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$\widehat{A}$-GENUS ON NON-SPIN MANIFOLDS WITH S^1 ACTIONS AND THE CLASSIFICATION OF POSITIVE QUATERNION-KÄHLER 12-MANIFOLDS

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Abstract

We prove that the $\widehat{A}$-genus vanishes on certain non-spin manifolds. Namely, $\widehat{A}(M)$ vanishes on any oriented, compact, connected, smooth manifold M with finite second homotopy group and endowed with non-trivial (isometric) smooth S^1 actions. This result extends that of Atiyah and Hirzebruch on spin manifolds endowed with smooth S^1 actions [1] to manifolds which are not necessarily spin.

We prove such vanishing by means of the elliptic genus defined by Ochanine [23, 24], showing that it also has the special property of being "rigid under S^1 actions" on these (not necessarily spin) manifolds.

We conclude with a non-trivial application of this new vanishing theorem by classifying the positive quaternion-Kähler 12-manifolds. Namely, we prove that every quaternion-Kähler 12-manifold with a complete metric of positive scalar curvature must be a symmetric space.

Introduction

This article is divided into two parts. The first part is devoted to proving the following vanishing theorem:

Theorem 1. *Let M be a $2n$-dimensional, oriented, compact, connected, smooth manifold with finite second homotopy group and endowed with a smooth S^1 action. Then,*

$$\widehat{A}(M) = 0.$$

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We prove this theorem by means of the "rigidity under S^1 actions" of the elliptic genus [23, 24, 6, 14, 18, 31, 32] on these manifolds. This, in fact, consists of showing the rigidity of certain elliptic operators (see Theorem 3). This vanishing is new since it *does not* follow from results on spin (spin^c or spin^h) manifolds. The manifolds under consideration are not necessarily spin (nor spin^c, nor spin^h). Hence, Theorem 1 extends that of Atiyah and Hirzebruch on spin manifolds endowed with smooth S^1 actions [1].

The second part of the paper is devoted to a classification problem in Riemannian Geometry. Namely, the classification of the 12-dimensional quaternion-Kähler manifolds with a complete metric of positive scalar curvature. Complete quaternion-Kähler manifolds with positive scalar curvature have only been classified in 4 dimensions by Hitchin [16], and in 8 dimensions by Poon and Salamon [27]. In this paper, we show that all such manifolds belong to the list of 12-dimensional symmetric spaces given by Wolf in [33], which is accomplished by applying the vanishing Theorem 1 at a crucial point.

Theorem 2. *A complete* 12*-dimensional quaternion-Kähler manifold with positive scalar curvature is isometric to one of the following symmetric spaces:*

1. *The quaternionic projective space* $\mathbb{HP}^3$.
2. *The complex Grassmannian* $\mathbb{G}\mathrm{r}_2(\mathbb{C}^5)$.
3. *The real Grassmannian* $\mathbb{G}\mathrm{r}_4(\mathbb{R}^7)$.

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Part I

1. $\widehat{A}$ and elliptic genera on non-spin manifolds

The main theorem of this part, Theorem 1, states the vanishing of the characteristic number $\widehat{A}(M)$ on certain non-spin manifolds. Such manifolds admit no spin structure and, therefore, have neither spin bundle, nor spinors, nor Dirac operator. This means that the characteristic number $\widehat{A}(M)$ is, a priori, a rational number and that we cannot estimate it in the usual index-theoretical way.

Instead, by noticing that the elliptic genus defined by Ochanine [23, 24] has two standard representations as Laurent series, one involving indices of (well-defined) signature operators on the manifold and the other twisted $\widehat{A}$-genera, we study it in our context and prove a rigidity result under circle actions which eventually leads to the vanishing theorem.

The elliptic genus has been well studied on spin manifolds. Witten gave an interpretation of the elliptic genus [31, 32] as the localized value of the equivariant signature (to the fixed point set of a circle action) on the loop space of the manifold, and predicted various rigidity theorems for elliptic operators on the original spin manifold. These theorems were proved by Taubes [30], Bott and Taubes [6], and Liu [21], and have been generalized further by others [13, 17, 9].

In the same vein of ideas, we shall prove a rigidity theorem for certain elliptic operators on π_2-finite manifolds with circle actions (see Theorem 3), a case that has not been studied so far. The proof is carried out along the lines of [6] and the key point is to prove a couple of spin-like properties of these manifolds in Lemmas 1 and 2 by using [7].

This part is organized as follows: In §§1.1 we review the definition of the elliptic genus and state the rigidity theorem. In §§1.2-1.3 we prove the rigidity theorem, and in §§1.4 we prove our main result, Theorem 1.

1.1 Elliptic genus and rigidity

Let $D\colon \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator acting on sections of the vector bundles E and F over a compact manifold M. The index of D is the virtual vector space

$$\mathrm{ind}(D) = \ker(D) - \mathrm{coker}(D).$$

If M admits a circle action preserving D, i.e. such that S^1 acts on E and F, and commutes with D, $\mathrm{ind}(D)$ admits a Fourier decomposition into complex 1-dimensional irreducible representations of S^1

$$\mathrm{ind}(D) = \sum a_m \, L^m,$$

where $a_m \in \mathbb{Z}$ and L^m is the representation of S^1 on $\mathbb{C}$ given by $e^{i\theta} \mapsto e^{im\theta}$.

Definition 1. The elliptic operator D is called *rigid* if $a_m = 0$ for all $m \neq 0$, i.e., $\mathrm{ind}(D)$ consists of the trivial representation with multiplicity a_0.

The elliptic operator D is called *universally rigid* if it is rigid under any S^1 action on M by isometries.

We shall be concerned with the elliptic operators associated with the signature operator. Let $\bigwedge_c^{\pm}$ be the even and odd complex differential forms on an oriented, compact, $2n$-dimensional, smooth manifold M under the Hodge $*$-operator, respectively. The signature operator

$$d_s \colon \textstyle\bigwedge_c^{+} \longrightarrow \bigwedge_c^{-}$$

is elliptic and the virtual dimension of its index equals the signature of M, $\tau(M)$. If W is a complex vector bundle on M endowed with a connection, we can *twist* the signature operator to forms with values in W

$$d_s \otimes W \colon \textstyle\bigwedge_c^{+}(W) \longrightarrow \bigwedge_c^{-}(W).$$

This operator is also elliptic and the virtual dimension of its index is denoted by $\tau(M, W)$.

Definition 2. Let M be an oriented, compact, $2n$-dimensional smooth manifold, and $T = TM \otimes \mathbb{C}$ its tangent bundle. Let R_i be the sequence of bundles defined by the formal series

$$R(q,T) = \sum_{i=0}^{\infty} q^i R_i = \bigotimes_{i=1}^{\infty} \textstyle\bigwedge_{q^i} T \otimes \displaystyle\bigotimes_{j=1}^{\infty} S_{q^j} T,$$

where $S_t T = \sum_{k=0}^{\infty} t^k S^k T$, $\bigwedge_t T = \sum_{k=0}^{\infty} t^k \bigwedge^k T$, and $S^k T$, $\bigwedge^k T$ denote the k-th symmetric and exterior tensor powers of T, respectively. The *elliptic genus* of M is defined as

$$\tau_q(M) = \tau(M, R(q,T)) = \sum_{i=0}^{\infty} q^i \cdot \tau(M, R_i). \tag{1}$$

The first few terms of the sequence $R(q,T)$ are

$$R_0 = 1, \quad R_1 = 2T \quad R_2 = 2(T^{\otimes 2} + T), \ldots$$

so that, in particular, the constant term of $\tau_q(M)$ is $\tau(M)$.

Theorem 3 (Rigidity Theorem). *Let M be an oriented, compact, connected, smooth $2n$-manifold endowed with smooth S^1 actions. In addition, assume that $\pi_2(M)$ is finite. Then, each of the operators $d_s \otimes R_i$ is universally rigid.*

1.2 Proof of the rigidity theorem

The manifolds under consideration posses a very special property induced by the π_2-finiteness, which is proved in [7] and which is used at a key point in the proof to replace the spin condition. We recall the notation from [6] and their main line of argument for the convenience of the reader and to give appropriate context to Lemmas 1 and 2 below.

Applying the Atiyah-Segal G-signature theorem [2], we have

$$\tau_q(M) = \sum_{\{P\}} \mu(P)$$

where P runs over the connected components of the fixed point set of the S^1 action (cf. [6, p. 155]). The contribution $\mu(P)$ of P to $\tau_q(M)$ is given by the index of the signature operator on P twisted by an appropriate power series of vector bundles on P (see [6] for details).

The contributions $\mu(P)$ are meromorphic functions on the 2-dimensional torus $\mathbb{T}_{q^2} = \mathbb{C}^*/q^2$ (the quotient of the multiplicative group of non-zero complex numbers $\mathbb{C}^*$ by the subgroup generated by the element $q^2 \neq 0$). The proof of the rigidity theorem is equivalent to showing that $\tau_q(M) = \sum_{\{P\}} \mu(P)$ has no poles at all on $\mathbb{T}_{q^2}$.

Define the translation $t_a\tau_q(M)$ of $\tau_q(M)$ by $a \in \mathbb{C}^*$, to be given by the map at the character level $\lambda \mapsto a\lambda$. The rigidity theorem for $\tau_q(M)$ will follow from showing that none of the translations $t_a\tau_q(M)$, by points $a \in \mathbb{T}_{q^2}$ of finite order, has a pole on the circle $|\lambda| = 1$. It is enough to consider k ranging over $\mathbb{N}$ and a ranging over the roots of the form

$$a = \alpha^s, \qquad \alpha^k = q,$$

with k and s relatively prime. The translations $t_{\alpha^s}\tau_q(M)$ can be expressed as *twists* of τ_q on the connected components M_k of the fixed point submanifold of the subgroup $\mathbb{Z}_k \subset S^1$, generated by $e^{2\pi i/k} \in S^1$, which *do contain fixed points of the S^1 action.* In fact, the translation $t_{\alpha^s}\tau_q(M)$ is the index of the signature operator on M_k' twisted by an appropriate power series in certain bundles T_r, where M'_k is the submanifold M_k with a specific orientation. We explain the meaning of r and T_r, and the choice of orientation of M_k below (cf. [6, (8.13)]).

Remark. The submanifolds M_k are intermediate steps between M and the fixed point set $\{P\}$ of the S^1 action. Therefore, we shall be interested only in those connected components of M_k that contain connected components P of fixed points of S^1 (cf. [6, p. 153]). Moreover, the submanifolds M_k must be orientable in order to define a signature operator and the translations $t_a\tau_q(M)$. We ensure the orientability of these components by the following lemma whose proof is postponed until Subsection 1.3:

Lemma 1. *Let M be an oriented, $2n$-dimensional, smooth manifold endowed with a smooth S^1 action. Consider $\mathbb{Z}_k \subset S^1$ and its corresponding action on M. If k is odd then the fixed point set M_k of the $\mathbb{Z}_k$ action is orientable. If k is even and M_k contains a fixed point of the S^1 action, M_k is also orientable.*

The translations $t_{\alpha^s}\tau_q(M)$ converge on some annulus containing the unit circle $|\lambda| = 1$ to the Laurent series of a meromorphic function on $\mathbb{T}_{q^2}$ which has no poles on the unit circle.

The bundles T_r

The subgroup $\mathbb{Z}_k$ acts on the normal bundle of M_k in M so that $T = TM$ splits over M_k as

$$T|_{M_k} = TM_k \oplus T_1^\# \oplus \ldots \oplus T_{k/2-1}^\# \oplus T_{[k/2]}^\#, \tag{2}$$

where each $T_r^\#$ is an irreducible real representation of $\mathbb{Z}_k$, and $[k/2]$ is the greatest integer smaller than or equal to $k/2$. The S^1 action on M induces an S^1 action on M_k, whose differential induces an action on $T|_{M_k}$, preserving the decomposition, and making each $T_r^\#$ an S^1 bundle over M_k, for $r = 1, \ldots, [(k-1)/2]$. Each $T_r^\#$, with $r \neq k/2$ if k is even, is endowed with a complex structure such that $\lambda \in S^1$ acts by λ^r, for $r = 1, \ldots, [(k-1)/2]$. Hence, $T_r^\#$ comes from a complex vector bundle T_r. For k even, the action on $T_{k/2}^\# = T_{k/2}$ is multiplication by -1, and it does not necessarily come from a complex vector bundle.

The $T_r^\#$ inherit an orientation from the complex structure on T_r, for $r = 1, \ldots, [(k-1)/2]$. Hence, if k is odd, TM_k has an induced orientation. If k is even, however, we only know that $TM_k \oplus T_{k/2}$ is orientable. On the other hand, Lemma 1 guarantees that M_k is also orientable. Let us, therefore, choose an orientation. In this way, $T_{k/2}$ inherits an orientation from M and M_k.

The orientation of M_k'

Observe that M is oriented and that the submanifolds P and M_k are orientable. The orientation of M_k should be chosen to be compatible with the orientations of the components P as follows: Let P be a connected component of the fixed point set of the S^1 action. Along P

$$TM|_P = TP \oplus \bigoplus_{i=1}^{l} E_i^{\#},$$

where $E_i^{\#}$ denotes the canonical underlying real bundle of the complex bundle E_i on which S^1 acts by sending λ to λ^{m_i} ($|m_i| \neq |m_j|$ unless $i = j$). Let $d_i = \operatorname{rank}_{\mathbb{C}}(E_i)$.

When k is odd, the decomposition in (2) determines an orientation on TM_k denoted by $+1$. If $P \subset M_k$, choose the exponents along P so that each $m_j \not\equiv 0 \,(\operatorname{mod} k)$ is congruent to some $r \in \{1, \ldots, (k-1)/2\}$. Choose the orientation of TP and the sign of those $m_j \equiv 0 \,(\operatorname{mod} k)$ so that the induced orientation on $TM|_P$ is the given one. The induced orientation on $TM_k|_P$ will be the $+1$ orientation. For each m_j, let $(l_j, \omega_j) \in \mathbb{Z} \times \{1, \ldots, k-1\}$ be such that

$$s \cdot m_j = l_j \cdot k + \omega_j, \tag{3}$$

and define

$$\varepsilon(P) = \sum_j d_j \cdot l_j. \tag{4}$$

The orientation for M_k' is now defined as $+1 \cdot (-1)^{\varepsilon(P)}$, if $M_k \supseteq P$. Lemma 2 below ensures that this orientation is well defined.

When k is even, $TM|_{M_k}$ decomposes according to (2). Since $\bigoplus_{r=1}^{k/2-1} T_r^{\#}$ is naturally oriented, $TM_k \oplus T_{k/2}$ inherits an orientation. Let us choose an orientation for TM_k and call it $+1$, which induces an orientation on $T_{k/2}$. If $P \subset M_k$, select the exponents at P as follows: If $m_j \not\equiv 0, k/2 \,(\operatorname{mod} k)$, make the choice as before so that $(m_j)_{\operatorname{mod} k} \in \{1, \ldots k/2-1\}$. Choose the signs for those $m_j \equiv 0, k/2 \,(\operatorname{mod} k)$ and the orientation of TP to make the induced orientation of $(TM_k \oplus T_{k/2})|_P$

correct. This ensures that the induced orientation of $TM|_P$ is correct. The induced orientation of $TM_k|_P$, however, *may not* be the correct one $(+1)$. Let $\varepsilon_0 = 0, 1$, with $\varepsilon_0 = 0$ if the induced orientation on $TM_k|_P$ is correct, and $\varepsilon_0 = +1$ if the induced orientation on $TM_k|_P$ is incorrect. For each m_j, define (l_j, ω_j) by (3) and set

$$\varepsilon(P) = \varepsilon_0 + \sum_j d_j \cdot l_j. \tag{5}$$

The orientation of M_k' is again $+1 \cdot (-1)^{\varepsilon(P)}$, if $M_k \supset P$. This orientation is well defined by the following lemma, whose proof is postponed until Subsection 1.3:

Lemma 2. *Let M be an oriented, compact, $2n$-dimensional, smooth manifold with $\pi_2(M)$ finite and endowed with a smooth S^1 action. Let $k \in \mathbb{N}$ and M_k be a connected component of the fixed point set of $\mathbb{Z}_k \subset S^1$. Let $s \in \mathbb{Z}$ be relatively prime to k, and P, $P' \subset M_k$ be connected fixed point submanifolds of the S^1 action. Use the prescription above, (4) or (5) respectively, to define the numbers $\varepsilon(P)$ and $\varepsilon(P')$. Then $(-1)^{\varepsilon(P)} = (-1)^{\varepsilon(P')}$.*

Provided with all the conditions above, the function of q

$$t_{\alpha^s} \tau_q(M),$$

is regular on an annulus containing the unit circle for all $k \in \mathbb{N}$, so that $t_{\alpha^s}\tau_q(M)$ has no poles on the unit circle $|\lambda| = 1$. Hence, $\tau_q(M)$ has no poles at all on $\mathbb{T}_{q^2}$, and must be constant.

1.3 The lemmas

Remark. Notice that Lemma 1 is valid on any compact smooth manifold with a circle action. It does not require any special condition on neither the fundamental group, nor (co)homology, nor homotopy type.

Remark. The content of the Lemma 2 essentially says that a compact smooth *non-spin* manifold with π_2 finite behaves under circle actions in the same way as (or better than) spin manifolds.

Proof of Lemma 1. We begin by recalling an argument from [30, pp. 488–489]. First, note that the case when k is odd follows from (2).

Let $k=2$ and denote the S^1 action on M by $\varphi\colon S^1\times M\longrightarrow M$. Let $T_0=TM_2$ and T_1 be the normal bundle of M_2 (with respect to an appropriate metric on the fibers). Since $\varphi(\pi,\cdot)$ is the identity on M_2, $\varphi(\pi,\cdot)_*\colon T_0\longrightarrow T_0$ is the identity map, and $\varphi(\pi,\cdot)_*\colon T_1\longrightarrow T_1$ is the involution induced by the $\mathbb{Z}_2$ action. Given an S^1 invariant metric on M, there is a metric on T_1 with respect to which $\varphi(\pi,\cdot)_*=-I$, where I is the identity automorphism.

Suppose that T_0 is not orientable. Then there exists a loop in M_2, $f\colon S^1\longrightarrow M_2$, on which T_0 is not trivial, i.e. $w_1(f^*T_0)\neq 0$ and $f^*T_0\cong \left(S^1\times_{\mathbb{Z}_2}\mathbb{R}\right)\times\mathbb{R}^{l-1}$, where $l=\dim M_2$. Similarly, since $w_1(T_0)=w_1(T_1)$ $f^*T_1\cong\left(S^1\times_{\mathbb{Z}_2}\mathbb{R}\right)\times\mathbb{R}^{m-1}$, where $m=\operatorname{rank}(T_1)$.

With the help of φ and f we define another map

$$f_1\colon S^1\times S^1\longrightarrow M_2,$$

$$(t,s)\mapsto f_1(t,s)=\varphi(t/2,f(s)),$$

which represents a 2-torus mapped into $M_2\subset M$, by letting the original loop follow its orbit under the S^1 action. For fixed s, $f_1(\cdot,f(s))^*T_0=f(s)^*\varphi_*T_0$, so that

$$f_1^*T_0\cong S^1\times\left(S^1\times_{\mathbb{Z}_2}\mathbb{R}\right)\times\mathbb{R}^{l-1}. \tag{6}$$

For the bundle T_1, $f_1(\cdot,f(s))^*T_1=f(s)^*\varphi_*T_1$, so that

$$f_1^*T_1\cong S^1\times_{\mathbb{Z}_2}\left(\left(S^1\times_{\mathbb{Z}_2}\mathbb{R}\right)\times\mathbb{R}^{m-1}\right), \tag{7}$$

where $\mathbb{Z}_2$ acts on $\left(\left(S^1\times_{\mathbb{Z}_2}\mathbb{R}\right)\times\mathbb{R}^{m-1}\right)$ as multiplication by ±1.

The $\mathbb{Z}_2$ cohomology of $S^1\times S^1$ has generators z_1, z_2 which restrict trivially to the second S^1 and to the first S^1, respectively. The total Stiefel-Whitney class of $f_1^*T_0$ can be obtained from (6)

$$w(f_1^*T_0)=1+z_2, \tag{8}$$

and that of $f_1^*T_1$ from (7)

$$w(f_1^*T_1)=1+m_{\mathrm{mod}(2)}\cdot z_1+z_2+(m-1)_{\mathrm{mod}(2)}\cdot z_1\wedge z_2. \tag{9}$$

From (8) and (9), the total Stiefel-Whitney class of $f_1^*(TM)$

$$w(f_1^*TM)=1+m_{\mathrm{mod}(2)}\cdot z_1+\left(m_{\mathrm{mod}(2)}+(m-1)_{\mathrm{mod}(2)}\right)\cdot z_1\wedge z_2, \tag{10}$$

which implies

$$w_2(f_1^*TM) \neq 0. \tag{11}$$

On the other hand, the 2-torus is homotopic to a sphere with S^1 acting on it by rotations. Indeed, let $p \in M_2$ be a fixed point of the S^1 action, and let p' be a point in the original loop $f(S^1)$. Consider a path γ in M_2 joining p and p'. Let S^1 act on γ to generate a disk, whose "central" point is p. This disk is attached to the 2-torus by a circle (its border) which is an orbit of the S^1 action. We can think of it as the image of an equivariant map from the 2-sphere to M, where S^1 acts on S^2 by rotations with its north and south poles as fixed points being mapped to the same point p. Let us denote this map by $f_2 \colon S^2 \longrightarrow M_2$, and note that

$$w_2(f_2^*TM) = w_2(f_1^*TM) \neq 0 \tag{12}$$

by (11).

Observe that $w_2(f_2^*TM)$ is now the reduction mod 2 of the Chern class $c_1(f_2^*TM)$ on S^2 (cf. [6, Lemmas 9.1]). Furthermore, $c_1(f_2^*TM)$ is even by [6, Lemmas 9.2], since the north and south pole of our sphere are mapped to the same point. This contradicts (12) (and (11)).

For even $k > 2$, use the same arguments together with the fact that $c_1(E) = 0$ for any bundle $E \longrightarrow S^1 \times S^1$ to which the rotations around an S^1 lift. q.e.d.

Sketch of proof of Lemma 2. Since Lemma 2 is the analogue of [6, Lemma 8.1] in our set-up, we shall only point out the relevant change in the proof.

Let $k \in \mathbb{N}$, P and P' be two distinct connected fixed point submanifolds of S^1 in M_k. Let $p \in P$ and $p' \in P'$. Consider a path joining p and p' which avoids other fixed points of S^1. Let S^1 act on it to generate a sphere with "north" and "south" poles p and p' respectively. Let the sets of integers $\{m_i\}$ and $\{m_i'\}$ denote the exponents of the S^1 action on T_pM and $T_{p'}M$ respectively.

By [6, Lemma 9.1], the number

$$\varepsilon(P) - \varepsilon(P') \equiv c \cdot \left(\sum_i (m_i - m_i') \right) \pmod 2,$$

where c is a constant. In other words, we must check that the parity of this sum of exponents is even. This is an immediate consequence of the following theorem by G. Bredon:

Theorem 4 ([7]). *Let N be a smooth manifold with a smooth S^1 action and assume that $\pi_{2i}(N)$ is finite for $1 \leq i \leq r-1$. Let x and y be two fixed points of the S^1 action. Then $T_xN - T_yN$ is divisible by $(1-L)^r$ in the representation ring $R(S^1)$, where L is the standard representation of S^1.*

In our set up $\pi_2(M)$ is finite, then the virtual representation $T_pM - T_{p'}M$ decomposes as follows:

$$T_pM - T_{p'}M = (1-L)^2 \otimes \left(\sum_j b_j L^j \right),$$

where $L \cong \mathbb{C}$ is the standard representation of S^1 and $\{b_j \in \mathbb{Z}\}$ is a finite set of integers. This means that

$$\sum_i (m_i - m_i') = \sum_j b_j \cdot j - 2 \sum_j b_j \cdot (j+1) + \sum_j b_j \cdot (j+2) = 0,$$

which proves the assertion. q.e.d.

Let us contrast the content of the previous paragraph with the following example of a manifold with smooth S^1 actions, but with infinite π_2:

Example. Given our interest in 12-dimensional manifolds in Part II, let us consider the complex S^1 representation $V = L^0 \oplus L^1 \oplus L^2 \oplus L^3 \oplus L^4$, with the obvious S^1 action on each complex line. The complex Grassmannian $\mathbb{G}\mathrm{r}_2(V)$ is non-spin and has an induced S^1 action with fixed points. We have, for instance, the following fixed points with their corresponding exponents:

- The complex 2-plane $p_1 = L^0 \oplus L^1$ with exponents $(1,2,2,3,3,4)$, which add up to 15.
- The complex 2-plane $p_2 = L^0 \oplus L^2$ with exponents $(-1,1,1,2,3,4)$, which add up to 10.

The difference of the sums of the exponents is $5 \not\equiv 0 \pmod 2$, failing to satisfy Lemma 2. This is due to the fact that $\mathbb{G}\mathrm{r}_2(V)$ is neither spin nor has finite π_2. In fact, $\pi_2(\mathbb{G}\mathrm{r}_2(V)) = \mathbb{Z}$. Furthermore, one can check directly that $\mathbb{G}\mathrm{r}_2(V)$ does not satisfy Theorem 1

$$\widehat{A}(\mathbb{G}\mathrm{r}_2(V)) \neq 0.$$

1.4 Vanishing of the $\widehat{A}$-genus

As a consequence of the rigidity theorem, we get the vanishing Theorem 1.

Remark. The manifold M may be neither spin, nor spin^c, nor spin^h. In such a case, $\widehat{A}(M)$ is only defined as a characteristic number and does not represent the index of an elliptic operator. Thus, $\widehat{A}(M)$ does not even have to be an integer. The prescribed homotopy type, however, yields this number as zero.

Proof of Theorem 1. We can assume that $\dim(M) = 4n$. Given that S^1 acts on M, the equivariant genus $\tau_q(M)_g$ is defined for any $g \in S^1$ as

$$\tau_q(M)_g = \sum \tau(M, R_i)_g \cdot q^i,$$

where $\tau(M, R_i)_g = \mathrm{tr}|_g \ker(d_s \otimes R_i) - \mathrm{tr}|_g \mathrm{coker}(d_s \otimes R_i)$. The coefficients of its q-development are now equivariant twisted signatures. Thus, according to Theorem 3, the value of $\tau_q(M)_g$ does not depend on g.

Applying the Atiyah-Segal G-signature theorem [2], $\tau_q(M)_g$ can be expressed in terms of the fixed point set M^g of g and the action of g on the normal bundle of $M^g \subset M$. In particular, let g be the orientation preserving involution in $\mathbb{Z}_2 \subset S^1$. Note that the submanifold $M^g = M_2$ may not be connected. We denote the transversal self-intersection of M_2 by $M_2 \circ M_2$. In [15, p. 315], Hirzebruch and Slodowy showed that

$$\tau_q(M)_g = \tau_q(M_2 \circ M_2).$$

On the other hand, applying the rigidity theorem (Theorem 3), $\tau_q(M) = \tau_q(M)_g$, i.e.

$$\tau_q(M) = \tau_q(M_2 \circ M_2). \tag{13}$$

The codimension of M_2 is positive and even, so that the elliptic genus $\tau_q(M)$ can be computed from the elliptic genera of submanifolds of M of codimension at least 4.

Let us now recall the expansion of $\tau_q(M)$ at the other cusp [14]

$$\widetilde{\tau_q}(M) = \frac{1}{q^{n/2}} \sum_{j=0}^{\infty} \widehat{A}(M, R'_j) \cdot q^j,$$

where R'_j is the sequence of virtual tensor bundles given by

$$R'(q,T) = \bigotimes_{k=2m+1} \bigwedge\nolimits_{-q^k} T \otimes \bigotimes_{k=2m+2} S_{q^k} T,$$

and $\widehat{A}(M, R'_j) = \langle \widehat{A}(M) \cdot \mathrm{ch}(R'_j), [M] \rangle$ may only be defined as a characteristic number (since M will not, in general, be spin). The first few terms of the sequence are

$$R'_0 = 1, \quad R'_1 = -T, \quad R'_2 = \bigwedge\nolimits^2 T + T, \quad \ldots$$

This expansion is obtained by considering $q = e^{\pi i t}$ and changing the t coordinate in (1) by $t \to -1/t$, and then by $t \to 2t$ (cf. [14]). This expansion has, a priori, a pole of order $n/2$.

On the other hand, by (13) we also have the following:

$$\widetilde{\tau}_q(M) = \widetilde{\tau}_q(M_2 \circ M_2), \tag{14}$$

whose right hand side has a pole of order at most $(n/2 - 1/2)$, since the dimension of any connected component of $M_2 \circ M_2$ is at most $4n - 4$. Therefore (14) implies that the first coefficient on the left hand side vanishes, i.e.,

$$\widehat{A}(M) = 0.$$

q.e.d.

Example. It is known that for every $m \in \mathbb{N}$, the real $8m+4$-dimensional Grassmannians $\mathbb{G}\mathrm{r}_4(\mathbb{R}^{2m+5})$ of 4-planes in $\mathbb{R}^{2m+5}$ are non-spin and $\pi_2(\mathbb{G}\mathrm{r}_4(\mathbb{R}^{2m+5})) = \mathbb{Z}_2$. The $\widehat{A}$-genus can be computed explicitly

$$\widehat{A}(\mathbb{G}\mathrm{r}_4(\mathbb{R}^{2m+5})) = 0,$$

in accordance with our theorem. In fact, not only the $\widehat{A}$-genus vanishes on these manifolds, but the entire elliptic genus [12]

$$\tau_q(\mathbb{G}\mathrm{r}_4(\mathbb{R}^{2m+5})) = 0.$$

Example. Borel and Hirzebruch proved in [5, Theorm 23.3.(iii)] the following result: Let G be a compact connected Lie group and U

a closed connected subgroup of G containing a maximal torus. If the second Betti number of G/U is zero, then

$$\widehat{A}(G/U) = 0.$$

Notice that the space G/U is not necessarily spin. Hence, it remains an interesting problem to determine whether the other coefficients of the elliptic genus have any integrality properties on the non-spin spaces.

Furthermore, given the subject of interest in Part II we have the following corollary:

Corollary 1. *Let M be an oriented, compact, connected, smooth 12-dimensional manifold with $\pi_2(M)$ finite and endowed with a smooth S^1 action. Suppose that M_2, the fixed point manifold of $\mathbb{Z}_2 \subset S^1$, contains no connected component of dimension 8. Then,*

$$\tau_q(M) \equiv 0,$$

and, in particular,

$$\tau(M) = 0.$$

Proof. This follows from (14) since the Laurent expansion on the right hand side has no half-integral powers of q at all, which means that the genus must vanish identically. q.e.d.

The theorems in this first part can be generalized further. For instance, one can consider other versions of the elliptic genus *twisted* with more general vector bundles on the manifolds and also prove rigidity theorems with respect to smooth S^1 actions. Furthermore, it is clear that the more information one obtains at the level of fixed point sets, the more vanishing theorems one may deduce. Along these lines, there is room for an analysis similar to that of Hirzebruch and Slodowy [15] for homogeneous spaces with vanishing second Betti number.

We shall explore these possibilities in the near future. In the meantime, let us pass to Part II to give a non-trivial application of the vanishing Theorem 1.

Part II

2. Quaternion-Kähler 12-manifolds

Let us consider $\mathbb{R}^{4n} \cong \mathbb{H}^n$ as a right module over the quaternions $\mathbb{H}$, whose elements are column vectors with entries in $\mathbb{H}$. Let $A \in \mathrm{Sp}(n)$, $q \in \mathrm{Sp}(1)$ and

$$\mu\colon \mathrm{Sp}(n) \times \mathrm{Sp}(1) \longrightarrow \mathrm{GL}(n, \mathbb{H}) \subseteq \mathrm{GL}(4n, \mathbb{R})$$

be the representation defined by $\mu(A, q)(X) = AXq^{-1}$, for $X \in \mathbb{H}^n$. Since $\ker(\mu) = \mathbb{Z}_2$, we define $\mathrm{Sp}(n)\mathrm{Sp}(1) = \mathrm{Sp}(n) \times_{\mathbb{Z}_2} \mathrm{Sp}(1)$, where we identify $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ with subgroups of $\mathrm{GL}(4n, \mathbb{R})$ so that

$$\mathrm{Sp}(n)\mathrm{Sp}(1) \subset \mathrm{SO}(4n) \subset \mathrm{GL}(4n, \mathbb{R}).$$

An oriented, connected, irreducible, Riemannian $4n$-manifold M is called a *quaternion-Kähler manifold*, $n \geq 2$, if its linear holonomy is contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$. We shall call M *positive* if its metric is complete and has positive scalar curvature. When $n = 1$ we add the condition that the manifold M must be Einstein and self-dual, since $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$.

A quaternion-Kähler manifold is Einstein, which renders three different types according to the scalar curvature being positive, negative or zero. The quaternion-Kähler manifolds with zero scalar curvature happen to be locally hyperkähler.

In the case of positive scalar curvature, Wolf showed in [33] that each compact centerless Lie group G is the isometry group of a quaternion-Kähler symmetric space given as the conjugacy class of a copy of $\mathrm{Sp}(1)$ in G, determined by a highest root of G. They are called "Wolf spaces" and are the only known examples with complete metrics of positive scalar curvature. Moreover, we know that there are only finitely many positive quaternion-Kähler manifolds for each n (cf. [20]). These facts have given some support to the following conjecture:

Conjecture 1. *Every positive quaternion-Kähler manifold is isometric to a* (*symmetric*) *Wolf space.*

Two decades ago, Hitchin proved in [16] that this conjecture is true in 4 dimensions ($n = 1$), namely that the 4-manifold must be isometric to S^4 or $\mathbb{CP}^2$. A decade ago, Poon and Salamon proved it in 8 dimensions ($n = 2$, see [27]). They proved that a positive 8-dimensional quaternion-Kähler manifold M is isometric to the quaternionic projective space $\mathbb{HP}^2$, or the complex Grassmannian $\mathbb{G}\mathrm{r}_2(\mathbb{C}^4)$, or the exceptional space $G_2/\mathrm{SO}(4)$. They carried out a careful study of the standard twistor space Z of M as a polarized algebraic variety, and were able to pin down the few candidates of polarized varieties that can occur as twistor spaces. In the last few years, more supporting evidence for the conjecture has been produced, such as the results in [20, 4, 3, 26].

By using Theorem 1, we are now able to produce a proof of the conjecture in 12 dimensions ($n = 3$). Theorem 2 states that a positive quaternionic Kähler 12-manifold must be a Wolf space.

Our approach is very different from those in [16, 27], since the main ingredients turn out to be topological and differential geometric, rather than algebraic geometric. Moreover, the study is carried out almost entirely on the (Riemannian) quaternion-Kähler manifolds, not on their twistor spaces.

Part II is organized as follows: In §§2.1 we review preliminaries of quaternionic Kähler geometry. In §§2.2 we review the index theory of certain elliptic operators on quaternion-Kähler manifolds. In §§2.3 we give the proof of Theorem 2, and some immediate applications.

2.1 Preliminaries on quaternion-Kähler geometry

The existence of the Sp(3)Sp(1)-structure induces an isomorphism

$$T^*M \otimes \mathbb{C} \cong E \otimes H,$$

where E and H denote the locally defined vector bundles over M associated to the standard (faithful) complex representations of Sp(3) and Sp(1) on $E = \mathbb{C}^6$ and $H = \mathbb{C}^2$, respectively.

Consider the globally defined bundle $Z = \mathbb{P}(H) \xrightarrow{\pi} M$ with fiber $\mathbb{CP}^1$, usually called the *standard twistor space* of M. It parameterizes orthogonal almost-complex structures at each tangent space of M, and Z itself is a complex manifold. When the scalar curvature is positive, Z has a canonical Kähler structure.

The quaternionic structure of M is characterized by a 4-form u coming from the second Chern class of the quaternionic line bundle H, i.e., $u = -c_2(H)$. The multiple $4u = -c_2(S^2H)$ is integral and nondegenerate [28], and we shall call it the *quaternionic class*. There is a complex line bundle $L \longrightarrow Z$ such that locally $\pi^*H \cong L^{1/2} \oplus L^{-1/2}$ and $l^2 = 4\pi^*u$, where $l = c_1(L)$. Such a class l is represented by a Kähler form on Z, which gives the following result to be used in Theorem 2, communicated to the authors by S. Salamon (see [22, 29] for similar results):

Lemma 3. *Let M be a compact connected quaternion-Kähler 12-manifold of positive scalar curvature. The symmetric bilinear form Q on $H^4(M)$ defined by*

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge (4u),$$

where $\alpha, \beta \in H^4(M)$, is positive definite.

Proof. This result follows from the Hodge-Riemann bilinear relations [11] on the twistor space Z of M. Indeed, since Z is a $\mathbb{CP}^1$-fibration over M

$$H^4(Z) \cong \pi^*(H^4(M)) \oplus \{l\} \otimes \pi^*(H^2(M))$$

and, at the same time,

$$H^4(Z) = P^{2,2} \oplus L^1(P^{1,1}) \oplus \{l^2\}$$

is the Lefschetz decomposition into primitive subspaces, where L^1 represents the map given by wedging with the Kähler form. Clearly, $\pi^*(H^4(M)) \cong P^{2,2} \oplus \{l^2\}$ so that by twistor transform

$$\int_M \alpha \wedge \beta \wedge (4u) = \frac{1}{2} \int_Z \pi^*(\alpha) \wedge \pi^*(\beta) \wedge l^3,$$

for $\alpha, \beta \in H^4(M)$, and the result follows from the fact that this bilinear form is positive definite on $P^{2,2} \oplus \{l^2\}$. q.e.d.

Remark. Here, we have restricted the statement to dimension 12 although it is valid in higher degrees and dimensions.

Given the content of Conjecture 1, it is clear that the (real) dimension d of the isometry group G of M is a key differential geometric invariant to be determined. Indeed, in 8 dimensions, d played a central role in the classification and it was seen to be the only relevant characteristic number of M. Similarly, d also plays a central role in 12 dimensions, as well as the pairing

$$\mathbf{v}(M) = \left\langle (4u)^3, [M] \right\rangle$$

which we shall call the *quaternionic volume*, and which now constitutes a separate parameter.

2.2 Quaternionic spinors

Let Δ be the 2^6-dimensional faithful spin representation of $\mathrm{Spin}(12)$. The representation Δ splits as

$$\Delta = \Delta_+ \oplus \Delta_-,$$

where $\Delta_\pm$ are two copies of the 2^5-dimensional irreducible representation of $\mathrm{Spin}(11) \subset \mathrm{Spin}(12)$. Clifford multiplication of an element of Δ_+ by an element of $T = T^*M$ gives an element of Δ_-.

There is an anti-linear mapping

$$\sigma\colon \Delta \longrightarrow \Delta \quad \text{with} \quad \sigma^2 = (-1)^3 = -1$$

which allows us to see Δ as a complex space underlying a quaternionic one ($\sigma = j$). Combining σ with an invariant Hermitian metric on Δ, we get an equivariant isomorphism $\Delta \cong \Delta^*$.

The irreducible representations of $\mathrm{Sp}(1)$ are the symmetric tensor powers S^qH of $H = \mathbb{C}^2$, with $\dim(S^qH) = q+1$. The tensor products of these representations behave according to the Clebsch-Gordan formula

$$S^jH \otimes S^kH \cong \sum_{r=0}^{\min(j,k)} S^{j+k-2r}H,$$

and to the K-theory formula

$$(H-2)^{\otimes m} = \sum_{j=0}^{m} (-1)^j \left\{ \binom{2m}{j} - \binom{2m}{j-2} \right\} S^{m-j}H, \tag{17}$$

which can be easily proved by induction.

We can define the Dirac operator with coefficients in S^qH by (cf. [20])

$$D(S^qH)\colon \Gamma(\Delta_+ \otimes S^qH) \longrightarrow \Gamma(\Delta_- \otimes S^qH),$$

with index

$$f(q) = \mathrm{ind} D(S^qH) = \left\langle \widehat{A}(M) \cdot \mathrm{ch}(S^qH), [M] \right\rangle, \tag{18}$$

provided that $q \geq 0$ and $3+q$ is even. The parity condition ensures that the corresponding coupled Dirac operator is *globally defined.* Moreover, $f(q)$ is a polynomial in q.

Theorem 5 ([20, 28]). *Let M be a 12-dimensional positive quaternion-Kähler manifold. Then*

$$\pi_1(M) = 0,$$

$$\pi_2(M) = \begin{cases} 0 & \textit{iff } M \textit{ is homothetic to } \mathbb{HP}^3, \\ \mathbb{Z} & \textit{iff } M \textit{ is homothetic to } \mathrm{Gr}_2(\mathbb{C}^5), \\ \textit{finite with 2-torsion} & \textit{otherwise.} \end{cases}$$

and M is spin if and only if M is homothetic to $\mathbb{HP}^3$.

The polynomial $f(q)$ satisfies

$$f(-q-2) = -f(q),$$

and

$$f(q) \quad = \quad \begin{cases} 0 & \textit{if } q = -3, -1, 1, \\ 1 & \textit{if } q = 3, \\ d = \dim(G) & \textit{if } q = 5. \end{cases}$$

2.3 Proof of the classification theorem

Note that Theorem 5 does not provide any relation between the parameters d and $\mathbf{v}$. Thus, the strategy of the proof is to pin down the pairs $(d, \mathbf{v})$ that can occur for actual positive quaternion-Kähler manifolds. Applying Theorem 5, we can assume that $\pi_2(M)$ is finite, i.e., $M \ncong \mathrm{Gr}_2(\mathbb{C}^5)$. We know that $d \geq 5$ (cf. [28]), so that M admits smooth S^1 actions.

We know that $\mathbf{v}$ is an integer and $\mathbf{v} \geq 1$ since $4u = -c_2(S^2H)$ is an integral cohomology class. Let us consider the (locally defined) virtual bundle $H - 2$ whose Chern character is

$$\mathrm{ch}(H-2) = u + \frac{1}{12}u^2 + \frac{1}{360}u^3,$$

since $-u = c_2(H)$. Clearly,

$$\left\langle \widehat{A}(M) \cdot \mathrm{ch}((H-2)^{\otimes 3}), [M] \right\rangle = \frac{\mathbf{v}}{64},$$

$$\left\langle \widehat{A}(M) \cdot \mathrm{ch}((H-2)^{\otimes 4}), [M] \right\rangle = 0,$$

and

$$\left\langle \widehat{A}(M) \cdot \mathrm{ch}((H-2)^{\otimes 5}), [M] \right\rangle = 0.$$

Applying formula (17) to these identities, gives the following system of equations:

$$\begin{aligned} \frac{\mathbf{v}}{64} &= f(3) - 6f(2) + 14f(1) - 14f(0), \\ 0 &= f(4) - 8f(3) + 27f(2) - 48f(1) + 42f(0), \\ 0 &= f(5) - 10f(4) + 44f(3) - 110f(2) + 165f(1) - 132f(0). \end{aligned} \tag{19}$$

The key point in the proof is to find another zero of $f(q)$.

Lemma 4. *Let M be a positive quaternion-Kähler 12-dimensional manifold different from $\mathbb{G}\mathrm{r}_2(\mathbb{C}^5)$. Then*

$$f(0) = 0.$$

Proof. We know that $\pi_2(M)$ is finite and that M admits smooth S^1 actions. Thus, by Theorem 1

$$\widehat{A}(M) = 0,$$

which means $f(0) = 0$. q.e.d.

Remark. Clearly, the conclusion of Lemma 4 also holds for any positive quaternion-Kähler manifold different from the complex Grassmannian and which admits a smooth S^1 action.

The system (19) therefore reduces to

$$\mathbf{v} = \frac{12}{5}\, d - \frac{112}{5}. \tag{20}$$

This relation between d and $\mathbf{v}$ greatly reduces the problem of determining the possible values of $(d, \mathbf{v})$. In particular, it implies that $d \geq 11$, already hinting that the manifold must be homogeneous.

Hence, Equation (20) gives us the following list of integral pairs:

(i) $(d, \mathbf{v}) = (11, 4)$,

(ii) $(d, \mathbf{v}) = (16, 16)$,

(iii) $(d, \mathbf{v}) = (21, 28)$,

(iv) $(d, \mathbf{v}) = (26, 40)$,

(v) $(d, \mathbf{v}) = (31, 52)$,

(vi) $(d, \mathbf{v}) = (36, 64)$.

The pair (vi) must give rise to the quaternionic projective space, since the isotropy group K at any point of M must have dimension at least 24 and is contained in $\mathrm{Sp}(3)\mathrm{Sp}(1)$.

The pairs (i) and (ii) are ruled out since the quadratic form Q in Lemma 3 is positive definite, i.e.,

$$\mathbf{v}^2 - 64\mathbf{v} - 16\mathbf{v}d + 576 - 288d + 36d^2 < 0.$$

The pairs (iv) and (v) are ruled out since there are no semi-simple Lie groups of the given dimension and rank less than or equal to 3, the upper bound found by Bielawski [4].

The pair (iii) is the only one in which we have to do more work. Since $d = 21$, the isotropy group K at any point must have dimension $9 \leq \dim(K) \leq 21$. By [28], the curvature tensor R of M splits as $R = tR_0 + R_1$, where R_0 is the curvature tensor of quaternionic projective space, t is the scalar curvature, and R_1 is a section of S^4E. Therefore, in order to measure the covariant derivative ∇R it suffices to look at ∇R_1 as a section of $S^5E \otimes H$, which must be invariant under K.

Consider the map given by the composition of the isotropy representation and projection onto the second factor

$$\mathrm{Lie}(K) \longrightarrow \mathfrak{sp}(3) \oplus \mathfrak{sp}(1) \longrightarrow \mathfrak{sp}(1).$$

There are two cases: Either the map is surjective or is identically zero. By checking the possible groups K, we see that if the map is surjective, there are no groups K with invariants in $S^5E \otimes H$, so $\nabla R_1 = 0$. On the other hand, if the map is zero, the only candidate for K with possible invariants in $S^5 \otimes H$ is $\mathrm{Sp}(2)$, so that the orbits of G on M have dimension 11. This is not possible due to the classification of cohomogeneity one quaternion-Kähler manifolds in [8, 25]. q.e.d.

As we have seen, the new vanishing Theorem 1 has a powerful application to geometry. The keys to the proof of the vanishing are the rigidity property under smooth S^1 actions and the dimension of the fixed point manifolds of S^1. This situation provides strong evidence for

Conjecture 1 since it is feasible to gain very detailed information on fixed point manifolds of S^1 actions (see [3, 26]) in the quaternion-Kähler set-up. It is therefore feasible to obtain further vanishing theorems on these manifolds, which will completely pin down their topology and geometry (see Corollary 1).

We conclude by simply stating immediate applications of our result to certain manifolds which fiber over positive quaternion-Kähler 12-manifolds. By using [19, 10] we have the following corollaries:

Corollary 2. *Let Z be a (compact) Fano contact manifold of complex dimension 7, which admits a Kähler-Einstein metric. Then Z is a homogeneous space, isometric to the twistor space of a Wolf space.*

Corollary 3. *Let S be a compact regular 3-Sasakian 15-manifold. Then S is homogeneous.*

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