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THE NORMALIZED MEAN CURVATURE FLOW FOR A SMALL BUBBLE IN A RIEMANNIAN MANIFOLD

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Abstract

The evolution of an embedded surface under the normalized mean curvature flow is the result of a complicated interaction between the geometry of the evolving surface and the geometry of the ambient space, and is not well understood in the context of a general Riemannian manifold. In the present paper we identify a class of initial conditions, that we call "bubbles", whose dynamics is primarily determined by the ambient space. A bubble is an embedded surface that is close to a small geodesic ball; we find that its shape is robust along the evolution. Moreover, under a relatively tight condition relating shape to size, we show that the velocity of the center of the bubble is given, to principal order, by the gradient of the scalar curvature. Finally under natural conditions of compactness and nondegeneracy we show that such solutions converge, as t tends to infinity, to surfaces of constant mean curvature.

0. Introduction

In this paper we consider the motion of hypersurfaces $\Sigma(t)$ in a compact n-dimensional Riemannian manifold M by normalized mean curvature, that is, with normal velocity given by the equation:

$$\langle X_t, \hat{N}\rangle = H^{\Sigma} - H, \tag{0.1}$$

where $\hat{N}$ denotes the unit outward normal, H is the mean curvature of Σ and H^{Σ} is the average mean curvature:

$$H^{\Sigma} = \frac{1}{A(\Sigma)} \int_{\Sigma} H d\sigma,$$

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with $A(\Sigma) = \int_\Sigma d\sigma$ the $(n-1)$-dimensional volume of Σ in the metric induced from M. Topologically, our hypersurfaces are embedded spheres, parametrized by a time-dependent smooth embedding $X(t) : S \to M$ from the $(n-1)$-sphere to M. It is easily verified that the left-hand side of (0.1) is, in fact, independent of parametrization.

The basic property of the law of evolution (0.1) is its isoperimetric nature. The volume $V(\Sigma)$ of the region enclosed by Σ is preserved:

$$\frac{dV(\Sigma(t))}{dt} = \int_\Sigma \langle X_t, \hat{N}\rangle d\sigma = \int_\Sigma (H^\Sigma - H) d\sigma = 0,$$

while the area $A(\Sigma)$ ($(n-1)$-dimensional volume) of Σ strictly decreases, unless H is constant:

$$\begin{aligned} \frac{dA(\Sigma(t))}{dt} &= \int_\Sigma \langle X_t, H\hat{N}\rangle d\sigma = \int_\Sigma (H^\Sigma - H) H d\sigma \\ &= -\int_\Sigma (H - H^\Sigma)^2 d\sigma. \end{aligned} \tag{0.2}$$

In particular, it is natural to expect that the hypersurface will tend to move towards regions of larger scalar curvature.

The evolution (0.1) was originally considered by M. Gage ($n = 2$) [6] and G. Huisken [8] ($n \geq 3$), when the ambient manifold M is euclidean space $\mathbb{R}^n$. Huisken proved that, in this case, if the initial hypersurface is smooth and uniformly convex, the evolution is defined for all time and the hypersurface remains convex and converges to a round sphere as $t \to \infty$. More recently Escher and Simonett [5] were able to prove this for initial hypersurfaces close to spheres (including some non-convex ones), by a different method, closer in spirit to ours in the present paper.

Relative to the much better understood evolution by mean curvature, the normalized problem presents several difficulties. The main one is that, the law being non-local, the maximum principle for parabolic equations does not apply. One consequence of this is that initially embedded hypersurfaces may develop self-intersections. An example for curves was described by Owen and Sternberg [13], and considered in more detail by Mayer and Simonett [12]. Another consequence is that, for the evolution on manifolds (even of constant curvature), convexity of the initial hypersurface may not be preserved, as was pointed out by Huisken. In spite of this difficulty, Huisken and S.-T. Yau [9] were able to use (0.1) to construct foliations by spheres of constant mean curvature — the steady-states of (0.1) — of asymptotically flat ends of positive mass (for 3-manifolds of positive scalar curvature). Their motivation was to develop a coordinate-independent definition of 'center of mass' of an isolated gravitational system. The problem of existence of local foliations by hypersurfaces of constant mean curvature was also considered by R. Ye [14]. He proved such foliations exist in a neighborhood of any nondegenerate critical point of the scalar curvature; and, conversely, that such foliations are unique in a certain class, and exist only in a neighborhood of a critical point.

R. Ye [14] used a perturbation of geodesic spheres of small radius to construct the local foliation. For the dynamical problem (0.1), this points to a potential problem when trying to construct solutions by a perturbation argument: except for special cases ('harmonic manifolds', including constant curvature manifolds), geodesic spheres *are not equilibria* of (0.1); in fact, on a general manifold the existence of time-independent solutions is a nontrivial issue. Nevertheless, we will show in this paper that if the initial hypersurface is sufficiently close to a small geodesic sphere (i.e., is a small '*bubble*'), the evolution is defined for all time and (under certain conditions) will converge to a leaf of one of these foliations, as $t \to \infty$. Clearly to obtain such a result one must be able to track the evolution of the hypersurface 'in the large' on the ambient manifold M, a problem not considered by the authors above. Our idea is to 'decouple' the effect of the ambient from the effect of the geometry of the interface: while the hypersurfaces start and remain very close to small geodesic spheres, we can describe how an appropriately defined 'barycenter' moves on M (at least asymptotically in a perturbation parameter).

The usual line of argument in studying existence and development of singularities for the mean-curvature flow proceeds in intrinsic fashion, by studying the evolution of geometric quantities on Σ (such as the trace-free second fundamental form). This approach is very effective and geometric; but in order to follow the large-scale motion of Σ on M we are led, instead, to work directly with the parametrizations $X(t)$, i.e., to study a motion in a manifold of embeddings. This introduces some technical difficulties-for example, there is no uniquely defined law of motion for embeddings corresponding to (0.1)-but, on the other hand, allows for the introduction of an infinite-dimensional dynamical systems approach to the problem. At least in the present 'perturbation' setting, we are able to draw on results from semigroup theory ('maximal regularity') and known results on infinite dimensional systems to give a fairly non-technical proof. It is our hope that such an approach will also prove fruitful for other geometric evolution equations.

The evolution we construct takes place in a submanifold of the manifold $\mathcal{E}$ of $C^{2+\alpha}$ 'small quasispherical embeddings' $X : S \to M$, which are radial graphs over a small geodesic sphere in M with center $\xi \in M$, radius $R > 0$. To define such an embedding, we need a diffeomorphism F from S to the unit tangent sphere at ξ and a 'shape function' $\psi : C^{2+\alpha} \to \mathbb{R}$, which we take to be a $C^{2+\alpha}$ function on S, with zero average on S. Introducing two small scale parameters $\delta \in (0, \delta_0)$, $\epsilon \in (0, \epsilon_0)$, we let $\mathcal{E} = \mathcal{E}_{\delta_0,\epsilon_0}$ be the space of embeddings which can be written in the form:

$$X_{(R,\xi,F,\psi)}(u) = \exp_\xi[\delta R(1 + \epsilon\psi(u))F(u)],$$

where $0 < R < 1$, $\psi \in C^{2+\alpha}(S)$ satisfies $\|\psi\|_{C^{2+\alpha}} < 1$ and $\mathrm{ave}_S[\psi] = 0$ and F is a $C^{2+\alpha}$ diffeomorphism from S to the unit tangent sphere $S_\xi \subset T_\xi M$. We take δ_0, ϵ_0 small enough that the open set $\mathrm{int}\,(X)$ bounded by the hypersurface $\Sigma = \mathrm{image}(X)$ (and containing ξ) is contained in a totally convex normal neighborhood of ξ, and is uniformly convex.

The same embedding $X \in \mathcal{E}$ can be written in the form $X_{(R,\xi,F,\psi)}$ in different ways, parametrized by $\xi \in \mathrm{int}(X)$. We need a choice of ξ that is as canonical as possible, given X; that is, a 'barycenter' for X. While a general Riemannian notion of 'barycenter' exists (see [10]), here we find it useful to work with the notion of '*analytic barycenter*', which appears already in [1]. This is the unique $\xi \in \mathrm{int}\,(X)$ for which X may be written as $X_{(R,\xi,F,\psi)}$, with ψ taken in the space:

$$\begin{aligned}
&K^{2+\alpha} = C^{2+\alpha}(S) \cap C_0(S),\\
&C_0(S) = \Big\{\psi \in C^0(S);\;\; \mathrm{ave}_S[\psi] = 0 = \mathrm{ave}_S[\psi u^i], i = 1, \dots n\Big\}.
\end{aligned}$$

In Section 1 we prove (in Lemma 1.1) that for X in a neighborhood $\mathcal{N}$ of the submanifold of 'standard parametrizations of geodesic spheres' $\mathcal{E}_0 \subset \mathcal{E}$, there is a unique $\xi = \mathcal{B}(X)$ with this property; $\mathcal{N} = \mathcal{N}(\delta_1, \epsilon_1)$ is open in $\mathcal{E}_{\delta_1,\epsilon_1}$.

The existence of the analytic barycenter allows us to consider evolution equations on the submanifold $\mathcal{N}_{\mathrm{std}} = \mathcal{N}_{\mathrm{std}}(\delta_1, \epsilon_1) \subset \mathcal{N}$, defined as the set of embeddings $X \in \mathcal{N}$ of the form:

$$X = X_{(R,\xi,e,\psi)} = \exp_\xi[R(1+\psi(\cdot))e(\cdot,\xi)],$$
$$\xi = \mathcal{B}(X), \quad \mathrm{ave}_S[\psi] = \mathrm{ave}_S[\psi u^i] = 0, i = 1, \dots, n,$$

where $e : S \to S_\xi$ is an isometry, defined by an orthonornal frame at ξ. $\mathcal{N}_{\mathrm{std}}$ is the image under a smooth injective map Φ of the manifold:

$$\mathcal{M}_0 = \mathcal{M}_0(\delta_1, \epsilon_1) = (0, \delta_1) \times \mathbb{F}M \times K^{2+\alpha}_{\epsilon_1},$$

where $\mathbb{F}M$ denotes the orthonormal frame bundle of M and the last factor is the ϵ_1-ball in $K^{2+\alpha}$. The next step is to find an evolution equation for $(R, \xi, e, \psi) \in \mathcal{M}_0$, the solutions of which map under Φ to parametrized solutions of (0.1) in $\mathcal{N}_{\mathrm{std}}$. Since the barycenter $\mathcal{B}(X)$ depends on the parametrization X (and not just on its image Σ), we need to fix an equation of motion in the space of embeddings. As a first attempt, one might expect that the equations on $\mathcal{M}_0$ would be induced by the evolution on $\mathcal{N}$:

$$X_t = (H^\Sigma - H)\hat{N}. \tag{0.3}$$

It turns out, however, that $\mathcal{N}_{\mathrm{std}}$ is not invariant under (0.3), which therefore does not induce a system on $\mathcal{M}_0$. Fortunately it is possible to compute a 'tangential correction' to (0.3) which does preserve $\mathcal{N}_{\mathrm{std}}$. This is explained in Section 1, where we find (in Lemma 1.7) a system on $\mathcal{M}_0$ whose solutions map to parametrized solutions of (0.1); and conversely, any motion $\Sigma(t)$ of 'small bubbles' by normalized mean curvature can be parametrized by $X_{(R,\xi,e,\psi)} \in \mathcal{N}_{\mathrm{std}}$, so that $(R, \xi, e, \psi)(t)$ is a solution of the system on $\mathcal{M}_0$. For each choice of the scale parameters δ, ϵ, we obtain the system on $\mathcal{M}_0$:

$$\begin{aligned} \delta R_t &= \mathrm{ave}_S[v_N - E] \\ \xi_t &= n\ \mathrm{ave}_S[(v_N - E)e] \\ \nabla_{\xi_t} e &= 0 \\ \delta\epsilon R\psi_t &= (v_N - E)_K - (\delta\psi)\mathrm{ave}_S[v_N - E]. \end{aligned} \tag{0.4}$$

Here $v_N = (H^\Sigma - H)\|N\|$ (where N is a particular normal vector to Σ) and $E = E(v_N)$ corresponds to the tangential correction referred to above. v_N and E are computed at $X_{(\delta R,\xi,e,\epsilon\psi)}$. (In particular, e is simply obtained by parallel transport of $e(0)$ along $\xi(t)$). It is for this system on $\mathcal{M}_0$ that our main result is proved.

Given a totally convex open set $U \subset M$ and positive numbers $\delta < \delta_1, \epsilon < \epsilon_1$, define the open subset of $\mathcal{M}_0$:

$$\mathcal{O}^{2+\alpha}_{\delta,\epsilon}(U) = \{(R, \xi, e, \psi) \in \mathcal{M}_0; 0 < R < \delta, (\xi, e) \in \mathbb{F}U, \|\psi\|_{C^{2+\alpha}} < \epsilon\};$$

we define analogously the open subset $\mathcal{O}^{\alpha}_{\delta,\epsilon}(U)$ of:

$$\mathcal{M}^\alpha = \mathcal{M}^\alpha(\delta_1, \epsilon_1) = (0, \delta_1) \times \mathbb{F}M \times K^\alpha_{\epsilon_1},$$

where $K^\alpha = C^\alpha \cap C_0(S)$. We find local solutions in the open subset:

$$W^T_{\delta,\epsilon}(U) = C^0([0,T], \mathcal{O}^{2+\alpha}_{\delta,\epsilon}(U)) \cap C^1((0,T], \mathcal{O}^{\alpha}_{\delta,\epsilon}(U))$$

of the space:

$$W^T_{\delta,\epsilon} = C^0([0,T], \mathcal{M}_0(\delta, \epsilon)) \cap C^1((0,T], \mathcal{M}^\alpha(\delta, \epsilon)).$$

In general we denote $\mathcal{N}_{\mathrm{std}}(\delta, \epsilon) = \Phi(\mathcal{M}_0(\delta, \epsilon))$. Our main result follows.

Main Theorem.

(i) (Local existence) *There exist constants $\delta_2 \in (0, \delta_1/2), \epsilon_2 \in (0, \epsilon_1/2)$ depending only on M, and $T = T(\delta_2, \epsilon_2) > 0$ so that for any $\Psi(0) = (\delta R(0), \xi(0), e(0), \epsilon\psi(0)) \in \mathcal{M}_0(\delta_2, \epsilon_2)$ with $\psi(0) \in C^\infty(S)$, there exists a unique solution $\Psi(t) = (\delta R(t), \xi(t), e(t), \epsilon\psi(t))$ of* $(0.4)_{\delta,\epsilon}$ *in $W^T_{\delta_2,\epsilon_2}(U_0)$ (where U_0 is a totally convex neighborhood of $\xi(0)$). The solution $\Psi(t)$ is smooth for $t > 0$. The hypersurfaces $\Sigma(t)$ parametrized by $X(t) \in \mathcal{N}_{\mathrm{std}}$:*

$$X(t)(u) = \exp_{\xi(t)}[\delta R(1 + \epsilon\psi(t,u))e(t,u)]$$

are smooth for $t \geq 0$ and satisfy Equation (0.1); *in particular, we have smooth local solutions for any initial embedding $X(0) \in \mathcal{N}_{\mathrm{std}}(\delta_2, \epsilon_2)$.*

(ii) (Global existence) *For a given $\Psi \in \mathcal{M}_0(\delta_2, \epsilon_2)$, define T^*_Ψ as the supremum of all $T > 0$ such that the solution $\Psi(t)$ found in* (i) *with initial value Ψ is in $W^T_{\delta_2,\epsilon_2}$. There exist constants $\delta_3 > 0, \epsilon_3 > 0$ depending only on M, so that if Ψ is in $\mathcal{M}_0(\delta, \epsilon)$, with $\psi(0) \in C^\infty(S)$, $\delta < \delta_3, \epsilon < \epsilon_3$ and $\delta^2 \ll \epsilon$, then $T^*_\Psi = \infty$. Thus, the solution $\Psi(t)$ defines a parametrization $X(t) \in \mathcal{N}_{\mathrm{std}}$ of a global smooth solution $\Sigma(t)$ of the normalized mean curvature flow* (0.1). *In particular, we have a global solution for arbitrary initial embedding $X(0) \in \mathcal{N}_{\mathrm{std}}(\delta, \epsilon)$ under the same conditions on δ, ϵ, and the solution stays in $\mathcal{N}_{\mathrm{std}}(\delta_2, \epsilon_2)$.*

(iii) (Motion of the barycenter) *In addition to the assumptions in* (ii), *suppose the scale parameters* δ, ϵ *satisfy:*

$$\delta^2 \ll \epsilon \ll \delta^{3/2}.$$

Then the leading term in the equation of motion for ξ *in* $(0.4)_{\delta,\epsilon}$ *is:*

$$\xi_t = c_n \nabla^M \mathrm{Scal}(\xi) R^2 \delta^2 + O(\epsilon^2 \delta^{-1}),$$

where $c_n = 2n/3(n+2)$. *Here* $\nabla^M \mathrm{Scal}$ *denotes the gradient of scalar curvature.*

(iv) (Asymptotic behavior) *Any accumulation point along a sequence* $t_n \to \infty$ *of a solution* $X(t)$ *of* (0.1) *in* $\mathcal{N}_{\mathrm{std}}$, $X(t_n) \to X_* \in \mathcal{N}_{\mathrm{std}}$, *parametrizes a hypersurface* Σ_* *of constant mean curvature. In addition, assume all critical points of the scalar curvature function* Scal *are nondegenerate. Then there are disjoint open neighborhoods* $V_p \subset M$ *of the* (*finitely many*) *critical points* p *of* Scal *and constants* $\delta_4 > 0, \epsilon_4 > 0$, *so that any global solution* $\Psi(t)$ *of* $(0.4)_{\delta,\epsilon}$ *with* $\delta < \delta_4, \epsilon < \epsilon_4, \epsilon \ll \delta^{3/2}$, *converges as* $t \to \infty$ *to some* $\Psi_* \in \mathcal{M}_0(\delta, \epsilon)$. *The embedding* $X_* \in \mathcal{N}_{\mathrm{std}}$ *corresponding to* Ψ_* *has barycenter* $\xi_* \in V_p$ *for some critical point* p *of* Scal, *and parametrizes a hypersurface of constant mean curvature* $\Sigma_* \subset V_p$, *which is the unique leaf of the local c.m.c foliation* ([14]) *at* p *enclosing the same volume as* $\Sigma(0)$.

In Parts (ii) and (iii) of the statement, $\delta^2 \ll \epsilon$ means $\delta^2 = \epsilon^\gamma$ for some $\gamma > 1$; $\epsilon \ll \delta^{3/2}$ is understood analogously.

Local existence (Part (i)) is well-known for (0.1). We include a proof for the equivalent system (0.4) in the framework of semigroup theory, since this leads to the continuation criterion we use for global existence. In fact, neither the decomposition $\varphi = R(1+\psi)$ of the 'shape function' nor detailed asymptotics are needed, and the proof we give (in Section 3, Lemma 3.2) works for more general initial data than stated in (i). We use results from 'maximal regularity theory' ([2], [3], [11]); all that is needed is to verify that the hypothesis of Theorem 2.14 in [3] are satisfied.

In Section 2 we develop the asymptotic expansions of equations (0.4) that are needed for global existence. The starting point are standard Taylor expansions of Jacobi fields in Riemannian normal coordinates, from which expansions for the mean curvature and its average (for a radial graph) follow easily. Another consequence is an *a priori* estimate for the radius $R(t)$, which follows from the conservation of the enclosed volume.

After giving a proof of local existence (Lemma 3.2), we prove global existence in Section 3 by an argument involving the variation of constants representation formula and maximal regularity estimates. Here the scale parameters and asymptotics are needed, and the restriction $\delta^2 \ll \epsilon$ is required. It guarantees that, in the equation for ψ_t, the 'euclidean' term $\delta^{-2}R^{-2}A\psi$ (where A is the linearized operator for euclidean ambient, $A\psi = \Delta_S\psi + (n-1)\psi$) dominates the largest Riemannian term, which is of order ϵ^{-1}. The fact that the spectrum of A is bounded above by a negative constant is crucial, and motivates our definition of analytic barycenter. A similar argument was used in [1].

We conclude in Section 4 with the proof of asymptotic convergence to a geodesic sphere. Since we assume the critical points of the scalar curvature function are nondegenerate, a small constant mean curvature sphere near a critical point must be a leaf of the local foliation at that critical point ([14]), and there is only one of those enclosing a given volume. We may then appeal to general results on infinite-dimensional dynamical gradient systems to conclude.

We close the introduction with a few heuristic remarks. If we consider the motion of small geodesic spheres on M as 'approximate solutions' of the flow, it is not hard to understand the leading term in the equation of motion for the barycenter from isoperimetric considerations. The (n-1)-dimensional area ('perimeter') $A(\xi, R)$ of a geodesic sphere with center ξ, radius R, is given by the classical Riemannian formula:

$$A(\xi, R) = R^{n-1}\int_S \left[1 - \frac{1}{6}\mathrm{Ric}\,(\xi, u)R^2 + O(R^3)\right] du,$$

where $\mathrm{Ric}\,(\xi, u)$ denotes the Ricci curvature of M at ξ in the direction u, if we identify S with the unit tangent sphere at ξ. (This is just the Gauss curvature at ξ when $n = 2$.) Differentiating in ξ for fixed R, we obtain:

$$\begin{aligned}(0.6)\quad \partial_\xi(A(\xi,R)) &= -R^{n-1}\left(\int_S\left[\frac{1}{6}\nabla_u \mathrm{Ric}\,(\xi,u)u^i R^2 + O(R^3)\right]du\right)e_i\\ &= -\frac{2\omega_{n-1}}{3(n+2)}R^{n+1}\nabla^M \mathrm{Scal}(\xi)+\cdots,\end{aligned}$$

by the calculation in [14] ($\omega_{n-1} = \mathrm{vol}\,(S^{n-1})$).

Consider the space of embeddings $\mathcal{H}$ of S^{n-1} into M, endowed with the Hilbert manifold structure defined by the L^2 inner product on $T_X\mathcal{H} = \{\text{vector fields along } X\}$. From (0.2), we see that for the perimeter functional $A(X)$:

$$\mathrm{grad}_{\mathcal{H}} A(X) = H\hat{N},$$

and for the enclosed volume functional $V(X)$:

$$\mathrm{grad}_{\mathcal{H}} V(X) = \hat{N}.$$

Denoting by $\mathcal{H}_0$ the 'submanifold' of embeddings enclosing a fixed volume, it follows formally that:

$$\mathrm{grad}_{\mathcal{H}_0} A(X) = (H - H^{\Sigma})\hat{N},$$

so that the evolution law (0.1) may be written as:

$$\frac{\partial X}{\partial t} = -\mathrm{grad}_{\mathcal{H}_0} A(X). \tag{0.7}$$

Consider now an approximate solution given by moving (parametrized) geodesic spheres $X_{\xi(t),R(t)}$. The volume preservation condition implies $R(t)$ is approximately constant, so the projection of $\partial_R X$ on $T_X\mathcal{H}_0$ is small, and may be ignored to lowest order. Thus, to lowest order, Equation (0.7) takes the form:

$$\xi_t \partial_\xi X \sim -\mathrm{grad}_{\mathcal{H}_0} A(X).$$

Denoting by $\xi_t^T(u) = \xi_t - \langle \xi_t, u\rangle u$ the 'tangential ' component of ξ_t, we have:

$$\langle \xi_t\partial_\xi X, \partial_\xi X\rangle_{\mathcal{H}_0} \sim R^{n-1}\int_S \xi_t^T(u)du.$$

With $\langle \mathrm{grad}_{\mathcal{H}_0} A(X), \partial_\xi X\rangle_{\mathcal{H}_0} = \partial_\xi A(\xi,R)$, this implies:

$$\omega_{n-1}\mathrm{ave}_S[\xi_t^T]R^{n-1} \sim -\partial_\xi A(\xi,R).$$

Noting that $\xi_t = n\ \mathrm{ave}_S[\xi_t^T]$, this combines with (0.6) to yield:

$$\xi_t \sim \frac{2n}{3(n+2)} R^2 \nabla^M \mathrm{Scal}(\xi) + \cdots ,$$

as claimed in Part (iii) of the main theorem.

This formal argument also suggests that our main result enjoys a certain universality, and is probably valid for a large class of geometric evolution laws that preserve the enclosed volume and reduce perimeter, and may be realized as the gradient of the perimeter functional in an appropriate Hilbert manifold $\mathcal{H}$. For the flow considered in this paper, the asymptotic behavior of 'quasi-spherical' solutions is controlled by the finite-dimensional gradient flow:

$$\xi_t = c_n \nabla^M \mathrm{Scal}\,(\xi).$$

The scalar curvature increases along solutions; $\xi(t)$ 'climbs' towards peaks of maximal scalar curvature. The ω-limit sets of orbits consist of critical points of Scal, with stable equilibria corresponding to local maxima. If all critical points of Scal are nondegenerate (or if M is real-analytic), solutions converge to a critical point as $t \to \infty$, and most orbits (in a topological sense) converge to stable equilibria. This behavior is reflected in the claim of Part (iv) in the main theorem.

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1. The manifold of small bubbles and the barycentric system

1.1 Existence of the analytic barycenter

Let M be a compact smooth oriented n-dimensional Riemannian manifold without boundary, $S \subset \mathbb{R}^n$ the unit sphere. In this paper we consider the motion by normalized mean curvature flow in the manifold $\mathcal{E}^{2+\alpha}$ of 'small, strictly convex, $C^{2+\alpha}$ embeddings' X, defined by the conditions:

(i) $X : S \to M$ is an embedding of class $C^{2+\alpha}$;

(ii) The image $\Sigma = \mathrm{im}\,(X)$ lies in a geodesically convex neighborhood of any point in $\mathrm{int}\,(X)$, the 'small' open subset of M bounded by Σ;

(iii) $\Sigma = \mathrm{im}\,(X)$ is a strictly convex hypersurface in M; in particular (given (ii)), $\mathrm{int}\,(X)$ is a geodesically convex subset of M.

$\mathcal{E}^{2+\alpha}$ is an open subset in the Banach manifold of $C^{2+\alpha}$ embeddings $S \to M$.

Any $\xi \in \mathrm{int}\,(X)$ determines a $C^{2+\alpha}$ function $\varphi_\xi : S \to \mathbb{R}^+$ and a $C^{2+\alpha}$ diffeomorphism $F(\cdot, \xi) : S \to S_\xi$ (the unit sphere in the tangent space $T_\xi M$) such that $\Sigma = \mathrm{im}(X)$ admits the parametrization:

$$X_\xi(u) = \exp_\xi[\varphi_\xi(u) F(u, \xi)].$$

Conversely (by compactness of M), there exists a constant $\delta_0 > 0$ depending only on M so that, for any $\xi \in M$, $F(\cdot, \xi) \in \mathrm{Diff}^{2+\alpha}(S; S_\xi)$ and $\varphi_\xi \in C^{2+\alpha}(S; \mathbb{R}^+)$ such that $\|\varphi_\xi\|_{C^0} < \delta_0$, the embedding X_ξ defined by this formula satisfies (i) and (ii). Moreover, there exists $\epsilon_0 \in (0, 1/10)$ depending only on M so that if we write $\varphi_\xi = R_\xi(1 + \psi_\xi)$ with $0 < R_\xi < \delta_0$, $\|\psi_\xi\|_{C^{2+\alpha}} < \epsilon_0$ and $\int_S \psi_\xi = 0$, then (iii) also holds and $X_\xi \in \mathcal{E}^{2+\alpha}$

The set of $C^{2+\alpha}$ embeddings $S \to M$ which can be written in this form is an open subset of $\mathcal{E}^{2+\alpha}$, which we will denote by $\mathcal{E}^{2+\alpha}_{\delta_0, \epsilon_0}$. From now on we let $\mathcal{E} = \mathcal{E}^{2+\alpha} = \mathcal{E}^{2+\alpha}_{\delta_0, \epsilon_0}$.

Consider a motion $\Sigma(t)$ of hypersurfaces by normalized mean curvature flow, parametrized by $X(t) \in \mathcal{E}^{2+\alpha}$. To follow the motion globally on M, we need a notion of 'barycenter' of $\Sigma(t)$. While a geometric notion of barycenter of sets in Riemannian manifolds exists (see [10]), we will need a different one; our 'analytic barycenter' will be defined only for small convex hypersurfaces which are sufficiently $C^{2+\alpha}$ close to a geodesic sphere.

We denote by $\mathcal{E}_0$ the manifold of standard parametrizations of small geodesic spheres:

$$\mathcal{E}_0 = \{X : S \to M; X(u) = \exp_\xi[R u^i e_i], \xi \in M, R \in [0, \delta_0/2], \mathbf{e} \in F_\xi M\},$$

where $F_\xi M$ denotes the space of oriented orthonormal frames of $T_\xi M$. (We do not exclude embeddings which degenerate to a point.) $\mathcal{E}_0$ is a submanifold of $\mathcal{E}^{2+\alpha}$, diffeomorphic to the orthonormal frame bundle of M cross an interval, $\mathbb{F}M \times [0, \delta_0/2]$. We will sometimes identify $X \in \mathcal{E}_0$ with the ordered triple $(\xi, R, \mathbf{e})$, and identify the frame $\mathbf{e}$ with the isometry $e(\cdot, \xi) : S \to S_\xi$ it defines via $e(u, \xi) = u^i e_i$ (written $e(u)$ if ξ is understood).

If $\bar{X} = (\bar{\xi}, \bar{R}, \bar{\mathbf{e}}) \in \mathcal{E}_0$, a basis of neighborhoods of $\bar{X}$ in $\mathcal{E}^{2+\alpha}$ is given by sets of the form:

$$\mathcal{N}^{2+\alpha}_{\delta,\epsilon}(\bar{X}) = \Big\{ X \in \mathcal{E}^{2+\alpha}; \bar{\xi} \in \operatorname{int}(X), \\ \|\psi_{\bar{\xi}}\|_{C^{2+\alpha}} < \epsilon,\ |\bar{R} - R_{\bar{\xi}}| < \delta,\ \|F(\cdot, \bar{\xi}) - e(\cdot, \bar{\xi})\|_{C^{2+\alpha}} < \epsilon \Big\}.$$

For X in a neighborhood $\mathcal{N}^{2+\alpha}_{\delta_M, \epsilon_M}$ of $\mathcal{E}_0$ in $\mathcal{E}^{2+\alpha}$, we wish to define a unique 'analytic barycenter'. Let $\mathcal{U}^{2+\alpha} \subset \mathcal{E}^{2+\alpha} \times M$ be the open subset defined by:

$$\mathcal{U}^{2+\alpha} = \{(X, \xi); \xi \in \operatorname{int}(X)\}.$$

We define a smooth map $\mathcal{P} : \mathcal{U}^{2+\alpha} \to \mathbb{R}^n$ by:

$$\mathcal{P}^i(X, \xi) = \int_S \varphi_\xi(u) u^i du = \int_S d_\xi(X(u)) u^i du, \quad i = 1, \ldots, n,$$

where $d_\xi(x) = \operatorname{dist}_M(\xi, x)$.

Lemma 1.1. *There exist $\delta_M \in (0, \delta_0/2), \epsilon_M \in (0, \epsilon_0/2)$ such that, defining:*

$$\mathcal{N}^{2+\alpha}_{\delta_M,\epsilon_M} = \left\{ X \in \mathcal{E}^\alpha; \exists \bar{X} = (\bar{\xi}, \bar{R}, \mathbf{e}) \in \mathcal{E}_0 \text{ with } X \in \mathcal{N}^{2+\alpha}_{\delta_M,\epsilon_M}(\bar{X}) \right\},$$

one has a smooth map $\mathcal{B} : \mathcal{N}^{2+\alpha}_{\delta_M,\epsilon_M} \to M$ with the properties:

(i) $\mathcal{P}(X, \mathcal{B}(X)) = 0, \quad X \in \mathcal{N}^{2+\alpha}_{\delta_M,\epsilon_M}$.

(ii) *$\mathcal{B}(X)$ is the only solution of $\mathcal{P}(X, \cdot) = 0$ in the ball with center $\mathcal{B}(X)$, radius ϵ_M.*

Proof.

Step (i). In the first step we work in $\mathcal{E}^{2+\alpha'}$, for a fixed $0 < \alpha' < \alpha$. Fix $\bar{X} \in \mathcal{E}_0$ parametrizing $S_{\bar{R}}(\bar{\xi})$. Clearly $\mathcal{P}(\bar{X}, \bar{\xi}) = 0$. By the Implicit Function Theorem (in the Banach manifold $\mathcal{E}^{2+\alpha'}$), to solve the problem locally near $(\bar{X}, \bar{\xi})$, it is enough to show that the partial differential $(D_\xi \mathcal{P})(\bar{X}, \bar{\xi}) : T_{\bar{\xi}} M \to \mathbb{R}^n$ is an isomorphism.

Writing $\bar{X}(u) = \exp_{\bar{\xi}}[\bar{R}e(u,\bar{\xi})]$ as above, this follows from the calculation:

$$
\begin{aligned}
(D_\xi \mathcal{P}^i)(\bar{X},\bar{\xi})v &= \int_S \left\langle \left(\nabla d_{\bar{X}(u)}\right)(\bar{\xi}), v\right\rangle u^i du \\
&= -\int_S \langle e(u,\bar{\xi}), v\rangle u^i du \\
&= -\int_S u^j v^j u^i du = -\frac{\mathrm{vol}\,(S)}{n} v^i,
\end{aligned}
$$

for each $v \in T_{\bar{\xi}}M$, where $v = v^j e_j$.

Thus we obtain a neighborhood of $(\bar{X},\bar{\xi})$ in $\mathcal{U}^{2+\alpha'}$ (which we may assume to have the form $\mathcal{N}^{2+\alpha'}_{\bar{\delta},\bar{\epsilon}}(\bar{X}) \times B_{\bar{\epsilon}/2}(\bar{\xi})$, with $\bar{\epsilon} < \epsilon_0/2 < 0.1$, $\delta < \delta_0/2 < 0.1$), and a smooth map

$$
\mathcal{B}_{\bar{X}} : \mathcal{N}^{2+\alpha'}_{\bar{\delta},\bar{\epsilon}}(\bar{X}) \to B_{\bar{\epsilon}/2}(\bar{\xi})
$$

such that $\mathcal{P}(X, \mathcal{B}_{\bar{X}}) = 0$. Furthermore, by reducing $\bar{\epsilon}$ if needed we may arrange that, for each $X \in \mathcal{N}^{2+\alpha'}_{\bar{\delta},\bar{\epsilon}}(\bar{X})$, $\mathcal{B}_{\bar{X}}(X)$ is the only solution of $\mathcal{P}(X,\cdot) = 0$ in $B_{\bar{\epsilon}}(\bar{\xi})$.

Step (ii). To define $\mathcal{B}$ globally, we use a standard covering argument. By compactness of $\mathcal{E}_0$ we find $\bar{X}_i = (\bar{\xi}_i, \bar{R}_i, \bar{e}(\cdot,\bar{\xi}_i))$, $i = 1,\ldots,N$, and $\bar{\delta} \in (0,\delta_0/2)\bar{\epsilon}_i \in (0,\epsilon_0/2)$ so that:

$$
\mathcal{E}_0 = \bigcup_{i=1}^N \mathcal{N}^{2+\alpha'}_{\bar{\delta}_i/L,\bar{\epsilon}_i/L}(\bar{X}_i) \cap \mathcal{E}_0
$$

and maps $\mathcal{B}_i = \mathcal{B}_{\bar{X}_i} : \mathcal{N}^{2+\alpha'}_{\bar{\delta}_i,\bar{\epsilon}_i}(\bar{X}_i) \to B_{\bar{\epsilon}_i/2}(\bar{\xi}_i)$ as in Step (i). Here $L > 0$ is defined by $L^{-1} = \min\{1/10, (10C_0)^{-1}\}$, with $C_0 > 0$ a constant depending only on M, defined below, in Step (iii) of this proof. We need to check compatibility: $\mathcal{B}_i(X) = \mathcal{B}_j(X)$ if $X \in \mathcal{N}^{2+\alpha'}_{\bar{\delta}_i,\bar{\epsilon}_i}(\bar{X}_i) \cap \mathcal{N}^{2+\alpha'}_{\bar{\delta}_j,\bar{\epsilon}_j}(\bar{X}_j)$. It is easy to show that:

$$
\max\{d(\xi_i,\xi_j), |\bar{R}_i - \bar{R}_j|\} \le \bar{\epsilon}_i \bar{R}_i + \bar{\epsilon}_j \bar{R}_j < \frac{1}{10}(\bar{\epsilon}_i + \bar{\epsilon}_j),
$$

since $\max_i\{\bar{R}_i\} < i_M/2 < 1/10$. In particular (choosing the indices so that $\bar{\epsilon}_i < \bar{\epsilon}_j$):

$$
\mathcal{B}_i(X) \in B_{\frac{\bar{\epsilon}_i}{2}}(\bar{\xi}_i) \subset B_{\frac{\bar{\epsilon}_i}{2} + \frac{1}{10}(\bar{\epsilon}_i + \bar{\epsilon}_j)}(\bar{\xi}_j) \subset B_{\bar{\epsilon}_j}(\bar{\xi}_j).
$$

By the uniqueness in Step (i), we must have $\mathcal{B}_i(X) = \mathcal{B}_j(X)$. Thus, letting:

$$\mathcal{W}^{2+\alpha'} = \bigcup_{i=1}^{N} \mathcal{N}^{2+\alpha'}_{\bar{\delta}_i,\bar{\epsilon}_i}(\bar{X}_i) \subset \mathcal{E}^{2+\alpha'},$$

we have a smooth map:

$$\mathcal{B} : \mathcal{W}^{2+\alpha'} \to M, \quad \mathcal{B}(X) \in \operatorname{int}(X),$$

such that $\mathcal{P}(X, \mathcal{B}(X)) = 0$. Furthermore, letting $\epsilon_{\min} = \frac{1}{4}\min\{\bar{\epsilon}_i; 1 \leq i \leq N\}$, $\mathcal{B}(X)$ is the unique solution to $\mathcal{P}(X,\xi) = 0$ in the ball $B_{\epsilon_{\min}}(\mathcal{B}(X))$.

Step (iii). We still need to show that, for some $\epsilon_M > 0$ depending only on M and $\{\bar{\epsilon}_1, \dots, \bar{\epsilon}_N\}$, $\mathcal{W}^{2+\alpha'}$ contains a subset $\mathcal{N}^{2+\alpha}_{\delta_M,\epsilon_M}$ as in the statement of the lemma. That is, assuming $X \in \mathcal{E}^{2+\alpha}$ satisfies, for some $(\bar{\xi}, \bar{R}, \bar{e}) \in \mathcal{E}_0$:

$$\bar{\xi} \in \operatorname{int}(X), \quad |R_{\bar{\xi}} - \bar{R}| < \delta_M,$$
$$\|\varphi_{\bar{\xi}} - \bar{R}\|_{C^{2+\alpha}} < \epsilon_M \bar{R}, \quad \|F(\cdot,\bar{\xi}) - \bar{e}(\cdot,\bar{\xi})\|_{C^{2+\alpha}} < \epsilon_M,$$

we must show that for some $i \in \{1, \dots, N\}$:

$$\bar{\xi}_i \in \operatorname{int}(X), \quad |R_{\bar{\xi}_i} - \bar{R}_i| < \bar{\delta}_i,$$
$$\|\varphi_{\bar{\xi}_i} - \bar{R}_i\|_{C^{2+\alpha'}} < \bar{\epsilon}_i \bar{R}_i, \quad \|F(\cdot,\bar{\xi}) - \bar{e}(\cdot,\bar{\xi}_i)\|_{C^{2+\alpha'}} < \bar{\epsilon}_i.$$

For some i (say $i = 1$) we have:

$$|d(x,\bar{\xi}_1) - \bar{R}_1| \leq \frac{\bar{\epsilon}_1}{L}\bar{R}_1 \leq \frac{\bar{\epsilon}_1}{10}\bar{R}_1 \quad \forall x \in S_{\bar{R}}(\bar{\xi}),$$
$$\|e(\cdot,\bar{\xi}) - \bar{e}(\cdot,\bar{\xi}_1)\|_{C^{2+\alpha'}} \leq \frac{\bar{\epsilon}_1}{L} \leq \frac{\bar{\epsilon}_1}{10}.$$

From this it is not hard to show that:

$$d(\bar{\xi},\bar{\xi}_1) \leq \frac{\bar{\epsilon}_1}{L}\bar{R}_1 \leq \frac{\bar{\epsilon}_1}{10}\bar{R}_1 \text{ and } |\bar{R} - \bar{R}_1| \leq \frac{2\bar{\epsilon}_1}{L}\bar{R}_1 \leq \frac{\bar{\epsilon}_1}{5}\bar{R}_1,$$

which easily implies $|R_{\bar{\xi}} - \bar{R}_1| < \bar{\delta}_1$, if δ_M is chosen small enough. Since $d(X(u),\bar{\xi}) \geq \bar{R}(1-\epsilon_M)$ for all $u \in S$, we have $\bar{\xi}_1 \in \operatorname{int}(X)$ provided $\bar{R}(1-\epsilon_M) > (\bar{\epsilon}_1/10)\bar{R}_1$, which holds if $\epsilon_M < 1 - \bar{\epsilon}_1/(10 - 2\bar{\epsilon}_1)$. We also have the estimate:

$$\begin{aligned}
&\|F(\cdot,\bar{\xi}) - \bar{e}(\cdot,\bar{\xi}_1)\|_{C^{2+\alpha'}} \\
&\leq \|F(\cdot,\bar{\xi}) - e(\cdot,\bar{\xi})\|_{C^{2+\alpha'}} + \|e(\cdot,\bar{\xi}) - \bar{e}(\cdot,\bar{\xi}_1)\|_{C^{2+\alpha'}} \\
&\leq \epsilon_M + \bar{\epsilon}_1/10 < \bar{\epsilon}_1.
\end{aligned}$$

To show $\|\varphi_{\bar{\xi}} - \bar{R}_1\|_{C^{2+\alpha'}}$ is small, we use:

$$\|\varphi_{\bar{\xi}_1} - \bar{R}_1\|_{C^{2+\alpha'}} < |\bar{R}_1 - \bar{R}| + \|\bar{R} - \varphi_{\bar{\xi}}\|_{C^{2+\alpha'}} + \|\varphi_{\bar{\xi}} - \varphi_{\bar{\xi}_1}\|_{C^{2+\alpha'}}$$

and the following fact:

$$C_0 := \sup \Big\{ \|F(\cdot, \xi)\|_{C^{2+\alpha'}}; X \in \mathcal{E}^{2+\alpha} \text{ and } \exists \bar{X} = (\bar{\xi}, \bar{R}, \bar{e}) \text{ s.t. } X \in \mathcal{N}^{2+\alpha}_{1/10,1/10}(\bar{X}), d(\xi, \bar{\xi}) < \frac{1}{4}\bar{R} \Big\} < \infty,$$

and C_0 depends only on M. This follows from compactness of the inclusion $C^{2+\alpha} \subset C^{2+\alpha'}$, by a short argument which will be omitted. Given this bound, one has:

$$\begin{aligned}
\|\varphi_{\bar{\xi}} - \varphi_{\bar{\xi}_1}\|_{C^{2+\alpha'}} &= \left\| \int_0^t \frac{d}{ds} \mathrm{dist}_M(X(\cdot), \xi(s)) ds \right\|_{C^{2+\alpha'}} \\
&\leq \left\| \int_0^t \langle \nabla_M d_{X(u)}(\xi(s)), \xi'(s) \rangle ds \right\|_{C^{2+\alpha'}} \\
&\leq \sup_{s \in [0,t]} \|\nabla_M d_{X(u)}(\xi(s))\|_{C^{2+\alpha'}} d(\bar{\xi}, \bar{\xi}_1) \\
&= \sup_{s \in [0,t]} \|F(\cdot, \xi(s))\|_{C^{2+\alpha'}} d(\bar{\xi}, \bar{\xi}_1) \\
&\leq C_0 \frac{\bar{\epsilon}_1}{L} \bar{R}_1 \leq \frac{\bar{\epsilon}_1}{10} \bar{R}_1,
\end{aligned}$$

since $d(\bar{\xi}, \bar{\xi}_1) \leq \frac{\bar{\epsilon}_1}{L} \bar{R}_1 \leq \frac{1}{4}\bar{R}$. Thus we have (bearing in mind that $\bar{R} \leq \bar{R}_1 + (\bar{\epsilon}_1/5)\bar{R}_1$):

$$\|\varphi_{\bar{\xi}_1} - \bar{R}_1\|_{C^{2+\alpha'}} < \frac{\epsilon_1}{5} \bar{R}_1 + \epsilon_M \bar{R} + \frac{\bar{\epsilon}_1}{10} \bar{R}_1 < \bar{\epsilon}_1 \bar{R}_1$$

for ϵ_M sufficiently small, depending only on $\{\bar{\epsilon}_1, \dots, \bar{\epsilon}_N\}$. This concludes the proof.

Since α is fixed throughout the paper, from now on we will often omit the superscript in the notation for $\mathcal{E}^{2+\alpha}$ and $\mathcal{N}^{2+\alpha}_{\delta_M, \epsilon_M}$.

Definition. For $X \in \mathcal{N}_{\delta_M, \epsilon_M}$, we refer to $\xi_X = \mathcal{B}(X) \in \mathrm{int}\,(X)$ as the *analytic barycenter* of X. The global motion we construct will exist in the open subset $\mathcal{N}_{\delta_M, \epsilon_M} \subset \mathcal{E}_{\delta_0, \epsilon_0}$, the manifold of small, almost-spherical embeddings, or parametrized '*bubbles*'.

It is important to note that the analytic barycenter $\mathcal{B}(X)$ depends on the parametrization X, and not just on the image Σ. In particular, if we wish to find an equation of motion for the barycenter, we must fix an evolution equation for the parametrization $X(t)$; the geometric evolution law (0.1), however, does not fix such an equation (only up to a tangential component). We take up these issues in the next subsection.

1.2 Equations of motion for the barycentric system

From this point on we set $\delta_1 = \delta_M, \epsilon_1 = \epsilon_M$ and introduce the notations:

$$C^{2+\alpha}_{\delta_1,\epsilon_1} = \{\varphi \in C^{2+\alpha}(S); 0 < \mathrm{ave}_S[\varphi] < \delta_1, \|\varphi - \mathrm{ave}_S[\varphi]\|_{C^{2+\alpha}} < \epsilon_1\};$$
$$C^{k+\gamma}_0 = \{\varphi \in C^{k+\gamma}(S); \mathrm{ave}_S[\varphi u^i] = 0, i = 1,\dots,n\} \ \ (k \in \mathbb{N}, \gamma \in (0,1));$$
$$B^{2+\alpha}_{\delta_1,\epsilon_1} = C^{2+\alpha}_{\delta_1,\epsilon_1} \cap C^{2+\alpha}_0.$$

For parametrized motions $\bar{X}(t)$ in $\mathcal{N}_{\delta_1,\epsilon_1}$, the barycenter map makes it possible to choose a parametrization $X(t)$ of $\Sigma(t) = \mathrm{im}\,(\bar{X}(t))$ in the space (a submanifold of $\mathcal{N}$):

$$\mathcal{N}_{\mathrm{std}} = \mathcal{N}_{\mathrm{std}}(\delta_1,\epsilon_1) = \{X \in \mathcal{N}; X = \exp_\xi[\varphi e(\cdot,\xi)], \varphi \in B^{2+\alpha}_{\delta_1,\epsilon_1}\}.$$

$X \in \mathcal{N}_{\mathrm{std}}$ is 'standard' in two ways: ξ is the barycenter of X and $F_\xi : S \to S_\xi$ is given by an isometry. Any $X \in \mathcal{N}$ can be reparametrized into $\mathcal{N}_{\mathrm{std}}$ (with $\xi = \mathcal{B}(X)$). In this subsection we derive equations of motion on a 'simpler' manifold $\mathcal{M}_0$, the solutions of which correspond precisely to solutions of normalized mean curvature flow parametrized by maps in $\mathcal{N}_{\mathrm{std}}$.

Definition 1.2.

$$\mathcal{M} = \mathcal{M}(\delta_1,\epsilon_1) = \mathbb{F}M \times C^{2+\alpha}_{\delta_1,\epsilon_1}, \quad \mathcal{M}_0 = \mathcal{M}_0(\delta_1,\epsilon_1) = \mathbb{F}M \times B^{2+\alpha}_{\delta_1,\epsilon_1}.$$

The standard connection on the frame bundle $\mathbb{F}M$ allows one to identify the tangent space to $\mathcal{M}_0$ at $(\xi, e, \varphi) \in \mathcal{M}_0$ with the vector space:

$$T_{(\xi,e,\varphi)}\mathcal{M}_0 = \Big\{(w,z,\chi); \\ w \in T_\xi M, z \in C^{2+\alpha}(S, T_\xi M) \ e\text{-skew-symmetric }, \chi \in C^{2+\alpha}_0\Big\}$$

(where e-skew-symmetric means: $\langle z(u), e(u')\rangle = -\langle z(u'), e(u)\rangle$, for all $u, u' \in S$). There is a natural smooth map $\Phi : \mathcal{M} \to \mathcal{N}$, mapping $\mathcal{M}_0$ onto $\mathcal{N}_{\mathrm{std}}$:

$$\Phi(\xi, e, \varphi) = X, \quad X(u) = \exp_\xi[\varphi(u)e(u)].$$

In the following lemma we compute the differential of Φ. This of course maps $T_{(\xi,e,\varphi)}\mathcal{M}_0$ to the tangent space $T_X\mathcal{N}$, the space of $C^{2+\alpha}$ vector fields $\mathbb{Z}$ on S along X ($\mathbb{Z}(u) \in T_{X(u)}M$); but we'll also need to consider the space of C^α vector fields along X:

$$\widetilde{T}_X\mathcal{N} := \Big\{\mathbb{Z} : S \to TM;\\ \mathbb{Z}(u) \in T_{X(u)}M, u \mapsto \exp_\xi^{-1}[\mathbb{Z}(u)] \in C^\alpha(S, T_\xi M) \ \ \forall \xi \in \mathrm{int}\,(X)\Big\},$$

as well as $\widetilde{T}_{(\xi,e,\varphi)}\mathcal{M}_0$, defined as $T_{(\xi,e,\varphi)}\mathcal{M}_0$ but with z and χ in C^α.

In the statement of the next lemma, for $v \in S_\xi$ $\mathbb{Y}_v(s)$, $s \geq 0$, denotes the Jacobi operator with initial conditions $\mathbb{Y}_v(0) = \mathbb{I}_{T_\xi M}, \dot{\mathbb{Y}}_v(0) = 0$ (covariant derivative) along the geodesic $\gamma_v(s) = \exp_\xi[sv]$; so $\mathbb{Y}_v(s)w \in T_{\gamma_v(s)}M$ is the value at s of the Jacobi field along γ_v with initial value w, initial covariant derivative 0. $\mathbb{J}_v(s)$ is the Jacobi operator along the same geodesic with initial conditions $\mathbb{J}_v(0) = 0, \dot{\mathbb{J}}_v(0) = \mathbb{I}_{T_\xi M}$. Both $\mathbb{Y}_v(s)$ and $\mathbb{J}_v(s)$ preserve the direction spanned by the tangent vector $\gamma_v'(s)$ (and its orthogonal complement) along the geodesic γ_v. We have $\mathbb{Y}_v(s)v = \gamma_v'(s)$, $\mathbb{J}_v(s)v = 0$.

Lemma 1.3. *The differential of Φ is given by:*

$$d\Phi(\xi, e, \varphi)[w, z, \chi] = \mathbb{Y}_e(\varphi)w + \chi\gamma_e'(\varphi) + \mathbb{J}_e(\varphi)z.$$

Here we identify the orthonormal frame e at ξ with the isometry $e : S \to S_\xi$ it defines, and for simplicity omit $u \in S$ from the notation.

Proof. Since $\Phi(\xi, e, \varphi) = \exp_\xi[\varphi e]$,we have:

$$d\Phi(\xi, e, \varphi)[w, z, \chi] = d_\xi(\exp_\xi[sv])_{|s=\varphi,v=e}w \\ + d_s(\exp_\xi[sv])_{|s=\varphi,v=e}\chi + d_v(\exp_\xi[sv])_{|s=\varphi,v=e}z.$$

Since $d_\xi(\exp_\xi[sv])w = \mathbb{Y}_v(s)w$, $d_s(\exp_\xi[sv]) = \gamma_v'(s)$ and $d_v(\exp_\xi[sv])z = \mathbb{J}_v(s)z$, the lemma follows.

The operators $\mathbb{Y}_u(s)$ are invertible for s sufficiently small (say, $s \in (0, \delta_0/2)$). It is often convenient to work with vectors in $T_\xi M$ (or maps $S \to T_\xi M$); so in general for vectors $\mathbb{Z}(u) \in T_{X(u)}M$ (where $X \in \mathcal{N}_{\mathrm{std}}, X = \exp_\xi[\varphi e]$), we set:

$$z(u) = \mathbb{Y}_{e(u)}(\varphi)^{-1}\mathbb{Z}(u) \in T_\xi M.$$

For a given frame $e : S \to S_\xi$ and $z \in T_\xi M$ (possibly depending on $u \in S$), we introduce the orthogonal decomposition:

$$z = \langle z, e\rangle e + z^\perp, \quad z^\perp : S \to T_\xi M.$$

As a final bit of notation, we denote by $\nu(u) \in T_\xi M$ the vector defined by the property:

$$\langle w, \nu(u)\rangle = \langle \mathbb{Y}_{e(u)}(\varphi) w, N(u)\rangle \quad \forall w : S \to T_\xi M,$$

where $N(u)$ is the normal vector to Σ defined from X by:

$$N(u) = \gamma'_{e(u)}(\varphi) - \varphi_k (h^{kl} J_l) \circ X,$$

with J_i the Jacobi fields defined in Section 2, (2.0). Since $\langle \gamma'_{e(u)}(\varphi), N\rangle = 1$, it follows that $\langle e, \nu\rangle \equiv 1$ on S.

We now observe a geometric constraint on the velocity vector of a curve in $\mathcal{N}_{\mathrm{std}}$.

Lemma 1.4. *Let $X(t), t \in [0, T)$ be a curve in $\mathcal{N}_{\mathrm{std}}$, $X(t) = \Phi(\xi(t), e(t), \varphi(t))$. Assume $\xi(t)$ is a C^1 curve on M, $e(t)$ is parallel along $\xi(t)$ and $\varphi_t \in C([0,T), C_0^\alpha(S))$ satisfies $\mathrm{ave}_S[\varphi_t u^i] = 0, i = 1, \dots, n$. Then the velocity vector $\mathbb{Z}(t) = X_t \in \widetilde{T}_{X(t)}\mathcal{N}$ satisfies:*

$$z^\perp = (\mathrm{ave}_S[z])^\perp.$$

(*Note both sides of this equation depend on t and on u.*)

Proof. From Lemma 1.3, we have:

$$\begin{aligned}\mathbb{Z}(t) = X_t &= d\Phi(\xi(t), e(t), \varphi(t))(\xi_t, \nabla_{\xi_t} e, \varphi_t)\\ &= \mathbb{Y}_e(\varphi)\xi_t + \varphi_t \gamma'_e(\varphi),\end{aligned}$$

since $\nabla_{\xi_t} e \equiv 0$. Pulling back to $T_\xi M$ via $\mathbb{Y}_e(\varphi)^{-1}$, this gives:

$$z(t) = \xi_t + \varphi_t e.$$

Thus $z^\perp = \xi_t^\perp$. Since $\mathrm{ave}_S[\varphi_t e] = \mathrm{ave}_S[\varphi_t u^i] e_i = 0$ we obtain $(\mathrm{ave}_S[z])^\perp = (\mathrm{ave}_S[\xi_t])^\perp = \xi_t^\perp$, proving the claim.

This lemma motivates the definition (for $X = \Phi(\xi, e, \varphi) \in \mathcal{N}_{\mathrm{std}}$):

$$\widetilde{T}_X \mathcal{N}_{\mathrm{std}} = \{\mathbb{Z} \in \widetilde{T}_X \mathcal{N}; z^\perp = (\mathrm{ave}_S z)^\perp, z = \mathbb{Y}_e^{-1}(\varphi)\mathbb{Z}\}.$$

Lemmas 1.3 and 1.4 show $d\Phi$ maps the subspace of $\widetilde{T}_{(\xi,e,\varphi)}\mathcal{M}_0$ defined by $z = 0$ into $\widetilde{T}_X\mathcal{N}_{\rm std}$.

We face the following *problem*: if we attempt to solve the parametrized normalized mean curvature flow in $\mathcal{N}_{\rm std}$ by setting $X_t = \mathbb{Z}(X) = (H^\Sigma - H)\hat{N} \in \widetilde{T}_X\mathcal{N}$, in general we should not expect that $\mathbb{Z}(X) \in \widetilde{T}_X\mathcal{N}_{\rm std}$. Thus we will not find $X(t)$ in $\mathcal{N}_{\rm std}$ solving $X_t = (H^\Sigma - H)\hat{N}$.

Fortunately for 'radial vector fields' $\mathbb{Z}(X) \in \widetilde{T}_X\mathcal{N}$ it is possible to correct this by adding a 'tangential component'. Precisely, given $X \in \mathcal{N}_{\rm std}$, $X = \exp_\xi[\varphi e]$, define the linear map:

$$S : T_\xi M \to T_\xi M, \quad Sw := n(w - \mathrm{ave}_S[w^T]), \quad w^T := w - \langle w, \nu\rangle e.$$

The notation w^T is justified by the fact that $\mathbb{Y}_e(\varphi)w^T$ is tangential to Σ: with $\mathbb{W} = \mathbb{Y}_e(\varphi)w$,

$$\langle \mathbb{Y}_e(\varphi)w^T, N\rangle = \langle \mathbb{W}, N\rangle - \langle w, \nu\rangle\langle \gamma_e'(\varphi), N\rangle = 0,$$

since $\langle w, \nu\rangle = \langle \mathbb{W}, N\rangle$ and $\langle \gamma_e'(\varphi), N\rangle = 1$.

The map S is invertible, provided δ_1, ϵ_1 are small enough: since $w^T = w^\perp - \langle w, \nu - e\rangle e$ and $\mathrm{ave}_S[w^\perp] = \frac{n-1}{n}w$ for $w \in T_\xi M$, we have:

$$Sw = w + \mathrm{ave}_S[n\langle w, \nu - e\rangle e].$$

Since $\langle w, \nu - e\rangle = \langle \mathbb{Y}_e(\varphi)w, N - \gamma_e'(\varphi)\rangle$ and:

$$\|\mathbb{Y}_e(\varphi)w\|\|\gamma_e'(\varphi) - N\| < (1/2n)\|w\|$$

for all $w \in T_\xi M$ (for sufficiently small ϵ_1, δ_1), it follows that $S = \mathbb{I} + E$, with $E \in \mathcal{L}(T_\xi M), \|E\| < 1/2$.

We use S to define for $X \in \mathcal{N}_{\rm std}$ a linear map $\mathcal{P}_X$ on vector fields $\mathbb{Z} \in \widetilde{T}_X\mathcal{N}$ of the form:

$$\mathbb{Z} = a\gamma_e'(\varphi), \quad a \in C^\alpha(S),$$

by setting:

$$\begin{aligned} &\mathcal{P}_X(\mathbb{Z}) = \mathbb{Z} + \mathbb{T}, \\ &\qquad \mathbb{T} = \mathbb{Y}_e(\varphi)w - \langle \mathbb{Y}_e(\varphi)w, N\rangle\gamma_e'(\varphi), \\ &\qquad w = nS^{-1}(\mathrm{ave}_S[z]), \quad z = ae \in C^\alpha(S, T_\xi M). \end{aligned}$$

Lemma 1.5. *Let $X \in \mathcal{N}_{\text{std}}, X = \exp_\xi[\varphi e]$. Let $\mathbb{Z} = a\gamma_e'(\varphi) \in \widetilde{T}_X\mathcal{N}$. Then:*

(i) $\widetilde{\mathbb{Z}} := \mathcal{P}_X(\mathbb{Z}) = \mathbb{Z} + \mathbb{T} \in \widetilde{T}_X\mathcal{N}_{\text{std}}$.

(ii) *Conversely, assume $\widetilde{\mathbb{Z}} = \mathbb{Z} + \mathbb{T} \in \widetilde{T}_X\mathcal{N}_{\text{std}}$ (with $\mathbb{T}(u) \in T_{X(u)}\Sigma$) has the form:*

$$\widetilde{\mathbb{Z}} = \mathbb{Y}_e(\varphi)w + \chi\gamma_e'(\varphi),$$

for some $w \in T_\xi M$ and $\chi \in C_0^\alpha$. Then $w = nS^{-1}(\text{ave}_S[z])$ (with $z = ae$), $\chi = \langle z, e\rangle - \langle w, \nu\rangle$ and $\widetilde{\mathbb{Z}} = \mathcal{P}_X(\mathbb{Z})$.

Proof.

(i) Let $\widetilde{z} = z + w - \langle w, \nu\rangle e = z + w^T$. Then $\widetilde{z}^\perp = w^\perp$ and:

$$\text{ave}_S[\widetilde{z}] = \text{ave}_S[z] + \text{ave}_S[w^T] = \text{ave}_S[z] - \frac{1}{n}Sw + w = w,$$

since $(1/n)Sw = \text{ave}_S[z]$. Thus $(\text{ave}_S[\widetilde{z}])^\perp = \widetilde{z}^\perp$, showing $\widetilde{\mathbb{Z}} \in \widetilde{T}_X\mathcal{N}_{\text{std}}$.

(ii) We have:

$$w + \chi e = \widetilde{z} = z + t,$$
$$\text{where } t := \mathbb{Y}_e(\varphi)^{-1}\mathbb{T}, \text{ave}_S[\chi e] = \text{ave}_S[\chi u^i]e_i = 0 \text{ and } z = ae,$$

and must show:

$$t = w - \langle w, \nu\rangle e, \quad \chi = \langle z, e\rangle - \langle w, \nu\rangle \quad \text{and}$$
$$n(w - \text{ave}_S[w^T]) = n\text{ave}_S[z].$$

The observation that $\langle t, \nu\rangle = \langle \mathbb{T}, N\rangle = 0$ implies $\langle w, \nu\rangle + \chi = a = \langle z, e\rangle$ and the first equality: $t = w + \chi e - ae = w - \langle w, \nu\rangle e = w^T$. But then:

$$w - \text{ave}_S[w^T] = \text{ave}_S[w - t] = \text{ave}_S[z - \chi e] = \text{ave}_S[z],$$

as desired.

We use this lemma to define a motion on $\mathcal{M}_0$ which corresponds to parametrized solutions of normalized mean curvature flow in $\mathcal{N}_{\text{std}}$.

Definition 1.6. The *barycentric system* on $\mathcal{M}_0$ is defined by the equations of motion:

$$\begin{aligned} (1.1) \qquad & \xi_t = n\ \text{ave}_S[(H^\Sigma - H)\|N\|e] - n\ \text{ave}_S[\langle w, \nu - e\rangle e] \\ & \nabla_{\xi_t} e = 0 \\ & \varphi_t = (H^\Sigma - H)\|N\| - n\langle \text{ave}_S[(H^\Sigma - H)\|N\|e], e\rangle \\ & \qquad - \langle w, \nu - e\rangle + n\langle \text{ave}_S[\langle w, \nu - e\rangle e], e\rangle, \end{aligned}$$

where:

$$w = nS^{-1}(\mathrm{ave}_S[(H^\Sigma - H)\|N\|e]), \quad \text{with}$$
$$S := \mathbb{I} + n\ \mathrm{ave}_S[\langle ., \nu - e\rangle e] \in \mathcal{L}(T_\xi M).$$

(Here $H^\Sigma - H, N, \nu$, are computed at the embedding $\Phi(\xi, e, \varphi)$.)

Lemma 1.7.

(i) *Let $(\xi(t), e(t), \varphi(t)) \in \mathcal{M}_0$ be a solution of the barycentric system* (1.1) *in $[0, T)$. Then $X(t) = \Phi(\xi(t), e(t), \varphi(t)) \in \mathcal{N}_{\mathrm{std}}$ is a solution of $\langle X_t, \hat{N}\rangle = H^\Sigma - H$ in $[0, T)$.*

(ii) *Conversely, let $X(t) \in \mathcal{N}$ be a solution of the geometric equation $\langle X_t, \hat{N}\rangle = H^\Sigma - H$ in $[0, T)$, with $X(0) \in \mathcal{N}_{\mathrm{std}}$. Then by reparametrizing $X(t)$ we may obtain $\bar{X}(t) \in \mathcal{N}_{\mathrm{std}}$, solution of:*

$$\bar{X}_t = \mathbb{Z}(\bar{X}) + \mathbb{T}_1, \quad \mathbb{Z}(\bar{X}) := (H^\Sigma - H)\|N\|\gamma_e'(\varphi),$$

where $\mathbb{Z}(\bar{X}) + \mathbb{T}_1 = \mathcal{P}_{\bar{X}}(\mathbb{Z}(\bar{X})) \in \widetilde{T}_{\bar{X}}\mathcal{N}_{\mathrm{std}}$. In particular, writing $\bar{X}(t) = \Phi(\xi(t), e(t), \bar{\varphi}(t))$ with $\xi = \mathcal{B}(\bar{X})$ and $e(t)$ parallel along $\xi(t)$, we obtain a solution of the barycentric system (1.1) *in $\mathcal{M}_0$.*

Remarks.

(i) $\nu - e \equiv 0$ for geodesic spheres, so the terms containing $\nu - e$ will be treated as 'error terms'.

(ii) The factors of n in the equation for φ_t in (1.1) can be understood as follows: if f satisfies $f_t = f - n\langle \mathrm{ave}_S[fe], e\rangle = f - n\mathrm{ave}_S[fu^i]u^i$, we have:

$$\begin{aligned}(\mathrm{ave}_S[fu^j])_t &= \mathrm{ave}_S[f_t u^j] \\ &= \mathrm{ave}_S[fu^j] - n\mathrm{ave}_S[fu^i]\mathrm{ave}_S[u^i u^j] = 0,\end{aligned}$$

since $\mathrm{ave}_S[u^i u^j] = \delta_{ij}/n$. Thus the condition $\mathrm{ave}_S[fu^j] = 0$ is preserved.

Proof. Throughout the proof we let $v_N(X) = (H^\Sigma - H)\|N\|, \mathbb{H}(X) = (H^\Sigma - H)\hat{N}$.

(i) Let $z = v_N e$, $\widetilde{z} = z + w - \langle w, \nu\rangle e$. Since $\langle \widetilde{z}, e\rangle = v_N - \langle w, \nu - e\rangle$, the barycentric system can be written as:

$$\xi_t = n\ \mathrm{ave}_S[\langle \widetilde{z}, e\rangle e], \quad \varphi_t = \langle \widetilde{z}, e\rangle - n\langle \mathrm{ave}_S[\langle \widetilde{z}, e\rangle e], e\rangle.$$

As shown in Lemma 1.5(i), $\widetilde{z}$ satisfies $\widetilde{z}^{\perp} = (\mathrm{ave}_S[\widetilde{z}])^{\perp}$. This implies:

$$\begin{aligned}\mathrm{ave}_S[\widetilde{z}^{\perp}] &= \mathrm{ave}_S[(\mathrm{ave}_S[\widetilde{z}])^{\perp}] \\ &= \frac{n-1}{n}\mathrm{ave}_S[\widetilde{z}] \text{ and } \mathrm{ave}_S[\langle \widetilde{z}, e\rangle e] = (1/n)\mathrm{ave}_S[\widetilde{z}].\end{aligned}$$

Thus, we have:

$$\begin{aligned}\xi_t + \varphi_t e &= n\ \mathrm{ave}_S[\langle \widetilde{z}, e\rangle e] + \langle \widetilde{z}, e\rangle e - n\langle \mathrm{ave}_S[\langle \widetilde{z}, e\rangle e], e\rangle e \\ &= n(\mathrm{ave}_S[\langle \widetilde{z}, e\rangle e])^{\perp} + \langle \widetilde{z}, e\rangle e = (\mathrm{ave}_S[\widetilde{z}])^{\perp} + \langle \widetilde{z}, e\rangle e \\ &= \widetilde{z}^{\perp} + \langle \widetilde{z}, e\rangle e = \widetilde{z}.\end{aligned}$$

Thus $X(t)$ satisfies the equation:

$$\begin{aligned}X_t &= \mathbb{Y}_e(\varphi)(\xi_t + \varphi_t e) = \mathbb{Y}_e(\varphi)\widetilde{z} \\ &= v_N \gamma_e'(\varphi) + \mathbb{T} \quad (\text{ with } \mathbb{T} = \mathbb{Y}_e(\varphi)(w - \langle w, \nu\rangle e)) \\ &= (H^{\Sigma} - H)\hat{N} + \mathbb{T}_1,\end{aligned}$$

where $\mathbb{T}_1 = \mathbb{T} + (H^{\Sigma} - H)\|N\|\gamma_e'(\varphi) - (H^{\Sigma} - H)\hat{N}$ satisfies $\langle \mathbb{T}_1, N\rangle \equiv 0$.

Remark. Note that the condition $\widetilde{z}^{\perp} = (\mathrm{ave}_S[\widetilde{z}])^{\perp}$ is used essentially in the proof.

(ii) $X(t)$ satisfies, for some tangential vector $\mathbb{T}(t, u)$:

$$X_t = \mathbb{H}(X) + \mathbb{T}.$$

Leting $\xi(t) = \mathcal{B}(X(t))$ and $e(t)$ be the parallel transport of the frame $e(0)$ along $\xi(t)$, we may reparametrize Σ in the form:

$$\bar{X}(t, \bar{u}) = \exp_{\xi(t)}[\varphi(t, \bar{u})e(t, \bar{u})],$$

via a diffeomorphism $G(t, \cdot) \in \mathrm{Diff}^{2+\alpha}(S)$, so $u = G(t, \bar{u})$ and $\bar{X}(t, \bar{u}) = X(t, G(t, \bar{u}))$. For the velocity vector:

$$\begin{aligned}\bar{X}_t(t, \bar{u}) &= X_t(t, G(t, \bar{u})) + X_u(t, G(t, \bar{u}))G_t(t, \bar{u}) \\ &= \mathbb{H}(\bar{X}) + \mathbb{T}(t, G(t, \bar{u})) + \bar{\mathbb{T}}(t, \bar{u}) \\ &= v_N(\bar{X})\gamma_e'(\varphi) + \mathbb{T}_2(t, \bar{u}) \\ &= \mathbb{Z}(\bar{X}) + \mathbb{T}_2(t, \bar{u}),\end{aligned}$$

where:

$$\begin{aligned}\mathbb{T}_2(t, \bar{u}) = \mathbb{T}(t, G(t, \bar{u})) + \bar{\mathbb{T}}(t, \bar{u}) + \mathbb{H}(\bar{X})(t, \bar{u}) \\ - v_N(\bar{X})(t, \bar{u})\gamma_e'(\varphi) \in T_{\bar{X}(t,\bar{u})}\Sigma\end{aligned}$$

and we used the fact that $\mathbb{H}$ is 'geometric':

$$\mathbb{H}(X)(t, G(t, \bar{u})) = \mathbb{H}(\bar{X})(t, \bar{u}).$$

Since we also have:

$$\bar{X}_t = \mathbb{Y}_e(\varphi)w + \chi\gamma_e'(\varphi) \text{ with } w = \xi_t \in T_\xi M \text{ and } \chi = \varphi_t \in C_0^{2+\alpha}(S),$$

we may apply Lemma 1.5(ii) to conclude $\mathbb{Z}(\bar{X}) + \mathbb{T}_2 = \mathcal{P}_{\bar{X}}(\mathbb{Z}(\bar{X}))$ and:

$$\xi_t = nS^{-1}(\mathrm{ave}_S[z]) \text{ with } z = v_N(\bar{X})e, \quad \varphi_t = v_N(\bar{X}) - \langle \xi_t, \nu\rangle.$$

From $n\ \mathrm{ave}_S[z] = S\xi_t = \xi_t + n\ \mathrm{ave}_S[\langle w, \nu - e\rangle e]$ (where $w = \xi_t$), we obtain:

$$\begin{aligned}\xi_t &= n\ \mathrm{ave}_S[v(\bar{X})e] - n\ \mathrm{ave}_S[\langle w, \nu - e\rangle e]\\ \varphi_t &= v_N(\bar{X}) - \langle \xi_t, e\rangle - \langle w, \nu - e\rangle,\end{aligned}$$

which is the barycentric system.

Summary. For $f \in C^{2+\alpha}(S)$, denote by $^\circ$ the L^2 projection on the subspace $C_0^{2+\alpha}$ defined by $\mathrm{ave}_S[fu^i] = 0$:

$$f^\circ = f - n\langle \mathrm{ave}_S[fe], e\rangle = f - \mathrm{ave}_S[nfu^i]u^i.$$

Then the barycenter system has the structure:

$$\begin{aligned}\xi_t &= n\ \mathrm{ave}_S[(v_N - E)e]\\ \nabla_{\xi_t} e &= 0\\ \varphi_t &= (v_N - E)^\circ;\end{aligned} \tag{1.2}$$

here $E = E(v_N)$ is uniquely defined so that:

$$\widetilde{z} := \xi_t + \varphi_t e = \text{ satisfies } \langle \widetilde{z}, \nu\rangle = v_N,$$

or equivalently for $X(t) = \Phi(\xi(t), e(t), \varphi(t)) \in \mathcal{N}_{\mathrm{std}}$:

$$X_t = \mathbb{Y}_e\widetilde{z} = v_N\gamma_e'(\varphi) + \mathbb{T} = (H^\Sigma - H)\hat{N} + \mathbb{T}_1.$$

Since $\widetilde{z}$ is a 'tangential correction' of $v_N e$: $\widetilde{z} = v_N e + w - \langle w, \nu\rangle e = v_N e + w^\perp - Ee$, we see that $\langle \widetilde{z}, e\rangle = v_N - E$, and the system can be written in the alternative form:

$$\begin{aligned}\xi_t &= \mathrm{ave}_S[\widetilde{z}]\\ \varphi_t &= \langle \widetilde{z}, e\rangle^\circ;\end{aligned}$$

note $\mathrm{ave}_S[\widetilde{z}] = n\ \mathrm{ave}_S[\langle \widetilde{z}, e\rangle e]$. From the point of view of the system on $\mathcal{M}_0$, the reason for the correction is that we want to prescribe $\langle \widetilde{z}, \nu\rangle$, not $\langle \widetilde{z}, e\rangle$. In fact we have:

$$\langle \widetilde{z}, \nu\rangle = v_N \Leftrightarrow \langle \widetilde{z}, e\rangle = v_N - E.$$

In the proof of global existence, we use a slightly different form of the barycentric system. Any $0 \le \varphi \in C_0^{2+\alpha}(S)$ can be written in the form $\varphi = R(1+\psi)$, where $R = \mathrm{ave}_S[\varphi] \ge 0$ and $\psi = (1/R)(\varphi - R)$ ($\psi \equiv 0$ if $\varphi \equiv 0$) is in the subspace $K^{2+\alpha}$ of $C_0^{2+\alpha}$, where we denote:

$$\begin{aligned} K^{2+\alpha} &:= \{\psi \in C_0^{2+\alpha}(S); \mathrm{ave}_S[\psi] = 0\}, \\ K^{2+\alpha}_{\epsilon_1} &= \{\psi \in K^{2+\alpha}; \|\psi\|_{C^{2+\alpha}} < \epsilon_1\}. \end{aligned}$$

Thus we may write:

$$\mathcal{M}_0 = \mathcal{M}_0(\delta_1, \epsilon_1) = (0, \delta_1) \times \mathbb{F}M \times K^{2+\alpha}_{\epsilon_1},$$

and derive differential equations for R and ψ via:

$$\varphi_t = (1+\psi)R_t + R\psi_t, \quad R_t = \mathrm{ave}_S[\varphi_t], \quad R\psi_t = \varphi_t - (1+\psi)\mathrm{ave}_S[\varphi_t].$$

Denoting by $f \mapsto (f)_K$ the L^2 projection $C^{2+\alpha} \to K^{2+\alpha}$, the R, ψ equations may be written in the form:

$$\begin{aligned} R_t &= \mathrm{ave}_S[v_N - E] \\ R\psi_t &= (v_N - E)_K - \psi\ \mathrm{ave}_S[v_N - E] \end{aligned}$$

Thus the equations of motion take the form:

$$\begin{aligned} R_t &= \mathrm{ave}_S[v_N - E] \\ \xi_t &= \mathrm{ave}_S[n(v_N - E)e] \\ \nabla_{\xi_t} e &= 0 \\ R\psi_t &= (v_N - E)_K - \psi\ \mathrm{ave}_S[v_N - E]. \end{aligned} \tag{1.3}$$

2. Asymptotics in Riemannian normal coordinates

2.1 Second fundamental form and mean curvature

In this section we apply some classical expansions in Riemannian normal coordinates to obtain the leading terms in the barycentric system on $\mathcal{M}_0(\delta_0, \epsilon_0)$. The main results are the asymptotics of the normal velocity $v = H^\Sigma - H$ (Lemma 2.1), an *a priori* estimate for $R(t)$ (Lemma 2.2), the asymptotics of the tangential correction E (Lemma 2.4), and, finally, of the barycentric system (Lemma 2.5). Only the statements of these lemmas are needed in Section 3; the expansions themselves are obtained in a completely standard way.

Ultimately we would like to identify the terms of order up to R^2, and up to first order jointly in $(\psi, \nabla^S\psi, D^2_S\psi)$ explicitly. Intermediate expansions will be obtained to varying orders; for instance, since v is of order R^{-1}, in Lemma 2.1 we actually state the asymptotics of Rv to order 3 in R.

We begin by considering the second fundamental form and mean curvature of a hypersurface $\Sigma = \mathrm{im}(X)$ in a Riemannian n-manifold M, parametrized by a 'radial embedding':

$$X : S \to M, \quad X(u) = \exp_\xi[\varphi(u)u_\xi],$$

where $\xi \in M$ is fixed and $u \mapsto u_\xi$ is a fixed isometry from S to the unit tangent sphere $S_\xi \subset T_\xi M$. We assume $\Sigma \subset C^*_\xi := C_\xi - \{\xi\}$, where C_ξ is a totally convex open neighborhood of ξ in M. (In other sections of the paper u_ξ is denoted $e(u, \xi)$ or $e(u)$; but in this section-except for Subsection 2.4, when we estimate E- we will not keep track of the frame e explicitly.)

It is easy to describe a local basis of tangent vectors and an outward normal vector for Σ. Let (e_i) be a local orthonormal frame on S, $(\bar{e}_i)$ the corresponding (under the isometry fixed above) local frame of S_ξ. We assume $(\bar{e}_i)$ is extended to an open sector $U_\xi \subset C^*_\xi$ by parallel translation along radial geodesics from ξ. Denoting by (ρ, ω) polar normal coordinates in C^*_ξ ($\rho > 0, \omega \in S_\xi$), and by $\gamma_\omega(\rho)$ the unit speed geodesic from ξ with initial tangent vector ω, we define n vector fields in U_ξ:

$$e_0(\rho, \omega) = d\exp_\xi(\rho\omega)\omega = \gamma'_\omega(\rho);$$

$$J_i(\rho, \omega) = d\exp_\xi(\rho\omega)[\rho\bar{e}_i],$$

the Jacobi field along $\gamma_\omega(\rho)$ with initial conditions $J_i(0,\omega) = 0, J_i'(0,\omega) = \bar{e}_i(\omega) \in T_\omega S_\xi$. By the Gauss lemma $< J_i, e_0 > \equiv 0$ in U_ξ. A basis of tangent vector fields to Σ is given by:

$$\begin{aligned} E_i = dXe_i &= d\exp_\xi(\varphi u_\xi)[\varphi_i u_\xi] + d\exp_\xi(\varphi u_\xi)[\varphi \bar{e}_i] \\ &= \varphi_i(e_0 \circ X) + J_i \circ X, \end{aligned}$$

where we have set $\varphi_i = (d^S\varphi)e_i$ (differential on S). The E_i are 'vector fields on M along X', in the sense that $E_i(u) \in T_{X(u)}M, i = 1, \ldots n-1$. Setting $h_{ij} = \langle J_i, J_j\rangle : U_\xi \to \mathbb{R}$, we obtain an outward normal vector to Σ:

$$\begin{aligned} N &= e_0 \circ X - \varphi_k(h^{kl}J_l) \circ X, \\ \|N\|^2 &= 1 + (h^{ij} \circ X)\varphi_i\varphi_j : U_\xi \to \mathbb{R}^+. \end{aligned} \tag{2.0}$$

N is also a vector field along X, and it is easy to check that it is orthogonal to the E_i.

The *second fundamental form* of Σ with respect to N is defined by:

$$A^N(E_i, E_j) = \langle \nabla_{E_i} N, E_j\rangle.$$

Using the above definitions for N and E_i, one easily obtains the expression:

$$\begin{aligned} A^N(E_i, E_j) = H(J_i, J_j) &+ h^{kl}(H(J_i, J_l)\varphi_j + H(J_j, J_l)\varphi_i)\varphi_k \\ &- J_i(\widetilde{\varphi}_j) + h^{kl}\langle \nabla_{J_i} J_j, J_l\rangle, \end{aligned}$$

where:

$H(X, Y) = \langle \nabla_X e_0, Y\rangle$ is the Hessian of the distance function d_ξ ($\nabla d_\xi = e_0$);

$\widetilde{\varphi}_j$ is the function on U_ξ defined by:

$$\widetilde{\varphi}_j(\rho, u_\xi) = \varphi_j(u), \quad \text{or} \quad \widetilde{\varphi}_j(\exp_\xi[\rho u_\xi]) = \varphi_j(u).$$

A little more precisely, the functions $H(J_i, J_j)$, h^{kl} and $\langle \nabla_{J_i} J_j, J_l\rangle$ on U_ξ define by composition with the embedding X the functions A_{ij}^N on S:

$$\begin{aligned} A_{ij}^N = H(J_i, J_j) \circ X &+ \varphi_k\varphi_j[h^{kl}H(J_i, J_k)] \circ X \\ &+ \varphi_k\varphi_i[h^{kl}H(J_i, J_l)] \circ X + \varphi_k[h^{kl}\langle \nabla_{J_i} J_j, J_l\rangle] \circ X \\ &- \langle \nabla^S_{e_i} e_j, \varphi_k e_k\rangle_S - (H^S\varphi)(e_i, e_j), \end{aligned} \tag{2.1}$$

where H^S denotes the Hessian on S:

$$(H^S\varphi)(e_i, e_j) = (d^S\varphi_j)e_i - \langle \nabla^S_{e_i} e_j, \varphi_k e_k\rangle_S.$$

We also used the fact that:

$$(d^S\varphi_j)e_i = d\widetilde{\varphi}_j \circ d\exp_\xi(\rho u_\xi)[\rho\bar{e}_i] = J_i(\widetilde{\varphi}_j).$$

We also need an expression for the *induced metric* on Σ:

$$\begin{aligned} g_{ij} &= \langle E_i, E_j\rangle = \varphi_i\varphi_j + h_{ij} \\ &= \varphi_i\varphi_j + \varphi^2\left[\delta_{ij} - \frac{1}{3}R_{ij}\varphi^2 - \frac{1}{12}R'_{ij}\varphi^3\right] + O(|\varphi|^6), \end{aligned} \tag{2.2}$$

using the expression for h_{ij} given below.

The functions on $U_\xi \subset M$ defined above have well-known Taylor expansions in normal polar coodinates; in order to state them, we introduce the notation:

$$R_{ij}(\omega) = \langle \mathrm{Rm}^M(\xi)(\omega, \bar{e}_i)\omega, \bar{e}_j\rangle, \quad R'_{ij}(\omega) = \langle (\nabla_\omega \mathrm{Rm}^M)(\xi)(\omega, \bar{e}_i)\omega, \bar{e}_j\rangle.$$

The following Taylor expansions in ρ, at a fixed $\omega \in S_\xi$ and $\rho = 0$, are easily obtained from the Jacobi equation:

$$\begin{aligned} J_i(\rho,\omega) &= \rho\bar{e}_i - \frac{1}{6}\rho^3\mathrm{Rm}^M(\xi)(\omega,\bar{e}_i)\omega \\ &\qquad - \frac{1}{24}\rho^4(\nabla_\omega \mathrm{Rm}^M(\xi))(\omega,\bar{e}_i)\omega + O(\rho^5); \\ H(J_i,J_j)(\rho,\omega) &= \delta_{ij}\rho - \frac{2}{3}R_{ij}(\omega)\rho^3 - \frac{5}{12}R'_{ij}(\omega)\rho^4 + O(\rho^5). \end{aligned}$$

Rm^M denotes the $(3,1)$ Riemann curvature tensor of M. (Here we used:

$$H(J_i,J_j) = \langle \nabla_{J_i} e_0, J_j\rangle = \langle J'_i, J_j\rangle + \langle [J_i, e_0], J_j\rangle = \langle J'_i, J_j\rangle,$$

since for the geodesic variation $f(\rho,\tau) := \exp_\xi(\rho\phi^i_\tau(\omega))$, with ϕ^i_τ the local flow of $\bar{e}_i$ on S_ξ:

$$\begin{aligned} [e_0, J_i] &= [d\exp_\xi(\rho\omega)\omega, d\exp_\xi(\rho\omega)\bar{e}_i] \\ &= \left[\frac{\partial f}{\partial\rho}, \frac{\partial f}{\partial\tau}\right]_{|\tau=0} = df\left[\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\tau}\right]_{|\tau=0} = 0.\Bigg) \end{aligned}$$

$$h_{ij}(\rho,\omega)=\rho^2\left[\delta_{ij}-\frac{1}{3}R_{ij}(\omega)\rho^2-\frac{1}{12}R'_{ij}(\omega)\rho^3\right]+O(\rho^6);$$
$$h^{ij}(\rho,\omega)=\rho^{-2}\left[\delta_{ij}+\frac{1}{3}R_{ij}(\omega)\rho^2+\frac{1}{12}R'_{ij}(\omega)\rho^3\right]+O(\rho^2);$$
$$\langle\nabla_{J_i}J_j,J_l\rangle(\rho,\omega)=\bar{\Gamma}_{ijl}(\omega)\rho^2+T_{4ijl}(\omega)\rho^4+T_{5ijl}(\omega)\rho^5+O(\rho^6),$$

where in the last line we set:

$$\bar{\Gamma}_{ijl}(\omega)=\langle\nabla^{S_\xi}_{\bar{e}_i}\bar{e}_j,\bar{e}_l\rangle_{S_\xi}(\omega),\ \text{so}\ \langle\nabla^{S}_{e_i}e_j,e_l\rangle_S(u)=\bar{\Gamma}_{ijl}(u_\xi).$$

These are the only expansions in normal coordinates we'll need; all other expansions obtained below follow from these and algebraic computation.

Introducing the decomposition $\varphi=R(1+\psi)$ with $\int_S\psi=0$, we define smooth functions on $SM\times[0,\delta_0)\times[0,\epsilon_0)\times\mathbb{R}^n\times\mathrm{Sym}^2(\mathbb{R}^n)$:

$$A_{ij}(\xi,\omega,r,z,p,q),\quad g_{ij}(\xi,\omega,r,z,p),\quad \mathcal{N}(\xi,\omega,r,z,p),$$

so that the second fundamental form and induced metric of $\Sigma=\mathrm{im}(X(\xi,R,\psi))$ and the length squared of N are given by:

$$A^N(E_i,E_j)=A_{ij}(\xi,u_\xi,R,\psi,\nabla^S\psi,H^S\psi)$$
$$g_{ij}=g_{ij}(\xi,u_\xi,R,\psi,\nabla^S\psi),\quad \|N\|^2=\mathcal{N}(\xi,u_\xi,R,\psi,\nabla^S\psi).$$

In terms of the Riemannian functions defined above:

$$\begin{aligned}A_{ij}&=H(J_i,J_j)(r(1+z),\omega)+r^2p_kp_j[h^{kl}H(J_i,J_k)](r(1+z),\omega)\\&\quad+r^2p_kp_i[h^{kl}H(J_j,J_k)](r(1+z),\omega)\\&\quad+rp_k\langle h^{kl}\nabla_{J_i}J_j,J_l\rangle(r(1+z),\omega)-rp_k\bar{\Gamma}_{ijk}(\omega)-rq(e_i,e_j),\\g_{ij}&=r^2p_ip_j+h_{ij}(r(1+z),\omega),\\\mathcal{N}&=1+r^2p_ip_jh^{ij}(r(1+z),\omega).\end{aligned}$$

The functions A_{ij},g_{ij},g^{ij} and $\mathcal{N}$ have Taylor expansions (at fixed (ξ,ω) and $(r,z,p,q)=(0,0,0,0)$) given as follows.

$$\begin{aligned}A_{ij}(r,z,p,q)&=\delta_{ij}r-\frac{2}{3}R_{ij}r^3-\frac{5}{12}R'_{ij}r^4+\delta_{ij}rz-2R_{ij}r^3z-\frac{5}{3}R'_{ij}r^4z\\&\quad+\left(T_{4ijk}+\frac{1}{3}R_{kl}\Gamma_{ijl}\right)r^3p_k+\left(T_{5ijk}+\frac{1}{6}R'_{kl}\Gamma_{ijl}\right)r^4p_k\\&\quad-rq+B_{ij}(u,r,z,p),\end{aligned}$$

where B_{ij} satisfies the estimate:

$$|B_{ij}(u,r,z,p)| \leq C[r^5 + |z|^2 + |p|^2],$$

provided $r \in (0, \delta_0)$ and $|z| + |p| < \epsilon_0$, for a constant C depending only on M.

$$\begin{aligned} g_{ij}(r,z,p) &= r^2\left(\delta_{ij} - \frac{1}{3}R_{ij}r^2 - \frac{1}{12}R'_{ij}r^3\right) \\ &\quad + \left[2\delta_{ij} - \frac{4}{3}R_{ij}r^2 - \frac{5}{12}R'_{ij}r^3\right] r^2 z + b_{ij}, \end{aligned}$$

where we have the estimate:

$$|b_{ij}(u,r,z,p)| \leq Cr^2(r^4 + |z|^2 + |p|^2),$$

for $r \in (0, \delta_0)$ and $|z| + |p| < \epsilon_0$, with C depending only on M. For the inverse metric tensor g^{ij}, we have:

$$\begin{aligned} & g^{ij}(r,z,p) \\ & = r^{-2}\left[\delta_{ij} + \frac{1}{3}R_{ij}r^2 + \frac{1}{12}R'_{ij}r^3 - 2\delta_{ij}z + \frac{4}{3}R_{ij}r^2 z + \frac{5}{12}R'_{ij}r^3 z\right] + b^{ij}, \\ & |b^{ij}(r,z,p)| \leq C(r^4 + |z|^2 + |p|^2), \end{aligned}$$

for C, r, z, p as before.

$$\begin{aligned} (2.2') \qquad \mathcal{N}(r,z,p) &= 1 + p_i p_j\left(\delta_{ij} + \frac{1}{3}R_{ij}r^2 + \frac{1}{6}R'_{ij}r^3 + B'_N(r,z)\right) \\ &= 1 + B_N(r,z,p), \end{aligned}$$

where $|B'_N| \leq C(r^4 + |z|)$, $|B_N| \leq C|p|^2$ for C, r, z, p as before. (In particular, we see that for expansions up to first order in (z,p), the superscript N in A^N may be supressed.)

The mean curvature of Σ with respect to the unit normal vector $\hat{N}$ is the trace:

$$H = \frac{1}{\|N\|} g^{ij} A^N(E_i, E_j).$$

As before, we use this expression to define a smooth function $\mathcal{H} : SM \times [0, \delta_0) \times [0, \epsilon_0) \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n) \to \mathbb{R}$ so that:

$$(RH)_{|X = X_{(R,\xi,\psi)}} = \mathcal{H}(\xi, u_\xi, R, \psi, \nabla^S\psi, H^S\psi).$$

$\mathcal{H}$ is linear in q and admits the Taylor expansion at $(\xi, \omega, 0, 0, 0, 0)$ (to third order in r and first order jointly in (z, p, q):

(2.3)

$$\begin{aligned}\mathcal{H}(\xi, \omega, r, z, p, q) &= n - 1 - [(n-1)z + \mathrm{tr}\,(q)] \\ &\quad - \frac{1}{3}(\mathrm{Ric}\,)r^2 + (\mathrm{Ric}\,)r^2 z + \langle T_4', p\rangle r^2 - \frac{1}{3}\langle \mathrm{Ric}\,, q\rangle r^2 \\ &\quad - \frac{1}{3}(\mathrm{Ric}\,')r^3 - \frac{1}{3}(\mathrm{Ric}\,')r^3 z + \langle T_5', p\rangle r^3 - \frac{1}{12}\langle \mathrm{Ric}\,', q\rangle r^3 \\ &\quad + B(\xi, \omega, r, z, p) + \langle C(\xi, \omega, r, z, p), q\rangle \\ &:= h_0 + h_2 r^2 + h_3 r^3 + B + \langle C, q\rangle,\end{aligned}$$

where $q_{ij} = q(e_i, e_j)$ and we have set:

$$\begin{aligned}\mathrm{tr}\, R_{ij} &= \mathrm{Ric}\,(\xi)(\omega, \omega) = \mathrm{Ric} \\ \mathrm{tr}\, R'_{ij} &= (\nabla_\omega \mathrm{Ric}\,)(\xi)(\omega, \omega) = \mathrm{Ric}\,', \\ \langle T_4', p\rangle &= (T_{4iik} + \frac{1}{3}R_{kl}\Gamma_{iil})p_k, \\ \langle T_5', p\rangle &= (T_{5iik} + \frac{1}{6}R'_{kl}\Gamma_{iil})p_k \\ \langle \mathrm{Ric}\,, q\rangle &= \langle \mathrm{Rm}^M(\xi)(\omega, \bar{e}_i)\omega, \bar{e}_j\rangle q(e_i, e_j), \\ \langle \mathrm{Ric}\,', q\rangle &= \langle (\nabla_\omega \mathrm{Rm}^M)(\xi)(\omega, \bar{e}_i)\omega, \bar{e}_j\rangle q(e_i, e_j).\end{aligned}$$

Here $\mathrm{Ric}\,(\xi)(\omega, \omega)$ denotes the Ricci curvature of M at ξ in the direction ω.

The remainder terms satisfy the estimates (for $r \in (0, \delta_0)$, $|z| + |p| < \epsilon_0$):

$$\begin{aligned}|B(\xi, \omega, r, z, p)| &\leq C(r^4 + |z|^2 + |p|^2), \\ |C^{ij}(\xi, \omega, r, z, p)| &\leq C(r^4 + |z| + |p|).\end{aligned}$$

$$(2.3') \qquad \begin{aligned}|d_z B| + |d_p B| &\leq C(r^4 + |z| + |p|), \\ |d_z C^{ij}| + |d_p C^{ij}| &\leq C(r^4 + 1).\end{aligned}$$

We record the lowest order contribution (in R) to the linear term in ψ in $\mathcal{H}(\xi, u_\xi, R, \psi, \nabla^S\psi, H^S\psi)$, the linear operator in $C^{2+\alpha}$:

$$A\psi = \Delta^S\psi + (n-1)\psi,$$

where Δ^S is the Laplace-Beltrami operator in S.

Remark. In the case $M = \mathbb{R}^n$, we have the exact expression for $\mathcal{H}$:

$$\begin{aligned}\mathcal{H} &= [(1+z)^2 + |p|^2]^{-1/2}[(1+z)^2\delta_{ij} + p_ip_j]^{-1} \\ &\quad \cdot [(1+z)^2\delta_{ij} + 2p_ip_j - (1+z)q_{ij}] \\ &= n - 1 - [(n-1)z + tr(q)] + O(|p|^2 + |z|^2 + |p||q| + |z||q|),\end{aligned}$$

where the remainder term is independent of r.

2.2 The Jacobian of X and the average mean curvature

The Jacobian of the embedding $X = X(\xi, R, \psi) : S \to M$ is given in terms of the induced metric by:

$$\mathrm{Jac}(X) = [\det(g_{ij})]^{1/2}.$$

As before, there is a smooth function $\mathcal{J} : SM \times [0, \delta_0) \times [0, \epsilon_0) \times \mathbb{R}^n \to \mathbb{R}^+$ so that:

$$\mathcal{J}(\xi, u_\xi, R, \psi, \nabla^S\psi) = \mathrm{Jac}(X_{(R,\xi,\psi)}).$$

We are interested in the Taylor expansion of $\mathcal{J}$ at $(r, z, p) = (0, 0, 0)$, for fixed $(\xi, \omega) \in SM$, to third order in r and first order jointly in (z, p). From (2.2), it is easy to see that we may write:

$$\det(g_{ij}) = r^{2(n-1)}[\det(I + \widetilde{h}) + O(|p|^2)],$$

where:

$$\begin{aligned}\widetilde{h}_{ij} &= -\frac{1}{3}R_{ij}r^2 - \frac{1}{12}R'_{ij}r^3 + \left[2\delta_{ij} - \frac{4}{3}R_{ij}r^2 - \frac{5}{12}R'_{ij}r^3\right] z + \widetilde{b}_{ij}, \\ |\widetilde{b}_{ij}| &\leq C(r^4 + |z|^2).\end{aligned}$$

From the well-known formula:

$$\frac{d}{dz}\det(I + \widetilde{h})_{|z=0} = \det(I + \widetilde{h})_{|z=0}\,\mathrm{tr}\,\left(\frac{d}{dz}\widetilde{h}\right)_{|z=0},$$

one easily computes:

$$\begin{aligned}\frac{\mathcal{J}}{r^{n-1}} &= 1 + (n-1)z - \left[\frac{1}{6}(\mathrm{Ric}\,) + \frac{2}{3}(\mathrm{Ric}\,)z\right] r^2 \\ &\quad - \left[\frac{1}{24}(\mathrm{Ric}\,') + \frac{5}{24}(\mathrm{Ric}\,')z\right] r^3 + B_1(\xi, \omega, r, z, p) \\ &:= J_0 + J_2r^2 + J_3r^3 + B_1,\end{aligned} \tag{2.4}$$

where the estimate:

$$|B_1| \leq C(r^4 + |z|^2 + |p|^2)$$

holds whenever $r \in (0, \delta_0)$, $|z| + |p| \leq \epsilon_0$ (with C depending only on M).

We now use (2.4) to compute the average mean curvature:

$$H^\Sigma = \frac{1}{\mathrm{vol}\,(\Sigma)} \int_\Sigma H d\sigma = \frac{\mathrm{ave}_S[H\mathrm{Jac}(X)]}{\mathrm{ave}_S[\mathrm{Jac}(X)]},$$

or equivalently:

$$\begin{aligned}(RH^\Sigma)_{|X_{(R,\xi,\psi)}} &= \frac{\mathrm{ave}_S[(\mathcal{H}\mathcal{J})(\xi, u_\xi, R, \psi, \nabla^S\psi, H^S\psi)]}{\mathrm{ave}_S[\mathcal{J}(\xi, u_\xi, R, \psi, \nabla^S\psi, H^S\psi)]} \\ &= \frac{\mathrm{ave}_S[\mathcal{H}\mathcal{J}]/R^{n-1}}{\mathrm{ave}_S[\mathcal{J}]/R^{n-1}}.\end{aligned}$$

From now on when evaluating averages on S (denoted by bar) we always assume $(\xi, u_\xi, R, \psi, \nabla^S\psi, H^S\psi)$ as the argument of $\mathcal{J}, \mathcal{J}\mathcal{H}$, etc; and we also assume $\psi \in K^{2+\alpha}(S)$. From (2.4) we compute:

$$\begin{aligned}\frac{1}{R^{n-1}}\mathrm{ave}_S[\mathcal{J}] &= 1 - \frac{1}{6n}\mathrm{Scal}(\xi)R^2 - \frac{2}{3}\mathrm{ave}_S[\mathrm{Ric}\,(u_\xi)\psi(u)]R^2 \\ &\quad - \frac{5}{24}\mathrm{ave}_S[\mathrm{Ric}\,'(u_\xi)\psi(u)]R^3 + \bar{B}_1 \\ &:= \bar{J}_0 + \bar{J}_2(\xi)R^2 + \bar{J}_3(\xi)R^3 + \bar{B}_1,\end{aligned}$$

where $\mathrm{Scal}(\xi)$ denotes the scalar curvature at ξ and we used the easily verified fact: $\mathrm{ave}_S[\mathrm{Ric}\,'(u_\xi)] = 0$.

Similarly, for $\mathcal{H}$ we have:

$$\mathrm{ave}_S[\mathcal{H}] = \bar{h}_0 + \bar{h}_2R^2 + \bar{h}_3R^3 + \mathrm{ave}_S[B + \langle C, q\rangle],$$

where:

$$\begin{aligned}\bar{h}_0 &= n - 1 \\ \bar{h}_2 &= -\frac{1}{3n}\mathrm{Scal}(\xi) \\ &\quad + \mathrm{ave}_S[(\mathrm{Ric}\,(u_\xi)\psi(u)] + \mathrm{ave}_S[\langle T_4', \nabla^S\psi\rangle] - \frac{1}{3}\mathrm{ave}_S[\langle R_{ij}, H^S_{ij}\psi\rangle] \\ \bar{h}_3 &= -\frac{1}{3}\mathrm{ave}_S[(\mathrm{Ric}\,'(u_\xi)\psi] + \mathrm{ave}_S[\langle T_5', \nabla^S\psi\rangle] - \frac{1}{12}\mathrm{ave}_S[\langle R_{ij}', H^S_{ij}\psi\rangle].\end{aligned}$$

We are interested in the difference:

$$
\begin{aligned}
&\mathrm{ave}_S\left[\frac{\mathcal{H}\mathcal{J}}{R^{n-1}}\right] - \mathrm{ave}_S\left[\frac{\mathcal{J}}{R^{n-1}}\right]\mathrm{ave}_S[\mathcal{H}]\\
&= (\overline{h_0J_2} - \bar{h}_0\bar{J}_2 + \overline{h_2J_0} - \bar{h}_2\bar{J}_0)R^2\\
&\quad + (\overline{h_0J_3} - \bar{h}_0\bar{J}_3 + \overline{h_3J_0} - \bar{h}_3\bar{J}_0)R^3 + \bar{B}_2\\
&:= \mathcal{C} + \bar{B}_2.
\end{aligned}
$$

A straightforward computation yields for the 'correction' $\mathcal{C}$:

$$
\begin{aligned}
\mathcal{C} &= \mathrm{ave}_S[(\mathrm{Ric}\,)\Delta\psi]\frac{R^2}{6} + \mathrm{ave}_S[(\mathrm{Ric}\,')(-3(n-1)\psi + \Delta\psi)]\frac{R^3}{24}\\
&:= \mathcal{C}_1[\psi]R^2 + \mathcal{C}_2[\psi]R^3,
\end{aligned}
\tag{2.5}
$$

while $\bar{B}_2$ satisfies the estimate:

$$|\bar{B}_2| \le C(R^4 + \|\psi\|_{C^1}\|\psi\|_{C^2}).$$

For the average mean curvature we obtain the expression:

$$RH^{\Sigma} = \mathrm{ave}_S[\mathcal{H}] + \mathcal{C} + B_2, \tag{2.6}$$

where $B_2 = \bar{B}_2 R^{n-1}/\mathrm{ave}_S[\mathcal{J}]$ satisfies the same estimate as $\bar{B}_2$.

Evaluated at $(R, \psi, \nabla^S\psi, H^S\psi)$, the expression (2.3) for $\mathcal{H}$ may be written as:

$$
\begin{aligned}
\mathcal{H} = n - 1 - \frac{1}{3}(\mathrm{Ric}\,)R^2 - \frac{1}{3}(\mathrm{Ric}\,')R^3 - A\psi&\\
+ L_1[\psi]R^2 + L_2[\psi]R^3 + B + \langle C, q\rangle&,
\end{aligned}
\tag{2.6$'$}
$$

where:

$$
\begin{aligned}
L_1[\psi] &:= (\mathrm{Ric}\,)\psi + \langle\nabla\psi, T_4'\rangle - \frac{1}{3}\langle R_{ij}, H_{ij}\psi\rangle\\
L_2[\psi] &:= -\frac{1}{3}(\mathrm{Ric}\,')\psi + \langle\nabla\psi, T_5'\rangle - \frac{1}{12}\langle R_{ij}', H_{ij}\psi\rangle.
\end{aligned}
$$

Combining the above (and using $\psi \in K^{2+\alpha}$), we obtain for the average mean curvature the expansion:

$$RH^{\Sigma} = n - 1 - \frac{1}{3n}\mathrm{Scal}(\xi)R^2 + \bar{L}_1[\psi]R^2 + \bar{L}_2[\psi]R^3 + B_3, \tag{2.7}$$

where $\bar{L}_1, \bar{L}_2$ denote averages over S of L_1, L_2 and B_3 satisfies the estimate:

$$|B_3| \le C(R^4 + \|\psi\|_{C^1}^2) + C(R^4 + \|\psi\|_{C^1})\|D^2\psi\|_{C^0}, \tag{2.7'}$$

provided $0 < R < \delta_0$ and $\|\psi\|_{C^1} < \epsilon_0$.

Combining (2.3) and (2.7), we obtain an expansion for the 'speed' $H^\Sigma - H$, which we state as a lemma:

Lemma 2.1. *There exist constants δ_0, ϵ_0 depending only on M with the following property. The expression $Rv = R(H^\Sigma - H)$ evaluated for the embedding $X_{(R,\xi,e,\psi)}$ defines a continuous (in fact, smooth) nonlinear map:*

$$R(H^\Sigma - H) : \mathcal{M}_0(\delta_0, \epsilon_0) \to C^\alpha(S),$$

which admits the expression:

$$\begin{aligned} Rv = R(H^\Sigma - H) = A\psi + \frac{1}{3}(\mathrm{Ric}_0)R^2 + \frac{1}{3}(\mathrm{Ric}')R^3 \\ + (\bar{L}_1 - L_1)[\psi]R^2 + (\bar{L}_2 - L_2)[\psi]R^3 + \mathcal{C} + B_v, \end{aligned} \tag{2.8}$$

where $\mathcal{C}$ is given in (2.5), *L_1, L_2 are given in* (2.6′) *and* Ric_0 *denotes the trace-free Ricci tensor:* $\mathrm{Ric}_0(\xi, u_\xi) = \mathrm{Ric}(\xi, u_\xi) - \frac{1}{n}\mathrm{Scal}(\xi)$.

The term B_v satisfies the estimates:

$$\begin{aligned} \|B_v\|_{C^0} &\le C(R^4 + \|\psi\|_{C^1}^2) + C(R^4 + \|\psi\|_{C^1})\|D^2\psi\|_{C^0}, \\ [B_v]_{C^\alpha} &\le C(R^4 + \|\psi\|_{C^2}\|\psi\|_{C^{1+\alpha}}) + C(R^4 + \|\psi\|_{C^1})[D^2\psi]_{C^\alpha}, \end{aligned} \tag{2.8'}$$

provided $0 < R < \delta_0$ and $\|\psi\|_{C^1} < \epsilon_0$.

Proof. Only the C^α estimate remains to be shown. We have:

$$B_v = B_3 - B - C^{ij}(H_{ij}^S\psi),$$

where $B(\psi, \nabla^S\psi), C^{ij}(\psi, \nabla^S\psi)$ are functions on S and B_3 is a real number satisfying the desired estimate (2.7). For B we have:

$$\begin{aligned} &|B(\psi, \nabla^S\psi)(u) - B(\psi, \nabla^S\psi)(\bar{u})| \\ &\le \sup_{|z|+|p|\le\epsilon_0} (|d_zB| + |d_pB|)\|\psi\|_{C^{1+\alpha}}|u - \bar{u}|^\alpha, \end{aligned}$$

so from (2.3′):

$$[B(\psi, \nabla^S\psi)]_{C^\alpha} \le C(R^4 + \|\psi\|_{C^1})\|\psi\|_{C^{1+\alpha}}.$$

The estimate of the remaining term is similar, again using (2.3′):

$$
\begin{aligned}
&|C^{ij}(\psi,\nabla^S\psi)(u)H^S_{ij}\psi(u) - C^{ij}(\psi,\nabla^S\psi)H^S_{ij}\psi(\bar{u})| \\
&\leq \sup_{|z|+|p|\leq\epsilon_0}(|d_zC^{ij}|+|d_pC^{ij}|)\|\psi\|_{C^{1+\alpha}}\|D^2\psi\|_{C^0}|u-\bar{u}|^\alpha \\
&\quad + (\sup_S|C^{ij}|)|H^S\psi(u)-H^S\psi(\bar{u})| \\
&\leq \{C(1+R^4)\|D^2\psi\|_{C^0}\|\psi\|_{C^{1+\alpha}} + C(R^4+\|\psi\|_{C^1})[D^2\psi]_{C^\alpha}\}|u-\bar{u}|^\alpha.
\end{aligned}
$$

We conclude:

$$
[C^{ij}H^S_{ij}\psi]_{C^\alpha} \leq C\|\psi\|_{C^{1+\alpha}}\|D^2\psi\|_{C^0} + C(R^4+\|\psi\|_{C^1})[D^2\psi]_{C^\alpha},
$$

as desired.

Remark. We have the bounds:

$$
\begin{aligned}
R\|v\|_{C^0} &\leq C(R^2+\|A\psi\|_{C^0}+R^2\|\psi\|_{C^2}+\|B_v\|_{C^0}), \\
R[v]_{C^\alpha} &\leq C(R^2+[A\psi]_{C^\alpha}+R^2\|\psi\|_{C^{2+\alpha}}+[B_v]_{C^\alpha})
\end{aligned}
\tag{2.9}
$$

2.3 The enclosed volume and an 'a priori estimate' for *R*.

The volume enclosed by $\Sigma = \mathrm{im}(X_{(R,\xi,e,\psi)})$ may be computed from the expression:

$$
\begin{aligned}
V(\Sigma) := \mathrm{vol}_n(\mathrm{int}\,(X)) &= \int_0^R\int_S \mathrm{Jac}(X)(\xi,\tau,\psi)du d\tau \\
&= \omega_{n-1}\int_0^R \mathrm{ave}_S[\frac{\mathcal{J}}{\tau^{n-1}}(\tau,\psi,\nabla^S\psi)]\tau^{n-1}d\tau,
\end{aligned}
$$

where $\omega_{n-1} = \mathrm{vol}\,(S)$. From (2.4), we have:

$$
\begin{aligned}
\frac{V(\Sigma)}{\omega_{n-1}} &= \int_0^R(1+\bar{J}_2\tau^2+\bar{J}_3\tau^3+\bar{B}_1)\tau^{n-1}d\tau \\
&= R^n\left(\frac{1}{n}+\frac{1}{n+1}\bar{J}_2R^2+\frac{1}{n+2}\bar{J}_3R^3\right)+R^n\bar{B}_5,
\end{aligned}
$$

where $\bar{B}_5$ satisfies the estimate:

$$
R^n|\bar{B}_5| = \left|\int_0^R\bar{B}_1\tau^{n-1}d\tau\right| \leq C(R^{n+4}+\|\psi\|^2_{C^1}R^n).
$$

This gives the expansion for $V(\Sigma)$:

$$\begin{aligned}\frac{V(\Sigma)}{R^n\omega_{n-1}/n} &= 1-\frac{1}{6n(n+1)}\mathrm{Scal}(\xi)R^2-\frac{1}{3}\mathrm{ave}_S[\mathrm{Ric}\,(u_\xi)\psi(u)]R^2\\ &\quad-\frac{2n+3}{12(n+2)}\mathrm{ave}_S[\mathrm{Ric}\,'(u_\xi)\psi(u)]R^3+\bar{B}_5\\ &= 1+\bar{B}_6,\end{aligned}$$

where $|\bar{B}_6|\leq CR^2$ if $R<\delta_0, \|\psi\|_{C^1}<\epsilon_0$. Thus, assuming we have $V(\Sigma)=V(\Sigma_0)$ for two such hypersurfaces, we obtain:

$$R^n(1+\bar{B}_6(R))=R_0^n(1+\bar{B}_6(R_0)).$$

This easily implies the following lemma.

Lemma 2.2. *Assume that for hypersurfaces* $\Sigma=\mathrm{im}(X), \Sigma_0=\mathrm{im}(X_0)$ *as above we have equality of the enclosed volumes:* $V(\Sigma)=V(\Sigma_0)$*. Then:*

$$\frac{R}{R_0}=1+B_7(R,R_0),$$

assuming $\max\{R,R_0\}<\delta_0$*, where* $|B_7|\leq C(R_0^2+R^2)$*, for* C *depending only on* M*. Moreover, there exist functions* $\bar{R}_{\min}(V_0,\epsilon_0)<\bar{R}_{\max}(V_0)$*, taking values in* $(0,\delta_0)$ *and defined for* $V_0\in\{\mathrm{vol}\,(X); X=X_{(R,\xi,e,\psi)}, 0<R<\delta_0, \|\psi\|_{C^0}<\epsilon_0\}$ *so that if* $V(X)=V_0, X=X_{(R,\xi,e,\psi)}\in\mathcal{E}_{\delta_0,\epsilon_0}$*, then:*

$$\bar{R}_{\min}(V_0,\epsilon_0)\leq R\leq\bar{R}_{\max}(V_0).$$

Proof. Only the last claim remains to be shown. Since the volume enclosed by geodesic spheres $V(S_\xi(\rho))$ is a monotone increasing function of ρ (for fixed ξ), for each $0<v\leq\max\{V(S_\xi(\rho)); 0<\rho\leq\delta_0\}$ there exists a unique $\rho=\rho(v,\xi)\leq\delta_0$ such that $V(S_\xi(\rho))=v$. In particular, we may define $R_{\min}(V_0,\epsilon,\xi)<R_{\max}(V_0,\epsilon,\xi)$ by:

$$(1+\epsilon)R_{\min}=\rho(V_0,\xi),\quad (1-\epsilon)R_{\max}=\rho(V_0,\xi),$$

and then minimize $R_{\min}$ and maximize $R_{\max}$ over ξ and ϵ to obtain:

$$\bar{R}_{\min}=(1+\epsilon_0)^{-1}\min_{\xi\in M}\rho(V_0,\xi)>0,\quad \bar{R}_{\max}=\max_{\xi\in M}\rho(V_0,\xi)<\delta_0.$$

2.4 The 'tangential correction'

In this subsection we estimate the 'correction term' in the barycenter system. Recall from Section 2.1 that, given $(\xi, e, \varphi) \in \mathcal{M}_0$, we define:

$$N(u) = \gamma'_{e(u)}(\varphi) - h^{kl}\varphi_k J_l(\varphi),$$

where φ_k are components of the gradient $\nabla^S \varphi$ with respect to a local orthonormal frame $(\bar{e}_i)$ on S (defined in a neighborhood of u) and the J_l are Jacobi fields along the geodesic $\gamma_{e(u)}$, orthogonal to $\gamma'_{e(u)}$. We then set:

$$E = \langle w, \nu - e\rangle, \quad w = nS^{-1}(\mathrm{ave}_S[v_N e]), \quad v_N = v\|N\|,$$

where $v = H^\Sigma - H$ and S is the linear map:

$$S := \mathbb{I} + \mathrm{ave}_S[n\langle \cdot, \nu - e\rangle e] : T_\xi M \to T_\xi M.$$

Define the functions of $u \in S$:

$$w_j = \langle \mathbb{Y}_e(\varphi)w, \mathbb{Y}_e(\varphi)e_j\rangle, \quad y_{ij} = \langle \mathbb{Y}_e(\varphi)e_i, \mathbb{Y}_e(\varphi)e_j\rangle;$$

Here (e_i) is the o.n. frame on S_ξ, image of $(\bar{e}_i)$ under the isometry $e : S \to S_\xi$. With (y^{ij}) the inverse matrix, we have:

$$w = y^{kl}w_l e_k, \quad \mathbb{Y}_e(\varphi)w = y^{kl}w_l\mathbb{Y}_e(\varphi)e_k.$$

Then, from:

$$\langle \mathbb{Y}_e Sw, \mathbb{Y}_e e_j\rangle = n\ \mathrm{ave}_S[v_N u^i]\langle \mathbb{Y}_e e_i, \mathbb{Y}_e e_j\rangle$$

and:

$$\begin{aligned}\langle \mathbb{Y}_e Sw, \mathbb{Y}_e e_j\rangle &= \langle \mathbb{Y}_e w, \mathbb{Y}_e e_j\rangle + \mathrm{ave}_S[n\langle \mathbb{Y}_e w, N - \gamma'_e\rangle u^i]\langle \mathbb{Y}_e e_i, \mathbb{Y}_e e_j\rangle \\ &= w_j + \mathrm{ave}_S[n\langle \mathbb{Y}_e e_k, N - \gamma'_e\rangle u^i y^{kl} w_l]y_{ij},\end{aligned}$$

we obtain:

$$w_j + \mathrm{ave}_S[n\mathcal{E}_k u^i y^{kl} w_l]y_{ij} = \mathrm{ave}_S[nv_N u^i]y_{ij},$$

where:

$$\mathcal{E}_k := \langle \mathbb{Y}_e e_k, N - \gamma'_e\rangle.$$

Thus, defining $w^m = y^{jm} w_j$ and $\widetilde{v}_m = \mathrm{ave}_S[v_N u^m]$, we conclude (w^m) is independent of $u \in S$, and is a solution of the linear system:

$$w^m + \mathrm{ave}_S[n\mathcal{E}_k u^m] w^k = n\widetilde{v}_m.$$

We are interested in:

$$E = \langle \mathbb{Y}_e w, N - \gamma_e' \rangle = w^k \mathcal{E}_k.$$

Note that $N - \gamma_e'(\varphi) = -h^{kl}\varphi_k J_l(\varphi)$. This motivates the definition in the following lemma.

Lemma 2.3. *Given a frame (e_i) on $S_\xi M$, defined in a neighborhood of $\omega \in S_\xi$, let $\mathcal{E}_i(\xi, \omega, r, z, p) := -\langle \mathbb{Y}_\omega(\rho) e_i, h^{kl} p_k J_l(\rho)\rangle_{|\rho = r(1+z)}$. We have the Taylor expansion:*

$$\begin{aligned} -\mathcal{E}_i &= p_i - \frac{R_{ik}}{3} p_k r^2 - \frac{R'_{ik}}{24} p_k r^3 + \beta_i \\ &:= \widetilde{p}_i + \beta_i, \end{aligned}$$

where $|\beta_i| \le C(r^4 + |z|^2 + |p|^2)$ for $r < \delta_0, |z| + |p| < \epsilon_0$.

Proof. With $Y_i = \mathbb{Y}_\omega e_i$, we have the expansion:

$$Y_i(\rho, \omega) = \rho e_i - \frac{1}{2}\rho^2 \mathrm{Rm}^M(\omega, e_i)\omega - \frac{1}{6}\rho^3 (\nabla_\omega \mathrm{Rm}^M)(\omega, e_i)\omega + O(\rho^4).$$

Combining this with the previously obtained expansions of h^{kl} and J_l, we easily obtain:

$$h^{kl}\langle Y_i(\rho), J_l(\rho)\rangle = \frac{\delta_{ik}}{r(1+z)} - \frac{R_{ik}}{3} r(1+z) - \frac{R'_{ik}}{24} r^2 (1+z)^2 + O(r^3).$$

Since $-\mathcal{E}_i = r p_k h^{kl} \langle Y_i(\rho), J_l(\rho)\rangle$, this yields the result.

Remark. For the evolution in euclidean space ($M = \mathbb{R}^n$), we have:

$$-\mathcal{E}_i = \frac{p_i}{1+z}$$

(in particular, $\mathcal{E}_i$ is independent of r).

The lemma implies:

$$|\mathcal{E}_i| < C|p|(1 + r^3) + O(r^4 + |z|^2 + |p|^2).$$

In particular, for a given (ξ, e, R, ψ) and evaluating $\mathcal{E}_i$ at $(\xi, e(u), R, \psi, \nabla^S\psi)$, we have $n|\mathcal{E}_i| < 1/2$ for ϵ_0, δ_0 small enough, and the system for (w^m) is uniquely solvable:

$$w^k = (\delta_{mk} - \mathrm{ave}_S[n\mathcal{E}_k u^m])n\widetilde{v}_m + \widetilde{v}_m B_{mk},$$

where B_{mk} is independent of u and $|B_{mk}| < C(R^4 + \|\psi\|^2_{C^1})$.

For the expression E, evaluated at (ξ, e, R, ψ), we obtain:

$$\begin{aligned} E = w^k\mathcal{E}_k &= -n(\delta_{mk} - \mathrm{ave}_S[n\mathcal{E}_k u^m])\widetilde{v}_m(\widetilde{\psi}_k + \beta_k) \\ &= -n\widetilde{v}_k\widetilde{\psi}_k + \widetilde{B}_E, \end{aligned}$$

where:

$$\widetilde{\psi}_i = \psi_k\left(\delta_{ik} - \frac{R_{ik}}{3}R^2 - \frac{R'_{ik}}{24}R^3\right)$$

and:

$$\widetilde{B}_E = -n\widetilde{v}_k\beta_k - n\ \mathrm{ave}_S[n(\widetilde{\psi}_k + \beta_k)u^m]\widetilde{v}_m(\widetilde{\psi}_k + \beta_k) \in C^\alpha(S)$$

satisfies:

$$\begin{aligned} \|\widetilde{B}_E\|_{C^0} &\le C\|v\|_{C^0}(\|\psi\|^2_{C^1} + R^4) \\ \|\widetilde{B}_E\|_{C^\alpha} &\le C\|v\|_{C^0}(\|\psi\|^2_{C^{1+\alpha}} + R^4). \end{aligned}$$

From the expression (2.12) for $\mathrm{ave}_S[vu^k]$ obtained in the next subsection, we obtain for $\widetilde{v}_k = \mathrm{ave}_S[v_N u^k]$:

$$\begin{aligned} \widetilde{v}_k = {} & \frac{2}{3(n+2)}e_k(\mathrm{Scal})R^2 + R\mathrm{ave}_S[L_1u^k] + R^2\mathrm{ave}_S[L_2u^k] \\ & + \frac{1}{R}\mathrm{ave}_S[B_4u^k] + \mathrm{ave}_S[vB_Nu^k]. \end{aligned}$$

Thus we have our final expression for E, which we state as a lemma.

Lemma 2.4. *$E = -S_k\psi_kR^2 + B_E$, where $S_k = \frac{2n}{3(n+2)}e_k(\mathrm{Scal})(\xi)$ and B_E, defined as:*

$$\begin{aligned} B_E = {} & \frac{2nR^2}{3(n+2)}e_k(\mathrm{Scal})\left(\frac{R_{ik}}{3}R^2\psi_i + \frac{R'_{ik}}{24}R^3\psi_i\right) \\ & - n\widetilde{\psi}_k(ave[L_1u^k]R + ave[L_2u^k]R^2) \\ & - (n\widetilde{\psi}_k)\frac{1}{R}\mathrm{ave}_S[B_4u^k] - n\widetilde{\psi}_k\mathrm{ave}_S[v\widetilde{B}_Nu^k] + \widetilde{B}_E \end{aligned}$$

satisfies:

$$\begin{aligned}\|B_E\|_{C^0} &\le C(R^4\|\psi\|_{C^1} + \|\psi\|_{C^1}\|\psi\|_{C^2}R + \|v\|_{C^0}(\|\psi\|^2_{C^1} + R^4)\\ &\quad + R^{-1}\|\psi\|_{C^1}\|B_v\|_{C^0}),\\ \|B_E\|_{C^\alpha} &\le C(R^4 + \|\psi\|_{C^{1+\alpha}}\|\psi\|_{C^2}R + \|v\|_{C^0}(\|\psi\|^2_{C^{1+\alpha}} + R^4)\\ &\quad + R^{-1}\|\psi\|_{C^{1+\alpha}}\|B_v\|_{C^0}).\end{aligned}$$

2.5 Asymptotics for the barycenter system

Using $\|N\| = 1 + B_N$, we recall the *barycenter system* (1.3):

$$\begin{aligned}&\xi_t = n\ \mathrm{ave}_S[vu^i]e_i - \mathrm{ave}_S[nEu^i]e_i + n\ \mathrm{ave}_S[vB_Nu^i]e_i\\ &\nabla_{\xi_t}e = 0\\ &R_t = \mathrm{ave}_S[v] - \mathrm{ave}_S[E] + \mathrm{ave}_S[vB_N]\\ &R\psi_t = (v - E + vB_N)_K - \psi\mathrm{ave}_S[v - E + vB_N],\end{aligned} \tag{2.10}$$

where $f \mapsto (f)_K$ is the L^2 projection $C^{2+\alpha} \to K^{2+\alpha}$. Our goal in this subsection is to identify the main terms in the asymptotics of (2.10), with estimates for the error terms.

From (2.9) it follows easily that:

$$\begin{aligned}R_t &= \mathcal{C}R^{-1} + \mathrm{ave}_S[S_k\psi_k]R + B_R\\ &= \mathcal{C}_1[\psi]R + \mathrm{ave}_S[S_k\psi_k]R + \mathcal{C}_2[\psi]R^2 + B_R\end{aligned} \tag{2.11}$$

where $S_k := \frac{2n}{3(n+2)}e_k(\mathrm{Scal})(\xi)$ and B_R, given by:

$$B_R = R^{-1}\bar{B}_v - \mathrm{ave}_S[B_E] + \mathrm{ave}_S[vB_N]$$

satisfies the estimate:

$$\begin{aligned}|B_R| &\le CR^{-1}\|B_v\|_{C^0} + C\|v\|_{C^0}(\|\psi\|^2_{C^1} + R^4)\\ &\quad + C\|\psi\|_{C^1}\|\psi\|_{C^2}R + CR^4\|\psi\|_{C^1}.\end{aligned} \tag{2.11$'$}$$

Furthermore, we have the estimate:

$$|R_t| < C\|\psi\|_{C^2}R + |B_R|.$$

Turning to ξ_t, we consider the average (using (2.8′)):

(2.12)

$$\begin{aligned}\mathrm{ave}_S[vu^i] &= \frac{R}{3}\mathrm{ave}_S[\mathrm{Ric}\,_0u^i] + \frac{R^2}{3}\mathrm{ave}_S[\mathrm{Ric}\,'u^i] + \frac{1}{R}\mathrm{ave}_S[(A\psi)u^i]\\ &\quad - R\mathrm{ave}_S[L_1[\psi]u^i] - R^2\mathrm{ave}_S[L_2[\psi]u^i] + \frac{1}{R}\mathrm{ave}_S[B_4u^i].\end{aligned}$$

It is easy to see that the first term vanishes (since it is the average of an odd function of u), and so does the third (since $\psi \in K^{2+\alpha}$). For the second term we use:

$$\begin{aligned}
\operatorname{ave}_S[\operatorname{Ric}' u^l] &= \frac{1}{\omega_{n-1}} \int_S \langle \nabla_{u_\xi}(\operatorname{Riem}^M)(u_\xi, \bar{e}_i) u_\xi, \bar{e}_i \rangle u^l du \\
&= \frac{1}{\omega_{n-1}} R_{jimi;n}(\xi) \int_S u^n u^j u^m u^l du \\
&= \frac{2}{n+2} \bar{e}_l(\operatorname{Scal})(\xi),
\end{aligned}$$

by the calculation in Lemma 1.2 in [14].

We conclude:

$$\begin{aligned}
\xi_t = \frac{2n}{3(n+2)} \nabla^M \operatorname{Scal}(\xi) R^2 &- nR \operatorname{ave}_S[L_1[\psi] u^i] e_i \\
&- nR^2 \operatorname{ave}_S[L_2[\psi] u^i] e_i + R^2 \operatorname{ave}_S[n S_k \psi_k u^i] e_i + B_\xi,
\end{aligned} \tag{2.13}$$

where $B_\xi = n\operatorname{ave}_S[vB_N u^i]e_i - \operatorname{ave}_S[nB_E u^i]e_i + nR^{-1}\operatorname{ave}_S[B_v u^i]e_i$ satisfies the bound:

$$\begin{aligned}
|B_\xi| \leq C\|v\|_{C^0}(\|\psi\|_{C^1}^2 + R^4) &+ CR^{-1}\|B_v\|_{C^0} \\
&+ C\|\psi\|_{C^1}\|\psi\|_{C^2} R + CR^4 \|\psi\|_{C^1}.
\end{aligned} \tag{2.13$'$}$$

With (2.11) and (2.12) we obtain the equation for ψ_t:

$$\begin{aligned}
R\psi_t = \frac{1}{R} A\psi + \frac{1}{3}\operatorname{Ric}_0(\xi, u) R &+ \frac{1}{3}\operatorname{Ric}'(\xi, u) R^2 - S_k u^k R^2 \\
- (L_1[\psi])_K R &- (L_2[\psi])_K R^2 + (S_k \psi_k)_K R^2 + B_\psi,
\end{aligned} \tag{2.14}$$

with B_ψ given by:

$$B_\psi = R^{-1}(B_v)_K - (B_E)_K + (vB_N)_K - \psi \operatorname{ave}_S[v - E + vB_N].$$

From this definition, (2.2$'$), (2.8$'$) and Lemma 2.4 one sees easily that:

$$\begin{aligned}
\|B_\psi\|_{C^0} &\leq C\|v\|_{C^0}(\|\psi\|_{C^1}^2 + R^4) + CR^{-1}\|B_v\|_{C^0} \\
&\qquad + CR\|\psi\|_{C^1}\|\psi\|_{C^2} + CR^4\|\psi\|_{C^1}; \\
\|B_\psi\|_{C^\alpha} &\leq C\|v\|_{C^\alpha}(\|\psi\|_{C^{1+\alpha}}^2 + R^4) + CR^{-1}\|B_v\|_{C^\alpha} \\
&\qquad + CR\|\psi\|_{C^{1+\alpha}}\|\psi\|_{C^2} + CR^4\|\psi\|_{C^{1+\alpha}},
\end{aligned} \tag{2.14$'$}$$

(still under the assumptions $R < \delta_0$, $\|\psi\|_{C^{1+\alpha}} < \epsilon_0$).

Combining (2.11′), (2.13′) and (2.14′) with the estimates (2.8) and (2.8′) for v and B_v, we obtain a more useful estimate for the remainder terms, stated in the final lemma of this section.

Lemma 2.5. *The barycentric system* (2.10) *in* $\mathcal{M}_0(\delta_0, \epsilon_0)$ *may be written in the form* (2.11), (2.13), (2.14), *plus the parallel transport equation for the frame* e. *The remainder terms satisfy the estimates:*

(2.15)
$$\begin{aligned} |B_R| + |B_\xi| + \|B_\psi\|_{C^0} &\leq cR^3 + cR^{-1}\|\psi\|_{C^1}\|\psi\|_{C^2} + cR^3\|D^2\psi\|_{C^0}, \\ \|B_\psi\|_{C^\alpha} &\leq cR^3 + cR^{-1}\|\psi\|_{C^{1+\alpha}}\|\psi\|_{C^{2+\alpha}} + cR^3\|D^2\psi\|_{C^\alpha}. \end{aligned}$$

3. Local and global existence

3.1 Local existence via maximal regularity

Our goal in this subsection is to prove local existence for the barycenter system on $\mathcal{M}_0$:

(3.1)
$$\begin{aligned} &\xi_t = \mathrm{ave}_S[n(v_N - E)e] \\ &\nabla_{\xi_t} e = 0 \\ &\varphi_t = (v_N - E)^\circ, \end{aligned}$$

where $v_N = (H^\Sigma - H)\|N\|$ and $f \mapsto f^\circ$ is the L^2 projection $C^\alpha \to C_0^\alpha$. In Section 2 we derived the expression for $H^N = \|N\|H$:

$$H^N = -P(\varphi, \nabla^S\varphi)[\varphi] + B(\varphi, \nabla^S\varphi),$$

where:

$$P(\varphi, \nabla^S\varphi)[\chi] = g^{ij}(H^S\chi)(e_i, e_j),$$

$g_{ij} = \varphi_i\varphi_j + h_{ij}$ is the induced metric on Σ and H^S is the Hessian on S. Defining:

$$P^\Sigma[\chi] := \frac{\|N\|}{\mathrm{vol}\,(\Sigma)}\int_\Sigma \|N\|^{-1}P[\chi]d\sigma, \quad B^\Sigma := \frac{\|N\|}{\mathrm{vol}\,(\Sigma)}\int_\Sigma \|N\|^{-1}Bd\sigma,$$

we may write:

$$v_N = P[\varphi] - P^\Sigma[\varphi] - B + B^\Sigma.$$

To apply the results in [3] on local existence, we introduce appropriate function spaces. We denote by $h^{k+\gamma}(S)$ (for an integer $k \geq 0$ and $\gamma \in (0,1)$) the closure of smooth functions in $C^{k+\gamma}(S)$, with the standard $C^{k+\gamma}$ norm. $h_0^{k+\gamma}(S)$ denotes the corresponding subspace of $C_0^{k+\gamma}(S)$. We also introduce local coordinates on the manifold $\mathbb{F}M$, locally modelled on $\mathbb{R}^n \times \mathrm{so}(n)$. For fixed $0 < \alpha < \beta < \beta_0 < 1$, define the Banach spaces:

$$E_1 = \mathbb{R}^n \times \mathrm{so}(n) \times h_0^{2+\alpha}(S), \quad E_0 = \mathbb{R}^n \times \mathrm{so}(n) \times h_0^{\alpha}(S)$$
$$E_\theta = \mathbb{R}^n \times \mathrm{so}(n) \times h_0^{1+\beta}(S), \quad E_\sigma = \mathbb{R}^n \times \mathrm{so}(n) \times h_0^{1+\beta_0}(S).$$

In the notation of [3], we seek a solution in the space:

$$W^T(\mathbb{R}^n) = C([0,T], E_\sigma) \cap C_\sigma([0,T], E_1) \cap C_\sigma^1((0,T], E_0).$$

The initial data $(\xi(0), e(0), \varphi(0))$ will be taken in the open subset of E_σ:

$$\mathcal{O}_{\delta_2,\epsilon_2}^{1+\beta_0}(U_0) = \left\{(x, A, \varphi) \in E_\sigma; x \in U_0, e(x) = e^A x \in \mathbb{F}_g(U_0), \varphi \in B_{\delta_2,\epsilon_2}^{1+\beta_0}\right\}.$$

Here $U_0 \subset R^n$ is the image of a normal coordinate neighborhood of $\xi(0)$ in M, and we identify U_0 and this neighborhood in the notation. $\mathbb{F}_g U_0$ is the orthonormal frame bundle of U_0, with respect to the metric g pulled back from M via the coordinate chart. We also use the fact that any $0 \leq \varphi \in C^0(S)$ may be written uniquely in the form $\varphi = R(1+\psi)$ with $\mathrm{ave}_S[\psi] = 0$.

Prior to proving the local existence lemma, we need an observation regarding the term E.

Lemma 3.1. *The linear assignment $v_N \mapsto E$ defines a smooth map:*

$$\mathbb{E} : \mathcal{O}_{\delta_1,\epsilon_1}^{1+\beta}(U) \to \mathcal{L}(C^\alpha, h^{1+\beta}).$$

Proof. Given $(\xi, e, \varphi) \in \mathcal{O}_{\delta_1,\epsilon_1}^{1+\beta}(U)$, the map $\mathbb{E}_{\xi,e,\varphi}$ is the composition $\mathbb{E} = \mathcal{Y} \circ \mathcal{S}$ of:

$$\text{(i)} \qquad v_N \mapsto \mathcal{S}[v_N] \in T_\xi M,$$

where $\mathcal{S}[w] = nS_{\xi,e,\varphi}^{-1}(\mathrm{ave}_S[we])$ defines a smooth map $\mathcal{O}_{\delta_1,\epsilon_1}^{1+\beta}(U) \to \mathcal{L}(C^\alpha, TM)$, preserving $\xi \in M$;

$$\text{(ii)} \qquad \mathcal{Y}_{(\xi,e,\varphi)} : \eta \mapsto \langle \mathbb{Y}_e(\varphi)\eta, N - \gamma_e'(\varphi)\rangle,$$

which defines a smooth map $\mathcal{O}^{1+\beta}_{\epsilon_1,\delta_1}(U) \to \mathcal{L}(T_\xi M, h^{1+\beta}(S))$. These observations prove the lemma.

Denoting $\mathcal{P}_{(\xi,e,\varphi)}[\chi] = (P - P^\Sigma)[\chi]$, $\quad \mathcal{B}(\xi,e,\varphi) = B - B^\Sigma$, we may write:

$$v_N - E = (v_N - E)_1[\varphi] + (v_N - E)_2,$$

where

$$(v_N - E)_1[\chi] = (\mathcal{P}_{(\xi,e,\varphi)} - \mathbb{E}_{(\xi,e,\varphi)} \circ \mathcal{P}_{(\xi,e,\varphi)})[\chi],$$
$$(v_N - E)_2(\xi,e,\varphi) = -\mathcal{B}(\xi,e,\varphi) + \mathbb{E}_{(\xi,e,\varphi)}(\mathcal{B}(\xi,e,\varphi)).$$

In local coordinates $(x^a, e^a_i, \varphi) \in U \times \mathbb{R}^{n^2} \times h^{2+\alpha}_0(S)$, $e_i = e^a_i \frac{\partial}{\partial x^a}$, the barycentric system is written:

$$\begin{aligned} x^a_t &= \mathrm{ave}_S[n(v_N - E)u^i]e^a_i \\ (e^a_i)_t &= -\Gamma^a_{bc}(x)\mathrm{ave}_S[n(v_N - E)u^j]e^b_j e^c_i \\ \varphi_t &= (v_N - E)^\circ, \end{aligned}$$

where $\Gamma^a_{bc}(x)$ are the Christoffel symbols of the Riemannian connection. We may write this as the quasilinear system in $U \times \mathbb{R}^{n^2} \times h^{2+\alpha}_0(S)$:

$$(x^a, e^a_i, \varphi)_t = \mathbb{P}_{(x,e_i,\varphi)}[\varphi] + \mathbb{B}(x, e^a_i, \varphi), \tag{3.2}$$

with:

$$\begin{aligned} \mathbb{P}_{(x,e,\varphi)}[\chi] := \Big(&\mathrm{ave}_S[n(v_N - E)_1[\chi]u^i]e^a_i, \\ &- \Gamma^a_{bc}(x)\mathrm{ave}_S[n(v_N - E)_1[\chi]u^j]e^b_j e^c_i, (v_N - E)^\circ_1[\chi]\Big) \end{aligned}$$

and:

$$\begin{aligned} \mathbb{B}(x, e^a_i, \varphi) = \Big(&\mathrm{ave}_S[n(v_N - E)_2 u^i]e^a_i, \\ &- \Gamma^a_{bc}\mathrm{ave}_S[n(v_N - E)_2 u^j]e^b_j e^c_i, (v_N - E)^\circ_2\Big). \end{aligned}$$

(Since parallel translation preserves orthonormality, it is harmless to consider $\mathbb{R}^{n^2}$ instead of so(n).)

To state the local and global existence results for (3.1), we introduce the space:

$$W^T_{\delta_1,\epsilon_1} = C^0([0,T], \mathcal{M}_0(\delta_1,\epsilon_1)) \cap C^1((0,T], \mathcal{M}^\alpha(\delta_1,\epsilon_1))$$

and the open subset:

$$W^T_{\delta_1,\epsilon_1}(U) = C^0([0,T], \mathcal{O}^{2+\alpha}_{\delta_1,\epsilon_1}(U)) \cap C^1((0,T], \mathcal{O}^\alpha_{\delta_1,\epsilon_1}(U)).$$

Lemma 3.2. *Let* $0 < \alpha < \beta_0 < 1$.

(i) *There exist constants* $\delta_2 \in (0, \delta_1/2), \epsilon_2 \in (0, \epsilon_1/2)$ *depending only on* M *and* $T = T(\delta_2, \epsilon_2) > 0$ *so that for any* $(\xi(0), e(0), \varphi(0)) \in \mathcal{M}_0^{1+\beta_0}(\delta_2, \epsilon_2)$ *such that* $\varphi(0) \in C^\infty(S)$*, there exists a unique solution* $(\xi(t), e(t), \varphi(t))$ *of* (3.1) *in* $W^T_{\delta_1,\epsilon_1}(U_0)$ *(where* U_0 *is a normal coordinate neighborhood of* $\xi(0)$*).*

(ii) $\varphi(t, \cdot) \in C^\infty(S)$ *and* $(\xi(t), e(t))$ *is a smooth curve in* $\mathbb{F}_g U_0$ *for* $t \in [0, T]$.

(iii) *The hypersurfaces* $\Sigma(t)$, $t \in [0, T]$ *parametrized by* $X(t) \in \mathcal{N}_{\text{std}}(\delta_1, \epsilon_1)$*:*

$$X(t, u) = \exp_{\xi_0}[\varphi(t, u)e(t, u)]$$

are smooth for $t > 0$ *and satisfy:*

$$X_t = (H^\Sigma - H)\hat{N} + \mathbb{T},$$

where $\mathbb{T}(t, u)$ *is tangent to* $\Sigma(t)$ *at* $X(t, u)$*. Thus* $\Sigma(t)$ *is a motion by normalized mean curvature.*

Proof.

(i) We apply Theorem 2.11 in [3] to the quasilinear system (3.2). There are three conditions to verify.

Claim (a): $\mathbb{P}$ defines a smooth map $\mathcal{O}^{1+\beta}_{\delta_2,\epsilon_2}(U_0) \to \mathcal{L}(E_1, E_0)$.

In Section 2 we showed that:

$$g_{ij}(\varphi, \nabla_S \varphi) = \varphi^2 \left(\delta_{ij} + \frac{\varphi_i \varphi_j}{\varphi^2} + \widetilde{h}_{ij}(\varphi) \right), \quad \|N\| = (1 + h^{ij}\varphi_i\varphi_j)^{1/2},$$

where $\widetilde{h}_{ij}$, the smooth function of $\tau > 0$ defined by:

$$h_{ij} = \langle J_i(\tau), J_j(\tau) \rangle = \tau^2(\delta_{ij} + \widetilde{h}_{ij})$$

satisfies $|\widetilde{h}_{ij}(\varphi)| \le c|\varphi|^2$ for $|\varphi| < \delta_2$, for some $c > 0$ depending only on M. From the definitions of P and P^Σ, it is clear that for $\delta_1 > 0$ sufficiently small, $\mathcal{P}$ defines a smooth map $\mathcal{P} : \mathcal{O}^{1+\beta}_{\delta_2,\epsilon_2}(U_0) \to \mathcal{L}(h_0^{2+\alpha}, h_0^\alpha)$. From Lemma 3.1, $\mathbb{E}$ defines a smooth map $\mathcal{O}^{1+\beta}_{\delta_2,\epsilon_2}(U) \to \mathcal{L}(h_0^\alpha, h^{1+\beta}(S))$. This shows $(v_N - E)_1 = \mathcal{P} - \mathbb{E} \circ \mathcal{P}$ defines a smooth map from the same set to $\mathcal{L}(h_0^{2+\alpha}, h^\alpha)$. It is then clear that $\mathbb{P}$ defines a smooth map into $\mathcal{L}(h_0^{2+\alpha}, E_0)$, which may be regarded as a map into $\mathcal{L}(E_1, E_0)$.

Claim (b): $\mathbb{B}$ defines a smooth map $\mathcal{O}^{1+\beta}_{\delta_2,\epsilon_2} \to E_0$.

Since $Hd_{\xi_0}(\tau)(J_i(\tau), J_j(\tau))$ (where Hd_{ξ_0} denotes the Hessian of the distance function to ξ_0) is a smooth function of τ satisfying:

$$|Hd_{\xi_0}(\varphi)(J_i(\varphi), J_j(\varphi))| \leq c|\varphi| \text{ for } |\varphi| < \delta_0,$$

with c depending only on M, the expression for $B(\varphi, \nabla_S\varphi)$ (see Section 2):

$$B = g^{ij}\Big[Hd_{\xi_0}(J_i, J_j) + h^{kl}\varphi_k(\varphi_j Hd_{\xi_0}(J_i, J_l) + \varphi_i Hd_{\xi_0}(J_j, J_l)) \\ + h^{kl}\varphi_k\langle\nabla_{J_i}J_j, J_l\rangle\Big]$$

and the expressions above for g_{ij} immediately imply that $\mathcal{B} = B - B^{\Sigma}$ defines a smooth map $\mathcal{B} : \mathcal{O}^{1+\beta}_{\delta_2,\epsilon_2} \to h_\alpha$, for $\delta_2 \in (0, \delta_0)$ sufficiently small. Given that $\mathbb{E}$ is smooth into $\mathcal{L}(h^\alpha_0, h^{1+\beta})$ (Lemma 3.1), this shows $(v_N - E)_2$ is smooth in h^α, and hence $\mathbb{B}$ is smooth from the same set into E_0.

Claim (c): $\mathbb{P}_{(\xi,e,\varphi)} \in \mathcal{M}_\sigma(E_1, E_0)$ for $(\xi, e, \varphi) \in \mathcal{O}^{1+\beta_0}_{\delta_2,\epsilon_2}(U)$.

It follows from theorem (2.14) in [3] (which generalizes [4]) that $\mathcal{P}_{(\xi,e,\varphi)} \in \mathcal{M}_\sigma(h^{2+\alpha}, h^\alpha)$, provided we can verify that $\mathcal{P}_{(\xi,e,\varphi)}$ generates an analytic semigroup in (F_1, F_0), where we set $F_1 = h^{2+\alpha_0}, F_0 = h^{\alpha_0}$ (for some $\alpha_0 \in (0, \alpha)$):

$$\mathcal{P}(\varphi, \nabla_S\varphi) \in \mathrm{Hol}(h^{2+\alpha_0}, h^{\alpha_0}), \text{ for all } \varphi \in \mathcal{O}^{1+\beta_0}_{\delta_1}.$$

This follows from the fact that, for such φ, $P(\varphi, \nabla_S\varphi)[.]$ is a second order, uniformly elliptic operator in S, and P^{Σ} defines a perturbation with zero relative norm: since $P^{\Sigma}[\chi] \in \mathbb{R}$ with $|P^{\Sigma}[\chi]| \leq \|\chi\|_{C^2}$, we have for each $\epsilon > 0$:

$$\|P^{\Sigma}[\chi]\|_{h^{\alpha_0}} = |P^{\Sigma}[\chi]| < \epsilon\|\chi\|_{h^{2+\alpha_0}} + C_\epsilon\|\chi\|_{h^{\alpha_0}}.$$

Since, in the notation of Lemma 3.1 above:

$$\begin{aligned}\|E \circ \mathcal{P}[\chi]\|_{h^{1+\beta}} &\leq c\|(\mathcal{Y} \circ S)[\mathcal{P}\chi]\|_{h^{1+\beta}} \\ &\leq c|\mathcal{S}[\mathcal{P}\chi]|_{TM} \leq c\|\mathcal{P}\chi\|_{C^0} \leq \|\chi\|_{C^2},\end{aligned}$$

the same argument shows:

$$(v_N - E)_1 = \mathcal{P} - \mathbb{E} \circ \mathcal{P} \in \mathcal{M}_\sigma(h^{2+\alpha}, h^\alpha).$$

Likewise, since:

$$|\text{ave}_S[n(v_N - E)_1[\chi]u^i]e^a_i| + |\Gamma^a_{bc}(x)\text{ave}_S[n(v_N - E)[\chi]u^j]e^b_j e^c_i| \\ \le c(x, e, \varphi)\|\chi\|_{C^2},$$

$\mathbb{P}_{(x,e,\varphi)}$ may be regarded as a 'lower order perturbation' of $(0, 0, (v_N - E)_1) \in \mathcal{M}_\sigma(E_1, E_0)$, which suffices for the claim.

(ii) (Higher regularity.) This follows from a standard 'bootstrapping' argument. From Part (i), $\varphi(t) \in h^{2+\alpha}$ for each $t > 0$. Fix $s > 0$ and apply Theorem 2.11 in [3] with initial data $\Psi(s)$ and spaces $E = E_0, E_\sigma, E_\theta, E_1$ defined as $E = \mathbb{R} \times \text{so}(n) \times h$, where $h = h_0^{1+\alpha}$ for E_0, $h = h_0^{2+\alpha}$ for E_σ, $h = h_0^{2+\alpha_0}$ for E_θ, $h = h_0^{3+\alpha}$ for E_1. It follows as above that $\mathcal{P}(\varphi, \nabla_S\varphi) \in \text{Hol}(F_1; F_0)$, where $F_1 = h^{3+\alpha_0}, F_0 \in h^{1+\alpha_0}$. We obtain a local solution Ψ_1:

$$\Psi_1 \in C^0([s, T_1]; \mathcal{M}_0^{2+\alpha}) \cap C^0((s, T_1]; \mathcal{M}_0^{3+\alpha}) \cap C^1((s, T_1]; \mathcal{M}_0^{1+\alpha}),$$

which must coincide with Ψ on some interval (s, s_1). Since $s \in (0, T)$ is arbitrary, this shows $\Psi \in C^0((0, T); \mathcal{M}_0^{3+\alpha}) \cap C^1((0, T); \mathcal{M}_0^{1+\alpha})$. Iterating this argument yields the conclusion of (ii).

(iii) This follows directly from Lemma 1.7.

The same argument yields local existence in a slightly different space.

Lemma 3.3. *There exist constants $\delta_2 \in (0, \delta_1/2), \epsilon_2 \in (0, \epsilon_1/2)$ depending only on M, and $T = T(\delta_2, \epsilon_2) > 0$ so that if $(\xi(0), e(0), \varphi(0) \in \mathcal{M}_0(\delta_2, \epsilon_2)$ with $\varphi(0)$ smooth, there exists a unique solution of* (3.1) *in $W^T_{\delta_1,\epsilon_1}(U_0)$, where U_0 is a normal coordinate neighborhood of $\xi(0)$. Statements* (ii) *and* (iii) *of Lemma* 3.2 *still hold.*

Proof. Identical to Lemma 3.2. Since no 'smoothing effect' is needed, Theorem 2.7 in [3] could also be used.

3.2 Global existence for the scaled system

Prior to establishing global existence, we introduce two small scale parameters $\epsilon \in (0, \epsilon_1), \delta \in (0, \delta_1)$ and write a general $X \in \mathcal{N}_{\text{std}}$ in the form:

$$X = \Phi(\delta R, \xi, e, \epsilon\psi), \quad X(u) = \exp_\xi[\delta R(1 + \epsilon\psi(u))e(u)],$$

where we always assume $0 < R < 1, \psi \in K^{2+\alpha}, \|\psi\|_{C^{2+\alpha}} < 1$. (Thus δ measures the 'size' of the bubble and ϵ the 'deviation from sphericity relative to size'.) With these scale parameters, the equations of motion take the form:

$$
(3.3)_{\delta,\epsilon}\qquad
\begin{aligned}
&\delta R_t = \mathrm{ave}_S[v_N - E]\\
&\xi_t = \mathrm{ave}_S[n(v_N - E)e]\\
&\nabla_{\xi_t} e = 0\\
&\delta\epsilon R\psi_t = (v_N - E)_K - \delta\psi\ \mathrm{ave}_S[v_N - E],
\end{aligned}
$$

with v_N and E computed at $(\delta R, \xi, e, \epsilon\psi) \in \mathcal{M}_0$.

To prove global existence, we use the expansions obtained in Section 2 (Lemma 2.5) to write the system in the form:

(3.4)

$$
\begin{aligned}
\delta R_t &= \mathcal{C}_1[\psi]\delta\epsilon R + \mathcal{C}_2[\psi]\delta^2\epsilon R^2 + \mathrm{ave}_S[S_k\psi_k]\delta\epsilon R + B_R(\xi, e, \delta R, \epsilon\psi)\\
\xi_t &= c_n\nabla^M \mathrm{Scal}(\xi)\delta^2 R^2 - \mathrm{ave}_S[nL_1[\psi]e]\delta\epsilon R - \mathrm{ave}_S[nL_2[\psi]e]\delta^2\epsilon R^2\\
&\quad + \mathrm{ave}_S[nS_k\psi_k e]\delta^2\epsilon R^2 + B_\xi(\xi, e, \delta R, \epsilon\psi)\\
\nabla_{\xi_t} e &= 0\\
\delta\epsilon R\psi_t &= \delta^{-1}R^{-1}\epsilon A\psi + \frac{1}{3}(\mathrm{Ric}_0)\delta R + \frac{1}{3}(\mathrm{Ric}')\delta^2 R^2 + S_k u^k \delta^2 R^2\\
&\quad - (L_1[\psi])_K\epsilon\delta R - (L_2[\psi])_K\epsilon\delta^2 R^2 + B_\psi(\xi, e, \delta R, \epsilon\psi).
\end{aligned}
$$

For the remainder terms, we have the estimates:

$$
\begin{aligned}
|B_R| + |B_\xi| + \|B_\psi\|_{C^0} &\le c\delta^3 R^3 + c\delta^{-1}R^{-1}\|\psi\|_{C^1}\|\psi\|_{C^2}\epsilon^2\\
&\quad + cR^3\delta^3\epsilon\|D^2\psi\|_{C^0},\\
\|B_\psi\|_{C^\alpha} &\le c\delta^3 R^3 + c\delta^{-1}R^{-1}\|\psi\|_{C^{1+\alpha}}\|\psi\|_{C^{2+\alpha}}\epsilon^2 + cR^3\delta^3\epsilon\|D^2\psi\|_{C^\alpha}.
\end{aligned}
$$

Remark. The traceless Ricci curvature Ric_0 vanishes identically on surfaces, so the δR term is not present in the equation for ψ_t when $n = 2$.

We re-state Lemma 3.3 as follows:

Lemma 3.4. *There exist $\delta_2 > 0, \epsilon_2 > 0$ depending only on M and $T = T(\delta_2, \epsilon_2) > 0$, with the following property. For each $\Psi(0) = (\delta R(0), \xi(0), e(0), \epsilon\psi(0))$ in $\mathcal{M}_0(\delta_2, \epsilon_2)$ with $\epsilon < \epsilon_2, \delta < \delta_2$, there exists a unique solution $\Psi(t) = (\delta R(t), \xi(t), e(t), \epsilon\psi(t)) \in \mathcal{M}_0(\delta_2, \epsilon_2)$ to $(3.3)_{\delta,\epsilon}$ in the space $W^T_{\delta_1,\epsilon_1}(U_0)$, where U_0 is a normal coordinate neighborhood of ξ_0. In particular, we have $\|\psi(t)\|_{C^{1+\alpha}}(S) < 1$ in $[0, T]$.*

From Lemma 2.2, the following estimates hold for $R(t)$, $t \in [0,T]$.

$$\delta R_{\min} \leq \delta R(t) \leq \delta R_{\max}, \tag{3.4$'$}$$

where $R_{\max}$ and $R_{\min}$, given by:

$$\begin{aligned} R_{\max} &= \bar{R}_{\max}(\mathrm{vol}\,(X(0))) \leq \bar{R}_{\max}(\mathrm{vol}\,(X_{(\delta_2,\xi(0),e(0),\epsilon_2)}) \\ R_{\min} &= \bar{R}_{\min}(\mathrm{vol}\,(X(0)),\epsilon_2) \geq \bar{R}_{\min}(\mathrm{vol}\,(X_{(\delta_2,\xi(0),e(0),\epsilon_2)}) \end{aligned}$$

may be taken to depend only on M.

For $t \in [0, T_{\delta,\epsilon})$, we have the estimates for remainder terms (using the bound $\|\psi\|_{C^{1+\alpha}} < 1$):

$$\begin{aligned} |B_R| + |B_\xi| + \|B_\psi\|_{C^0} &\leq c\delta^3 R_{\max}^3(1+\epsilon\|D^2\psi\|_{C^0}) \\ &\quad + c\delta^{-1}R_{\min}^{-1}(\epsilon^2+\epsilon^2\|D^2\psi\|_{C^0}), \\ \|B_\psi\|_{C^\alpha} &\leq R_{\max}^3\delta^3(1+\epsilon\|D^2\psi\|_{C^\alpha}) \\ &\quad + c\delta^{-1}R_{\min}^{-1}(\epsilon^2+\epsilon^2\|D^2\psi\|_{C^\alpha}). \end{aligned}$$

We will need some facts from semigroup theory. Consider the operator $A\psi = \Delta^S\psi + (n-1)\psi$ on the Hilbert space:

$$X = \{\psi \in L^2(S); \mathrm{ave}_S[\psi] = 0 = \mathrm{ave}_S[\psi u^i], i = 1,\dots,n\}.$$

A is self-adjoint on X with eigenvalues $\lambda_n < \lambda_{\max} < 0$. $A \in \mathcal{M}_1(h_0^{2+\alpha}, h_0^\alpha)$, i.e., A has the maximal regularity property in the Banach couple $(h_0^{2+\alpha}, h_0^\alpha)$ (see [3]). Denote by $e^{sA}, s > 0$, the analytic semigroup generated by A.

Lemma 3.5. *There exist constants $\mu > 0, M_0 > 0$ such that the semigroup generated by A in $h_0^{2+\alpha}$ satisfies the estimates:*

(1) $\|e^{sA}\phi\|_{C^{2+\alpha}} \leq M_0 e^{-\mu s}\|\phi\|_{C^{2+\alpha}}, \quad \phi \in h_0^{2+\alpha};$

(2) $\|e^{sA}\phi\|_{C^{2+\alpha}} \leq \dfrac{M_0}{s} e^{-\mu s}\|\phi\|_{C^\alpha}, \quad \phi \in h_0^\alpha;$

(3) $\displaystyle\sup_{0<s\leq\bar{s}} \left\| \int_0^s e^{(s-\sigma)A}\phi(\sigma)d\sigma \right\|_{C^{2+\alpha}(S)} \leq M_0 \sup_{0<s\leq\bar{s}} \|\phi(s)\|_{C^\alpha(S)},$

if $\phi : (0,\bar{s}] \to h_0^\alpha$ is continuous (where M_0 is independent of $\bar{s}$).

Proof. The first two estimates are well-known consequences of the fact that A generates an analytic semigroup on $h_0^{2+\alpha}$. Estimate (3) is equivalent to the statement $A \in \mathcal{M}_1(h_0^{2+\alpha}, h_0^{\alpha})$. The fact that M_0 may be taken independent of $\bar{s}$ uses the negative upper bound on the spectrum of A.

We may now state and prove the global existence lemma. For $\Psi \in \mathcal{M}_0(\delta_2, \epsilon_2)$, define $T^*_{\delta,\epsilon}(\Psi)$ as the supremum (possibly infinite) of all $T > 0$ such that the solution of $(3.3)_{\delta,\epsilon}$ found in Lemma 3.4, with initial condition Ψ, is in $W^T_{\delta_2,\epsilon_2}$.

Lemma 3.6. *There exist $\delta_3 \in (0, \delta_2), \epsilon_3 \in (0, \epsilon_2)$ so that if $0 < \delta < \delta_3, 0 < \epsilon < \epsilon_3$ and $\delta^2 \ll \epsilon$, we have $T^*_{\delta,\epsilon}(\Psi) = \infty$, for any $\Psi \in \mathcal{M}_0(\delta_3, \epsilon_3)$. In addition, $\|\epsilon\psi(t)\|_{C^{2+\alpha}} < \epsilon_2$, for all $t \geq 0$.*

Proof. By contradiction, suppose $T^*_{\delta,\epsilon}$ is finite. Let $\xi_\infty \in M$ be a limit point of $\xi(t)$ as $t \nearrow T^*_{\delta,\epsilon}$; choose a normal coordinate neighborhood $U_\infty = B_{\xi_\infty}(d_\infty)$ of ξ_∞ in M (identified with the open set U_∞ in $\mathbb{R}^n$). Now take $\tau_{\delta,\epsilon} \in (0, T^*_{\delta,\epsilon})$ so that $\xi(\tau_{\delta,\epsilon}) \in U_\infty$. Our goal is to show that $\Psi(\cdot)$ extends continuously to $[\tau_{\delta,\epsilon}, T^*_{\delta,\epsilon}]$, taking values in $\mathcal{O}^{2+\alpha}_{\delta_2,\epsilon_2}(U_\infty)$. Then local existence and continuity (Lemma 3.4) contradict the maximality of $T^*_{\delta,\epsilon}$ (note that in particular we would have $\psi(T^*_{\delta,\epsilon}) \in h_0^{2+\alpha}$). Thus, we must show:

(i) $R(t)$ extends to a C^1 function $[\tau_{\delta,\epsilon}, T^*_{\delta,\epsilon}] \to (0, \delta_2)$.

(ii) $(\xi(t), e(t))$ extends continuously to a C^1 curve $[\tau_{\delta,\epsilon}, T^*_{\delta,\epsilon}] \to \mathbb{F}_g U_\infty$; (by standard results in differential geometry, it is enough to show this for $\xi(t)$).

(iii) $\psi(t)$ extends to a C^0 map $[\tau_{\delta,\epsilon}, T^*_{\delta,\epsilon}] \to \mathcal{O}^{2+\alpha}_{\delta_2,\epsilon_2}$.

We begin by re-scaling time, setting

$$\frac{ds}{dt} = \delta^{-2} R^{-2}(t).$$

Since $R(t)$ is uniformly bounded from above and from below, if we let $S^*_{\delta,\epsilon} < \infty$ correspond to $T^*_{\delta,\epsilon}$ (and $\sigma_{\delta,\epsilon}$ correspond to $\tau_{\delta,\epsilon}$), we are assuming $S^*_{\delta,\epsilon} < \infty$, and must show (i)-(iii) for $\Psi(t(s))$ and $[\sigma_{\delta,\epsilon}, S^*_{\delta,\epsilon}]$. We abuse notation by writing $\psi(s) = \psi(t(s))$,etc. Omitting the parallel transport equation for the frame from now on, we write the system in the s variable as:

$$\begin{aligned} R_s &= \rho(R(s), \xi(s), e(s), \psi(s)) \\ \xi_s &= \alpha(R(s), \xi(s), e(s), \psi(s)) \\ \psi_s &= A\psi + \phi(R(s), \xi(s), e(s), \psi(s)), \end{aligned}$$

where:

$$\begin{aligned} \rho &= \mathcal{C}_1[\psi]\delta^2\epsilon^2 R^3 + \mathcal{C}_2[\psi]\delta^3\epsilon R^4 + \mathrm{ave}_S[S_k\psi_k]\delta^2\epsilon R^3 + \delta R^2 B_R \\ \alpha &= c_n\nabla^M \mathrm{Scal}(\xi)\delta^4 R^4 - \mathrm{ave}_S[nL_1[\psi]e]\delta^3\epsilon R^3 - \mathrm{ave}_S[nL_2[\psi]e]\delta^4\epsilon R^4 \\ &\quad + \mathrm{ave}_S[nS_k\psi_k]\delta^4\epsilon R^4 + \delta^2 R^2 B_\xi \\ \phi &= \frac{1}{3}(\mathrm{Ric}_0)\delta^2 R^2\epsilon^{-1} + \frac{1}{3}(\mathrm{Ric}')\delta^3\epsilon^{-1}R^3 - S^k u^k \delta^3\epsilon^{-1}R^3 \\ &\quad - (L_1[\psi])_K\delta^2 R^2 - (L_2[\psi])_K\delta^3 R^3 + (S_k\psi_k)_K\delta^3 R^3 + \delta\epsilon^{-1}RB_\psi. \end{aligned}$$

The main step in showing (i)-(iii) consists of estimating:

$$Z_{\delta,\epsilon} := \sup_{s\in[0,S^*_{\delta,\epsilon})} \|\psi(s)\|_{C^{2+\alpha}}.$$

(This supremum may in principle be infinite.) Choosing local coordinates x on U_∞, with ξ_∞ corresponding to $x = 0$, we have the *representation formulas* for the solution:

$$\begin{aligned} R(s) &= R(0) + \int_0^s \rho(\sigma)d\sigma, \quad s \in [0, S^*_{\delta,\epsilon}); \\ x(s) &= x(\sigma_{\delta,\epsilon}) + \int_{\sigma_{\delta,\epsilon}}^s \alpha(\sigma)d\sigma, \text{ as long as } x(s) \in U_\infty; \\ \psi(s) &= e^{sA}\psi(0) + \int_0^s e^{(s-\sigma)A}\phi(\sigma)d\sigma, s \in [0, S^*_{\delta,\epsilon}). \end{aligned}$$

Claim 3.7. $\sup_{s\in[0,S^*_{\delta,\epsilon})}\|\phi\|_{C^\alpha} \le C(\delta,\epsilon)(1+Z_{\delta,\epsilon})$, *where* $C(\delta,\epsilon) \to 0$ *for* $\delta \to 0_+, \epsilon, \to 0_+$.

Assuming the claim, from Lemma 3.5(1) and (3) we obtain, for all $0 \le s < S^*_{\delta,\epsilon}$:

$$\begin{aligned} \|\psi(s)\|_{C^{2+\alpha}} &\le M_0 e^{-\mu s}\|\psi(0)\|_{C^{2+\alpha}} + M_0 \sup_{s\in 0,S^*_{\delta,\epsilon})} \|\phi(s)\|_{C^\alpha} \\ &\le M_0\|\psi(0)\|_{C^{2+\alpha}} + M_0 C(\delta,\epsilon)(1+Z_{\delta,\epsilon}). \end{aligned}$$

Thus:

$$Z_{\delta,\epsilon} \leq M_0 C(\delta,\epsilon)(Z_{\delta,\epsilon}+1) + M_0\|\psi(0)\|_{C^{2+\alpha}}.$$

If δ, ϵ are chosen small enough that $C(\delta,\epsilon) < 1$ and $C(\delta,\epsilon) < \epsilon_2/4M_0 < 1/2M_0$, we have:

$$Z_{\delta,\epsilon} \leq 2M_0\|\psi(0)\|_{C^{2+\alpha}} + \frac{\epsilon_2}{2}. \tag{3.5}$$

Thus $Z_{\delta,\epsilon} < \infty$. This and the bound on ϕ (from the claim) easily imply (iii). In particular, one sees immediately from the expressions for α, ρ, B_R and B_ξ (and still assuming $\epsilon^2 \ll \delta$) that:

$$\sup_{s\in(\sigma_{\delta,\epsilon}, S^*_{\delta,\epsilon})} (|\rho| + |\alpha|_{\mathbb{R}^n}) < c_{\delta,\epsilon}, \tag{3.5$'$}$$

where $c_{\delta,\epsilon}$ depends only on the manifold M, the constants M_0, R_{min} and $R_{\max}$, and on $\|\psi(0)\|_{C^{2+\alpha}}$. Thus we may choose $\widetilde{\sigma}_{\delta,\epsilon} \in (\sigma_{\delta,\epsilon}, S_{\delta,\epsilon})$ so that $x(\widetilde{\sigma}_{\delta,\epsilon}) < d_\infty/3$ and

$$|x(s) - x(\widetilde{\sigma}_{\delta,\epsilon})| = \left|\int_{\widetilde{\sigma}_{\delta,\epsilon}}^s \alpha(\sigma)d\sigma\right| < d_\infty/3,$$

as long as $x(s) \in U_\infty$. In particular $x(s) \in U_\infty$ for *all* $s \in [\widetilde{\sigma}_{\delta,\epsilon}, S^*_{\delta,\epsilon})$. This bound on $|x(s)|$, the bound $R_{min} < R(s) < R_{\max} < \delta_2$ and the estimates (3.5), (3.5′) easily imply (i) and (ii).

In addition, assume $\psi(0)$ satisfies:

$$2M_0\|\psi(0)\|_{C^{2+\alpha}} < \epsilon_2/2.$$

Then (reverting to the original time variable t) $\|\psi(T^*_{\delta,\epsilon})\|_{C^{2+\alpha}} < \epsilon_2$. By the local existence Lemma 3.4, this contradicts the maximality of $T^*_{\delta,\epsilon}$ (and also implies the last claim in the lemma).

Proof of Claim 3.7.

The claim follows directly from the estimate:

(3.6)

$$\begin{aligned}
\|\phi\|_{C^\alpha} &\leq c\delta^2\epsilon^{-1}R^2_{\max} + c\delta^3\epsilon^{-1}R^3_{\max} + c\|\psi\|_{C^{2+\alpha}}\delta^2 R^2_{\max} \\
&\quad + c\|\psi\|_{C^{2+\alpha}}\delta^3 R^3_{\max} \\
&\quad + c\delta^2\epsilon^{-1}R_{\max}[R^3_{\max}\delta^3(1+\epsilon\|\psi\|_{C^{2+\alpha}}) + \delta^{-1}R^{-1}_{\min}\epsilon^2(1+\|\psi\|_{C^{2+\alpha}})] \\
&\leq c\delta^2\epsilon^{-1}R^2_{\max} + c\delta^2 R^2_{\max}\|\psi\|_{C^{2+\alpha}} \\
&\quad + c\delta^4\epsilon^{-1}R^4_{\max} + c\delta^4 R^4_{\max}\|\psi\|_{C^{2+\alpha}} + c\delta\epsilon(1+\|\psi\|_{C^{2+\alpha}}) \\
&\leq c\delta^2\epsilon^{-1}R^2_{\max} + c\delta\epsilon + c(\delta^2R^2_{\max} + \delta\epsilon)\|\psi\|_{C^{2+\alpha}} \\
&\leq c(\delta,\epsilon)(1+\|\psi\|_{C^{2+\alpha}}) \leq c(\delta,\epsilon)(1+Z(\delta,\epsilon)),
\end{aligned}$$

assuming (on the last step) $\delta^2 \ll \epsilon$.

Remark. One can easily trace the origin of the condition $\delta^2 \ll \epsilon$ to the equation for ψ_t: it is needed so that the (euclidean) linear operator term $\frac{1}{\delta^2 R^2} A\psi$ dominates the largest Riemannian term $\frac{1}{3\epsilon}\mathrm{Ric}_0$.

Motion of the analytic barycenter.

Under the assumption $\delta^2 \ll \epsilon$, the largest terms in the equation for ξ_t are:

(a) the ψ-independent Riemannian term $c_n \nabla^M \mathrm{Scal}(\xi)\delta^2 R^2$;

(b) the Riemannian term linear in ψ: $-\mathrm{ave}_S[nL_1[\psi]e]\delta\epsilon R$;

(c) the Euclidean term $R^{-1}\|\psi\|_{C^1}\|\psi\|_{C^2}\delta^{-1}\epsilon^2$.

We may choose the relative scales so that the term (a) dominates the euclidean term (c) by imposing $\delta^{-1}\epsilon^2 \ll \delta^2$. This also implies (a) dominates (b). Thus we have the lemma (Part (iii) of the main theorem):

Lemma 3.8. *Assume the scale parameters ϵ and δ satisfy:*

$$\delta^2 \ll \epsilon \ll \delta^{3/2}.$$

Then the leading terms in the equations of motion for ξ and ψ are:

$$\begin{aligned}
\xi_t &= c_n \nabla^M \mathrm{Scal}(\xi) R^2 \delta^2 + O(\epsilon^2\delta^{-1}),\\
\psi_t &= \delta^{-2}R^{-2}A\psi + O(\epsilon^{-1}).
\end{aligned}$$

4. Asymptotic behavior

In this section we discuss the asymptotic behavior of a global solution of (0.1), and prove Part (iv) of the main theorem, asserting the convergence of solutions to parametrized hypersurfaces of constant mean curvature.

It is simpler to argue *subconvergence* of solutions, based on the fact that (0.4) is a gradient system on $\mathcal{M}_0$, satisfying (0.2):

$$\frac{dA(\Sigma(t))}{dt} = -\int_\Sigma (H - H^\Sigma)^2 d\sigma \le 0.$$

By parabolic regularity the solution curves of (0.4) are relatively compact in $C^{2+\alpha}(S)$ (cf. the last statement in Lemma 3.6), and so by the Invariance Principle ([7]) the $\omega-$limit set of a complete orbit is contained in the maximal invariant set of:

$$\Omega = \left\{(R,\xi,e,\psi) \in \mathcal{M}_0; H \equiv H^{\Sigma}, \Sigma = \mathrm{im}(X_{(R,\xi,e,\psi)})\right\}. \tag{4.1}$$

Thus, for any initial embedding X_0 for which a global solution exists, the $\omega-$limit set $\omega(X_0)$ is contained in the set of hypersurfaces (topological spheres) in M with constant mean curvature.

In the proof of convergence we will need a lemma contained in R. Ye's paper [14]. We need to point out a slight difference between Ye's parametrization of quasi-spherical hypersurfaces and ours. Fix $p \in M$ and a totally convex neighborhood $U_p \subset M$. For $\tau \in U_p, \chi \in C^2(S)$ and $0 < r < \delta_0$, consider the embedding:

$$Y_{(r,\tau,\chi)} : S \to M, \quad Y_{(r,\tau,\chi)}(u) = \exp_\tau[r(1-\chi)\bar{e}(u,\tau)],$$
$$\Sigma_{(r,\tau,\chi)} = \mathrm{im}(Y_{(r,\tau,\chi)}),$$

where the frame $\bar{e}(\cdot,\tau)$ is defined by radial parallel translation of a fixed frame at p. (In contrast with the notation in the present paper, it is *not* assumed that $\mathrm{ave}_S[\chi] = 0$.) To obtain a local foliation by constant mean curvature spheres, R. Ye considers the function:

$$\mathcal{H}(r,\tau,\chi) = r \times (\text{mean curvature of } \Sigma_{(r,\tau,\chi)}),$$

defined on an open subset of $(0,\delta_0) \times U_p \times \mathcal{K}^\perp$, taking values in $C^\alpha(S)$; here $\mathcal{K} = C_0^{2+\alpha} \subset C^{2+\alpha}(S)$ denotes the kernel of $A = \Delta + (n-1)$, and $\mathcal{K}^\perp$ its L^2 orthogonal complement.

Lemma 4.1 ([14]).

(i) *If $p \in M$ is a nondegenerate critical point of scalar curvature, there exists $\bar{\delta} > 0$ and smooth functions $(\tau,\chi) : [0,\bar{\delta}) \to U_p \times \mathcal{K}^\perp$ with $\tau(0) = p$, such that $\mathcal{H}(r,\tau(r),r^2\chi(r)) \equiv n-1$ for $0 \le r < \bar{\delta}$. The family $\mathcal{F} = (\Sigma_r)_{r\in(0,\bar{\delta})}, \Sigma_r = \Sigma_{(r,\tau(r),r^2\chi(r))}$ defines a foliation of a deleted neighborhood of p by hypersurfaces of constant mean curvature $(n-1)/r$.*

(ii) *Conversely, there is a neighborhood $V_p \subset U_p$ of p and $\hat{\delta} \in (0,\bar{\delta})$ so that if $\Sigma = \mathrm{im}(X), X = X_{(R,\xi,\psi)} \in \mathcal{N}_{\mathrm{std}}(\delta_1,\epsilon_1), \xi \in V_p$, is a hypersurface of constant mean curvature $(n-1)/r$ with $r \in (0,\hat{\delta})$, then $\Sigma = \Sigma_r$ is a leaf of the local foliation at p.*

Proof. R. Ye [14] uses the implicit function theorem to argue the existence of functions $\tau(r), \chi(r)$ such that $\mathcal{H}(r, \tau(r), r^2\chi(r)) \equiv n-1$. Conversely (as observed in the 'intermediate remark' on p. 389 and in the proof of Theorem 2.1 in [14]), there exist $\bar{\delta} > 0, V_p \subset U_p$ so that if, for some fixed $r_0 \in (0, \bar{\delta})$ and some $\xi \in V_p$ we have $\Sigma = \Sigma_{(r_0, \tau, r_0^2\chi)}$ with constant mean curvature $(n-1)/r_0$ contained in U_p, then $\chi = \chi(r_0)$ and $\xi = \tau(r_0)$, and Σ is a leaf of the local foliation at p.

Switching to our parametrization, we observe that if

$$\Sigma = \mathrm{im}(X_{(R,\xi,e,\psi)}) = \mathrm{im}(Y_{(r_0,\xi,r_0^2\chi)}),$$

we have: $R(1+\psi) = r_0(1+r_0^2\chi)$. To conclude (ii), we just have to observe that $r_0 \in (0, \bar{\delta})$ (defined as $(n-1)$ times the reciprocal of the mean curvature of Σ) satisfies $r_0 \to 0$ as $R \to 0$ (uniformly for ξ and χ varying in compact sets). This follows immediately from the expression $R = r_0(1 + r_0^2 \mathrm{ave}_S[\chi])$.

We now conclude the proof of Part (iv) of the main theorem

Lemma 4.2. *There exist constants $\delta_4 > 0, \epsilon_4 > 0$ depending only on M so that any limit point:*

$$\lim \Psi(t_i) = \Psi_* = (\delta R_*, \xi_*, e_*, \epsilon\psi_*), \quad t_i \to \infty,$$

of $(3.3)_{\delta,\epsilon}$ with $\epsilon < \epsilon_4, \delta < \delta_4, \epsilon^2 \ll \delta^3$ corresponds to $X_* \in \mathcal{N}_{\mathrm{std}}(\delta_4, \epsilon_4)$ parametrizing a hypersurface Σ_* of constant mean curvature, which is contained in some V_p as in Lemma 4.1 (*with p a critical point of* Scal) *and is a leaf of the local c.m.c. foliation at p.*

In particular, since the set of leaves of local c.m.c. foliations enclosing a given volume is finite (with at most one in each V_p, p a critical point of Scal), we must have *convergence* $\Psi(t) \to \Psi_*$, where Ψ_* corresponds to X_* parametrizing a c.m.c hypersurface Σ_* enclosing the same volume as $X(0)$ — as claimed in Part (iv) of the main theorem.

Proof. We first argue that $\xi_* \in \bigcup_p V_p$ if δ_4, ϵ_4 are small enough and $\epsilon_4^2 \ll \delta_4^3$. Otherwise we can find sequences $\delta_k \to 0, \epsilon_k \to 0$ with $\epsilon_k^2 \ll \delta_k^3$, and solutions $\Psi^k(t)$ of $(3.3)_{(\delta_k,\epsilon_k)}$ admitting a limit point $\Psi_*^k = (\delta_k R_*^k, \xi_*^k, e_*^k, \epsilon_k\psi_*^k)$ as $t \to \infty$, where $\xi_* \notin \bigcup_{p \in \mathrm{Crit(Scal)}} V_p$ for any $k \geq 1$.

The corresponding embedding X_*^k parametrizes a hypersurface of constant mean curvature- a stationary solution of (0.1). From (3.3) we see that $\mathrm{ave}_S[n(v_N - E)e] = 0$ (evaluated at $\Psi_* \in \mathcal{M}_0$.) From (3.4) (after division by δ_k^2), we obtain:

$$c_n\nabla^M \mathrm{Scal}(\xi_*^k) = \mathrm{ave}_S[nL_1[\psi_*^k]e_*^k]\frac{\epsilon_k}{\delta_k}R_*^k + \mathrm{ave}_S[nL_2[\psi_*^k]e_*^k]\epsilon_k(R_*^k)^2$$
$$+ \mathrm{ave}_S[nS_{*j}\psi_{*j}^k e_*^k]\epsilon_k(R_*^k)^2$$
$$+ (\delta_k)^{-2}B_\psi(\xi_*^k, e_*^k, \delta_k R_*^k, \epsilon_k\psi_*^k).$$

Since $\epsilon_k^2 \ll \delta_k^3$ (in particular also $\epsilon_k \ll \delta_k$) and $\|\psi_*^k\|_{C^{2+\alpha}}, |R_*^k|$ are bounded independently of k, we see that $\xi_*^k \notin \bigcup_{p\in\mathrm{Crit(Scal)}}$ would lead to a contradiction for k large enough.

In particular, for δ_4, ϵ_4 sufficiently small, given the estimate (3.5) we obtain $\xi \in V_p$ and $\Sigma_* \subset V_p$ for some $p \in \mathrm{Crit(Scal)}$. It then follows from Lemma 4.1.(ii) that Σ_* is a leaf of the local c.m.c. foliation at p.

Remark. We expect a stronger result linking the dynamics of (0.4) to the gradient flow of scalar curvature on M, namely that there exists an n-dimensional attracting invariant submanifold of $\mathcal{M}_0$ on which (3.3) is *conjugate* to the gradient flow of Scal via a time-preserving homeomorphism (for δ, ϵ small enough).

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