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ON THE ZARISKI CLOSURE OF THE LINEAR PART OF A PROPERLY DISCONTINUOUS GROUP OF AFFINE TRANSFORMATIONS

H. ABELS, G.A. MARGULIS & G.A. SOIFER

Abstract

Let Γ be a subgroup of the group of affine transformations of the affine space $\mathbb{R}^{2n+1}$. Suppose Γ acts properly discontinuously on $\mathbb{R}^{2n+1}$. The paper deals with the question which subgroups of $\mathrm{GL}(2n+1, \mathbb{R})$ occur as Zariski closure $\overline{\ell(\Gamma)}$ of the linear part of such a group Γ. The two main results of the paper say that $\mathrm{SO}(n+1, n)$ does occur as $\overline{\ell(\Gamma)}$ of such a group Γ if n is odd, but does not if n is even.

1. Introduction

A well known classical theorem due to Bieberbach says that every discrete group Γ of isometries of the n-dimensional Euclidean space $\mathbb{R}^n$ with compact quotient $\Gamma \setminus \mathbb{R}^n$ contains a subgroup of finite index consisting of translations. Hence such a group Γ is virtually abelian, i.e., Γ contains an abelian subgroup of finite index.

Let us now look at the group of affine transformations instead of the group of isometries of $\mathbb{R}^n$. More precisely, let $G_n = \mathrm{Aff}(\mathbb{R}^n)$ denote the group of affine transformations of $\mathbb{R}^n$. The group G_n is the semidirect product $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with its group of translations. A subgroup Γ of G_n is said to act *properly discontinuously* on $\mathbb{R}^n$ if for every compact subset K of $\mathbb{R}^n$ the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ of recurrencies is finite. If a discrete group Γ consists of isometries then Γ acts properly on $\mathbb{R}^n$. But this is not true for all discrete subgroups of G_n, e.g., for an infinite discrete subgroup of $\mathrm{GL}_n(\mathbb{R})$.

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A subgroup Γ of G_n will be called *crystallographic* if Γ acts properly discontinuously on $\mathbb{R}^n$ and the orbit space $\Gamma \setminus \mathbb{R}^n$ is compact. In [4] Auslander conjectured that every crystallographic subgroup Γ of G_n is virtually solvable, i.e., contains a solvable subgroup of finite index. In [19] Milnor asked whether Auslander's conjecture is true without the assumption that the orbit space $\Gamma \setminus \mathbb{R}^n$ be compact. Milnor's question has a positive answer for $n \leq 2$ (easy for $n = 2$, trivial for $n = 1$). But for $n = 3$ the answer to Milnor's question is negative. In fact, the second named author proved that there is a nonabelian free subgroup Γ of $\mathrm{Aff}(\mathbb{R}^3)$ acting properly discontinuously on $\mathbb{R}^3$ ([15, 16]). On the other hand, D. Fried and W. Goldman [11] proved Auslander's conjecture for $n = 3$ using cohomological arguments. For higher dimensions the existing results confirming the Auslander conjecture are proved under the assumption that the linear part $l(\Gamma)$ of Γ belongs to some special subgroup of $\mathrm{GL}_n(\mathbb{R})$ where $l : G_n \to \mathrm{GL}_n(\mathbb{R})$ denotes the natural homomorphism ([12, 14]). For a survey on existing results see [1].

The two main results of the present paper are:

Theorem A. *For n even there is no properly discontinuous subgroup Γ of G_{2n+1} with linear part $l(\Gamma)$ Zariski dense in* $\mathrm{SO}(n+1, n)$.

Theorem B. *For n odd there are properly discontinuous free subgroups Γ of G_{2n+1} with linear part $l(\Gamma)$ Zariski dense in* $\mathrm{SO}(n+1, n)$.

Theorem A, its proof and the statement were contained in the preprint [17] of Margulis circulated in 1991. The proof of Theorem B is based on our results [2] and the strategy is essentially the same as in [15, 16]. The results of the present paper and more were announced in [3].

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2. A sufficient condition for failure of proper discontinuity

In this section we prove a general criterion for certain pairs of affine transformations to generate a subgroup of the affine group which is not properly discontinuous. Recall that $G_n = \mathrm{Aff}(\mathbb{R}^n)$ and that $l : G_n \longrightarrow \mathrm{GL}_n(\mathbb{R})$ is the natural epimorphism. For any $g \in G_n$ let us consider the decomposition of $\mathbb{R}^n$ into a direct sum of the $l(g)$-invariant linear subspaces

$$\mathbb{R}^n = A^-(g) \oplus A^0(g) \oplus A^+(g), \tag{2.1}$$

where $A^-(g), A^0(g)$ and $A^+(g)$ are determined by the condition that their sum is $\mathbb{R}^n$ and all eigenvalues of the restriction $l(g) \mid A^-(g)$ (resp. $l(g) \mid A^0(g),\ l(g) \mid A^+(g)$) are of modulus less than 1 (resp. equal to 1, greater than 1). We set $D^-(g) = A^-(g) \oplus A^0(g)$ and $D^+(g) = A^+(g) \oplus A^0(g)$. It is clear that $A^0(g) = D^-(g) \cap D^+(g)$. Let us call an element $g \in G_n$ *pseudohyperbolic* if $\dim A^0(g) = 1$ and the only eigenvalue of $l(g) \mid A^0(g)$ is 1 (not -1). Let Ω be the set of pseudohyperbolic elements of G_n and let $\Omega_0 = \{g \in \Omega \mid g\,x \neq x \text{ for every } x \in \mathbb{R}^n\}$ be the subset of pseudohyperbolic fixed–point–free elements of G.

For $g \in \Omega$ there is exactly one g–invariant line C_g and C_g is parallel to $A^0(g)$. We call C_g the *axis* of g. The restriction of g to C_g is a parallel translation by a vector $\tau(g) \in A^0(g)$. The vector $\tau(g)$ is the $A^0(g)$-component with respect to the decomposition (2.1) of $g\,x - x$ for any $x \in \mathbb{R}^n$. We call $\tau(g)$ the *translational part* of g. We have $g \in \Omega_0$ if and only if $\tau(g) \neq 0$.It is easy to see that

$$C_{g^m} = C_g \quad \text{and} \quad \tau\left(g^m\right) = m\,\tau(g) \ \text{ for } m \neq 0.$$

If $g \in \Omega_0$ then every $x \in D^+(g)$ has a unique decomposition $x = \lambda(x)\tau(g) + a(x)$ where $\lambda(x) \in \mathbb{R}$ and $a(x) \in A^+(g)$. We say that a vector $x \in D^+(g)$ is *positive with respect to* $g \in \Omega_0$ if $\lambda(x) > 0$.

Two elements $g_1, g_2 \in \Omega_0$ will be called *transversal* if

$$\mathbb{R}^n = A^+\left(g_1\right) \oplus D^+\left(g_2\right) = D^+\left(g_1\right) \oplus A^+\left(g_2\right)$$

or, equivalently, if $\dim\left(D^+\left(g_1\right) \cap D^+\left(g_2\right)\right) = 1$ and

$$\mathbb{R}^n = A^+\left(g_1\right) \oplus A^+\left(g_2\right) \oplus \left(D^+\left(g_1\right) \cap D^+\left(g_2\right)\right).$$

For two transversal elements $g_1, g_2 \in \Omega_0$, we say that g_1 and g_2 *form a positive pair* if the semiline $\{x \in D^+\left(g_1\right) \cap D^+\left(g_2\right) \mid x$ is positive with respect to $g_1\}$ coincides with the semiline $\{x \in D^+(g_1) \cap D^+(g_2) \mid x$ is positive with respect to $g_2\}$.

We shall denote by E_g^- and E_g^+ the affine subspaces containing C_g and parallel to $D^-(g)$ and $D^+(g)$, respectively. We will use throughout the whole paper the following notational conventions: affine subspaces corresponding to g will be denoted by attaching g as lower index, like C_g, E_g^+ etc., whereas vector subspaces corresponding to g will be denoted by putting g into brackets, like $A^+(g)$, $D^-(g)$, etc. Let us introduce an order on E_g^+ by saying that $u \in E_g^+$ is greater than $v \in E_g^+$ if $u - v$ is positive with respect to g. The condition that g_1 and g_2 form a positive pair is equivalent to the condition that the orders on $E_{g_1}^+ \cap E_{g_2}^+$ induced from $E_{g_1}^+$ and $E_{g_2}^+$ coincide.

Lemma 2.2. *Let $g_1, g_2 \in \Omega_0$ be two transversal elements which form a positive pair. Then there exist a compact set $K \subset \mathbb{R}^n$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that*

$$m_i \to \infty \quad \textit{and} \quad n_i \to \infty \quad \textit{as} \quad i \to \infty \quad \textit{and}$$

$$\left(g_1^{-m_i} g_2^{n_i} K\right) \cap K \neq \emptyset.$$

Proof. For $x \in \mathbb{R}^n$ and $g \in \Omega_0$, let $B_g^+(x)$ denote the affine subspace containing x and parallel to $A^+(g)$. Let us define the projection $P_g : E_g^+ \to C_g$ by the equality

$$P_g(x) = C_g \cap B_g^+(x) \text{ for } x \in E^+(g).$$

$C_g \cap B_g^+(x)$ is a point since $D^+(g) = A^+(g) \oplus A^0(g)$ and C_g is parallel to $A^0(g)$. The subspace $A^+(g)$ is $l(g)$-invariant. Therefore,

$$P_g(g\,x) = P_g(x) + \tau(g) \quad \text{for } x \in E_g^+. \tag{1}$$

Let d denote the Euclidean metric on $\mathbb{R}^n$. The absolute values of all eigenvalues of the restriction $l\left(g^{-1}\right) \mid A^+(g)$ are less than 1. This easily implies that there exist positive constants c and b such that for any $x \in E_g^+$ and $n \in \mathbb{N}^+$

$$d\left(g^{-n}x, P_g\left(g^{-n}x\right)\right) \leq \mathrm{c}e^{-bn} d(x, P_g(x)). \tag{2}$$

Let us fix a point $r(g) \in C_g$ and let $R(g)$ denote the interval $[r(g), r(g) + \tau_g) \subset C_g$. It follows from (1) that, for every $x \in E^+(g)$, there is a unique integer $k(x, g)$ such that

$$P_g\left(g^{k(x,g)}x\right) \in R(g).$$

Note that if x is greater than $r(g)$ in $E^+(g)$ then $k(x,g) \leq 0$.

Since g_1 and g_2 form a positive pair one can find a vector $v \in D^+(g_1) \cap D^+(g_2)$ such that v is positive with respect to both g_1 and g_2. Let us fix $x_0 \in E^+_{g_1} \cap E^+_{g_2}$ and set $x_i = x_0 + i\,v, i \in \mathbb{N}^+$. We have $x_i \in E^+_{g_1} \cap E^+_{g_2}$. Let

$$m_i = -k(x_i, g_1) \quad \text{and} \quad n_i = -k(x_i, g_2).$$

Let $v_1 \in A^0(g_1)$ and $v_2 \in A^0(g_2)$ be such that $v \in v_1 + A^+(g_1)$ and $v \in v_2 + A^+(g_2)$. Since v is positive with respect to both g_1 and g_2 we have that $v_1 = \lambda_1 \tau(g_1)$ and $v_2 = \lambda_2 \tau(g_2)$ where $\lambda_1 > 0$ and $\lambda_2 > 0$. Then it is easy to see that

$$\lim_{i\to\infty} \frac{i}{m_i} = \lambda_1 \quad \text{and} \quad \lim_{i\to\infty} \frac{i}{n_i} = \lambda_2. \tag{3}$$

It is clear that the growth of the functions $f_j(i) = d\left(x_i, P_{g_j}(x_i)\right), j = 1, 2$, is asymptotically linear. Thus (1), (2) and (3) imply that

$$\lim_{i\to\infty} d\left(g_1^{-m_i} x_i, R(g_1)\right) = \lim_{i\to\infty} d\left(g_2^{-n_i} x_i, R(g_2)\right) = 0. \tag{4}$$

We get from (4) that there exists a compact set $K \subset \mathbb{R}^n$ such that for all i

$$g_1^{-m_i} x_i \in K \quad \text{and} \quad g_2^{-n_i} x_i \in K. \tag{5}$$

We have $g_1^{-m_i} x_i = \left(g_1^{-m_i} g_2^{n_i}\right) g_2^{-n_i} x_i$. This and (5) imply that K is a compact set as claimed. q.e.d.

Corollary 2.3. *Let $g_1, g_2 \in \Omega_0$ be two transversal elements which form a positive pair. Then the subgroup of G_n generated by g_1 and g_2 is not properly discontinuous.*

Proof. Let $\{m_i\}$ and $\{n_i\}$ be as in Lemma 2.2. Then it is enough to check that the set $\left\{g_1^{-m_i} g_2^{n_i} \mid i \in \mathbb{N}^+\right\}$ is infinite. We may assume that $m_i < m_{i+1}$ and $n_i < n_{i+1}$ for every i since $m_i \to +\infty$ and $n_i \to +\infty$ as $i \to \infty$. Then for $i < j$ we have $g_1^{-m_i} g_2^{n_i} \neq g_1^{-m_j} g_2^{n_j}$ because otherwise $g_1^{m_j - m_i} = g_2^{n_j - n_i}$. But $g_1^m = g_2^n$ for positive m and n implies $A^+(g_1) = A^+(g_1^m) = A^+(g_2^n) = A^+(g_2)$ contradicting transversality. q.e.d.

3. Orientations on subspaces of $\mathbb{R}^{2n+1}$

In this section we introduce orientations on certain lines. In the next section we shall compare these orientations with the ones defined in the last section.

We fix a positive natural number n. Let $X = \mathbb{R}^{n+1}$ and $Y = \mathbb{R}^n$ and let B be the quadratic form on $X \times Y = \mathbb{R}^{2n+1}$ given by

$$B\left(x_1, \dots, x_{n+1}, y_1, \dots, y_n\right) = x_1^2 + \cdots + x_{n+1}^2 - y_1^2 - \cdots - y_n^2,$$

where $(x_1, \dots, x_{n+1}) \in X$ and $(y_1, \dots y_n) \in Y$. Consider the set Ψ of all maximal B-isotropic subspaces V of $\mathbb{R}^{2n+1}$. We have the two projections

$$p : \mathbb{R}^{2n+1} \longrightarrow X \quad \text{and} \quad q : \mathbb{R}^{2n+1} \longrightarrow Y.$$

The restriction of q to $V \in \Psi$ is a linear isomorphism $V \longrightarrow Y$. Hence if we fix an orientation on Y we have also fixed an orientation on each $V \in \Psi$. For $V \in \Psi$ let us denote the B-orthogonal complement of V by $V^\perp = \{z \in \mathbb{R}^{2n+1};\ B(z, V) = 0\}$. We have $V \subset V^\perp$ since V is B-isotropic. We also have

$$\dim V^\perp = 1 + \dim V = n + 1.$$

The restriction of p to $V^\perp$ is a linear isomorphism $V^\perp \longrightarrow X$. Hence if we fix an orientation on X we have also fixed an orientation on $V^\perp$ for each $V \in \Psi$. Thus we have orientations on both V and $V^\perp$ and we can single out one of the two halfspaces of $V^\perp \setminus V$ as positive by the following definition: if $(v_1, \dots, v_n)$ is a positively oriented basis of V then a vector v in $V^\perp$ not in V is *positive* (with respect to V) if the basis $(v_1, \dots, v_n, v)$ of $V^\perp$ is positively oriented. Accordingly, every affine line L in $V^\perp$ transversal to V is oriented: For two points x_0, x_1 of L we have $x_0 < x_1$ if $v = x_1 - x_0$ belongs to the positive half of $V^\perp \setminus V$.

If $V_1 \in \Psi$ and $V_2 \in \Psi$ are transversal then $V_1^\perp \cap V_2^\perp$ is a line which is transversal to both V_1 and V_2. So there are two orientations ω_1 and ω_2 on L, where ω_i is defined if we consider L as a line in $V_i^\perp$. We have:

Lemma 3.1. *The orientations defined above on L are the same if n is even and are opposite if n is odd.*

Proof. One can easily show that there exists a basis $(e_1, \dots, e_{2n+1})$ in $\mathbb{R}^{2n+1}$ such that V_1 (resp. V_2) is the linear span of $(e_1, \dots, e_n)$ (resp. of $(e_{n+1}, \dots, e_{2n})$) and the form B with respect to this basis is given by the equation

$$B\left(x_1, \dots, x_{2n+1}\right) = -x_1 x_{n+1} \cdots - x_n x_{2n} + x_{2n+1}^2.$$

Let us note that $V_1^\perp$ (resp. $V_2^\perp$) is the linear span of $(e_1, \dots, e_n, e_{2n+1})$ (resp. of $(e_{n+1}, \dots, e_{2n+1})$).

Let us denote by H_B the group of elements of $\mathrm{GL}_{2n+1}(\mathbb{R})$ preserving the form B, and let H_B^0 denote the connected component (in the Euclidean topology) of the identity in H_B. Direct calculations show that, for every $j, 1 \le j \le n$, and every $t \in \mathbb{R}$, the transformation $h_j(t)$ defined by

$$e_i \to e_i, e_{n+i} \to e_{n+i} \quad \text{if} \quad 1 \le i \le n,\ i \ne j$$

$$e_j \to \frac{e_j + e_{n+j}}{2} + \frac{\cos t(e_j - e_{n+j})}{2} + \frac{\sin t e_{2n+1}}{2}$$

$$e_{n+j} \to \frac{e_j + e_{n+j}}{2} - \frac{\cos t(e_j - e_{n+j})}{2} - \frac{\sin t e_{2n+1}}{2}$$

$$e_{2n+1} \to -\sin t(e_j - e_{n+j}) + \cos t e_{2n+1}$$

belongs to H_B. We have

$$h_j(\pi)e_i, = e_i, h_j(\pi)e_{n+i} = e_{n+i} \quad \text{if} \quad 1 \le i \le n,\ i \ne j,$$
$$h_j(\pi)e_j = e_{n+j}, h_j(\pi)e_{n+j} = e_j,$$
$$h_j(\pi)e_{2n+1} = -e_{2n+1}.$$

Therefore, if we denote $h_1(\pi) \dots h_n(\pi)$ by h_π, we have

$$h_\pi(e_i) = e_{n+i}, h_\pi(e_{n+i}) = e_i \quad \text{and} \quad h_\pi(e_{2n+1}) = (-1)^n e_{2n+1}. \tag{1}$$

Since $h_j(0) = \mathrm{Id}$ and $h_j(t) \in H_B$ depends continuously on t we have

$$h_\pi \in H_B^0. \tag{2}$$

The orientations on V and $V^\perp$ defined above depend continuously on $V \in \Psi$. Hence these orientations are invariant under the action of the group H_B^0. Now assuming that the basis $(e_1, \dots, e_n)$ (resp. $(e_1, \dots, e_n, e_{2n+1})$) is positively oriented in V_1 (resp. in $V_1^\perp$) we get from (1) and (2) that the basis $(e_{n+1}, \dots, e_{2n})$ (resp. $(e_{n+1}, \dots, e_{2n}, (-1)^n e_{2n+1})$) is positively oriented in V_2 (resp. in $V_2^\perp$). This immediately implies the desired statement. q.e.d.

4. The sign of an affine transformation

In this section we define the notion of sign for certain affine transformations g. It tells us whether the translational part of g is positive or negative with respect to the orientation of the preceding section. We translate this into the orientations of the second section thus obtaining a criterion for (non–) proper discontinuity. The proof of Theorem A will follow.

Let $H_B = \mathrm{SO}(n+1,n)$ be the special orthogonal group of the quadratic form B on $\mathbb{R}^{2n+1}$ and let G_B be the subgroup of those elements of G_{2n+1} whose linear part is in H_B. Let g be a pseudohyperbolic element of G_B, i.e., $g \in G_B \cap \Omega$, then $\dim A^0(g) = 1$ and $\dim A^-(g) = \dim A^+(g) = n$. The subspaces $A^-(g)$ and $A^+(g)$ are isotropic, thus $A^-(g)$ and $A^+(g)$ are in Ψ. In the preceding section we fixed orientations on $A^+(g)$ and on $D^+(g) = A^+(g) \oplus A^0(g) = (A^+(g))^\perp$ and similarly on $A^-(g)$ and on $D^-(g) = A^-(g) \oplus A^0(g) = (A^-(g))^\perp$. We define the *sign* $\sigma(g)$ of $g \in G_B \cap \Omega$ as $+1$ or -1 or 0 according to whether the translational part $\tau(g)$ of g is a positive or a negative vector of $D^+(g) \setminus A^+(g)$ or is zero. Recall that $g \in \Omega$ is fixed point free, i.e., $g \in \Omega_0$, iff $\tau(g) = 0$. For $g \in G_B \cap \Omega$ and $m > 0$ we have

$$\sigma(g^m) = \sigma(g) \tag{4.1}$$

since $\tau(g^m) = m\,\tau(g)$ and

$$\begin{aligned} A^\circ(g) &= A^\circ(g^m) \\ A^+(g) &= A^+(g^m) \\ A^-(g) &= A^-(g^m). \end{aligned}$$

For $g \in G_B \cap \Omega$ we have

$$\sigma\left(g^{-1}\right) = (-1)^{n+1}\sigma(g). \tag{4.2}$$

This follows from $\tau\left(g^{-1}\right) = -\tau(g) \in A^0(g) = A^0\left(g^{-1}\right)$ and Lemma 3.1 since $A^+(g)$ and $A^-(g) = A^+\left(g^{-1}\right)$ are transversal.

We shall need to compute $\sigma(g)$. Let $g \in G_B \cap \Omega$. Note that B restricts to a positive definite form on $A^0(g)$. Let $e^0(g)$ be the unique vector of $A^0(g)$ with $B\left(e^0(g), e^0(g)\right) = 1$ and positive with respect to the orientations of $A^+(g)$ and $D^+(g)$. Define $\alpha(g) \in \mathbb{R}$ by

$$\tau(g) = \alpha(g)e^0(g). \tag{4.3}$$

Then

$$\sigma(g) = \operatorname{sgn}\alpha(g). \tag{4.4}$$

We have

$$\alpha(g) = B\left(e^0(g), g\,x - x\right) \tag{4.5}$$

for every point x of the affine space. To see this recall the decomposition (2.1):

$$\mathbb{R}^n = A^-(g) \oplus A^0(g) \oplus A^+(g). \tag{4.6}$$

Now (4.5) follows from (4.3) since the translational part $\tau(g)$ is the $A^0(g)$–component of $g\,x - x$ for every point x and $A^0(g)$ is orthogonal to both $A^+(g)$ and $A^-(g)$. Note that

$$\alpha\left(g^m\right) = m\,\alpha(g) \quad \text{for} \quad m > 0 \tag{4.7}$$

and

$$\alpha\left(g^{-1}\right) = (-1)^{n+1}\alpha(g) \qquad , \tag{4.8}$$

which is proved like (4.2).

The following proposition is basic for our approach.

Proposition 4.9. *Let g_1, g_2 be transversal pseudohyperbolic elements of G_B. If $\sigma(g_1)\,\sigma\left(g_2^{-1}\right) < 0$ then the group generated by g_1 and g_2 is not properly discontinuous.*

Proof. Note that $g_1, g_2 \in G_B \cap \Omega_0$ since their signs are not zero. The proposition now follows from Corollary 2.3 and the following lemma.

Lemma 4.10. *Let g_1, g_2 be transversal elements of $G_B \cap \Omega_0$. If $\sigma(g_1)\,\sigma\left(g_2^{-1}\right) < 0$ then g_1 and g_2 form a positive pair.*

Proof. Given $g_i \in G_B \cap \Omega_0$ we have defined orientations on $V_i = A^+(g_i)$ and $V_i^\perp = D^+(g_i)$ and thus can talk about the positive half space of $V_i^\perp \setminus V_i$. The vector $e^0(g_i)$ defined before (4.3) is positive with respect to V_i. Let L be the line $V_1^\perp \cap V_2^\perp$. An element $\ell \neq 0$ in L is positive with respect to V_1 iff $(-1)^n\ell$ is positive with respect to V_2, by Lemma 3.1. Write

$$\begin{aligned} \ell &= v_1 + \beta_1\tau(g_1) \\ &= v_2 + \beta_2\tau(g_2), \end{aligned}$$

$v_i \in V_i, \beta_i \in \mathbb{R}$. Then g_1, g_2 form a positive pair iff $\beta_1\beta_2 > 0$. But

$$\tau(g_i) = \alpha(g_i)e^0(g_i).$$

Thus, ℓ is positive for V_1 iff $\beta_1\alpha(g_1) > 0$ iff $(-1)^n\ell = (-1)^n v_2 + (-1)^n\beta_2\alpha(g_2)e^0(g_2)$ is positive for V_2 iff $\beta_2(-1)^n\alpha(g_2) = -\beta_2\alpha(g_2^{-1}) > 0$, by (4.8). Thus g_1, g_2 form a positive pair iff $\alpha(g_1)\alpha(g_2^{-1}) < 0$, which proves our claim in view of (4.4). q.e.d.

Proof of Theorem A. Let Γ be a subgroup of G_n and let $H(\Gamma)$ be the Zariski closure of its linear part $\ell(\Gamma)$. Suppose we have $H(\Gamma)^\circ = H_B^\circ$ for the connected components with respect to the Euclidean topology. It then follows from the main algebraic result of [13] that Γ contains an element γ_0 such that $\ell(\gamma_0)$ is pseudohyperbolic. To see this look at the natural representation ρ of H_B on $\bigwedge^n \mathbb{R}^{2n+1}$. The representation ρ restricted to H_B° is irreducible. Then $\rho(h)$ is proximal in the sense of [2] iff there is only one eigenvalue of h of modulus 1, counting multiplicities, see [2, 5.1]. The main algebraic result of [13] implies that $\rho(\ell(\Gamma))$ contains a proximal element since $\rho(H_B^\circ)$ does and the representation $\rho \mid H_B^\circ$ is irreducible, see [2, Theorem 4.1].

We have $\gamma_0 \in \Omega_0$ if Γ is properly discontinuous because otherwise γ_0 has a fixed point and hence the cyclic subgroup of Γ generated by γ_0 does not act properly discontinuously on $\mathbb{R}^{2n+1}$. We can find an element $\gamma \in \Gamma$ such that

$$(A^+(\gamma_0) \cup A^-(\gamma_0)) \cap \ell(\gamma)(A^+(\gamma_0) \cup A^-(\gamma_0)) = 0 \tag{4.11}$$

since $H(\Gamma)^0 = H_B^\circ$. Put $\gamma_1 = \gamma\gamma_0\gamma^{-1}$. Then $A^+(\gamma_1) = \gamma A^+(\gamma_0)$ and $A^-(\gamma_1) = \gamma A^-(\gamma_0)$. Then (4.11) implies that γ_1 is transversal to both γ_0 and γ_0^{-1}. On the other hand we see from (4.2) for n even that the signs of γ_0 and γ_0^{-1} are different. It now remains to apply Proposition 4.9 either to the pair $(g_1 = \gamma_0, g_2 = \gamma_1)$ or to the pair $(g_1 = \gamma_0^{-1}, g_2 = \gamma_1)$. q.e.d.

5. Pseudohyperbolicity of products

We are now heading for a proof of Theorem B. In this section we give a sufficient condition for when the product of two linear maps is pseudohyperbolic. In the next section we prove a generalization of the basic additivity lemma.

We need the following concept of pseudohyperbolicity which is both more general and more precise.

Before we give the actual definition which is somewhat technical we give some explanation. Let V be a real vector space of dimension m.

We fix a natural number p with $1 \leq p \leq m-1$. Let $g \in \mathrm{GL}(V)$. We order the complex eigenvalues of g according to their moduli $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$. The condition of (p, ε)-pseudohyperbolicity stipulates among other things that $|\lambda_p| > |\lambda_{p+1}|$. We thus can talk about the p eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$ of maximal modulus and also of the $m-p$ eigenvalues of minimal modulus $\{\lambda_{p+1}, \ldots, \lambda_m\}$. Then V is the direct sum of two g-invariant subspaces V^+ and V^- such that every eigenvalue of $g \mid V^+$ resp. $g|V^-$ is a λ_i with $i \leq p$ resp. $i > p$. In particular V^+ and V^- have dimension p and $m-p$, resp. The number ε in the following definition of pseudohyperbolicity denotes the distance between these two subspaces. The number s is related to the size of the gap between $|\lambda_p|$ and $|\lambda_{p+1}|$, more precisely, s is an upper bound for $|\lambda_{p+1}| \cdot |\lambda_p|^{-1}$, see (5.3). The actual definition of pseudohyperbolicity is formulated not in terms of eigenvalues but in terms of norms. The reason is that we want to have pseudohyperbolicity of products of pseudohyperbolic elements, see Lemma 5.6.

Here is the definition. Let V be a real vector space of dimension m. Fix a norm $\|\cdot\|$ on V and let d be the corresponding metric. Let $S(V)$ be the sphere $\{x \in V; \|x\| = 1\}$. For $p \in \{1, \ldots, m-1\}$, $0 < s < 1$, $\varepsilon > 0$ let $\Omega(p, s, \varepsilon)$ be the set of elements $g \in \mathrm{GL}(V)$ with the following properties: There is a p-dimensional linear subspace V^+ of V and a complementary linear subspace V^- of V, both g-invariant, such that

$$\|g|V^-\| \leq s \cdot \|g^{-1}|V^+\|^{-1} \tag{5.1}$$

and $d(v^-, v^+) \geq \varepsilon$ for any two vectors $v^- \in S(V^-)$, $v^+ \in S(V^+)$, where, of course, $S(V^-) = S(V) \cap V^-$ etc. Note that

$$\|g^{-1}|V^+\|^{-1} = \min\{\|gv\|; v \in S(V^+)\}. \tag{5.2}$$

We have for every eigenvalue λ^- of g on V^- and λ^+ of g on V^+:

$$|\lambda^-| \leq \|g|V^-\| \leq s\|g^{-1}|V^+\|^{-1} \leq s \cdot |\lambda^+|. \tag{5.3}$$

Since $s < 1$, it follows that the eigenvalues of $g \mid V^+$ are the p eigenvalues of g of maximal modulus and the eigenvalues of $g \mid V^-$ are the $m-p$ eigenvalues of g of minimal modulus. In particular, the g-invariant subspaces V^+ and V^- are uniquely determined by this condition.

5.4 (Inverses of pseudohyperbolic elements)**.** Note that $g \in \Omega(p, s, \varepsilon)$ iff $g^{-1} \in \Omega(n-p, s, \varepsilon)$ with $V^-(g^{-1}) = V^+(g)$ and $V^+(g^{-1}) = V^-(g)$.

An element $g \in \bigcup_{s<1} \Omega(p,s,\varepsilon)$ will be called (p,ε)-pseudohyperbolic and we put for such g

$$s_p(g) := \|g|V^-\| \cdot \|g^{-1}|V^+\|. \tag{5.5}$$

So $s_p(g)$ is the smallest s such that $g \in \Omega(p,s,\varepsilon)$ for some $\varepsilon > 0$. Note that g is (r,ε)-proximal in the terminology of [2] iff g is $(1,\varepsilon)$-pseudohyperbolic with $s_1(g) \le r^{-1}$. Two (p,ε)-pseudohyperbolic elements g,h are called (p,ε)-*transversal* if $d(v,w) \ge \varepsilon$ whenever $v \in S(V^-(g)) \cup S(V^-(h))$ and $w \in S(V^+(g)) \cup S(V^+(h))$. In the following lemma we will use the Hausdorff distance associated with $\|\cdot\|$, also denoted by d. Thus, for any two compact subsets A,B of V put

$$d(A,B) = \max\left\{\max_{b\in B}\min_{a\in A} d(a,b), \max_{a\in A}\min_{b\in B} d(a,b)\right\}.$$

So $d(A,B)$ is the minimum of the numbers r such that $A \subset B_r(B)$ and $B \subset B_r(A)$, where $B_r(X) = \bigcup_{x\in X} B_r(x)$ and $B_r(x) = \{y \in V; d(x,y) \le r\}$ is the ball of radius r with center x. For two linear subspaces V_1, V_2 of V put

$$d(V_1,V_2) = d(S(V_1), S(V_2)).$$

Assume now, that $g \in H_B$, where $H_B = O(B)$ and B is the quadratic form of the signature $(n+1,n)$. Let g be (n,ε)-pseudohyperbolic, then g^{-1} is also (n,ε)-pseudohyperbolic and $V^+(g) = A^+(g)$, $V^+(g^{-1}) = A^-(g)$, $V^-(g) = D^-(g)$, $V^-(g^{-1}) = D^+(g)$. In that case we will define $s(g)$ as $s(g) = \max\{s_n(g),\ s_n(g^{-1})\}$.

Lemma 5.6. *For any $\varepsilon > 0$ there are two real numbers $a(\varepsilon)$ and $s(\varepsilon)$, $s(\varepsilon) > 1$ such that if g and h are (p,ε)-transversal, $s_p(g) < s(\varepsilon)^{-1}$ and $s_p(h) < s(\varepsilon)^{-1}$ then:*

(1) *gh is $\left(p,\frac{\varepsilon}{2}\right)$-pseudohyperbolic.*

(2) $a(\varepsilon)^{-1}s_p(g)s_p(h) < s_p(gh) < a(\varepsilon)s_p(g)s_p(h)$.

(3) $d(V^+(gh), V^+(g)) < a(\varepsilon)s_p(g)$.

(4) $d(V^-(gh), V^-(h)) < a(\varepsilon)s_{n-p}(h)$.

We often call real numbers like $a(\varepsilon)$ constants because, although they do depend on ε, they do not depend on the other variables, in particular not on g, $V^+(g), \ldots$.

Proof. We start with the case $p = 1$. We only show the claims (1), (3) and the second inequality of (2). The remaining inequality of (2) and also (4) will be obtained later by passing to inverses, using 5.4, but for a different dimension. If g is $(1, \varepsilon)$-pseudohyperbolic then $V^+(g)$ is a g-invariant line corresponding to a real eigenvalue of g which we call $\lambda(g)$, so $gv = \lambda(g)v$ for $v \in V^+(g)$.

Lemma 5.7. *For every $\varepsilon > 0$ there is a constant $a(\varepsilon) > 1$ with the following property. For any two $(1, \varepsilon)$-transversal elements g, h in* $\mathrm{GL}(V)$ *with $s_1(g) < a(\varepsilon)^{-1}$ and $s_1(h) < a(\varepsilon)^{-1}$ we have:*

(1) *gh is $(1, \frac{\varepsilon}{2})$-proximal.*

(2) $s_1(gh) < a(\varepsilon)s_1(g)s_1(h)$.

(3) $a(\varepsilon)^{-1}|\lambda(g)\lambda(h)| < |\lambda(gh)| < a(\varepsilon)|\lambda(g)\lambda(h)|$.

(4) $d(V^+(gh), V^+(g)) < a(\varepsilon)s_1(g)$.

Proof. Let $B(x, r) = \{y \in V; d(x, y) \leq r\}$ be the ball of radius r with center x. For any $(1, \varepsilon)$-pseudohyperbolic element g fix one vector $x^+(g)$ in $S(V^+(g))$. Put

$$g_1(x) = \frac{g(x)}{\|g(x)\|} \quad \text{for } x \neq 0$$

and

$$U(g) = B\left(x^+(g), \frac{\varepsilon}{2}\right).$$

It is easy to see that there is a constant $a(\varepsilon) > 0$ for every $\varepsilon > 0$ such that if g and h are $(1, \varepsilon)$-transversal with $s_1(g) \leq a(\varepsilon)^{-1}$ and $s_1(h) \leq a(\varepsilon)^{-1}$ then

$$(5.8) \quad h_1 U(g) \subset B(x^+(g), r) \text{ or } h_1 U(g) \subset B(-x^+(g), r) \quad \text{with } r = \min\left(a(\varepsilon)s_1(h), \frac{\varepsilon}{2}\right)$$

$$(5.9) \quad a(\varepsilon)^{-\frac{1}{2}}|\lambda(h)| \cdot \|x\| < \|h(x)\| < a(\varepsilon)^{\frac{1}{2}}|\lambda(h)| \cdot \|x\| \quad \text{for } x \in U(g)$$

$$(5.10) \qquad \|h_1(x) - h_1(y)\| < a(\varepsilon)^{\frac{1}{2}}s_1(h)\|x - y\| \quad \text{for } x, y \in U(g).$$

The same holds with g und h interchanged. So we have by (5.8)

$$h_1 U(g) \subset \pm \overset{\circ}{U}(h) \tag{5.11}$$

and

$$g_1 U(h) \subset \pm \overset{\circ}{U}(g). \tag{5.12}$$

Thus $g_1 \circ h_1 : U(g) \to \pm \overset{\circ}{U}(g)$ and

$$\|g_1 h_1(x) - g_1 h_1(y)\| < a(\varepsilon) s_1(g) s_1(h) \|x - y\| \tag{5.13}$$

for $x, y \in U(g)$. We then obtain the following facts from Tits's lemma [2, 2.1] and thus basically from the Banch fixed point theorem: gh is proximal, $V^-(g\,h)$ does not intersect $\overset{\circ}{U}(g)$ and the projective map induced by $g\,h$ on the projective space $\mathbb{P}$ has a unique fixed point in the image of $U(g)$ in $\mathbb{P}$ and this fixed point is $V^+(g\,h)$. This proves Claim (1). Claim (4) follows from inequality (5.8) and the corresponding inequality with g and h interchanged, and Claim (3) follows from inequality (5.9) for g and h. Finally inequality (5.13) implies that $s_1(gh) < b(\varepsilon) s_1(g) s_1(h)$ for some constant $b(\varepsilon)$ depending only on ε and of course the given norm. We may assume that $b(\varepsilon) \geq a(\varepsilon)$ and thus replace $a(\varepsilon)$ by $b(\varepsilon)$ to retain everything and also obtain Claim (2). q.e.d.

5.14. We shall constantly use the following notion. Fix p. Let $k \geq 1$ be a natural number and let two functions f_1, f_2 be given on some subset E of the direct product of k copies of $\mathrm{GL}(V)$. We say that f_1 is *dominated* by f_2 and write $f_1 \ll f_2$ if for every $\varepsilon > 0$ there is a $c(\varepsilon) > 0$ such that $f_1(g_1, \ldots, g_k) \leq c(\varepsilon) f_2(g_1, \ldots, g_k)$ whenever $(g_1, \ldots, g_k) \in E$ and the elements $g_i, i = 1, \ldots, k$, are pairwise (p, ε)-transversal. If $f_1 \ll f_2$ and $f_2 \ll f_1$ then we write $f_1 \sim f_2$ and say that f_1 and f_2 are *equivalent*.

Proof of Lemma 5.6 *for arbitrary p.* Note that if $g \in \Omega(p, s, \varepsilon)$ then $\bigwedge^p g : \bigwedge^p V \longrightarrow \bigwedge^p V$ is proximal, i.e., has a unique eigenvalue of maximal modulus and this eigenvalue has multiplicity one. The corresponding line $V^+(\bigwedge^p g)$ is $\bigwedge^p V^+(g)$, the hyperplane $V^<(\bigwedge^p g)$ is $\{x \in \bigwedge^p V; x \wedge \bigwedge^{n-p} V^-(g) = 0\}$, the eigenvalue $\lambda^+(\bigwedge^p g)$ of maximal modulus is the product of the p eigenvalues of g of largest modulus and all the eigenvalues of $\bigwedge^p g$ on $V^<(\bigwedge^p g)$ are of modulus $\leq s \cdot |\lambda^+(\bigwedge^p g)|$.

In order to proceed we need the quantitative specifications of proximality as follows:

$$d\left(x^{+}\left(\bigwedge^{p} g\right),\ S\left(V^{<}\left(\bigwedge^{p} g\right)\right)\right) \sim 1 \tag{5.15}$$

$$\frac{\|g \mid V^{<}(\bigwedge^{p} g)\|}{|\lambda^{+}(\bigwedge^{p} g)|} \sim s(g). \tag{5.16}$$

To see this we may assume that our norm comes from a positive definite quadratic form Q on V. Now change the form Q to the form Q_g with the following properties: $V^- := V^-(g)$ is orthogonal to $V^+ := V^+(g)$ with respect to Q_g, $Q_g|V^+ = Q|V^+$, $Q_g|V^- = Q|V^-$. Then $\sup_{Q(x)=1} Q_g(x)$ and $\inf_{Q(x)=1} Q_g(x)$ are ~ 1 as functions of g, i.e., there is a compact set $K = K(\varepsilon)$ of quadratic forms, depending on ε, such that $Q_g \in K$ for every (p,ε)-pseudohyperbolic g. For the induced form on $\bigwedge^p V$, also denoted Q_g, and which also belongs to a compact set $K'(\varepsilon)$ of quadratic forms, we then have $V^{+}(\bigwedge^{p} g) \perp V^{<}(\bigwedge^{p} g)$, which implies (5.15).

For orthonormal vectors $x_1, \dots, x_r$ in V^+, $x_{r+1}, \dots, x_p$ in V^- we have for the norm $\|\cdot\|$ corresponding to Q_g:

$$\|g(x_1 \wedge \cdots \wedge x_p)\| = \|g(x_1 \wedge \cdots \wedge x_r)\| \cdot \|g(x_{r+1} \wedge \cdots \wedge x_p)\|. \tag{5.17}$$

The second factor can be estimated by

$$\|gx_{r+1} \wedge \cdots \wedge gx_p\| \le \|gx_{r+1}\| \cdot \|gx_{r+2}\| \cdot \cdots \cdot \|gx_p\| \le \|g \mid V^-\|^{p-r}, \tag{5.18}$$

since

$$\|z_1 \wedge \cdots \wedge z_{t+1}\| = \|z_1 \wedge \cdots \wedge z_t\| \cdot \|\pi(z_{t+1})\|$$

where π is the orthogonal projection of z_{t+1} onto the span of $z_1, \dots, z_t$.

As to the first factor in (5.17), take the Cartan decomposition of $g \mid V^+ = k_1 \cdot d \cdot k_2$ where k_1, k_2 are unitary with respect to $Q_g \mid V^+ = Q \mid V^+$ and d is diagonal with entries $d_1 \ge d_2 \ge \cdots \ge d_p > 0$ with respect to an orthonormal basis $y_1, \dots, y_p$ of V^+. Assuming that $x_1, \dots, x_r$ forms part of the basis $k_2^{-1} y_1, \dots, k_2^{-1} y_p$, as we may, we obtain

$$\|g(x_1 \wedge \cdots \wedge x_r)\| = \prod_i d_i \le d_1 \dots d_r, \tag{5.19}$$

where the index i runs through an r–element subset of $\{1, \dots p\}$. We thus obtain

$$\left\|g \,|\, V^{<}\left(\bigwedge^{p} g\right)\right\| < C \cdot s \cdot \left|\lambda^{+}\left(\bigwedge^{p}(g)\right)\right|$$

for some constant C, since we have a corresponding inequality for every element of an orthonormal basis of $V^{<}(\bigwedge^p g)$ by (5.17)-(5.19). Changing back to the original quadratic form Q gives the inequality $\ll$ in (5.16). The opposite inequality is easily proved using the Cartan decompositions of $g \mid V^+$ and $g \mid V^-$ with respect to Q_g.

It follows now from Lemma 5.7, the case $p = 1$, that there is a constant $s(\varepsilon) > 1$ such that if g and h are (p, ε)-transversal and $s(g) < s(\varepsilon)^{-1}$, $s(h) < s(\varepsilon)^{-1}$ then $\bigwedge^p(gh)$ is proximal since (p, ε)-transversality of g and h implies $(1, c(\varepsilon))$-transversality of $\bigwedge^p g$ and $\bigwedge^p h$ for some $c(\varepsilon) > 0$. The proof of 5.7 and of the Banach fixed point theorem show furthermore that then $V^+(\bigwedge^p(gh)) = \lim (gh)^n V^+(\bigwedge^p g)$ which implies that the sequence $(gh)^n\ V^+(g)$ converges to a p-dimensional linear subspace of V which we call V^+. Similarly, using inverses and 5.4, the sequence $(gh)^{-n}\ V^-(h)$ converges to an $n-p$-subspace V^- of V. By 5.7(4) we have

$$d(V^+(g), V^+) < a_2(\varepsilon) s_p(g) \tag{5.20}$$

and

$$d(V^-(h), V^-) < a_2(\varepsilon) s_{n-p}(h). \tag{5.21}$$

Finally, for $x \in U(g) := B(S(V^+(g)), \varepsilon/2)$ we have

(5.22)
$$a_3(\varepsilon)^{-1} \|\pi_h^+(x)\| \cdot \|h^{-1} \mid V^+(h)\|^{-1} \leq \|hx\| \leq a_3(\varepsilon) \|\pi_h^+(x)\| \cdot \|h \mid V^+(h)\|$$

where π_h^+ and π_h^- are the projections in $V = V^+(h) \oplus V^-(h)$, and similarly with g and h interchanged. This implies

$$\|(gh)(x)\| \geq a_4(\varepsilon)^{-1} \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|x\|$$

for $x \in V^+$ by (5.20) and that there is a point $x \in S(V^+)$ with

$$\|gh(x)\| \leq a_4(\varepsilon) \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|h^{-1} \mid V^+(h)\|^{-1} \cdot \|x\|,$$

since $\pi_h^+ : V^+ \longrightarrow V^+(h)$ is surjective and $\inf\limits_{x \in S(V^+)} \|\pi_h^+(x)\| \geq a_5(\varepsilon)\|x\|$, and similarly for g. Thus

$$a_4(\varepsilon)^{-1} \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|h^{-1} \mid V^+(h)\|^{-1} \leq \|(gh)^{-1} \mid V^+\|^{-1} \leq a_4(\varepsilon) \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|h^{-1} \mid V^+(h)\|^{-1}.$$

Similarly for $(gh)^{-1}$:

$$\begin{aligned}&a_4(\varepsilon)^{-1}\cdot\|g\mid V^-(g)\|\cdot\|h\mid V^-(h)\|\\&\le\|gh\mid V^-\|\\&\le a_4(\varepsilon)\cdot\|g\mid V^-(g)\|\cdot\|h\mid V^-(h)\|.\end{aligned}$$

Hence for $a(\varepsilon)=\max(2,a_i(\varepsilon))$ and $s_p(g)\le a(\varepsilon)^{-1}$, $s_{n-p}(h)\le a(\varepsilon)^{-1}$, we obtain that gh is pseudohyperbolic with $V^+(gh)=V^+$, $V^-(gh)=V^-$,

$$s_p(gh)\sim s_p(g)s_p(h)$$

and $d(V^+(gh),V^+(g))\ll s_p(g)$, $d(V^-(gh),\,V^-(h))\ll s_p(h)$. q.e.d.

We obtain:

Corollary 5.23. *Let g and h be ε-hyperbolic, ε-transversal elements then there is a constant $s(\varepsilon)<1$ such that if $s(g)<s(\varepsilon)$, $s(h)<s(\varepsilon)$ then gh is $\varepsilon/2$-hyperbolic and gh and h, gh and g are $\varepsilon/2$-transversal.*

Proof. By Lemma 5.6 there is a constant $s(\varepsilon)$, such that for $s(g)<s(\varepsilon)$, $s(h)<s(\varepsilon)$ we have $d(A^+(g),A^+(gh))=d(V^+(g),V^+(gh))<\varepsilon/4$ and therefore $d(A^+(gh),D^-(h))\ \ge\ -d(A^+(gh),A^+(g))+d(A^+(g),D^-(h))\ge\varepsilon/2$. The same is true for the other subspaces. q.e.d.

The next lemma will be applied in the following situation. Let $h_1,\ldots,h_m$ be ε-hyperbolic, pairwise ε-transversal elements and let H be the group generated by $h_1,\ldots,h_m$. Under certain hypotheses for the h_i's we can guarantee that certain elements h of H are still hyperbolic, but h will not be ε-hyperbolic, in general. E.g., think of $h_1^{-n}\ h_2\ h_1^n$. The lemma shows that we can keep our elements ε-hyperbolic and ε-transversal, namely by multiplying by a fixed element g_0.

Lemma 5.24. *Let $g_0,h_1,\ldots,h_m$ be ε-hyperbolic, pairwise ε-transversal elements and let $s=\max\{s(g_0),s(h_1),\ldots,s(h_m)\}$. Let $g_\ell=g_0\cdot h_{i_1}^{n_1}\ldots h_{i_\ell}^{n_\ell}$, $i_k\ne i_{k+1}$, $\ell\in\mathbb{Z}$, $\ell>0$, $n_i\in\mathbb{Z}$, $n_i\ne 0$ for $i=1,\ldots,\ell$ and $M_\ell=|n_1|+\cdots+|n_\ell|$. Then there is a constant $b(\varepsilon)>1$, such that if $s<b(\varepsilon)^{-2}$, then for every s_0, such that $s<s_0^2<b(\varepsilon)^{-2}$, and $\ell\in\mathbb{Z}$, $\ell>0$, we have:*

(1) $s(g_\ell)\le s_0^{M_\ell+1}$.

(2) $d(A^+(g_{\ell-1}),\,A^+(g_\ell))\le\dfrac{\varepsilon}{2}s_0^{M_{\ell-1}}$.

(3) $d(A^+(g_0),\ A^+(g_\ell)) \leq \frac{\varepsilon}{2}$.

(4) $d(A^-(g_\ell),\ A^+(h_i) \cup A^-(h_i)) \geq \frac{\varepsilon}{2}$ *for* $i \neq i_\ell$.

(5) g_ℓ *is* $\varepsilon/2$*-hyperbolic.*

Proof. Let us consider $a(\varepsilon/2)$, given by Lemma 5.6 and put $b(\varepsilon) = 4\varepsilon^{-1}a\,(\varepsilon/2)$. Let us assume that $s < b(\varepsilon)^{-2}$ and let s_0 be any number such that $s < s_0^2 < b(\varepsilon)^{-2}$. Let us first to show, that the statements of the the lemma follow from the following inequalities:

$$\begin{aligned} s(g_\ell) &\leq \frac{\varepsilon}{4} s_0^{M_\ell+2} \\ d(A^+(g_\ell), A^+(g_{\ell+1})) &\leq \frac{\varepsilon}{4} s_0^{M_\ell+1} \\ d(A^-(g_\ell)), A^-(h_{i_\ell})) &\leq \frac{\varepsilon}{4} s_0^{|n_\ell|}. \end{aligned}$$

In fact,

$$\begin{aligned} d(A^+(g_0), A^+(g_\ell)) &\leq \sum_{i=1}^{\ell} d(A^+(g_{i-1}), A^+(g_i)) \\ &\leq \frac{\varepsilon}{4} s_0 + \frac{\varepsilon}{4} s_0^{M_1+1} + \cdots + \frac{\varepsilon}{4} s_0^{M_{\ell-1}+1} \leq \frac{\varepsilon}{4}. \end{aligned}$$

and

$$d(A^-(g_\ell)), A^-(h_{i_\ell})) \leq \varepsilon/4.$$

Therefore, for all ℓ, $\ell \geq 0$

$$\begin{aligned} d(A^+(g_\ell), A^-(g_\ell)) &\geq d(A^+(g_0), A^-(h_{i_\ell})) - d(A^+(g_\ell), A^+(g_0)) \\ &\quad - d(A^-(g_\ell), A^-(h_{i_\ell})) \\ &\geq \varepsilon - \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \frac{\varepsilon}{2} \end{aligned}$$

which means, that for all ℓ, $\ell \geq 0$, g_ℓ is $\varepsilon/2$-hyperbolic.

Additionally, if $i \neq i_\ell$ we have

$$\begin{aligned} &d(A^-(g_\ell), A^+(h_i) \cup A^-(h_i)) \\ &\geq d(A^-(h_{i_\ell}), A^+(h_i) \cup A^-(h_i)) - d(A^-(g_\ell), A^-(h_{i_\ell})) \\ &\geq \varepsilon - \frac{\varepsilon}{4} > \frac{\varepsilon}{2}, \end{aligned}$$

for all ℓ and i, $i_\ell \neq i$. So for all ℓ and i if $i_\ell \neq i$ elements the g_ℓ and h_i are $\varepsilon/2$-transversal.

We will prove the inequalities above by induction on the index ℓ. We will assume, that they are true if the index is $\leq \ell$. Therefore, the elements g_ℓ and h_i are $\varepsilon/2$-hyperbolic and $\varepsilon/2$-transversal, if $i_\ell \neq i$, therefore we can use the statements of Lemma 5.6. First of all, because of (2) in Lemma 5.6,

$$s(g_{\ell+1}) \leq a(\varepsilon/2)s(g_\ell) \cdot s^{|n_{\ell+1}|} \leq a(\varepsilon/2)s_0^{M_\ell+2} \cdot s_0^{2|n_{\ell+1}|} \leq \frac{\varepsilon}{4}s_0^{M_{\ell+1}+2}.$$

Now, by (3) of Lemma 5.6,

$$d(A^+(g_\ell), A^+(g_{\ell+1})) \leq a(\varepsilon/2)s(g_\ell) \leq a(\varepsilon/2)s_0^{M_\ell+2} \leq \frac{\varepsilon}{4}s_0^{M_\ell+1}.$$

$$d(A^-(g_\ell), A^-(h_{i_\ell})) \leq \alpha(\varepsilon/2)s_0^{2|n_\ell|} \leq \frac{\varepsilon}{4}s_0^{|n_\ell|}.$$

q.e.d.

6. The Basic Lemma

The main claim of the Basic Lemma states additivity of the function α of (4.3).

So in this section we return to affine maps. Recall that $H_B = \mathrm{SO}(n+1,n)$ is the orthogonal group of the form B of signature $(n+1,n)$ on $\mathbb{R}^{2n+1}$ and G_B is the subgroup of $G_{2n+1} = \mathrm{Aff}(\mathbb{R}^{2n+1})$ of those elements with linear part in H_B. Also, Ω was the set of those $g \in G_n$ for which $\dim A^0(g) = 1$ and the eigenvalue of $\ell(g) \mid A^0(g)$ is $+1$, see paragraphs following (2.1). These elements were called pseudohyperbolic in Sections 2-4. Let us clarify how this notion of pseudohyperbolicity is related to the one of Section 5. If $\ell(g)$ is in $\Omega(n,s,\varepsilon)$ of Section 5 for some $s < 1$ and $\varepsilon > 0$ and $g \in G_B^\circ$ then g is pseudohyperbolic. Conversely, if $g \in \Omega$ then some power of $\ell(g)$ is in $\Omega(n,s,\varepsilon)$ for some $s < 1$ and $\varepsilon > 0$. Cf. the corresponding discussion for proximality in [2, Section 2].

We need some auxiliary lemmas before we can prove the Basic Lemma.

Lemma 6.1.

(a) *Let $g \in G_B^\circ$, $\ell(g) \in \Omega(n,s,\varepsilon)$. Then $\ell(g) \in \Omega(n+1,s',b(\varepsilon))$ for some constant $b(\varepsilon)$ and $\frac{s'}{s}$ and $\frac{s}{s'}$ are bounded, independently of ε, if s is sufficiently small.*

(b) *$d(v,w) \geq c(\varepsilon)$ for every $g \in \Omega(n,s,\varepsilon), v \in S(A^0(g))$ and every isotropic vector w with $\|w\| = 1$.*

(c) *For the given norm $\|\cdot\|$ and the norm $\|\cdot\|_B$ induced by B on $A^0(g)$ we have $\|\cdot\| \sim \|\cdot\|_B$, i.e.,*

$$c_1(\varepsilon) > \sup_{x \in A^0(g)} \frac{\|x\|}{\|x\|_B} > \inf_{x \in A^0(g)} \frac{\|x\|}{\|x\|_B} > c_1(\varepsilon)^{-1}$$

for some $c_1(\varepsilon)$ and every $g \in \Omega(n,s,\varepsilon)$.

(d) *$\|g \mid A^0(g)\| \sim 1$ and $\|g^{-1} \mid A^0(g)\| \sim 1$.*

(e)

$$\begin{aligned} s(g) &= \|g \mid D^-(g)\| \cdot \|g^{-1} \mid A^+(g)\| \\ &\sim \|g \mid A^-(g)\| \cdot \|g^{-1} \mid D^+(g)\| = s(g^{-1}). \end{aligned}$$

Proof.

(a) Take a basis $e_1, \dots e_{2n+1}$ of V such that $B(e_i, e_j) = 0$ for $i \neq j$ and $B(e_i, e_i) = 1$ for $i \leq n+1$, $B(e_i, e_i) = -1$ for $i > n+1$. Suppose our norm $\|\cdot\|$ is given by the Euclidean form Q with respect to this basis. Then for $a \in H_B$ we have $a^{-1} = P\, a^t\, P^{-1}$. Write $a = k_1 d k_2$ in

Cartan form, so $k_1, k_2 \in O(Q)$, $d = (d_1, \dots, d_{2n+1})$ is diagonal with $d_1 \geq d_2 \geq \cdots \geq d_{2n+1} > 0$. Then a is (n,s,ε_1)-pseudohyperbolic iff $d_{n+1} \leq s d_n$ iff a^t is (n,s,ε_2)-pseudohyperbolic iff $a^{-1} = P\, a^t\, P^{-1}$ is (n,s,ε_3)-pseudohyperbolic, since $P \in O(Q)$ iff a is $(n+1,s,\varepsilon_4)$-pseudohyperbolic, by 5.4.

To see the claim concerning $b(\varepsilon)$, look at the map $\perp$ which associates to every subspace W of V its orthogonal space with respect to B. Regard it as a map from a subset of a Grassmannian to another Grassmannian. Then $\perp$ is continuous and the claim follows from $D^+(g) = A^+(g)^\perp$ and $D^-(g)^\perp = A^-(g)$.

(b) Otherwise there are sequences g_i of (n,ε)-pseudohyperbolic elements such that $D^+(g_i)$ and $D^-(g_i)$ converge to spaces D^+, D^- containing an isotropic vector w. Then $A^+(g_i) = D^+(g_i)^\perp$ and $A^-(g_i) = D^-(g_i)^\perp$ converge to spaces $A^+ = D^{+\perp}$ and $A^- = D^{-\perp}$ containing w, contradicting (n,ε)-pseudohyperbolicity.

(c) By part (b) of the proof the set of pairs of spaces $\{(A^+(g), A^-(g)),$ $g\,(n,\varepsilon)$-pseu-dohyperbolic$\}$ in the product of Grassmannians is compact, hence so is the set of $A^0(g)'$s since $A^0(g) = D^+(g) \cap D^-(g)$.

(d) follows from (c) since g is a unitary operator on $A^0(g)$ with respect to $B \mid A^0(g)$.

(e) is proved by the same method as (a). q.e.d.

We also need the following inequality:

$$d(x, E_g^+) \ll s(g) d(x, g^{-1}x) \tag{6.2}$$

for $s(g) \ll 1$.

To see this first note that $d(x, E_g^+) \sim \|p^-(x-c)\|$ for arbitrary $c \in C(g)$ where p^- is the projection to $A^-(g)$ in the decomposition $V = A^-(g) \oplus A^0(g) \oplus A^+(g)$, by (n, s, ε)-pseudohyperbolicity. Thus

$$d(x, E_g^+) \ll s(g) d(g^{-1}x, E_g^+)$$

by 6.1 (c) and (a). The triangle inequality gives

$$s(g)^{-1} d(x, E_g^+) \ll d(g^{-1}x, x) + d(x, E_g^+)$$

which yields (6.2).

Lemma 6.3. *Let g and h be two ε-hyperbolic ε-transversal elements, such that gh is $\varepsilon/2$-hyperbolic, gh and g, gh and h are $\varepsilon/2$-transversal. Let q be any point in the affine space. Then there is a constant $d(\varepsilon)$ such that*

$$\begin{aligned} d(q, E_{gh}^+) \le{} & d(q, E_g^+) + d(\varepsilon) s(g) \left[|\alpha(g)| \right. \\ & \left. + d(q, C_g) + s(h)(|\alpha(h)| + d(q, C_h)) \right], \\ d(q, E_{gh}^-) \le{} & d(q, E_h^-) + d(\varepsilon) s(h) \left[|\alpha(h)| \right. \\ & \left. + d(q, C_h) + s(g)(|\alpha(g)| + d(q, C_g)) \right], \\ d(q, C_{gh}) \le{} & d(\varepsilon)(s(h) + s(g)) d(q, C_g) \\ & + d(\varepsilon)(d(q, E_g^+) + d(q, C_h)) + s(g)|\alpha(g)| + s(h)|\alpha(h)|. \end{aligned}$$

Proof. We know that $|\alpha(g)| = |\alpha(g^{-1})|$, $s(g) = s(g^{-1})$, $d(q, C_g) = d\left(q, C_{g^{-1}}\right)$ and $E_{g^{-1}}^+ = E_g^-$ for every hyperbolic element g, then the second inequality follows from the first one, and because $d(q, C_{gh}) \ll d(q, E_{gh}^+) + d(q, E_{gh}^-)$, the third inequality is just a corollary of the first and the second. So we have to prove just the first inequality.

Let now q_0 be a point on $L = E_h^+ \cap E_g^-$ such that $d(q, q_0) = d(q, L)$. Let $y = h^{-1}q_0$ and $x = gq_0$, then $d(y, E_{gh}^+) \ll s(gh)d(ghy, y) = s(gh)d(x, y)$. We have the inequalities

$$\begin{aligned} d(q_0, y) &\ll d(q_0, C_h) + |\alpha(h)| \\ d(x, q_0) &\ll d(q_0, C_g) + |\alpha(g)|. \end{aligned}$$

Therefore,

$$d(x, y) \ll d(q_0, C_h) + \alpha(q_0, C_g) + |\alpha(h)| + |\alpha(g)|.$$

Now,

$$d(q_0, C_h) \ll d(q, C_h) + d(q, q_0) \ll d(q, C_h) + d(q, C_g).$$

The same for $d(q_0, C_g)$ i.e.,

$$d(q_0, C_g) \ll d(q, C_g) + d(q, C_h).$$

Then

$$d(y, E_{gh}^+) \ll s(gh)\left[\, |\alpha(h)| + |\alpha(g)| + d(q, C_g) + d(q, C_h)\right],$$

and

$$\begin{aligned} d(q, E_{gh}^+) &\ll d(q, q_0) + d(q_0, y) + d(y, E_{gh}^+) \\ &\ll d(q, C_g) + d(q, C_h) + \alpha(h) \\ &\quad + s(gh) \cdot \left[\, |\alpha(h)| + |\alpha(g)| + d(q, C_g) + d(q, C_h)\right]. \end{aligned}$$

An elementary geometrical fact says, that

$$|d(q, E_{gh}^+) - d(q, E_g^+)| \ll \sin\left(\measuredangle\, E_{gh}^+, E_g^+\right)\left[d\left(q, E_{gh}^+\right) + d\left(q, E_g^+\right)\right].$$

Now, using Lemma 5.6, we immediately have

$$|d(q, E_{gh}^+) - d(q, E_g^+)| \ll s(g)\left[d(q, E_{gh}^+) + d(q, E_g^+)\right]$$

and

$$s(gh) \ll s(g) \cdot s(h)$$

with $d(q, E_g^+) \leq d(q, C_g)$.

Combining these two inequalities we have

$$d(q, E^+_{gh}) \le d(q, E^+_g) + d(\varepsilon)s(g)\,[\,|\alpha(g)| + d(q, C_g) \\ + s(h)\cdot(|\alpha(h)| + d(q, C_h))]$$

for some constant $d(\varepsilon)$. q.e.d.

6.4. Basic Lemma. *Let g and h be ε-hyperbolic, ε-transversal elements such that gh is $\varepsilon/2$-hyperbolic, gh and g, gh and h are $\varepsilon/2$-transversal. Let q be any point in affine space. Then there is a constant $c(\varepsilon)$, such that*

$$|\alpha(gh) - \alpha(g) - \alpha(h)| \le c(\varepsilon)\,(s(g)|\alpha(g)| + s(h)|\alpha(h)| \\ + d(q, C_g) + d(q, C_h))\,.$$

Proof. Let q_0 be a point on $L = E^+_h \cap E^-_g$, such that $d(q, q_0) = d(q, L)$. The elements g and h are ε-transversal, so $d(q, q_0) \ll d(q, E^+_h) + d(q, E^-_g)$, but $d(q, E^+_h) \le d(q, C_h)$, $d(q, E^-_g) \le d(q, C_g)$, thus

(i) $$d(q, q_0) \ll d(q, C_g) + d(q, C_h).$$

Let us consider the two vectors $v_g = g\,q_0 - q_0$ and $v_h = q_0 - h^{-1}q_0$. By definition,

(ii) $$\alpha(gh) = B(gq_0 - h^{-1}q_0, e^0(gh)) = B(v_g, e^0(gh)) + B^0(v_h, e^0(gh)).$$

We have $v_g \in D^-(g)$ and $v_g = \alpha(g)e^0(g) + w_1$, $w_1 \in A^-(g)$, and for the vector v_h we have $v_h = \alpha(h)e^0(h) + w_2$, $w_2 \in A^+(h)$.

Now by ε-hyperbolicity of g and h, we have $\|w_1\| \ll d(q_0, C_g)$ and $\|w_2\| \ll d(q_0, C_h)$. Let now w and $\overline{w}$ be the projections of the vector $\alpha(g)e^0(g)$ onto $D^-(gh)$ parallel to $A^+(g)$ and $A^+(gh)$ respectively. Then $w = \alpha(g)e^0(g) + u_1$, $u_1 \in A^+(g)$ and $\overline{w} = \alpha(g)e^0(g) + u_2$, $u_2 \in A^+(gh)$. Let us show that $w = \alpha(g)e^0(gh) + u_3$, where $u_3 \in A^-(gh)$. In fact, $w \in D^-(gh)$, and $w = \alpha\, e^0(gh) + u_3$, $u_3 \in A^-(gh)$. So we have to prove that $\alpha = \alpha(g)$. We have $B(w, w) = B(\alpha\, e^0(gh) + u_e,\ \alpha\, e^0(gh) + u_e) = \alpha^2 B(e^0(gh), e^0(gh)) = \alpha^2$ and on the other hand $B(w, w) = B(\alpha(g)e^0(g) + u_1, \alpha(g)e^0(g) + u_1) = \alpha(g)^2 B(e^0(g), e^0(g)) = \alpha(g)^2$. Then $\alpha = \alpha(g)$ or $-\alpha(g)$ and because of the orientation procedure $\alpha = \alpha(g)$. By (3) of Lemma 5.6 $d(A^+(gh), A^+(g)) \ll s(g)$ therefore $\|w - \overline{w}\| \ll$

$s(g)(\|w\| + \|\overline{w}\|)$. It is easy to see, using $\varepsilon/2$-hyperbolicity of g and gh, that $\|w\| \ll |\alpha(g)|$ and $\|\overline{w}\| \ll |\alpha(g)|$ and so

$$\|w - \overline{w}\| \ll s(g)|\alpha(g)|.$$

Then

$$\text{(iii)} \qquad |B(\overline{w}, e^0(gh)) - B(w, e^0(gh))| = |B(w - \overline{w}, e^0(gh)| \\ \ll \|w - \overline{w}\| \ll s(g)|\alpha(g)|.$$

We have, by definition of v_g,

$$|\alpha(g) - B(v_g, e^0(gh))| \\ \leq |\alpha(g) - B(\alpha(g)e^0(g),\ e^0(gh))| + |B(w_1, e^0(gh)|,$$

but

$$B(w, e^0(gh) = B(\alpha(g)e^0(gh) + u_3,$$

$e^0(gh)) = \alpha(g)$ and $|B(w_1, e^0(gh))| \ll \|w_1\| \ll d(q_0, C_g)$. Therefore, $|\alpha(g) - B(v_g, e^0(gh))| \ll |B(w, e^0(gh) - B(\alpha(g)e^0(g), e^0(gh))| + d(q_0, C_g)$ $= |B(w, e^0(gh) - B(\overline{w}, e^0(gh)| + d(q_0, C_g)$. Now by (iii)

$$|\alpha(g) - B(v_g, e^0(gh)| \ll s(g)|\alpha(g)| + d(q_0, C_g).$$

The same is true for h, namely,

$$|\alpha(h) - B(v_h, e^0(gh)| \ll s(h)|\alpha(h)| + d(q_0, C_h).$$

Then by (ii)

$$|\alpha(gh) - \alpha(g) - \alpha(h)| \leq |\alpha(g) - B(v_g, e^0(gh))| + |\alpha(h) - B(v_h, e^0(gh))|.$$

Now from (i)

$$d(q_0, C_g) \leq d(q_0, q) + d(q, C_g) \ll d(q, C_h) + d(q, C_g)$$

and

$$d(q_0, C_h) \ll d(q, C_h) + d(q, C_g).$$

Therefore,

$$|\alpha(gh) - \alpha(g) - \alpha(h)| \ll s(g)|\alpha(g)| + s(h)|\alpha(h)| + d(q, C_g) + d(q, C_h).$$

Which proves the statement of the Basic Lemma. q.e.d.

Let g_0, $h_1, \dots, h_m$ be ε-hyperbolic ε-transversal elements. Let us now put $g_\ell = g_0 h_{i_1}^{n_1} \dots h_{i_\ell}^{n_\ell}$, $i_k \neq i_{k+1}$, $\ell \in \mathbb{Z}$, $\ell > 0$, $M_\ell = |n_1| + \dots + |n_\ell|$. We will fix a point $q \in \mathbb{R}^{2n+1}$ and put $a_\ell = d(q, E^+_{g_\ell})$, $b_\ell = d(q, C_{g_\ell})$, $C = \max\{d(q, C_{g_0}),\ d(q, C_{h_1}), \dots, d(q, C_{h_m})\}$. Let now $t_\ell = \alpha(g_\ell)$, $T = \max\{|\alpha(g_0)|,\ |\alpha(h_1)|, ..., \alpha(h_m)|\}$ and $s = \max\{s(g_0), s(h_1), ..., s(h_m)\}$. Let $a(\varepsilon)$, $b(\varepsilon)$, $c(\varepsilon)$ and $d(\varepsilon)$ be the constants given by Lemmas 5.6, 5.24, 6.3 and 6.4, respectively. Put $a_0(\varepsilon) = \max\{a(\varepsilon), b(\varepsilon), c(\varepsilon), d(\varepsilon)\}$. We can rewrite the results of Lemmas 6.3 and 6.4 in the following form.

$$(6.5)\quad a_{\ell+1} \leq a_\ell + a_0(\varepsilon) s(g_\ell) \left(|t_\ell| + b_\ell + s\left(h_{i_{\ell+1}}\right)^{|n_{i_{\ell+1}}|} \left[|n_{i_{\ell+1}}| T + C \right] \right).$$

$$(6.6)\quad b_{\ell+1} \leq a_0(\varepsilon) \left(s(h_{i_{\ell+1}})^{|n_{i_{\ell+1}}|} + s(g_\ell) \right) b_\ell + a_0(\varepsilon)(a_\ell + C + s(g_\ell)|t_\ell| + s(h_{i_{\ell+1}})^{|n_{i_{\ell+1}}|} |n_{i_{\ell+1}}| T).$$

$$(6.7)\quad |t_{\ell+1} - t_\ell - n_{i_{\ell+1}} \alpha\left(h_{i_{\ell+1}}\right)| \leq a_0(\varepsilon) \left(a_\ell + C + s\left(h_{i_{\ell+1}}\right)^{|n_{i_{\ell+1}}|} |n_{i_{\ell+1}}| T + s(g_\ell)|t_\ell| \right).$$

We would like to explain now the final steps in the proof of Theorem B. The goal of the first step is to show that $a_\ell \ll 1$, $b_\ell \ll 1$ and $|t_\ell| \ll M_\ell$ for all ℓ.

The value of $a_{\ell+1}$ depends by (6.5) on the values of a_ℓ, b_ℓ, t_ℓ and the same for $b_{\ell+1}$, and $t_{\ell+1}$. So the idea is as follows. Let $r_\ell = \max\{10 a_0(\varepsilon) a_\ell$, b_ℓ, $10 a_0(\varepsilon) C$, $10 a_0(\varepsilon) T$, $|t_\ell|/10\ M_\ell a_0(\varepsilon)\}$. We will show, see (6.18), that

$$r_{\ell+1} \leq r_\ell \left(1 + 2 a_0^2(\varepsilon) s_0^{M_\ell} M_\ell \right),$$

and then, because $s_0 < 1$, we will have that

$$Q = \prod \left(1 + 2 a_0^2(\varepsilon) s_0^{M_\ell} M_\ell \right) < \infty$$

and therefore $r_\ell \leq r_0 Q$ for all ℓ, i.e., $r_\ell \ll 1$. Then, from the definition of r_ℓ, we have $a_e \ll 1$, $b_\ell \ll 1$, $|t_\ell| \ll M_\ell$.

The fact that $b_\ell \ll 1$ has a clear geometrical meaning: Let H be the group generated by the hyperbolic transversal affine transformations $h_1, \dots, h_s$. Then under certain conditions, there is a universal constant $C(H)$, such that for every $h \in H$, $d(q, C_h) \leq C(H)$. Then we will show that $|t_\ell| \sim M_\ell$, so one can interpret the statement of the Basic Lemma as an additive property of the function α.

Lemma 6.8. *Let* $r_\ell = \max\{10a_0(\varepsilon)a_\ell,\ b_\ell,\ 10a_0(\varepsilon)C,\ 10a_0(\varepsilon)T,\ |t_\ell|/10M_\ell a_0(\varepsilon)\}$. *There are constants* $b_0(\varepsilon) > 1$ *and* $Q_0(\varepsilon)$ *such that if* $s < b_0(\varepsilon)^{-1}$, *then for all* $\ell, \ell \in \mathbb{Z}$, $\ell \geq 1$ *we have* $a_\ell \leq Q_0(\varepsilon)$, $b_\ell \leq Q_0(\varepsilon)$, $|t_\ell| \leq Q_0(\varepsilon)M_\ell$, $r_\ell \leq Q_0(\varepsilon)$.

Proof. Let us first take $s_0 \in \mathbb{R}$, $s_0 < 1$ such that $a_0(\varepsilon/2)^2 s_0^m\, m \leq \frac{1}{10}$ for all natural numbers m. Note that in particular $s_0 < b(\varepsilon)^{-1}$. Put $b_0(\varepsilon) = s_0^{-2}$ and let $s < b_0^{-1}(\varepsilon)$. Then in view of Lemma 5.24 we can assume that the elements g_ℓ are $\varepsilon/2$-hyperbolic and $\varepsilon/2$-transversal for all ℓ. Using again Lemma 5.24, we can rewrite inequalities (6.5)-(6.7) as follows:

$$a_{\ell+1} \leq a_\ell + a_0(\varepsilon/2)s_0^{M_\ell+1}\left[|t_\ell| + b_\ell + s_0^{2|n_{i_{\ell+1}}|}|n_{i_{\ell+1}}|T + C\right], \tag{6.9}$$

$$\begin{aligned} b_{\ell+1} \leq a_0(\varepsilon/2)\left(s_0^{2|n_{i_{\ell+1}}|} + s_0^{M_\ell+1}\right) b_\ell \\ + a_0(\varepsilon/2)\left[C + a_\ell + s_0^{2|n_{i_{\ell+1}}|}|n_{i_{\ell+1}}|T + s_0^{M_\ell+1}|t_\ell|\right], \end{aligned} \tag{6.10}$$

$$\begin{aligned} &\left|t_{\ell+1} - t_\ell - |n_{i_{\ell+1}}|\alpha\left(h_{i_{\ell+1}}\right)\right| \\ &\qquad \leq a_0(\varepsilon/2)\left[s_0^{M_\ell+1}|t_\ell| + s^{2|n_{i_{\ell+1}}|}|n_{i_{\ell+1}}|T + C + b_\ell\right]. \end{aligned} \tag{6.11}$$

Now using the fact that $a_0(\varepsilon/2)^2 s_0^m m \leq s_0/10$ in particular $a_0(\varepsilon/2)^2 s_0 \leq 1/10$, we have from (6.9)-(6.11)

$$a_{\ell+1} \leq a_\ell + \frac{s_0^{M_\ell}}{10}\left[|t_\ell| + b_\ell + \frac{T}{10a_0(\varepsilon/2)^2} + C\right], \tag{6.12}$$

$$b_{\ell+1} \leq \frac{b_\ell}{10} + \frac{T}{10} + s_0^{M_\ell}|t_\ell| + a_0(\varepsilon/2)[C + a_\ell], \tag{6.13}$$

$$|t_{\ell+1}| \leq |t_\ell| + |n_{i_{\ell+1}}|T + s_0^{M_\ell}|M_\ell|T + \frac{s_0^{|n_{i_{\ell+1}}|}\cdot T}{10} + a_0(\varepsilon/2)[C + b_\ell]. \tag{6.14}$$

Because of our notation we have

$$\begin{aligned} a_{\ell+1} &\leq \frac{r_\ell}{10a_0(\varepsilon/2)} + s_0^{M_\ell}M_\ell a_0(\varepsilon/2)r_\ell + \frac{s_0^{M_\ell}r_\ell}{1000} + \frac{s_0^{M_\ell}r_\ell}{100} \\ &\leq \frac{r_\ell}{10a_0(\varepsilon/2)} + 2s_0^{M_\ell}M_\ell a_0(\varepsilon/2)r_\ell \leq \\ &\leq \frac{r_\ell}{10a_0(\varepsilon/2)}\left[1 + 2a_0^2(\varepsilon/2)s_0^{M_\ell}M_\ell\right]. \end{aligned} \tag{6.15}$$

From (6.13) we have

$$b_{\ell+1} \le r_\ell \tag{6.16}$$

and from (6.14) we have

(6.17)
$$\begin{aligned}|t_{\ell+1}| &\le 10M_\ell a_0(\varepsilon/2)r_\ell + \frac{|n_{i_{\ell+1}}|r_\ell}{10d_0(\varepsilon/2)} + \frac{r_\ell}{100d_0(\varepsilon/2)} + \frac{r_\ell}{10} + a_0(\varepsilon/2)r_\ell \\ &\le 10a_0(\varepsilon/_2)r_\ell(M_\ell + |n_{i_{\ell+1}}|) \\ &= 10a_0(\varepsilon/2)M_{\ell+1}r_\ell.\end{aligned}$$

Therefore

$$r_{\ell+1} \le r_\ell(1 + 2a_0^2(\varepsilon/2)s_0^{M_\ell}M_\ell). \tag{6.18}$$

If $Q = \prod(1 + 2a_0^2(\varepsilon/2)s_0^{m_\ell}M_\ell) < \infty$, then $a_\ell \le Qr_1/10a_0(\varepsilon/2)$, $b_\ell \le Qr_1$, $|t_\ell| \le 10M_\ell a_0(\varepsilon/2)Qr$. Let $Q_0 = 10M_0(\varepsilon/2)Qr_1$ then $a_\ell \le Q_0$, $b_\ell \le Q_0|t_\ell| \le Q_0M_\ell$. q.e.d.

Lemma 6.19. *Assume that $d(q, S_{g_0}) = \alpha(g_0) = 0$ and $\alpha(h_i) > 0$ for all $i = 1, \dots, m$. There is a constant $c_0(\varepsilon) > 1$ such that if $s < c_0(\varepsilon)^{-1}$ and $C \le c_0(\varepsilon)^{-1}T$, then $t_\ell \ge \frac{2}{3}M_\ell T$.*

Proof. Let s_0 be a positive real number < 1 such that $a_0(\varepsilon/2)^2 s_0 < 1/10$ and $s_0 < b_0(\varepsilon)^{-1}$. Put $c_0(\varepsilon) = s_0^{-2}$. For $s < c_0(\varepsilon)^{-1}$ we have $s < b_0(\varepsilon)^{-1}$ and we can therefore use the statements of Lemma 6.8. So $a_\ell \le Q_0(\varepsilon)$, $b_\ell \le Q_0(\varepsilon)$, $|t_\ell| \le M_\ell Q_0(\varepsilon)$. We have from (6.11)

$$\begin{aligned}t_{\ell+1} \ge t_\ell + |n_{i_{\ell+1}}|T - a_0(\varepsilon/2)C - a_0(\varepsilon/2)b_\ell \\ - s_0^{m_\ell}|t_\ell| - s^{|n_{i_{\ell+1}}|}|n_{i_{\ell+1}}|T.\end{aligned} \tag{6.20}$$

Now from (6.12) we have $a_{\ell+1} \le a_\ell + s_0^{M_\ell}M_\ell Q_0$, therefore for some constant $Q_1 = Q_1(\varepsilon)$, $a_{\ell+1} \le a_0 + s_0Q_1(\varepsilon)$, but $a_0 = 0$ then

$$a_{\ell+1} \le s_0Q_1(\varepsilon). \tag{6.21}$$

Assume that $c \le T/10a_0^2(\varepsilon/2)$ then from (6.10) we have

$$b_{\ell+1} \le \frac{s_0b_\ell}{10} + \frac{s_0T}{10} + a_0(\varepsilon/2)C + a_0(\varepsilon/2)s_0Q_1(\varepsilon).$$

Then, because $b_0 = 0$, there is a constant $Q_2(\varepsilon)$ such that

$$b_{\ell+1} \leq s_0 Q_2(\varepsilon) + a_0(\varepsilon/2)C.$$

Let us now assume that $s_0 Q_2(\varepsilon) a_0(\varepsilon/2) \leq \frac{T}{10}$. We know that $C \leq \frac{T}{10a_0^2(\varepsilon/2)}$, then from (6.20) we have

$$\begin{aligned} t_{\ell+1} &\geq t_\ell + |n_{i_{\ell+1}}|T - \frac{T}{10a^2(\varepsilon/2)} - \frac{T}{10} - \frac{T}{10} \\ &\geq t_\ell + \left(|n_{i_{\ell+1}}| - \frac{3}{10}\right) T \\ &\geq t_\ell + \frac{2}{3}|n_{i_{\ell+1}}|T. \end{aligned}$$

Now $t_0 = 0$, hence $t_{\ell+1} \geq \frac{2}{3} M_\ell$. q.e.d.

Proof of Theorem B. Using [2], we can find elements $\widetilde{h}_1, \ldots, \widetilde{h}_m$ in $\mathrm{Aff}\,(\mathbb{R}^{2n+1})$ such that:

(1) $\ell\left(\widetilde{h}_1\right), \ldots, \ell\left(\widetilde{h}_m\right)$ are hyperbolic and pairwise transversal.

(2) The group $\left\langle \ell(\widetilde{h}_1), \ldots, \ell(\widetilde{h}_m) \right\rangle$ generated by $\ell\left(\widetilde{h}_1\right), \ldots, \ell\left(\widetilde{h}_m\right)$ is free and Zariski dense in $O(n+1, n)$.

(3) The Zariski closure of every group $\left\langle \ell\left(\widetilde{h}_i\right) \right\rangle$, $i = 1, \ldots, m$ is connected.

We will assume that $\widetilde{h}_i = \ell\left(\widetilde{h}_i\right)$, $i = 1, \ldots, m$, in other words, that all these elements fix the point zero. Let now $g_0 \in \mathrm{Aff}\,(\mathbb{R}^{2n+1})$ be a hyperbolic element transversal to every $\widetilde{h}_i$, $i = 1, \ldots, m$. We can assume that we choose g_0 such that there is a point q in $\mathbb{R}^{2n+1}$ such that $g_0(q) = q$. Let now ε, $\varepsilon \in \mathbb{R}$, be a positive number such that $g_0, \widetilde{h}_1, \ldots, \widetilde{h}_m$ are ε-transversal and ε-hyperbolic. Then there are vectors $v_1, \ldots, v_m$ such that if $h_i = \widetilde{h}_i v_i$, $i = 1, \ldots, m$ then $\alpha(h_i) > 0$, for all $i = 1, \ldots, m$, see (4.3) and (4.5) for the definition of α.

It is clear that for every n,$n \in \mathbb{Z}$, $n > 0$ and hyperbolic element h we have: $s(h^n) = s(h)^n$, $d(q, C_{h^n}) = d(q, C_h)$, $\alpha(h^n) = n\alpha(h)$. So, there is N, $N \in \mathbb{Z}$, $N > 0$ such that the elements g_0^N, $h_1^N, \ldots, h_m^N$ satisfy the requirements of Lemma 6.8, but on the other hand for every n, $n > N$, the elements $h_1^n, \ldots, h_m^n$ have Property (2).

We will assume therefore that the elements g_0, $h_1, \ldots, h_m$ satisfy Lemma 6.19 and so for $g_\ell = g_0 h_{i_1}^{n_1} \ldots h_{i_\ell}^{n_\ell}$, we have $\alpha(g_\ell) \geq \frac{2}{3} M_\ell T$, where $M_\ell = |n_1| + \cdots + |h_\ell|$ and $T = \max\{\alpha(h_1), \ldots, \alpha(h_m)\}$.

Let us now show that group $\langle h_1, \ldots, h_m \rangle = H$ acts properly discontinuously on $\mathbb{R}^{2n+1}$. Let K be any compact subset in $\mathbb{R}^{2n+1}$ then $\{h \in H : h\,K \cap K \neq \emptyset\} = \{h \in H : g_0 h\,K \cap g_0 K \neq \emptyset\}$.

There is a constant $c = c(K)$, such that if

$$g_0 h\,K \cap g_0 K \neq \emptyset, \quad \text{then} \quad d(g_0 h\,x, g_0 x) \leq c(K)$$

for some point $x \in K$, hence

$$\{h \in H : g_0 h\,K \cap g_0 K \neq \emptyset\} \subseteq \{h \in H : \alpha(g_0 h) \ll c(K)\}.$$

As we explained above, if $h = h_{i_1}^{n_1}, \ldots, h_{i_\ell}^{n_\ell}$ is in the latter set, then $M_\ell \leq \dfrac{3\alpha(g_0 h)}{2T} \ll \dfrac{3c(K)}{2T}$ and therefore

$$\#\{h \in H : h\,K \cap K \neq \emptyset\} \leq \#\left\{h \in H : M_\ell \leq \frac{3c(K)}{2T}\right\} < \infty.$$

q.e.d.

References

[1] H. Abels, *Properly discontinuous groups of affine transformations, a survey*, Geom. Dedicata **87** (2001) 309–333.

[2] H. Abels, G.A. Margulis & G.A. Soifer, *Semigroups containing proximal linear maps*, Israel J. of Math. **91** (1995) 1–30.

[3] H. Abels, G.A. Margulis & G.A. Soifer, *Properly discontinuous groups of affine transformations with orthogonal linear part*, Comptes Rendus Acad. Sci. Paris **324**(**I**) (1997) 253–258.

[4] L. Auslander. *The structure of compact locally affine manifolds*, Topology **3** (1964) 131–139.

[5] L. Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume* (Erste Abhandlung), Math. Ann. **70** 297–336.

[6] L. Bieberbach, *Über die Bewegungsgruppen der Euklidischen Räume* (Zweite Abhandlung), Math. Ann. **72** 400–412.

[7] T.A. Drumm, *Fundamental polyhedra for Margulis space–times,* Doctoral dissertation University of Maryland 1990.

[8] T.A. Drumm, *Examples of nonproper affine actions*, Mich. Math. J. **39** (1992) 435–442.

[9] T.A. Drumm, *Translations and the holonomy of complete affine flat manifolds*, Math. Res. Letters **1** (1994) 757–764.

[10] T.A. Drumm and W. Goldman, *Complete flat Lorentz 3-manifolds with free fundamental group*, Int. J. of Math. **1** (1990) 149–161.

[11] D. Fried and W.M. Goldman, *Three-dimensional affine crystallographic groups*, Adv. in Math. **47** (1983) 1–49.

[12] W.M. Goldman and Y. Kamishima, *The fundamental group of a compact flat Lorentz space form is virtually polycyclic*, J. Differential Geom. **19** (1984), 233–240.

[13] I.Ya. Goldsheid and G.A. Margulis, *Lyapunov exponents of a product of random matrices*, Uspekhi Mat. Nauk **44(5)** (1989), 13–60; English translation in Russian Math. Surveys **44(5)** (1989) 11–71.

[14] F. Grunewald and G.A. Margulis, *Transitive and quasitransitive actions of affine groups preserving a generalized Lorentz–structure,* J. of Geometry and Physics **5** (1988) 493–531.

[15] G.A. Margulis, *Free properly discontinuous groups of affine transformations*, Dokl. Akad. Nauk SSSR **272** (1983) 937–940.

[16] G.A. Margulis, *Complete affine locally flat manifolds with a free fundamental group*, J. Soviet Math. **134** (1987) 129–134; translated from Zapiski NS LOMI **134** (1984) 190–205.

[17] G.A. Margulis, *On the Zariski closure of the linear part of a properly discontinuous group of affine transformations*, preprint (1991).

[18] G. Mess, *Flat Lorentz spacetimes,* preprint, February 1990.

[19] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Adv. in Math. **25** (1977) 178–187.

[20] M.S. Raghunathan, *Discrete subgroups of Lie groups*, Springer 1972.

[21] J. Tits, *Free subgroups in linear groups*, J. of Algebra **20** (1972) 250–270.

Universität Bielefeld
33501 Bielefeld, Germany

Yale University
New Haven, CT 06520

Bar-Ilan University
52900 Ramat-Gan, Israel