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MELROSE-UHLMANN PROJECTORS, THE METAPLECTIC REPRESENTATION AND SYMPLECTIC CUTS

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Abstract

By applying the symplectic cutting operation to cotangent bundles, one can construct a large number of interesting symplectic cones. In this paper we show how to attach algebras of pseudodifferential operators to such cones and describe the symbolic properties of the algebras.

0. Introduction

The Melrose-Uhlmann projectors which we refer to in the title of this article are projection operators which look microlocally like the standard Szegö projectors on $L^2(S^1)$. They belong to a class of pseudodifferential operators with singular symbols which were studied by Melrose-Uhlmann in [8] and by one of us in [2]. One of the main goals of this paper will be to give a microlocal description of the algebra of classical pseudodifferential operators which commute with such a projection operator.

Another of the main goals of this paper will be to examine some microlocal aspects of a basic operation in cobordism theory: the *cutting* operation. Let M be a C^∞ manifold, $\tau : S^1 \times M \to M$ an action of S^1 on M and $\Phi : M \to \mathbb{R}$ an S^1-invariant function. If zero is a regular value of Φ the set

$$W = \{p \in M\ ,\ \Phi(p) \geq 0\}$$

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is a manifold with boundary, and if S^1 acts freely on the boundary, one gets a C^∞ manifold without boundary by collapsing the circle orbits in the boundary to points. This new manifold, which we will denote by M, is the disjoint union of the manifold, $M_{\text{red}} = \Phi^{-1}(0)/S^1$ and the interior, W^0, of W; and M_{red} sits inside M as a codimension 2 submanifold. For example let $M = \mathbb{C}^n$ and let τ be multiplication by unit complex numbers. If $\Phi(z) = |z|^2 - 1$, then M is the blow up of $\mathbb{C}^n$ at 0 and $M_{\text{red}} = \mathbb{C}P^{n-1}$ is the exceptional divisor. On the other hand if $\Phi(z) = 1 - |z|^2$, then M is $\mathbb{C}P^n$ and $M_{\text{red}} = \mathbb{C}P^{n-1}$.

It was observed several years ago by one of us (see [5]) that this cutting operation can be symplecticized. Namely suppose that $M = (M, \omega)$ is a symplectic manifold, τ a Hamiltonian action and Φ the moment map associated with this action. Then the symplectic form on $W^0 = M - M_{\text{red}}$ extends smoothly to a symplectic form on M_{cut} and so also does the action τ and moment map, Φ. Moreover, M_{red} is a symplectic submanifold of M_{cut} and, as an abstract symplectic manifold, is isomorphic to the usual symplectic reduction of M by τ.

To prove these assertions one needs a somewhat different description of M_{cut}. Consider the product manifold, $M \times \mathbb{C}$, with the product symplectic form, $\omega_M - \omega_{\mathbb{C}}$, and the action on it of $S^1 \times S^1$. The moment map for this product action is $(\Phi(m), -|z|^2)$; so if we restrict to the diagonal subgroup of $S^1 \times S^1$ we get a Hamiltonian action of S^1 on $M \times \mathbb{C}$ with moment map, $\Psi(m, z) = \Phi(m) - |z|^2$, and it is not hard to see that M_{cut} can be identified with the reduced space

$$(M \times \mathbb{C})_{\text{red}} = \Psi^{-1}(0)/S^1 \,. \tag{0.1}$$

Moreover this space has a residual action on it of S^1, and it is not hard to see that this action coincides with the action of S^1 described above.

Suppose now that the action τ can be quantized; i.e., suppose that one can associate with (M, τ) a representation, $\tau^{\#}$, of S^1 on a Hilbert space, $Q(M)$, by some kind of "quantization" procedure. Then, in view of the fact that the symplectic form on $M \times \mathbb{C}$ defined above is the product of the symplectic form on M and on $\mathbb{C}$, one gets for the quantization of $M \times \mathbb{C}$

$$Q(M) \otimes Q(\mathbb{C})^*$$

or equivalently

$$\operatorname{Hom}(Q(\mathbb{C}), Q(M)) \,.$$

Thus by the "quantization commutes with reduction" principle one gets for the reduced space, $M_{\text{cut}} = (M \times \mathbb{C})_{\text{red}}$ the quantization

$$\operatorname{Hom}(Q(\mathbb{C}),\, Q(M))^{S^1}\,.$$

To complete this quantum description of M_{cut} we still have to specify a quantization $Q(\mathbb{C})$, of the action of S^1 on $\mathbb{C}$ and for this there is a more or less canonical candidate, the oscillator representation of S^1 on $L^2(\mathbb{R})$. Thus the Hilbert space

$$\operatorname{Hom}(L^2(\mathbb{R})\,,Q(M))^{S^1} \tag{0.2}$$

is an obvious candidate for $Q(M_{\text{cut}})$.[1]

To see how this construction is related to the theory of Melrose-Uhlmann projectors let H_n, $n = 0, 1, \dots$ be the one-dimensional subspace of $L^2(\mathbb{R})$ spanned by the n^{th} Hermite function, h_n. This subspace transforms as $e^{in\theta}$ under the action of $\theta \in S^1$. Therefore the space

$$\operatorname{Hom}(H_n,\, Q(M))^{S^1}$$

can be identified with the space

$$Q_n(M) = \{f \in Q(M)\,,\, \tau^{\#}(e^{i\theta})f = e^{in\theta} f\}$$

via the map

$$T \mapsto Th_n\,,$$

and the space (0.2) can be identified with the direct sum

$$\bigoplus_{n=0}^{\infty} Q_n(M)\,. \tag{0.3}$$

Let us denote by Π_+ the orthogonal projection of $Q(M)$ onto the space (0.3). The examples we will be interested in in this paper with be quantizations defined using microlocal analysis, and for these examples Π_+ will be a projector of Melrose-Uhlmann type. Moreover in these examples there will be a natural algebra of "quantum observables" on M: either pseudodifferential operators or Toeplitz operators, and hence

[1] To make (0.2) into a Hilbert space we will take the intertwining operators in this "Hom" to be Hilbert-Schmidt.

a natural algebra of quantum observables on M_{cut}, namely the operators which commute with Π_+.

Finally we'll explain why the metaplectic representation is involved in the construction we've just described. The oscillator representations of S^1 on $L^2(\mathbb{R})$ is unfortunately not a representation of S^1 itself but of its metaplectic double cover. This double cover is just another copy of S^1; so there would seem to be no problem in substituting it for S^1 in the definition (0.2). However, if one wants to attach symbols to the quantum observables we just defined, the fact that the S^1 acting on $\mathbb{C}$ is not the same S^1 as that acting on $L^2(\mathbb{R})$ causes some unpleasant parity complications and one has to make use of metaplectic techniques to deal with these complications.

A few words about the contents of this article. For simplicity we will henceforth assume that the manifold M above is the cotangent bundle of a compact manifold, X, and that the algebra of "quantum observables" is the algebra of pseudodifferential operators, $\Psi(X)$.[2] As for the action, τ we will assume it is a *canonical* action, i.e., each of the symplectomorphisms, $\tau(e^{i\theta})$, is a canonical transformation. By a theorem of de la Harpe-Karoubi [4] every such action can be quantized by a unitary representation

$$\tau^{\#} : S^1 \to U(H)\,,\; H = L^2(X)\,,$$

by Fourier integral operators; and for this representation the projector Π_+ is of Melrose-Uhlmann type. (See [2] Theorem 4.4. We will also prove this explicitly in §4 by showing that Π_+ is microlocally conjugate to the standard Szegö projector.) The main result of this article is the following:

"**Theorem.**" *Let Ψ_+ be the algebra of pseudodifferential operators which commute with Π_+. Then the algebra $\Pi_+\Psi_+\Pi_+$ quantizes the algebra of classical observables, $C^\infty(M_{\text{cut}})$.*

The second statement needs some amplification (which will be supplied in §5); however the reason for the quotation marks is the parity complications we referred to above. We will discuss this "metaplectic glitch" in more detail in §1 and will show that there are two ways of dealing with it: one by making the action of S^1 on M a "metaplectic"

[2]However, most of the results below are true, mutatis mutandis, for the algebra of Toeplitz operators on a strictly pseudoconvex domain.

action and the other by making the action of S^1 on $\mathbb{C}^1$ a "metaplectic" action. We will show that both these alternatives give rise to an interesting symbol calculus for operators in $\Pi_+\Psi_+\Pi_+$.

In Section 2 we will discuss a differential operator version of the "Theorem" above for the manifold $X = S^1$ and the standard Szegö projector, and then in Section 3 we will extend this result to the algebra of pseudodifferential operators on product manifolds of the form, $X = Y \times S^1$. In Section 4 we will show that it suffices to prove our "Theorem" in this case by showing that there exists a Fourier integral operator locally conjugating the general case to this case. Finally in Section 5 we will discuss the symbolic calculus of the algebra $\Pi\Psi_+\Pi$. We will show that an operator of degree r in this algebra has a leading symbol which is an homogeneous function of degree r on M_{cut} and that products and Poisson brackets of symbols correspond to products and commutators of operators. We will also show that this algebra can be equipped with a residue trace which, for operators of degree $-d$, $d = \dim M/2$, is given by integrating the leading symbol of the operator over M_+, and will deduce from this a Weyl law for operators of elliptic type.

Finally in Section 6 we will discuss what happens when one starts with a cotangent bundle and applies to it repeated symplectic cuts. One can construct in this way a lot of interesting symplectic cones, and by the techniques of this paper one gets (modulo the $\mathbb{Z}_2$ problems discussed above) algebras of classical polyhomogeneous pseudodifferential operators quantizing these cones. The details of this construction will be spelled out elsewhere but in Section 6 we will indicate (roughly) how to quantize in this way the cones over the classical three dimensional lens spaces.

1. The metaplectic glitch

Let M be a manifold with an action τ of a circle S^1 and an S^1 invariant function $\Phi : M \to \mathbb{R}$. Suppose S^1 acts freely on the level set $\Phi^{-1}(0)$. Then the quotient $M_{\mathrm{red}} := \Phi^{-1}(0)/S^1$ is a manifold. Consider the manifold with boundary $\{m \in M \mid \Phi(m) \geq 0\}$, and collapse the circle orbits in the boundary to points. The resulting space

$$M_{\mathrm{cut}} := \{m \in M \mid \Phi(m) \geq 0\}/\!\sim,$$

where $\sim$ is the relation described above (cf. (1.1) below), is a C^0 manifold. The manifold M_{red} embeds naturally in M_{cut} as a codimension 2 submanifold and the difference $M_{\text{cut}} \smallsetminus M_{\text{red}}$ is homeomorphic to $\{m \in M \mid \Phi(m) > 0\}$.

If, in addition, M is a symplectic manifold, the action τ is Hamiltonian and $\Phi : M \to \mathbb{R}$ is the corresponding moment map then M_{cut} is symplectic. More specifically:

Proposition 1.1. *Let (M, ω) be a symplectic manifold with a Hamiltonian action τ of S^1; let $\Phi : M \to \mathbb{R}$ denote a corresponding moment map. Suppose S^1 acts freely on $\Phi^{-1}(0)$. Define an equivalence relation $\sim$ on $\{m \in M \mid \Phi(m) \geq 0\}$ for $m \neq m'$ by the identification:*

(1.1)
$$m \sim m' \iff \Phi(m) = \Phi(m') = 0 \quad \textit{and} \quad m = \lambda \cdot m' \quad \textit{for some} \quad \lambda \in S^1.$$

Then:

(1) *The C^0 manifold M_{cut} can be given the structure of a C^∞ symplectic manifold (M_+, ω_+) so that the reduced space $M_{\text{red}} = \Phi^{-1}(0)/S^1$ embeds symplecticly and the difference $M_+ \smallsetminus M_{\text{red}}$ is symplectomorphic to $\{m \in M \mid \Phi(m) > 0\}$.*

(2) *Alternatively, the C^0 manifold M_{cut} can be given the structure of a C^∞ symplectic orbifold (M_{++}, ω_{++}) so that the set of regular points is symplectomorphic to $\{m \in M \mid \Phi(m) > 0\}$, the set of singular points is symplectomorphic to the reduced space M_{red}, and the structure group of all points in M_{red} is $\mathbb{Z}_2$.*

Remark 1.2. Even though M_+ and M_{++} are the same as topological spaces, namely M_{cut}, they are not the same as orbifolds. In particular $C^\infty(M_+) \neq C^\infty(M_{++})$.

Remark 1.3. One readily sees from the proof below that the Hamiltonian action τ of S^1 on (M, ω) descends to a Hamiltonian action of S^1 on (M_+, ω_+) which fixes M_{red} pointwise and makes the embedding $\{\Phi > 0\} \hookrightarrow M_+$ equivariant. The same statement holds for M_{++}.

Proof of Proposition 1.1. Consider the diagonal action of S^1 on $(M \times \mathbb{C}, \omega - i dz \wedge d\bar{z})$. The map $\widetilde{\Phi}(m, z) = \Phi(m) - |z|^2$ is a corresponding moment map. Since S^1 acts freely on $\Phi^{-1}(0)$ it acts freely on $\widetilde{\Phi}^{-1}(0)$. Hence $M_+ := \widetilde{\Phi}^{-1}(0)/S^1$ is a symplectic manifold. The composition of the embedding $j : \{\Phi \geq 0\} \hookrightarrow \widetilde{\Phi}^{-1}(0)$, $j(m) = (m, \sqrt{\Phi(m)})$ with the orbit map $\widetilde{\Phi}^{-1}(0) \to \widetilde{\Phi}^{-1}(0)/S^1 = M_+$ is onto. It induces a homeomorphism $\varphi : M_{\text{cut}} = \{\Phi \geq 0\}/{\sim} \to M_+$. Note that $\varphi|_{\{\Phi > 0\}}$ is an open embedding. Moreover, since $j^*(\omega - i dz \wedge d\bar{z}) = \omega$, it is symplectic. Similarly one checks that the difference $M_+ \smallsetminus \varphi(\{\Phi > 0\})$ is the reduced space M_{red}. This proves the first part of the theorem.

Denote elements of $\mathbb{C}/\mathbb{Z}_2$ by $[z]$, so that $[z] = [-z]$. Consider the S^1 action on $\mathbb{C}/\mathbb{Z}_2$ given by $\mu \cdot [z] = [\sqrt{\mu} z]$. This action is well-defined and preserves the symplectic form $-[i\, dz \wedge d\bar{z}]$ on $\mathbb{C}/\mathbb{Z}_2$ corresponding to $-i\, dz \wedge d\bar{z}$; the moment map for this action is the map, $[z] \mapsto -|z|^2$. Now consider the diagonal action of S^1 on $(M \times \mathbb{C}/\mathbb{Z}_2, \omega - [i\, dz \wedge d\bar{z}])$ and proceed as in the first part of the proof, denoting the reduction of $M \times \mathbb{C}/\mathbb{Z}_2$ at zero by M_{++}. q.e.d.

Now let M be a cotangent bundle of a compact manifold, X, of dimension $n > 1$; and let the action, τ, above be a canonical action (an action preserving the canonical cotangent one-form, $\Sigma \xi_i\, dx_i$). By the theorem of de la Harpe-Karoubi-Weinstein that we cited in the introduction, there exists a representation, $\tau^{\#}$, of S^1 on $L^2(X)$ which quantizes τ in the sense that for each $e^{i\theta} \in S^1$, $\tau^{\#}(e^{i\theta})$ is a unitary Fourier integral operator with $\tau(e^{i\theta})$ as its underlying canonical transformation. Let

$$\Pi : L^2(X) \to L^2(X)$$

be orthogonal projection onto the space

$$\mathrm{span}\{f \in L^2(X) \mid \quad \tau^{\#}(e^{i\theta}) f = e^{in\theta} f, \quad n \geq 0\}\,, \tag{1.2}$$

and let Ψ_+ be the algebra of classical pseudodifferential operators which commute with Π. The main result of this paper (modulo a few qualifications which we will explain shortly) asserts:

(∗) Π *is a projector of Melrose-Uhlmann type,*
and the algebra $\Pi \Psi_+ \Pi$ *quantizes*
the algebra of classical observables, $C^\infty(M_+)$.

As we remarked in the introduction there is a metaplectic glitch involved in making the statement above correct. To explain this glitch we note that, since $M = T^*X$, the obvious candidate for the quantum Hilbert space to associate with M is $L^2(X)$; and since

$$\mathbb{C} = \mathbb{R}^2 = T^*\mathbb{R}$$

the obvious candidate for the quantum Hilbert space to associate with $\mathbb{C}$ is $L^2(\mathbb{R})$. Thus, if one subscribes to the principle that "quantization commutes with reduction" one should associate with M_+ the quantum Hilbert space[3]

$$\mathrm{Hom}(L^2(\mathbb{R}), L^2(X))^{S^1}. \tag{1.3}$$

We must, of course, still specify how S^1 is to act on $L^2(\mathbb{R})$ for (1.3) to make sense; and this, we will see, is the source of the "metaplectic glitch" that we referred to above. Let's briefly review how the metaplectic (or Segal-Shale-Weil) representation of S^1 on $L^2(\mathbb{R})$ is defined: Let x and y be the standard Darboux coordinates on $\mathbb{R}^2$ and let $\mathfrak{h}_3 = \mathrm{span}\{x, y, 1\}$. This space sits inside the Poisson algebra, $C^\infty(\mathbb{R}^2)$, as a three-dimensional Heisenberg algebra, and can be represented on $L^2(\mathbb{R})$ by the standard Schroedinger representation

$$x \to x, \quad y \to \tfrac{\partial}{\partial x}, \quad 1 \to I\,. \tag{1.4}$$

This exponentiates to a representation, κ, of the Heisenberg group, H_3, on $L^2(\mathbb{R})$; and, by the Stone-Von Neumann theorem, κ is the unique irreducible representation of H_3 for which the center, $\mathbb{R}$, of H_3 acts as $e^{it}I$. Consider now the symplectic action of S^1 on $\mathbb{R}^2$ given by $\theta \to e^{i\theta}$. Being a linear action this preserves $\mathfrak{h}_3$, and being symplectic, acts on $\mathfrak{h}_3$ by Lie algebra automorphisms. Hence, since H_3 is simply connected, this action can be exponentiated to an action, ρ, of S^1 on H_3 by Lie group automorphisms; and this enables one to define, for every θ, a new representation, κ_θ, of H_3 on $L^2(X)$ by setting

$$\kappa_\theta(h) = \kappa(h_\theta), \quad h_\theta = \rho(e^{i\theta})h\,.$$

This representation is identical with κ on the center of H_3; so by the Stone-Von Neumann theorem κ and κ_θ are isomorphic: there exists a unitary operator

$$\gamma_\theta : L^2(X) \to L^2(X)$$

such that $\gamma_\theta^{-1}\kappa\gamma_\theta = \kappa_\theta$. Moreover, since κ is irreducible, this operator is unique up to a constant multiple of module one. From this uniqueness it is easy to see that $\gamma_{\theta_1+\theta_2}$ is a constant multiple of $\gamma_{\theta_1}\gamma_{\theta_2}$; i.e., the map

$$e^{i\theta} \to \gamma(\theta) \tag{1.5}$$

[3]There is a "Hom" rather than a tensor product here because the symplectic cutting procedure requires one to take the symplectic form on $\mathbb{C}$ to be the negative of the usual symplectic form (cf. proof of Proposition 1.1 above).

is a *projective* representation of S^1 on $L^2(X)$. The problem of converting this projective representation into a bona fide *linear* representation is a standard problem in representation theory and involves an obstruction which sits in the group cohomology of the group, S^1. For (1.5) this obstruction unfortunately doesn't vanish; but one can make it vanish by pulling it back to the metaplectic double cover, $\widetilde{S}^1$, of S^1. Since $\widetilde{S}^1$ is just the group S^1 itself, double covering itself by the map

$$e^{i\theta} \mapsto e^{2i\theta} \tag{1.6}$$

one gets a linear representation, $\widetilde{\gamma}$, of S^1 on $L^2(\mathbb{R})$ by composing (1.5) with (1.6) and adjusting constant multiples. This is, by definition, the metaplectic representation of S^1 on $L^2(\mathbb{R})$; and its clear from this definition that its the *only* representation of S^1 on $L^2(\mathbb{R})$ compatible with (1.4).

Coming back to the space of intertwining operators (1.3), if the representation of S^1 on $L^2(\mathbb{R})$ is the metaplectic representation, the space (1.3) is not strictly speaking well-defined since the "S^1" acting on $L^2(\mathbb{R})$ is *not* the same group as the "S^1" acting on $L^2(X)$ and on $M \times \mathbb{C}$: it is the metaplectic double cover of this group. This is the "metaplectic glitch" which we referred to above. We will discuss below two ways of dealing with it, one of which leads to an interesting quantization of M_+ and the other to an interesting quantization of M_{++}.

The first way is to make the action of S^1 on the second factor of (1.3) a metaplectic action. Namely let $\mathbb{Z}_2 = \{\pm 1\} = \{\lambda \in S^1, \lambda^2 = 1\}$. Then $S^1/\mathbb{Z}_2$ acts on $M/\mathbb{Z}_2$, and the quantization of this action is the action of $S^1/\mathbb{Z}_2$ on $L^2(X/\mathbb{Z}_2) = L^2(X)^{\mathbb{Z}_2}$. Let's temporarily relabel the groups, S^1 and $S^1/\mathbb{Z}_2$, letting S^1 temporarily be labeled $\widetilde{S}^1$ and $S^1/\mathbb{Z}_2$, temporarily labeled S^1; and let's replace the space of intertwining operators, (1.3), by

$$\operatorname{Hom}(L^2(\mathbb{R}), L^2(X)^{\mathbb{Z}_2})^{\widetilde{S}^1}\,, \tag{1.7}$$

which is now well-defined since the same group is acting on both factors. Let $h_i \in L^2(\mathbb{R})$ be the i^{th} Hermite function, normalized to have L^2-norm one. Then if T is an intertwining operator belonging to the space (1.7), $Th_n = 0$ for all n odd and the map

$$T \to \sum_{n=0}^{\infty} Th_{2n} \tag{1.8}$$

maps the space (1.7) bijectively onto the $\mathbb{Z}_2$-invariant part of the space (1.2). Let Π^{even} be the orthogonal projection of $L^2(X)^{\mathbb{Z}_2}$ onto this space and let Ψ_+^{even} be the ring of $\mathbb{Z}_2$-invariant classical pseudodifferential operators which commute with Π^{even}. Then the following even version of assertion $(*)$ above is true:

Theorem 1. Π^{even} *is a projector of Melrose-Uhlmann type and the algebra* $\Pi^{\text{even}}\Psi_+^{\text{even}}\Pi^{\text{even}}$ *quantizes the algebra of classical observables,* $C^\infty(M_+)_{\text{even}}$.

The second way of dealing with this "metaplectic glitch" is to make the action of $\widetilde{S}^1$ on the first factor of (1.3) an action of S^1 by noting that one gets from the metaplectic representation a representation of $\widetilde{S}^1/\mathbb{Z}_2 = S^1$ on $L^2(\mathbb{R})^{\mathbb{Z}_2}$. This makes the space of intertwining operators

$$\mathrm{Hom}(L^2(\mathbb{R})^{\mathbb{Z}_2}, L^2(X))^{S^1} \tag{1.9}$$

well-defined. Moreover, the action of S^1 on $L^2(\mathbb{R})^{\mathbb{Z}_2}$ is given by $e^{i\theta} \bullet h_{2n} = e^{in\theta}h_{2n}$, so the mapping (1.8) maps the space (1.9) bijectively onto the space (1.2), and the projector, Π, is projection onto its image. Now, however, the classical counterpart of the space (1.9) is no longer M_+ but M_{++}. Indeed, the first factor in (1.9) is the space, $L^2(\mathbb{R})^{\mathbb{Z}_2}$, which one can think of as the quantization of the orbifold, $(T^*\mathbb{R})/\mathbb{Z}_2$. Therefore, by the principle of "quantization commutes with reduction" the classical counterpart of (1.9) is the symplectic reduction at zero of the orbifold, $M \times ((T^*\mathbb{R})/\mathbb{Z}_2)$, i.e., it is M_{++}. Our second version of the "Theorem" above states:

Theorem 2. $\Pi\Psi_+\Pi$ *is the quantization of the algebra of classical observables,* $C^\infty(M_{++})$.

We now sketch the proof of Theorems 1 and 2 for the space $M = T^*S^1$ (with pseudodifferential operators replaced by differential operators). By definition $(T^*S^1)_+$ is the reduction at zero of the manifold $(T^*S^1 \times \mathbb{C} = \mathbb{R} \times S^1 \times \mathbb{C}, ds \wedge \frac{d\lambda}{i\lambda} - idz \wedge d\bar{z})$ (where $(s, \lambda = e^{i\theta}, z) \in \mathbb{R} \times S^1 \times \mathbb{C}$), by the S^1 action

$$\mu \cdot (s, \lambda, z) = (s, \mu\lambda, \mu z)$$

with moment map

$$\widetilde{\Phi}(s, \lambda, z) = s - |z|^2.$$

The set $\{(|z|^2, 1, z) \in \mathbb{R} \times S^1 \times \mathbb{C} \mid z \in \mathbb{C}\}$ parameterizes S^1 orbits in $\Psi^{-1}(0)$. Hence the map $\pi : \widetilde{\Phi}^{-1}(0) \to \mathbb{C}$, $\pi(s = |z|^2, \lambda, z) = \lambda^{-1}z$

induces a diffeomorphism $\widetilde{\Phi}^{-1}(0)/S^1 \to \mathbb{C}$. The embedding $j : [0,\infty) \times S^1 \to \widetilde{\Phi}^{-1}(0)$, $j(s,\lambda) = (s,\lambda,\sqrt{s})$ has the property that the composition $\sigma = \pi \circ j : [0,\infty) \times S^1 \to \mathbb{C}$ is onto; it is one-to-one on $(0,\infty) \times S^1$ and maps $\{0\} \times S^1$ to 0. Note that $\sigma(s,\lambda) = \lambda^{-1}\sqrt{s}$ and that σ induces a homeomorphism $\varphi : \big(([0,\infty) \times S^1)/\sim\big) \to \mathbb{C}$.

Now consider the ring of real $\mathbb{C}$-valued polynomials on $\mathbb{R}^2 = \mathbb{C}$ invariant under the $\mathbb{Z}_2$ action $z \mapsto -z$. It is generated by $z^2, |z|^2$ and $\bar{z}^2$. Note that $\sigma^* z^2 = \lambda^{-2} s = e^{-2i\theta} s$, $\sigma^*|z|^2 = s$ and $\sigma^* \bar{z}^2 = e^{2i\theta} s$. On the other hand we will show in §2 that the ring of differential operators on S^1 which commute with the projector, Π^{even}, is generated by the operators

$$\frac{1}{i}\frac{d}{d\theta}e^{2i\theta}, \quad \frac{1}{i}e^{-2i\theta}\frac{d}{d\theta} \text{ and } \frac{1}{i}\frac{d}{d\theta}\,;$$

and the symbols of these operators are exactly $se^{2i\theta}$, $se^{-2i\theta}$ and s. Thus the ring of even polynomial functions on $(T^*S^1)_+$ is exactly the ring of the symbols of differential operators which commute with Π^{even}.

The proof of Theorem 2 is similar. First note that

$$(T^*\mathbb{R})/\mathbb{Z}_2 = \mathbb{R}^2/\mathbb{Z}_2 = \mathbb{C}/\mathbb{Z}_2\,.$$

Let's again denote elements of $\mathbb{C}/\mathbb{Z}_2$ by $[z]$, so that $[z] = [-z]$. Consider the S^1 action on $\mathbb{C}/\mathbb{Z}_2$ given by $\mu \cdot [z] = [\sqrt{\mu} z]$. As we noted previously this action is well-defined and preserves the symplectic form on $\mathbb{C}/\mathbb{Z}_2$ corresponding to $-i\, dz \wedge d\bar{z}$; and the moment map for this action is the map, $[z] \mapsto -|z|^2$.

Now let's check what $(T^*S^1)_{++}$ looks like. By definition $(T^*S^1)_{++}$ is the reduction at zero of the orbifold

$$(T^*S^1) \times (\mathbb{C}/\mathbb{Z}_2) = \mathbb{R} \times S^1 \times (\mathbb{C}/\mathbb{Z}_2)$$

by the circle action

$$\mu \cdot (s, \lambda = e^{i\theta}, [z]) = (s, \mu\lambda, [\sqrt{\mu} z])\,,$$

the moment map for this action being the function, $\widetilde{\Phi}(s,\lambda,[z]) = s - |z|^2$. Arguing as above we get a surjective map $\sigma : [0,\infty) \times S^1 \to \mathbb{C}/\mathbb{Z}_2$ which is one-to-one on $(0,\infty) \times S^1$ and sends $\{0\} \times S^1$ to $[0]$. The only difference is that now σ is given by

$$\sigma(s,\lambda) = [\lambda^{-1/2}\sqrt{s}].$$

Consider now the ring of (complex valued) "polynomial" functions on $\mathbb{C}/\mathbb{Z}_2$. This ring, by definition, is the ring of $\mathbb{Z}_2$-invariant polynomial functions on $\mathbb{C}$; and, as we noted above, this ring is generated by z^2, $\bar{z}^2$ and $|z|^2$. By abuse of notation we can think of these functions as living on $\mathbb{C}/\mathbb{Z}_2$. Now note that now $\sigma^* z^2 = \lambda^{-1} s = e^{-i\theta} s$, $\sigma^* \bar{z}^2 = e^{i\theta} s$ and $\sigma^* |z|^2 = s$. On the other hand we will prove in §2 that the ring of differential operators on S^1 which commute with Π is generated by

$$\frac{1}{i}\frac{d}{d\theta}e^{i\theta}, \quad \frac{1}{i}e^{-i\theta}\frac{d}{d\theta} \text{ and } \frac{1}{i}\frac{d}{d\theta};$$

and the symbols of these operators are exactly the functions $se^{i\theta}$, $se^{-i\theta}$ and s above. Thus the ring of polynomial functions on $(T^*S^1)_{++}$ is exactly the ring of symbols of differential operators which commute with Π.

2. The Szegö projector on S^1

The classical Szegö projector

$$\Pi : L^2(S^1) \to L^2(S^1)$$

is the orthogonal projection of the space $L^2(S^1)$ onto the space

$$\operatorname{span}\{e^{in\theta} \mid n \geq 0\}.$$

Our goal in this section is to determine all differential operators on S^1 which commute with Π. It is easy to check that the operators

$$\frac{1}{i}\frac{d}{d\theta}, \ \frac{1}{i}\frac{d}{d\theta}e^{i\theta} \quad e^{-i\theta}\frac{1}{i}\frac{d}{d\theta} \tag{2.1}$$

have this property, and we will prove that the only differential operators that commute with Π are sums and products of these operators.

Theorem 2.1. *The algebra of differential operators on the circle S^1 which commute with the Szegö projector Π is generated by the operators* (2.1).

Proof. We will first prove that

$$\left(\frac{1}{i}\frac{d}{d\theta}e^{i\theta}\right)^k = e^{ik\theta}p_k\left(\frac{1}{i}\frac{d}{d\theta}\right) \tag{2.2}$$

where

$$p_k(x) = (x+1)\dots(x+k). \tag{2.3}$$

Assume by induction that this holds for $k-1$. Then

$$\begin{aligned}\left(\frac{1}{i}\frac{d}{d\theta}e^{i\theta}\right)^k &= \frac{1}{i}\frac{d}{d\theta}e^{i\theta}\left(e^{i(k-1)\theta}p_{k-1}\left(\frac{1}{i}\frac{d}{d\theta}\right)\right) \\ &= \frac{1}{i}e^{ik\theta}p_{k-1}\left(\frac{1}{i}\frac{d}{d\theta}\right) \\ &= \frac{1}{i}e^{ik\theta}\left(\frac{1}{i}\frac{d}{d\theta}+k\right)p_{k-1}\left(\frac{1}{i}\frac{d}{d\theta}\right) \\ &= \frac{1}{i}e^{ik\theta}p_k\left(\frac{1}{i}\frac{d}{d\theta}\right).\end{aligned}$$

q.e.d.

Now let Q be a differential operator of degree d which commutes with Π and transforms under the action τ of S^1 by

$$\tau_\theta^* Q = e^{ik\theta} Q \tau_\theta^*, \qquad k \geq 0. \tag{2.4}$$

Such an operator has to be of the form $e^{ik\theta}q(\frac{1}{i}\frac{d}{d\theta})$ for some d-th degree polynomial $q(x)$. The commutator condition $[Q, \Pi] = 0$ implies that

$$Qe^{im\theta} = \Pi Q e^{im\theta} Q \Pi e^{im\theta} = 0$$

for $m = -k, -k+1, \dots, -1$, so the integers $m = -k+s$, $s = 0, \dots, k-1$ are roots of q. Thus $p_k(x)$ divides $q(x)$; and letting $r(x) = q(x)/p_k(x)$, one has:

$$Q = r\left(\frac{1}{i}\frac{d}{d\theta}\right)\left(\left(\frac{1}{i}\frac{d}{d\theta}\right)e^{i\theta}\right)^k \tag{2.5}$$

by (2.2).

If Q transforms under the action τ of S^1 by

$$\tau_\theta^* Q = e^{-ik\theta} Q \tau_\theta^*, \qquad k \geq 0, \tag{2.6}$$

the transpose of Q transforms by (2.4). Therefore the transpose of Q has to be of the form (2.5), and Q itself of the form

$$Q = \left(\frac{1}{i}e^{-i\theta}\right)^k r\left(\frac{1}{i}\frac{d}{d\theta}\right). \tag{2.7}$$

Finally let Q be any differential operator on the circle commuting with Π. Explicitly let

$$Q = \sum_{r=0}^{d} f_r(\theta) \left(\frac{1}{i} \frac{d}{d\theta} \right)^r ,$$

and let $c_{k,r}$ be the kth Fourier coefficient of $f_r(\theta)$. Then

$$Q = \sum_k Q_k \tag{2.8}$$

with

$$Q_k = e^{ik\theta} \sum_{r=0}^{d} c_{k,r} \left(\frac{1}{i} \frac{d}{d\theta} \right)^r .$$

Each of the Q_k's commute with Π and transform under the action of S^1 by (2.4) or by (2.6); hence it has to be of the form (2.5) or (2.7). In particular $Q_k = 0$ for $|k| > d$; so the sum (2.8) is finite, and every summand is in the algebra generated by the operators (2.1). q.e.d.

We will need in §3 an "even" variant of Theorem 2.1 (whose proof we will omit since it is essentially the same as the proof above).

Theorem 2.2. *Let Π^{even} be the orthogonal projection from $L^2(S^1)$ onto the space*

$$\operatorname{span}\{e^{2in\theta} \mid n \geq 0\}. \tag{2.9}$$

The algebra of differential operators on S^1 which commute with Π^{even} is generated by

$$\frac{1}{i}\frac{d}{d\theta}e^{2i\theta}, \quad \frac{1}{i}\frac{d}{d\theta}e^{-2i\theta}, \quad \frac{1}{i}\frac{d}{d\theta}. \tag{2.10}$$

The symbols of these operators are a Poisson subalgebra of the algebra of C^∞ functions on T^*S^1, and as we saw in the introduction this algebra can be identified with the algebra of "polynomials" on the space $\mathbb{C}/\mathbb{Z}_2 = (T^*S^1)_{++}$. This proves:

Theorem 2.3. *The algebra of differential operators on S^1 which commute with the even Szegö projector Π^{even} has for its symbol algebra the algebra of polynomials on the cut space $\mathbb{C} = (T^*S^1)_+$.*

What about the algebra of differential operators which commute with the usual Szegö projector? The same argument gives:

Theorem 2.4. *The algebra of differential operators on S^1 which commute with the Szegö projector Π has for its symbol algebra the algebra of polynomials on the cut space $\mathbb{C}/\mathbb{Z}_2 = (T^*S^1)_{++}$.*

Finally we characterize smooth functions on the cut space $(T^*S^1)_+ = \mathbb{C}$ which can be extended to smooth functions on T^*S^1.

Theorem 2.5. *A function $f \in C^\infty((T^*S^1)_+)$ has the property that $\sigma^* f \in C^0([0,\infty) \times S^1)$ is the restriction of a smooth function on T^*S^1 iff the infinite jet of f at 0 is even, i.e., is invariant under $z \mapsto -z$. Here, as before, $\sigma(s,\lambda) = \lambda^{-1}\sqrt{s}$.*

Proof. If $f \in C^\infty(\mathbb{C})$ vanishes at zero to infinite order, then $\sigma^* f$ can be extended by zero to a smooth function on $\mathbb{R} \times S^1 = T^*S^1$. Therefore the condition on $\sigma^* f$ to extend is the condition on the infinite jet of f at 0. We can write the the jet $j^\infty f(0)$ as

$$j^\infty f(0) = \sum_{n=0}^{\infty} \sum_{k+l=n} a_{kl} z^k \bar{z}^l$$

for some $a_{kl} \in \mathbb{C}$. Since $\sigma^* z = \lambda^{-1} s^{1/2}$,

$$\sigma^* \left(\sum_{k+l=n} a_{kl} z^k \bar{z}^l \right) = \left(\sum_{k+l=n} a_{kl} \lambda^{l-k} \right) s^{n/2}.$$

Since $\sigma^* f$ extends to a smooth function on T^*S^1 iff $\sigma^*(j^\infty f(0))$ has no fractional powers of s, we must have

$$j^\infty f(0) = \sum_{m=0}^{\infty} \sum_{k+l=2m} a_{kl} z^k \bar{z}^l,$$

i.e., $j^\infty f(0)$ is a power series in z^2, $\bar{z}^2$ and $|z|^2$. The latter is true iff $j^\infty f(0)(z,\bar{z}) = j^\infty f(0)(-z,-\bar{z})$. q.e.d.

3. The Szegö projector on $\mathbb{R}^n \times S^1$

Let Π_1 be the Szegö projector on $L^2(S^1)$ (the operator we called Π in §2). From Π_1 one gets a projection operator,

$$I_{\mathbb{R}^n} \otimes \Pi_1$$

on $L^2(\mathbb{R}^*) \otimes L^2(S^1)$ which extends by continuity to a projection operator

$$\Pi : L^2(\mathbb{R}^n \times S^1) \to L^3(\mathbb{R}^n \times S^1)\,.$$

Our goal in this section will be to determine the commutator of Π in the algebra of pseudodifferential operators on $\mathbb{R}^n \times S^1$. For simplicity we will only consider pseudodifferential operators of the form

$$Qf = \sum_m e^{im\theta} \int q(x,\xi,\theta,m) e^{ix\cdot\xi} \hat{f}(\xi,m)\, d\xi \tag{3.1}$$

$\hat{f}$ being the Fourier transform of f:

$$\hat{f}(\xi,m) = \left(\frac{1}{2\pi}\right)^{n+1} \int e^{-ix\cdot\xi} e^{-im\theta} f(x,\theta)\, dx\, d\theta \tag{3.2}$$

and $q(x,\xi,\theta s)$ being a classical polyhomogeneous symbol of compact support in x. We can decompose q into its Fourier modes

$$q(x,\xi,\theta,s) = \sum e^{ik\theta} q_k(x,\xi,s) \tag{3.3}$$

with

$$q_k(x,\xi,s) = \frac{1}{2\pi} \int q(x,\xi,\theta,s) e^{-ik\theta}\, d\theta\,; \tag{3.4}$$

and from (3.3) we get a corresponding decomposition of Q:

$$Q = \sum Q_k \tag{3.5}$$

Q_k being the operator with symbol

$$q_k(x,\xi,s) e^{ik\theta}\,. \tag{3.6}$$

Letting $p_k(x,y,m)$ be the conormal distribution

$$p_k(x,y,m) = \left(\frac{1}{2\pi}\right)^{n+1} \int q_k(x,\xi,m) e^{i(x-y)\cdot\xi}\, d\xi \tag{3.7}$$

we can, by (3.1)–(3.2), write the Schwartz kernel of Q_k as a sum:

$$\sum_m e^{i(k+m)\theta} e^{-im\psi} p_k(x,y,m)\,. \tag{3.8}$$

From (3.8) we will deduce:

Lemma 3.1. *For k positive the Schwartz kernel of $[\Pi, Q_k]$ is*

$$\sum_{-k\le m<0} e^{i(k+m)\theta}e^{-im\psi}p_k(x,y,m). \tag{3.9}$$

Proof. The Schwartz kernel of $\Pi Q_k - Q_F\Pi$ is

$$\sum_{m+k\ge 0} e^{i(k+m)\theta}e^{-im\psi}p_k(x,y,m) - \sum_{m\le 0} e^{i(k+m)\theta}e^{-im\psi}p_k(x,ym)$$

and this difference is the same as the finite sum (3.9). Similarly for k negative one has q.e.d.

Lemma 3.2. *The Schwartz kernel of $[\Pi, Q_k]$ is*

$$\sum_{k\le m<0} e^{im\theta}e^{-i(m-k)\psi}p_k(x,y,m). \tag{3.10}$$

From these results we can easily read off necessary and sufficient conditions for Q and Π to commute.

Theorem 3.3. *Q and Π commute if and only if, for all k,*

$$q_k(x,\xi,m) = 0 \tag{3.11}$$

for $-|k| \le m < 0$.

In particular for $k > 0$ this implies that there exists a classical polyhomogeneous symbol $q_k^{\#}(x,\xi,s)$ with

$$q_k(x,\xi,s) = q_k^{\#}(x,\xi,s)\Pi_{m=1}^{k}(s+m). \tag{3.12}$$

Let $Q_k^{\#}$ be the pseudodifferential operator with $q_k^{\#}$ as symbol. Since $q_k^{\#}$ doesn't depend on θ this operator commutes with the action of S^1 on $\mathbb{R}^n \times S^1$ and by (3.6) and (2.2)

$$Q_k = Q_k^{\#}\left(\frac{1}{\sqrt{-1}}\frac{d}{d\theta}\right)^k. \tag{3.13}$$

Similarly

$$Q_{-k} = Q_{-k}^{\#}\left(e^{-i\theta}\frac{1}{\sqrt{-1}}\frac{d}{d\theta}\right)^k \tag{3.14}$$

so we have proved:

Theorem 3.4. *A necessary and sufficient condition for Q to commute with Π is that, for every k, Q_k have a factorization of the form,* (3.13)–(3.14), *the operator $Q_k^{\#}$ being a classical polyhomogeneous pseudodifferential operator on $\mathbb{R}^n \times S^1$ which is S^1 invariant.*

As another application of Lemmas 3.1-3.2 we will prove:

Theorem 3.5. *If the symbol $q(x, \xi, \theta, s)$ of Q vanishes to infinite order on the set $\xi \neq 0$, $s = 0$ the operator, $[\Pi, Q]$ is a smoothing operator.*

Remark. If $[\Pi, Q]$ is a smoothing operator the operator

$$\Pi Q \Pi + (I - \Pi) Q (I - \Pi)$$

differs from Q by a smoothing operator and commutes with Π. In other words Q is the sum of an operator which commutes with Π and a smoothing operator.

Proof. It suffices to show that each of the operators $[\Pi, Q_k]$ is smoothing and hence, by (3.9), that $p_k(x, y, m)$ is smooth. But $p_k(x, y, m)$ is defined by the integral (3.7), and we can expand the integrand in a finite Taylor series

$$q_k(x, \xi, m) = \sum_{\ell=0}^{N} \frac{1}{\ell!} \left(\frac{d}{ds} \right)^{\ell} q_k(x\xi, 0) m^{\ell} + r_N(x, \xi, m)$$

where

$$r_N(x, \xi, s) = \frac{1}{N!} \int_0^1 (1-t)^N \left(\frac{d}{ds} \right)^N q_k(x, \xi, ts)\, dt$$

is a classical polyhomogeneous symbol of degree equal to $\deg Q - N$. Thus if $q_k(x; \xi, s)$ vanishes to infinite order at $s = 0$

$$q_k(x, \xi, m) = r_N(x, \xi, m)$$

for all N; so by (3.7)

$$p_k(x, y, m) = \left(\frac{1}{2n} \right)^{n+1} \int r_N(x, \xi, m) e^{i(x-y)\cdot\xi}\, d\xi$$

and, for all integers, ℓ, the right side is in C^ℓ for $N \geq n + \deg Q + \ell$. Hence the left-hand side is in C^∞. q.e.d.

Let Q be a pseudodifferential operator of order m which commutes with Π, and let $\sigma = \sigma(Q)(x, \xi, \theta, s)$ be its leading symbol. By (3.4) this leading symbol only depends on the variables s and θ, as a smooth function of s, $se^{i\theta}$ and $se^{-i\theta}$. We will prove that the converse is true.

Theorem 3.6. *Let σ be a smooth function on the complement of the zero section in $T^*(\mathbb{R}^n \times S^1)$ which is homogeneous of degree m and only depends on s and θ as a smooth function of s, $se^{i\theta}$ and $se^{-i\theta}$. Then there exists an m^{th} order pseudodifferential operator, Q, which commutes with Π and has leading symbol, σ.*

Proof. Let $\sigma = \sigma_+ + \sigma_- + \sigma_0$, σ_+ being the sum of the positive Fourier modes of σ and σ_- the sum of the negative Fourier modes. Since σ_- is the complex conjugate of $\bar{\sigma}_+$, it suffices to prove the theorem for σ_+. Let σ_k, $k > 0$, be the k^{th} Fourier mode of σ. By hypothesis

$$\sigma_k(x, \xi, \theta, s) = \sigma_k^{\#}(x, \xi, s) s^k e^{ik\theta}.$$

Let $Q_k^{\#}$ be an S^1 invariant pseudodifferential operator with leading symbol equal to σ_k, and let

$$Q_k = Q_k^{\#} \left(\frac{1}{\sqrt{-1}} \frac{d}{d\theta} e^{i\theta} \right)^k .$$

Then Q_k commutes with Π and has σ_k as its leading symbol. Let H be the pseudodifferential operator on $\mathbb{R}^n \times S^1$ with symbol $(\xi^2 + s^2)^{-\frac{1}{2}}$, let $\rho(s)$ be a compactly supported function which is 1 on the interval, $|s| < 1$ and let

$$N_1 < N_2 < \dots$$

be an increasing sequence of positive integers. Then the sum

$$Q_+ = \sum_{k>0} \rho \left(N_k \left(\frac{1}{\sqrt{-1}} \frac{d}{d\theta} \right) H \right) Q_k$$

is well-defined and (provided that the N_k's go to infinity fast enough) is a classical pseudodifferential operator which commutes with Π and has leading symbol

$$\sum_{k>0} \rho \left(N_k \frac{s}{(\xi^2 + s^2)^{\frac{1}{2}}} \right) \sigma_k(x, \xi, s, \theta).$$

In particular this symbol has the same formal power series expansion on the set $s = 0$ as does σ_+. Hence one can find an m^{th} order pseudodifferential operator, R_+, whose total symbol vanishes to infinite order on $s = 0$ and whose leading symbol is $\sigma_+ - \sigma(Q_+)$. Thus $Q_+ + R_+$ has leading symbol, σ_+, and commutes with Π modulo smoothing operators. Therefore, as we pointed out above, it is the sum of an operator which commutes with Π and a smoothing operator. q.e.d.

Let Q be a pseudodifferential operator which commutes with Π. We will show that the operator

$$\Pi Q = Q\Pi = \Pi Q \Pi$$

"lives microlocally" on the set $s > 0$.

Theorem 3.7. *ΠQ is smoothing if and only if the symbol $q(x, \xi, \theta, s)$ of Q is of order $-\infty$ on the set $s \geq 0$.*

Proof. By (3.1)

$$\Pi Q f = \sum_{m \geq 0} e^{im\theta} \int q(x, \xi, \theta, m) e^{ix\cdot\xi} \hat{f}(\xi, m)\, d\xi$$

and this is smoothing if and only if q is a symbol of order $-\infty$ on the set $s \geq 0$.

Let $(\Pi_1)_{\text{even}}$ be the even Szegö projector on $L^2(S^1)$ (the operator we called Π_{even} in §2) and let

$$\Pi_{\text{even}} = I_{\mathbb{R}^n} \otimes (\Pi_1)_{\text{even}}\,.$$

For this projector there are obvious analogues of Theorems (3.3)–(3.7). We will content ourselves with describing the even analogue of Theorem 3.4. q.e.d.

Theorem 3.8. *A necessary and sufficient condition for Q to commute with Π_{even} is that, for all k, $Q_{2k+1} = 0$, and for all positive k*

$$Q_{2k} = Q_{2k}^{\#} \left(\frac{1}{\sqrt{-1}} \frac{d}{d\theta} e^{2i\theta} \right)^k \tag{3.15}$$

and

$$Q_{-2k} = Q_{-2k}^{\#} \left(e^{-2i\theta} \frac{1}{\sqrt{-1}} \frac{d}{d\theta} \right)^k , \tag{3.16}$$

$Q_{2k}^{\#}$ and $Q_{-2k}^{\#}$ being pseudodifferential operators which are S^1-invariant.

4. Canonical forms for circle actions

The first of the canonical forms which we will discuss in this section is an equivariant Darboux theorem for symplectic cones. We recall that a symplectic cone is a symplectic manifold (M, ω) equipped with a free proper action ρ of $\mathbb{R}$ which satisfies

$$\rho_a{}^*\omega = e^a\omega. \tag{4.1}$$

Let Ξ be the vector field generating the action, $\Xi(m) = \frac{d}{dt}\big|_{t=0}\rho_t(m)$. The infinitesimal version of (4.1) is

$$\omega = L_\Xi\omega = d(\iota(\Xi)\omega). \tag{4.2}$$

Suppose now that in addition to the $\mathbb{R}$ action one has a free action τ of S^1 on M which preserves the symplectic form ω and commutes with ρ, hence preserves

$$\alpha := \iota(\Xi)\omega. \tag{4.3}$$

Then, if we denote the generator of the S^1 action by V,

$$0 = L_V\alpha = \iota(V)d\alpha + d\iota(V)\alpha.$$

Since $\omega = d\alpha$, we get

$$\iota(V)\omega = -d(\alpha(V)). \tag{4.4}$$

In other words τ is a Hamiltonian action with moment map

$$\Phi = \alpha(V). \tag{4.5}$$

Let $d = \dim M/2 = n+1$. A simple canonical model for a $2d$ dimension symplectic cone with a homogeneous symplectic action of S^1 is the complement M_0 of the zero section in $T^*(\mathbb{R}^n \times S^1)$. In this model

$$\omega_0 = \sum d\xi_i \wedge dx_i + ds \wedge d\theta$$

is the symplectic form,

$$\alpha_0 = \sum \xi_i dx_i + s d\theta$$

is the Liouville one-form (so that $d\alpha_0 = \omega_0$),

$$\Xi_0 = \sum \xi_i \frac{\partial}{\partial \xi_i} + s\frac{\partial}{\partial \theta}$$

is the generator of the $\mathbb{R}$ action (so that $\iota(\Xi_0)\omega_0 = \alpha_0$),

$$V_0 = \frac{\partial}{\partial\theta}$$

is the generator of the S^1 action and

$$\Phi_0 = s$$

is the corresponding moment map.

Theorem 4.1. *Let $(M, \omega, \Phi : M \to \mathbb{R})$ and $(M_0, \omega_0, \Phi_0 : M_0 \to \mathbb{R})$ be as above. Let p and p_0 be points in M and M_0 respectively. If $\Phi(p) = \Phi_0(p_0)$, there exist $S^1 \times \mathbb{R}$ invariant neighborhoods U and U_0 of p and p_0 respectively, and an $S^1 \times \mathbb{R}$ equivariant symplectomorphism γ of (U, p) onto (U_0, p_0).*

Proof. Let $\xi = \Xi(p)$, $v = V(p)$, $\xi_0 = \Xi_0(p_0)$ and $v_0 = v(p_0)$. By definition of α (equation (4.3))

$$\omega_p(\xi, v) = \alpha_p(v) = \Phi(p)$$

and

$$(\omega_0)_p(\xi_0, v_0) = (\alpha_0)_p(v_0) = \Phi_0(p_0)$$

so

$$\omega_p(\xi, v) = (\omega_0)_p(\xi_0, v_0).$$

Hence there exists a linear symplectic mapping $A : T_pM \to T_{p_0}M_0$ mapping ξ to ξ_0 and v to v_0 (note that there are two cases to consider: $\omega_p(\xi, v) = 0$ and $\omega_p(\xi, v) \neq 0$). Let X and X_0 be the $S^1 \times \mathbb{R}$ orbits through p and p_0, and i and i_0 the inclusions of X and X_0 into M and M_0 respectively. The map A above extends uniquely to an $S^1 \times \mathbb{R}$ equivariant isomorphism of symplectic vector bundles

$$A : i^*TM \to i_0^*TM_0,$$

and this can be exponentiated to an $S^1 \times \mathbb{R}$ equivariant map

$$\Gamma_A : W \to W_0$$

of an $S^1 \times \mathbb{R}$ neighborhood W of X onto an $S^1 \times \mathbb{R}$ invariant neighborhood W_0 of X_0 with the property that $d\Gamma_A = A$ at the points of X. Indeed, since the action of $S^1 \times \mathbb{R}$ is proper, the orbits X and X_0 are embedded, and there is an $S^1 \times \mathbb{R}$-invariant metrics on M and M_0.

Use the exponential maps for these invariant metrics. By construction of Γ_A, the form $\widetilde{\omega} := \Gamma_A^*\omega_0$ is equal to ω at all points of X. To conclude the proof of the theorem, we will show that there exists an $S^1 \times \mathbb{R}$ invariant neighborhood U of X and an $S^1 \times \mathbb{R}$ equivariant open embedding $f : U \to W$ such that $f = id$ on X and $f^*\widetilde{\omega} = \omega$. The proof will be the standard Moser deformation argument. However we must check that it produces an $S^1 \times \mathbb{R}$ equivariant deformation.

Let $\omega_t = (1-t)\omega + t\widetilde{\omega}$ and let W' be the open subset of W on which ω_t is symplectic for $0 \leq t \leq 1$. Since ω_t is $S^1 \times \mathbb{R}$ invariant, W' is $S^1 \times \mathbb{R}$ invariant. Since $\omega_t = \omega$ on X, W' contains X. Since ω_t is nondegenerate on W' there exists a vector field y_t on W' satisfying

$$\iota(y_t)\omega_t = \alpha - \widetilde{\alpha}, \tag{4.6}$$

where $\widetilde{\alpha} := \iota(\Xi)\widetilde{\omega}$. Moreover, since $\alpha - \widetilde{\alpha} = \iota(\Xi)(\omega - \widetilde{\omega})$, the vector field y_t is zero at points of X. Let U be the subset of W' consisting of all the points q at which y_t has an integral curve $\gamma_q(t)$ with $\gamma_q(0) = q$ and $\gamma_q(t)$ is defined for all $t \in [0,1]$. Let $f_t : U \to W'$ be the map $f_t(q) = \gamma_q(t)$, the isotopy generated by y_t. Then by Moser's trick $f_t^*\omega_t = \omega$, and in particular $f_1^*\widetilde{\omega} = \omega$. Note that by definition U and W' are $S^1 \times \mathbb{R}$ invariant. Moreover, by (4.1) $\rho_a^*\omega_t = e^a\omega_t$ and $\rho_a^*(\alpha - \widetilde{\alpha}) = e^a(\alpha - \widetilde{\alpha})$; so the vector field y_t defined by (4.6) is $\mathbb{R}$ invariant. Thus the isotopy it generates is $\mathbb{R}$ equivariant. q.e.d.

The second canonical form is for S^1 representations quantizing canonical actions. Let M be the cotangent bundle of a compact manifold X with the zero section removed: $M = T^*X \smallsetminus X$, and τ be an action of S^1 on M which preserves the canonical one form. Let $X_0 = \mathbb{R}^n \times S^1$ ($n = \dim X - 1$), $M_0 = T^*X_0 \smallsetminus X_0$, and τ_0 the obvious action of S^1 on M_0.

These actions quantize to give representation $\tau^\#$ and $\tau_0^\#$, of S^1 on $L^2(X)$ and $L^2(\mathbb{R}^n \times S^1)$. Let p and p_0 be points of M and M_0 with $\Phi(p) = \Phi_0(p_0)$, where Φ and Φ_0 are the corresponding moment maps, and let $\gamma : (U,p) \to (U_0,p_0)$ be a canonical transformation mapping an $S^1 \times \mathbb{R}$ -invariant neighborhood U of p onto an $S^1 \times \mathbb{R}$ -invariant neighborhood U_0 of p_0.

Theorem 4.2. *The transformation γ can be implemented by a Fourier integral operator of order zero*

$$F : C^\infty(X) \to C^\infty(\mathbb{R}^n \times S^1)$$

with the properties:

(1) $F^*F = I$ *on* U,

(2) $FF^* = I$ *on* U_0,

(3) $\tau_0^{\#}(e^{i\theta})F = F\tau^{\#}(e^{i\theta})$.

Proof. Let F_0 be the zeroth order Fourier integral operator with compact support which implements γ on U and has the following three properties:

(1) $F_0^*F_0 = I + R_0$ on U, R_0 being a pseudodifferential operator of order -1.

(2) $F_0F_0^* = I + S_0$ on U_0, S_0 being a pseudodifferential operator of order -1.

(3) The symbol of F_0 is S^1 invariant.

By averaging F_0 by the action

$$\theta \mapsto \tau_0^{\#}(e^{-i\theta})F_0\tau_0^{\#}(e^{i\theta})$$

one gets a Fourier integral operator F_1 which implements γ, has the same leading symbol as F_0 and intertwines $\tau^{\#}$ and $\tau_0^{\#}$. In particular, since it has the same a leading symbol as F_0 it continues to satisfy $F_1^*F_1 = I + R$, $FF^* = I + S$ with pseudodifferential operators of order -1 R and S. Now define F to be the operator

$$F_1(I + R)^{-\frac{1}{2}} = (I + S)^{-\frac{1}{2}}F_1.$$

q.e.d.

Let Π and Π_0 be the Szegö projections associated with the representations $\tau^{\#}$ and $\tau_0^{\#}$. One consequence of Theorem 4.2 is that if Q is a pseudodifferential operator with microsupport on U which commutes with Π modulo smoothing operators, FQF^* is a pseudodifferential operator with microsupport in U_0 which commutes with Π_0 modulo smoothing operators. Hence many of the results which we proved in §3 for the commutator ring of Π_0 are valid for the commutator ring of Π as well. We will describe a number of such results in the next section.

5. The algebra, Ψ^{+}, and its symbol calculus

Let M be the cotangent bundle of a compact manifold, X, with its zero section deleted, let τ be an action of S^1 on M by canonical transformations, and let $\tau^{\#}$ be a representation of S^1 on $L^2(X)$ compatible with τ. Let Π^{even} be the "even" Szegö projector (defined in §1) and Ψ_+^{even} the algebra of $\mathbb{Z}_2$-invariant pseudodifferential operators on M which commute with Π^{even}. As we pointed put in the introduction, the complement, U, of M_{red} in M_+ can be identified with the open set, U in M where the moment map of τ is positive. Therefore, if A is a pseudodifferential operator of order m in Ψ_+^{even}, the restriction of its leading symbol to U can be regarded as a homogeneous function of degree m on the open dense subset, U, of M_+. By Theorems 2.3, 4.2 and the even version of Theorem 3.6, this function extends to a smooth even function on M_+. Thus one has a symbol map

$$\left(\Psi_+^{\text{even}}\right)_m \to C^\infty(M_+)_m^{\text{even}} \tag{5.1}$$

from the space of m^{th} order pseudodifferential operators in Ψ_+^{even} to the space of even homogeneous functions of degree m on M_+. Let $\mathcal{A}$ be the algebra of operators

$$\Pi^{\text{even}}\Psi_+^{\text{even}}\Pi^{\text{even}}$$

and let

$$\mathcal{A}^m = \Pi^{\text{even}}\left(\Psi_+^{\text{even}}\right)_m \Pi^{\text{even}}\,.$$

Theorem 5.1. *From the map* (5.1) *one gets a short exact sequence*

$$0 \to \mathcal{A}^{m-1} \to \mathcal{A}^m \xrightarrow{\sigma} C^\infty(M_+)_m^{\text{even}} \to 0\,. \tag{5.2}$$

Proof. Given $A \in (\Psi_+^{\text{even}})_m$, suppose the leading symbol of A vanishes on U. Then one can find a pseudodifferential operator, A', whose total symbol vanishes on U and whose leading symbol is identical with $\sigma(A)$. Thus by Theorem 3.5 A' commutes with Π^{even} modulo smoothing operators. Hence, by the remark following Theorem 3.5, one can modify A' by adding to it a smoothing operator, so it actually does commute with Π^{even}. Moreover, since the total symbol of A' vanishes on U $\Pi^{\text{even}}A'$ is smoothing by the even version of Theorem 3.7; so by replacing A' by $A' - \Pi^{\text{even}\,A'}$, one can assume not only that A' commutes with Π^{even} but that $\Pi^{\text{even}}A' = 0$. Since $\sigma(A) = \sigma(A')$, the operator, $A - A'$ is of order $m-1$ and

$$\Pi^{\text{even}}A\Pi^{even} = \Pi^{\text{even}}(A - A')\Pi^{\text{even}}\,.$$

This proves that the map, σ, in (5.2) is injective; and that it is surjective follows from (the even version of) Theorem 3.6. q.e.d.

We claim next

Theorem 5.2. *If A_1 and A_2 are in $\mathcal{A}$, $\sigma(A_1A_2) = \sigma(A_1)\sigma(A_2)$ and $\sigma([A_1, A_2]) = -\sqrt{-1}\{\sigma(A_1), \sigma(A_2)\}$. Moreover if $A \in \mathcal{A}$, $A^* \in \mathcal{A}$ and $\sigma(A^*) = \overline{\sigma(A)}$.*

Proof. Microlocally on U these are standard identities for leading symbols of pseudodifferential operators. Therefore, since U is a dense subset of M_+ they hold globally on all of M_+. q.e.d.

An operator, $A \in \mathcal{A}^m$ is *elliptic* if $\sigma(A)$ is everywhere nonzero. We will show that these operators have the usual properties of elliptic operators:

Theorem 5.3. *If $A \in \mathcal{A}^m$ is elliptic, it is invertible modulo smoothing operators, i.e., there exists a $B \in \mathcal{A}^{-m}$ such that $I - BA$ and $I - AB$ are smoothing.*

Proof. Replacing A by A^*A we can assume that A is self-adjoint and that $\sigma(A) > 0$. Let $A = \Pi^{\text{even}}Q\Pi^{\text{even}}$, $Q \in \Psi_+^{\text{even}}$. Since $\sigma(Q) = \sigma(A)$ on U we an assume that $\sigma(Q) > 0$ on an open conic set, V in M containing the closure of U. Let P be a pseudodifferential operator of order m whose total symbol is supported in the complement of the closure of U and whose leading symbol is nonnegative and strictly greater than zero on the complement of V. By Theorems 3.5 and 3.6 $[P, \Pi^{\text{even}}]$ and $\Pi^{\text{even}}P$ are smoothing, so, by modifying P by a smoothing operator, we can assume that $[P, \Pi^{\text{even}}]$ and $\Pi^{\text{even}}P$ are zero. Replacing Q by $Q + \lambda P$, $\lambda \gg 0$, we can assume that the symbol of Q is positive everywhere, and hence that Q is invertible modulo smoothing operators, i.e., there exists a pseudodifferential operator, Q_1, of order $-m$, with $Q_1Q - I$ and $QQ_1 - I$ smoothing. It is easy to see that $[\Pi, Q_1]$ is smoothing; and hence Q_1 can be modified by adding to it a smoothing operator such that $[\Pi, Q_1] = 0$. Now set $B = \Pi Q_1 \Pi$. q.e.d.

The results above justify to some extent the assertion in Theorem 5.1 that the algebra $\mathcal{A}$ "quantizes" the algebra of classical observables, $C^\infty(M_+)^{\text{even}}$. A slightly more compelling justification is the following.

Theorem 5.4. *If $A \in \mathcal{A}_m$, $m > 0$, is elliptic and self-adjoint and $\sigma(A)$ is everywhere-positive, the spectrum of A is discrete, and its eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \dots$$

satisfy the Weyl law

$$N(\lambda) \sim \mathrm{vol}\{m \in M_+, \sigma(A)(m) < \lambda\}.$$

Here $N(\lambda)$ is the Weyl counting function

$$N(\lambda) = \#\{\lambda_i < \lambda\}$$

and "vol" means symplectic volume.

It is shown in [3] that a Weyl law for an algebra of operators of the type above is implied by the existence of a "residue trace"; and the following theorem asserts that a "residue trace" exists on the algebra $\mathcal{A}$.

Theorem 5.5. *There exists a linear map*

$$\mathrm{res} : \mathcal{A} \to \mathbb{C}$$

with the following properties:

(a) $\mathrm{res}\, A = 0$ *if and only if A can be written as a sum of commutators*

$$A = \sum_{i=1}^{N} [A_i, B_i],$$

$A_i, B_i \in \mathcal{A}$.

(b) *If A is of degree $-n$*

$$\mathrm{res}(A) = \int_{M_+} \sigma(A) \omega_+^m \tag{5.3}$$

ω_+ being the symplectic form on M_+.

Remark. If (M, ω) is a symplectic cone of dimension $2n$, and f a homogeneous function of degree $-n$, the form $f\omega^n$ is a $2n$ form of degree of homogeneity zero; so

$$L_\Xi f\omega^n = 0 = d(\iota(\Xi) f\omega^n).$$

Thus $\iota(\Xi) f\omega^n$ is closed. Let Γ be a compact $2n-1$ dimensional submanifold of M. Γ is called a *contour* if it intersects every ray of the cone, M, in exactly one point. It is very easy to see that if Γ and Γ_1 are contours, Γ can be smoothly deformed into Γ_1 and hence the integral

$$\int_\Gamma \iota(\Xi) f\omega^n$$

is independent of the choice of Γ; and this integral is *defined* to be the integral

$$\int f\omega^n.$$

We won't give the proof of the existence of this residue trace here. Details can be found in [2].

We will next describe some analogous results for the Szegö projector, Π, and the algebra of the pseudodifferential operators, Ψ_+, commuting with Π. Let

$$\mathcal{B} = \Pi\Psi_+\Pi.$$

It is clear from Theorems 2.4 and 3.6 (and the canonical form Theorem 4.2) that the leading symbol of an operator, $B \in \mathcal{B}^n$ can be interpreted as a function on M_{++}; and the following is proved by the same proof as that of Theorem 5.1.

Theorem 5.6. *There exists a short exact sequence*

$$0 \to \mathcal{B}^{m-1} \to \mathcal{B}^n \xrightarrow{\sigma} C^\infty(M_{++})_m. \tag{5.4}$$

Notice, by the way, that if $\widetilde{U}$ is the complement of the cut locus, M_{red}, in M_{++}, one has a map

$$\gamma : \mathcal{B}^m \to C^\infty(M_{++})_{\mathrm{even}} \to C^\infty(U). \tag{5.5}$$

We claim:

Theorem 5.7. *If $B \in \mathcal{B}^m$ is of the form $B = \Pi Q \Pi$, with $Q \in \Psi_+^m$, then $\gamma(B)$ is the restriction to U of the usual pseudodifferential symbol of Q.*

In other words on the complement of the cut locus in M the symbol calculus for the algebra, $\mathcal{B}$, is identical with the usual symbol calculus for pseudodifferential operators on the open subset, U, of M.

Proof. It suffices to check this in the model case, $X = \mathbb{R}^n \times S^1$; and in this model case, it is a consequence of Theorem 3.5 and Theorem 3.7. q.e.d.

6. Application: Toric symplectic cones

In this section we apply our microlocal version of symplectic cuts to the punctured cotangent bundle $T_0^*S^2 := T^*S^2 \smallsetminus S^2$ of the two-sphere to obtain symplectic cones over lens spaces. We then show that by applying symplectic cuts repeatedly to the punctured cotangent bundle of an n-torus one can obtain almost all symplectic toric cones.

As a preparation for the argument to follow, we generalize Proposition 1.1(1) (see [5] for details). Suppose we have a Hamiltonian action of an n-torus $G \simeq \mathbb{R}^n/\mathbb{Z}^n$ on a symplectic manifold (M, ω) with an associated moment moment map $\Phi : M \to \mathfrak{g}^*$. Pick a primitive vector λ in the integral lattice $\mathbb{Z}_G$ of G. Then the group $H_\lambda := \{\exp t\lambda \mid t \in \mathbb{R}\}$ is a closed subgroup of G isomorphic to S^1. The restriction of the action of G on M to H_λ is Hamiltonian with a corresponding moment map

$$\Phi_\lambda = \langle \Phi, \lambda \rangle,$$

where, as usual, $\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the canonical pairing. If H_λ acts freely on the set $\Phi_\lambda^{-1}(0)$, then the cut of M with respect to the action of H_λ makes sense. We denote the resulting space by $M_{+\lambda}$. Since the actions of G and H_λ on M commute, the action of G on M descends to a Hamiltonian action of G on $M_{+\lambda}$. The moment map Φ descends to a map $\Phi_{+\lambda}$ on $M_{+\lambda}$; it is an associated moment map for the action of G. Finally, it is not hard to see that

$$\Phi_{+\lambda}(M_{+\lambda}) = \Phi(M) \cap \{\eta \in \mathfrak{g}^* \mid \langle \eta, \lambda \rangle \geq 0\}.$$

In other words the moment image of $M_{+\lambda}$ is cut out from the moment image of M by the half-space $\{\eta \mid \langle \eta, \lambda \rangle \geq 0\}$.

Another ingredient that we will need is an analogue of the Delzant's theorem for toric symplectic cones. Recall that a toric symplectic cone is a symplectic manifold (M, ω) with a free proper action $\{\rho_t\}$ of $\mathbb{R}$ making it a symplectic cone and with an effective symplectic action of a torus

G commuting with $\{\rho_t\}$ and satisfying $2\dim G = \dim M$. (Note that such an action of G is automatically Hamiltonian and that there is a naturally associated moment map $\Phi : M \to \mathfrak{g}^*$ with $\Phi(\rho_t(m)) = e^t\Phi(m)$ for all $m \in M$, $t \in \mathbb{R}$.) We will further assume throughout that the base $M/\mathbb{R}$ of our symplectic cone $(M, \omega, \rho_t, \Phi : M \to \mathfrak{g}^*)$ is compact and connected. Note that the base $M/\mathbb{R}$ is naturally contact; more or less by definition it is a contact toric manifold.

Remark 6.1. The classification of compact connected contact toric manifolds (equivalently, of symplectic toric cones over a compact connected base) is somewhat more complicated than Delzant's classification of compact symplectic toric manifolds; see [6] and references therein. There is, however, a class of symplectic toric cones for which the classification is particularly nice. Namely assume in addition, the moment image $\Phi(M)$ lies in an open half-space in $\mathfrak{g}^*$, i.e., that there is a vector $X \in \mathfrak{g}$ such that the function $\langle \Phi, X\rangle$ is strictly positive. Then $\Phi(M) \cup \{0\}$ is a strictly convex rational[4] polyhedral cone (the result is implicit in [1]; cf. [7, Theorem 4.3]). Moreover, the polyhedral cone $\Phi(M) \cup \{0\}$ uniquely determines the symplectic toric cone $(M, \omega, \rho_t, \Phi : M \to \mathfrak{g}^*)$. In particular, if $(M_i, \omega_i, \rho^i_t, \Phi_i : M_i \to \mathfrak{g}^*)$, $i = 1, 2$, are two symplectic toric G-cones (over a compact connected base) whose moment images are the same convex polyhedral cones, then M_1 and M_2 are isomorphic as symplectic toric G-cones [6, Theorem 2.18 (4)].

In what follows we take the standard n torus $\mathbb{T}^n$ to be the Lie group $\mathbb{R}^n/\mathbb{Z}^n$. Thus the Lie algebra of $\mathbb{T}^n$ is $\mathbb{R}^n$. The identification of $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ by way of the standard basis identifies the weight lattice of $\mathbb{T}^n$ with $\mathbb{Z}^n$.

Consider the action of the torus $\mathbb{T}^2$ on the punctured cotangent bundle $T^*_0S^2$ generated by the normalized geodesic flow for the round metric and by the lift of a rotation of S^2 about an axis. It is not hard to see that the image of the associated homogeneous moment map

$$\Phi : T^*_0S^2 \to \mathbb{R}^2$$

is the cone C spanned by the vectors $(-1, 1)$ and $(1, 1)$ with the vertex at the origin deleted:

$$C = \{t_1(-1, 1) + t_2(1, 1) \in \mathbb{R}^2 \mid t_1, t_2 \geq 0\},$$

[4] "rational" means that the supporting hyperplanes are cut out by vectors in the integral lattice of the torus G

so the manifold $T_0^*S^2$ is a symplectic cone over $\mathbb{R}P^3$.

More generally there is a natural action of $\mathbb{T}^2$ on the symplectic cone over any lens space $L(p,q)$. Fix two positive relatively prime integers p and q. The map $\mathbb{T}^2 \to S^1 \times S^1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 = |z_2|^2 = 1\}$, $[\theta_1, \theta_2] \mapsto (e^{2\pi i\theta_1}, e^{2\pi i\theta_2})$ identifies $\mathbb{T}^2$ with $S^1 \times S^1$. The group $\Gamma := \{(\mu_1, \mu_2) \in S^1 \times S^1 \mid \mu_1 \mu_2^q = 1,\ \mu_1^p = 1\}$ is cyclic of order p. The quotient of $\mathbb{C}^2 \smallsetminus \{0\}$ by the natural action of Γ ($(\mu_1, \mu_2) \cdot (z_1, z_2) = (\mu_1 z_1, \mu_2 z_2)$) is, more or less by definition, the symplectic cone on the lens space $L(p,q)$:

$$(\mathbb{C}^2 \smallsetminus \{0\})/\Gamma = L(p,q) \times \mathbb{R}.$$

The natural action of $\mathbb{T}^2 \simeq S^1 \times S^1$ on $\mathbb{C}^2$ descends to an effective Hamiltonian action of $\mathbb{T}^2/\Gamma$ on the cone $L(p,q) \times \mathbb{R}$. We compute the image of the associated moment map as follows. The natural action of $\mathbb{T}^2 \simeq S^1 \times S^1$ on $\mathbb{C}^2$ descends to an effective Hamiltonian action of $\mathbb{T}^2/\Gamma$ on $\mathbb{C}^2 \smallsetminus \{0\})/\Gamma$. The kernel of the surjective map $\varphi : S^1 \times S^1 \to S^1 \times S^1$, $\varphi(\mu_1, \mu_2) = (\mu_1 \mu_2^q, \mu_2^{-p})$ is exactly Γ. This gives us an isomorphism $\overline{\varphi} : \mathbb{T}^2/\Gamma \to \mathbb{T}^2$. With this identification the image of the moment map for the action of $\mathbb{T}^2 \simeq \mathbb{T}^2/\Gamma$ is

$$C_{p,q} := \{t_1(1,0) + t_2(p,q) \in \mathbb{R}^2 \mid t_1, t_2 \geq 0\}.$$

Note that if we pick a different basis of the weight lattice of $\mathbb{T}^2$, the moment cone $C_{p,q}$ will change by an action of an element of $\mathrm{SL}(2,\mathbb{Z})$.

We claim that we can obtain the cone $C_{p,q}$ (up to a change of lattice basis) by cutting the image of $T_0^*S^2$ with a half-space. Indeed the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ maps $C_{p,q}$ onto

$$C'_{p,q} := \{t_1(1,1) + t_2(p+2q, p+q) \in \mathbb{R}^2 \mid t_1, t_2 \geq 0\}$$

and

$$C'_{p,q} = C \cap \{\eta \in \mathbb{R}^2 \mid \langle \eta, (p+2q, -p-q)\rangle \geq 0\},$$

where, as above, C denotes the moment image of $T_0^*S^2$. We conclude that there is a Hamiltonian action of $\mathbb{T}^2$ on $L(p,q) \times \mathbb{R}$ such that the moment map image is the cut of the moment map image of $T_0^*S^2$ by a half-space. It follows from Remark 6.1 that

$$\left(T_0^*S^2\right)_{+(p+2q,-p-q)} = L(p,q) \times \mathbb{R},$$

i.e., that we can obtain the symplectic cone on the lens space $L(p,q)$ by cutting the punctured cotangent bundle of S^2.

More generally almost all toric symplectic cones can be obtained by iterated cuts starting with the cotangent bundle of the standard n-torus $\mathbb{T}^n$. Indeed, as remarked above, strictly convex rational polyhedral cones in $\mathbb{R}^n$ (satisfying certain integrality conditions) classify, as moment map images, a large class of symplectic toric cones. Each of these polyhedral cones is the intersection of finitely many half-spaces with primitive integral normals. Therefore these moment map images can be obtained from $\mathbb{R}^n \setminus \{0\}$ by repeated cuts by half-spaces. Consequently the corresponding symplectic cones can be obtained from the punctured cotangent bundle of the standard torus $\mathbb{T}^n$ by repeated symplectic cuts.

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