1 = u_2$ $\partial\Omega$.

Let

$$N(x, z) = \frac{(\nabla u_2, -1)}{\sqrt{1 + |\nabla u_2|^2}}, \quad (x, z) \in \overline{\Omega} \times \mathbb{R}.$$

denote the downward unit normal vector to D_2 at $(x, u_2(x))$. Thus $\operatorname{div} N(x, z)$, the distributional mean curvature of D_2 at the point $(x, u_2(x))$ with respect to the upward normal vector, is nonnegative almost everywhere since u_2 is a convex function. Let

$$\omega = \{(x, z) \in \mathbb{R}^{n+1} : u_1(x) < z < u_2(x),\ x \in \Omega\}.$$

By the divergence theorem we have

$$\begin{aligned} 0 \le \int_\omega \mathrm{div} N dv &= \int_{D_1} N \cdot \nu_1 d\sigma - \int_{D_2} d\sigma \\ &= \mathrm{Vol}(D_1) - \mathrm{Vol}(D_2) + \int_{D_1} (N \cdot \nu_1 - 1) d\sigma \end{aligned}$$

where ν_1 is the downward unit normal vector to D_1. Since $0 \le N \cdot \nu_1 \le 1$ on D_1 we have $\mathrm{Vol}(D_1) - \mathrm{Vol}(D_2) \ge 0$; obviously, the equality holds only when $D_1 = D_2$. q.e.d.

We need one more lemma which states that volume is continuous under uniform convergence of uniformly Lipschitz convex functions.

Lemma 4.3. *Let w_k be a sequence of uniformly Lipschitz convex functions on Ω converging uniformly to w. Then*

$$\int_\Omega \sqrt{1 + |\nabla w|^2}\, dx = \lim_{k \to \infty} \int_\Omega \sqrt{1 + |\nabla w_k|^2}\, dx. \tag{4.4}$$

Proof. Let $W_k = \sqrt{1 + |\nabla w_k|^2}$; then $|\nabla W_k| \le |\nabla^2 w_k| \le \Delta w_k$ a.e. in Ω since w_k is convex. Therefore,

$$\int_\Omega |\nabla W_k|\, dx \le \sup_\Omega |\nabla w_k| |\partial \Omega|.$$

Hence W_k are uniformly bounded in $W^{1,1}$ and so converge in L^1 to $\sqrt{1 + |\nabla w|^2}$. q.e.d.

We are now ready to prove the main result of this section. Let $\mathcal{L}$ be the collection of liftings of Σ and set

$$\mu := \inf_{L \in \mathcal{L}} \mathrm{Vol}(L).$$

Theorem 4.4. *Suppose Σ_δ is C^2 and locally strictly convex up to the boundary for some fixed $\delta > 0$. There exists a locally convex hypersurface M in $\mathbb{R}^{n+1}$ of class $C^{0,1}$ up to the boundary with $\partial M = \partial \Sigma$ and $K_M \equiv K$. Moreover, M is homeomorphic to Σ and $\mathrm{Vol}(M) = \mu$.*

Proof. For each $k \ge 1$ choose $\Sigma_k \in \mathcal{L}$ such that

$$\mathrm{Vol}(\Sigma_k) \le \mu - \frac{1}{k}.$$

By Lemmas 4.1 and 4.2 we may assume $\Sigma_k \preceq \Sigma_{k+1}$ for all $k \ge 1$. According to Theorem 3.4 after passing to a subsequence we may assume

$\{\Sigma_k\}$ converges in Hausdorff metric to a locally convex hypersurface M which, in addition, is homeomorphic to each Σ_k. Clearly $\partial M = \partial\Sigma$. It remains to show $\mathrm{Vol}(M) = \mu$ and $K_M \equiv K$.

Consider a point $p \in M$. There exists a sequence $p_k \in \Sigma_k$, $k = 1, 2, \ldots$, converging to p (in $\mathbb{R}^{n+1}$) such that $\Sigma_k \cap_{p_k} B_R(p_k)$ converges to $M \cap_p B_R(p)$ in Hausdorff metric where $R > 0$. According to Theorem 3.1, when R is chosen sufficiently small each $\Sigma_k \cap_{p_k} B_R(p_k)$ can be represented as the graph of a convex function w_k with a uniform $C^{0,1}$ norm bound (independent of k). By compactness we may choose a coordinate system in $\mathbb{R}^{n+1}$ such that, after possibly passing to subsequences, all the functions w_k are defined in a fixed domain $\Omega \in \mathbb{R}^n$ satisfying

$$\|w_k\|_{C^{0,1}(\overline{\Omega})} \leq C_0 \quad \text{independent of } k \tag{4.5}$$

and w_k converges uniformly to a function $w \in C^{0,1}(\overline{\Omega})$ whose graph obviously locally represents M. Hence by Lemma 4.3 and a covering argument, $\mathrm{Vol}(M) = \mu$.

Consider now the Dirichlet problem for the Gauss curvature equation (2.1) in $\overline{\Omega}$. Using w_k as a subsolution for each $k \geq 1$, by Theorem 2.1 we obtain a unique convex solution $u_k \in C^{0,1}(\overline{\Omega})$ of (2.1) satisfying $u_k = w_k$ on $\partial\Omega$. We have $u_k \geq w_k$ on $\overline{\Omega}$ and by (4.5)

$$\|u_k\|_{C^{0,1}(\overline{\Omega})} \leq C_0 \quad \text{independent of } k.$$

Thus there exists a subsequence, which we still denote by $\{u_k\}$, converging to a convex function u in $C^{0,1}(\overline{\Omega})$. We see u satisfies (2.1) and $u \geq w$ on $\overline{\Omega}$ with $u = w$ on $\partial\Omega$.

On the other hand, for each $k \geq 1$ let $\widetilde{\Sigma}_k$ be the lifting of Σ_k obtained by replacing D_k with $\widetilde{D}_k$, where D_k and $\widetilde{D}_k$ are the graphs of w_k and u_k over Ω, respectively. Similarly, let $\widetilde{M}$ be the locally convex hypersurface obtained from M by replacing the graph of w over Ω by that of u. Clearly $\widetilde{\Sigma}_k$ converges to $\widetilde{M}$ as u_k converges uniformly to u on $\overline{\Omega}$. Since by Lemma 4.2 $\mu \leq \mathrm{Vol}(\widetilde{\Sigma}_k) \leq \mathrm{Vol}(\Sigma_k)$ for each k it follows that $\mathrm{Vol}(\widetilde{M}) = \mu$ and therefore $\mathrm{Vol}(\widetilde{D}) = \mathrm{Vol}(D)$. As both u and w are convex functions, this implies $u \equiv w$ on $\overline{\Omega}$ by the proof of Lemma 4.2. Since u satisfies (2.1), M has constant Gauss curvature K in a neighborhood of p. q.e.d.

5. Regularity

In this section we study the regularity of the hypersurface M constructed in the previous section to complete our proof of Theorems 1.1 and 1.2. Throughout this section, we assume that Σ is a locally convex immersed hypersurface which is C^2 and locally strictly convex along its boundary $\partial\Sigma$. Thus Σ is C^2 and locally strictly convex in a neighborhood of, and up to, $\partial\Sigma$. In addition, we assume $\partial\Sigma$ to be embedded and smooth. Let $K \leq \min K_\Sigma$ be a nonnegative constant and let M the locally convex hypersurface with $K_M \equiv K$ and $\partial M = \partial\Sigma$ constructed in Section 4. By Theorem 4.4, M is $C^{0,1}$ up to the boundary.

Theorem 5.1. *If $K > 0$ then M is smooth up to the boundary and locally strictly convex.*

Proof. Consider an interior point $p \in M$ which we assume to be the origin of $\mathbb{R}^{n+1}$. Since M is of class $C^{0,1}$, M locally near p can be represented as a convex graph $x_{n+1} = u(x) \geq 0$ over a domain $\Omega_1 \subset \mathbb{R}^n \equiv \{x_{n+1} = 0\}$ with a $C^{0,1}$ norm bound

$$\|u\|_{C^{0,1}(\overline{\Omega_1})} \leq C_1.$$

It follows that u satisfies the inequalities in the viscosity sense

$$K \leq \det(u_{ij}) \leq K(1 + C_1^2)^{\frac{n+2}{2}} \quad \text{in } \Omega_1.$$

We may assume $M \cap_p \{x_{n+1} = 0\} \subset \Omega_1$. By a theorem of Caffarelli [2], the nodal set $\{u = 0\}$ either is a single point, in which case M is smooth and strictly convex at p (see [4]), or does not contain any interior extreme points. So we will be done if we can show that $\{u = 0\} = \{0\}$. Suppose this is not the case. Then we can find two points $q_1, q_2 \in \partial M$ such that $\overline{q_1 q_2} \subseteq M \cap \{x_{n+1} = 0\}$ and $x_{n+1} = 0$ is a local supporting plane of M at every point on $\overline{q_1 q_2}$. By the proof (Step 3) of Theorem 3.1, $\overline{q_1 q_2}$ is transversal to ∂M at the endpoints. Without loss of generality we may assume

$$q_i = (0, \ldots, 0, (-1)^i a, 0), \quad i = 1, 2,$$

where $a > 0$. Consequently, there exists a constant $\delta > 0$ such that, in a neighborhood of $\overline{q_1 q_2}$, M is given as a convex graph $x_{n+1} = u(x) \geq 0$ over a domain

$$\Omega_0 := \{x := (x', x_n) \in \mathbb{R}^n | \varphi_1(x') < x_n < \varphi_2(x') \text{ for } |x'| < \delta\}$$

where φ_1, φ_2 are smooth functions since ∂M is smooth and transversal to $\overline{q_1q_2}$. Let ψ be a smooth function defined on ∂B_r, where $B_r \subset \Omega_0$ is the n-ball of radius $r \leq \delta$ centered at the origin, satisfying $\psi(0, \pm r) = 0$ and

$$\psi(x', x_n) \geq \max\{u(x', \varphi_1(x')), u(x', \varphi_2(x'))\}, \quad \forall\, (x', x_n) \in \partial B_r.$$

This is possible since both $u(x', \varphi_1(x'))$ and $u(x', \varphi_2(x'))$ are smooth in x' as ∂M is smooth and tangential to $x_{n+1} = 0$. By [6] there exists a unique strictly convex solution $v \in C^\infty(\overline{B_r})$ to the Dirichlet problem of the Monge-Ampère equation

$$\det(v_{ij}) = K \ \text{ in } \overline{B_r}, \quad v = \psi \ \text{ on } \partial B_r.$$

Since $\det(v_{ij}) = K \leq \det(u_{ij})$ in $\overline{B_r}$ and, by the convexity of u,

$$u(x', x_n) \leq \max\{u(x', \varphi_1(x')), u(x', \varphi_2(x'))\}, \quad \forall\, (x', x_n) \in \Omega_0$$

which implies $v \geq u$ on ∂B_r, we have $v \geq u \geq 0$ on $\overline{B_r}$ by the comparison principle. Since v is strictly convex and $v(0, a) = v(0, -a) = 0$, however, we have $v(0) < 0$ which is a contradiction. This proves that M is strictly convex and smooth in any interior point. Finally the boundary regularity follows from [6]. The proof is thus complete. q.e.d.

This completes the proof of Theorem 1.1. Turning to the case $K = 0$ we first prove the following lemma.

Lemma 5.2. *Let N be a locally convex hypersurface with $K_N \equiv 0$. Let p be an interior point of N and P a local supporting hyperplane to N at p. Then p is contained in a k-dimensional subsimplex of $N \cap_p P$ with vertices on ∂N for some $1 \leq k \leq n$.*

Proof. This follows form the argument in Step 3 of the proof of Theorem 3.1 as there is no hyperplane through p satisfying assumption (3.11). We redo the proof here for the reader's convenience. Since N is locally convex, P is a local supporting hyperplane to N at every point on $N \cap_p P$. Let D be the set of points on ∂N that (intrinsically) belong to $N \cap_p P$. It suffices to show that any point in $N \cap_p P$ is contained in the convex hull of D. If this is not the case, there is a point $q \in N \cap_p P$ which is separated by a hyperplane from D. We may assume $P = \{x_{n+1} = 0\}$ and q lies in $x_n > \varepsilon$ while D lies in $x_n < -\varepsilon$ for some $\varepsilon > 0$. It then follows that $N \cap_q \{x_{n+1} < \delta x_n\}$ is contained in the interior of N when δ is sufficiently small. This is a contradiction as the Gauss curvature of

$N \cap_q \{x_{n+1} < \delta x_n\}$ is zero everywhere while its boundary is contained in the hyperplane $x_{n+1} = \delta x_n$. q.e.d.

Theorem 5.3. *If $K = 0$, then M is $C^{1,1}$ up to the boundary.*

Proof. Let p be an interior point of M. From Step 3 of the proof of Theorem 3.1 we see that M has a tangent hyperplane at p. Suppose M locally (near p) is written as a convex graph $x_{n+1} = u(x)$ with $u \geq 0$ over $T_pM := \{x_{n+1} = 0\}$. Since T_pM is the tangent hyperplane to M at every point on $M \cap_p T_pM$, u is defined in a domain $\Omega \subset \mathbb{R}^n$ such that $M \cap_p T_pM \subset \{(x, 0) : x \in \overline{\Omega}\}$. By [7], in order to prove that M is $C^{1,1}$ it suffices to show that there exists a constant C, depending only on ∂M, and $\epsilon = \epsilon(p) > 0$ such that

$$u(x) \leq C|x|^2 \tag{5.1}$$

for all $x \in B_\epsilon(x^0) \subset \mathbb{R}^n$ where $p = (x^0, 0)$.

By Lemma 5.2, p is contained in a k-dimensional subsimplex, which we denote as S, of $M \cap_p T_pM$ with vertices on ∂M for some $1 \leq k \leq n$. According to [7], in order to prove (5.1) it suffices to consider the case $k = 1$. Suppose now that S is a segment with end points $q_1 := (x^1, 0)$, $q_2 := (x^2, 0)$ on ∂M. By the proof of Theorem 3.1, S is transversal to $\partial\Omega$ at the end points and both $\partial\Omega$ and $u|_{\partial\Omega}$ are smooth in a neighborhood of x^i $(i = 1, 2)$. Of the two end points, suppose that q_2 is the closer to p. We may assume $x_2 = 0$ and e_n to be the interior unit normal to $\partial\Omega$ at 0 where e_k $(1 \leq k \leq n + 1)$ is the unit vector in the positive x_k-axis direction. Since $x_{n+1} = 0$ is a local supporting hyperplane to M at q_2, $e_{n+1} \cdot \mathit{\Pi}(X, X) \geq 0$ for any $X \in T_{q_2}\partial M$. On the other hand, from the proof of Theorem 3.1 we see that the angle between $\mathit{\Pi}(X, X)$ and ν_Σ at q_2 does not exceed $\frac{\pi}{2} - 2\beta$ for some uniform constant $\beta > 0$. It follows that

$$\nu_\Sigma(q_2) = e_n \cos\alpha + e_{n+1} \sin\alpha \tag{5.2}$$

where $2\beta - \frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. We distinguish two cases: (i) $\alpha \leq \beta$ and (ii) $\alpha > \beta$.

If $\alpha \leq \beta$, then for any $X \in T_p\partial M$, the angle between $\mathit{\Pi}(X, X)$ and e_n is less than or equal to $\frac{\pi}{2} - \beta$ and, therefore, $e_n \cdot \mathit{\Pi}(X, X) \geq \sin\beta > 0$. This implies that $\partial\Omega \cap B_\delta(0)$ is uniformly strictly convex where $\delta > 0$ is a uniform constant. We therefore may follow the proof of [7] to derive (5.1).

We now suppose $\alpha > \beta$. Then locally Σ is a strictly convex graph $x_{n+1} = \underline{u}(x)$ over $\Omega \cap B_\delta(0)$ for some uniform constant $\delta > 0$. To prove (5.1) we then can follow the proof of Theorem 3.2 in [11]. This proves Theorem 5.3. q.e.d.

Remark 5.4. Theorem 1.2 follows from Lemma 5.2 and Theorem 5.3. If $\min K_\Sigma > 0$, Theorem 5.3 may be proved by approximation as follows. For any positive constant $\varepsilon \le \min K_\Sigma$, by Theorem 5.1 there exists a smooth locally strictly convex hypersurface M^ε with constant Gauss curvature ε and $\partial M^\varepsilon = \partial\Sigma$. By Theorem 3.5 (see Remark 3.6) we obtain a subsequence $\varepsilon_k \to 0$ such that $\{M^{\varepsilon_k}\}$ is convergent in local $C^{1,1}$ norms. Clearly, the limiting hypersurface must be M. Consequently, M is $C^{1,1}$ up to the boundary.

Remark 5.5. We have the following characterization of M at boundary (for $K = 0$). Let $p \in \partial M$ and choose coordinates of $\mathbb{R}^{n+1}$ with origin at p such that e_{n+1} and e_n are the unit normal and interior conormal to T_pM, respectively, where e_k as before is the unit vector in the positive direction of x_k axis.

Proposition 5.6. *Let $K = 0$ and $p \in M$. Suppose $M \cap_p T_pM$ does not contain any point in $\partial M \cap \{x_n > 0\}$. Then there exists some unit vector $X \in T_p\partial M$ such that $\theta(X) = 0$ where $\theta(X)$ is defined by*

$$\mathit{\Pi}(X,X) = |\mathit{\Pi}(X,X)|(e_n \cos\theta(X) + e_{n+1}\sin\theta(X)), \quad X \in T_p\partial\Sigma, \quad X \neq 0. \tag{5.3}$$

Proof. Suppose

$$\min\{\theta(X) : X \in T_p\partial\Sigma,\ |X| = 1\} > 0.$$

Then $M \cap_p \{0 \le x_{n+1} \le \lambda x_n\}$ does not contain any point on $\partial M \cap \{x_n > 0\}$ when $\lambda > 0$ is sufficiently small. By Lemma 5.2 this implies

$$(M \cap_p \{0 \le x_{n+1} \le \lambda x_n\}) \cap \{x_n > 0\} = \emptyset,$$

contradicting the fact that $T_pM = \{x_{n+1} = 0\}$. q.e.d.

6. Locally convex hypersurfaces with extreme boundary

We first give a brief proof of Corollary 1.3.

Proof of Corollary 1.3. By Theorem 1.2 we obtain a globally convex hypersurface M with $K_M \equiv 0$ and $\partial M = \Gamma$. Moreover, M is on the inner side of Σ along the boundary. Consider now an arbitrary interior point $q \in \Sigma$ and let P be a local supporting hyperplane to Σ at q. Since Σ is C^2 and locally strictly convex near the boundary, $\Sigma \cap_q P$ does not (intrinsically) contain points on $\partial\Sigma$. Let $t_0 > 0$ be the smallest value such that Σ^{t_0} contains a point p on $\partial\Sigma$, where

$$\Sigma^t := \Sigma \cap_q \{z \in \mathbb{R}^{n+1} : (z-q) \cdot \nu_P \leq t\}, \quad t \geq 0.$$

We see that $\partial\Sigma$ locally near p lies in the half space $\{z \in \mathbb{R}^{n+1} : (z-p) \cdot \nu_P \geq 0\}$. Since M is globally convex, ∂M lies on one side of T_pM. Let us assume $T_pM = \{x_{n+1} = 0\}$ and that ∂M lies in $x_{n+1} \geq 0$.

We now choose coordinates in $\mathbb{R}^{n+1}$ such that e_{n+1} and e_n are the unit normal and interior conormal to ∂M at p, respectively. We claim that

$$\nu_P = e_n \cos\alpha + e_{n+1} \sin\alpha, \quad \text{for some } 0 \leq \alpha \leq \tfrac{\pi}{2}. \tag{6.1}$$

Note that this implies $\Sigma^{t_0} \subset \{x_{n+1} \leq 0\}$ and thus completes the proof.

For any $0 < t < t_0$, since Σ^t does not contain points on ∂M, it is easy to see that $(\Sigma \backslash \Sigma^t) \cup D_t$ is a lifting of Σ with respect to $K = 0$, where D_t is the region on the hyperplane $P_t := \{z \in \mathbb{R}^{n+1} : (z-q) \cdot \nu_P = t\}$ bounded by $\Sigma^t \cap P_t$. Thus $M \cap_p \{z \in \mathbb{R}^{n+1} : (z-q) \cdot \nu_P \leq t_0\}$ does not intersect the region bounded by $\Sigma^t \cup D_t$ for any $0 < t < t_0$. Consequently, T_pM does not intersect the interior of Σ^{t_0}. This proves (6.1). q.e.d.

We next construct a smooth locally strictly convex, non-embedded, surface M of Gauss curvature one in $\mathbb{R}^3$ such that ∂M is strictly extreme. Let S_1 be the unit sphere centered at $(\frac{1}{2}, 0, 0)$. Cut off a small cap from the top of S_1 using a plane perpendicular to the line through $(0, 0, 2)$ and the center of S_1. Let Σ_1 be the resulting spherical cap and Σ_2 the reflection of Σ_1 with respect to $x_1 = 0$. Now, connecting the boundary circles of Σ_1 and Σ_2 by a thin convex bridge, we obtain a locally strictly convex surface Σ with self-intersection. Moreover, $\partial\Sigma$ is strictly extreme. According to the bridge principle of Hauswirth [16] there exists a locally strictly convex surface M of constant Gauss curvature one with the same boundary. It follows from [16] that M is a small perturbation of Σ and therefore has self-intersection.

If we start with cutting a small cap from the top of S_1 using a horizontal plane and repeat the rest of the above procedure, we get a non-embedded locally strictly convex K-surface M such that ∂M is extreme while the boundary of the convex hull of ∂M has interior singularities along the bottom edges.

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