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SPECIAL LAGRANGIAN SUBMANIFOLDS WITH ISOLATED CONICAL SINGULARITIES. V. SURVEY AND APPLICATIONS

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Abstract

This is the last in a series of five papers studying compact *special Lagrangian submanifolds* (SL *m-folds*) X in (almost) Calabi-Yau m-folds M with *singularities* $x_1, \ldots, x_n$ locally modelled on special Lagrangian cones $C_1, \ldots, C_n$ in $\mathbb{C}^m$ with isolated singularities at 0. Readers are advised to begin with this paper.

We survey the major results of the previous four papers, giving brief explanations of the proofs. We apply the results to describe the *boundary* of a moduli space of compact, nonsingular SL m-folds N in M. We prove the existence of special Lagrangian *connected sums* $N_1 \# \cdots \# N_k$ of SL m-folds $N_1, \ldots, N_k$ in M. We also study SL 3-folds with T^2-cone singularities, proving results related to ideas of the author on invariants of Calabi-Yau 3-folds, and the SYZ Conjecture.

Let X be a compact SL m-fold with isolated conical singularities x_i and cones C_i for $i = 1, \ldots, n$. The first paper studied the *regularity* of X near its singular points, and the the second the *moduli space of deformations* of X. The third and fourth papers construct *desingularizations* of X, realizing X as a limit of a family of compact, nonsingular SL m-folds N^t in M for small $t > 0$. Let L_i be an *asymptotically conical* SL m-fold in $\mathbb{C}^m$ asymptotic to C_i at infinity. We make N^t by gluing tL_i into X at x_i for $i = 1, \ldots, n$.

1. Introduction

Special Lagrangian m-folds (SL *m-folds*) are a distinguished class of real m-dimensional minimal submanifolds which may be defined in $\mathbb{C}^m$, or in *Calabi-Yau m-folds*, or more generally in *almost Calabi-Yau m-folds* (compact Kähler m-folds with trivial canonical bundle). We write an almost Calabi-Yau m-fold as M or (M, J, ω, Ω), where the manifold

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M has complex structure J, Kähler form ω and holomorphic volume form Ω.

This is the fifth in a series of five papers [17, 18, 19, 20] studying SL m-folds with *isolated conical singularities*. That is, we consider an SL m-fold X in an almost Calabi-Yau m-fold M for $m > 2$ with singularities at $x_1, \ldots, x_n$ in M, such that for some special Lagrangian cones C_i in $T_{x_i}M \cong \mathbb{C}^m$ with $C_i \setminus \{0\}$ nonsingular, X approaches C_i near x_i, in an asymptotic C^1 sense.

New readers of the series are advised to begin with this paper. We shall survey the major results of [17, 18, 19, 20], giving explanations, but avoiding the long, technical analytic proofs of previous papers. We also integrate the results to give an (incomplete) description of the *boundary* of a moduli space of compact SL m-folds, and apply them to prove some conjectures in [9, 13] on *connected sums* of SL m-folds, and T^2-*cone singularities* of SL 3-folds.

Having a good understanding of the singularities of special Lagrangian submanifolds will be essential in clarifying the Strominger-Yau–Zaslow conjecture on the mirror symmetry of Calabi-Yau 3-folds [29], and also in resolving conjectures made by the author [9] on defining new invariants of Calabi-Yau 3-folds by counting special Lagrangian homology 3-spheres with weights. The series aims to develop such an understanding for simple singularities of SL m-folds.

We begin in §2 with an introduction to almost Calabi-Yau and special Lagrangian geometry, and the *deformation theory* of compact SL m-folds. Section 3 defines SL *m-folds with conical singularities*, our subject, gives examples of *special Lagrangian cones*, and some basics on homology and cohomology.

Section 4 describes the first paper [17] on the *regularity* of SL m-folds X with conical singularities $x_1, \ldots, x_n$. We study the asymptotic behaviour of X and its derivatives near x_i, how quickly it converges to the cone C_i.

In §5 we discuss the second paper [18] on the *deformation theory* of compact SL m-folds X with conical singularities in an almost Calabi-Yau m-fold M. We find that the *moduli space* $\mathcal{M}_X$ of deformations of X in M is locally homeomorphic to the zeroes of a smooth map $\Phi : \mathcal{I}_{X'} \to \mathcal{O}_{X'}$ between finite-dimensional vector spaces, and if the *obstruction space* $\mathcal{O}_{X'}$ is zero then $\mathcal{M}_X$ is a smooth manifold. We also study deformations in *smooth families* of almost Calabi-Yau m-folds $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F} \subset \mathbb{R}^d$.

Section 6 is an aside on *asymptotically conical* SL *m-folds* (AC SL

m-folds) in $\mathbb{C}^m$, that is, nonsingular, noncompact SL m-folds L in $\mathbb{C}^m$ which are asymptotic at infinity to an SL cone C at a prescribed *rate* λ. Our main sources are [17, §7] and Marshall [23]. The theories of AC SL m-folds and SL m-folds with conical singularities are similar in many respects.

Section 7 explains the third and fourth papers [19, 20] on *desingularizations* of a compact SL m-fold X with conical singularities x_i with cones C_i for $i = 1, \ldots, n$ in an almost Calabi-Yau m-fold M. We take AC SL m-folds L_i in $\mathbb{C}^m$ asymptotic to C_i at infinity, and glue tL_i into X at x_i for small $t > 0$ to get a smooth family of compact, *nonsingular* SL m-folds $\widetilde{N}^t$ in M, with $\widetilde{N}^t \to X$ as $t \to 0$. We also study desingularizations in *families* of almost Calabi-Yau m-folds $(M, J^s, \omega^s, \Omega^s)$ for $s \in \mathcal{F}$.

The new material of the paper is §8–§10. We study the moduli space $\mathcal{M}_N$ of compact, nonsingular SL m-folds N in §2, the moduli space $\mathcal{M}_X$ of compact SL m-folds X with conical singularities in §5, and the moduli space $\mathcal{M}_L^\lambda$ of AC SL m-folds in $\mathbb{C}^m$ with rate λ in §6. Section 8 explains how these three kinds of moduli space fit together.

The idea is that $\mathcal{M}_N$ has a *compactification* $\overline{\mathcal{M}}_N$ with *boundary* $\partial\mathcal{M}_N = \overline{\mathcal{M}}_N \setminus \mathcal{M}_N$ consisting of singular SL m-folds. Suppose N is constructed as in §7 by desingularizing X with conical singularities $x_1, \ldots, x_n$ by gluing in AC SL m-folds $L_1, \ldots, L_n$. Then in good cases we expect $\mathcal{M}_X \subseteq \partial\mathcal{M}_N$, and $\mathcal{M}_N$ may be locally modelled on a subset of $\mathcal{M}_X \times \mathcal{M}_{L_1}^0 \times \cdots \times \mathcal{M}_{L_n}^0$ near X.

Section 9 considers *connected sums* of SL m-folds. Suppose X is a compact, *immersed* SL m-fold in M with *transverse self-intersection points* $x_1, \ldots, x_n$. This includes the case where X is a union of $q > 1$ embedded SL m-folds $X_1, \ldots, X_q$, and the x_i are intersections between X_j and X_k. Then we can consider X to be an *SL m-fold with conical singularities*, with each cone C_i the union of two transverse *SL m-planes* $\Pi_i^+, \Pi_i^- \cong \mathbb{R}^m$ in $\mathbb{C}^m$.

When $\Pi_i^\pm$ satisfy an *angle criterion*, Lawlor [21] constructed a family of AC SL m-folds $L_i^{\pm,A}$ for $A > 0$ with cone $\Pi_i^+ \cup \Pi_i^-$, diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}$. We apply the results of §7 to construct SL m-folds $\widetilde{N}^t$ by gluing $tL_i^{\pm,A}$ into X at x_i. These $\widetilde{N}^t$ are *multiple connected sums* of X with itself. In this way we re-prove and extend a result of Lee [8].

Finally, §10 studies SL 3-folds X with conical singularities with cone C the $\mathrm{U}(1)^2$-invariant SL T^2-cone due to Harvey and Lawson [5, §III.3.A]. These have particularly nice properties. For instance, the moduli space $\mathcal{M}_X$ is always smooth, and under topological conditions the compactified moduli space $\overline{\mathcal{M}}_N$ of desingularizations N is near X a nonsingular manifold with boundary $\mathcal{M}_X$. We prove several conjectures from [9, 13].

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2. Special Lagrangian geometry

We begin with some background from symplectic geometry. Then special Lagrangian submanifolds (SL m-folds) are introduced both in $\mathbb{C}^m$ and in *almost Calabi-Yau m-folds*. We also describe the *deformation theory* of compact SL m-folds. Some references for this section are McDuff and Salamon [24], Harvey and Lawson [5], McLean [26], and the author [16].

2.1 Background from symplectic geometry

We start by recalling some elementary symplectic geometry, which can be found in McDuff and Salamon [24]. Here are the basic definitions.

Definition 2.1. Let M be a smooth manifold of even dimension $2m$. A closed 2-form ω on M is called a *symplectic form* if the $2m$-form ω^m is nonzero at every point of M. Then (M, ω) is called a *symplectic manifold*. A submanifold N in M is called *Lagrangian* if $\dim N = m = \frac{1}{2} \dim M$ and $\omega|_N \equiv 0$.

The simplest example of a symplectic manifold is $\mathbb{R}^{2m}$.

Definition 2.2. Let $\mathbb{R}^{2m}$ have coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_m)$, and define the standard metric g' and symplectic form ω' on $\mathbb{R}^{2m}$ by

$$g' = \sum_{j=1}^{m} (\mathrm{d}x_j^2 + \mathrm{d}y_j^2) \quad \text{and} \quad \omega' = \sum_{j=1}^{m} \mathrm{d}x_j \wedge \mathrm{d}y_j. \tag{1}$$

Then $(\mathbb{R}^{2m}, \omega')$ is a symplectic manifold. When we wish to identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$, we take the complex coordinates $(z_1, \ldots, z_m)$ on $\mathbb{C}^m$ to be $z_j = x_j + iy_j$. For $R > 0$, define B_R to be the open ball of radius R about 0 in $\mathbb{R}^{2m}$.

Darboux's Theorem [24, Th. 3.15] says that every symplectic manifold is locally isomorphic to $(\mathbb{R}^{2m}, \omega')$. Our version easily follows.

Theorem 2.3. *Let* (M,ω) *be a symplectic* $2m$*-manifold and* $x \in M$*. Then there exist* $R > 0$ *and an embedding* $\Upsilon : B_R \to M$ *with* $\Upsilon(0) = x$ *such that* $\Upsilon^*(\omega) = \omega'$*, where* ω' *is the standard symplectic form on* $\mathbb{R}^{2m} \supset B_R$*. Given an isomorphism* $\upsilon : \mathbb{R}^{2m} \to T_xM$ *with* $\upsilon^*(\omega|_x)=\omega'$*, we can choose* Υ *with* $\mathrm{d}\Upsilon|_0=\upsilon$*.*

Let N be a real m-manifold. Then its tangent bundle T^*N has a *canonical symplectic form* $\hat\omega$, defined as follows. Let $(x_1,\ldots,x_m)$ be local coordinates on N. Extend them to local coordinates $(x_1,\ldots,x_m, y_1,\ldots,y_m)$ on T^*N such that $(x_1,...,y_m)$ represents the 1-form $y_1\mathrm{d}x_1 + \cdots + y_m\mathrm{d}x_m$ in $T^*_{(x_1,...,x_m)}N$. Then $\hat\omega = \mathrm{d}x_1\wedge\mathrm{d}y_1 + \cdots + \mathrm{d}x_m\wedge\mathrm{d}y_m$.

Identify N with the zero section in T^*N. Then N is a *Lagrangian submanifold* of T^*N. The *Lagrangian Neighbourhood Theorem* [24, Th. 3.33] shows that any compact Lagrangian submanifold N in a symplectic manifold looks locally like the zero section in T^*N.

Theorem 2.4. *Let* (M,ω) *be a symplectic manifold and* $N \subset M$ *a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood* U *of the zero section* N *in* T^*N*, and an embedding* $\Phi : U \to M$ *with* $\Phi|_N = \mathrm{id} : N \to N$ *and* $\Phi^*(\omega) = \hat\omega$*, where* $\hat\omega$ *is the canonical symplectic structure on* T^*N*.*

We shall call U, Φ a *Lagrangian neighbourhood* of N. Such neighbourhoods are useful for parametrizing nearby Lagrangian submanifolds of M. Suppose that $\tilde N$ is a Lagrangian submanifold of M which is C^1-close to N. Then $\tilde N$ lies in $\Phi(U)$, and is the image $\Phi\bigl(\Gamma(\alpha)\bigr)$ of the graph $\Gamma(\alpha)$ of a unique C^1-small 1-form α on N.

As $\tilde N$ is Lagrangian and $\Phi^*(\omega) = \hat\omega$ we see that $\hat\omega|_{\Gamma(\alpha)} \equiv 0$. But one can easily show that $\hat\omega|_{\Gamma(\alpha)} = -\pi^*(\mathrm{d}\alpha)$, where $\pi : \Gamma(\alpha) \to N$ is the natural projection. Hence $\mathrm{d}\alpha = 0$, and α is a *closed* 1*-form.* This establishes a 1-1 correspondence between small closed 1-forms on N and Lagrangian submanifolds $\tilde N$ close to N in M, which is an essential tool in proving the results of §5 and §7.

2.2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We define *calibrations* and *calibrated submanifolds*, following [5].

Definition 2.5. Let (M, g) be a Riemannian manifold. An *oriented tangent* k*-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [5, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [5, §III].

Definition 2.6. Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g', a real 2-form ω' and a complex m-form Ω' on $\mathbb{C}^m$ by

$$\begin{aligned} g' &= |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \\ \omega' &= \tfrac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \qquad \text{and} \\ \Omega' &= \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{aligned} \tag{2}$$

Then g', ω' are as in Definition 2.2, and $\mathrm{Re}\,\Omega'$ and $\mathrm{Im}\,\Omega'$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if L is calibrated with respect to $\mathrm{Re}\,\Omega'$, in the sense of Definition 2.5.

Harvey and Lawson [5, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds:

Proposition 2.7. *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if $\omega'|_L \equiv 0$ and $\mathrm{Im}\,\Omega'|_L \equiv 0$.*

Thus special Lagrangian submanifolds are *Lagrangian* submanifolds satisfying the extra condition that $\mathrm{Im}\,\Omega'|_L \equiv 0$, which is how they get their name.

2.3 Almost Calabi-Yau m-folds and SL m-folds

We shall define special Lagrangian submanifolds not just in Calabi-Yau manifolds, as usual, but in the much larger class of *almost Calabi-Yau manifolds*.

Definition 2.8. Let $m \geqslant 2$. An *almost Calabi-Yau m-fold* is a quadruple (M, J, ω, Ω) such that (M, J) is a compact m-dimensional complex manifold, ω is the Kähler form of a Kähler metric g on M, and Ω is a non-vanishing holomorphic $(m, 0)$-form on M.

We call (M, J, ω, Ω) a *Calabi-Yau m-fold* if in addition ω and Ω satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}. \tag{3}$$

Then for each $x \in M$ there exists an isomorphism $T_xM \cong \mathbb{C}^m$ that identifies g_x, ω_x and Ω_x with the flat versions g', ω', Ω' on $\mathbb{C}^m$ in (2). Furthermore, g is Ricci-flat and its holonomy group is a subgroup of $\mathrm{SU}(m)$.

This is not the usual definition of a Calabi-Yau manifold, but is essentially equivalent to it.

Definition 2.9. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold, and N a real m-dimensional submanifold of M. We call N a *special Lagrangian submanifold*, or *SL m-fold* for short, if $\omega|_N \equiv \operatorname{Im} \Omega|_N \equiv 0$. It easily follows that $\operatorname{Re} \Omega|_N$ is a nonvanishing m-form on N. Thus N is orientable, with a unique orientation in which $\operatorname{Re} \Omega|_N$ is positive.

Again, this is not the usual definition of SL m-fold, but is essentially equivalent to it. In Definition 9.9 we give a more general definition of SL m-fold involving a *phase* $\mathrm{e}^{i\theta}$. Suppose (M, J, ω, Ω) is an almost Calabi-Yau m-fold, with metric g. Let $\psi : M \to (0, \infty)$ be the unique smooth function such that

$$\psi^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}, \tag{4}$$

and define $\widetilde{g}$ to be the conformally equivalent metric $\psi^2 g$ on M. Then $\operatorname{Re} \Omega$ is a *calibration* on the Riemannian manifold $(M, \widetilde{g})$, and SL m-folds N in (M, J, ω, Ω) are calibrated with respect to it, so that they are minimal with respect to $\widetilde{g}$.

If M is a Calabi-Yau m-fold then $\psi \equiv 1$ by (3), so $\widetilde{g} = g$, and an m-submanifold N in M is special Lagrangian if and only if it is calibrated w.r.t. $\operatorname{Re} \Omega$ on (M, g), as in Definition 2.6. This recovers the usual definition of special Lagrangian m-folds in Calabi-Yau m-folds.

2.4 Deformations of compact SL m-folds

The *deformation theory* of special Lagrangian submanifolds was studied by McLean [26, §3], who proved the following result in the Calabi-Yau case. The extension to the almost Calabi-Yau case is described in [16, §9.5].

Theorem 2.10. *Let N be a compact SL m-fold in an almost Calabi-Yau m-fold (M, J, ω, Ω). Then the moduli space $\mathcal{M}_N$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.*

Here is a sketch of the proof of Theorem 2.10. Let g be the Kähler metric on M, and define $\psi : M \to (0,\infty)$ by (4). Applying Theorem 2.4 gives an open neighbourhood U of N in T^*N and an embedding $\Phi : U \to M$. Let $\pi : U \to N$ be the natural projection. Define an m-form β on U by $\beta = \Phi^*(\operatorname{Im}\Omega)$. If α is a 1-form on N let $\Gamma(\alpha)$ be the graph of α in T^*N, and write $C^\infty(U) \subset C^\infty(T^*N)$ for the subset of 1-forms whose graphs lie in U.

Then each submanifold $\widetilde{N}$ of M which is C^1-close to N is $\Phi\bigl(\Gamma(\alpha)\bigr)$ for some small $\alpha \in C^\infty(U)$. Here is the condition for $\widetilde{N}$ to be special Lagrangian.

Lemma 2.11. *In the situation above, if $\alpha \in C^\infty(U)$ then $\widetilde{N} = \Phi\bigl(\Gamma(\alpha)\bigr)$ is an* SL *m-fold in M if and only if $\mathrm{d}\alpha = 0$ and $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = 0$.*

Proof. By Definition 2.9, $\widetilde{N}$ is an SL m-fold if and only if $\omega|_{\widetilde{N}} \equiv \operatorname{Im}\Omega|_{\widetilde{N}} \equiv 0$. Pulling back by Φ and pushing forward by $\pi : \Gamma(\alpha) \to N$, we see that $\widetilde{N}$ is special Lagrangian if and only if $\pi_*\bigl(\hat{\omega}|_{\Gamma(\alpha)}\bigr) \equiv \pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) \equiv 0$, since $\Phi^*(\omega) = \hat{\omega}$ and $\Phi^*(\operatorname{Im}\Omega) = \beta$. But $\pi_*\bigl(\hat{\omega}|_{\Gamma(\alpha)}\bigr) = -\mathrm{d}\alpha$, and the lemma follows. q.e.d.

We rewrite the condition $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = 0$ in terms of a function F.

Definition 2.12. Define $F : C^\infty(U) \to C^\infty(N)$ by $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = F(\alpha)\,\mathrm{d}V_g$, where $\mathrm{d}V_g$ is the volume form of $g|_N$ on N. Then Lemma 2.11 shows that if $\alpha \in C^\infty(U)$ then $\Phi(\Gamma(\alpha))$ is special Lagrangian if and only if $\mathrm{d}\alpha = F(\alpha) = 0$.

In [18, Prop. 2.10] we compute the expansion of F up to first order in α.

Proposition 2.13. *This function F may be written*

$$F(\alpha)[x] = -\mathrm{d}^*\bigl(\psi^m\alpha\bigr) + Q\bigl(x, \alpha(x), \nabla\alpha(x)\bigr) \quad \textit{for } x \in N, \tag{5}$$

where $Q : \bigl\{(x,y,z) : x \in N,\ y \in T_x^*N \cap U,\ z \in \otimes^2 T_x^*N\bigr\} \to \mathbb{R}$ is smooth and $Q(x,y,z) = O(|y|^2 + |z|^2)$ for small y, z.

From Definition 2.12 and Proposition 2.13 we see that the moduli space $\mathcal{M}_N$ of special Lagrangian deformations of N is locally approximately isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}^*(\psi^m\alpha) = 0$. But by Hodge theory, this is isomorphic to the de Rham cohomology group $H^1(N,\mathbb{R})$, and is a manifold with dimension $b^1(N)$.

To carry out this last step rigorously requires some technical machinery: one must work with certain *Banach spaces* of sections of $\Lambda^k T^*N$ for $k = 0,1,2$, use *elliptic regularity results* to prove that the map $\alpha \mapsto \bigl(\mathrm{d}\alpha, \mathrm{d}F|_0(\alpha)\bigr)$ is *surjective* upon the appropriate Banach spaces, and then use the *Implicit Mapping Theorem for Banach spaces* to show that the kernel of the map is what we expect. This concludes our sketch of the proof of Theorem 2.10.

We extend Theorem 2.10 to *families* of almost Calabi-Yau m-folds.

Definition 2.14. Let (M,J,ω,Ω) be an almost Calabi-Yau m-fold. A *smooth family of deformations of* (M,J,ω,Ω) is a connected open set $\mathcal{F} \subset \mathbb{R}^d$ for $d \geqslant 0$ with $0 \in \mathcal{F}$ called the *base space*, and a smooth family $\bigl\{(M,J^s,\omega^s,\Omega^s) : s \in \mathcal{F}\bigr\}$ of almost Calabi-Yau structures with $(J^0,\omega^0,\Omega^0) = (J,\omega,\Omega)$.

If N is a compact SL m-fold in (M,J,ω,Ω), the moduli of deformations of N in each $(M,J^s,\omega^s,\Omega^s)$ for $s \in \mathcal{F}$ make up a big moduli space $\mathcal{M}_N^{\mathcal{F}}$.

Definition 2.15. Let $\bigl\{(M,J^s,\omega^s,\Omega^s) : s \in \mathcal{F}\bigr\}$ be a smooth family of deformations of an almost Calabi-Yau m-fold (M,J,ω,Ω), and N be a compact SL m-fold in (M,J,ω,Ω). Define the *moduli space* $\mathcal{M}_N^{\mathcal{F}}$ *of deformations of* N *in the family* $\mathcal{F}$ to be the set of pairs $(s,\hat{N})$ for which $s \in \mathcal{F}$ and $\hat{N}$ is a compact SL m-fold in $(M,J^s,\omega^s,\Omega^s)$ which is diffeomorphic to N and isotopic to N in M. Define a *projection* $\pi^{\mathcal{F}} : \mathcal{M}_N^{\mathcal{F}} \to \mathcal{F}$ by $\pi^{\mathcal{F}}(s,\hat{N}) = s$. Then $\mathcal{M}_N^{\mathcal{F}}$ has a natural topology, and $\pi^{\mathcal{F}}$ is continuous.

The following result is proved by Marshall [23, Th. 3.2.9], using similar methods to Theorem 2.10.

Theorem 2.16. *Let* $\bigl\{(M,J^s,\omega^s,\Omega^s) : s \in \mathcal{F}\bigr\}$ *be a smooth family of deformations of an almost Calabi-Yau m-fold* (M,J,ω,Ω)*, with base space* $\mathcal{F} \subset \mathbb{R}^d$*. Suppose N is a compact* SL *m-fold in* (M,J,ω,Ω) *with* $[\omega^s|_N] = 0$ *in* $H^2(N,\mathbb{R})$ *and* $[\operatorname{Im}\Omega^s|_N] = 0$ *in* $H^m(N,\mathbb{R})$ *for all* $s \in \mathcal{F}$*. Let* $\mathcal{M}_N^{\mathcal{F}}$ *be the moduli space of deformations of N in* $\mathcal{F}$*, and* $\pi^{\mathcal{F}} : \mathcal{M}_N^{\mathcal{F}} \to \mathcal{F}$ *the natural projection.*

Then $\mathcal{M}_N^{\mathcal{F}}$ *is a smooth manifold of dimension* $d + b^1(N)$*, and* $\pi^{\mathcal{F}}$:

$\mathcal{M}_N^{\mathcal{F}} \to \mathcal{F}$ a smooth submersion. For small $s \in \mathcal{F}$ the moduli space $\mathcal{M}_N^s = (\pi^{\mathcal{F}})^{-1}(s)$ of deformations of N in $(M, J^s, \omega^s, \Omega^s)$ is a nonempty smooth manifold of dimension $b^1(N)$, with $\mathcal{M}_N^0 = \mathcal{M}_N$.

Here a necessary condition for the existence of an SL m-fold $\hat{N}$ isotopic to N in $(M, J^s, \omega^s, \Omega^s)$ is that $[\omega^s|_N] = [\operatorname{Im}\Omega^s|_N] = 0$ in $H^*(N, \mathbb{R})$, since $[\omega^s|_N]$ and $[\omega^s|_{\hat{N}}]$ are identified under the natural isomorphism between $H^2(N, \mathbb{R})$ and $H^2(\hat{N}, \mathbb{R})$, and similarly for $\operatorname{Im}\Omega^s$.

The point of the theorem is that these conditions $[\omega^s|_N] = [\operatorname{Im}\Omega^s|_N] = 0$ are also *sufficient* for the existence of $\hat{N}$ when s is close to 0 in $\mathcal{F}$. That is, the only *obstructions* to existence of compact SL m-folds when we deform the underlying almost Calabi-Yau m-fold are the obvious cohomological ones.

3. SL cones and conical singularities

We begin in §3.1 with some definitions on *special Lagrangian cones*. Section 3.2 gives *examples* of SL cones, and §3.3 defines SL *m-folds with conical singularities*, the subject of the paper. Section 3.4 discusses *homology* and *cohomology* of SL m-folds with conical singularities.

3.1 Preliminaries on special Lagrangian cones

We define *special Lagrangian cones*, and some notation.

Definition 3.1. A (singular) SL m-fold C in $\mathbb{C}^m$ is called a *cone* if $C = tC$ for all $t > 0$, where $tC = \{t\,\mathbf{x} : \mathbf{x} \in C\}$. Let C be a closed SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a compact, nonsingular $(m-1)$-submanifold of $\mathcal{S}^{2m-1}$, not necessarily connected. Let g_Σ be the restriction of g' to Σ, where g' is as in (2).

Set $C' = C \setminus \{0\}$. Define $\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then ι has image C'. By an abuse of notation, *identify* C' with $\Sigma \times (0, \infty)$ using ι. The *cone metric* on $C' \cong \Sigma \times (0, \infty)$ is $g' = \iota^*(g') = \mathrm{d}r^2 + r^2 g_\Sigma$.

For $\alpha \in \mathbb{R}$, we say that a function $u : C' \to \mathbb{R}$ is *homogeneous of order* α if $u \circ t \equiv t^\alpha u$ for all $t > 0$. Equivalently, u is homogeneous of order α if $u(\sigma, r) \equiv r^\alpha v(\sigma)$ for some function $v : \Sigma \to \mathbb{R}$.

In [17, Lem. 2.3] we study *homogeneous harmonic functions* on C'.

Lemma 3.2. *In the situation of Definition* 3.1, *let $u(\sigma, r) \equiv r^\alpha v(\sigma)$ be a homogeneous function of order α on $C' = \Sigma \times (0, \infty)$, for $v \in C^2(\Sigma)$.*

Then

$$\Delta u(\sigma,r) = r^{\alpha-2}\big(\Delta_\Sigma v - \alpha(\alpha+m-2)v\big), \tag{6}$$

where Δ, Δ_Σ are the Laplacians on (C',g') and (Σ,g_Σ). Hence, u is harmonic on C' if and only if $\Delta_\Sigma v = \alpha(\alpha+m-2)v$.

Following [17, Def. 2.5], we define:

Definition 3.3. In the situation of Definition 3.1, suppose $m > 2$ and define

$$\mathcal{D}_\Sigma = \big\{\alpha \in \mathbb{R} : \alpha(\alpha+m-2) \text{ is an eigenvalue of } \Delta_\Sigma\big\}. \tag{7}$$

Then $\mathcal{D}_\Sigma$ is a countable, discrete subset of $\mathbb{R}$. By Lemma 3.2, an equivalent definition is that $\mathcal{D}_\Sigma$ is the set of $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic function u of order α on C'.

Define $m_\Sigma : \mathcal{D}_\Sigma \to \mathbb{N}$ by taking $m_\Sigma(\alpha)$ to be the multiplicity of the eigenvalue $\alpha(\alpha+m-2)$ of Δ_Σ, or equivalently the dimension of the vector space of homogeneous harmonic functions u of order α on C'. Define $N_\Sigma : \mathbb{R} \to \mathbb{Z}$ by

$$N_\Sigma(\delta) = -\sum_{\alpha\in\mathcal{D}_\Sigma\cap(\delta,0)} m_\Sigma(\alpha) \quad \text{if } \delta < 0, \quad \text{and}$$
$$N_\Sigma(\delta) = \sum_{\alpha\in\mathcal{D}_\Sigma\cap[0,\delta]} m_\Sigma(\alpha) \quad \text{if } \delta \geqslant 0.$$

Then N_Σ is monotone increasing and upper semicontinuous, and is discontinuous exactly on $\mathcal{D}_\Sigma$, increasing by $m_\Sigma(\alpha)$ at each $\alpha \in \mathcal{D}_\Sigma$. As the eigenvalues of Δ_Σ are nonnegative, we see that $\mathcal{D}_\Sigma \cap (2-m,0) = \emptyset$ and $N_\Sigma \equiv 0$ on $(2-m,0)$.

We define the *stability index* of C, and *stable* and *rigid* cones [18, Def. 3.6].

Definition 3.4. Let C be an SL cone in $\mathbb{C}^m$ for $m > 2$ with an isolated singularity at 0, let G be the Lie subgroup of $\mathrm{SU}(m)$ preserving C, and use the notation of Definitions 3.1 and 3.3. Then [18, eq. (8)] shows that

$$m_\Sigma(0) = b^0(\Sigma), \quad m_\Sigma(1) \geqslant 2m \quad \text{and} \quad m_\Sigma(2) \geqslant m^2 - 1 - \dim G. \tag{8}$$

Define the *stability index* s-ind(C) to be

$$\text{s-ind}(C) = N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \dim G. \tag{9}$$

Then s-ind$(C) \geqslant 0$ by (8), as $N_\Sigma(2) \geqslant m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$ by definition. We call C *stable* if s-ind$(C) = 0$.

Following [17, Def. 6.7], we call C *rigid* if $m_\Sigma(2) = m^2 - 1 - \dim G$. As

$$\text{s-ind}(C) \geqslant m_\Sigma(2) - (m^2 - 1 - \dim G) \geqslant 0,$$

we see that *if C is stable, then C is rigid.*

We shall see in §5 that s-ind(C) is the dimension of an obstruction space to deforming an SL m-fold X with a conical singularity with cone C, and that if C is *stable* then the deformation theory of X simplifies. An SL cone C is *rigid* if all infinitesimal deformations of C as an SL cone come from $\mathrm{SU}(m)$ rotations of C. This will be useful in the geometric measure theory material of §4.2.

3.2 Examples of special Lagrangian cones

In our first example we can compute the data of §3.1 very explicitly.

Example 3.5. Here is a family of special Lagrangian cones constructed by Harvey and Lawson [5, §III.3.A]. For $m \geqslant 3$, define

$$C^m_{\mathrm{HL}} = \bigl\{(z_1, \dots, z_m) \in \mathbb{C}^m : i^{m+1} z_1 \cdots z_m \in [0, \infty),\ |z_1| = \cdots = |z_m|\bigr\}.$$

Then C^m_{HL} is a *special Lagrangian cone* in $\mathbb{C}^m$ with an isolated singularity at 0, and $\Sigma^m_{\mathrm{HL}} = C^m_{\mathrm{HL}} \cap \mathcal{S}^{2m-1}$ is an $(m-1)$-torus T^{m-1}. Both C^m_{HL} and Σ^m_{HL} are invariant under the $\mathrm{U}(1)^{m-1}$ subgroup of $\mathrm{SU}(m)$ acting by

(10)
$$(z_1, \dots, z_m) \mapsto (\mathrm{e}^{i\theta_1} z_1, \dots, \mathrm{e}^{i\theta_m} z_m) \quad \text{for } \theta_j \in \mathbb{R} \text{ with } \theta_1 + \cdots + \theta_m = 0.$$

In fact $\pm C^m_{\mathrm{HL}}$ for m odd, and $C^m_{\mathrm{HL}}, iC^m_{\mathrm{HL}}$ for m even, are the unique SL cones in $\mathbb{C}^m$ invariant under (10), which is how Harvey and Lawson constructed them.

The metric on $\Sigma^m_{\mathrm{HL}} \cong T^{m-1}$ is flat, so it is easy to compute the eigenvalues of $\Delta_{\Sigma^m_{\mathrm{HL}}}$. This was done by Marshall [23, §6.3.4]. There is a 1-1 correspondence between $(n_1, \dots, n_{m-1}) \in \mathbb{Z}^{m-1}$ and eigenvectors of $\Delta_{\Sigma^m_{\mathrm{HL}}}$ with eigenvalue

$$m \sum_{i=1}^{m-1} n_i^2 - \sum_{i,j=1}^{m-1} n_i n_j. \tag{11}$$

Using (11) and a computer we can find the eigenvalues of $\Delta_{\Sigma^m_{\mathrm{HL}}}$ and their multiplicities. The Lie subgroup G^m_{HL} of $\mathrm{SU}(m)$ preserving C^m_{HL} has identity component the $\mathrm{U}(1)^{m-1}$ of (10), so that $\dim G^m_{\mathrm{HL}} = m-1$. Thus we can calculate $\mathcal{D}_{\Sigma^m_{\mathrm{HL}}}$, $m_{\Sigma^m_{\mathrm{HL}}}$, $N_{\Sigma^m_{\mathrm{HL}}}$, and the stability index s-ind(C^m_{HL}). This was done by Marshall [23, Table 6.1] and the author [18, §3.2]. Table 1 gives the data $m, N_{\Sigma^m_{\mathrm{HL}}}(2), m_{\Sigma^m_{\mathrm{HL}}}(2)$ and s-ind(C^m_{HL}) for $3 \leqslant m \leqslant 12$.

m	3	4	5	6	7	8	9	10	11	12
$N_{\Sigma^m_{\mathrm{HL}}}(2)$	13	27	51	93	169	311	331	201	243	289
$m_{\Sigma^m_{\mathrm{HL}}}(2)$	6	12	20	30	42	126	240	90	110	132
s-ind(C^m_{HL})	0	6	20	50	112	238	240	90	110	132

Table 1: Data for $\mathrm{U}(1)^{m-1}$-invariant SL cones C^m_{HL} in $\mathbb{C}^m$.

One can also prove that

$$N_{\Sigma^m_{\mathrm{HL}}}(2) = 2m^2+1 \;\text{ and }\; m_{\Sigma^m_{\mathrm{HL}}}(2) = \text{s-ind}(C^m_{\mathrm{HL}}) = m^2-m \;\text{ for } m \geqslant 10. \tag{12}$$

As C^m_{HL} is *stable* when s-ind$(C^m_{\mathrm{HL}}) = 0$ we see from Table 1 and (12) that C^3_{HL} is a *stable* cone in $\mathbb{C}^3$, but C^m_{HL} is *unstable* for $m \geqslant 4$. Also C^m_{HL} is *rigid* when $m_{\Sigma^m_{\mathrm{HL}}}(2) = m^2-m$, as $\dim G^m_{\mathrm{HL}} = m-1$. Thus C^m_{HL} is *rigid* if and only if $m \neq 8, 9$, by Table 1 and (12).

Here is an example taken from [10, Ex. 9.4], chosen as it is easy to write down.

Example 3.6. Let $a_1, \ldots, a_m \in \mathbb{Z}$ with $a_1 + \cdots + a_m = 0$ and highest common factor 1, such that $a_1, \ldots, a_k > 0$ and $a_{k+1}, \ldots, a_m < 0$ for $0 < k < m$. Define

$$\begin{aligned} L_0^{a_1,\ldots,a_m} = \bigl\{ \bigl(i\mathrm{e}^{ia_1\theta}x_1, \mathrm{e}^{ia_2\theta}x_2, \ldots, \mathrm{e}^{ia_m\theta}x_m \bigr) : \theta \in [0, 2\pi), \\ x_1, \ldots, x_m \in \mathbb{R}, \qquad a_1x_1^2 + \cdots + a_mx_m^2 = 0 \bigr\}. \end{aligned} \tag{13}$$

Then $L_0^{a_1,\ldots,a_m}$ is an *immersed* SL *cone* in $\mathbb{C}^m$, with an isolated singularity at 0.

Define $C^{a_1,\ldots,a_m} = \bigl\{ (x_1, \ldots, x_m) \in \mathbb{R}^m : a_1x_1^2 + \cdots + a_mx_m^2 = 0 \bigr\}$. Then $C^{a_1,\ldots,a_m}$ is a quadric cone on $\mathcal{S}^{k-1} \times \mathcal{S}^{m-k-1}$ in $\mathbb{R}^m$, and $L_0^{a_1,\ldots,a_m}$ is the image of an immersion $\Phi : C^{a_1,\ldots,a_m} \times \mathcal{S}^1 \to \mathbb{C}^m$, which is generically 2:1. Therefore $L_0^{a_1,\ldots,a_m}$ is an immersed SL cone on $(\mathcal{S}^{k-1} \times \mathcal{S}^{m-k-1} \times \mathcal{S}^1)/\mathbb{Z}_2$.

Further examples of SL cones are constructed by Harvey and Lawson [5, §III.3], Haskins [6], the author [10, 11], and others. Special Lagrangian cones in $\mathbb{C}^3$ are a special case, which may be treated using the theory of *integrable systems*. In principle this should yield a *classification* of all SL cones on T^2 in $\mathbb{C}^3$. For more information see McIntosh [25] or the author [15].

3.3 Special Lagrangian m-folds with conical singularities

Now we can define *conical singularities* of SL m-folds, following [17, Def. 3.6].

Definition 3.7. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold for $m > 2$, and define $\psi : M \to (0, \infty)$ as in (4). Suppose X is a compact singular SL m-fold in M with singularities at distinct points $x_1, \ldots, x_n \in X$, and no other singularities.

Fix isomorphisms $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ for $i = 1, \ldots, n$ such that $\upsilon_i^*(\omega) = \omega'$ and $\upsilon_i^*(\Omega) = \psi(x_i)^m\Omega'$, where ω', Ω' are as in (2). Let $C_1, \ldots, C_n$ be SL cones in $\mathbb{C}^m$ with isolated singularities at 0. For $i = 1, \ldots, n$ let $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$, and let $\mu_i \in (2, 3)$ with

$$(2, \mu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset, \quad \text{where } \mathcal{D}_{\Sigma_i} \text{ is defined in (7).} \tag{14}$$

Then we say that X has a *conical singularity* or *conical singular point* at x_i, with *rate* μ_i and *cone* C_i for $i = 1, \ldots, n$, if the following holds.

By Theorem 2.3 there exist embeddings $\Upsilon_i : B_R \to M$ for $i = 1, \ldots, n$ satisfying $\Upsilon_i(0) = x_i$, $\mathrm{d}\Upsilon_i|_0 = \upsilon_i$ and $\Upsilon_i^*(\omega) = \omega'$, where B_R is the open ball of radius R about 0 in $\mathbb{C}^m$ for some small $R > 0$. Define $\iota_i : \Sigma_i \times (0, R) \to B_R$ by $\iota_i(\sigma, r) = r\sigma$ for $i = 1, \ldots, n$.

Define $X' = X \setminus \{x_1, \ldots, x_n\}$. Then there should exist a compact subset $K \subset X'$ such that $X' \setminus K$ is a union of open sets $S_1, \ldots, S_n$ with $S_i \subset \Upsilon_i(B_R)$, whose closures $\bar{S}_1, \ldots, \bar{S}_n$ are disjoint in X. For $i = 1, \ldots, n$ and some $R' \in (0, R]$ there should exist a smooth $\phi_i : \Sigma_i \times (0, R') \to B_R$ such that $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \to M$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i$, and

$$\left|\nabla^k(\phi_i - \iota_i)\right| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \to 0 \text{ for } k = 0, 1. \tag{15}$$

Here ∇ is the Levi-Civita connection of the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$, $|\,.\,|$ is computed using $\iota_i^*(g')$. If the cones $C_1, \ldots, C_n$ are *stable* in the sense of Definition 3.4, then we say that X has *stable conical singularities*.

We will see in Theorem 4.4 that if (15) holds for $k = 0, 1$ and some μ_i satisfying (14), then we can choose a natural ϕ_i for which (15) holds for *all* $k \geqslant 0$, and for *all* rates μ_i satisfying (14). Thus the number of derivatives required in (15) and the choice of μ_i both make little difference. We choose $k = 0, 1$ in (15), and some μ_i in (14), to make the definition as weak as possible.

We suppose $m > 2$ for two reasons. Firstly, the only SL cones in $\mathbb{C}^2$ are finite unions of SL planes $\mathbb{R}^2$ in $\mathbb{C}^2$ intersecting only at 0. Thus any SL 2-fold with conical singularities is actually *nonsingular* as an immersed 2-fold, so there is really no point in studying them. Secondly, $m = 2$ is a special case in the analysis of [17, §2], and it is simpler to exclude it. Therefore we will suppose $m > 2$ throughout the paper.

Here are the reasons for the conditions on μ_i in Definition 3.7:

- We need $\mu_i > 2$, or else (15) does not force X to approach C_i near x_i.
- The definition involves a choice of $\Upsilon_i : B_R \to M$. If we replace Υ_i by a different choice $\widetilde{\Upsilon}_i$ then we should replace ϕ_i by $\widetilde{\phi}_i = (\widetilde{\Upsilon}_i^{-1} \circ \Upsilon_i) \circ \phi_i$ near 0 in B_R. Calculation shows that as $\Upsilon_i, \widetilde{\Upsilon}_i$ agree up to second order, we have $\left|\nabla^k(\widetilde{\phi}_i - \phi_i)\right| = O(r^{2-k})$.

  Therefore we choose $\mu_i < 3$ so that these $O(r^{2-k})$ terms are absorbed into the $O(r^{\mu_i - 1 - k})$ in (15). This makes the definition independent of the choice of Υ_i, which it would not be if $\mu_i > 3$.
- Condition (14) is needed to prove the regularity result Theorem 4.4, and also to reduce to a minimum the obstructions to deforming compact SL m-folds with conical singularities studied in §5.

3.4 Homology and cohomology

Next we discuss *homology* and *cohomology* of SL m-folds with conical singularities, following [17, §2.4]. For a general reference, see for instance Bredon [2]. When Y is a manifold, write $H^k(Y, \mathbb{R})$ for the k^{th} *de Rham cohomology group* and $H^k_{\text{cs}}(Y, \mathbb{R})$ for the k^{th} *compactly-supported de Rham cohomology group* of Y. If Y is compact then $H^k(Y, \mathbb{R}) = H^k_{\text{cs}}(Y, \mathbb{R})$. The *Betti numbers* of Y are $b^k(Y) = \dim H^k(Y, \mathbb{R})$ and $b^k_{\text{cs}}(Y) = \dim H^k_{\text{cs}}(Y, \mathbb{R})$.

Let Y be a topological space, and $Z \subset Y$ a subspace. Write $H_k(Y, \mathbb{R})$ for the k^{th} *real singular homology group* of Y, and $H_k(Y; Z, \mathbb{R})$ for the k^{th}

real singular relative homology group of $(Y;Z)$. When Y is a manifold and Z a submanifold we define $H_k(Y,\mathbb{R})$ and $H_k(Y;Z,\mathbb{R})$ using *smooth* simplices, as in [2, §V.5]. Then the pairing between (singular) homology and (de Rham) cohomology is defined at the chain level by integrating k-forms over k-simplices.

Suppose X is a compact SL m-fold in M with conical singularities $x_1,\ldots,x_n$ and cones $C_1,\ldots,C_n$, and set $X'=X\setminus\{x_1,\ldots,x_n\}$ and $\Sigma_i=C_i\cap\mathcal{S}^{2m-1}$, as in §3.3. Then X' is the interior of a compact manifold $\overline{X}'$ with boundary $\coprod_{i=1}^n\Sigma_i$. Using this we show in [17, §2.4] that there is a natural long exact sequence

$$\cdots\rightarrow H^k_{\rm cs}(X',\mathbb{R})\rightarrow H^k(X',\mathbb{R})\rightarrow\bigoplus_{i=1}^n H^k(\Sigma_i,\mathbb{R})\rightarrow H^{k+1}_{\rm cs}(X',\mathbb{R})\rightarrow\cdots, \tag{16}$$

and natural isomorphisms

$$\begin{aligned} H_k\bigl(X;\{x_1,\ldots,x_n\},\mathbb{R}\bigr)^*\cong H^k_{\rm cs}(X',\mathbb{R})&\cong H_{m-k}(X',\mathbb{R})\\ &\cong H^{m-k}(X',\mathbb{R})^*\quad\text{for all } k\geqslant 0\end{aligned} \tag{17}$$

$$\text{and}\quad H^k_{\rm cs}(X',\mathbb{R})\cong H_k(X,\mathbb{R})^*\quad\text{for all } k>1. \tag{18}$$

The inclusion $\iota: X\rightarrow M$ induces homomorphisms $\iota_*: H_k(X,\mathbb{R})\rightarrow H_k(M,\mathbb{R})$.

4. The asymptotic behaviour of X near x_i

We now review the work of [17] on the *regularity* of SL m-folds with conical singularities. Let M be an almost Calabi-Yau m-fold and X an SL m-fold in M with conical singularities at $x_1,\ldots,x_n$, with identifications υ_i and cones C_i. We study how quickly X converges to the cone $\upsilon(C_i)$ in $T_{x_i}M$ near x_i.

We start in §4.1 by writing X in a special coordinate system near x_i, as the graph of an exact 1-form $\eta_i=\mathrm{d}A_i$ on $C'_i=C_i\setminus\{0\}$. The special Lagrangian condition reduces to a *nonlinear elliptic p.d.e.* on the function A_i. In §4.2 we explain how *elliptic regularity* of this p.d.e. implies that A_i and its derivatives decay quickly near x_i.

4.1 Lagrangian neighbourhood theorems

In [17, Th. 4.3] we extend the *Lagrangian Neighbourhood Theorem*, Theorem 2.4, to special Lagrangian cones.

Theorem 4.1. *Let C be an* SL *cone in $\mathbb{C}^m$ with isolated singularity at* 0*, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Define $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$, with image $C \setminus \{0\}$. For $\sigma \in \Sigma$, $\tau \in T^*_\sigma \Sigma$, $r \in (0,\infty)$ and $u \in \mathbb{R}$, let (σ, r, τ, u) represent the point $\tau + u\,\mathrm{d}r$ in $T^*_{(\sigma,r)}\bigl(\Sigma \times (0,\infty)\bigr)$. Identify $\Sigma \times (0,\infty)$ with the zero section $\tau = u = 0$ in $T^*\bigl(\Sigma \times (0,\infty)\bigr)$. Define an action of $(0,\infty)$ on $T^*\bigl(\Sigma \times (0,\infty)\bigr)$ by*

$$t : (\sigma, r, \tau, u) \longmapsto (\sigma, tr, t^2\tau, tu) \quad \text{for } t \in (0,\infty), \tag{19}$$

so $t^*(\hat\omega) = t^2\hat\omega$ for $\hat\omega$ the canonical symplectic structure on $T^*\bigl(\Sigma \times (0,\infty)\bigr)$.

Then there exists an open neighbourhood U_C of $\Sigma \times (0,\infty)$ in $T^*\bigl(\Sigma \times (0,\infty)\bigr)$ invariant under (19) *given by*

$$U_C = \bigl\{(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (0,\infty)\bigr) : \bigl|(\tau, u)\bigr| < 2\zeta r\bigr\} \quad \text{for some } \zeta > 0, \tag{20}$$

where $|\,.\,|$ is calculated using the cone metric $\iota^*(g')$ on $\Sigma \times (0,\infty)$, and an embedding $\Phi_C : U_C \to \mathbb{C}^m$ with $\Phi_C|_{\Sigma \times (0,\infty)} = \iota$, $\Phi_C^*(\omega') = \hat\omega$ and $\Phi_C \circ t = t\,\Phi_C$ for all $t > 0$, where t acts on U_C as in (19) *and on $\mathbb{C}^m$ by multiplication.*

These U_C, Φ_C are a *Lagrangian neighbourhood* of C' in $\mathbb{C}^m$ which is *equivariant under the action of dilations*. Effectively they are a special coordinate system on $\mathbb{C}^m$ near C', in which ω' assumes a simple form. In [17, Th. 4.4] we use U_{C_i}, Φ_{C_i} to construct a particular choice of ϕ_i in Definition 3.7.

Theorem 4.2. *Let (M, J, ω, Ω), $\psi, X, n, x_i, \upsilon_i, C_i, \Sigma_i, \mu_i, R, \Upsilon_i$ and ι_i be as in Definition* 3.7*. Theorem* 4.1 *gives $\zeta > 0$, neighbourhoods U_{C_i} of $\Sigma_i \times (0,\infty)$ in $T^*\bigl(\Sigma_i \times (0,\infty)\bigr)$ and embeddings $\Phi_{C_i} : U_{C_i} \to \mathbb{C}^m$ for $i = 1, \dots, n$.*

Then for sufficiently small $R' \in (0, R]$ there exist unique closed 1-*forms η_i on $\Sigma_i \times (0, R')$ for $i = 1, \dots, n$ written $\eta_i(\sigma, r) = \eta_i^1(\sigma, r) + \eta_i^2(\sigma, r)\mathrm{d}r$ for $\eta_i^1(\sigma, r) \in T^*_\sigma \Sigma_i$ and $\eta_i^2(\sigma, r) \in \mathbb{R}$, and satisfying $|\eta_i(\sigma, r)| < \zeta r$ and*

$$\bigl|\nabla^k \eta_i\bigr| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \to 0 \text{ for } k = 0, 1, \tag{21}$$

computing $\nabla, |\,.\,|$ using the cone metric $\iota_i^*(g')$, such that the following holds.

Define $\phi_i : \Sigma_i \times (0, R') \to B_R$ by $\phi_i(\sigma, r) = \Phi_{C_i}\bigl(\sigma, r, \eta_i^1(\sigma, r), \eta_i^2(\sigma, r)\bigr)$. Then $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \to M$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i$

for open sets $S_1, \ldots, S_n$ *in* X' *with* $\bar{S}_1, \ldots, \bar{S}_n$ *disjoint, and* $K = X' \setminus (S_1 \cup \cdots \cup S_n)$ *is compact. Also* ϕ_i *satisfies* (15), *so that* R', ϕ_i, S_i, K *satisfy Definition* 3.7.

We explained in §2.1 that in a Lagrangian neighbourhood U, Φ of a Lagrangian m-fold N gives a 1-1 correspondence between nearby Lagrangian m-folds $\widetilde{N}$ and closed 1-forms on N. Theorem 4.2 uses this correspondence for the Lagrangian neighbourhoods U_{C_i}, Φ_{C_i} of Theorem 4.1. This is why the 1-forms η_i are closed. We can extend Theorem 2.4 to SL m-folds with conical singularities [17, Th. 4.6], in a way compatible with Theorems 4.1 and 4.2.

Theorem 4.3. *Suppose* (M, J, ω, Ω) *is an almost Calabi-Yau* m-*fold and* X *a compact* SL m-*fold in* M *with conical singularities at* $x_1, \ldots, x_n$. *Let the notation* $\psi, \upsilon_i, C_i, \Sigma_i, \mu_i, R, \Upsilon_i$ *and* ι_i *be as in Definition* 3.7, *and let* $\zeta, U_{C_i}, \Phi_{C_i}, R', \eta_i, \eta_i^1, \eta_i^2, \phi_i, S_i$ *and* K *be as in Theorem* 4.2.

Then making R' *smaller if necessary, there exists an open tubular neighbourhood* $U_{X'} \subset T^*X'$ *of the zero section* X' *in* T^*X', *such that under* $\mathrm{d}(\Upsilon_i \circ \phi_i) : T^*\big(\Sigma_i \times (0, R')\big) \to T^*X'$ *for* $i = 1, \ldots, n$ *we have*

(22)
$$\big(\mathrm{d}(\Upsilon_i \circ \phi_i)\big)^*(U_{X'}) = \big\{(\sigma, r, \tau, u) \in T^*\big(\Sigma_i \times (0, R')\big) : \big|(\tau, u)\big| < \zeta r\big\},$$

and there exists an embedding $\Phi_{X'} : U_{X'} \to M$ *with* $\Phi_{X'}|_{X'} = \mathrm{id} : X' \to X'$ *and* $\Phi_{X'}^*(\omega) = \hat{\omega}$, *where* $\hat{\omega}$ *is the canonical symplectic structure on* T^*X', *such that*

(23)
$$\Phi_{X'} \circ \mathrm{d}(\Upsilon_i \circ \phi_i)(\sigma, r, \tau, u) \equiv \Upsilon_i \circ \Phi_{C_i}\big(\sigma, r, \tau + \eta_i^1(\sigma, r), u + \eta_i^2(\sigma, r)\big)$$

for all $i = 1, \ldots, n$ *and* $(\sigma, r, \tau, u) \in T^*\big(\Sigma_i \times (0, R')\big)$ *with* $\big|(\tau, u)\big| < \zeta r$. *Here* $|(\tau, u)|$ *is computed using the cone metric* $\iota_i^*(g')$ *on* $\Sigma_i \times (0, R')$.

This is an essential tool in the deformation theory of §5 and desingularization results of §7, as it gives a special coordinate system on M near X' in which ω assumes a simple form. In these coordinate, deformations or desingularizations of X become *graphs of closed* 1-*forms* on X' away from x_i, as in §2.1.

4.2 Regularity of X near x_i

The results of §4.1 used only the fact that X' is *Lagrangian* in (M, ω). Our next theorems make essential use of the *special* Lagrangian condition. In [17, §5] we study the asymptotic behaviour of the maps ϕ_i of Theorem 4.2. Combining [17, Th. 5.1], [17, Lem. 4.5] and [17, Th. 5.5] proves:

Theorem 4.4. *In the situation of Theorem* 4.2 *we have* $\eta_i = \mathrm{d}A_i$ *for* $i = 1, \dots, n$*, where* $A_i : \Sigma_i \times (0, R') \to \mathbb{R}$ *is given by* $A_i(\sigma, r) = \int_0^r \eta_i^2(\sigma, s)\mathrm{d}s$*. Suppose* $\mu_i' \in (2, 3)$ *with* $(2, \mu_i'] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ *for* $i = 1, \dots, n$*. Then*

$$\begin{aligned} &\left|\nabla^k(\phi_i - \iota_i)\right| = O(r^{\mu_i'-1-k}), \quad \left|\nabla^k \eta_i\right| = O(r^{\mu_i'-1-k}) \quad \textit{and} \\ &\left|\nabla^k A_i\right| = O(r^{\mu_i'-k}) \quad \textit{as } r \to 0 \textit{ for all } k \geqslant 0 \textit{ and } i = 1, \dots, n. \end{aligned} \tag{24}$$

Hence X *has conical singularities at* x_i *with cone* C_i *and rate* μ_i'*, for all possible rates* μ_i' *allowed by Definition* 3.7*. Therefore, the definition of conical singularities is essentially independent of the choice of rate* μ_i*.*

Theorem 4.4 in effect *strengthens* the definition of SL m-folds with conical singularities, Definition 3.7, as it shows that (15) actually implies the much stronger condition (24) on all derivatives.

The proof works by treating X' near x_i as a deformation of the SL cone C_i in $\mathbb{C}^m$. Thus we can apply Proposition 2.13 with N replaced by $\Sigma_i \times (0, R')$ and $\alpha = \eta_i = \mathrm{d}A_i$, and we find that A_i satisfies the second-order nonlinear p.d.e.

$$\mathrm{d}^*\big(\psi^m \mathrm{d}A_i\big)(\sigma, r) = Q\big(\sigma, r, \mathrm{d}A_i(\sigma, r), \nabla^2 A_i(\sigma, r)\big) \tag{25}$$

for $(\sigma, r) \in \Sigma_i \times (0, R')$, where Q is a smooth nonlinear function.

When r is small the Q term in (25) is also small and (25) approximates $\Delta_i A_i = 0$, where Δ_i is the Laplacian on the cone C_i. Therefore (25) is *elliptic* for small r. Using known results on the regularity of solutions of nonlinear second-order elliptic p.d.e.s, and a theory of analysis on weighted Sobolev spaces on manifolds with cylindrical ends developed by Lockhart and McOwen [22], we can then prove (24).

Our next result [17, Th. 6.8] is an application of *geometric measure theory*. For an introduction to the subject, see Morgan [27]. Geometric measure theory studies measure-theoretic generalizations of submanifolds called *integral currents*, which may be very singular, and is particularly powerful for *minimal* submanifolds. As from §2 SL m-folds are minimal submanifolds w.r.t. an appropriate metric, many major results of geometric measure theory immediately apply to *special Lagrangian integral currents*, a very general class of singular SL m-folds with strong compactness properties.

Theorem 4.5. *Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold and define $\psi : M \to (0, \infty)$ as in* (4). *Let $x \in M$ and fix an isomorphism $\upsilon : \mathbb{C}^m \to T_x M$ with $\upsilon^*(\omega) = \omega'$ and $\upsilon^*(\Omega) = \psi(x)^m \Omega'$, where ω', Ω' are as in* (2).

Suppose that T is a special Lagrangian integral current in M with $x \in T^\circ$, where $T^\circ = \operatorname{supp} T \setminus \operatorname{supp} \partial T$, and that $\upsilon_*(C)$ is a multiplicity 1 *tangent cone to T at x, where C is a rigid special Lagrangian cone in $\mathbb{C}^m$, in the sense of Definition* 3.4. *Then T has a conical singularity at x, in the sense of Definition* 3.7.

This is a *weakening* of Definition 3.7 for *rigid* cones C. Theorem 4.5 also holds for the larger class of *Jacobi integrable* SL cones C, [17, Def. 6.7].

Basically, Theorem 4.5 shows that if a singular SL m-fold T in M is locally modelled on a rigid SL cone C in only a very weak sense, then it necessarily satisfies Definition 3.7. One moral of Theorems 4.4 and 4.5 is that, at least for rigid SL cones C, more-or-less *any* sensible definition of SL m-folds with conical singularities is equivalent to Definition 3.7.

Theorem 4.5 is proved by applying regularity results of Allard and Almgren, and Adams and Simon, mildly adapted to the special Lagrangian situation, which roughly say that if a tangent cone C_i to X at x_i has an isolated singularity at 0, is multiplicity 1, and rigid, then X has a parametrization ϕ_i near x_i which satisfies (15) for some $\mu_i > 2$. It then quickly follows that X has a conical singularity at x_i, in the sense of Definition 3.7.

As discussed in [17, §6.3], one can use other results from geometric measure theory to argue that for tangent cones C which are not Jacobi integrable, Definition 3.7 may be *too strong*, in that there could exist examples of singular SL m-folds with tangent cone C which are not covered by Definition 3.7, as the decay conditions (15) are too strict.

5. Moduli of SL m-folds with conical singularities

Next we review the work of [18] on *deformation theory* for compact SL m-folds with conical singularities. Following [18, Def. 5.4], we define the space $\mathcal{M}_X$ of compact SL m-folds $\hat{X}$ in M with conical singularities deforming a fixed SL m-fold X with conical singularities.

Definition 5.1. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, \ldots, x_n$ with identifications $\upsilon_i : \mathbb{C}^m \to T_{x_i} M$ and cones $C_1, \ldots, C_n$. Define the *moduli space* $\mathcal{M}_X$ *of deformations of* X to be the set of $\hat{X}$ such that:

(i) $\hat{X}$ is a compact SL m-fold in M with conical singularities at $\hat{x}_1, \ldots, \hat{x}_n$ with cones $C_1, \ldots, C_n$, for some $\hat{x}_i$ and identifications $\hat{\upsilon}_i : \mathbb{C}^m \to T_{\hat{x}_i}M$.

(ii) There exists a homeomorphism $\hat{\imath} : X \to \hat{X}$ with $\hat{\imath}(x_i) = \hat{x}_i$ for $i = 1, \ldots, n$ such that $\hat{\imath}|_{X'} : X' \to \hat{X}'$ is a diffeomorphism and $\hat{\imath}$ and ι are isotopic as continuous maps $X \to M$, where $\iota : X \to M$ is the inclusion.

In [18, Def. 5.6] we define a *topology* on $\mathcal{M}_X$, and explain why it is the natural choice. We will not repeat the complicated definition here; readers are referred to [18, §5] for details.

In [18, Th. 6.10] we describe $\mathcal{M}_X$ near X, in terms of a smooth map Φ between the *infinitesimal deformation space* $\mathcal{I}_{X'}$ and the *obstruction space* $\mathcal{O}_{X'}$.

Theorem 5.2. *Suppose* (M, J, ω, Ω) *is an almost Calabi-Yau* m*-fold and* X *a compact* SL m*-fold in* M *with conical singularities at* $x_1, \ldots, x_n$ *and cones* $C_1, \ldots, C_n$*. Let* $\mathcal{M}_X$ *be the moduli space of deformations of* X *as an* SL m*-fold with conical singularities in* M*, as in Definition* 5.1*. Set* $X' = X \setminus \{x_1, \ldots, x_n\}$*.*

Then there exist natural finite-dimensional vector spaces $\mathcal{I}_{X'}$*,* $\mathcal{O}_{X'}$ *with* $\mathcal{I}_{X'}$ *isomorphic to the image of* $H^1_{\mathrm{cs}}(X', \mathbb{R})$ *in* $H^1(X', \mathbb{R})$ *and* $\dim \mathcal{O}_{X'} = \sum_{i=1}^n \text{s-ind}(C_i)$*, where* $\text{s-ind}(C_i)$ *is the stability index of Definition* 3.4*. There exists an open neighbourhood* U *of* 0 *in* $\mathcal{I}_{X'}$*, a smooth map* $\Phi : U \to \mathcal{O}_{X'}$ *with* $\Phi(0) = 0$*, and a map* $\Xi : \{u \in U : \Phi(u) = 0\} \to \mathcal{M}_X$ *with* $\Xi(0) = X$ *which is a homeomorphism with an open neighbourhood of* X *in* $\mathcal{M}_X$*.*

Here is a sketch of the proof. For simplicity, consider first the subset of $\hat{X} \in \mathcal{M}_X$ which have the same singular points $x_1, \ldots, x_n$ and identifications $\upsilon_1, \ldots, \upsilon_n$ as X. If $\hat{X}$ is C^1 close to X in an appropriate sense then $\hat{X}' = \Phi_{X'}\bigl(\Gamma(\alpha)\bigr)$, where $U_{X'}, \Phi_{X'}$ is the Lagrangian neighbourhood map of Theorem 4.3, and $\Gamma(\alpha) \subset U_{X'}$ is the graph of a small 1-form α on X'.

Since $\hat{X}'$ is Lagrangian, α is *closed*, as in §2.1. Also, if ϕ_i, η_i and $\hat{\phi}_i, \hat{\eta}_i$ are as in Theorem 4.2 for $X, \hat{X}$ then $(\Upsilon_i \circ \phi_i)^*(\alpha) = \hat{\eta}_i - \eta_i$ on $\Sigma_i \times (0, R')$, so applying Theorem 4.4 to $X, \hat{X}$ shows that if $i = 1, \ldots, n$ and $\mu'_i \in (2, 3)$ with $(2, \mu'_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$ then

$$\bigl|\nabla^k \alpha(x)\bigr| = O\bigl(d(x, x_i)^{\mu'_i - 1 - k}\bigr) \quad \text{near } x_i \text{ for all } k \geqslant 0. \tag{26}$$

As α is closed it has a cohomology class $[\alpha] \in H^1(X', \mathbb{R})$. Since (26) implies that α decays quickly near x_i, it turns out that α must be *exact* near x_i. Therefore $[\alpha]$ can be represented by a compactly-supported form on X', and lies in the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$.

Choose a vector space $\mathcal{I}_{X'}$ of compactly-supported 1-forms on X' representing the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$. Then we can write $\alpha = \beta + \mathrm{d}f$, where $\beta \in \mathcal{I}_{X'}$ with $[\alpha] = [\beta]$ is unique, and $f \in C^\infty(X')$ is unique up to addition of constants. As $\hat{X}'$ is special Lagrangian we find that f satisfies a *second-order nonlinear elliptic p.d.e.* similar to (25):

$$\mathrm{d}^*\big(\psi^m(\beta + \mathrm{d}f)\big)(x) = Q\big(x, (\beta + \mathrm{d}f)(x), (\nabla\beta + \nabla^2 f)(x)\big) \tag{27}$$

for $x \in X'$. The *linearization* of (27) at $\beta = f = 0$ is $\mathrm{d}^*\big(\psi^m(\beta + \mathrm{d}f)\big) = 0$.

We study the family of small solutions β, f of (27) for which f has the decay near x_i required by (26). There is a ready-made theory of analysis on manifolds with cylindrical ends developed by Lockhart and McOwen [22], which is well-suited to this task. We work on certain *weighted Sobolev spaces* $L^p_{k,\boldsymbol{\mu}}(X')$ of functions on X'.

By results from [22] it turns out that the operator $f \mapsto \mathrm{d}^*(\psi^m \mathrm{d}f)$ is a *Fredholm* map $L^p_{k,\boldsymbol{\mu}}(X') \to L^p_{k-2,\boldsymbol{\mu}-2}(X')$, with cokernel of dimension $\sum_{i=1}^n N_{\Sigma_i}(2)$. This cokernel is in effect the *obstruction space* to deforming X with x_i, υ_i fixed, as it is the obstruction space to solving the linearization of (27) in f at $\beta = f = 0$.

By varying x_i, υ_i, and allowing f to converge to different constants on the ends of X' rather than zero, we overcome many of these obstructions. This reduces the dimension of the obstruction space $\mathcal{O}_{X'}$ from $\sum_{i=1}^n N_{\Sigma_i}(2)$ to $\sum_{i=1}^n \text{s-ind}(C_i)$. The obstruction map Φ is constructed using the Implicit Mapping Theorem for Banach spaces. This concludes our sketch.

If the C_i are *stable* then $\mathcal{O}_{X'} = \{0\}$ and we deduce [18, Cor. 6.11]:

Corollary 5.3. *Suppose (M, J, ω, Ω) is an almost Calabi-Yau m-fold and X a compact* SL *m-fold in M with stable conical singularities, and let $\mathcal{M}_X$ and $\mathcal{I}_{X'}$ be as in Theorem* 5.2*. Then $\mathcal{M}_X$ is a smooth manifold of dimension* $\dim \mathcal{I}_{X'}$.

This has clear similarities with Theorem 2.10. Here is another simple condition for $\mathcal{M}_X$ to be a manifold near X, [18, Def. 6.12].

Definition 5.4. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold and X a compact SL m-fold in M with conical singularities, and let

$\mathcal{I}_{X'}, \mathcal{O}_{X'}, U$ and Φ be as in Theorem 5.2. We call X *transverse* if the linear map $\mathrm{d}\Phi|_0 : \mathcal{I}_{X'} \to \mathcal{O}_{X'}$ is surjective.

If X is transverse then $\{u \in U : \Phi(u) = 0\}$ is a manifold near 0, so Theorem 5.2 yields [18, Cor. 6.13]:

Corollary 5.5. *Suppose (M, J, ω, Ω) is an almost Calabi-Yau m-fold and X a transverse compact* SL *m-fold in M with conical singularities, and let $\mathcal{M}_X, \mathcal{I}_{X'}$ and $\mathcal{O}_{X'}$ be as in Theorem* 5.2. *Then $\mathcal{M}_X$ is near X a smooth manifold of dimension* $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$.

In [18, §7] we extend all this to *families* of almost Calabi-Yau m-folds. Combining Definitions 2.15 and 5.1, we define moduli spaces in families:

Definition 5.6. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold and X a compact SL m-fold in M with conical singularities at $x_1, ..., x_n$. Suppose $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ is a smooth family of deformations of (M, J, ω, Ω). Define the *moduli space $\mathcal{M}_X^{\mathcal{F}}$ of deformations of X in the family $\mathcal{F}$* to be the set of pairs $(s, \hat{X})$ such that:

(i) $s \in \mathcal{F}$ and $\hat{X}$ is a compact SL m-fold in $(M, J^s, \omega^s, \Omega^s)$ with conical singularities at $\hat{x}_1, \ldots, \hat{x}_n$ with cones $C_1, \ldots, C_n$, for some $\hat{x}_i$.

(ii) There exists a homeomorphism $\hat{\imath} : X \to \hat{X}$ with $\hat{\imath}(x_i) = \hat{x}_i$ for $i = 1, \ldots, n$ such that $\hat{\imath}|_{X'} : X' \to \hat{X}'$ is a diffeomorphism and $\hat{\imath}$ and ι are isotopic as continuous maps $X \to M$, where $\iota : X \to M$ is the inclusion.

Define a *projection* $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ by $\pi^{\mathcal{F}}(s, \hat{X}) = s$. In [18, Def. 7.5] we define a natural *topology* on $\mathcal{M}_X^{\mathcal{F}}$, for which $\pi^{\mathcal{F}}$ is continuous.

Here [18, Th. 7.9] is the families analogue of Theorem 5.2.

Theorem 5.7. *Suppose (M, J, ω, Ω) is an almost Calabi-Yau m-fold and X a compact* SL *m-fold in M with conical singularities at $x_1, \ldots, x_n$. Let $\mathcal{M}_X, X', \mathcal{I}_{X'}, \mathcal{O}_{X'}, U, \Phi$ and Ξ be as in Theorem* 5.2.

Suppose $\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\}$ is a smooth family of deformations of (M, J, ω, Ω), in the sense of Definition 2.14, *such that $\iota_*(\gamma) \cdot [\omega^s] = 0$ for all $\gamma \in H_2(X, \mathbb{R})$ and $s \in \mathcal{F}$, where $\iota : X \to M$ is the inclusion, and $[X] \cdot [\mathrm{Im}\, \Omega^s] = 0$ for all $s \in \mathcal{F}$, where $[X] \in H_m(M, \mathbb{R})$ and $[\mathrm{Im}\, \Omega^s] \in H^m(M, \mathbb{R})$. Let $\mathcal{M}_X^{\mathcal{F}}$ and $\pi^{\mathcal{F}} : \mathcal{M}_X^{\mathcal{F}} \to \mathcal{F}$ be as in Definition* 5.6.

Then there exists an open neighbourhood $U^{\mathcal{F}}$ of $(0,0)$ in $\mathcal{F} \times U$, a smooth map $\Phi^{\mathcal{F}} : U^{\mathcal{F}} \to \mathcal{O}_{X'}$ with $\Phi^{\mathcal{F}}(0, u) \equiv \Phi(u)$, and a map $\Xi^{\mathcal{F}}$:

$\{(s,u)\in U^{\mathcal F}:\Phi^{\mathcal F}(s,u)=0\}\to\mathcal M_X^{\mathcal F}$ with $\Xi^{\mathcal F}(0,u)\equiv\bigl(0,\Xi(u)\bigr)$ and $\pi^{\mathcal F}\circ\Xi^{\mathcal F}(s,u)\equiv s$, which is a homeomorphism with an open neighbourhood of $(0,X)$ in $\mathcal M_X^{\mathcal F}$.

The conditions $\iota_*(\gamma)\cdot[\omega^s]=0$ for all γ and $[X]\cdot[\operatorname{Im}\Omega^s]=0$ are *necessary conditions* for the existence of any SL m-fold $\hat X$ with conical singularities isotopic to X in $(M,J^s,\omega^s,\Omega^s)$. Here are the families analogues of Definition 5.4 and Corollaries 5.3 and 5.5, taken from [18, Def. 7.11 & Cor.s 7.10 & 7.12].

Corollary 5.8. *In the situation of Theorem* 5.7, *suppose X has stable singularities. Then $\mathcal M_X^{\mathcal F}$ is a smooth manifold of dimension $d+\dim\mathcal I_{X'}$ and $\pi^{\mathcal F}:\mathcal M_X^{\mathcal F}\to\mathcal F$ a smooth submersion. For small $s\in\mathcal F$ the fibre $(\pi^{\mathcal F})^{-1}(s)$ is a nonempty smooth manifold of dimension $\dim\mathcal I_{X'}$, with $(\pi^{\mathcal F})^{-1}(0)=\mathcal M_X$.*

Note the similarity of Corollary 5.8 and Theorem 2.16.

Definition 5.9. In the situation of Definition 5.7, we call X *transverse in $\mathcal F$* if the linear map $\mathrm d\Phi^{\mathcal F}|_{(0,0)}:\mathbb R^d\times\mathcal I_{X'}\to\mathcal O_{X'}$ is surjective. If X is transverse in the sense of Definition 5.4 then it is also transverse in $\mathcal F$.

Corollary 5.10. *In the situation of Theorem* 5.7, *suppose X is transverse in $\mathcal F$. Then $\mathcal M_X^{\mathcal F}$ is near $(0,X)$ a smooth manifold of dimension $d+\dim\mathcal I_{X'}-\dim\mathcal O_{X'}$, and $\pi^{\mathcal F}:\mathcal M_X^{\mathcal F}\to\mathcal F$ is a smooth map near $(0,X)$.*

Now there are a number of well-known moduli space problems in geometry where in general moduli spaces are obstructed and singular, but after a generic perturbation they become smooth manifolds. For instance, moduli spaces of instantons on 4-manifolds can be made smooth by choosing a generic metric, and similar things hold for Seiberg–Witten equations, and moduli spaces of pseudo-holomorphic curves in symplectic manifolds.

In [18, §9] we try (but do not quite succeed) to replicate this for moduli spaces of SL m-folds with conical singularities, by choosing a *generic Kähler metric* in a fixed Kähler class. Our first result is [18, Th. 9.1]:

Theorem 5.11. *Let (M,J,ω,Ω) be an almost Calabi-Yau m-fold, X a compact* SL *m-fold in M with conical singularities, and $\mathcal I_{X'},\mathcal O_{X'}$ as in Theorem* 5.2. *Then there exists a smooth family of deformations $\bigl\{(M,J,\omega^s,\Omega):s\in\mathcal F\bigr\}$ of (M,J,ω,Ω) with $[\omega^s]=[\omega]\in H^2(M,\mathbb R)$ for* *all $s \in \mathcal{F}$, such that X is transverse in $\mathcal{F}$ in the sense of Definition* 5.9, *and $d = \dim \mathcal{F} = \dim \mathcal{O}_{X'}$.*

Combining this with Corollary 5.10 we see that $\mathcal{M}_X^{\mathcal{F}}$ is a manifold near $(0, X)$ and $\pi^{\mathcal{F}}$ a smooth map near $(0, X)$. It then follows from Sard's Theorem that for small generic $s \in \mathcal{F}$, the moduli space $(\pi^{\mathcal{F}})^{-1}(s)$ of deformations of X in (M, J, ω^s, Ω) is a smooth manifold near X.

Thus, given a compact SL m-fold X in (M, J, ω, Ω) with conical singularities, we can perturb ω a little bit in its Kähler class to ω^s, and the moduli space $\mathcal{M}_X^s$ in (M, J, ω^s, Ω) will be a smooth manifold near X. More generally [18, Th. 9.3], if $W \subseteq \mathcal{M}_X$ is a compact subset then we can perturb ω to ω^s so $\mathcal{M}_X^s$ is a smooth manifold near W.

We would like to conclude that by choosing a sufficiently generic perturbation ω^s we can make $\mathcal{M}_X^s$ smooth everywhere. This is the idea of the following conjecture, [18, Conj. 9.5]:

Conjecture 5.12. *Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold, X a compact* SL *m-fold in M with conical singularities, and $\mathcal{I}_{X'}$, $\mathcal{O}_{X'}$ be as in Theorem* 5.2. *Then for a second category subset of Kähler forms $\check{\omega}$ in the Kähler class of ω, the moduli space $\check{\mathcal{M}}_X$ of compact* SL *m-folds $\hat{X}$ with conical singularities in $(M, J, \check{\omega}, \Omega)$ isotopic to X is a manifold of dimension $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$.*

If we could treat the moduli spaces $\mathcal{M}_X$ as compact, say if we had a good understanding of the compactification $\bar{\mathcal{M}}_X$ of $\mathcal{M}_X$ in §8, then this would follow from [18, Th. 9.3]. However, without knowing $\mathcal{M}_X$ is compact, the condition that $\check{\mathcal{M}}_X$ is smooth everywhere is in effect the intersection of an infinite number of genericity conditions on $\check{\omega}$, and we do not know that this intersection is dense (or even nonempty) in the Kähler class.

Notice that Conjecture 5.12 constrains the topology and cones of SL m-folds X with conical singularities that can occur in a generic almost Calabi-Yau m-fold, as we must have $\dim \mathcal{I}_{X'} \geqslant \dim \mathcal{O}_{X'}$.

6. Asymptotically conical SL m-folds

We now discuss *asymptotically conical* SL m-folds L in $\mathbb{C}^m$, [17, Def. 7.1].

Definition 6.1. Let C be a closed SL cone in $\mathbb{C}^m$ with isolated singularity at 0 for $m > 2$, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$, so that Σ is a compact, nonsingular $(m-1)$-manifold, not necessarily connected. Let

g_Σ be the metric on Σ induced by the metric g' on $\mathbb{C}^m$ in (2), and r the radius function on $\mathbb{C}^m$. Define $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then the image of ι is $C \setminus \{0\}$, and $\iota^*(g') = r^2 g_\Sigma + \mathrm{d}r^2$ is the cone metric on $C \setminus \{0\}$.

Let L be a closed, nonsingular SL m-fold in $\mathbb{C}^m$. We call L *asymptotically conical* (AC) with *rate* $\lambda < 2$ and *cone* C if there exists a compact subset $K \subset L$ and a diffeomorphism $\varphi : \Sigma \times (T,\infty) \to L \setminus K$ for some $T > 0$, such that

$$\left|\nabla^k(\varphi - \iota)\right| = O(r^{\lambda-1-k}) \quad \text{as } r \to \infty \text{ for } k = 0, 1. \tag{28}$$

Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota^*(g')$.

This is very similar to Definition 3.7, and in fact there are strong parallels between the theories of SL m-folds with conical singularities and of asymptotically conical SL m-folds. We continue to assume $m > 2$ throughout.

In §6.1–§6.2 we review the results of [17, §7] on AC SL m-folds. Section 6.3 covers the *deformation theory* of AC SL m-folds in $\mathbb{C}^m$ following Marshall [23] and Pacini [28], and §6.4 discusses *examples* of AC SL m-folds.

6.1 Cohomological invariants of AC SL m-folds

Let L be an AC SL m-fold in $\mathbb{C}^m$ with cone C, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Using the notation of §3.4, as in (16) there is a long exact sequence

$$\cdots \to H^k_{\mathrm{cs}}(L,\mathbb{R}) \to H^k(L,\mathbb{R}) \to H^k(\Sigma,\mathbb{R}) \to H^{k+1}_{\mathrm{cs}}(L,\mathbb{R}) \to \cdots . \tag{29}$$

Following [17, Def. 7.2] we define *cohomological invariants* $Y(L), Z(L)$ of L.

Definition 6.2. Let L be an AC SL m-fold in $\mathbb{C}^m$ with cone C, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. As $\omega', \operatorname{Im}\Omega'$ in (2) are closed forms with $\omega'|_L \equiv \operatorname{Im}\Omega'|_L \equiv 0$, they define classes in the relative de Rham cohomology groups $H^k(\mathbb{C}^m; L, \mathbb{R})$ for $k = 2, m$. But for $k > 1$ we have the exact sequence

$$0 = H^{k-1}(\mathbb{C}^m,\mathbb{R}) \to H^{k-1}(L,\mathbb{R}) \xrightarrow{\cong} H^k(\mathbb{C}^m; L,\mathbb{R}) \to H^k(\mathbb{C}^m,\mathbb{R}) = 0.$$

Let $Y(L) \in H^1(\Sigma,\mathbb{R})$ be the image of $[\omega']$ in $H^2(\mathbb{C}^m; L,\mathbb{R}) \cong H^1(L,\mathbb{R})$ under $H^1(L,\mathbb{R}) \to H^1(\Sigma, R)$ in (29), and $Z(L) \in H^{m-1}(\Sigma,\mathbb{R})$ be the image of $[\operatorname{Im}\Omega']$ in $H^m(\mathbb{C}^m; L,\mathbb{R}) \cong H^{m-1}(L,\mathbb{R})$ under $H^{m-1}(L,\mathbb{R}) \to H^{m-1}(\Sigma, R)$ in (29).

Here are some conditions for $Y(L)$ or $Z(L)$ to be zero, [17, Prop. 7.3]:

Proposition 6.3. *Let L be an* AC SL *m-fold in $\mathbb{C}^m$ with cone C and rate λ, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. If $\lambda < 0$ or $b^1(L) = 0$ then $Y(L) = 0$. If $\lambda < 2 - m$ or $b^0(\Sigma) = 1$ then $Z(L) = 0$.*

6.2 Lagrangian neighbourhood theorems and regularity

Next we give versions of parts of §4.1–§4.2 for AC SL m-folds rather than SL m-folds with conical singularities. Here are the analogues of Theorems 4.2 and 4.3, proved in [17, Th.s 7.4 & 7.5].

Theorem 6.4. *Let C be an* SL *cone in $\mathbb{C}^m$ with isolated singularity at* 0*, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Define $\iota : \Sigma \times (0,\infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Let ζ, $U_C \subset T^*\bigl(\Sigma \times (0,\infty)\bigr)$ and $\Phi_C : U_C \to \mathbb{C}^m$ be as in Theorem* 4.1.

Suppose L is an AC SL *m-fold in $\mathbb{C}^m$ with cone C and rate $\lambda < 2$. Then there exists a compact $K \subset L$ and a diffeomorphism $\varphi : \Sigma \times (T,\infty) \to L \setminus K$ for some $T > 0$ satisfying* (28)*, and a closed* 1*-form χ on $\Sigma \times (T,\infty)$ written $\chi(\sigma,r) = \chi^1(\sigma,r) + \chi^2(\sigma,r)\mathrm{d}r$ for $\chi^1(\sigma,r) \in T^*_\sigma\Sigma$ and $\chi^2(\sigma,r) \in \mathbb{R}$, satisfying*

$$\begin{aligned} &\bigl|\chi(\sigma,r)\bigr| < \zeta r, \quad \varphi(\sigma,r) \equiv \Phi_C\bigl(\sigma, r, \chi^1(\sigma,r), \chi^2(\sigma,r)\bigr) \\ &\textit{and} \quad \bigl|\nabla^k\chi\bigr| = O(r^{\lambda-1-k}) \quad \textit{as } r \to \infty \textit{ for } k = 0, 1, \end{aligned} \tag{30}$$

computing $\nabla, |\,.\,|$ using the cone metric $\iota^*(g')$.

Theorem 6.5. *Suppose L is an* AC SL *m-fold in $\mathbb{C}^m$ with cone C. Let $\Sigma, \iota, \zeta, U_C, \Phi_C, K, T, \varphi, \chi, \chi^1, \chi^2$ be as in Theorem* 6.4*. Then making T, K larger if necessary, there exists an open tubular neighbourhood $U_L \subset T^*L$ of the zero section L in T^*L, such that under $\mathrm{d}\varphi : T^*\bigl(\Sigma \times (T,\infty)\bigr) \to T^*L$ we have*

$$(\mathrm{d}\varphi)^*(U_L) = \bigl\{(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (T,\infty)\bigr) : \bigl|(\tau, u)\bigr| < \zeta r\bigr\}, \tag{31}$$

and there exists an embedding $\Phi_L : U_L \to \mathbb{C}^m$ with $\Phi_L|_L = \mathrm{id} : L \to L$ and $\Phi_L^*(\omega') = \hat\omega$, where $\hat\omega$ is the canonical symplectic structure on T^*L, such that

$$\Phi_L \circ \mathrm{d}\varphi(\sigma, r, \tau, u) \equiv \Phi_C\bigl(\sigma, r, \tau + \chi^1(\sigma,r), u + \chi^2(\sigma,r)\bigr) \tag{32}$$

for all $(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (T,\infty)\bigr)$ with $|(\tau, u)| < \zeta r$, computing $|\,.\,|$ using $\iota^*(g')$.

Combining [17, Prop. 7.6] and [17, Th.s 7.7 & 7.11] gives an analogue of Theorem 4.4, on the *regularity* of L near infinity in $\mathbb{C}^m$. As in [17, Th. 7.11], the theorem can be strengthened when $0 \leqslant \lambda < \min\bigl(\mathcal{D}_\Sigma \cap (0,\infty)\bigr)$.

Theorem 6.6. *In Theorem* 6.4 *we have* $[\chi] = Y(L)$ *in* $H^1\bigl(\Sigma \times (T,\infty),\mathbb{R}\bigr) \cong H^1(\Sigma,\mathbb{R})$, *where* $Y(L)$ *is as in Definition* 6.2. *Let* γ *be the unique* 1*-form on* Σ *with* $\mathrm{d}\gamma = \mathrm{d}^*\gamma = 0$ *and* $[\gamma] = Y(L) \in H^1(\Sigma,\mathbb{R})$, *which exists by Hodge theory. Then* $\chi = \pi^*(\gamma) + \mathrm{d}E$, *where* $\pi : \Sigma \times (T,\infty) \to \Sigma$ *is the projection and* $E \in C^\infty\bigl(\Sigma \times (T,\infty)\bigr)$.

If either $\lambda = \lambda'$, *or* λ, λ' *lie in the same connected component of* $\mathbb{R} \setminus \mathcal{D}_\Sigma$, *then* L *is an* AC SL m*-fold with rate* λ' *and*

$$\begin{gathered} \bigl|\nabla^k(\varphi - \iota)\bigr| = O(r^{\lambda'-1-k}), \qquad \bigl|\nabla^k\chi\bigr| = O(r^{\lambda'-1-k}), \\ \bigl|\nabla^{k+1}E\bigr| = O(r^{\lambda'-1-k}) \quad \textit{for all } k \geqslant 0, \textit{ and} \\ |E| = \begin{cases} O(r^{\lambda'}), & \lambda' \neq 0, \\ O\bigl(|\log r|\bigr), & \lambda' = 0. \end{cases} \end{gathered} \tag{33}$$

Here $\nabla, |\,.\,|$ *are computed using the cone metric* $\iota^*(g')$ *on* $\Sigma \times (T,\infty)$.

6.3 Moduli spaces of AC SL m-folds

The deformation theory of asymptotically conical SL m-folds in $\mathbb{C}^m$ has been studied independently by Pacini [28] and Marshall [23]. Pacini's results are earlier, but Marshall's are more complete.

Definition 6.7. Suppose L is an asymptotically conical SL m-fold in $\mathbb{C}^m$ with cone C and rate $\lambda < 2$, as in Definition 6.1. Define the *moduli space* $\mathcal{M}_L^\lambda$ *of deformations of* L *with rate* λ to be the set of AC SL m-folds $\hat{L}$ in $\mathbb{C}^m$ with cone C and rate λ, such that $\hat{L}$ is diffeomorphic to L and isotopic to L as an asymptotically conical submanifold of $\mathbb{C}^m$. One can define a natural *topology* on $\mathcal{M}_L^\lambda$, in a similar way to the conical singularities case of [18, Def. 5.6].

Note that if L is an AC SL m-fold with rate λ, then it is *also* an AC SL m-fold with rate λ' for any $\lambda' \in [\lambda, 2)$. Thus we have defined a 1-*parameter family* of moduli spaces $\mathcal{M}_L^{\lambda'}$ for L, and not just one. Since we did not impose any condition on λ in Definition 6.1 analogous to (14) in the conical singularities case, it turns out that $\mathcal{M}_L^\lambda$ depends nontrivially on λ.

The following result can be deduced from Marshall [23, Th. 6.2.15] and [23, Table 5.1]. (See also Pacini [28, Th. 2 & Th. 3].) It implies conjectures by the author in [9, Conj. 2.12] and [16, §10.2].

Theorem 6.8. *Let L be an asymptotically conical* SL *m-fold in $\mathbb{C}^m$ with cone C and rate $\lambda < 2$, and let $\mathcal{M}_L^\lambda$ be as in Definition* 6.7. *Set $\Sigma = C \cap \mathcal{S}^{2m-1}$, and let $\mathcal{D}_\Sigma, N_\Sigma$ be as in* §3.1 *and $b^k(L), b^k_{\mathrm{cs}}(L)$ as in* §3.4. *Then:*

(a) *If $\lambda \in (0,2) \setminus \mathcal{D}_\Sigma$ then $\mathcal{M}_L^\lambda$ is a manifold with*

$$\dim \mathcal{M}_L^\lambda = b^1(L) - b^0(L) + N_\Sigma(\lambda). \tag{34}$$

Note that if $0 < \lambda < \min\bigl(\mathcal{D}_\Sigma \cap (0,\infty)\bigr)$ then $N_\Sigma(\lambda) = b^0(\Sigma)$.

(b) *If $\lambda \in (2-m, 0)$ then $\mathcal{M}_L^\lambda$ is a manifold of dimension $b^1_{\mathrm{cs}}(L) = b^{m-1}(L)$.*

This is the analogue of Theorems 2.10 and 5.2 for AC SL m-folds. If $\lambda \in (2-m, 2) \setminus \mathcal{D}_\Sigma$ then the deformation theory for L with rate λ is *unobstructed* and $\mathcal{M}_L^\lambda$ is a *smooth manifold* with a given dimension. This is similar to the case of nonsingular compact SL m-folds in Theorem 2.10, but different to the conical singularities case in Theorem 5.2.

6.4 Examples

Examples of AC SL m-folds L are constructed by Harvey and Lawson [5, §III.3], the author [10, 11, 12, 14], and others. Nearly all the known examples (up to translations) have minimum rate λ either 0 or $2 - m$, which are topologically significant values by Proposition 6.3. For instance, all examples in [11] have $\lambda = 0$, and [10, Th. 6.4] constructs AC SL m-folds with $\lambda = 2 - m$ in $\mathbb{C}^m$ from any SL cone C in $\mathbb{C}^m$. The only explicit, nontrivial examples known to the author with $\lambda \neq 0, 2-m$ are in [12, Th. 11.6], and have $\lambda = \frac{3}{2}$.

We shall give three families of examples of AC SL m-folds L in $\mathbb{C}^m$ explicitly. The first family is adapted from Harvey and Lawson [5, §III.3.A].

Example 6.9. Let C^m_{HL} be the SL cone in $\mathbb{C}^m$ of Example 3.5. We shall define a family of AC SL m-folds in $\mathbb{C}^m$ with cone C^m_{HL}. Let $a_1, \ldots, a_m \geqslant 0$ with exactly two of the a_j zero and the rest positive. Write $\mathbf{a} = (a_1, \ldots, a_m)$, and define

$$\begin{aligned} L^{\mathbf{a}}_{\mathrm{HL}} = \bigl\{(z_1, \ldots, z_m) \in \mathbb{C}^m : i^{m+1} z_1 \cdots z_m \in [0,\infty),& \\ |z_1|^2 - a_1 = \cdots = |z_m|^2 - a_m\bigr\}.& \end{aligned} \tag{35}$$

Then $L^{\mathbf{a}}_{\rm HL}$ is an AC SL m-fold in $\mathbb{C}^m$ diffeomorphic to $T^{m-2}\times\mathbb{R}^2$, with cone $C^m_{\rm HL}$ and rate 0. It is invariant under the $\mathrm{U}(1)^{m-1}$ group (10). It is surprising that equations of the form (35) should define a nonsingular submanifold of $\mathbb{C}^m$ *without boundary*, but in fact they do.

Now suppose for simplicity that $a_1,\ldots,a_{m-2}>0$ and $a_{m-1}=a_m=0$. As $\Sigma^m_{\rm HL}\cong T^{m-1}$ we have $H^1(\Sigma^m_{\rm HL},\mathbb{R})\cong\mathbb{R}^{m-1}$, and calculation shows that $Y(L^{\mathbf{a}}_{\rm HL})=(\pi a_1,\ldots,\pi a_{m-2},0)\in\mathbb{R}^{m-1}$ in the natural coordinates. Since $L^{\mathbf{a}}_{\rm HL}\cong T^{m-2}\times\mathbb{R}^2$ we have $H^1(L^{\mathbf{a}}_{\rm HL},\mathbb{R})=\mathbb{R}^{m-2}$, and $Y(L^{\mathbf{a}}_{\rm HL})$ lies in the image $\mathbb{R}^{m-2}\subset\mathbb{R}^{m-1}$ of $H^1(L^{\mathbf{a}}_{\rm HL},\mathbb{R})$ in $H^1(\Sigma^m_{\rm HL},\mathbb{R})$, as in Definition 6.2. As $b^0(\Sigma^m_{\rm HL})=1$, Proposition 6.3 shows that $Z(L^{\mathbf{a}}_{\rm HL})=0$.

Take $C=C^m_{\rm HL}$, $\Sigma=\Sigma^m_{\rm HL}$ and $L=L^{\mathbf{a}}_{\rm HL}$ in Theorem 6.8, and let $0<\lambda<\min\bigl(\mathcal{D}_\Sigma\cap(0,\infty)\bigr)$. Then $b^1(L)=m-2$, $b^0(L)=1$ and $N_\Sigma(\lambda)=b^0(\Sigma)=1$, so part (a) of Theorem 6.8 shows that $\dim\mathcal{M}^\lambda_L=m-2$. This is consistent with the fact that L depends on $m-2$ real parameters $a_1,\ldots,a_{m-2}>0$.

The family of all $L^{\mathbf{a}}_{\rm HL}$ has $\frac{1}{2}m(m-1)$ connected components, indexed by which two of $a_1,\ldots,a_m$ are zero. Using the theory of §7, these can give many *topologically distinct* ways to desingularize SL m-folds with conical singularities with these cones.

Our second family, from [10, Ex. 9.4], was chosen for its simplicity.

Example 6.10. Let $m,a_1,\ldots,a_m,k$ and $L^{a_1,\ldots,a_m}_0$ be as in Example 3.6. For $0\neq c\in\mathbb{R}$ define

$$(36)\quad L^{a_1,\ldots,a_m}_c=\bigl\{\bigl(i\mathrm{e}^{ia_1\theta}x_1,\mathrm{e}^{ia_2\theta}x_2,\ldots,\mathrm{e}^{ia_m\theta}x_m\bigr):\theta\in[0,2\pi),\\ x_1,\ldots,x_m\in\mathbb{R},\qquad a_1x_1^2+\cdots+a_mx_m^2=c\bigr\}.$$

Then $L^{a_1,\ldots,a_m}_c$ is an AC SL m-fold in $\mathbb{C}^m$ with rate 0 and cone $L^{a_1,\ldots,a_m}_0$. It is diffeomorphic as an immersed SL m-fold to $(\mathcal{S}^{k-1}\times\mathbb{R}^{m-k}\times\mathcal{S}^1)/\mathbb{Z}_2$ if $c>0$, and to $(\mathbb{R}^k\times\mathcal{S}^{m-k-1}\times\mathcal{S}^1)/\mathbb{Z}_2$ if $c<0$.

Our third family was first found by Lawlor [21], made more explicit by Harvey [4, p. 139–140], and discussed from a different point of view by the author in [11, §5.4(b)]. Our treatment is based on that of Harvey.

Example 6.11. Let $m>2$ and $a_1,\ldots,a_m>0$, and define polynomials p,P by

$$p(x)=(1+a_1x^2)\cdots(1+a_mx^2)-1\quad\text{and}\quad P(x)=\frac{p(x)}{x^2}.$$

Define real numbers $\phi_1, \dots, \phi_m$ and A by

$$\phi_k = a_k \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1 + a_k x^2)\sqrt{P(x)}} \quad \text{and} \quad A = \omega_m (a_1 \cdots a_m)^{-1/2}, \tag{37}$$

where ω_m is the volume of the unit sphere in $\mathbb{R}^m$. Clearly $\phi_k, A > 0$. But writing $\phi_1 + \cdots + \phi_m$ as one integral gives

$$\phi_1 + \cdots + \phi_m = \int_0^{\infty} \frac{p'(x)\mathrm{d}x}{(p(x)+1)\sqrt{p(x)}} = 2\int_0^{\infty} \frac{\mathrm{d}w}{w^2+1} = \pi,$$

making the substitution $w = \sqrt{p(x)}$. So $\phi_k \in (0, \pi)$ and $\phi_1 + \cdots + \phi_m = \pi$. This yields a 1-1 correspondence between m-tuples $(a_1, \dots, a_m)$ with $a_k > 0$, and $(m+1)$-tuples $(\phi_1, \dots, \phi_m, A)$ with $\phi_k \in (0, \pi)$, $\phi_1 + \cdots + \phi_m = \pi$ and $A > 0$.

For $k = 1, \dots, m$ and $y \in \mathbb{R}$, define a function $z_k : \mathbb{R} \to \mathbb{C}$ by

$$\begin{aligned} z_k(y) &= \mathrm{e}^{i\psi_k(y)} \sqrt{a_k^{-1} + y^2}, \qquad \text{where} \\ \psi_k(y) &= a_k \int_{-\infty}^{y} \frac{\mathrm{d}x}{(1 + a_k x^2)\sqrt{P(x)}}. \end{aligned}$$

Now write $\boldsymbol{\phi} = (\phi_1, \dots, \phi_n)$, and define a submanifold $L^{\boldsymbol{\phi},A}$ in $\mathbb{C}^m$ by

$$L^{\boldsymbol{\phi},A} = \big\{ (z_1(y)x_1, \dots, z_m(y)x_m) : y \in \mathbb{R},\ x_k \in \mathbb{R},\ x_1^2 + \cdots + x_m^2 = 1 \big\}. \tag{38}$$

Then $L^{\boldsymbol{\phi},A}$ is closed, embedded, and diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}$, and Harvey [4, Th. 7.78] shows that $L^{\boldsymbol{\phi},A}$ is *special Lagrangian*. One can also show that $L^{\boldsymbol{\phi},A}$ is *asymptotically conical*, with rate $2 - m$ and cone the union $\Pi^0 \cup \Pi^{\boldsymbol{\phi}}$ of two special Lagrangian m-planes $\Pi^0, \Pi^{\boldsymbol{\phi}}$ in $\mathbb{C}^m$ given by

$$\Pi^0 = \big\{ (x_1, \dots, x_m) : x_j \in \mathbb{R} \big\},\ \Pi^{\boldsymbol{\phi}} = \big\{ (\mathrm{e}^{i\phi_1}x_1, \dots, \mathrm{e}^{i\phi_m}x_m) : x_j \in \mathbb{R} \big\}. \tag{39}$$

As $\lambda = 2 - m < 0$ we have $Y(L^{\boldsymbol{\phi},A}) = 0$ by Proposition 6.3. Now $L^{\boldsymbol{\phi},A} \cong \mathcal{S}^{m-1} \times \mathbb{R}$ so that $H^{m-1}(L^{\boldsymbol{\phi},A}, \mathbb{R}) \cong \mathbb{R}$, and $\Sigma = (\Pi^0 \cup \Pi^{\boldsymbol{\phi}}) \cap \mathcal{S}^{2m-1}$ is the disjoint union of two unit $(m-1)$-spheres $\mathcal{S}^{m-1}$, so $H^{m-1}(\Sigma, \mathbb{R}) \cong \mathbb{R}^2$. The image of $H^{m-1}(L^{\boldsymbol{\phi},A}, \mathbb{R})$ in $H^{m-1}(\Sigma, \mathbb{R})$ is $\big\{ (x, -x) : x \in \mathbb{R} \big\}$ in the natural coordinates. Calculation shows that $Z(L^{\boldsymbol{\phi},A}) = (A, -A) \in H^{m-1}(\Sigma, \mathbb{R})$, which is why we defined A this way in (37).

Apply Theorem 6.8 with $L = L^{\phi,A}$ and $\lambda \in (2-m, 0)$. As $L \cong \mathcal{S}^{m-1} \times \mathbb{R}$ we have $b^1_{\mathrm{cs}}(L) = 1$, so part (b) of Theorem 6.8 shows that $\dim \mathcal{M}^\lambda_L = 1$. This is consistent with the fact that when $\boldsymbol{\phi}$ is fixed, $L^{\phi,A}$ depends on one real parameter $A > 0$. Here $\boldsymbol{\phi}$ is fixed in $\mathcal{M}^\lambda_L$ as the cone $C = \Pi^0 \cup \Pi^\phi$ of L depends on $\boldsymbol{\phi}$, and all $\hat{L} \in \mathcal{M}^\lambda_L$ have the same cone C, by definition.

7. Desingularizing singular SL m-folds

We now discuss the work of [19, 20] on *desingularizing* compact SL m-folds with conical singularities. Here is the basic idea. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold, and X a compact SL m-fold in M with conical singularities $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Suppose $L_1, \ldots, L_n$ are AC SL m-folds in $\mathbb{C}^m$ with the same cones $C_1, \ldots, C_n$ as X.

If $t > 0$ then $tL_i = \{t\,\mathbf{x} : \mathbf{x} \in L_i\}$ is also an AC SL m-fold with cone C_i. We construct a 1-parameter family of compact, nonsingular *Lagrangian* m-folds N^t in (M, ω) for $t \in (0, \delta)$ by gluing tL_i into X at x_i, using a partition of unity.

When t is small, N^t is *close to special Lagrangian* (its phase is nearly constant), but also *close to singular* (it has large curvature and small injectivity radius). We prove using analysis that for small $t \in (0, \delta)$ we can deform N^t to a *special* Lagrangian m-fold $\widetilde{N}^t$ in M, using a small Hamiltonian deformation. The proof involves a delicate balancing act, showing that the advantage of being close to special Lagrangian outweighs the disadvantage of being nearly singular.

Doing this in full generality is rather complex. There are two kinds of *obstructions* to the existence of $\widetilde{N}^t$. Firstly, if $Y(L_i) \neq 0$ then N^t may not exist as a *Lagrangian* m-fold. Secondly, if X' is not connected then we may not be able to deform N^t to a *special* Lagrangian m-fold $\widetilde{N}^t$ because of problems with small eigenvalues of the Laplacian Δ on N^t. In each case, $\widetilde{N}^t$ exists for small t if the $Y(L_i)$ or $Z(L_i)$ satisfy an equation.

We also extend the results to desingularization in *families* of almost Calabi-Yau m-folds $(M, J^s, \omega^s, \Omega^s)$. The cohomology classes $[\omega^s], [\mathrm{Im}\,\Omega^s]$ contribute to the obstruction equations in $Y(L_i)$ and $Z(L_i)$ for the existence of $\widetilde{N}^t$. Thus, a singular SL m-fold X which has no desingularizations $\widetilde{N}^t$ in (M, J, ω, Ω) can still admit desingularizations $\widetilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ for small $s \neq 0$.

We begin in §7.1 by explaining desingularization in the simplest case, in one almost Calabi-Yau m-fold (M, J, ω, Ω) when $Y(L_i) = 0$ and X' is connected. Section 7.2 extends this to X' not connected, and §7.3 to $Y(L_i) \neq 0$, introducing the two kinds of obstructions. Section 7.4 discusses desingularization in families $(M, J^s, \omega^s, \Omega^s)$.

7.1 Desingularization in the simplest case

Our simplest desingularization result is [19, Th. 6.13].

Theorem 7.1. *Suppose (M, J, ω, Ω) is an almost Calabi-Yau m-fold and X a compact* SL *m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $L_1, \ldots, L_n$ be asymptotically conical* SL *m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \ldots, n$, and $X' = X \setminus \{x_1, \ldots, x_n\}$ is connected.*

Then there exists $\epsilon > 0$ and a smooth family $\{\widetilde{N}^t : t \in (0, \epsilon]\}$ of compact, nonsingular SL *m-folds in (M, J, ω, Ω), such that $\widetilde{N}^t$ is constructed by gluing tL_i into X at x_i for $i = 1, \ldots, n$. In the sense of currents, $\widetilde{N}^t \to X$ as $t \to 0$.*

Here is a sketch of the proof, divided into seven steps.

Step 1. Apply Theorem 4.1 to C_i for $i = 1, \ldots, n$, and Theorem 4.3 to X, and Theorem 6.5 to L_i for $i = 1, \ldots, n$. This gives *Lagrangian neighbourhoods* U_{C_i}, Φ_{C_i} for C_i, and $U_{X'}, \Phi_{X'}$ for X', and U_{L_i}, Φ_{L_i} for L_i.

Moreover $U_{X'}, \Phi_{X'}$ and U_{C_i}, Φ_{C_i} are related via Υ_i and an exact 1-form η_i on $\Sigma_i \times (0, R')$ from Theorem 4.2, and U_{L_i}, Φ_{L_i} and U_{C_i}, Φ_{C_i} are related via an exact 1-form χ_i on $\Sigma_i \times (T, \infty)$ from Theorem 6.4.

Step 2. Let $t > 0$ be small. We define a nonsingular Lagrangian m-fold N^t in (M, ω), roughly as follows. Choose $\tau \in (0, 1)$ satisfying certain conditions. At distance at least $2t^\tau$ from $x_1, \ldots, x_n$ we define N^t to be X'. At distance up to t^τ from x_i we define N^t to be $\Upsilon_i(tL_i \cap B_R)$.

Between distances t^τ and $2t^\tau$ from x_i we define N^t to be a Lagrangian annulus $\Sigma_i \times (t^\tau, 2t^\tau)$ interpolating between X' and $\Upsilon_i(tL_i \cap B_R)$, using the Lagrangian neighbourhoods U_{C_i}, Φ_{C_i}, $U_{X'}, \Phi_{X'}$ and U_{L_i}, Φ_{L_i}. This is equivalent to choosing a closed 1-form $\xi_i^t(\sigma, r)$ on $\Sigma_i \times [t^\tau, 2t^\tau]$ interpolating between $t^2\chi_i(\sigma, t^{-1}r)$ at $r = t^\tau$ and $\eta_i(\sigma, r)$ at $r = 2t^\tau$.

Step 3. Let $e^{i\theta^t}$ be the phase function of N^t, so that N^t is special Lagrangian if $\sin\theta^t \equiv 0$. We bound various norms of $\psi^m \sin\theta^t$ in terms of powers of t. These bounds imply that N^t is *close to special Lagrangian* when t is small. We also estimate other geometrical quantities, like the curvature and injectivity radius of N^t, in terms of powers of t.

Step 4. We glue together the Lagrangian neighbourhoods U_{C_i}, Φ_{C_i}, $U_{X'}$, $\Phi_{X'}$ and U_{L_i}, Φ_{L_i} to define a Lagrangian neighbourhood U_{N^t}, Φ_{N^t} for N^t.

Step 5. Let $f \in C^\infty(N^t)$. Then $\mathrm{d}f$ is a 1-form on N^t, and the graph $\Gamma(\mathrm{d}f)$ is a submanifold of T^*N^t. If f is small in C^1 then $\Gamma(\mathrm{d}f) \subset U_{N^t} \subset T^*N^t$, and then $\widetilde{N}^t = \Phi_{N^t}\bigl(\Gamma(\mathrm{d}f)\bigr)$ is a nonsingular Lagrangian m-fold in (M,ω). Every small Hamiltonian deformation of N^t can be written in this way.

We show that $\widetilde{N}^t$ is *special* Lagrangian if and only if

$$\mathrm{d}^*(\psi^m \cos\theta^t \mathrm{d}f)(x) = \psi^m \sin\theta^t + Q^t\bigl(x, \mathrm{d}f(x), \nabla^2 f(x)\bigr) \tag{40}$$

for all $x \in N^t$, as in (25), (27), where Q^t is smooth and $Q^t(x,y,z) = O\bigl(t^{-2}|y|^2 + |z|^2\bigr)$ for small y,z.

Step 6. Working in the Sobolev space $L^{2m}_3(N^t)$, we show that the operator

$$\begin{gathered} P^t : \bigl\{u \in L^{2m}_3(N^t) : \textstyle\int_{N^t} u \,\mathrm{d}V^t = 0\bigr\} \to \bigl\{v \in L^{2m}_1(N^t) : \textstyle\int_{N^t} v \,\mathrm{d}V^t = 0\bigr\} \\ \text{given by} \quad P^t(u) = \mathrm{d}^*(\psi^m \cos\theta^t \mathrm{d}u) \end{gathered} \tag{41}$$

has an inverse $(P^t)^{-1}$ which is (in a rather weak sense) bounded independently of t. The restriction to u with $\int_{N^t} u \,\mathrm{d}V^t = 0$ is necessary as $P^t(1) = 0$, so P^t is not invertible on spaces including 1.

Step 7. We inductively construct a sequence $(f_k)_{k=0}^\infty$ in $L^{2m}_3(N^t)$ with $f_0 = 0$, $\int_{N^t} f_k \,\mathrm{d}V^t = 0$ and $f_k = (P^t)^{-1}\bigl(\psi^m \sin\theta^t + Q^t(x, \mathrm{d}f_{k-1}, \nabla^2 \mathrm{d}f_{k-1})\bigr)$, so

$$\mathrm{d}^*(\psi^m \cos\theta^t \mathrm{d}f_k) \equiv \psi^m \sin\theta^t + Q^t\bigl(x, \mathrm{d}f_{k-1}(x), \nabla^2 f_{k-1}(x)\bigr). \tag{42}$$

Using the bounds on $\psi^m \sin\theta^t$ from Step 3 and on $(P^t)^{-1}$ from Step 6 we show that $(f_k)_{k=0}^\infty$ exists and converges in $L^{2m}_3(N^t)$

for small t. The limit f satisfies (40), and is smooth by elliptic regularity. Then $\tilde{N}^t = \Phi_{N^t}\bigl(\Gamma(\mathrm{d}f)\bigr)$ is the SL m-fold we seek.

The condition $\lambda_i < 0$ in Theorem 6.3 is there for two reasons. Firstly, it forces $Y(L_i) = 0$ by Proposition 6.3, and therefore χ_i is an *exact* 1-form on $\Sigma_i \times (T, \infty)$, since $[\chi_i] = Y(L_i) \in H^1(\Sigma_i, \mathbb{R})$ by Theorem 6.6. This exactness makes it possible to define the closed 1-form ξ_i^t in Step 2.

Secondly, we need $\lambda_i < 0$ so that the contributions to $\psi^m \sin\theta^t$ from tapering χ_i off to zero on the annulus $\Sigma_i \times [t^\tau, 2t^\tau]$ are small enough for the method to work. If $\lambda_i > 0$ then $\|\psi^m \sin\theta^t\|_{L^2}$ is too large, and we cannot prove that the sequence $(f_k)_{k=0}^\infty$ in Step 7 converges.

7.2 Desingularization when X' is not connected

In [19, Th. 7.10] we extend Theorem 7.1 to X' not connected.

Theorem 7.2. *Suppose (M, J, ω, Ω) is an almost Calabi-Yau m-fold and X a compact* SL *m-fold in M with conical singularities at $x_1, \dots, x_n$ and cones $C_1, \dots, C_n$. Define $\psi : M \to (0, \infty)$ as in (4). Let $L_1, \dots, L_n$ be asymptotically conical* SL *m-folds in $\mathbb{C}^m$ with cones $C_1, \dots, C_n$ and rates $\lambda_1, \dots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \dots, n$. Write $X' = X \setminus \{x_1, \dots, x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$.*

Set $q = b^0(X')$, and let $X'_1, \dots, X'_q$ be the connected components of X'. For $i = 1, \dots, n$ let $l_i = b^0(\Sigma_i)$, and let $\Sigma_i^1, \dots, \Sigma_i^{l_i}$ be the connected components of Σ_i. Define $k(i,j) = 1, \dots, q$ by $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr) \subset X'_{k(i,j)}$ for $i = 1, \dots, n$ and $j = 1, \dots, l_i$. Suppose that

$$\sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j) = k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] = 0 \quad \textit{for all } k = 1, \dots, q. \tag{43}$$

Suppose also that the compact m-manifold N obtained by gluing L_i into X' at x_i for $i = 1, \dots, n$ is connected. A sufficient condition for this to hold is that X and L_i for $i = 1, \dots, n$ are connected.

Then there exists $\epsilon > 0$ and a smooth family $\bigl\{\tilde{N}^t : t \in (0, \epsilon]\bigr\}$ of compact, nonsingular SL *m-folds in (M, J, ω, Ω) diffeomorphic to N, such that $\tilde{N}^t$ is constructed by gluing tL_i into X at x_i for $i = 1, \dots, n$. In the sense of currents in geometric measure theory, $\tilde{N}^t \to X$ as $t \to 0$.*

The new issues when X' is not connected occur in Steps 6 and 7 of §7.1. Suppose $b^0(X') = q > 1$, so that X' has q connected components $X'_1, \dots, X'_q$. Then the operator P^t of (41) turns out to have $q - 1$ *small* *positive eigenvalues* $\lambda_1, \ldots, \lambda_{q-1}$ of size $O(t^{m-2})$. The corresponding eigenfunctions $w_1, \ldots, w_{q-1}$ are approximately constant on the parts of N^t coming from each X'_k, and change rapidly on the 'small necks' in between.

As $(P^t)^{-1} w_k = \lambda_k^{-1} w_k$ and $\lambda_k = O(t^{m-2})$ we see that $(P^t)^{-1}$ is $O(t^{2-m})$ on $\langle w_1, \ldots, w_{q-1} \rangle$, and so cannot be bounded independently of t. To repair the proof, roughly speaking we set $W^t = \langle 1, w_1, \ldots, w_{q-1} \rangle$, and let $(W^t)^\perp$ be the orthogonal subspace to W^t in $L^2(N^t)$. Then P^t maps

$$P^t : L^{2m}_3(N^t) \cap (W^t)^\perp \to L^{2m}_1(N^t) \cap (W^t)^\perp \tag{44}$$

and has an inverse $(P^t)^{-1}$ bounded independently of t on these spaces, in a weak sense. (Actually, we do something more complicated than this, in which W^t is an approximation to $\langle 1, w_1, \ldots, w_{q-1} \rangle$ defined explicitly in terms of bounded harmonic functions on $L_1, \ldots, L_n$.)

In Step 7, the sequence $(f_k)_{k=0}^\infty$ is constructed as before. The bound on the inverse of (44) can be used to inductively bound the components of f_k in $(W^t)^\perp$. But we still need to bound the components $\pi_{W^t}(f_k)$ of f_k in W^t. Since $f_0 = 0$ and $Q(x, 0, 0) = 0$, Equation (42) gives $P^t f_1 = \psi^m \sin \theta^t$, so that $f_1 = (P^t)^{-1}(\psi^m \sin \theta^t)$. It turns out that we need $\pi_{W^t}(f_1) = o(t^2)$ for f_k to remain small as $k \to \infty$.

As $(P^t)^{-1} = O(t^{2-m})$ on $\langle w_1, \ldots, w_{q-1} \rangle$, this holds if $\pi_{W^t}(\psi^m \sin \theta^t) = o(t^m)$. Calculation shows that the dominant term in $\pi_{W^t}(\psi^m \sin \theta^t)$ is $O(t^m)$, and proportional to the left-hand side of (43). Therefore $\pi_{W^t}(\psi^m \sin \theta^t) = o(t^m)$ if and only if (43) holds, and this is the condition for the sequence $(f_k)_{k=0}^\infty$ to remain bounded and converge to a small solution of (40).

If X' is connected, so that $q = 1$, then $k(i,j) \equiv 1$ and (43) becomes

$$\sum_{i=1}^n \psi(x_i)^m Z(L_i) \cdot \sum_{j=1}^{l_i} [\Sigma_i^j] = 0.$$

But $\sum_{j=1}^{l_i} [\Sigma_i^j] = [\Sigma_i]$, and $Z(L_i) \cdot [\Sigma_i] = 0$ as $Z(L_i)$ is the image of a class in $H^{m-1}(L_i, \mathbb{R})$, and Σ_i is the boundary of L_i. Therefore (43) holds automatically when X' is connected, and Theorem 7.2 reduces to Theorem 7.1 in this case.

7.3 Desingularization when $Y(L_i) \neq 0$

In [20, Th. 6.13] we extend Theorem 7.1 to $\lambda_i \leqslant 0$, allowing $Y(L_i) \neq 0$.

Theorem 7.3. *Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold for $2 < m < 6$, and X a compact* SL *m-fold in M with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $L_1, \ldots, L_n$ be asymptotically conical* SL *m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose that $\lambda_i \leqslant 0$ for $i = 1, \ldots, n$, that $X' = X \setminus \{x_1, \ldots, x_n\}$ is connected, and that there exists $\varrho \in H^1(X', \mathbb{R})$ such that $\bigl(Y(L_1), \ldots, Y(L_n)\bigr)$ is the image of ϱ under the map $H^1(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ in* (16)*, where $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$.*

Then there exists $\epsilon > 0$ and a smooth family $\bigl\{\widetilde{N}^t : t \in (0, \epsilon]\bigr\}$ of compact, nonsingular SL *m-folds in (M, J, ω, Ω), with $\widetilde{N}^t$ constructed by gluing tL_i into X at x_i for $i = 1, \ldots, n$. In the sense of currents, $\widetilde{N}^t \to X$ as $t \to 0$.*

There is also [20, Th. 6.12] an analogue of Theorem 7.2, combining the modifications of Theorems 7.2 and 7.3, but for brevity we will not give it.

The new issues when $Y(L_i) \neq 0$ come mostly in Step 2 of §7.1. As $[\chi_i] = Y(L_i) \in H^1(\Sigma_i, \mathbb{R})$ by Theorem 6.6, if $Y(L_i) \neq 0$ then χ_i is no longer an *exact* form. Therefore, in Step 2 we cannot choose a closed 1-form ξ_i^t on $\Sigma_i \times [t^\tau, 2t^\tau]$ interpolating between $t^2\chi_i$ at $r = t^\tau$ and η_i at $r = 2t^\tau$, since $t^2\chi_i$ and η_i have different cohomology classes.

Thus we cannot choose N^t to coincide with X away from x_i, and work locally near x_i, as we did in §7.1. Instead, we define N^t away from x_i to be $\Phi_{X'}\bigl(\Gamma(t^2\alpha)\bigr)$, where α is a 1-form on X' satisfying $\mathrm{d}\alpha = \mathrm{d}^*(\psi^m\alpha) = 0$, and $|\alpha| = O(r^{-1})$ near x_i. We show using analysis on manifolds with ends that there is a unique such 1-form α with $[\alpha] = \varrho$ for each $\varrho \in H^1(X', \mathbb{R})$.

To glue $\Upsilon_i(tL_i \cap B_R)$ and $\Phi_{X'}\bigl(\Gamma(t^2\alpha)\bigr)$ together as Lagrangian m-folds the cohomology classes of $t^2\chi_i$ and $t^2\alpha$ must agree in $H^1(\Sigma_i, \mathbb{R})$. This holds if the image of ϱ under the map $H^1(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ is $\bigl(Y(L_1), \ldots, Y(L_n)\bigr)$, as in the theorem. Thus, the existence of ϱ with this property is a *necessary condition* for the existence of N^t as a *Lagrangian* m-fold in (M, ω).

After constructing N^t we need to estimate norms of $\psi^m \sin\theta^t$ in Step 3. The condition $\mathrm{d}^*(\psi^m\alpha) = 0$ means that linear terms in $t^2\alpha$ contribute 0 to $\psi^m \sin\theta^t$. However, quadratic terms in $t^2\alpha$ contribute $O(t^4)$ to $\psi^m \sin\theta^t$ on most of N^t, so all norms of $\psi^m \sin\theta^t$ are at least $O(t^4)$.

Now to show that $(f_k)_{k=0}^\infty$ converges in Step 7 we need $\|\psi^m \sin\theta^t\|_{L^2} = o(t^{1+m/2})$ for small t. As $\|\psi^m \sin\theta^t\|_{L^2}$ has $O(t^4)$ contributions, this is possible only if $m < 6$. Therefore we have to restrict to complex dimension $m < 6$ when $Y(L_i) \neq 0$ for this method of proof to work.

7.4 Desingularization in families $(M, J^s, \omega^s, \Omega^s)$

Next we explain the work of [20, §7–§8] on desingularization in *families* of almost Calabi-Yau m-folds $(M, J^s, \omega^s, \Omega^s)$. The analogue of Theorem 7.1 is [20, Th. 7.15], but for brevity we will not give it. Here [20, Th. 7.14] is the families analogue of Theorem 7.2.

Theorem 7.4. *Suppose (M, J, ω, Ω) is an almost Calabi-Yau m-fold and X a compact* SL *m-fold in M with conical singularities at $x_1, \dots, x_n$ and cones $C_1, \dots, C_n$. Define $\psi : M \to (0, \infty)$ as in (4). Let $L_1, \dots, L_n$ be asymptotically conical* SL *m-folds in $\mathbb{C}^m$ with cones $C_1, \dots, C_n$ and rates $\lambda_1, \dots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \dots, n$. Write $X' = X \setminus \{x_1, \dots, x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$.*

Set $q = b^0(X')$, and let $X'_1, \dots, X'_q$ be the connected components of X'. For $i = 1, \dots, n$ let $l_i = b^0(\Sigma_i)$, and let $\Sigma_i^1, \dots, \Sigma_i^{l_i}$ be the connected components of Σ_i. Define $k(i,j) = 1, \dots, q$ by $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr) \subset X'_{k(i,j)}$ for $i = 1, \dots, n$ and $j = 1, \dots, l_i$. Suppose the compact m-manifold N obtained by gluing L_i into X' at x_i for $i = 1, \dots, n$ is connected. A sufficient condition for this to hold is that X and L_i for $i = 1, \dots, n$ are connected.

Suppose $\bigl\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\bigr\}$ is a smooth family of deformations of (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$. Let $\iota_* : H_2(X, \mathbb{R}) \to H_2(M, \mathbb{R})$ be the natural inclusion. Suppose that

$$[\omega^s] \cdot \iota_*(\gamma) = 0 \quad \text{for all } s \in \mathcal{F} \text{ and } \gamma \in H_2(X, \mathbb{R}). \tag{45}$$

Define $\mathcal{G} \subseteq \mathcal{F} \times (0,1)$ to be the subset of $(s,t) \in \mathcal{F} \times (0,1)$ with

$$\bigl[\operatorname{Im}\Omega^s\bigr] \cdot \bigl[\overline{X'_k}\bigr] = t^m \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j) = k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] \quad \text{for } k = 1, \dots, q. \tag{46}$$

Then there exist $\epsilon \in (0,1)$ and $\kappa > 1$ and a smooth family

$$\bigl\{\widetilde{N}^{s,t} : (s,t) \in \mathcal{G}, \quad t \in (0, \epsilon], \quad |s| \leqslant t^{\kappa + m/2}\bigr\}, \tag{47}$$

such that $\widetilde{N}^{s,t}$ is a compact, nonsingular SL *m-fold in $(M, J^s, \omega^s, \Omega^s)$ diffeomorphic to N, which is constructed by gluing tL_i into X at x_i for $i = 1, \dots, n$. In the sense of currents in geometric measure theory, $\widetilde{N}^{s,t} \to X$ as $s, t \to 0$.*

To prove it we modify Steps 1–7 of §7.1 in the following ways. In Step 1 we generalize Theorem 4.3 to give smooth families of maps $\Upsilon_i^s : B_R \to M$ and $\Phi_{X'}^s : U_{X'} \to M$ for small $s \in \mathcal{F}$ with $(\Upsilon_i^s)^*(\omega^s) = \omega'$, $(\Phi_{X'}^s)^*(\omega^s) = \hat{\omega}$ and $\Upsilon_i^0 = \Upsilon_i$, $\Phi_{X'}^0 = \Phi_{X'}$. Using these, in Step 2 we define a smooth family of Lagrangian m-folds $N^{s,t}$ in (M, ω^s) for small $s \in \mathcal{F}$ and $t \in (0, \delta)$. In the rest of the proof we make everything depend on $s \in \mathcal{F}$, and deform $N^{s,t}$ to an SL m-fold $\widetilde{N}^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ for small $s \in \mathcal{F}$ and $t \in (0, \delta)$.

To allow X' not connected, as in §7.2, we introduce a vector subspace $W^{s,t} \subset C^\infty(N^{s,t})$, and we need $\pi_{W^{s,t}}(\psi^m \sin\theta^{s,t}) = o(t^m)$. The dominant terms in $\pi_{W^{s,t}}(\psi^m \sin\theta^{s,t})$ are of two kinds: $O(t^m)$ terms involving the $Z(L_i)$, as in §7.2, and also terms in $\bigl[\operatorname{Im}\Omega^s\bigr] \cdot \bigl[\overline{X_k'}\bigr]$. Equation (46) requires these two terms to cancel, so that $\pi_{W^{s,t}}(\psi^m \sin\theta^{s,t}) = o(t^m)$, and the rest of the proof works.

Here [20, Th. 8.10] is the families analogue of Theorem 7.3.

Theorem 7.5. *Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold for $2 < m < 6$, and X a compact* SL *m-fold in M with conical singularities at $x_1, \dots, x_n$ and cones $C_1, \dots, C_n$. Let $L_1, \dots, L_n$ be asymptotically conical* SL *m-folds in $\mathbb{C}^m$ with cones $C_1, \dots, C_n$ and rates $\lambda_1, \dots, \lambda_n$. Suppose $\lambda_i \leqslant 0$ for $i = 1, \dots, n$, and $X' = X \setminus \{x_1, \dots, x_n\}$ is connected.*

Suppose $\bigl\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\bigr\}$ is a smooth family of deformations of (M, J, ω, Ω), with base space $\mathcal{F} \subset \mathbb{R}^d$, satisfying

$$[\operatorname{Im}\Omega^s] \cdot [X] = 0 \quad \textit{for all } s \in \mathcal{F}, \textit{ where } [X] \in H_m(M, \mathbb{R}). \tag{48}$$

Define $\varpi \in H^2_{\mathrm{cs}}(X', \mathbb{R})$ to be the image of $\bigl(Y(L_1), \dots, Y(L_n)\bigr)$ under the map $\bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R}) \to H^2_{\mathrm{cs}}(X', \mathbb{R})$ in (16). Define $\mathcal{G} \subseteq \mathcal{F} \times (0,1)$ to be

(49)
$$\mathcal{G} = \bigl\{(s,t) \in \mathcal{F} \times (0,1) : [\omega^s] \cdot \iota_*(\gamma) = t^2 \varpi \cdot \gamma \ \textit{ for all } \gamma \in H_2(X, \mathbb{R})\bigr\},$$

where $\iota_* : H_2(X, \mathbb{R}) \to H_2(M, \mathbb{R})$ is the natural inclusion.

Then there exist $\epsilon \in (0,1)$, $\kappa > 1$ and $\vartheta \in (0,2)$ and a smooth family

$$\bigl\{\widetilde{N}^{s,t} : (s,t) \in \mathcal{G}, \quad t \in (0, \epsilon], \quad |s| \leqslant t^\vartheta\bigr\}, \tag{50}$$

such that $\widetilde{N}^{s,t}$ is a compact, nonsingular SL *m-fold in $(M, J^s, \omega^s, \Omega^s)$, which is constructed by gluing tL_i into X at x_i for $i = 1, \dots, n$. In the sense of currents in geometric measure theory, $\widetilde{N}^{s,t} \to X$ as $s, t \to 0$.*

In §7.3 we saw that when $Y(L_i) \neq 0$ there is a topological obstruction to defining N^t as a Lagrangian m-fold in (M, ω), so that N^t exists only if the $Y(L_i)$ satisfy an equation. In this case there is also an obstruction to defining $N^{s,t}$ as a Lagrangian m-fold in (M,ω^s), but now the condition in (49) for $N^{s,t}$ to exist involves both $Y(L_i)$, which determine ϖ, and $[\omega^s]$.

Here is how to understand the relation between the conditions for N^t to exist in Theorem 7.3, and for $N^{s,t}$ to exist in Theorem 7.5. As (16) is exact, $\big(Y(L_1),\ldots,Y(L_n)\big)$ is the image of $\varrho \in H^1(X',\mathbb{R})$ if and only if the image ϖ of $\big(Y(L_1),\ldots,Y(L_n)\big)$ in $H^2_{\rm cs}(X',\mathbb{R})$ is zero.

Now $\omega^0 = \omega$ and $[\omega]\cdot\iota_*(\gamma) = 0$ for all $\gamma \in H_2(X,\mathbb{R})$, as X' is Lagrangian in (M,ω). Thus when $s = 0$, Equation (49) reduces to $t^2\varpi\cdot\gamma = 0$ for all γ. But $H^2_{\rm cs}(X',\mathbb{R}) \cong H_2(X,\mathbb{R})^*$ by (18). Thus when $s = 0$ Equation (49) is equivalent to $\varpi = 0$, which is equivalent to the existence of ϱ in Theorem 7.3.

Note that as the conditions (46) and (49) for the existence of $\widetilde{N}^{s,t}$ involve both s and t, it can happen that an SL m-fold X with conical singularities admits no desingularizations $\widetilde{N}^t$ in (M,J,ω,Ω), but does admit desingularizations $\widetilde{N}^{s,t}$ in $(M,J^s,\omega^s,\Omega^s)$ for small $s \neq 0$. Thus we can overcome obstructions to the existence of desingularizations by varying the underlying almost Calabi-Yau m-fold (M,J,ω,Ω).

We also prove a theorem [20, Th. 8.9] combining Theorems 7.4 and 7.5, desingularizing in families when $Y(L_i) \neq 0$ and X' is not connected. However, for technical reasons it is not as strong as the author would like, in that we must assume both sides of (46) are zero rather than just that (46) holds.

8. Discussion: How moduli spaces fit together

We now consider the *boundary* $\partial\mathcal{M}_N$ of a moduli space $\mathcal{M}_N$ of SL m-folds.

Definition 8.1. Let (M,J,ω,Ω) be an almost Calabi-Yau m-fold, N a compact, nonsingular SL m-fold in M, and $\mathcal{M}_N$ the moduli space of deformations of N in M. Then $\mathcal{M}_N$ is a smooth manifold of dimension $b^1(N)$, by Theorem 2.10. In general $\mathcal{M}_N$ will be a *noncompact* manifold, but we can construct a natural *compactification* $\overline{\mathcal{M}}_N$ as follows.

Regard $\mathcal{M}_N$ as a moduli space of special Lagrangian *integral currents* in the sense of geometric measure theory, as discussed in [17, §6]. An introduction to geometric measure theory can be found in Morgan [27]. Let $\overline{\mathcal{M}}_N$ be the closure of $\mathcal{M}_N$ in the space of integral currents. As elements of $\mathcal{M}_N$ have uniformly bounded volume, $\overline{\mathcal{M}}_N$ is *compact* by [27, 5.5].

Define the *boundary* $\partial\mathcal{M}_N$ to be $\overline{\mathcal{M}}_N \setminus \mathcal{M}_N$. Then elements of $\partial\mathcal{M}_N$ are *singular special Lagrangian integral currents*. Essentially, they are singular SL m-folds X in M which are limits of nonsingular $\hat{N} \in \mathcal{M}_N$ in an appropriate sense. By a result of Almgren [1], the singular set of each $X \in \partial\mathcal{M}_N$ has Hausdorff dimension at most $m-2$.

In good cases, say if (M, J, ω, Ω) is suitably generic, it seems reasonable that $\partial\mathcal{M}_N$ should be divided into a number of *strata*, each of which is a moduli space of singular SL m-folds with singularities of a particular type, and is itself a manifold with singularities. In particular, some or all of these strata could be moduli spaces $\mathcal{M}_X$ of SL m-folds with isolated conical singularities, as in §5.

In this case, using §7 for each $\hat{X} \in \mathcal{M}_X$ we can try to construct desingularizations $\widetilde{N}^t$ in $\mathcal{M}_N$ by gluing in AC SL m-folds $\hat{L}_i$ at the singular points $\hat{x}_i$ of $\hat{X}$ for $i = 1, \ldots, n$. In good cases, say when the cones C_i of $\hat{X}$ are *stable*, every $\hat{N} \in \mathcal{M}_N$ close to $\mathcal{M}_X$ might be constructed uniquely from some $\hat{X}, \hat{L}_1, \ldots, \hat{L}_n$, and so we could identify an open neighbourhood of $\mathcal{M}_X$ in $\mathcal{M}_N$ with a submanifold of the product $\mathcal{M}_X \times \mathcal{M}^0_{L_1} \times \cdots \times \mathcal{M}^0_{L_n}$.

The goal of this section is to work towards such a description of $\mathcal{M}_N$ near a boundary stratum $\mathcal{M}_X$ which is a moduli space of SL m-folds with conical singularities. Our treatment will be informal or conjectural in places, and is far from giving a complete picture of $\partial\mathcal{M}_N$.

8.1 Topological calculations and dimension counting

We shall consider the following situation.

Definition 8.2. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold for $2 < m < 6$. Here the assumption $m < 6$ is only so that we can apply Theorem 7.3 and [20, Th. 6.12], and all of the topological calculations below actually hold when $m > 2$. Define $\psi : M \to (0, \infty)$ as in (4).

Let X be a compact SL m-fold in M with conical singularities $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Write $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$ and $X' = X \setminus \{x_1, \ldots, x_n\}$. Let $\mathcal{M}_X$ be the moduli space of deformations of X in M, as in Definition 5.1. Write $\hat{X}$ for a general element of $\mathcal{M}_X$. Let $\mathcal{I}_{X'}$ be the image of $H^1_{\mathrm{cs}}(X', \mathbb{R})$ in $H^1(X', \mathbb{R})$, as in Theorem 5.2.

Let $L_1, \ldots, L_n$ be asymptotically conical SL m-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rate 0. Let $\mathcal{M}^0_{L_i}$ be the moduli space of deformations of L_i with rate 0, as in Definition 6.7. Write $\hat{L}_i$ for a general element of $\mathcal{M}^0_{L_i}$.

Let $q = b^0(X')$ and $X'_1, \dots, X'_q$ be the connected components of X'. For $i = 1, \dots, n$ let $l_i = b^0(\Sigma_i)$, and let $\Sigma_i^1, \dots, \Sigma_i^{l_i}$ be the connected components of Σ_i. Define $k(i,j) = 1, \dots, q$ by $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr) \subset X'_{k(i,j)}$, as usual.

Define $\mathcal{Y}_i \subset H^1(\Sigma_i, \mathbb{R})$ and $\mathcal{Z}_i \subset H^{m-1}(\Sigma_i, \mathbb{R})$ to be the images of the map $H^k(L_i, \mathbb{R}) \to H^k(\Sigma_i, \mathbb{R})$ of (29) for $k = 1, m-1$. Define maps $\pi_{\mathcal{Y}_i} : \mathcal{M}^0_{L_i} \to \mathcal{Y}_i$ and $\pi_{\mathcal{Z}_i} : \mathcal{M}^0_{L_i} \to \mathcal{Z}_i$ by $\pi_{\mathcal{Y}_i}(\hat{L}_i) = Y(\hat{L}_i)$ and $\pi_{\mathcal{Z}_i}(\hat{L}_i) = Z(\hat{L}_i)$. These are well-defined as $Y(L_i), Z(L_i)$ are images of classes in $H^k(L_i, \mathbb{R})$ by Definition 6.2. Write general elements of $\mathcal{Y}_i$ as γ_i, and of $\mathcal{Z}_i$ as δ_i.

Let the vector subspace $\mathcal{Y}$ in $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$ be the intersection of $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$ with the image of the map $H^1(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ in (16). Let the vector subspace $\mathcal{Z}$ in $\mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n$ be the set of all $(\delta_1, \dots, \delta_n)$ for which

$$\sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j) = k}} \psi(x_i)^m \delta_i \cdot [\Sigma_i^j\,] = 0 \quad \text{for all } k = 1, \dots, q. \tag{51}$$

Suppose $\bigl(Y(L_1), \dots, Y(L_n)\bigr) \in \mathcal{Y}$. This is equivalent to the existence of ϱ in Theorem 7.3. Suppose $\bigl(Z(L_1), \dots, Z(L_n)\bigr) \in \mathcal{Z}$. This is equivalent to Equation (43) of Theorem 7.2. Let N be the compact m-manifold obtained by gluing L_i into X' at x_i for $i = 1, \dots, n$, as in Theorem 7.2. Suppose N is connected.

Let $\widetilde{N}^t$ for $t \in (0, \epsilon]$ be the desingularizations of X constructed in Theorem 7.2 when $Y(L_i) = 0$ (as then L_i is actually AC with rate $\lambda_i < 0$ by [17, Th. 7.11(b)]), and in Theorem 7.3 when X' is connected, and in [20, Th. 6.12] in the general case. Then each $\widetilde{N}^t$ is a compact SL m-fold in M diffeomorphic to N. Let $\mathcal{M}_N$ be the moduli space of deformations of $\widetilde{N}^t$, which is independent of t. Then $\mathcal{M}_N$ is a smooth manifold of dimension $b^1(N)$, by Theorem 2.10.

The next four results compute the dimensions of various spaces.

Lemma 8.3. *In the situation above we have* $\dim \mathcal{I}_{X'} = b^1_{\mathrm{cs}}(X') + q - \sum_{i=1}^n l_i$.

Proof. From (16) we see that $\mathcal{I}_{X'}$ fits into an exact sequence

$$0 \to H^0(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^0(\Sigma_i, \mathbb{R}) \to H^1_{\mathrm{cs}}(X', \mathbb{R}) \to \mathcal{I}_{X'} \to 0.$$

The lemma follows by alternating sum of dimensions. q.e.d.

Proposition 8.4. *In the situation above, $\mathcal{M}^0_{L_i}$ is a smooth manifold with*

$$\begin{aligned} \dim \mathcal{Y}_i = b^1(L_i) - b^0(L_i) + l_i - b^1_{\mathrm{cs}}(L_i), \quad \dim \mathcal{Z}_i = l_i - b^0(L_i) \\ \textit{and} \quad \dim \mathcal{M}^0_{L_i} = b^1(L_i) - b^0(L_i) + l_i. \end{aligned} \tag{52}$$

Also the projection $\pi_{\mathcal{Y}_i} \times \pi_{\mathcal{Z}_i} : \mathcal{M}^0_{L_i} \to \mathcal{Y}_i \times \mathcal{Z}_i$ is a smooth submersion.

Proof. From (29) we see that $\mathcal{Y}_i$ fits into an exact sequence

$$0 \to H^0(L_i,\mathbb{R}) \to H^0(\Sigma_i,\mathbb{R}) \to H^1_{\mathrm{cs}}(L_i,\mathbb{R}) \to H^1(L_i,\mathbb{R}) \to \mathcal{Y}_i \to 0.$$

Taking alternating sums of dimensions gives $\dim \mathcal{Y}_i$ in (52). Similarly, $\mathcal{Z}_i$ fits into an exact sequence

$$0 \to \mathcal{Z}_i \to H^{m-1}(\Sigma_i,\mathbb{R}) \to H^m_{\mathrm{cs}}(L_i,\mathbb{R}) \to H^m(L_i,\mathbb{R}) \to 0.$$

But Poincaré duality gives $b^{m-1}(\Sigma_i) = b^0(\Sigma_i) = l_i$, $b^m_{\mathrm{cs}}(L_i) = b^0(L_i)$ and $b^m(L_i) = b^0_{\mathrm{cs}}(L_i) = 0$, so we deduce $\dim \mathcal{Z}_i$ in (52).

Suppose $0 < \lambda_i < \min\big(\mathcal{D}_{\Sigma_i} \cap (0,\infty)\big)$. Then [17, Th. 7.11(b)] shows that any AC SL m-fold $\hat{L}_i$ with cone C_i and rate λ_i is also asymptotically conical with rate 0. Hence $\mathcal{M}^{\lambda_i}_{L_i} = \mathcal{M}^0_{L_i}$, in the notation of Definition 6.7. Part (a) of Theorem 6.8 then shows that $\mathcal{M}^0_{L_i}$ is smooth with dimension given in (52).

We can deduce that $\pi_{\mathcal{Y}_i} \times \pi_{\mathcal{Z}_i}$ is a smooth submersion from the proof of Theorem 6.8 in Marshall [23, §6]. Smoothness holds for fairly general reasons. To show that $\pi_{\mathcal{Y}_i} \times \pi_{\mathcal{Z}_i}$ is a *submersion* we need to verify that the natural projection $T_{L_i}\mathcal{M}^0_{L_i} \to \mathcal{Y}_i \times \mathcal{Z}_i$ is *surjective*, and this follows from the determination of $T_{L_i}\mathcal{M}^0_{L_i}$ in [23, §5.2]. q.e.d.

Proposition 8.5. $\dim \mathcal{Z} = 1-q+\sum_{i=1}^n \dim \mathcal{Z}_i = 1-q+\sum_{i=1}^n l_i - \sum_{i=1}^n b^0(L_i)$.

Proof. Let $\delta_i \in \mathcal{Z}_i$ for $i = 1,\dots,n$. Then $\delta_i \cdot [\Sigma_i] = 0$, since δ_i is the image of a class in $H^1(L_i)$ and Σ_i is a boundary in L_i. As $[\Sigma_i] = \sum_{j=1}^{l_i} [\Sigma_i^j]$, summing the left hand side of (51) over $k = 1,\dots,q$ yields $\sum_{i=1}^n \psi(x_i)^m \delta_i \cdot [\Sigma_i]$, which is zero. Thus, for any $(\delta_1,\dots,\delta_n) \in \mathcal{Z}_1 \times \dots \times \mathcal{Z}_n$, the sum of (51) over $k = 1,\dots,q$ holds automatically. That is, the q Equations (51) on $\delta_1,\dots,\delta_n$ are *dependent*, and represent at most $q-1$ independent restrictions on $\delta_1,\dots,\delta_n$.

We claim that (51) is *exactly* $q-1$ independent restrictions on $\delta_1,\ldots,\delta_n$. Then $\dim\mathcal{Z} = 1-q+\sum_{i=1}^n \dim\mathcal{Z}_i$, and the proposition follows from (52). To see this, note that X' has q connected components $X'_1,\ldots,X'_q$, which are joined into one connected N by gluing in $L_1,\ldots,L_n$. Define a *link* to be a triple (X'_j, X'_k, L^l_i), where $1 \leqslant j < k \leqslant q$ and L^l_i is a connected component of some L_i which is glued into both X'_j and X'_k at x_i.

Then we can choose a *minimal set* of $q-1$ links which join $X'_1,\ldots,X'_q$ into one component. It is not difficult to show that from (X'_j, X'_k, L^l_i) we can construct $\delta_i \in \mathcal{Z}_i$ as the image of a class in $H^{m-1}(L^l_i,\mathbb{R})$, such that the $q-1$ classes δ_i obtained from the minimal set of $q-1$ links give linearly independent left-hand sides of (51), thought of as vectors in $\mathbb{R}^q$. Hence (51) is at least $q-1$ independent restrictions on $\delta_1,\ldots,\delta_n$, and the proof is complete. q.e.d.

Proposition 8.6. $b^1(N) = \dim\mathcal{Y} + 1 + b^1_{\mathrm{cs}}(X') + \sum_{i=1}^n b^1_{\mathrm{cs}}(L_i) - \sum_{i=1}^n l_i$.

Proof. Regard X' as the interior of a compact manifold $\overline{X}'$ with boundary $\coprod_{i=1}^m \Sigma_i$, and L_i as the interior of a compact manifold $\bar{L}_i$ with boundary Σ_i. Then N is constructed by gluing $\overline{X}'$ and $\bar{L}_1,\ldots,\bar{L}_n$ together along $\Sigma_1,\ldots,\Sigma_n$.

Thus the disjoint union $\coprod_{i=1}^n \Sigma_i$ is a subset of N, with $N \setminus \coprod_{i=1}^n \Sigma_i$ diffeomorphic to the disjoint union of X' and $L_1,\ldots,L_n$. The pair $(N;\coprod_{i=1}^n \Sigma_i)$ gives an exact sequence in cohomology:

(53)
$$\cdots \to H^{k-1}(N,\mathbb{R}) \to \bigoplus_{i=1}^n H^{k-1}(\Sigma_i,\mathbb{R}) \to H^k_{\mathrm{cs}}(X',\mathbb{R}) \oplus \bigoplus_{i=1}^n H^k_{\mathrm{cs}}(L_i,\mathbb{R}) \to H^k(N,\mathbb{R}) \to \cdots,$$

since $H^k(N;\coprod_{i=1}^n \Sigma_i,\mathbb{R}) \cong H^k_{\mathrm{cs}}(X',\mathbb{R}) \oplus \bigoplus_{i=1}^n H^k_{\mathrm{cs}}(L_i,\mathbb{R})$ by excision.

Now from the definitions of $\mathcal{Y}_i,\mathcal{Y}$ and exactness of (16) and (29) we find that the kernel of $\bigoplus_{i=1}^n H^1(\Sigma_i,\mathbb{R}) \to H^2_{\mathrm{cs}}(X',\mathbb{R}) \oplus \bigoplus_{i=1}^n H^2_{\mathrm{cs}}(L_i,\mathbb{R})$ in (53) is $\mathcal{Y}$. Thus as $b^0_{\mathrm{cs}}(X') = b^0_{\mathrm{cs}}(L_i) = 0$ we have an exact sequence

$$0 \to H^0(N,\mathbb{R}) \to \bigoplus_{i=1}^n H^0(\Sigma_i,\mathbb{R}) \to H^1_{\mathrm{cs}}(X',\mathbb{R}) \oplus \bigoplus_{i=1}^n H^1_{\mathrm{cs}}(L_i,\mathbb{R}) \to H^1(N,\mathbb{R}) \to \mathcal{Y} \to 0.$$

The result follows by alternating sums of dimensions, as $b^0(N) = 1$.
q.e.d.

8.2 Describing the moduli space $\mathcal{M}_N$ near its boundary

We continue to use the notation of §8.1. From Lemma 8.3 and Propositions 8.4–8.6 we deduce the following theorem. Smoothness of $\mathcal{F}^X_{L_1,\ldots,L_n}$ and the first line of (55) follow as $\pi_{\mathcal{Y}_i} \times \pi_{\mathcal{Z}_i}$ is a smooth submersion by Proposition 8.4, and the rest of (55) from the dimension formulae above.

Theorem 8.7. *In the situation of Definition* 8.2, *define a family* $\mathcal{F}^X_{L_1,\ldots,L_n}$ *of* n*-tuples of* AC SL m*-folds by*

(54)
$$\mathcal{F}^X_{L_1,\ldots,L_n} = \Big\{(\hat{L}_1,\ldots,\hat{L}_n) \in \mathcal{M}^0_{L_1} \times \cdots \times \mathcal{M}^0_{L_n} : \\ \big(Y(\hat{L}_1),\ldots,Y(\hat{L}_n)\big) \in \mathcal{Y}, \quad \big(Z(\hat{L}_1),\ldots,Z(\hat{L}_n)\big) \in \mathcal{Z}\Big\}.$$

Then $\mathcal{F}^X_{L_1,\ldots,L_n}$ *is a smooth manifold with*

(55)
$$\begin{aligned}\dim \mathcal{F}^X_{L_1,\ldots,L_n} &= \dim \mathcal{Y} + \dim \mathcal{Z} + \textstyle\sum_{i=1}^n (\dim \mathcal{M}^0_{L_i} - \dim \mathcal{Y}_i - \dim \mathcal{Z}_i) \\ &= \dim \mathcal{Y} + 1 - q + \textstyle\sum_{i=1}^n b^1_{\mathrm{cs}}(L_i) \\ &= b^1(N) - \dim \mathcal{I}_{X'}.\end{aligned}$$

The significance of the theorem is that $\mathcal{F}^X_{L_1,\ldots,L_n}$ is the family of n-tuples of AC SL m-folds $\hat{L}_1,\ldots,\hat{L}_n$ which can be used to desingularize X using the results of §7. Now $\mathcal{M}_N$ is smooth with $\dim \mathcal{M}_N = b^1(N)$ by Theorem 2.10. If the cones C_i are *stable* then Corollary 5.3 shows that $\mathcal{M}_X$ is smooth with $\dim \mathcal{M}_X = \dim \mathcal{I}_{X'}$. So we see from Theorem 8.7 that:

Corollary 8.8. *Suppose the* SL *cones* $C_1,\ldots,C_n$ *are stable, in the sense of Definition* 3.4. *Then the moduli spaces* $\mathcal{M}_X, \mathcal{M}_N$ *and* $\mathcal{F}^X_{L_1,\ldots,L_n}$ *are smooth manifolds with* $\dim \mathcal{M}_X + \dim \mathcal{F}^X_{L_1,\ldots,L_n} = \dim \mathcal{M}_N$.

We claim that in the stable singularities case, $\mathcal{M}_N$ is roughly speaking locally diffeomorphic to $\mathcal{M}_X \times \mathcal{F}^X_{L_1,\ldots,L_n}$ near $\mathcal{M}_X \subset \partial\mathcal{M}_N$. That is, each $\hat{N}$ in this region of $\mathcal{M}_N$ can be constructed from some unique $\hat{X} \in \mathcal{M}_X$ and $(\hat{L}_1,\ldots,\hat{L}_n) \in \mathcal{F}^X_{L_1,\ldots,L_n}$ by gluing $\hat{L}_i$ into $\hat{X}$ at $\hat{x}_i$. This is the reason for the formula $\dim \mathcal{M}_X + \dim \mathcal{F}^X_{L_1,\ldots,L_n} = \dim \mathcal{M}_N$. To explain why, we make the following definition:

Definition 8.9. Suppose (M, J, ω, Ω) is Calabi-Yau, so that $\psi \equiv 1$. Choose $\hat{X} \in \mathcal{M}_X$ with singular points $\hat{x}_1, \ldots, \hat{x}_n$ and $(\hat{L}_1, \ldots, \hat{L}_n) \in \mathcal{F}^X_{L_1,\ldots,L_n}$. Then the definition of $\mathcal{F}^X_{L_1,\ldots,L_n}$ implies that $\hat{X}, \hat{L}_i$ satisfy the hypotheses of Theorem 7.2 if $X(\hat{L}_i) = 0$, or Theorem 7.3 if $q = 1$, or [20, Th. 6.12] in the general case. Thus, these theorems give $\epsilon > 0$ such that for $t \in (0, \epsilon]$ there exists a compact SL m-fold $\tilde{N}^t$ in M constructed by gluing $\hat{L}_i$ into $\hat{X}$ at $\hat{x}_i$.

Observe that $(t\hat{L}_1, \ldots, t\hat{L}_n) \in \mathcal{F}^X_{L_1,\ldots,L_n}$ for $t > 0$. Define a subset U in $\mathcal{M}_X \times \mathcal{F}^X_{L_1,\ldots,L_n}$ and $\Psi : U \to \mathcal{M}_N$ by $(\hat{X}, (t\hat{L}_1, \ldots, t\hat{L}_n)) \in U$ if $t \in (0, \epsilon]$, where $\epsilon > 0$ depends on $\hat{X}, \hat{L}_i$ as above, and then $\Psi(\hat{X}, (t\hat{L}_1, \ldots, t\hat{L}_n)) = \tilde{N}^t$.

To make Ψ well-defined we have to ensure that $\tilde{N}^t$ is independent of choices made in its construction. Actually the choice of $\varrho \in H^1(X', \mathbb{R})$ in Theorem 7.3 does affect $\tilde{N}^t$. Now ϱ is unique up to addition of the kernel of $H^1(X', \mathbb{R}) \to \bigoplus_{i=1}^n H^1(\Sigma_i, \mathbb{R})$ in (16). Choose a vector subspace of $H^1(X', \mathbb{R})$ transverse to this kernel, and restrict ϱ to lie in this subspace.

This gives a way to choose ϱ uniquely. Once this is done the N^t are independent of choices up to a small Hamiltonian isotopy, and $\tilde{N}^t$ is the unique SL m-fold in this Hamiltonian isotopy class close to N^t, so $\tilde{N}^t$ is independent of the remaining choices.

Here is why we assumed M is Calabi-Yau above. If M is only *almost* Calabi-Yau, then ψ need not be constant. But $\mathcal{Z}$ depends on $\psi(x_i)$ by (51). Thus, if we vary X to $\hat{X} \in \mathcal{M}_X$ then we should define $\mathcal{F}^{\hat{X}}_{L_1,\ldots,L_n}$ using $\hat{\mathcal{Z}}$ defined with $\psi(\hat{x}_i)$ in (51) instead of $\mathcal{Z}$. So the family $\mathcal{F}^X_{L_1,\ldots,L_n}$ should vary with $\hat{X} \in \mathcal{M}_X$ rather than being constant, but only in a rather trivial way.

We claim that when the C_i are stable, the map Ψ is a *local diffeomorphism* from the interior U° of U to its image in $\mathcal{M}_N$. One can justify this as follows. By [16, §9.4] we can define *natural coordinates* on $\mathcal{M}_N$, local diffeomorphisms $\mathcal{M}_N \to H^1(N, \mathbb{R})$ defined uniquely up to translations in $H^1(N, \mathbb{R})$. In the same way [18, §6.5] defines local diffeomorphisms $\mathcal{M}_X \to \mathcal{I}_{X'}$ uniquely up to translations in $\mathcal{I}_{X'}$, and a similar thing applies for $\mathcal{M}^0_{L_i}$, so that we can construct natural coordinate systems on $\mathcal{F}^X_{L_1,\ldots,L_n}$.

Using the topological calculations of §8.1, one can show that for C_i stable the natural coordinate systems on $\mathcal{M}_N$ can be identified with products of the natural coordinate systems on $\mathcal{M}_X$ and $\mathcal{F}^X_{L_1,\ldots,L_n}$, and Ψ is just the product map in these coordinates. Thus Ψ is a local diffeomorphism on U°.

When the C_i are not stable, things are more complicated. Then $\mathcal{M}_X$ may be singular. If X is *transverse*, which we expect for *generic* (M, J, ω, Ω) by Conjecture 5.12, then $\mathcal{M}_X$ is smooth of dimension $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$ near X. In this case Theorem 8.7 implies that $\dim \mathcal{M}_X + \dim \mathcal{F}^X_{L_1,\ldots,L_n} = \dim \mathcal{M}_N - \dim \mathcal{O}_{X'}$. We expect Ψ to be a *smooth immersion* wherever $\mathcal{M}_X$ is smooth, with image of codimension $\dim \mathcal{O}_{X'}$ in $\mathcal{M}_N$.

Thus, when the C_i are not stable and X is transverse, the desingularization results of §7 do not yield the whole of $\mathcal{M}_N$ locally, but only a subset of codimension $\dim \mathcal{O}_{X'} = \sum_{i=1}^n \text{s-ind}(C_i)$. Where do these extra degrees of freedom in $\mathcal{M}_N$ come from? A rough answer is that the $\sum_{i=1}^n \text{s-ind}(C_i)$ reduction in $\dim \mathcal{M}_X$ reappears as an extra s-ind(C_i) degrees of freedom to deform each L_i as an AC SL m-fold, but with rate $\lambda_i < 2$ rather than rate 0.

Choose λ_i with $\max(\mathcal{D}_{\Sigma_i} \cap [0,2)) < \lambda_i < 2$. Then by part (a) of Theorem 6.8, the moduli space $\mathcal{M}^{\lambda_i}_{L_i}$ of deformations of L_i with rate λ_i is smooth with

$$\begin{aligned} \dim \mathcal{M}^{\lambda_i}_{L_i} &= b^1(L_i) - b^0(L_i) + N_{\Sigma_i}(\lambda_i) \\ &= \dim \mathcal{M}^0_{L_i} + N_{\Sigma_i}(2) - m_{\Sigma_i}(2) - b^0(\Sigma_i), \end{aligned} \tag{56}$$

using the notation of Definition 3.3, and the fact that N_{Σ_i} is monotone increasing and upper semicontinuous, and increases by $m_{\Sigma_i}(2)$ at 2.

Suppose C_i is *rigid*, as in Definition 3.4. Then (9) and (56) give

$$\dim \mathcal{M}^{\lambda_i}_{L_i} = \dim \mathcal{M}^0_{L_i} + \text{s-ind}(C_i) + 2m. \tag{57}$$

Thus, deforming L_i with rate λ_i rather than 0 gives an extra s-ind$(C_i)+2m$ degrees of freedom. Here the $2m$ comes from *translations* in $\mathbb{C}^m$, since the AC SL m-folds of rate $\lambda \geqslant 1$ are closed under translations, and the s-ind(C_i) from new, nontrivial deformations of L_i.

So when X is transverse and the C_i are rigid, we can understand the difference in dimension $\sum_{i=1}^n \text{s-ind}(C_i)$ between $\mathcal{M}_X \times \mathcal{F}^X_{L_1,\ldots,L_n}$ and $\mathcal{M}_N$ as coming from an extra s-ind(C_i) nontrivial deformations of L_i an an AC SL m-fold with rate λ_i rather than 0. If the C_i are not rigid, we should take into account also extra infinitesimal deformations of C_i as an SL cone.

8.3 The index of singularities of SL m-folds

We can now make more rigorous some speculations by the author in [16, §10.3]. Suppose, as above, that $\mathcal{M}_N$ is a moduli space of compact, nonsingular SL m-folds in (M, J, ω, Ω), and that $\mathcal{M}_X$ is a moduli space of singular SL m-folds in $\partial\mathcal{M}_N$ with singularities of a particular type, and $X \in \mathcal{M}_X$.

Define the *index* of the singularities of X to be $\mathrm{ind}(X) = \dim \mathcal{M}_N - \dim \mathcal{M}_X$, provided $\mathcal{M}_X$ is smooth near X so $\dim \mathcal{M}_X$ is well-defined. Note that $\mathrm{ind}(X)$ depends not just on X and its singularities, but also on N through $\dim \mathcal{M}_N = b^1(N)$. Thus there could be topologically distinct desingularizations $N_1, N_2, \ldots$ yielding different values of $\mathrm{ind}(X)$.

We can also work in *families* $\mathcal{F}$ of almost Calabi-Yau m-folds. Defining $\mathcal{M}_N^{\mathcal{F}}$ as in Definition 2.15 and $\mathcal{M}_X^{\mathcal{F}}$ as in Definition 5.6, the *index of X in $\mathcal{F}$* is $\mathrm{ind}^{\mathcal{F}}(X) = \dim \mathcal{M}_N^{\mathcal{F}} - \dim \mathcal{M}_X^{\mathcal{F}}$. Note that $\mathrm{ind}(X) \leqslant \dim \mathcal{M}_N = b^1(N)$, as $\dim \mathcal{M}_X \geqslant 0$, and $\mathrm{ind}^{\mathcal{F}}(X) \leqslant \dim \mathcal{M}_N^{\mathcal{F}}$.

Combining Corollary 5.5 and Theorem 8.7, we can compute $\mathrm{ind}(X)$ when X is *transverse* with conical singularities. In the families case, if X is *transverse in $\mathcal{F}$* then a similar proof shows that $\mathrm{ind}^{\mathcal{F}}(X)$ is given by the same formula (58).

Theorem 8.10. *Let X be a compact,* transverse *SL m-fold in (M, J, ω, Ω) with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Construct desingularizations N of X by gluing AC SL m-folds L_i in at x_i for $i = 1, \ldots, n$, as in §7. Let $q, \mathcal{Y}$ be as in Definition 8.2. Then*

$$\mathrm{ind}(X) = \dim \mathcal{Y} + 1 - q + \textstyle\sum_{i=1}^n b^1_{\mathrm{cs}}(L_i) + \sum_{i=1}^n \text{s-ind}(C_i). \tag{58}$$

When $n = 1$ this proves [9, Conj. 2.13], in the transverse case.

Suppose C_i is not *rigid*, for instance if Σ_i is not connected. Then C_i may lie in a smooth, connected moduli space $\mathcal{C}_i$ of SL cones in $\mathbb{C}^m$, upon which $\mathrm{SU}(m)$ does not act transitively. In this case, as in [18, §8.3], it is better to define $\mathcal{M}_X$ to be the moduli space of $\hat{X}$ with cones $\hat{C}_i \in \mathcal{C}_i$, rather than with fixed cones C_i. Under suitable transversality assumptions, this increases the dimension of $\mathcal{M}_X$ by the codimension of the $\mathrm{SU}(m)$ orbit of C_i in $\mathcal{C}_i$, so this codimension should be subtracted from the r.h.s. of (58).

Here is why the index is an important idea. As $\mathrm{ind}(X)$ is the codimension of $\mathcal{M}_X$ in $\mathcal{M}_N$, the *largest pieces* of $\partial\mathcal{M}_N$ are the $\mathcal{M}_X$ with *smallest index*. So we argue that singularities with *small index* are the most generic, and the most interesting. In good cases, we expect $\overline{\mathcal{M}}_N$ to be a *compact manifold with singular boundary*. Thus the largest pieces of $\partial\mathcal{M}_N$ should have codimension 1 in $\mathcal{M}_N$, and hence index 1. If $\partial\mathcal{M}_N$ has no index 1 strata, then $\overline{\mathcal{M}}_N$ is roughly a compact, singular manifold *without boundary*.

As tools, $\mathrm{ind}(X)$ or $\mathrm{ind}^{\mathcal{F}}(X)$ probably work best together with a *genericity assumption* on (M,J,ω,Ω) or $\mathcal{F}$. Suppose that ω is *generic in its Kähler class*. Then as in Conjecture 5.12, we expect SL m-folds X in (M,J,ω,Ω) with conical singularities to be *transverse*, so we can compute $\mathrm{ind}(X)$ using (58).

Since $\mathrm{ind}(X) \leqslant b^1(N)$, this places *strong restrictions* on the kinds of singularities that can occur in $\partial\mathcal{M}_N$. In the families case, when $\mathcal{F}$ is generic in a suitable sense we expect SL m-folds X in $(M,J^s,\omega^s,\Omega^s)$ with conical singularities to be *transverse in* $\mathcal{F}$. Then we can compute $\mathrm{ind}^{\mathcal{F}}(X)$ using (58). Since $\mathrm{ind}^{\mathcal{F}}(X) \leqslant \dim\mathcal{M}_N^{\mathcal{F}}$, this again places strong restrictions on the kinds of singularities that can occur in $\partial\mathcal{M}_N^{\mathcal{F}}$.

For some problems we only need to know about singularities with index up to a certain value. For example, in [9] the author proposed to define an invariant of almost Calabi-Yau 3-folds by a weighted count of SL homology 3-spheres in a given homology class. To understand how this invariant transforms as we deform (M,J,ω,Ω), we restrict to a generic 1-dimensional family $\mathcal{F}$, and then we meet only singularities with index 1.

The index may also be useful in the SYZ Conjecture [29]. This explains mirror symmetry of (almost) Calabi-Yau 3-folds $M,\check{M}$ in terms of *fibrations* by SL 3-tori N, $\check{N}$. The corresponding moduli spaces $\mathcal{M}_N$, $\mathcal{M}_{\check{N}}$ have dimension 3, so that $\partial\mathcal{M}_N$, $\partial\mathcal{M}_{\check{N}}$ can only contain singularities of index 1, 2 or 3.

9. Applications to connected sums

We shall now apply the results of §7 to the case where the SL m-fold X with conical singularities is actually a *nonsingular, immersed* SL m-*fold*, the singular points x_i are *self-intersection points* of X satisfying an *angle criterion*, and the AC SL m-folds L_i are chosen from the $L^{\phi,A}$ of Example 6.11 due to Lawlor [21]. The desingularizations N are *multiple connected sums* of X with itself.

For the connected sum $X_1\#X_2$ of two SL 3-folds at one point in a Calabi-Yau 3-fold, our results were conjectured by the author in [9,

§5–§6]. Butscher [3] proves existence of SL connected sums $X_1 \# X_2$ at one point by gluing in Lawlor necks $L^{\phi,A}$, where X_1, X_2 are compact SL m-folds in $\mathbb{C}^m$ with boundary.

Closer to our results, Lee [8] considers a compact, connected, immersed SL m-fold X in a Calabi-Yau m-fold M, whose self-intersection points x_i satisfy an angle criterion. She glues in $L^{\phi,A}$ at x_i to get a family of compact, *embedded* SL m-folds in M. Her result is re-proved in Theorem 9.5 below.

9.1 Transverse intersections of SL planes Π^+, Π^- in $\mathbb{C}^m$

Let Π^+, Π^- be two SL m-planes $\mathbb{R}^m$ in $\mathbb{C}^m$. We call $\Pi^\pm$ *transverse* if they intersect transversely, that is, if $\Pi^+ \cap \Pi^- = \{0\}$. Note that this has nothing to do with the use of 'transverse' in §5. We now classify transverse pairs of SL m-planes up to the action of $\mathrm{SU}(m)$. Related results may be found in [3, §1] and [8, §2], but we believe our approach is clearer.

Proposition 9.1. *Let* Π^+, Π^- *be transverse* SL *m-planes* $\mathbb{R}^m$ *in* $\mathbb{C}^m$. *Then there exists* $B \in \mathrm{SU}(m)$ *and* $\phi_1, \dots, \phi_m \in (0, \pi)$ *such that* $B(\Pi^+) = \Pi^0$ *and* $B(\Pi^-) = \Pi^{\boldsymbol{\phi}}$, *where* $\boldsymbol{\phi} = (\phi_1, \dots, \phi_m)$ *and*

(59)
$$\Pi^0 = \big\{(x_1, \dots, x_m) : x_j \in \mathbb{R}\big\},\ \Pi^{\boldsymbol{\phi}} = \big\{(\mathrm{e}^{i\phi_1} x_1, \dots, \mathrm{e}^{i\phi_m} x_m) : x_j \in \mathbb{R}\big\},$$

as in (39). *Moreover* $\phi_1, \dots, \phi_m$ *are unique up to order, so that we can make them unique by assuming that* $\phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_m$, *and* $\phi_1 + \cdots + \phi_m = k\pi$ *for some* $k = 1, \dots, m-1$.

To prove this, first find $B' \in \mathrm{SU}(m)$ with $B'(\Pi^+) = \Pi^0$, and then find $B'' \in \mathrm{SO}(m)$ which 'diagonalizes' $B'(\Pi^-)$ to get $\Pi^{\boldsymbol{\phi}}$, and set $B = B''B'$. The process is like diagonalizing a real quadatic form on $\mathbb{R}^m$ with an $\mathrm{SO}(m)$ matrix. We use the proposition to divide transverse pairs $\Pi^\pm$ into *types*.

Definition 9.2. Let Π^+, Π^- be transverse SL m-planes $\mathbb{R}^m$ in $\mathbb{C}^m$. Then Proposition 9.1 gives $B \in \mathrm{SU}(m)$ and unique $0 < \phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_m < \pi$ with $\phi_1 + \cdots + \phi_m = k\pi$ for some $k = 1, \dots, m-1$, such that $B(\Pi^+) = \Pi^0$ and $B(\Pi^-) = \Pi^{\boldsymbol{\phi}}$. We say that the intersection point 0 of Π^+, Π^- is of *type* k.

Note that this depends on the *order* of Π^+, Π^-. Exchanging $\Pi^\pm$ replaces ϕ_j by $\pi - \phi_{m+1-j}$ for $j = 1, \dots, m$, and therefore Π^+, Π^- intersect with type k if and only if Π^-, Π^+ intersect with type $m - k$.

When $k = 1$ in Proposition 9.1, Example 6.11 gives AC SL m-folds $L^{\phi,A}$ for $A > 0$ with cone $\Pi^0 \cup \Pi^\phi$. Thus $L^{\pm,A} = B^{-1}L^{\phi,A}$ is an AC SL m-fold with cone $\Pi^+ \cup \Pi^-$. When $k = m-1$ we can exchange $\Pi^\pm$ to get $k = 1$, and do the same thing. So combining Example 6.11 and Proposition 9.1 gives:

Proposition 9.3. *Let Π^+, Π^- be transverse* SL *m-planes $\mathbb{R}^m$ in $\mathbb{C}^m$, for $m > 2$. Regard $C = \Pi_+ \cup \Pi_-$ as an* SL *cone in $\mathbb{C}^m$ with isolated singularity at* 0. *Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a disjoint union $\Sigma^+ \cup \Sigma^-$, where $\Sigma^\pm$ are the unit spheres $\mathcal{S}^{m-1}$ in $\Pi^\pm$. Then:*

(a) *Suppose Π^+, Π^- intersect with type* 1. *Then there is a* 1*-parameter family of* AC SL *m-folds $L^{\pm,A}$ in $\mathbb{C}^m$ asymptotic to C with rate $2-m$ for $A > 0$, with $Z(L^{\pm,A})\cdot[\Sigma^+] = A$ and $Z(L^{\pm,A})\cdot[\Sigma^-] = -A$.*

(b) *Suppose Π^+, Π^- intersect with type $m-1$. Then there is a* 1*-parameter family of* AC SL *m-folds $L^{\pm,A}$ in $\mathbb{C}^m$ asymptotic to C with rate $2-m$ for $A > 0$, with $Z(L^{\pm,A})\cdot[\Sigma^+] = -A$ and $Z(L^{\pm,A})\cdot [\Sigma^-] = A$.*

The $L^{\pm,A}$ are images of the $L^{\phi,A}$ of Example 6.11 *under* $\mathrm{SU}(m)$ *rotations.*

We will call the $L^{\pm,A}$ *Lawlor necks*. They are diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}$.

9.2 Desingularizing immersed SL m-folds

Here is some notation for self-intersection points of immersed SL m-folds.

Definition 9.4. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold for $m > 2$, and define $\psi : M \to (0, \infty)$ as in (4). Let X be a compact, nonsingular immersed SL m-fold in M. That is, X is a compact m-manifold (not necessarily connected) and $\iota : X \to M$ an immersion, with special Lagrangian image.

Call $x \in M$ a *self-intersection point* of X if $\iota^*(x)$ is at least two points in X. Call such an x *transverse* if $\iota^*(x)$ is exactly two points x^+, x^- in X, and $\iota_*(T_{x^+}X) \cap \iota_*(T_{x^-}X) = \{0\}$ in T_xM.

Let x be a transverse self-intersection point of X, and $x^\pm$ as above. Choose an isomorphism $\upsilon : \mathbb{C}^m \to T_xM$ with $\upsilon^*(\omega) = \omega'$ and $\upsilon^*(\Omega) = \psi(x)^m\Omega'$, where ω', Ω' are as in (2). Define $\Pi^\pm = \upsilon^*\big(\iota_*(T_{x^\pm}X)\big)$. Then Π^+, Π^- are transverse SL planes $\mathbb{R}^m$ in $\mathbb{C}^m$.

Define the *type* $k = 1, \ldots, m-1$ of x to be the type of Π^+, Π^-, in the sense of Definition 9.2. This is independent of the choice of υ, but it *does* depend on the order of x^+, x^-, and exchanging $x^\pm$ replaces k by $m-k$.

We now apply Theorem 7.1 to desingularize *connected*, immersed SL m-folds X in M. Our result is equivalent to Lee's main result [8, Theorems 3 & 4], save that Lee also allows $m = 2$, and considers only the Calabi-Yau case.

Theorem 9.5. *Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold for $m > 2$, and X a compact, connected, immersed* SL *m-fold in M. Suppose $x_1, \ldots, x_n$ are transverse self-intersection points of X with type 1 or $m-1$.*

Then there exists $\epsilon > 0$ and a smooth family $\{\tilde{N}^t : t \in (0, \epsilon]\}$ of compact, immersed SL *m-folds in (M, J, ω, Ω), such that $\tilde{N}^t$ is constructed by gluing a Lawlor neck $L^{\pm, t^m A_i}$ into X at x_i for $i = 1, \ldots, n$, and so is a multiple connected sum of X with itself. In the sense of currents, $\tilde{N}^t \to X$ as $t \to 0$. If $x_1, \ldots, x_n$ are the only self-intersection points of X then $\tilde{N}^t$ is embedded.*

Proof. For each $i = 1, \ldots, n$ let $\iota^*(x_i)$ be x_i^+, x_i^-, and define $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ and $\Pi_i^\pm$ as in Definition 9.4. Then Π_i^+, Π_i^- are transverse SL planes with type 1 or $m-1$, by assumption. Therefore Proposition 9.3 gives a 1-parameter family $L_i^{\pm, A}$ for $A > 0$ of AC SL m-folds in $\mathbb{C}^m$ asymptotic to $\Pi_i^+ \cup \Pi_i^-$.

Choose some $A_1, \ldots, A_n > 0$, for instance $A_i \equiv 1$, and let $L_i = L_i^{\pm, A_i}$ for $i = 1, \ldots, n$. Apply Theorem 7.1 to X with conical singular points x_i, cones $C_i = \Pi_i^+ \cup \Pi_i^-$ and AC SL m-folds L_i for $i = 1, \ldots, n$. As X is connected $X' = X \setminus \{x_1^\pm, \ldots, x_n^\pm\}$ is connected, and L_i has rate $\lambda_i = 2 - m < 0$, so the hypotheses hold.

Thus Theorem 7.1 gives $\epsilon > 0$ and the family $\tilde{N}^t$, and most of the theorem follows. If X has other self-intersection points than $x_1, \ldots, x_n$ then the $\tilde{N}^t$ are immersed. But if $x_1, ..., x_n$ are the only self-intersection points then X' and the L_i are embedded, so the $\tilde{N}^t$ are embedded for small t. q.e.d.

When $m = 3$ the only possible types are 1 and $m-1$, giving:

Corollary 9.6. *Let X be a compact, connected, immersed* SL 3*-fold with transverse self-intersection points in an almost Calabi-Yau* 3*-fold M. Then X is a limit of embedded* SL 3*-folds.*

Next we apply Theorem 7.2 to desingularize non-connected X. If a self-intersection point x_i has type $m-1$ we can swap x_i^+, x_i^- to get type 1, so for simplicity we suppose the x_i all have type 1.

Theorem 9.7. *Let (M,J,ω,Ω) be an almost Calabi-Yau m-fold for $m>2$, and X a compact, immersed* SL *m-fold in M. Define $\psi: M \to (0,\infty)$ as in (4). Suppose $x_1,\ldots,x_n \in M$ are transverse self-intersection points of X with type 1, and let $x_i^\pm \in X$ be as in Definition 9.4. Set $q=b^0(X)$, and let $X_1,\ldots,X_q$ be the connected components of X. Suppose $A_1,\ldots,A_n>0$ satisfy*

$$\sum_{\substack{i=1,\ldots,n:\\ x_i^+\in X_k}} \psi(x_i)^m A_i = \sum_{\substack{i=1,\ldots,n:\\ x_i^-\in X_k}} \psi(x_i)^m A_i \quad \text{for all } k=1,\ldots,q. \tag{60}$$

Let N be the oriented multiple connected sum of X with itself at the pairs of points x_i^+, x_i^- for $i=1,\ldots,n$. Suppose N is connected.

Then there exists $\epsilon>0$ and a smooth family $\bigl\{\tilde N^t : t\in(0,\epsilon]\bigr\}$ of compact, immersed SL *m-folds in (M,J,ω,Ω) diffeomorphic to N, such that $\tilde N^t$ is constructed by gluing a Lawlor neck $L^{\pm,t^m A_i}$ into X at x_i for $i=1,\ldots,n$. In the sense of currents, $\tilde N^t \to X$ as $t\to 0$. If $x_1,\ldots,x_n$ are the only self-intersection points of X then $\tilde N^t$ is embedded.*

Proof. Use the notation of the proof of Theorem 9.5. The assumption that N is connected above is one of the hypotheses of Theorem 7.2. Part (a) of Proposition 9.3 gives $Z(L_i)\cdot[\Sigma_i^\pm] = \pm A_i$, and using this we find that (43) is equivalent to (60). The other hypotheses of Theorem 7.2 are established in the proof of Theorem 9.5. Thus Theorem 7.2 applies, and the rest of the proof follows Theorem 9.5. q.e.d.

To decide whether desingularizations $\tilde N^t$ of X exist, we need to know when (60) admits solutions $A_i>0$. Since $q-1$ of Equations (60) are independent as in the proof of Proposition 8.5, there can only exist *nonzero* solutions A_i if $n\geqslant q$.

Here is a *graphical method* for deciding. Draw q vertices, numbered $1,\ldots,q$. For $i=1,\ldots,n$ draw a directed edge from vertex j to vertex k, where $x_i^+\in X_j$ and $x_i^-\in X_k$. Then (60) admits solutions $A_i>0$ if and only if whenever we divide the q vertices into two nonempty disjoint subsets B and C, there is at least one directed edge going from B to C, and at least one going from C to B.

As in §8 we can compare the dimensions of moduli spaces $\mathcal{M}_X, \mathcal{M}_N$ in Theorem 9.7. Since $m>2$ it is easy to show that each connected sum *either* reduces b^0 by 1 and fixes b^1, *or* fixes b^0 and increases b^1 by

1. As $b^0(X)=q$, $b^0(N)=1$, and $\dim\mathcal{M}_X=b^1(X)$ by Theorem 2.10 in the immersed case, we see that

$$\dim\mathcal{M}_N=b^1(N)=n+1-q+b^1(X)=n+1-q+\dim\mathcal{M}_X. \tag{61}$$

If $n=q$ then $\dim\mathcal{M}_N=1+\dim\mathcal{M}_X$, and $\overline{\mathcal{M}}_N$ is a *manifold with boundary* $\mathcal{M}_X$ near X, and the singularities of X have *index* $\operatorname{ind}(X)=1$ in the sense of §8.3. Note that (58) does not give the right answer for $\operatorname{ind}(X)$ in this case, as $C_i=\Pi_i^+\cup\Pi_i^-$ is not *rigid*, and to get the right definition of $\mathcal{M}_X$ we have to allow singular cones $\hat C_i$ not just some fixed $\Pi_i^+\cup\Pi_i^-$, but any union $\hat\Pi_i^+\cup\hat\Pi_i^-$ of transverse SL m-planes $\hat\Pi_i^\pm$ in $\mathbb{C}^m$ with type 1.

9.3 Desingularizing immersed SL m-folds in families

Next we extend Theorem 9.7 to *families* of almost Calabi-Yau m-folds. Following the proof of Theorem 9.7 and using Theorem 7.4 we prove:

Theorem 9.8. *Let (M,J,ω,Ω) be an almost Calabi-Yau m-fold for $m>2$, and X a compact, immersed* SL *m-fold in M with immersion ι. Define $\psi:M\to(0,\infty)$ as in (4). Suppose $x_1,\dots,x_n\in M$ are transverse self-intersection points of X with type* 1*, and let $x_i^\pm\in X$ be as in Definition* 9.4*. Set $q=b^0(X)$, and let $X_1,\dots,X_q$ be the connected components of X. Let N be the oriented multiple connected sum of X with itself at the pairs of points x_i^+,x_i^- for $i=1,\dots,n$. Suppose N is connected.*

Suppose $\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\}$ is a smooth family of deformations of (M,J,ω,Ω) with $\iota^*([\omega^s])=0$ in $H^2(X,\mathbb{R})$ for all $s\in\mathcal{F}$. Let $A_1,\dots,A_n>0$. Define $\mathcal{G}\subseteq\mathcal{F}\times(0,1)$ to be the subset of $(s,t)\in\mathcal{F}\times(0,1)$ with

$$[\operatorname{Im}\Omega^s]\cdot[X_k]=t^m\sum_{\substack{i=1,\dots,n:\\ x_i^+\in X_k}}\psi(x_i)^mA_i-t^m\sum_{\substack{i=1,\dots,n:\\ x_i^-\in X_k}}\psi(x_i)^mA_i \tag{62}$$

for all $k=1,\dots,q$. Then there exist $\epsilon\in(0,1)$, $\kappa>1$ and a smooth family

$$\bigl\{\widetilde N^{s,t}:(s,t)\in\mathcal{G},\quad t\in(0,\epsilon],\quad |s|\leqslant t^{\kappa+m/2}\bigr\}, \tag{63}$$

such that $\widetilde N^{s,t}$ is a compact, nonsingular SL *m-fold in $(M,J^s,\omega^s,\Omega^s)$ diffeomorphic to N, constructed by gluing a Lawlor neck $L^{\pm,t^mA_i}$ into X*

at x_i for $i = 1, \dots, n$. In the sense of currents, $\widetilde{N}^{s,t} \to X$ as $s, t \to 0$. If $x_1, \dots, x_n$ are the only self-intersection points of X then $\widetilde{N}^{s,t}$ is embedded.

Thus, the main condition for the existence of desingularizations $\widetilde{N}^{s,t}$ of X in $(M, J^s, \omega^s, \Omega^s)$ is that there should exist solutions $A_1, \dots, A_n > 0$ to (62). Note that the sum of (62) over $k = 1, \dots, q$ gives $[\operatorname{Im}\Omega^s] \cdot [X] = 0$, which is clearly a necessary condition for $\widetilde{N}^{s,t}$ to exist with $[\widetilde{N}^{s,t}] = [X]$.

In [9] the author proposed to define an invariant $I_3 : H_3(M, \mathbb{Z}) \to \mathbb{Q}$ of almost Calabi-Yau 3-folds (M, J, ω, Ω) by counting SL *homology* 3-*spheres* in a given homology class with a topological weight. Theorem 9.8 will be important for this programme, because it determines the *transformation rules* I_3 satisfies as we deform (M, J, ω, Ω) so that J passes through certain real hypersurfaces in the complex structure moduli space.

To explain this we need the idea of SL *m-folds with phase* $\mathrm{e}^{i\theta}$, as in [9, 10].

Definition 9.9. Let (M, J, ω, Ω) be an almost Calabi-Yau m-fold, and N an oriented real m-dimensional submanifold of M. Fix $\theta \in \mathbb{R}$. We call N a *special Lagrangian submanifold*, or SL *m-fold* for short, with *phase* $\mathrm{e}^{i\theta}$ if

$$\omega|_N \equiv 0 \quad \text{and} \quad (\sin\theta \operatorname{Re}\Omega - \cos\theta \operatorname{Im}\Omega)|_N \equiv 0, \tag{64}$$

and $\cos\theta \operatorname{Re}\Omega + \sin\theta \operatorname{Im}\Omega$ is a positive m-form on the oriented m-fold N.

If N is *compact* it easily follows that $[\Omega] \cdot [N] = R\mathrm{e}^{i\theta}$, where $[\Omega] \in H^m(M, \mathbb{R})$ and $[N] \in H_m(M, \mathbb{Z})$, and $R = \int_N \cos\theta \operatorname{Re}\Omega + \sin\theta \operatorname{Im}\Omega > 0$. Thus the homology class $[N]$ determines the phase $\mathrm{e}^{i\theta}$ of N.

The definition of SL m-fold used in the rest of the paper, Definition 2.9, is of SL m-fold with phase 1. If N has phase $\mathrm{e}^{i\theta}$ in (M, J, ω, Ω) then it has phase 1 in $(M, J, \omega, \mathrm{e}^{-i\theta}\Omega)$, so if we are dealing with SL m-folds in only one homology class then we can rescale Ω to make the phase 1. But when we consider several SL m-folds $N_1, N_2, \dots$ we cannot always take them to have phase 1.

Using this notation, we rewrite Theorem 9.8 when $n = 1$ and $q = 2$, so that we take the *connected sum* $X_1 \# X_2$ of SL m-folds X_1, X_2 at one point x.

Theorem 9.10. *Let (M,J,ω,Ω) be an almost Calabi-Yau m-fold for $m>2$, and X_1,X_2 be compact, connected* SL *m-folds in M with the same phase $\mathrm{e}^{i\theta}$, which intersect transversely at $x\in M$ with type* 1*. Suppose $\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\}$ is a smooth family of deformations of (M,J,ω,Ω) with $\iota^*([\omega^s])=0$ in $H^2(X_k,\mathbb{R})$ for all $k=1,2$ and $s\in\mathcal{F}$. Write*

$$[\Omega^s]\cdot[X_k]=R^s_k\mathrm{e}^{i\theta^s_k}\quad\textit{for } k=1,2 \textit{ and}\quad[\Omega^s]\cdot\bigl([X_1]+[X_2]\bigr)=R^s\mathrm{e}^{i\theta^s},\tag{65}$$

where $R^s_k,R^s>0$ and $\theta^s_k,\theta^s\in\mathbb{R}$ depend continuously on s with $\theta^0_k=\theta^0=\theta$. Make $\mathcal{F}$ smaller if necessary so that $R^s_k,\theta^s_k,R^s,\theta^s$ are well-defined. Define

$$\mathcal{G}=\bigl\{(s,t)\in\mathcal{F}\times(0,1):R^s_1\sin(\theta^s_1-\theta^s)=t^m\psi(x)^m\bigr\}.\tag{66}$$

Then there exist $\epsilon\in(0,1)$, $\kappa>1$ and a smooth family

$$\bigl\{\widetilde{N}^{s,t}:(s,t)\in\mathcal{G},\quad t\in(0,\epsilon],\quad |s|\leqslant t^{\kappa+m/2}\bigr\},\tag{67}$$

such that $\widetilde{N}^{s,t}$ is a compact SL *m-fold with phase $\mathrm{e}^{i\theta^s}$ in $(M,J^s,\omega^s,\Omega^s)$ diffeomorphic to $X_1\#X_2$, constructed by gluing a Lawlor neck $L^{\pm,t^m}$ into $X_1\cup X_2$ at x. In the sense of currents, $\widetilde{N}^{s,t}\to X$ as $s,t\to 0$. If X_1,X_2 are embedded and x is their only intersection point then $\widetilde{N}^{s,t}$ is embedded.*

This follows from Theorem 9.8 with $X=X_1\cup X_2$, $n=1$, $x_1=x$ and $A_1=1$, replacing (M,J,ω,Ω), $(M,J^s,\omega^s,\Omega^s)$ by $(M,J,\omega,\mathrm{e}^{-i\theta}\Omega)$, $(M,J^s,\omega^s,\mathrm{e}^{-i\theta^s}\Omega^s)$ so that X_1,X_2 and $\widetilde{N}^{s,t}$ have phase 1. Equation (62) becomes

$$[\mathrm{e}^{-i\theta^s}\operatorname{Im}\Omega^s]\cdot[X_1]=t^m\psi(x)^m\quad\text{and}\quad[\mathrm{e}^{-i\theta^s}\operatorname{Im}\Omega^s]\cdot[X_2]=-t^m\psi(x)^m.$$

By (65) these are equivalent to

$$R^s_1\sin(\theta^s_1-\theta^s)=t^m\psi(x)^m\quad\text{and}\quad R^s_2\sin(\theta^s_2-\theta^s)=-t^m\psi(x)^m.\tag{68}$$

But $R^s_1\sin(\theta^s_1-\theta^s)=-R^s_2\sin(\theta^s_2-\theta^s)$ as $R^s_1\mathrm{e}^{i\theta^s_1}+R^s_2\mathrm{e}^{i\theta^s_2}=R^s\mathrm{e}^{i\theta^s}$ by (65). Thus both equations of (68) are equivalent, so we use only the first in (66).

We can interpret Theorem 9.10 like this. From (65) we see that θ^s always lies between θ^s_1 and θ^s_2 for small $s\in\mathcal{F}$. Thus making $\mathcal{F}$ smaller if necessary we can divide $\mathcal{F}$ into three regions:

$$\begin{gathered}\mathcal{F}^+=\{s\in\mathcal{F}:\theta^s_1>\theta^s>\theta^s_2\},\quad\mathcal{F}^-=\{s\in\mathcal{F}:\theta^s_1<\theta^s<\theta^s_2\},\\ \text{and}\quad\mathcal{F}^0=\{s\in\mathcal{F}:\theta^s_1=\theta^s=\theta^s_2\}.\end{gathered}\tag{69}$$

If $[X_1],[X_2]$ are linearly dependent in $H_m(M,\mathbb{R})$ then $\theta_1^s \equiv \theta_2^s \equiv \theta^s$, giving $\mathcal{G}=\emptyset$ in (66), and the theorem is trivial.

So suppose $[X_1],[X_2]$ are linearly independent. Then for $\mathcal{F}$ suitably generic $\mathcal{F}^0$ will be a *smooth real hypersurface* in $\mathcal{F}$, which divides $\mathcal{F}\setminus\mathcal{F}^0$ into two open regions $\mathcal{F}^\pm$. Call $\mathcal{F}^+$ the *positive side* and $\mathcal{F}^-$ the *negative side* of $\mathcal{F}^0$. Now $\theta_1^s-\theta^s$ is small close to $\mathcal{F}^0$ in $\mathcal{F}$, and so $\sin(\theta_1^s-\theta^s)$ has the same sign as $\theta_1^s-\theta^s$. Therefore $R_1^s\sin(\theta_1^s-\theta^s)=t^m\psi(x)^m$ in (66) admits a unique solution $t>0$ for small s if and only if $\theta_1^s>\theta^s$, that is, if and only if $s\in\mathcal{F}^+$.

We thus have the following picture, described when $m=3$ in Conjecture 6.5 of [9], which we have now proved. By Theorem 2.16 we can extend X_1,X_2 to families of SL m-folds X_k^s with phase $\mathrm{e}^{i\theta_k^s}$ in $(M,J^s,\omega^s,\Omega^s)$ for $k=1,2$ and small $s\in\mathcal{F}$, such that X_1^s,X_2^s intersect transversely with type 1 at $x^s\in M$ close to x. On the hypersurface $\mathcal{F}^0$ in $\mathcal{F}$ the phases of X_1^s,X_2^s are equal.

On the positive side $\mathcal{F}^+$ of $\mathcal{F}^0$ there exist SL m-fold connected sums $X_1^s\#X_2^s$ with phase $\mathrm{e}^{i\theta^s}$. On the negative side $\mathcal{F}^-$ there are no such $X_1^s\#X_2^s$. Thus, as we cross hypersurfaces $\mathcal{F}^0$ in $\mathcal{F}$ where the phases of two SL m-folds X_1,X_2 become equal, we create or destroy new SL m-folds $X_1\#X_2$ by connected sum at points x where X_1,X_2 intersect transversely with type 1 or $m-1$.

The conjectured invariant $I_3: H_3(M,\mathbb{Z})\to\mathbb{Q}$ of [9] should change in a predictable fashion as we cross hypersurfaces $\mathcal{F}^0$, owing to the creation and destruction of SL homology 3-spheres. Theorem 9.8 also gives criteria for the existence of multiple connected sums $X_1\#X_2\#\cdots\#X_q$ of SL m-folds. Using this I can derive a complete set of transformation rules for I_3, and also extend the programme to all $m\geqslant 3$. I am writing papers about this.

10. Stable T^2-cone singularities in SL 3-folds

We now study SL 3-folds with conical singularities modelled on the *stable T^2-cone* $C=C_{\rm HL}^3$ of Example 3.5, given by

$$C=\bigl\{(z_1,z_2,z_3)\in\mathbb{C}^m: z_1z_2z_3\in[0,\infty),\quad |z_1|=|z_2|=|z_3|\bigr\}. \tag{70}$$

Example 6.9 gives three families of AC SL 3-folds $L_{\rm HL}^{\mathbf{a}}$ with rate 0 and cone C, which we will write as L_j^a for $j=1,2,3$ and $a>0$, given by

$$L_1^a = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 z_3 \in [0, \infty),\ |z_1|^2 - a = |z_2|^2 = |z_3|^2\},$$
$$L_2^a = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 z_3 \in [0, \infty),\ |z_1|^2 = |z_2|^2 - a = |z_3|^2\},$$
$$L_3^a = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 z_3 \in [0, \infty),\ |z_1|^2 = |z_2|^2 = |z_3|^2 - a\}.$$

Then L_j^a is diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.

Identify $\Sigma = C \cap \mathcal{S}^5$ with $T^2 = \mathrm{U}(1)^2$ by the map

$$(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}) \longmapsto \left(\tfrac{1}{\sqrt{3}}\,\mathrm{e}^{i\theta_1}, \tfrac{1}{\sqrt{3}}\,\mathrm{e}^{i\theta_2}, \tfrac{1}{\sqrt{3}}\,\mathrm{e}^{-i\theta_1 - i\theta_2}\right). \tag{71}$$

This identifies $H^1(\Sigma, \mathbb{R}) \cong H^1(T^2, \mathbb{R}) = \mathbb{R}^2$. Under this identification, as in Example 6.9 we have $Z(L_j^a) = 0$ for all j, a and

$$Y(L_1^a) = (\pi a, 0),\ Y(L_2^a) = (0, \pi a) \ \text{ and } \ Y(L_3^a) = (-\pi a, -\pi a). \tag{72}$$

For all j, a define a *holomorphic disc* D_j^a with $\partial D_j^a \subset L_j^a$ and area πa by

$$D_j^a = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_j|^2 \leqslant a, \quad z_k = 0 \text{ for } j \neq k\}. \tag{73}$$

The cone C is interesting as it has three *topologically distinct* families of AC SL 3-folds asymptotic to it, giving three different ways to desingularize singularities of SL 3-folds with cone C. It is also significant as it is the *only* nontrivial example of a *stable* SL cone in $\mathbb{C}^m$ known to the author. Mark Haskins [7, Th. A] has proved that C is the only stable T^2-cone in $\mathbb{C}^3$ up to SU(3) isomorphisms. But SL m-folds with stable conical singularities have particularly good properties, as in Corollaries 5.3 and 8.8.

The author discussed singular SL 3-folds with cone C in [9, §3–§4] and [13]. We can now prove some of the conjectures in these papers, and give nontrivial applications of Theorems 7.3 and 7.5. For simplicity we consider only SL 3-folds with one or two singular points, but the results of §7 apply to arbitrarily many singularities $x_1, \ldots, x_n$.

10.1 SL 3-folds with one T^2-cone singularity

We shall need a lemma on 3-manifolds with boundary.

Lemma 10.1. *Let N be a compact, oriented 3-manifold with boundary Σ. Then the natural map $H^1(N, \mathbb{R}) \to H^1(\Sigma, \mathbb{R})$ has image of dimension $\frac{1}{2} b^1(\Sigma)$.*

Proof. Let $N^\circ = N \setminus \Sigma$. The pair (N, Σ) has an exact sequence in cohomology

$$(74)\quad \cdots \to H^1(N,\mathbb{R}) = H^1(N^\circ,\mathbb{R}) \to H^1(\Sigma,\mathbb{R}) \to H^2_{\mathrm{cs}}(N^\circ,\mathbb{R}) \to \cdots .$$

But $H^1(N,\mathbb{R}) \cong H^2_{\mathrm{cs}}(N^\circ,\mathbb{R})^*$ and $H^1(\Sigma,\mathbb{R}) \cong H^1(\Sigma,\mathbb{R})^*$ by Poincaré duality for N and Σ. These isomorphisms identify the map $H^1(N,\mathbb{R}) \to H^1(\Sigma,\mathbb{R})$ with the dual of the map $H^1(\Sigma,\mathbb{R}) \to H^2_{\mathrm{cs}}(N^\circ,\mathbb{R})$ in (74). Hence the image of $H^1(N,\mathbb{R}) \to H^1(\Sigma,\mathbb{R})$ has the same dimension as the cokernel of $H^1(\Sigma,\mathbb{R}) \to H^2_{\mathrm{cs}}(N^\circ,\mathbb{R})$, and the lemma follows by exactness in (74). q.e.d.

Consider the following situation, as in [9, §4].

Definition 10.2. Let (M, J, ω, Ω) be an almost Calabi-Yau 3-fold, and X a compact, connected SL 3-fold with exactly one conical singularity at x, with cone C in (70). Then $X' = X \setminus \{x\}$ is connected. Let $\Sigma = C \cap \mathcal{S}^5$, and identify $H^1(\Sigma,\mathbb{R}) \cong \mathbb{R}^2$ as above. Since X' is the interior of a compact, oriented 3-manifold $\overline{X}'$ with boundary $\Sigma \cong T^2$, Lemma 10.1 shows that the map $H^1(X',\mathbb{R}) \to H^1(\Sigma,\mathbb{R})$ of (16) has image $\mathbb{R}$.

Similarly, the map $H^1(X',\mathbb{Q}) \to H^1(\Sigma,\mathbb{Q})$ has image $\mathbb{Q} \subset \mathbb{Q}^2$. Thus there exist coprime integers k_1, k_2, such that $(k_2, -k_1) \in \mathbb{Z}^2 \subset \mathbb{Q}^2 \subset \mathbb{R}^2$ generates the images of $H^1(X',\mathbb{R}) \to H^1(\Sigma,\mathbb{R})$ and $H^1(X',\mathbb{Q}) \to H^1(\Sigma,\mathbb{Q})$. Define $k_3 = -k_1 - k_2$. By exactness in (16), the map $\mathbb{R}^2 = H^1(\Sigma,\mathbb{R}) \to H^2_{\mathrm{cs}}(X',\mathbb{R})$ has kernel $\langle (k_2, -k_1) \rangle$. Therefore this map is given by

$$(75)\qquad (y_1, y_2) \mapsto (k_1 y_1 + k_2 y_2)\chi \quad \text{for some nonzero } \chi \in H^2_{\mathrm{cs}}(X',\mathbb{R}).$$

Then k_1, k_2, k_3 and χ are unique up to an overall change of sign.

The integers k_1, k_2, k_3 were introduced in [9, Def. 4.3]. We now carry out the topological calculations of §8 for desingularizing X by gluing in L^a_j at x.

Proposition 10.3. *In the situation of Definition* 10.2, *fix* $j = 1, 2$ *or* 3 *and let* N_j *be the compact* 3*-manifold obtained by gluing* L^a_j *into* X' *at* x. *Use the notation of* §8, *with* $n = 1$ *and* $L_1 = L^a_j$. *Then*

$\mathcal{Z}_1 = \mathcal{Z} = \{0\}$ *and*

(76)

$$\dim \mathcal{I}_{X'} = b^1_{\mathrm{cs}}(X'), \quad \mathcal{Y}_1 = \begin{cases} \langle (1,0) \rangle, & j=1, \\ \langle (0,1) \rangle, & j=2, \\ \langle (1,1) \rangle, & j=3, \end{cases} \quad \mathcal{Y} = \begin{cases} \mathcal{Y}_1, & k_j = 0, \\ \{0\}, & k_j \neq 0, \end{cases}$$

(77)

$$\mathcal{F}^X_{L_1} = \begin{cases} \{L^a_j : a > 0\}, & k_j = 0, \\ \emptyset, & k_j \neq 0, \end{cases} \quad b^1(N_j) = \begin{cases} b^1_{\mathrm{cs}}(X') + 1, & k_j = 0, \\ b^1_{\mathrm{cs}}(X'), & k_j \neq 0. \end{cases}$$

Proof. In §8.1 as $L_1 \cong \mathcal{S}^1 \times \mathbb{R}^2$ and $\Sigma_1 \cong T^2$ we have $l_1 = 1$, $b^0(L_1) = 1$ and $b^1_{\mathrm{cs}}(L_1) = 0$, and $q = 1$ as X' is connected. Thus (52) gives $\dim \mathcal{Z}_1 = 0$, and $\mathcal{Z} \subseteq \mathcal{Z}_1$, so $\mathcal{Z}_1 = \mathcal{Z} = \{0\}$, and Lemma 8.3 gives $\dim \mathcal{I}_{X'} = b^1_{\mathrm{cs}}(X')$.

Now $\mathcal{Y}_1 \subset H^1(\Sigma, \mathbb{R}) = \mathbb{R}^2$ is the image of $H^1(L^a_j, \mathbb{R}) \rightarrow H^1(\Sigma, \mathbb{R})$ by Definition 8.2, and calculation shows it is as in (76). But the image of $H^1(X', \mathbb{R}) \rightarrow H^1(\Sigma, \mathbb{R})$ is $\langle (k_2, -k_1) \rangle$ by Definition 10.2, and $\mathcal{Y}$ is the intersection of this image with $\mathcal{Y}_1$. As $k_3 = -k_1 - k_2$, we see that $\mathcal{Y}$ is as given in (76).

Equation (52) gives $\dim \mathcal{M}^0_{L_1} = 1$, so $\mathcal{M}^0_{L_1} = \{L^a_j : a > 0\}$. But by (54), as $\mathcal{Z}_1 = \{0\}$ we see that $\mathcal{F}^X_{L_1}$ is the subset of $\hat{L}_1 \in \mathcal{M}^0_{L_1}$ with $Y(\hat{L}_1) \in \mathcal{Y}$, so (76) gives the first equation of (77). The second follows from Proposition 8.6, since $\dim \mathcal{Y}$ is 1 if $k_j = 0$ and 0 if $k_j \neq 0$ by (76). q.e.d.

Now from §8.2, $\mathcal{F}^X_{L_1}$ is the family of AC SL 3-folds L_1 which satisfy the hypotheses of the desingularization results of §7. Thus if $k_j = 0$, applying Theorem 7.3 with $L_1 = L^1_j$ gives:

Theorem 10.4. *Suppose* (M, J, ω, Ω) *is an almost Calabi-Yau* 3-*fold, and* X *a compact, connected* SL 3-*fold with exactly one conical singularity at* x, *with cone* C *in* (70). *Let* k_1, k_2, k_3 *be as in Definition* 10.2, *and suppose* $k_j = 0$ *for* $j = 1, 2$ *or* 3. *Then there exists* $\epsilon > 0$ *and a smooth family* $\{\tilde{N}^t_j : t \in (0, \epsilon]\}$ *of compact* SL 3-*folds in* (M, J, ω, Ω) *constructed by gluing* $L^{t^2}_j$ *into* X *at* x. *In the sense of currents,* $\tilde{N}^t_j \rightarrow X$ *as* $t \rightarrow 0$.

If $k_j \neq 0$ then for topological reasons there exist no *Lagrangian* 3-folds N^t_j constructed by gluing tL^1_j into X at x, and hence no SL 3-folds

$\tilde{N}_j^t$. As the k_j are not all zero and $k_1+k_2+k_3=0$ there can be at most one j with $k_j=0$.

In the situation of Theorem 10.4, by Corollary 5.3 and (76) the moduli space $\mathcal{M}_X$ of deformations of X is a smooth manifold of dimension $b^1_{\rm cs}(X')$, and by Theorem 2.10 and (77) the moduli space $\mathcal{M}_{N_j}$ of deformations of $\tilde{N}_j^t$ is a smooth manifold of dimension $b^1_{\rm cs}(X')+1$.

Hence the singularities of X have *index one* in the sense of §8.3, and $\overline{\mathcal{M}}_{N_j}$ is near X a *nonsingular manifold with boundary* $\mathcal{M}_X$. So in this case we have a very good understanding of the boundary $\partial\mathcal{M}_{N_j}$ of $\mathcal{M}_{N_j}$, as discussed in §8.

Following [9, §3.3] we can explain why $\tilde{N}_j^t$ becomes singular as $t\rightarrow 0$, using the D_j^a of (73). As $\tilde{N}_j^t$ is made by gluing $L_j^{t^2}$ into X at x, and there is a holomorphic disc $D_j^{t^2}$ with area πt^2 and $\partial D_j^{t^2}\subset L_j^{t^2}$, we expect there to exist a holomorphic disc $\tilde{D}^t$ with area πt^2 and $\partial\tilde{D}^t\subset\tilde{N}_j^t$, for small t.

As $t\rightarrow 0$ the area of $\tilde{D}^t$ goes to zero, and $\tilde{D}^t$ collapses to a point. Its boundary $\mathcal{S}^1$ in $\tilde{N}_j^t$ also collapses to a point, giving the singular SL 3-fold X. The author expects that singularities with cone C are the generic singularity of SL 3-folds occurring when areas of holomorphic discs become zero.

10.2 SL 3-folds with one T^2-cone singularity in families

Next we apply Theorem 7.5 to desingularize SL 3-folds with one singularity x with cone C in *families* of almost Calabi-Yau 3-folds $(M,J^s,\omega^s,\Omega^s)$.

Theorem 10.5. *Suppose* (M,J,ω,Ω) *is an almost Calabi-Yau* 3-*fold, and* X *a compact, connected* SL 3-*fold with exactly one conical singularity at* x*, with cone* C *in* (70). *Let* k_1,k_2,k_3,χ *be as in Definition* 10.2. *Suppose* $\{(M,J^s,\omega^s,\Omega^s):s\in\mathcal{F}\}$ *is a smooth family of deformations of* (M,J,ω,Ω) *with*

$$[\operatorname{Im}\Omega^s]\cdot[X]=0 \quad \textit{for all } s\in\mathcal{F}, \textit{ where } [X]\in H_3(M,\mathbb{R}). \tag{78}$$

Let $\iota_*:H_2(X,\mathbb{R})\rightarrow H_2(M,\mathbb{R})$ *be the inclusion, fix* $j=1,2$ *or* 3, *and define*

$$\mathcal{G}_j=\Big\{(s,t)\in\mathcal{F}\times(0,1):[\omega^s]\cdot\iota_*(\gamma)=\pi t^2k_j(\chi\cdot\gamma) \quad \textit{for all } \gamma\in H_2(X,\mathbb{R})\Big\}. \tag{79}$$

Then there exist $\epsilon \in (0,1)$, $\kappa > 1$ *and* $\vartheta \in (0,2)$ *and a smooth family*

$$\big\{\widetilde{N}_j^{s,t} : (s,t) \in \mathcal{G}_j, \quad t \in (0,\epsilon], \quad |s| \leqslant t^\vartheta\big\}, \tag{80}$$

such that $\widetilde{N}_j^{s,t}$ *is a compact* SL 3*-fold in* $(M, J^s, \omega^s, \Omega^s)$ *constructed by gluing* $L_j^{t^2}$ *into* X *at* x*. In the sense of currents,* $\widetilde{N}_j^{s,t} \to X$ *as* $s,t \to 0$.

Proof. We apply Theorem 7.5, with $L_1 = L_j^1$. As X is connected, X' is connected. Equation (78) gives (48). The values of $Y(L_j^a)$ in $H^1(\Sigma,\mathbb{R}) \cong \mathbb{R}^2$ are given in (72), and the map $H^1(\Sigma,\mathbb{R}) \to H^2_{\mathrm{cs}}(X',\mathbb{R})$ is given in (75). Combining these two shows that the image of $Y(L_j^a)$ in $H^2_{\mathrm{cs}}(X',\mathbb{R})$ is $\pi a k_j\,\chi$. Thus putting $a = 1$ as $L_1 = L_j^1$, we have $\varpi = \pi k_j\,\chi$ in Theorem 7.5, and so $\mathcal{G}_j$ in (79) agrees with $\mathcal{G}$ in (49). The result then follows from Theorem 7.5. q.e.d.

We now specialize to the case that $b^1_{\mathrm{cs}}(X') = 0$. Then $b^2(X') = 0$ by (17), so (16) gives an exact sequence

$$0 \to H^1(X',\mathbb{R}) \to H^1(\Sigma_i,\mathbb{R}) \cong \mathbb{R}^2 \to H^2_{\mathrm{cs}}(X',\mathbb{R}) \to 0.$$

As $b^1(X') = b^2_{\mathrm{cs}}(X')$ by (17), this gives $b^1(X') = b^2_{\mathrm{cs}}(X') = 1$.

Now $H_2(X,\mathbb{R}) \cong H^2_{\mathrm{cs}}(X',\mathbb{R})^* \cong \mathbb{R}$ by (18), and $\chi \neq 0$ by (75). Thus there exists a unique $\gamma_0 \in H_2(X,\mathbb{R})$ with $\chi \cdot \gamma_0 = 1$. The condition $[\omega^s] \cdot \iota_*(\gamma) = \pi t^2 k_j (\chi \cdot \gamma)$ for all $\gamma \in H_2(X,\mathbb{R})$ in (79) then becomes the single real equation $[\omega^s] \cdot \iota_*(\gamma_0) = \pi t^2 k_j$.

As in (69), let us divide the family $\mathcal{F}$ into three regions:

$$\begin{aligned} \mathcal{F}^+ &= \big\{s \in \mathcal{F} : [\omega^s] \cdot \iota_*(\gamma_0) > 0\big\}, \\ \mathcal{F}^- &= \big\{s \in \mathcal{F} : [\omega^s] \cdot \iota_*(\gamma_0) < 0\big\}, \\ \text{and} \quad \mathcal{F}^0 &= \big\{s \in \mathcal{F} : [\omega^s] \cdot \iota_*(\gamma_0) = 0\big\}. \end{aligned} \tag{81}$$

If $\iota_*(\gamma_0) \neq 0$ and $\mathcal{F}$ is sufficiently generic, then $\mathcal{F}^0$ will be a *smooth real hypersurface* in $\mathcal{F}$, which divides $\mathcal{F} \setminus \mathcal{F}^0$ into two open regions $\mathcal{F}^\pm$. Call $\mathcal{F}^+$ the *positive side* and $\mathcal{F}^-$ the *negative side* of $\mathcal{F}^0$.

We investigate the existence of deformations X^s and desingularizations $\widetilde{N}_j^{s,t}$ of X in $(M, J^s, \omega^s, \Omega^s)$, for small s in each of these regions.

(a) From §5, a *necessary condition* for the existence of any SL 3-fold X^s with a conical singularity isotopic to X in $(M, J^s, \omega^s, \Omega^s)$ is that $\iota_*(\gamma) \cdot [\omega^s] = 0$ for all $\gamma \in H_2(X,\mathbb{R})$. Thus, such X^s can exist only if $s \in \mathcal{F}^0$.

As $\dim \mathcal{I}_{X'} = b^1_{\rm cs}(X') = 0$, Corollary 5.8 then shows that for *small* $s \in \mathcal{F}^0$, there is a *unique* deformation X^s of X close to X in $(M, J^s, \omega^s, \Omega^s)$.

(b) If $s \in \mathcal{F}^0$ then $[\omega^s] \cdot \iota_*(\gamma_0) = \pi t^2 k_j$ has solutions $t > 0$ if and only if $k_j = 0$, and then any $t > 0$ is a solution. For small $s \in \mathcal{F}^0$ we may apply Theorem 10.4 to the unique X^s above to get a 1-parameter family of SL 3-folds $\widetilde{N}_j^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$ for small $t > 0$, with $b^1(\widetilde{N}_j^{s,t}) = b^1(N_j) = 1$.

(c) If $s \in \mathcal{F}^+$ then $[\omega^s] \cdot \iota_*(\gamma_0) = \pi t^2 k_j$ has a unique solution $t > 0$ if and only if $k_j > 0$. Theorem 10.5 then shows that a desingularization $\widetilde{N}_j^{s,t}$ exists provided $t \leqslant \epsilon$ and $|s| \leqslant t^\vartheta$, which is unique as $b^1(\widetilde{N}_j^{s,t}) = b^1(N_j) = 0$.

By applying Theorem 10.5 not just to X but to X^s for small $s \in \mathcal{F}^0$, one can show that such $\widetilde{N}_j^{s,t}$ actually exist for all small $s \in \mathcal{F}^+$.

(d) If $s \in \mathcal{F}^-$ then $[\omega^s] \cdot \iota_*(\gamma_0) = \pi t^2 k_j$ has a unique solution $t > 0$ if and only if $k_j < 0$. As for $s \in \mathcal{F}^+$, we find that a unique desingularization $\widetilde{N}_j^{s,t}$ then exists for small $s \in \mathcal{F}^-$.

As k_1, k_2, k_3 are not all zero with $k_1 + k_2 + k_3 = 0$, there is at least one $k_j < 0$, and at least one $k_j > 0$. Suppose $k_1 < 0$ and $k_2, k_3 > 0$, for example. Imagine $s \in \mathcal{F}$ moving along a curve near 0 starting in $\mathcal{F}^-$, crossing $\mathcal{F}^0$ and ending in $\mathcal{F}^+$. Then initially there is one SL homology 3-sphere $\widetilde{N}_1^{s,t}$ in $(M, J^s, \omega^s, \Omega^s)$. As s crosses $\mathcal{F}^0$ this SL 3-fold collapses to a singular SL 3-fold X^s, with a conical singularity with cone C.

As s moves into $\mathcal{F}^+$ it is desingularized in *two topologically distinct ways* to give two SL homology 3-spheres $\widetilde{N}_2^{s,t}, \widetilde{N}_3^{s,t}$. We have found a process by which *one* SL *homology* 3-*sphere can turn into two* SL *homology* 3-*spheres* as we deform the underlying almost Calabi-Yau 3-fold $(M, J^s, \omega^s, \Omega^s)$. This was described in [9, §4.2], and we have now proved [9, Conj. 4.4].

In [9, Prop. 4.5] we compute $H_1(X, \mathbb{Z})$, $H_1(N_j, \mathbb{Z})$ and show:

Proposition 10.6. *In the situation of Definition* 10.2, *let* N_j *be the compact, nonsingular* 3*-manifold obtained by gluing* L^a_j *into* X' *at* x, *for* $j = 1, 2, 3$. *Suppose* $b^1_{\rm cs}(X') = 0$. *Then* $H_1(X, \mathbb{Z})$ *is finite. If* $k_j \neq 0$ *then* $H_1(N_j, \mathbb{Z})$ *is also finite with* $|H_1(N_j, \mathbb{Z})| = |k_j| \cdot |H_1(X, \mathbb{Z})|$.

In the situation above with $k_1 < 0$ and $k_2, k_3 > 0$, as $k_1 + k_2 + k_3 = 0$ we have $|k_1| = |k_2| + |k_3|$, and therefore $|H_1(N_1, \mathbb{Z})| = |H_1(N_2, \mathbb{Z})| +$

$|H_1(N_3,\mathbb{Z})|$ by Proposition 10.6. Now $|H_1(N_j,\mathbb{Z})|$ is the *number of flat* U(1)-*connections* on N_j. Thus when $\widetilde{N}_1^{s,t}$ turns into $\widetilde{N}_2^{s,t}$ and $\widetilde{N}_3^{s,t}$, the number of SL homology 3-spheres with flat U(1)-connections *does not change*.

This is physically significant since 3-*branes* in string theory correspond in a classical limit to SL 3-folds with flat U(1)-connections, as in [29] for instance. The proposal of [9] is to count SL homology 3-spheres with flat U(1)-connections in a given homology class. We have shown that this number is *conserved* under a nontrivial kind of transition in the family of SL homology 3-spheres.

10.3 SL 3-folds with two T^2-cone singularities

Next we study SL 3-folds with *two* singularities with cone C.

Definition 10.7. Let (M,J,ω,Ω) be an almost Calabi-Yau 3-fold, and X a compact, connected SL 3-fold with exactly two conical singularities at x_1,x_2, both with cone C in (70). Then $X'=X\setminus\{x_1,x_2\}$ is connected. Write Σ_1,Σ_2 for the two copies of $\Sigma=C\cap\mathcal{S}^5$ at x_1,x_2, and identify $H^1(\Sigma_i,\mathbb{R})\cong\mathbb{R}^2$ as above. Write elements of $H^1(\Sigma_1,\mathbb{R})\oplus H^1(\Sigma_2,\mathbb{R})=\mathbb{R}^2\oplus\mathbb{R}^2$ as $\bigl((u,v),(y,z)\bigr)$.

Since X' is the interior of a compact, oriented 3-manifold $\overline{X}'$ with boundary $\Sigma_1\cup\Sigma_2$, the map $H^1(X',\mathbb{R})\to H^1(\Sigma_1,\mathbb{R})\oplus H^1(\Sigma_2,\mathbb{R})$ of (16) has image $\mathbb{R}^2$ by Lemma 10.1. Choose a basis $\bigl((u_1,v_1),(y_1,z_1)\bigr)$, $\bigl((u_2,v_2),(y_2,z_2)\bigr)$ for this image. As it is also a basis over $\mathbb{R}$ for the image of $H^1(X',\mathbb{Q})\to H^1(\Sigma_1,\mathbb{Q})\oplus H^1(\Sigma_2,\mathbb{Q})$ we can take $u_1,\ldots,z_2\in\mathbb{Q}$.

Let α_1,α_2 be closed 1-forms on $\overline{X}'$ such that the images of $[\alpha_1],[\alpha_2]$ in $H^1(\Sigma_1,\mathbb{R})\oplus H^1(\Sigma_2,\mathbb{R})$ are this basis. Then $\alpha_1\wedge\alpha_2$ is a closed 2-form on $\overline{X}'$, an oriented 3-manifold with boundary $\Sigma_1\cup\Sigma_2$, so by Stokes' Theorem we have $\int_{\Sigma_1\cup\Sigma_2}\alpha_1\wedge\alpha_2=0$. This gives the consistency condition

$$u_1v_2-u_2v_1+y_1z_2-y_2z_1=0. \tag{82}$$

Applying Theorem 7.3 gives a necessary and sufficient criterion for when we can desingularize X by gluing in AC SL 3-folds $L_{j_1}^{a_1},L_{j_2}^{a_2}$ at x_1,x_2.

Theorem 10.8. *Suppose* (M,J,ω,Ω) *is an almost Calabi-Yau* 3-*fold, and* X *a compact, connected* SL 3-*fold with exactly two conical singularities at* x_1, x_2, *both with cone* C *in* (70). *Let* $u_1,\dots,z_2$ *be as in Definition* 10.7. *Choose* $j_1, j_2 = 1,2,3$ *and* $a_1, a_2 > 0$, *and set* $L_i = L_{j_i}^{a_i}$ *for* $i = 1,2$. *Then there exists* $\epsilon > 0$ *and a smooth family* $\bigl\{\widetilde{N}^t : t \in (0,\epsilon]\bigr\}$ *of compact* SL 3-*folds in* (M,J,ω,Ω) *constructed by gluing* tL_i *into into* X *at* x_i *if and only if*

$$\bigl(Y(L_1),Y(L_2)\bigr) \in \bigl\langle \bigl((u_1,v_1),(y_1,z_1)\bigr), \bigl((u_2,v_2),(y_2,z_2)\bigr)\bigr\rangle \subset \mathbb{R}^2 \oplus \mathbb{R}^2, \tag{83}$$

where $Y(L_i)$ *are given in* (72). *In the sense of currents,* $\widetilde{N}^t \to X$ *as* $t \to 0$.

Here ϱ exists in Theorem 7.3 if and only if $\bigl(Y(L_1),Y(L_2)\bigr)$ lies in the image of $H^1(X',\mathbb{R}) \to H^1(\Sigma_1,\mathbb{R}) \oplus H^1(\Sigma_2,\mathbb{R})$, that is, if and only if (83) holds. We are interested in how many *topologically distinct* ways of desingularizing X there are, and in the *index* of the singularities of X.

Let us use the notation of §8, so that $\mathcal{I}_{X'}$ is the image of $H^1_{\mathrm{cs}}(X',\mathbb{R})$ in $H^1(X',\mathbb{R})$, and $\mathcal{M}_X$ the moduli space of deformations of X in (M,J,ω,Ω) as in §5, and N the compact 3-manifold obtained by gluing $L_i = L_{j_i}^{a_i}$ into X at x_i for $i = 1,2$, and $\mathcal{M}_N$ the moduli space of deformations of $\widetilde{N}^t$ in (M,J,ω,Ω).

Corollary 5.3 and Theorem 2.10 show that $\mathcal{M}_X, \mathcal{M}_N$ are smooth with

$$\dim \mathcal{M}_X = \dim \mathcal{I}_{X'} \quad \text{and} \quad \dim \mathcal{M}_N = b^1(N). \tag{84}$$

As $q = l_1 = l_2 = 1$, Lemma 8.3 gives $\dim \mathcal{I}_{X'} = b^1_{\mathrm{cs}}(X') - 1$. Equation (52) shows that $\mathcal{Z}_i = \mathcal{Z} = \{0\}$ and $\dim \mathcal{Y}_i = 1$, so that $\mathcal{Y}_i = \langle Y(L_i)\rangle$. Therefore

$$\mathcal{Y} = \bigl\langle \bigl(Y(L_1),0\bigr), \bigl(0,Y(L_2)\bigr)\bigr\rangle \cap \bigl\langle \bigl((u_1,v_1),(y_1,z_1)\bigr), \bigl((u_2,v_2),(y_2,z_2)\bigr)\bigr\rangle. \tag{85}$$

Proposition 8.6 then gives

$$b^1(N) = \dim \mathcal{Y} + b^1_{\mathrm{cs}}(X') - 1 = \dim \mathcal{Y} + \dim \mathcal{I}_{X'}, \tag{86}$$

so that $\dim \mathcal{M}_N = \dim \mathcal{M}_X + \dim \mathcal{Y}$, and $\mathrm{ind}(X) = \dim \mathcal{Y}$ in the sense of §8.3.

For generic $u_1,\dots,z_2$ there will be no choices of j_1, j_2, a_1, a_2 for which (83) holds, and so no way to desingularize X in (M,J,ω,Ω). Here are four examples where other things happen.

Example 10.9. Suppose $u_1,\dots,z_2$ are given by

(87)
$$\bigl((u_1,v_1),(y_1,z_1)\bigr)=\bigl((1,0),(0,0)\bigr),\ \ \bigl((u_2,v_2),(y_2,z_2)\bigr)=\bigl((0,0),(1,0)\bigr),$$

so that (82) holds. Then we can desingularize X using $L_1=L_1^{a_1}$ and $L_2=L_1^{a_2}$ for any $a_1,a_2>0$, with $\mathrm{ind}(X)=\dim\mathcal{Y}=2$. We must have $j_1=j_2=1$, so there is only one topological possibility.

Example 10.10. Let $\bigl((u_1,v_1),(y_1,z_1)\bigr)=\bigl((1,0),(r,0)\bigr)$ for $r>0$ in $\mathbb{Q}$, and $\bigl((u_2,v_2),(y_2,z_2)\bigr)$ be generic with $v_2+rz_2=0$. The only way to desingularize X is with $L_1=L_1^a$ and $L_2=L_1^{ra}$ for $a>0$, with $\mathrm{ind}(X)=\dim\mathcal{Y}=1$.

Example 10.11. Let $r>0$ be in $\mathbb{Q}$ with $r\neq 1$, and let

(88)
$$\bigl((u_1,v_1),(y_1,z_1)\bigr)=\bigl((1,0),(0,r)\bigr),\ \ \bigl((u_2,v_2),(y_2,z_2)\bigr)=\bigl((0,r),(1,0)\bigr).$$

Then there are exactly *two topologically distinct* ways to desingularize X:

(a) $j_1=1$, $j_2=2$, $a_1=a>0$, $a_2=ra>0$, $L_1=L_1^a$ and $L_2=L_2^{ra}$,

(b) $j_1=2$, $j_2=1$, $a_1=ra>0$, $a_2=a>0$, $L_1=L_2^{ra}$ and $L_2=L_1^a$.

Both have $\mathrm{ind}(X)=\dim\mathcal{Y}=1$.

Example 10.12. Suppose $u_1,\dots,z_2$ are given by

(89)
$$\bigl((u_1,v_1),(y_1,z_1)\bigr)=\bigl((1,0),(0,1)\bigr),\ \ \bigl((u_2,v_2),(y_2,z_2)\bigr)=\bigl((0,1),(1,0)\bigr),$$

which is the case $r=1$ in Example 10.11. Then there are exactly *three topologically distinct* ways to desingularize X, each with $\mathrm{ind}(X)=\dim\mathcal{Y}=1$:

(a) $j_1=1$, $j_2=2$, $a_1=a_2=a>0$, $L_1=L_1^a$ and $L_2=L_2^a$,

(b) $j_1=2$, $j_2=1$, $a_1=a_2=a>0$, $L_1=L_2^a$ and $L_2=L_1^a$,

(c) $j_1=j_2=3$, $a_1=a_2=a>0$, and $L_1=L_2=L_3^a$.

We can explain this using the *holomorphic discs* D_j^a of (73). For each desingularization $\tilde{N}^t$ when t is small, we expect there to exist unique holomorphic discs $\tilde{D}_1^t, \tilde{D}_2^t$ in (M, J), where $\tilde{D}_i^t$ is near x_i, has area πa_i and boundary $\partial\tilde{D}_i^t \subset \tilde{N}^t$ for $i = 1, 2$.

In Example 10.9 the homology classes $[\partial\tilde{D}_1^t], [\partial\tilde{D}_2^t] \in H_1(\tilde{N}^t, \mathbb{R})$ are *linearly independent*. Therefore by deforming $\tilde{N}^t$ as an SL 3-fold we can vary the areas of $\tilde{D}_1^t, \tilde{D}_2^t$ independently. These two areas give two real parameters for the desingularization of X, which is why $\operatorname{ind}(X) = 2$.

However, in Examples 10.10–10.12 the homology classes $[\partial\tilde{D}_1^t], [\partial\tilde{D}_2^t]$ are *proportional* in $H_1(\tilde{N}^t, \mathbb{R})$. This forces $\operatorname{area}(\tilde{D}_2^t) = c \cdot \operatorname{area}(\tilde{D}_1^t)$ to hold under deformations of $\tilde{N}^t$, where $c = r$ in Example 10.10 and part (a) of Example 10.11, $c = r^{-1}$ in part (b) of Example 10.11, and $c = 1$ in Example 10.12. In particular, the areas of $\tilde{D}_1^t, \tilde{D}_2^t$ can only become zero *simultaneously*.

Therefore the two singularities x_1, x_2 in X are not independent, but *coupled together*. For an SL m-fold X with conical singularities $x_1, \ldots, x_n$ one might naïvely expect each x_i to be desingularized separately, and $\operatorname{ind}(X)$ to be the sum of contributions from each x_i. But in Examples 10.10–10.12 we see that x_1, x_2 can only be desingularized together, and $\operatorname{ind}(X) = 1$ is not a sum of separate contributions from x_1, x_2.

In Examples 10.10–10.12 the moduli spaces $\mathcal{M}_X$, $\mathcal{M}_N$ are smooth with $\dim \mathcal{M}_N = \dim \mathcal{M}_X + 1$. Therefore $\overline{\mathcal{M}}_N$ is near X a *nonsingular manifold with boundary* $\mathcal{M}_X$. So we have a good understanding of the boundary $\partial\mathcal{M}_N$ of $\mathcal{M}_N$, as in §8. It is also interesting that in Examples 10.11 and 10.12 we have *two or three different moduli spaces* $\mathcal{M}_N$ *with common boundary* $\mathcal{M}_X$.

Finally we discuss the ideas of [13] on SL *fibrations* of (almost) Calabi-Yau 3-folds, as required by the SYZ Conjecture [29]. Let (M, J, ω, Ω) be an almost Calabi-Yau 3-fold and $f : M \to B$ an SL *fibration*. That is, B is a compact 3-manifold, f is continuous and piecewise smooth, and for some $\Delta \subset B$ with $B \setminus \Delta$ open and dense the fibres $X_b = f^{-1}(b)$ are SL 3-tori for $b \in B \setminus \Delta$, and singular SL 3-folds for $b \in \Delta$.

The idea of [13, §7–§8] is that for ω generic in its Kähler class, Δ should be of codimension 1 in B, and for $b \in \Delta$ generic X_b should have 2 (or $2n$) singularities with cone C, as in Definition 10.7. Then Δ is locally a hypersurface which divides $B \setminus \Delta$ into two pieces. These are *two different moduli spaces* of SL 3-tori with *common boundary* Δ, as in Examples 10.11 and 10.12. Our results provide a partial proof of speculations in [13, §8.2(a)].

References

[1] F.J. Almgren, *Almgren's big regularity paper. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension* 2, with preface by Jean E. Taylor and Vladimir Scheffer, World Scientific Monograph Series in Mathematics, **1**, World Scientific Publishing Co., Inc., River Edge, NJ, 2000, MR 2003d:49001, Zbl 0985.49001.

[2] G.E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, **139**, Springer-Verlag, Berlin, 1993, MR 94d:55001, Zbl 0791.55001.

[3] A. Butscher, *Regularizing a singular special Lagrangian variety*, math.DG/0110053, to appear in Communications in Analysis and Geometry.

[4] R. Harvey, *Spinors and calibrations*, Perspectives in Mathematics, **9**, Academic Press, Inc., Boston, 1990, MR 91e:53056, Zbl 0694.53002.

[5] R. Harvey & H.B. Lawson, *Calibrated geometries*, Acta Mathematica **148** (1982) 47–157, MR 85i:53058, Zbl 0584.53021.

[6] M. Haskins, *Special Lagrangian cones*, math.DG/0005164, to appear in American Journal of Mathematics.

[7] M. Haskins, *The geometric complexity of special Lagrangian T^2-cones*, math.DG/0307129.

[8] Y.-I. Lee, *Embedded Special Lagrangian Submanifolds in Calabi-Yau Manifolds*, to appear in Communications in Analysis and Geometry.

[9] D.D. Joyce, *On counting special Lagrangian homology* 3*-spheres*, in 'Topology and Geometry: Commemorating SISTAG', editors A.J. Berrick, M.C. Leung and X.W. Xu, 125–151, Contemporary Mathematics, **314**, AMS, Providence, RI, 2002, MR 2003i:53076.

[10] D.D. Joyce, *Special Lagrangian m-folds in $\mathbb{C}^m$ with symmetries*, Duke Math. J. **115** (2002) 1–51, MR 1 932 324.

[11] D.D. Joyce, *Constructing special Lagrangian m-folds in $\mathbb{C}^m$ by evolving quadrics*, Math. Ann. **320** (2001) 757–797, MR 2002j:53066.

[12] D.D. Joyce, *Evolution equations for special Lagrangian* 3*-folds in* $\mathbb{C}^3$, Ann. Global Anal. Geom. **20** (2001) 345–403, MR 2003a:53065, Zbl 1003.53040.

[13] D.D. Joyce, *Singularities of special Lagrangian fibrations and the SYZ Conjecture*, math.DG/0011179, version 2, 2002, to appear in Communications in Analysis and Geometry.

[14] D.D. Joyce, *Ruled special Lagrangian* 3*-folds in* $\mathbb{C}^3$, Proceedings of the London Mathematical Society **85** (2002) 233–256, MR 2003i:53077.

[15] D.D. Joyce, *Special Lagrangian* 3*-folds and integrable systems*, math.DG/0101249, to appear in volume 1 of the Proceedings of the Mathematical Society of Japan's 9th International Research Institute on 'Integrable Systems in Differential Geometry', Tokyo, 2000.

[16] D.D. Joyce, *Lectures on Calabi-Yau and special Lagrangian geometry*, math.DG/0108088, published, with extra material, as Part I of M. Gross, D. Huybrechts & D. Joyce, *Calabi-Yau Manifolds and Related Geometries*, Universitext series, Springer, Berlin, 2003, MR 1 963 559, Zbl 1001.00028.

[17] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities.* I. *Regularity*, math.DG/0211294.

[18] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities.* II. *Moduli spaces*, math.DG/0211295.

[19] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities.* III. *Desingularization, the unobstructed case*, math.DG/0302355.

[20] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities.* IV. *Desingularization, obstructions and families*, math.DG/0302356.

[21] G. Lawlor, *The angle criterion*, Invent. Math. **95** (1989) 437–446, MR 89m:49077, Zbl 0662.49018.

[22] R. Lockhart, *Fredholm, Hodge and Liouville Theorems on noncompact manifolds*, Trans. AMS **301** (1987) 1–35, MR 88g:58190, Zbl 0623.58019.

[23] S.P. Marshall, *Deformations of special Lagrangian submanifolds*, Oxford D.Phil. Thesis, 2002.

[24] D. McDuff & D. Salamon, *Introduction to symplectic topology*, Second edition, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998, MR 2000g:53098, Zbl 0844.58029.

[25] I. McIntosh, *Special Lagrangian cones in* $\mathbb{C}^3$ *and primitive harmonic maps*, J. London Math. Soc. **67** (2003) 769–789.

[26] R.C. McLean, *Deformations of calibrated submanifolds*, Communications in Analysis and Geometry **6** (1998) 705–747, MR 99j:53083, Zbl 0929.53027.

[27] F. Morgan, *Geometric measure theory, a beginner's guide*, Academic Press, San Diego, 1995, Zbl 0974.49025.

[28] T. Pacini, *Deformations of asymptotically conical special Lagrangian submanifolds*, math.DG/0207144.

[29] A. Strominger, S.-T. Yau & E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Physics **B479** (1996) 243–259, MR 97j:32022, Zbl 0896.14024.

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