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COHOMOLOGY THEORY IN BIRATIONAL GEOMETRY

CHIN-LUNG WANG

Abstract

This is a continuation of [9], where it was shown that K-equivalent complex projective manifolds have the same Betti numbers by using the theory of p-adic integrals and Deligne's solution to the Weil conjecture. The aim of this note is to show that with a little more book-keeping work, namely by applying Faltings' p-adic Hodge Theory, our p-adic method also leads to the equivalence of Hodge numbers — a result which was previously known via motivic integration.

1. Introduction

Mori's minimal model theory has proven to be important in various geometric problems and is also important in our philosophical view point of birational geometry. In order to understand the relation between birational but not isomorphic minimal models in dimensions bigger than two, the notion of K-equivalence was developed to serve as a formal analogue, but with the advantage of independence from the existence problem of minimal models. This applies to the most interesting case of birational Calabi-Yau manifolds which was studied extensively in the last decade.

In dimension three, any birational map between minimal models can be decomposed as a composite of flops [7]. This gives very precise information needed in analyzing birational minimal threefolds. However, the only known proof of this result relies on detailed classification of

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terminal singularities, hence is out of reach in higher dimensions. Because of this, new dimension-free approaches are needed for the study of K-equivalent manifolds.

In [9], an integration formalism was formulated to compare numerical invariants of K-equivalent manifolds. In particular, the p-adic integral has been used to prove the equivalence of Betti numbers and the motivic integral was used to prove the equivalence of Hodge numbers (cf. [9], §5.5, [4], [1], [2]). More recently, by formally viewing the intersection theory as an integration theory, the author has shown that the complex elliptic genera are the most general Chern numbers invariant under K-equivalence [10].

The aim of this note is to present a proof of the equivalence of Hodge numbers along the original line using p-adic integral and the Weil conjecture. This was announced with a sketched proof in [11]. The new input needed here is the so-called p-adic Hodge Theory developed by Fontaine and Messing [6] and completed by Faltings [5]. It turns out that one may apply the existing theorems quite straightforwardly except a few minor technicalities. One is related to the Cĕbotarev density theorem that the zeta functions determine *only* the semisimplifications of the p-adic étale cohomology as Galois representation. The other is the reduction procedure from finitely generated fields over $\mathbb{Q}$ to number fields. Fortunately, these problems can all be handled by quite standard tricks and the proof goes through.

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2. *K*-partial ordering in a birational class

For a birational map $f : X \dashrightarrow X'$ between two $\mathbb{Q}$-Gorenstein (complex projective) varieties, we say that $X \leq_K X'$ (resp. $X <_K X'$) if there is a birational correspondence $(\phi, \phi') : Y \to X \times X'$ extending f with Y smooth, such that $\phi^* K_X \leq_{\mathbb{Q}} \phi'^* K_{X'}$ (resp. $<_{\mathbb{Q}}$) as divisors. These relations are easily seen to be independent of the choice of Y. Notice that $X \leq_K X'$ and $X \geq_K X'$ imply $X =_K X'$, that is $\phi^* K_X =_{\mathbb{Q}} \phi'^* K_{X'}$. In this case, we say that X and X' are K-equivalent.

In this K-partial ordering, divisorial contractions and flips will decrease its K-level while flops induce K-equivalence. It is easy to see that

K-equivalent terminal varieties are isomorphic in codimension one. In fact, more is true in general:

Theorem 2.1. *Let $f : X \dashrightarrow X'$ be a birational map between two varieties with canonical singularities. Suppose that the exceptional locus $Z \subset X$ is proper and that K_X is nef along Z, then $X \le_K X'$. Moreover, If X' is terminal then* $\operatorname{codim}_X Z \ge 2$.

(This is Theorem 1.4 in [9]; a better presentation of its proof is included in the appendix.) In particular, birational minimal models are K-equivalent.

Conjecture 2.2. *For complex projective manifolds X and X' with $X \le_K X'$, the canonical morphism $T : H^k(X, \mathbb{Q}) \to H^k(X', \mathbb{Q})$ induced from the graph closure $\overline{\Gamma}_f \subset X \times X'$ is a monomorphism which preserves the rational Hodge structures. More generally, T induces a monomorphism of motives, e.g., in the sense of Galois representations or in the sense of Chow motives.*

While it is of fundamental importance to study the cycle $\overline{\Gamma}_f \subset X \times X'$ directly, we instead restrict ourselves to a numerical version in this note. Namely, we prove in §4 the main result of this note:

Theorem 2.3. *Let X and X' be two K-equivalent complex projective manifolds. Then $h^{p,q}(X) = h^{p,q}(X')$ for all p, q. More precisely, if $X =_K X'$ is defined over a finitely generated field F over $\mathbb{Q}$, then for any prime ℓ,*

$$H^j_{\mathrm{et}}(X_{\overline{F}}, \mathbb{Q}_\ell)^{\mathrm{ss}} \cong H^j_{\mathrm{et}}(X'_{\overline{F}}, \mathbb{Q}_\ell)^{\mathrm{ss}}$$

as $\operatorname{Gal}(\overline{F}/F)$ *representations.*

3. *p*-adic integration and étale cohomology

We start by recalling the construction in [9]. We will assume that X and X' are smooth projective and K-equivalent, though the construction also works for log-terminal varieties. Take an integral model of the K-equivalence diagram over $\operatorname{Spec} S$ with S a finitely generated

$\mathbb{Z}$-algebra:

$$\begin{tikzcd} & Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] \arrow[dd] & \\ X \arrow[dr] & & X' \arrow[dl] \\ & \operatorname{Spec} S & \end{tikzcd}$$

For *almost all* maximal ideals P in S, in fact a Zariski open dense set in the maximal spectrum of S, we have good reductions of X, X', Y, ϕ and ϕ'. In such cases, let $R = \hat{S}_P$ be the completion of S at P with residue field $k_P := R/P \cong \mathbb{F}_q$, $q = p^r$ for some r. For ease of notation, we use the same symbol to denote the corresponding object over $\operatorname{Spec} R$. Let $\{U_i\}$ be a Zariski open cover of X such that $K_X|_{U_i}$ is trivial for each i with generator Ω_i a regular n-form on U_i, where $n = \dim X$. Then for a compact open subset $A \subset U_i(R) \subset X(R)$, we define its p-adic measure by

$$\mu_X(A) \equiv \int_A |\Omega_i|_p.$$

This is independent of the choice of the generator Ω_i. The p-adic measure of $X(R)$ and $X'(R)$ are the same by the *change of variable formula* and $X =_K X'$. By a direct extension of *Weil's formula* [12], we see in [9] that (let $\overline{X}$ be the special fiber over $\operatorname{Spec} \mathbb{F}_q$)

$$\mu_X(X(R)) = \frac{|\overline{X}(\mathbb{F}_q)|}{q^n}.$$

By applying this to finite extensions of $\mathbb{F}_q$, we conclude that X and X' have the same local zeta functions $Z(\overline{X}, t) = Z(\overline{X}', t)$ with

$$Z(\overline{X}, t) := \exp\left(\sum\nolimits_{k \geq 1} |\overline{X}(\mathbb{F}_{q^k})| \frac{t^k}{k} \right).$$

Knowing this for one P already allows us to apply Grothendieck-Deligne's solution to the celebrated *Weil conjecture* [3] to conclude that K-equivalent manifolds have the same Betti numbers. Indeed, let

$$P_j(t) = \det\left(1 - t \operatorname{Fr}_q \,|\, H^j_{\text{et}}(\overline{X}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\right)$$

be the characteristic polynomial of the Frobenius map Fr_q acting on the ℓ-adic étale cohomologies of $\overline{X}_{\overline{\mathbb{F}}_q}$ for any fixed $\ell \neq p$, then Grothendieck's

Lefschetz trace formula implies that

$$Z(\overline{X}, t) = \frac{P_1(t) \ldots P_{2n-1}(t)}{P_0(t) P_2(t) \ldots P_{2n}(t)}.$$

Moreover, Deligne showed that $P_j(t) \in \mathbb{Z}[t]$ which is independent of the choices of ℓ and all roots of $P_j(t)$ have absolute value $q^{-j/2}$ (*Riemann Hypothesis*). This clearly implies that $\overline{X}$ and $\overline{X}'$ have the same ℓ-adic Betti numbers, hence by comparison theorem X and X' have the same ordinary Betti numbers.

4. Global ℓ-adic representations

In this section we prove Theorem 2.3 using two technical devices discussed in §5 and §6. The basic observation here is that in fact more is true by putting together the information provided by all such P's. Let us assume that $X =_K X'$ are defined over S such that the quotient field F of S is a number field. The general case can be reduced to the number field case by the standard trick in §6. Under this assumption we then know that X and X' have good reductions and have the same local zeta functions for all $P \in \operatorname{Spec} S \backslash A$ with A a finite set.

Consider the following two semisimplifications of ℓ-adic cohomologies

$$H^j_{\mathrm{et}}(X_{\overline{F}}, \mathbb{Q}_\ell)^{\mathrm{ss}} \quad \text{and} \quad H^j_{\mathrm{et}}(X'_{\overline{F}}, \mathbb{Q}_\ell)^{\mathrm{ss}}$$

as (global) ℓ-adic representations of $\mathrm{Gal}(\overline{F}/F)$, denoted by ρ and ρ'. In the language of [8], these are *integral representations*, meaning that the associate local representations are *unramified* and have integral characteristic polynomial for the Frobenius generator of $\mathrm{Gal}(\bar{k}_P/k_P)$ for all but finitely many P. This is indeed the case by Deligne's result since for P with good reduction, the local ℓ-adic representation is exactly $H^j_{\mathrm{et}}(\overline{X}_{\bar{k}_P}, \mathbb{Q}_\ell)^{\mathrm{ss}}$ with characteristic polynomial $P_j(t)$ as before.

By the *Cěbotarev density theorem* ([8], Ch1, §2) that $\bigcup_{P \notin A} \mathrm{Gal}(\bar{k}_P/k_P)$ is dense in $\mathrm{Gal}\,(\overline{F}/F)/(\ker \rho \cap \ker \rho')$ and the fact that rational semisimple representations are characterized by their trace functions (characters), this implies that

$$H^j_{\mathrm{et}}(X_{\overline{F}}, \mathbb{Q}_\ell)^{\mathrm{ss}} \cong H^j_{\mathrm{et}}(X'_{\overline{F}}, \mathbb{Q}_\ell)^{\mathrm{ss}}$$

as $\mathrm{Gal}(\overline{F}/F)$ representations.

Remark 4.1. From the point of view of motive theory, the above argument is simply the one showing that L function is equivalent to the semisimplification of the corresponding Galois representations. Also there is a *semisimplicity conjecture* stating that the cohomological ℓ-adic representations are always semisimple.

Now we select a prime $P \in \operatorname{Spec} S$ with $\operatorname{char} k_P = \ell$. Let K be the completion of F at P, then by *base change theorem* we also get

$$H^j_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{\text{ss}} \cong H^j_{\text{et}}(X'_{\overline{K}}, \mathbb{Q}_\ell)^{\text{ss}}$$

as $\operatorname{Gal}(\overline{K}/K)$ representations (usually highly ramified!) — here we do not even need to require X or X' to have good reductions at P. By Faltings' *Hodge-Tate decomposition theorem* in the next section, this then implies the equivalence of $\mathbb{Q}_\ell$ (and hence $\mathbb{Q}$) Hodge numbers.

5. p-adic Hodge theory

In this section we recall the p-adic Hodge Theory that we are going to apply. Following the usual convention, we will switch the prime number ℓ to p.

Let X a smooth projective manifold over a p-adic field K. Let $G = \operatorname{Gal}(\overline{K}/K)$ and $\mathbb{C}_p$ be the completion of $\overline{K}$. Then there exists a natural G-equivariant isomorphism [5], the so-called Hodge-Tate decomposition:

$$\bigoplus\nolimits_i \big(\mathbb{C}_p \otimes_K H^{m-i}(X_K, \Omega^i)(-i)\big) \cong \mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p),$$

where G acts on $H^{m-i}(X_K, \Omega^i)$ trivially and on the right hand side diagonally. Here (i) means the Tate twist by i-th power of cyclotomic character $(\varprojlim \mu_{p^n})^{\otimes i}$. Since $\mathbb{C}_p^G = K$ and $\mathbb{C}_p(i)^G = 0$ for $i \neq 0$, it is clear that

$$h^{i,m-i} = \dim_K \big(\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)(i)\big)^G.$$

Now the key observation is that the semisimplification is already enough to determine the Hodge numbers.

Proposition 5.1. *In the notation as above, then*

$$h^{i,m-i} = \dim_K \big(\mathbb{C}_p \otimes_{\mathbb{Q}_p} H^m_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)^{\text{ss}}(i)\big)^G.$$

Proof. Let $H^m_{\rm et}(X_{\overline{K}}, \mathbb{Q}_p) =: V = V_0 \supset V_1 \supset \cdots \supset V_k$ be a filtration of G-submodules such that V_j/V_{j+1}'s are simple G-modules. Then by definition $V^{\rm ss} = \bigoplus_j V_j/V_{j+1}$. Since $\mathbb{C}_p$ is a flat G-module and the functor of taking G-invariants $A \mapsto A^G$ is left exact, simple induction shows that

$$\dim_K(\mathbb{C}_p \otimes V)^G \le \dim_K(\mathbb{C}_p \otimes V^{\rm ss})^G.$$

The same inequality applies to $V(i)$ as well for any $i \in \mathbb{Z}$, hence

$$\begin{aligned}\sum_i h^{i,m-i} &= \sum_i \dim_K(\mathbb{C}_p \otimes V(i))^G \\ &\le \sum_i \dim_K(\mathbb{C}_p \otimes V^{\rm ss}(i))^G \le \dim_{\mathbb{Q}_p} V^{\rm ss},\end{aligned}$$

where the last inequality is a general fact about G-modules. Now since both ends are equal to $\dim_{\mathbb{Q}_p} V$, all the inequalities are equalities. In particular,

$$h^{i,m-i} = \dim_K \big(\mathbb{C}_p \otimes V(i)\big)^G = \dim_K \big(\mathbb{C}_p \otimes V^{\rm ss}(i)\big)^G.$$

q.e.d.

Remark 5.2 (Suggested by C.-L. Chai)**.** Proposition 5.1 follows immediately from the fact that Hodge-Tate representations form a Tannakian category, in particular it is stable under extensions.

6. Deformation to the number field case

The aim of this section is to explain that given a finite number of complex projective varieties and a finite number of morphisms between them, denoted by $\mathcal{X}$, one may always *deform* the system $\mathcal{X}$ a little bit to make it defined over a number field F. Moreover, when the issue of smoothness is imposed, one may find a $\mathbb{Z}$-algebra S integral over $\mathbb{Z}$ with F as its quotient field such that all the given objects are defined over S and the imposed smoothness condition is preserved under reductions over all but a finite number of primes.

As we will only apply it to the K-equivalence diagram here, we restrict ourself to this case for simplicity. We start with a model $\mathcal{X} \to \operatorname{Spec} S$ with S a finitely generated $\mathbb{Z}$-algebra. Namely, $(\phi, \phi') : Y \to X \times X' \to \operatorname{Spec} S$ with relation (E is a relative normal crossing divisor over $\operatorname{Spec} S$)

$$K_{Y/S} = \phi^* K_{X/S} + E = \phi'^* K_{X'/S} + E.$$

Consider a large enough number field F such that there exists an F-valued point $\eta : \operatorname{Spec} F \to \operatorname{Spec} S$ in the *regular values* of $\mathfrak{X} \to \operatorname{Spec} S$. Then, by considering the resulting fiber diagram over η, we get a K-equivalence diagram $\mathfrak{X}_\eta$ over F by taking base change of the above relation to the fiber over η. By selecting an integral model of $\mathfrak{X}_\eta$ again then we are done.

Notice that these algebraic (number field) points η are dense in $\operatorname{Spec} S$ and the union of Galois groups $\operatorname{Gal}(\overline{F}/F)$ among all η is dense in $\operatorname{Gal}(\overline{F}_S/F_S)$ for F_S the quotient field of S. This allows us to deduce that

$$H^j_{\mathrm{et}}(X_{\overline{F}_S}, \mathbb{Q}_\ell)^{\mathrm{ss}} \cong H^j_{\mathrm{et}}(X'_{\overline{F}_S}, \mathbb{Q}_\ell)^{\mathrm{ss}}$$

as $\operatorname{Gal}(\overline{F}_S/F_S)$ representations from the number field case.

Remark 6.1. For people not familiar with this procedure, the following example provides a trivial illustration. Let e and π be any two algebraically independent numbers. Take say two plane conics X and X' defined by $2x^2 + ey^2 + z^2 = 0$ and $\pi x^2 + y^2 + z^2 = 0$ with birational map (in fact an isomorphism) given by $(x, y, z) \mapsto \left(\sqrt{\frac{\pi}{2}}x, \frac{1}{\sqrt{e}}y, z\right)$. This system is defined over the ring $S = \mathbb{Z}\left[e, \pi, \frac{1}{\sqrt{e}}, \sqrt{\frac{\pi}{2}}\right] \cong \mathbb{Z}[u, v, w, s]/(uw^2 - 1, v - 2s^2)$. Take $F = \mathbb{Q}$, an F-rational point of $\operatorname{Spec} S$ could be taken to be $\eta = (u, v, w, s) = (1, 2, 1, 1)$. Over this point η, both X and X' are *deformed* into $2x^2 + y^2 + z^2 = 0$ and the birational map is *deformed* into the identity map.

7. Appendix: Proof of Theorem 2.1

Let us recall the proof briefly. Let $(\phi, \phi') : Y \to X \times X'$ be a resolution of f so that the union of the exceptional set of ϕ and ϕ' is a normal crossing divisor of Y. Let $K_Y =_{\mathbb{Q}} \phi^* K_X + E =_{\mathbb{Q}} \phi'^* K_{X'} + E'$. So

$$\phi'^* K_{X'} =_{\mathbb{Q}} \phi^* K_X + F, \quad \text{with } F := E - E'.$$

It suffices to show that $F \geq 0$. Let $F = \sum_{j=0}^{n-1} F_j$ with $\dim \phi'(\operatorname{Supp} F_j) = j$. We will show that $F_j \geq 0$ for $j = n-1, n-2, \ldots, 1, 0$ inductively. As E' is ϕ'-exceptional, $F_{n-1} \geq 0$ is clear. Suppose that we have already shown that $F_j \geq 0$ for $j \geq k + 1$.

Consider the surface $S_k := H^{n-2-k}.\phi'^* L^k$ on Y where H is very ample on Y and L is very ample on X'. We get a relations of divisors on S_k:

$$\phi'^* K_{X'}|_{S_k} =_{\mathbb{Q}} \phi^* K_X|_{S_k} + a - b,$$

where $H^{n-2-k}.\phi'^* L^k.F = a - b$ with both a and b effective. Notice that b can only come from F_k since $\sum_{j \geq k+1} F_j \geq 0$ and $L^k \cap \phi'(F_j) = \emptyset$ for $j < k$. Now we look at

$$b.\phi'^* K_{X'} =_{\mathbb{Q}} b.\phi^* K_X + b.a - b^2.$$

The left hand side is always zero since $\phi'(b) \subset L^k \cap \phi'(F_k)$ is zero dimensional. Moreover, since $\phi'^* K_{X'} =_{\mathbb{Q}} \phi^* K_X$ on $\phi^{-1}(X \backslash Z)$, we must have that $\phi(\operatorname{Supp} F) \subset Z$. In particular, $b.\phi^* K_X \geq 0$. It is also clear that $b.a \geq 0$. However, if $b \neq 0$ then it is a nontrivial combination of ϕ' exceptional curves in S_k. By the Hodge index theorem for surfaces we then have that $b^2 < 0$, a contradiction. So $b = 0$ and $F_k \geq 0$. The codimension statement is easy and we omit its proof.

References

[1] V. Batyrev, *On the Betti numbers of birationally isomorphic projective varieties with trivial canonical bundles*, math.AG/9710020.

[2] V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, math.AG/9711008.

[3] P. Deligne, *La conjecture de Weil* I, II, IHES Publ. Math. **43** (1974) 273–307; **52** (1980) 313–428.

[4] J. Denef & F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999) 201–232.

[5] G. Faltings, *p-adic Hodge thoery*, J. AMS **1**(**1**) (1988) 255–299.

[6] J.-M. Fontaine & W. Messing, *p-adic periods and p-adic étale cohomology*, Contemp. Math. **67** (1987) 179–207.

[7] J. Kollár, *Flops*, Nagoya Math. J. **113** (1989) 15–36.

[8] J.-P. Serre, *Abelian ℓ-adic Representations and Elliptic Curves*, W.A. Benjamin, Inc., 1968.

[9] C.-L. Wang, *On the topology of birational minimal models*, J. Differential Geom. **50** (1998) 129–146.

[10] C.-L. Wang, *K-equivalence in birational geometry and characterizations of complex elliptic genera*, to appear in J. Alg. Geom., 2002.

[11] C.-L. Wang, *Cohomology theory in birational geometry*, Report on NSC Project 89-2115-M-002-012, Taiwan 2000.

[12] A. Weil, *Adèle and Algebraic Groups*, Prog. Math., **23**, Birkhauser, Boston 1982.

National Tsing-Hua University, Hsinchu, Taiwan