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AN EIGENVALUE ESTIMATE FOR THE $\bar\partial$-LAPLACIAN

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Abstract

We study the $\bar\partial$ -Laplacian on forms taking values in L^k, a high power of a semipositive line bundle over a compact complex manifold, and give an estimate of the number of eigenvalues smaller than λ. Examples show that the estimate gives the right order of magnitude in terms of the two spectral parameters k and λ.

1. Introduction

Let X be an n-dimensional compact complex manifold and let F be a holomorphic line bundle over X. If we fix a hermitian metric on X and a line bundle metric on F we can define the formal adjoint of the $\bar\partial$-operator acting on F-valued differential forms and then form the corresponding Laplace operator

$$\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

Let

$$\mathcal{H}^{p,q}_{\leq\lambda}(F, X)$$

be the linear span of (p, q)-eigenforms of $\Box$, with corresponding eigenvalues smaller than or equal to λ. Thus in particular , $\mathcal{H}^{p,q}_{\leq 0} =: \mathcal{H}^{p,q}$ is the space of harmonic forms, which by Hodge's theorem is isomorphic to the Dolbeault cohomology groups. Put also

$$h^{p,q}_{\leq\lambda}(F) = \dim\mathcal{H}^{p,q}_{\leq\lambda}(F)$$

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so that $h^{p,q}_{\leq\lambda}$ is the number of eigenvalues, counting multiplicity, of $\Box$ acting on (p,q)-forms.

We now choose F to be of the special form

$$F = F_k = L^k \otimes E,$$

where $k \in N$, and L and E are holomorphic line bundles. The metric on F_k is taken to be the natural metric induced by given smooth metrics on E and L.

The main result of this paper is the following asymptotic estimate for the distribution of eigenvalues of $\Box$.

Theorem 1.1. *Assume L is given a metric of semipositive curvature. Take $q \geq 1$. Then, if $0 \leq \lambda \leq k$,*

$$h^{n,q}_{\leq\lambda}(L^k \otimes E, X) \leq C(\lambda+1)^q k^{n-q}. \tag{1.1}$$

If $1 \leq k \leq \lambda$, then

$$h^{n,q}_{\leq\lambda}(L^k \otimes E, X) \leq C\lambda^n. \tag{1.2}$$

Since E is allowed to be an arbitrary line bundle, we see by substituting $E \otimes K^{-1}$ (where K^{-1} is the dual of the canonical bundle of X) for E, that the same asymptotic estimate also holds for the numbers $h^{0,q}_{\leq\lambda}$.

In particular, we get the bound

$$h^{0,q}(F_k, X) =: h^{0,q}_{\leq 0}(F_k, X) \leq Ck^{n-q}$$

for the dimensions of the Dolbeault cohomology groups.

The first result of this type is due to Siu, [8], [9], in connection with his proof of the Grauert-Riemenschneider conjecture, [6]. Siu proved that for $q \geq 1$

$$h^{0,q}(L^k) = o(k^n).$$

He then went on to prove that if we assume that moreover the metric of L is strictly positive *somewhere*, one gets from this together with the Hirzebruch Riemann-Roch theorem a bound from below on the dimension of the space of holomorphic sections

$$h^{0,0}(L^k) \geq ck^n.$$

This in turn implies that a manifold X which admits a line bundle L having these properties (i.e., curvature semipositive everywhere and strictly positive somewhere), must be Moishezon, thus proving the conjecture of Grauert and Riemenschneider. A further consequence of all this is that we actually have

$$h^{n,q}(L^k) = 0$$

if q and k are greater than or equal to 1. It is important to realize however that if we drop the assumption that L be strictly positive somewhere, then the cohomology no longer needs to vanish, and the estimate that we get from Theorem 1.1 is in general of the best possible order of magnitude (see Section 4).

Very shortly after Siu's work Demailly found another approach to the Grauert-Riemenschneider conjecture based on his so called "Holomorphic Morse inequalities", see [3], [4], [5].

Here we no longer assume that L is even semipoistive. For an arbitrary (smooth) metric on L, Demailly found an asymptotic estimate for the alternating sums

$$\sum_{q=0}^{m}(-1)^{m-q}h^{0,q}(L^k \otimes E)$$

$(m \leq n)$ in terms of certain integrals involving L's curvature form. These estimates immediately imply Siu's upper estimate on the cohomology as well as the lower bound on the dimension on the space of holomorphis sections.

Demailly obtained his estimate from an asymptotic formula for the numbers $h^{p,q}_{\leq\lambda}$, for $\lambda = \lambda_0 k$, combined with an idea used in Witten's proof of the classical Morse inequalities, [12]. Demailly's formulas are however only exact up to an error term of size $o(k^n)$, so they appear to be less precise than Theorem 1.1 when L is semipositive.

After Demailly's work, estimates of lower order error terms were given by Bouche [1], [2]. Like Demailly, Bouche has no assumption on positivity of the line bundle, but instead supposes that the curvature form is everywhere degenerate. He then, among other things, obtained bounds on the dimensions of cohomology groups of the form $O(k^r)$ assuming that the curvature form everywhere has no more than r nonzero eigenvalues, and that moreover there exists a codimension r foliation of X such that the tangent space of the foliation is everywhere included in the kernel of the curvature form. The precise relation of Bouche's results and Theorem 1.1 is for the moment not clear to me, but one might note that the example we have of when Theorem 1.1 is optimal is of the type studied by Bouche.

Our proof of Theorem 1.1 is rather different from the proofs of both Siu and Demailly-Bouche. It is based on an estimate for the Bergman kernel for the spaces $\mathcal{H}^{n,q}_{\leq\lambda}$. By this we mean the function

$$B(x) = \sum |\alpha_j(x)|^2 \qquad x \in X,$$

where $\{\alpha_j\}$ is an orthonormal basis for $\mathcal{H}^{n,q}_{\leq\lambda}$, and the norm is the pointwise norm defined by the metrics on F and X. More precisely, $B(x)$ is the pointwise trace on the diagonal of the true Bergman kernel, defined as the reproducing kernel for $\mathcal{H}^{n,q}_{\leq\lambda}$.

The relevance of $B(x)$ for our problem lies in the formula

$$\int_X B(x) = h^{n,q}_{\leq\lambda},$$

which is evident since each term in the definition of B contributes a 1 to the integral. On the other hand, B is intimately related to the solution of the extremal problem

$$S(x) = \sup |\alpha(x)|^2/\|\alpha\|^2,$$

where the supremum is taken over all α in $\mathcal{H}^{n,q}_{\leq\lambda}$. In fact, it is easy to see, and well known, that

$$S(x) \leq B(x) \leq c_{n,q} S(x),$$

where $c_{n,q}$ are binomial coefficients. Theorem 1.1 therefore follows if we can prove a submeanvalue inequality that estimates the values of a form $\alpha \in \mathcal{H}^{n,q}_{\leq\lambda}$ at any point $x \in X$ by its norm in L^2 (see Theorem 2.3). We next explain how this is done for a harmonic form (i.e., for $\lambda = 0$), assuming X is Kähler. The general case is similar, modulo additional complications.

The first observation is that we always have, in local coordinates, z, where $z(x) = 0$ and the metric is approximately Euclidean, that

$$|\alpha(0)|^2 \leq Ck^n \int_{|z|\leq k^{-1/2}} |\alpha|^2. \tag{1.3}$$

This holds even if L is not semipositive. The reason is that on this small scale the metric on F_k is essentially independent of k, so the inequality follows from a submeanvalue inequality for a k-independent elliptic operator.

The next, and most important, step is to apply Siu's so called $\partial\bar\partial$-Bochner Kodaira formula, see [10]. To explain what this amounts to, first recall that the main idea in the classical Bochner method is to draw conclusions about, e.g., harmonic forms by considering

$$\Box|\alpha|^2,$$

i.e., the Laplacian of the *function* $|\alpha|^2$. In the $\partial\bar\partial$ Bochner formula we instead consider, for α an F-valued (n,q)-form, the new differential *form* locally defined as

$$T_\alpha = c_{n-q}\gamma\wedge\bar\gamma e^{-\phi},$$

where c_{n-q} is choosen so that T_α becomes a nonnegative $(n-q,n-q)$-form, $\gamma = *\alpha$ is the Hodge-$*$ of α, and $e^{-\phi}$ locally defines the metric on F. A similar construction is possible also for $(0,q)$-forms, and indeed for forms of arbitrary bidegree, but for semipositive bundles it is better to work with (n,q) and then draw conclusions about $(0,q)$ by tensoring with the inverse of the bundle of $(n,0)$-forms. Then

$$|\alpha|^2\omega_n = T_\alpha\wedge\omega_q$$

where ω is the metric form on X, so the norm of α is given by the trace of T_α. One computes $\partial\bar\partial T_\alpha$ and finds that

$$i\partial\bar\partial T_\alpha\wedge\omega_{q-1}\geq 0$$

if α is harmonic and the metric is Kähler. It then follows from a result of Skoda, [11], that the function $\sigma(r)/r^{2q}$, where

$$\sigma(r) = \int_{|z|<r}|\alpha|^2,$$

is essentially nondecreasing in r for r small. More precisely, Skoda's theorem says that if the metric is Euclidean in the coordinates z, then $\sigma(r)/r^{2q}$ is nondecreasing, and from this one can deduce that it is almost nondecreasing if ω is almost Euclidean. Combining with (1.3) we obtain,

$$|\alpha(0)|^2\leq Ck^{n-q}\int_{|z|<1}|\alpha|^2$$

and as explained above this submeanvalue inequality proves Theorem 1.1 in this case.

The plan of this paper is as follows. In Section 2 we prove the submeanvalue inequality for forms in $\mathcal{H}^{n,q}_{\leq\lambda}$ by a variant of Skoda's argument. The proof uses a generalization of Siu's $\partial\bar\partial$-Bochner Kodaira formula for non-Kähler manifolds, the proof of which we postpone to Section 3. (This replaces the use of the non-Kählerian Kodaira-Nakano formula of Griffiths-Demailly in Siu's and Demailly's arguments.) In Section 4 we complete the proof of Theorem 1.1, and give examples showing our estimates give the right order of magnitude. In the final section we relate Theorem 1.1 to L^2-estimates for $\bar\partial$, and show how these imply the Grauert-Riemenschneider conjecture.

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2. A submeanvalue inequality for (almost-)harmonic forms

Fix a point x in X and choose local coordinates, $z = (z_1, \dots z_n)$ near x such that $z(x) = 0$ and such that the metric form on X, ω, satisfies

$$\omega = i/2\partial\bar\partial|z|^2 = \beta$$

at the point x. The next proposition is the crucial step in our arguments. We formulate it for (n,q)-forms since that makes the proofs more natural, but the same statement also holds for $(0,q)$-forms as we can see by substituting

$$E = E' \otimes K_X^{-1},$$

where K_X is the canonical bundle.

Theorem 2.1. *Let $\alpha \in \mathcal{H}^{n,q}_{\leq\lambda}(L^k \otimes E, X)$ satisfy $\bar\partial\alpha = 0$. Assume the metric on L has semipositive curvature. Then, for $r < \lambda^{-1/2}$ and $r < c_0$*

$$\int_{|z|<r} |\alpha|^2 \leq Cr^{2q}(\lambda+1)^q \int_X |\alpha|^2.$$

The constants c_0 and C are independent of k, λ and the point x.

For the proof of Theorem 2.1 we associate to α a certain form, T_α, of bidegree $(n-q, n-q)$. To define T_α we represent α by a scalar valued form α_j with the aid of a local trivialization of $F = F_k = L^k \otimes E$. We then let $\gamma_j = *\alpha_j$, where $*$ is the Hodge operator of the metric ω on X, and put

$$T_\alpha = c_{n-q}\gamma_j \wedge \bar\gamma_j e^{-\phi_j},$$

where ϕ_j defines the metric on F, and $c_{n-q} = (i)^{(n-q)^2}$ is choosen so that $T_\alpha \geq 0$. It is easy to see that T_α is well defined globally. We shall now use the following differential inequality.

Proposition 2.2. *Let α be an F-valued (n,q)-form. If α is $\bar\partial$-closed*

$$i\partial\bar\partial T_\alpha \wedge \omega_{q-1} \geq -(2\mathrm{Re}\,\langle \Box\alpha, \alpha\rangle + \langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle - c|\alpha|^2)\omega_n. \tag{2.1}$$

The constant c is equal to zero if $\bar\partial\omega_{q-1} = \bar\partial\omega_q = 0$, hence in particular if ω is Kähler.

Here Λ is the adjoint of the operator exterior multiplication with ω, and Θ_F is the curvature form on F. Alternately the curvature term in (2.1) can be expressed as

$$\langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle\omega_n = \Theta_F \wedge c_{n-q}\gamma_j \wedge \bar\gamma_j \wedge \omega_{q-1} e^{-\phi_j}.$$

In particular, in our case $F = F_k = L^k \otimes E$ so $\Theta_F = k\Theta_L + \Theta_E$. From now on we assume that $\Theta_L \geq 0$ so the curvature term is always bigger than

$$\langle \Theta_E \wedge \Lambda\alpha, \alpha\rangle\omega_n.$$

This expression is independent of k and can be estimated from below by a constant times $|\alpha|^2$, so we get

$$i\partial\bar\partial T_\alpha \wedge \omega_{q-1} \geq -(2\mathrm{Re}\,\langle \Box\alpha, \alpha\rangle - c'|\alpha|^2)\omega_n. \tag{2.2}$$

Put, for r small

$$\sigma(r) = \int_{|z|<r} |\alpha|^2\omega_n = \int_{|z|<r} T_\alpha \wedge \omega_q.$$

In the proof of Theorem 1.1 we may clearly assume that $\lambda \geq 1$. Theorem 2.1 then says that the inequality

$$\sigma(r) \leq Cr^{2q}\lambda^q, \tag{2.3}$$

holds if we have normalized so that the L^2-norm of α is equal to 1.

From (2.2) we see that

$$\int_{|z|<r} (r^2 - |z|^2) i\partial\bar\partial T_\alpha \wedge \omega_{q-1} \geq -c'r^2\sigma(r) - 2r^2 \int_{|z|<r} |\Box\alpha||\alpha|\omega_n. \tag{2.4}$$

Put

$$\lambda(r) = \left(\int_{|z|<r} |\Box\alpha|^2\right)^{1/2},$$

and use Cauchy's inequality to obtain

$$\int_{|z|<r} |\Box\alpha||\alpha|\omega_n \leq \lambda(r)\sigma(r)^{1/2}.$$

Trivially $\lambda(r) \leq \lambda$ if $\alpha \in \mathcal{H}^{n,q}_{\leq\lambda}$, but we shall see later that we can get a better estimate. Applying Stokes' formula to the left hand side of (2.4) we get, since $\beta = i/2\partial\bar\partial|z|^2$,

$$\begin{aligned} 2\int_{|z|<r} & T_\alpha \wedge \omega_{q-1} \wedge \beta \\ &\leq \int_{|z|=r} T_\alpha \wedge \omega_{q-1} \wedge \partial|z|^2 + cr^2\sigma(r) + 2r^2\sigma(r)^{1/2}\lambda(r). \end{aligned} \tag{2.5}$$

By the choice of local coordinates we have, since ω is smooth,

$$(1 - O(r))\omega \leq \beta \leq (1 - O(r))\omega.$$

Hence

$$T_\alpha \wedge \omega_{q-1} \wedge \beta \geq q(1 - O(r))|\alpha|^2\omega_n.$$

Next, if $\omega = \beta$ the boundary integral in (2.5) can be estimated by an integral with respect to surface measure

$$r\int_{|z|=r} |\alpha|^2 dS,$$

and this implies that in our case

$$\int_{|z|=r} T_\alpha \wedge \omega_{q-1} \wedge \partial|z|^2 \leq r(1 - O(r)) \int_{|z|=r} |\alpha|^2(\omega_n/\beta_n)dS.$$

But

$$\int_{|z|=r} |\alpha|^2(\omega_n/\beta_n)dS = \sigma'(r),$$

so if we also incorporate the term $cr^2\sigma(r)$ in $O(r)\sigma(r)$, we get

$$2q(1 - O(r))\sigma(r) \leq r\sigma'(r) + 2r^2\sigma(r)^{1/2}\lambda(r).$$

Finally we substitute $\sigma(r) = s(r)^2$ and divide by $2rs(r)$ to obtain

$$q(1/r - O(1))s(r) \leq s'(r) + r\lambda(r). \tag{2.6}$$

We shall now prove

$$s(r) \leq Cr^m \lambda^{m/2}$$

for $m \leq q$ by induction over m. The statement is trivial for $m = 0$, since we have assumed that the L^2-norm of α is 1, so assume it has been proved for a certain value of $m < q$. Then (2.6) implies

$$(m+1)(1/r - O(1))s(r) \leq s'(r) + r\lambda(r).$$

The form $\Box\alpha$ also lies in $\mathcal{H}^{n,q}_{\leq\lambda}$ and has L^2 norm bounded by λ. By the induction hypothesis we then get that

$$\lambda(r) \leq Cr^m \lambda^{m/2+1}.$$

We therefore have

$$(m+1)(1/r - O(1))s(r) \leq s' + Cr^{m+1}\lambda^{m/2+1}.$$

Put

$$\Phi = (m+1)\int 1/r - O(1)dr \sim (m+1)\log r$$

and multiply by the integrating factor $e^{-\Phi}$. The result is that

$$(se^{-\Phi})' \geq -C\lambda^{m/2+1}.$$

Integrate this inequality from r to $\lambda^{-1/2}$. Since $e^{-\Phi} \sim 1/r^{m+1}$ we obtain

$$r^{-(m+1)}s(r) \leq C\lambda^{m/2+1/2} + s(\lambda^{-1/2})\lambda^{m/2+1/2} \leq C\lambda^{m/2+1/2}.$$

By induction we therefore have that

$$s(r) \leq Cr^q \lambda^{q/2},$$

which after squaring gives the desired estimate for $\sigma(r)$. This completes the proof of Theorem 2.1.

We are now ready to state and prove the submeanvalue inequality.

Theorem 2.3. *Let $\alpha \in \mathcal{H}^{n,q}_{\leq\lambda}(L^k \otimes E)$ satisfy $\bar\partial\alpha = 0$. Assume the line bundle L is equipped with a metric of semipositive curvature. Then for any $x \in X$*

$$|\alpha(x)|^2 \leq Ck^{n-q}(\lambda+1)^q \int_X |\alpha|^2\omega_n \tag{2.7}$$

if $\lambda \leq k$ and

$$|\alpha(x)|^2 \leq C\lambda^n \int_X |\alpha|^2\omega_n \tag{2.8}$$

if $\lambda \geq k \geq 1$. The constant C is independent of k, λ and x.

Proof. Assume first $\lambda \leq k$ and fix x. Choose as before local coordinates, z, near x such that $z(x) = 0$ and $\omega = i/2\partial\bar\partial|z|^2 = \beta$ at the point x. Choose also local trivializations of L and E near x. We may assume the local trivializations are chosen so that the metric of L has the form

$$\phi(z) = \sum \mu_j|z_j|^2 + o(|z|^2).$$

For any form α we express α in terms of the trivialization and local coordinates and put

$$\alpha^{(k)}(z) = \alpha(z/\sqrt{k}),$$

so that $\alpha^{(k)}$ is defined for $|z| < 1$ if k is large enough. We also scale the laplacian by putting

$$k\Box^{(k)}\alpha^{(k)} = (\Box\alpha)^{(k)}.$$

It is not hard to see that if $\Box$ is defined by the metric ψ on F_k, then $\Box^{(k)}$ is associated to the line bundle metric $\psi(z/\sqrt{k})$. In particular, if $F_k = L^k \otimes E$ and $\psi = k\phi + \psi_0$, then $\Box^{(k)}$ is associated to

$$\sum \mu_j|z_j|^2 + \psi_0(0) + o(1),$$

and hence converges to a k-independent elliptic operator. It therefore follows from Gårding's inequality together with Sobolev estimates that

$$|\alpha(0)|^2 \leq C\left(\int_{|z|<1} |\alpha^{(k)}|^2\omega_n + \int_{|z|<1} |\left(\Box^{(k)}\right)^m \alpha^{(k)}|^2\omega_n\right), \tag{2.9}$$

if $m > n/2$. Now

$$\int_{|z|<1} |\alpha^{(k)}|^2\omega_n = k^n \int_{|z|<1/\sqrt{k}} |\alpha|^2\omega_n$$

and

$$\int_{|z|<1} \left|\left(\Box^{(k)}\right)^m \alpha^{(k)}\right|^2 \omega_n = k^{n-2m} \int_{|z|<1/\sqrt{k}} |(\Box)^m \alpha|^2 \omega_n.$$

Normalize so that the L^2-norm of α is one. By Theorem 2.1

$$k^n \int_{|z|<1/\sqrt{k}} |\alpha|^2 \omega_n \le C k^{n-q} (\lambda+1)^q,$$

and

$$\begin{aligned} k^{n-2m} \int_{|z|<1/\sqrt{k}} |(\Box)^m \alpha|^2 \omega_n &\le C k^{n-q} (\lambda+1)^q (\lambda/k)^{2m} \\ &\le C k^{n-q} (\lambda+1)^q. \end{aligned}$$

We have thus proved the first part of Theorem 2.3. The second statement is much easier. We now apply (2.9) to the scaling $\alpha^{(\lambda)}$ instead, and get immediately (i.e., without using Theorem 2.1) that

$$|\alpha(0)|^2 \le C \lambda^n.$$

q.e.d.

3. The $\partial\bar{\partial}$-Bochner formula for non-Kähler manifolds

In this section we prove a differential inequality leading to Proposition 2.2. The basic idea is due to Siu [10], but we also need to control the extra terms that occur when the metric is not Kähler.

Let α be an F-valued (n, q)-form. As in Section 2 we define an associated $(n-q, n-q)$-form which in a local trivialization is defined as

$$T_\alpha = c_{n-q} \gamma \wedge \bar{\gamma} e^{-\psi},$$

where $\gamma = *\alpha$, $c_{n-q} = i^{(n-q)^2}$, and ψ defines the metric on F. (Since the computation is local we have dropped the subscripts on γ and α.) Here * denotes the Hodge operator of the hermitian manifold X, defined by the formula

$$\alpha \wedge \overline{*\alpha} = |\alpha|^2 \omega_n.$$

In particular, note that with the convention we use, * is a complex linear operator. The relation $*\alpha = \gamma$ can also be expressed as

$$\alpha = c_{n-q} \gamma \wedge \omega_q,$$

and we moreover have

$$*\gamma = (-1)^{n-q} c_{n-q} \gamma \wedge \omega_q.$$

In the following computations we shall also use the relations $ic_q = (-1)^q c_{q-1}$ and $c_{q-1} = c_{q+1}$. We first compute $\partial\bar\partial T_\alpha$. Put $\partial_\psi = e^\psi \partial e^{-\psi}$ and use

$$\bar\partial(\chi \wedge \bar\eta e^{-\psi}) = \bar\partial\chi \wedge \bar\eta e^{-\psi} + (-1)^s \chi \wedge \overline{\partial_\psi \eta} e^{-\psi}$$

and a similar formula for ∂, if χ and η are forms and s is the degree of χ. We get

$$\begin{aligned}\partial\bar\partial(\gamma \wedge \bar\gamma e^{-\psi}) = \Big(&\partial_\psi\bar\partial\gamma \wedge \bar\gamma + \gamma \wedge \overline{\bar\partial\partial_\psi\gamma} \\ &+ (-1)^{n-q}\partial_\psi\gamma \wedge \overline{\partial_\psi\gamma} + (-1)^{n-q+1}\bar\partial\gamma \wedge \overline{\bar\partial\gamma}\Big)e^{-\psi}\end{aligned}$$

which equals

$$\begin{aligned}(3.1)\quad (-\bar\partial\partial_\psi\gamma \wedge \bar\gamma + \gamma \wedge \overline{\bar\partial\partial_\psi\gamma} + \partial\bar\partial\psi \wedge \gamma \wedge \bar\gamma \\ + (-1)^{n-q}\partial_\psi\gamma \wedge \overline{\partial_\psi\gamma} + (-1)^{n-q+1}\bar\partial\gamma \wedge \overline{\bar\partial\gamma})e^{-\psi}\end{aligned}$$

if we also use the commutator formula

$$\partial_\psi\bar\partial + \bar\partial\partial_\psi = \partial\bar\partial\psi.$$

First note that

$$\bar\partial^*\alpha = - * \partial_\psi\gamma = (-1)^{n-q} c_{n-q-1} \partial_\psi\gamma \wedge \omega_{q-1},$$

which implies

$$\begin{aligned}\bar\partial\bar\partial^*\alpha &= (-1)^{n-q} c_{n-q-1}(\bar\partial\partial_\psi\gamma \wedge \omega_{q-1} + (-1)^{n-q-1}\partial_\psi\gamma \wedge \bar\partial\omega_{q-1}) \\ &= (-1)^{n-q} c_{n-q-1}(\bar\partial\partial_\psi\gamma \wedge \omega_{q-1} + O(|\bar\partial^*\alpha||\bar\partial\omega_{q-1}|)).\end{aligned}$$

Multiply (3.1) by $ic_{n-q}\omega_{q-1}$ and recall $ic_{n-q} = (-1)^{n-q} c_{n-q-1}$. We have five terms. By the last formula, the first term equals

$$-\bar\partial\bar\partial^*\alpha \wedge \bar\gamma e^{-\psi} = -\langle \bar\partial\bar\partial^*\alpha, \alpha\rangle \omega_n$$

up to an error of size $O(|\bar\partial^*\alpha||\bar\partial\omega_{q-1}||\alpha|)$. Since the entire expression is real, the second term must be the conjugate of the first one so these terms together give

$$-2\mathrm{Re}\,\langle \bar\partial\bar\partial^*\alpha, \alpha\rangle \omega_n.$$

The third term is the curvature term

$$\langle \Theta_F \wedge \Lambda\alpha, \alpha\rangle \omega_n.$$

Checking signs we see that the fourth term equals

$$\left|\bar{\partial}^*\alpha\right|^2 \omega_n,$$

so it only remains to analyse the last term, which is a bit more subtle, since it defines a non-definite quadratic form.

Consider the bilinear form on $(n-q,1)$-forms defined by

$$[\chi, \eta]\omega_n = ic_{n-q}(-1)^{n-q+1}\chi \wedge \bar{\eta} \wedge \omega_{q-1} = -c_{n-q+1}\chi \wedge \bar{\eta} \wedge \omega_{q-1}.$$

It is clear that this form is negative definite on the subspace, V, of forms that can be written $\chi = \chi_0 \wedge \omega$ (it then equals a negative multiple of the norm squared of χ_0). The annihilator of V with respect to $[\ ,\ ]$, V°, clearly consists precisely of forms satisfying $\chi \wedge \omega_q = 0$, and since $[\ ,\ ]$ is definite on V we have $V \cap V^\circ = \{0\}$. One can now verify by brute calculation in an orthonormal basis for ω that $[\ ,\ ]$ is positive definite on V°. Accepting this, it follows in particular that any form χ can be decomposed uniquely

$$\chi = \chi_1 + \chi_0 \wedge \omega$$

with χ_1 in V° (this is a special case of the Lefschetz primitive decomposition).

We now consider $\chi = \bar{\partial}\gamma$ and recall that

$$\alpha = c_{n-q}\gamma \wedge \omega_q.$$

From now on we assume that α is $\bar{\partial}$-closed, so

$$\bar{\partial}\gamma \wedge \omega_q = (-1)^{n-q-1}\gamma \wedge \bar{\partial}\omega_q.$$

It follows that when we decompose $\bar{\partial}\gamma = \chi_1 + \chi_0 \wedge \omega$ we have

$$\chi_0 \wedge \omega \wedge \omega_q = (-1)^{n-q-1}\gamma \wedge \bar{\partial}\omega_q.$$

Since χ_0 is of bidegree $(n-q-1,0)$ this means that $|\chi_0| = O(|\gamma||\bar{\partial}\omega_q|) = O(|\alpha||\bar{\partial}\omega_q|)$. This means that the only possible negative contribution of $[\bar{\partial}\gamma, \bar{\partial}\gamma]$ can be estimated by $c|\alpha|^2$. If we also estimate the earlier error term

$$O(|\bar{\partial}^*\alpha||\bar{\partial}\omega_{q-1}||\alpha|) \le C_\epsilon|\alpha|^2 + \epsilon|\bar{\partial}^*\alpha|^2,$$

and collect all the terms we get

(3.2)
$$i\partial\bar\partial T_\alpha \geq -2\mathrm{Re}\,\langle\Box\alpha,\alpha\rangle + \langle\Theta_F\wedge\Lambda\alpha,\alpha\rangle + (1-\epsilon)|\bar\partial^*\alpha|^2 - C_\epsilon|\alpha|^2,$$

which in particular proves Proposition 2.2. (Note that $\bar\partial\bar\partial^*\alpha = \Box\alpha$ if α is $\bar\partial$-closed.)

4. Proof of the main result

Let E be an hermitian vector bundle of rank N over a manifold X which is equipped with a positive measure, $d\mu$. Let V be a subspace of the space of continuous global sections in $L^2(X)$, and let α_j be an orthonormal basis for V. Define for x in X

$$B(x) = \sum |\alpha_j(x)|^2$$

and

$$S(x) = \sup_{\alpha\in V} |\alpha(x)|^2/\|\alpha\|^2.$$

The next lemma is classical in Bergman's theory of reproducing kernels, but is perhaps most well known when $N = 1$.

Lemma 4.1. *With assumptions as above*

$$S(x) \leq B(x) \leq NS(x). \tag{4.1}$$

In particular

$$\int_X S(x)d\mu \leq \dim(V) \leq N\int_X S(x)d\mu. \tag{4.2}$$

Proof. Let $\alpha\in V$ have norm less than one. Then

$$\alpha = \sum c_j\alpha_j, \qquad \sum |c_j|^2 \leq 1.$$

Hence $|\alpha(x)|^2 \leq \sum |\alpha_j(x)|^2$, so $S(x) \leq B(x)$. For the second inequality fix $x\in X$ and represent a section α at x by $(\alpha^1(x),\dots\alpha^N(x))$ with the help of a local trivialization chosen so that

$$|\alpha(x)|^2 = \sum_1^N |\alpha^l(x)|^2.$$

Then

$$B(x) = \sum_l \sum_j |\alpha_j^l(x)|^2.$$

For l fixed put $c_j = \overline{\alpha}_j^l(x)$. Then

$$\sum_j |\alpha_j^l(x)|^2 = \left|\sum_j c_j \alpha_j^l(x)\right| \leq \left|\sum c_j \alpha_j(x)\right| \leq \left(\sum |c_j|^2\right)^{1/2} S(x)^{1/2}.$$

Hence, for any l

$$\sum_j |\alpha_j^l(x)|^2 \leq S(x),$$

and summing over l we get (4.1). We then get (4.2) by integrating over X. q.e.d.

We now have all the ingredients for the proof of Theorem 1.1. Let first $\mathcal{Z}^{n,q}_{\leq\lambda}$ be the subspace of $\mathcal{H}^{n,q}_{\leq\lambda}$ consisting of $\bar{\partial}$-closed forms. We apply Lemma 4.1 with E equal to the bundle of (n,q)-forms with values in F_k. The estimate for $S(x)$ furnished by Theorem 2.3 together with Lemma 4.1 then immediately gives Theorem 1.1 for $\mathcal{Z}^{n,q}_{\leq\lambda}$. We shall now see that

$$h^{n,q}_{\leq\lambda} \leq \dim \mathcal{Z}^{n,q}_{\leq\lambda} + \dim \mathcal{Z}^{n,q+1}_{\leq\lambda}, \tag{4.3}$$

which completes the proof since our estimate for $\dim Z^{n,q+1}_{\leq\lambda}$ is better, or at least not worse, than our estimate for $\mathcal{Z}^{n,q}_{\leq\lambda}$.

First note that if α is an eigenform of $\Box$, so that

$$\Box\alpha = \lambda\alpha$$

and if we decompose $\alpha = \alpha^1 + \alpha^2$ where α^1 is $\bar{\partial}$-closed and α^2 is orthogonal to the space of $\bar{\partial}$-closed forms then the α^j's are also eigenforms with the same eigenvalue. To see this, note that $\Box$ commutes with $\bar{\partial}$, so $\bar{\partial}\Box\alpha^1 = 0$ and

$$\langle \Box\alpha^2, \eta\rangle = \langle \alpha^2, \Box\eta\rangle = 0$$

if $\bar{\partial}\eta = 0$. Hence

$$\Box\alpha^j = (\Box\alpha)^j = \lambda\alpha^j.$$

Now decompose

$$\mathcal{H}^{n,q}_{\leq\lambda} = \mathcal{Z}^{n,q}_{\leq\lambda} \oplus (\mathcal{H}^{n,q}_{\leq\lambda} \ominus \mathcal{Z}^{n,q}_{\leq\lambda}).$$

Since $\bar\partial$ maps $\mathcal{H}^{n,q}_{\leq\lambda} \ominus \mathcal{Z}^{n,q}_{\leq\lambda}$ injectively into $(Z)^{n,q+1}_{\leq\lambda}$ (4.3) follows and the proof of Theorem 1.1 is complete.

The next simple proposition shows that the order of magnitude given in Theorem 1.1 can not be improved in general.

Proposition 4.2. *For any $0 \leq q \leq n$ there exists a compact Kähler manifold X and a semipositive line bundle over X such that*

$$h^{n,q}_{\leq\lambda}(L^k) \geq c(\lambda+1)^q k^{n-q}$$

for large k.

Proof. We use two classical facts. The first one is that if L is a strictly positive line bundle over an m-dimensional manifold, then

$$h^{m,0}(L^k) \sim k^m$$

for large k, see, e.g., [5] for references. The second one is that if I denotes the trivial line bundle over a manifold that is still m-dimensional then

$$h^{m,m}_{\leq\lambda}(I) \sim \lambda^m.$$

This is the classical Weyl asymptotics for an elliptic operator, in this case $\Box$ on (m,m)-forms. Actually, for the sake of constructing an example we can take the line bundle $\mathcal{O}(1)$ on projective space in the first case, and a product of tori in the second case and it is then easy to verify both claims directly. Put

$$X = S \times V,$$

where S is q-dimensional and V is of dimension $n-q$, and let L be the pullback to X of a strictly positive line bundle, L' over V. Then L is a semipositive bundle over X. The laplacian on X acting on L^k-valued forms can be decomposed

$$\Box = \Box_S + \Box_V$$

with the obvious notation. Any form on X of the type $\alpha = h\eta$, where h is the pullback to X of a holomorphic $(n-q)$-form on V with values in $(L')^k$, and η is the pullback of an eigenform of $\Box_S$, is therefore an eigenform of $\Box$. Hence

$$h^{n,q}_{\leq\lambda}(L^k, X) \geq h^{q,q}(I, S) h^{n-q,0}((L')^k, V) \geq c(\lambda+1)^q k^{n-q}.$$

q.e.d.

5. L^2-estimates and the Grauert-Riemenschneider conjecture

In the case of a semipositive line bundle with curvature positive definite at some point, Theorem 1.1 combined with the Riemann-Roch formula of Hirzebruch leads to a proof of the Grauert-Riemenschneider conjecture as in Siu, [8],[9]. We shall now give another approach that does not depend on Riemann-Roch but instead uses L^2-estimates for solutions of the $\bar\partial$-equation.

It is well known that a $\bar\partial$-closed form is exact if and only if it is orthogonal to the space of harmonic forms. We shall now see that if the form is even orthogonal to $\mathcal{H}^{p,q}_{\leq\lambda}$ then we can solve the $\bar\partial$-equation with an estimate.

Proposition 5.1. *Let f be a $\bar\partial$-closed (p,q)-form with values in F with $q \geq 1$. Assume f is orthogonal to $\mathcal{H}^{p,q}_{\leq\lambda}(F)$. Then we can solve*

$$\bar\partial u = f$$

with

$$\|u\|^2 \leq \frac{1}{\lambda}\|f\|^2.$$

Proof. By the Hodge decomposition

$$f = \Box v = \bar\partial\bar\partial^* v + \bar\partial^*\bar\partial v,$$

since f is in particular orthogonal to the harmonic forms. Since $\bar\partial f = 0$ it follows that

$$0 = \langle \bar\partial\bar\partial^*\bar\partial v, v\rangle = \|\bar\partial^*\bar\partial v\|^2,$$

so $f = \bar\partial u$, $u = \bar\partial^* v$. Expand v in an orthonormal basis of eigenforms

$$v = \sum c_j v_j, \qquad \Box v_j = \lambda_j v_j.$$

Then

$$f = \sum c_j \lambda_j v_j,$$

so by the hypothesis on f, $c_j = 0$ if $\lambda_j \leq \lambda$. Then

$$\|u\|^2 = \langle \bar\partial^* v, \bar\partial^* v\rangle = \langle f, v\rangle = \sum |c_j|^2\lambda_j \leq 1/\lambda \sum |c_j|^2\lambda_j^2 = 1/\lambda\|f\|^2.$$

q.e.d.

Let us now return to the situation of the Grauert-Riemenschneider conjecture. Consider two open sets, U and V in X, with U compactly included in V. We say that a subspace W of sections to F over V is ϵ-concentrated on U if

$$\int_U |w|^2 \geq (1-\epsilon)\int_V |w|^2$$

for all $w \in W$. It follows from a result of Lindholm [7] that if V is pseudoconvex and $F = L^k$ with $L > 0$ then for any $\epsilon > 0$ there is a subspace W of holomorphic sections which is ϵ-concentrated on U with dimension asymptotic to

$$c_n k^n \int_U (\Theta_L)^n$$

as k tends to infinity. For us it will be enough to use the simpler fact that if L is strictly positive somewhere we can find U and V and a corresponding subspace W which is $1/2$-concentrated and has dimension bounded from below by a constant times k^n. For this it suffices to take as U and V concentric small balls inside the region where L:s curvature is positive. If we choose local coordinates and trivializations so that the metric on L has the form

$$\phi(z) = \sum \mu_j |z_j|^2 + o(|z|^2)$$

with $\mu_j > 0$ we can take W as the space of polynomials in z of degree bounded by, say, $n/2$.

For any $w \in W$ we form

$$f_w = \bar\partial \chi w,$$

where χ is smooth with support in V and equal to 1 in U. The dimension of the space of f_w's is then of the order k^n so by Theorem 1.1 and Proposition 5.1 we can for a subspace of this space of almost full dimension solve $\bar\partial u_w = f_w$ with $\|u_w\|^2 \leq 1/\lambda \|f_w\|^2$. Put $S(w) = \chi w - u_w$. Then $S(w)$ is a global holomorphic section to F, and if λ is chosen large enough, the map from w to $S(w)$ is injective. Hence it follows that

$$h^{0,0}(L^k) \geq c k^n$$

and the Grauert-Riemenschneider conjecture follows from this as in the papers of Siu and Demailly.

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