J. DIFFERENTIAL GEOMETRY
60 (2002) 155-175

ADDING HANDLES TO NADIRASHVILI'S SURFACES

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Abstract

We construct complete bounded minimal surfaces in $\mathbb{R}^3$ with arbitrary topological genus.

1. Introduction

The so called Calabi-Yau problem, which deals with the existence of complete nonflat minimal surfaces with bounded coordinate functions, has been the instigator of many interesting articles on the theory of minimal surfaces in $\mathbb{R}^3$ over the last few decades.

Two articles, in particular, have made very important, if not fundamental, contributions. The first one was by L.P. Jorge and F. Xavier [2], who constructed examples in a slab. The second one was by N. Nadirashvili [5], who recently produced examples contained in a ball. In both cases, the key step was the ingenious use of Runge's classical theorem.

In respect to complete bounded minimal surfaces, an open question still remains as to whether information about their geometry can be obtained [8]. One approach to this problem consists of deciding whether Nadirashvili's surfaces with nontrivial topology exist or not. The first such surface, with the topology of a cylinder, was obtained in [4].

However, in general, constructing examples with nontrivial topology is a difficult matter because of the period conditions. This problem has been dealt with in depth over the last few years for several families of

Research partially supported by MCYT-FEDER Grant No. BFM2001-3489.
Received March 13, 2002.

minimal surfaces, including the parabolic case [7] and the hyperbolic one [3].

In this paper, we have proved the following theorem:

> **Theorem.** *For any genus $\sigma \geq 1$, there exists a complete bounded minimal surface in $\mathbb{R}^3$ with genus σ and one end.*

Our procedure works as follows:

Firstly, we deform the Weierstrass data of a given minimal surface of genus σ and nonempty boundary, $\sigma \geq 1$. In order to do this, we use the Implicit Function Theorem and Runge's theorem, in such a way that the resulting surface has no periods. The second step consists of adapting Nadirashvili's techniques to this more general setting of nontrivial topology. Hence, our deformation increases the intrinsic diameter, but it controls the Euclidean diameter in $\mathbb{R}^3$. In this way we construct a sequence of genus σ minimal sufaces contained in a fixed ball, which converges to a complete genus σ minimal surface lying in the same ball.

The paper is structured as follows. In Section 2 we introduce all the notation and concepts that we have used throughout the paper. Section 3 sets out the principal results in this paper: two lemmas and the main theorem. In this section, the main theorem has been proved by using Lemma 2. The proof of this lemma is quite technical and has been given in Section 5. Lemma 1 is a tool for getting Lemma 2 and has been proved in Section 4.

2. Background and notation

Let $\mathcal{N}$ and $d\hat{s}^2$ be a Riemann surface and a Riemannian conformal metric on $\mathcal{N}$, respectively. Given a curve α in $\mathcal{N}$, by $\mathrm{length}(\alpha, d\hat{s})$ we mean the length of α with the metric $d\hat{s}^2$. Given a subset $W \subset \mathcal{N}$, we define:

- $\mathrm{dist}_{(d\hat{s},W)}(p,q) = \inf\{\mathrm{length}(\alpha, d\hat{s}) \mid \alpha : [0,1] \to W,\ \alpha(0) = p,\ \alpha(1) = q\}$, for $p, q \in W$,
- $\mathrm{dist}_{(d\hat{s},W)}(T_1,T_2) = \inf\{\mathrm{dist}_{(d\hat{s},W)}(p,q) : p \in T_1,\ q \in T_2\}$, for $T_1, T_2 \subset W$,
- $\mathrm{diam}_{d\hat{s}}(W) = \sup\{\mathrm{dist}_{(d\hat{s},W)}(p,q) : p,q \in W\}$.

The concepts of (multiplicative) divisor on $\mathcal{N}$, integral divisor on $\mathcal{N}$, and the natural partial ordering, $\geq$, on divisors can be found in [1].

Let ω be a meromorphic function or 1-form on $\mathcal{N}$. Let $W \subset \mathcal{N}$ and suppose that ω has a finite number of zeroes, $z_1, \dots, z_n$, and a finite number of poles, $p_1, \dots, p_n$, in W. We denote by $(\omega_{|W})_0 = z_1 \dots z_n$, $(\omega_{|W})_\infty = p_1 \dots p_n$, and $(\omega_{|W}) = (\omega_{|W})_0/(\omega_{|W})_\infty$, the zero divisor, the polar divisor, and the divisor of ω on W, respectively. When $W = \mathcal{N}$, we simply write (ω), $(\omega)_0$, and $(\omega)_\infty$, respectively.

Throughout this paper, $\beta_1, \dots, \beta_{2\sigma+1}$ will denote a sequence of pairwise distinct complex numbers, and $\overline{M}$ will be the algebraic hyperelliptic curve of genus σ given by:

$$\overline{M} = \left\{ (z,w) \in \overline{\mathbb{C}}^2 : w^2 = \prod_{i=1}^{2\sigma+1} (z - \beta_i) \right\}.$$

Let $A(z,w) = (z,-w)$ be the hyperelliptic involution on $\overline{M}$, and label $\infty = (\infty,\infty)$ and $M = \overline{M} - \{\infty\}$. If $h : \Omega \subset \mathbb{C} \to \mathbb{C}$ is a meromorphic function, we do not distinguish between h and $h \circ z : z^{-1}(\Omega) \subset M \to \mathbb{C}$.

Given $D \subset M$ a domain, we will say that a function, or a 1-form, is harmonic, holomorphic, meromophic, ... on $\overline{D}$, if it is harmonic, holomorphic, meromorphic, ... on a domain containing $\overline{D}$.

Let $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ be the Weierstrass representation of a minimal immersion

$$X : \overline{D} \to \mathbb{R}^3$$

where $D \subset M$ is a domain invariant under A. If $A^*\Phi = -\Phi$, then we can write $\Phi_j = \varphi_j(z)\frac{dz}{w}$, where φ_j is a holomorphic funtion on $z(\overline{D}) \subset \mathbb{C}$, $j = 1, 2, 3$. We will denote $\varphi \stackrel{\text{def}}{=} (\varphi_1, \varphi_2, \varphi_3)$.

With this notation, if we write the Riemannian metric induced by X as $ds_X^2 = \lambda_X^2 \|\frac{dz}{w}\|^2$, then

$$\lambda_X = \frac{1}{\sqrt{2}}\|\varphi\| = \frac{1}{\sqrt{2}}\sqrt{|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2}. \tag{1}$$

For the sake of simplicity, given $W \subset M$, $p, q \in W$ and $T \subset W$, we write $\mathrm{dist}_{(X,W)}(p,q)$ and $\mathrm{dist}_{(X,W)}(p,T)$ instead of $\mathrm{dist}_{(ds_X,W)}(p,q)$ and $\mathrm{dist}_{(ds_X,W)}(p,T)$, respectively.

Let P be a simple closed polygonal curve in $\mathbb{C}$. We denote $\mathrm{Int}(P)$ as the bounded connected component of $\mathbb{C} \setminus P$. Given $\xi > 0$, small enough, we define P^ξ as the parallel polygonal curve in $\mathrm{Int}(P)$, satisfying that the distance between parallel sides is equal to ξ. Whenever we write P^ξ in the paper we are assuming that ξ is small enough to define the polygon properly. If $D = z^{-1}(\mathrm{Int}(P)) \subset M$, then we write $D^\xi = z^{-1}(\mathrm{Int}(P^\xi))$.

3. The main theorem

In order to get the main theorem, we need the following two lemmas. We prove them in Sections 4 and 5.

Lemma 1. *Consider* $\Omega \subset \mathbb{C}$ *a simply connected domain with* $\{\beta_1, \ldots, \beta_{2\sigma+1}\} \subset \Omega$, $D = z^{-1}(\Omega)$ *and* $F : \overline{D} \to \mathbb{R}^3$ *a minimal immersion whose Weierstrass representation* Φ *satisfies* $A^*(\Phi) = -\Phi$, *i.e.,* $\Phi_3 = \varphi_3(z)\frac{dz}{w}$ *and* $g = G(z)$. *Then for any* K_1, K_2 *disjoint compact* 1-*connected sets of* $\mathbb{C}$ *with* $\beta_1, \ldots, \beta_{2\sigma+1} \in \overset{\circ}{K}_2$, *and any* $\alpha > 0$, *there exists* $h : \overline{\Omega} \to \mathbb{C}$, *a holomorphic function without zeroes, such that:*

1. $|h - \alpha| < 1/\alpha$ *in* K_1.
2. $|h - 1| < 1/\alpha$ *in* K_2.
3. *The minimal immersion* $\widetilde{F} : \overline{D} \to \mathbb{R}^3$ *with Weierstrass representation* $\widetilde{\Phi}$ *given by* $\widetilde{g} = g/h$ *and* $\widetilde{\Phi}_3 = \Phi_3$ *is well defined.*

Lemma 2. *Let* P *be a polygon on* $\mathbb{C}$ *satisfying* $\{\beta_1, \ldots, \beta_{2\sigma+1}\} \subset \mathrm{Int}(P)$ *and let* $r > 0$. *Consider* $D = z^{-1}(\mathrm{Int}(P))$ *and* $X : \overline{D} \to \mathbb{R}^3$ *a minimal immersion satisfying:*

1. $X = \mathrm{Re}\left(\displaystyle\int_{p_0} \Phi\right)$, *where* $p_0 = (\beta_1, 0)$ *and* $A^*\Phi = -\Phi$.
2. $\|X\| < r$ *in* $\overline{D}$.

Then, for any $\varepsilon, s > 0$ *such that* $\{\beta_1, \ldots, \beta_{2\sigma+1}\} \subset \mathrm{Int}(P^\varepsilon)$, *there exist a polygon* $\widetilde{P}$ *and a conformal minimal immersion* $Y : \overline{\widetilde{D}} \to \mathbb{R}^3$, $\widetilde{D} = z^{-1}(\mathrm{Int}(\widetilde{P}))$ *such that:*

1. $\overline{\mathrm{Int}(P^\varepsilon)} \subset \mathrm{Int}\,\widetilde{P} \subset \overline{\mathrm{Int}\,\widetilde{P}} \subset \mathrm{Int}(P)$.
2. $Y = \mathrm{Re}\left(\displaystyle\int_{p_0} \widetilde{\Phi}\right)$, *where* $\widetilde{\Phi}$ *satisfies* $A^*(\widetilde{\Phi}) = -\widetilde{\Phi}$.
3. $\mathrm{dist}_{(Y,\overline{\widetilde{D}})}(\partial(\widetilde{D}), \partial(D^\varepsilon)) > s$.
4. $Y(\widetilde{D}) \subset B_R$, $\quad R = \sqrt{r^2 + (2s)^2} + \varepsilon$.
5. $\|Y - X\| < \varepsilon$ *in* D^ε.

At this point, we state and prove our main result.

Theorem 1. *There exist a simply connected domain $\Sigma \subset \mathbb{C}$ containing $\{\beta_1, \ldots, \beta_{2\sigma+1}\}$ and a complete bounded minimal immersion $X : S = z^{-1}(\Sigma) \to \mathbb{R}^3$.*

Proof. Let $r_1 > 1$ and $\rho_1 > 0$ to be specified later, and define $r_n = \sqrt{r_{n-1}^2 + (2/n)^2} + 1/n^2$, and $\rho_n = \rho_1 + \sum_{i=2}^n 1/i$, $n \geq 2$. Our strategy consists of using Lemma 2 to define a sequence:

$$\chi_n = (X_n : \overline{D}_n \to \mathbb{R}^3, P_n, \varepsilon_n, \xi_n),$$

where X_n is a conformal minimal immersion, $D_n = z^{-1}(\mathrm{Int}(P_n))$, P_n is a polygon enclosing $\{\beta_1, \ldots, \beta_{2\sigma+1}\}$, $\{\varepsilon_n\}$, $\{\xi_n\}$ are decreasing sequences of nonvanishing terms satisfying $\varepsilon_n, \xi_n < 1/n^2$, and:

(A_n) $\rho_n < \mathrm{dist}_{(X_n, \overline{D_n^{\xi_n}})}(p_0, \partial(D_n^{\xi_n}))$.

(B_n) $X_n(D_n) \subset B_{r_n}$.

(C_n) $X_n(p) = \mathrm{Re}\left(\displaystyle\int_{p_0}^p \Phi^n\right)$, where $A^*(\Phi^n) = -\Phi^n$.

(D_n) $\|X_n - X_{n-1}\| < \varepsilon_n$ in $D_{n-1}^{\xi_{n-1}}$.

(E_n) $\lambda_{X_n} \geq \alpha_n \lambda_{X_{n-1}}$ in $D_{n-1}^{\xi_{n-1}}$, where $\{\alpha_i\}_{i\in\mathbb{N}}$ is a sequence of real numbers such that $0 < \alpha_i < 1$ and $\{\prod_{i=1}^n \alpha_i\}_n$ converges to $1/2$.

(F_n) $\overline{\mathrm{Int}(P_{n-1}^{\xi_{n-1}})} \subset \mathrm{Int}(P_n^{\xi_n}) \subset \overline{\mathrm{Int}(P_n)} \subset \mathrm{Int}(P_{n-1})$.

The choice of the first element of the sequence is not difficult. For instance, and just for completeness, we suggest the following. Take $\overline{M} = \left\{(z, w) \in \overline{\mathbb{C}}^2 \; w^2 = (z-2)^{2\sigma+1} + 1\right\}$, $g^1 = (z-2)^{2\sigma+1}$, $\Phi_3^1 = (z-2)^{4\sigma+1}\,\frac{dz}{w}$. Let P_1 be a polygon enclosing the zeroes $\{\beta_1, \ldots, \beta_{2\sigma+1}\}$ of $(z-2)^{2\sigma+1}+1$, but leaving 2 in the exterior domain. Note that Φ^1 is exact. So, if $D_1 \overset{\mathrm{def}}{=} z^{-1}(\mathrm{Int}(P_1))$ then $X_1(p) = \mathrm{Re}\left(\int_{p_0}^p \Phi^1\right)$, $p \in \overline{D_1}$, is well defined. Finally, we choose $\rho_1 < \mathrm{dist}_{(X_1, \overline{D_1})}(p_0, \partial(D_1))$ and $r_1 > 1$ such that $X_1(D_1) \subset B_{r_1}$. We also choose $\xi_1 < 1$ small enough satisfying (A_1). The choice of $\varepsilon_1 < 1$ is irrelevant.

Suppose that we have $\chi_1, \ldots, \chi_n$. Now, we construct the $(n+1)$-th term.

Take a sequence $\{\widehat{\varepsilon}_m\} \searrow 0$, with $\widehat{\varepsilon}_m < \frac{1}{(n+1)^2}$, $\forall m$. For each m, we consider $Y_m : \widetilde{D}_m \to \mathbb{R}^3$ and $\widetilde{P}_m$ given by Lemma 2, for the data:

$$X = X_n,\ P = P_n,\ r = r_n,\ s = 1/(n+1),\ \varepsilon = \widehat{\varepsilon}_m.$$

If m is large enough, Assertions 1 and 5 in Lemma 2 tell us that $\overline{D_n^{\xi_n}} \subset \widetilde{D}_m$ and the sequence $\{Y_m\}$ converges to X_n uniformly in $\overline{D_n^{\xi_n}}$. In particular, $\{\lambda_{Y_m}\}$ converges uniformly to λ_{X_n} in $\overline{D_n^{\xi_n}}$. Therefore there is a $m_0 \in \mathbb{N}$ such that:

$$\overline{D_n^{\xi_n}} \subset D_n^{\hat{\varepsilon}_{m_0}} \subset \widetilde{D}_{m_0}, \tag{2}$$

$$\rho_n < \mathrm{dist}_{(Y_{m_0}, \overline{D_n^{\xi_n}})}(p_0, \partial(D_n^{\xi_n})), \tag{3}$$

$$\lambda_{Y_{m_0}} \geq \alpha_{n+1}\lambda_{X_n} \qquad \text{in } D_n^{\xi_n}. \tag{4}$$

We define $X_{n+1} = Y_{m_0}$, $P_{n+1} = \widetilde{P}_{m_0}$, and $\varepsilon_{n+1} = \widehat{\varepsilon}_{m_0}$. From (2), (3) and Assertion 3 in Lemma 2, it is not hard to see that $\rho_{n+1} < \mathrm{dist}_{(X_{n+1}, \overline{D_{n+1}})}(p_0, \partial(D_{n+1}))$. Finally, take ξ_{n+1} small enough such that (A_{n+1}) and (F_{n+1}) hold. The remaining properties directly follow from (2), (4) and the aforementioned lemma. This concludes the construction of the sequence $\{\chi_n\}_{n\in\mathbb{N}}$.

Now, we define

$$\Sigma = \bigcup_{n=1}^{\infty} \mathrm{Int}(P_n^{\xi_n}).$$

Σ is a simply connected domain in $\mathbb{C}$ containing $\{\beta_1, \ldots, \beta_{2\sigma+1}\}$. Label $S = z^{-1}(\Sigma)$.

Properties (D_n) and the fact that $\varepsilon_n < 1/n^2$ give us that the sequence of minimal immersion $\{X_n\}$ is a Cauchy sequence, uniformly on compact sets of S, and so $\{X_n\}$ converges.

Let $X : S \to \mathbb{R}^3$ be the limit of $\{X_n\}$. X has the following properties:

- X is an immersion. Indeed, for any $p \in S$ there exists $n \in \mathbb{N}$ such that $p \in D_n^{\xi_n}$. From Properties (E_i), $i = k, \ldots, n+1$ we get:

  $$\begin{aligned}\lambda_{X_k}(p) &\geq \alpha_k \lambda_{X_{k-1}}(p) \geq \cdots \geq \alpha_k \ldots \alpha_{n+1}\lambda_{X_n}(p) \\ &\geq \alpha_k \ldots \alpha_1 \lambda_{X_n}(p), \quad \forall k > n.\end{aligned}$$

  Taking limit as $k \to \infty$, we deduce:

  $$\lambda_X(p) \geq \frac{1}{2}\lambda_{X_n}(p) > 0, \tag{5}$$

  and so X is an immersion.

- X is minimal and conformal.
- $X(S)$ is bounded in $\mathbb{R}^3$. Let $p \in S$ and $n \in \mathbb{N}$ such that $p \in D_n^{\xi_n}$, then
$$\|X(p)\| \leq \|X(p) - X_n(p)\| + \|X_n(p)\| \leq \frac{1}{2} + r_n,$$
for an n large enough. From the definition, the sequence $\{r_n\}$ is bounded in $\mathbb{R}$.
- The surface S is complete with the metric induced by X. Indeed, if n is large enough, and taking (5) and (A_n) into account, one has:
$$\operatorname{dist}_{(X,\overline{D_n^{\xi_n}})}(p_0, \partial D_n^{\xi_n}) > \frac{1}{2} \operatorname{dist}_{(X_n,\overline{D_n^{\xi_n}})}(p_0, \partial D_n^{\xi_n}) > \frac{1}{2}\rho_n.$$
The completeness is due to the fact that $\{\rho_n\}_{n\in\mathbb{N}}$ diverges.

This concludes the proof. q.e.d.

4. Proof of Lemma 1

Lemma 1 tells us that the set of funtions given by Runge's theorem on M is large enough to provide us with a solution to our period problem.

The proof of this lemma requires of several claims about meromorphic one forms on the surface $\overline{M}$.

Along this section, $\mathcal{B} = \{\gamma_1, \gamma_2, \ldots, \gamma_{2\sigma}\}$ will represent a basis of the homology of $\overline{M}$ contained in $z^{-1}(K_2)$. In Figure 1 you can see the z-projection of γ_i, that we have called δ_i, $i = 1, \ldots, 2\sigma$. Note that $\mathcal{B}$ is also an homology basis of M.

Let us define $\mathcal{H}_\infty$ as the complex vector space of the meromorphic 1-forms τ on $\overline{M}$ with poles only at ∞, and satisfying $\tau = -A^*\tau$. Notice that a nonexact element of $\mathcal{H}_\infty$ has the form $P(z)\frac{dz}{w}$, where $P(z)$ is a non-null polynomial.

Claim 1. *Consider* $(a_1, \ldots, a_{2\sigma}) \in \mathbb{C}^{2\sigma} - \{(0, \ldots, 0)\}$ *and* $c = \sum_{j=1}^{2\sigma} a_j\gamma_j$. *Then there exists* $\tau \in \mathcal{H}_\infty$ *satisfying* $\int_c \tau \neq 0$.

Proof. As a consequence of Riemann-Roch theorem, the first holomorphic De Rham cohomology group, $H^1_{\mathrm{hol}}(M)$, is generated by
$$\mathcal{V} = \left\{ \left[z^{j-1}\frac{dz}{w} \right],\ j = 1, \ldots, 2\,\sigma \right\}.$$

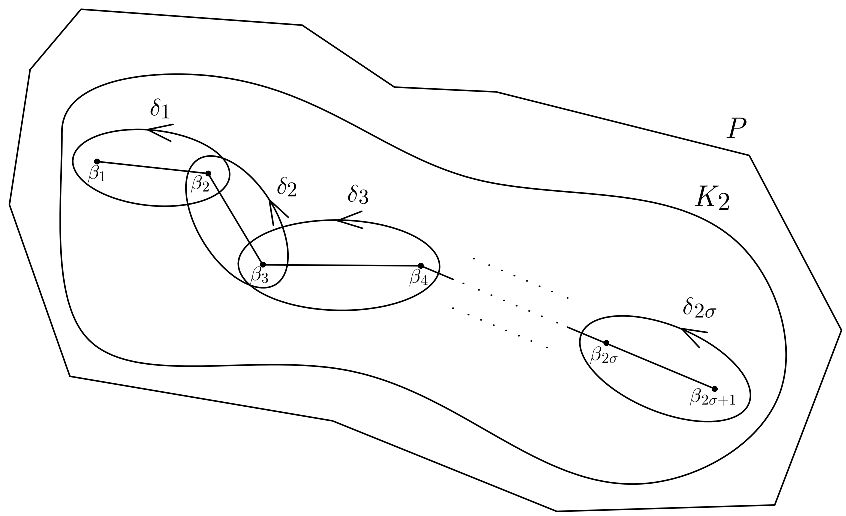

Figure 1: Curves $\delta_1, \delta_2, \ldots, \delta_{2\sigma}$.

See [1] for the details. Therefore, the map $I : H^1_{\text{hol}}(M) \longrightarrow \mathbb{C}^{2\sigma}$, given by $I([\psi]) = \left(\int_{\gamma_j} \psi\right)_{j=1,\ldots,2\sigma}$ is a linear isomorphism. Thus, there is $[\psi] \in H^1_{\text{hol}}(M)$ such that $I([\psi]) \notin \left\{(z_1, \ldots z_{2\sigma}) \in \mathbb{C}^{2\sigma} \ : \ \sum_{j=1}^{2\sigma} a_j z_j = 0\right\}$. As $\mathcal{V}$ is a basis of $H^1_{\text{hol}}(M)$, there is $\tau \in \mathcal{H}_\infty \cap [\psi]$, and so, $\int_c \tau \neq 0$. This proves the claim. q.e.d.

Furthermore, we are interested in controling the zeroes of the one-form τ given in the above claim. This is possible thanks to the next result.

Claim 2. *Let τ be a meromorphic 1-form in $\mathcal{H}_\infty$ and $p \in M$. Then there is a meromorphic function $H : M \to \mathbb{C}$ satisfying:*

(i) $H \circ A = -H$.

(ii) $(H)_\infty = \infty^k$, $k \in \mathbb{N}$.

(iii) $(\tau + d\,H)_0 \geq (\tau)_0 \cdot p \cdot A(p)$.

Proof. We know that $\tau = P(z)\frac{dz}{w}$, where $P(z)$ is a polynomial. Write $(\tau)_0 = p^{n(p)} \cdot A(p)^{n(p)} \cdot D$, where D is an integral divisor not containing either p or $A(p)$. Define

$$J = \begin{cases} \frac{P(z)^2}{(z-z(p))^{n(p)-1}} w, & p \neq A(p) \\ \\ \frac{P(z)^2}{(z-z(p))^{n(p)}} w, & p = A(p). \end{cases}$$

Notice that J satisfies (i) and (ii). Moreover $(J)_0 \geq p^{n(p)+1} \cdot A(p)^{n(p)+1} \cdot D^2$. As the order of p (and $A(p)$) as zero of $d(J)$ and τ is the same, then there exists $\lambda \in \mathbb{C}$ such that $(\tau + \lambda dJ)_0 \geq p^{n(p)+1} \cdot A(p)^{n(p)+1} \cdot D$. This concludes the claim. q.e.d.

Claim 3. *Let $H\left(\overline{\Omega}\right)$ be the* real *vector space of the holomorphic functions on $\overline{\Omega}$. Then the linear map $F : H\left(\overline{\Omega}\right) \to \mathbb{R}^{4\sigma}$, given by:*

$$F(t) = \left(\mathrm{Re}\left[\int_{\gamma_j} t\, \Phi_3 \left(\frac{1}{g} + g \right) \right]_{j=1,\dots,2\sigma}, \right.$$
$$\left. \mathrm{Im}\left[\int_{\gamma_j} t\, \Phi_3 \left(\frac{1}{g} - g \right) \right]_{j=1,\dots,2\sigma} \right)$$

is surjective.

Proof. We proceed by contradiction. Assume F is not onto. Then, there is $(\mu_1, \dots, \mu_{4\sigma}) \in \mathbb{R}^{4\sigma} - \{(0, \dots, 0)\}$, such that $F(H\left(\overline{\Omega}\right)) \subseteq \{(x_1, \dots, x_{4\sigma}) \in \mathbb{R}^{4\sigma} \ / \ \sum_{j=1}^{4\sigma} \mu_j x_j = 0\}$. This is equivalent to say that

$$\sum_{j=1}^{2\sigma} \left[u_j \int_{\gamma_j} \frac{t}{g} \Phi_3 + \overline{u_j} \int_{\gamma_j} t g \Phi_3 \right] = 0 \quad \forall t \in H\left(\overline{\Omega}\right), \tag{6}$$

where $u_j = \mu_j - i\mu_{2\sigma+j}$, $j = 1, \dots, 2\sigma$.

Claims 1 and 2 guarantee the existence of a differential $\tau \in \mathcal{H}_\infty$ satisfying:

(i) $(\tau)_0 \geq \left((\frac{1}{g}\Phi_3)_{|\overline{\Omega}} \right)_0^2 \left((g\ dg)_{|\overline{\Omega}} \right)_0$.

(ii) $\sum_{j=1}^{2\sigma} \overline{u_j} \int_{\gamma_j} \tau \neq 0$.

If we define $f \stackrel{\text{def}}{=} \frac{\tau}{2\, g\, dg}$, then $t = \frac{g\, d(f)}{\Phi_3}$ belongs to $H\left(\overline{\Omega}\right)$. In this case, and integrating by parts, (6) becomes

$$\sum_{j=1}^{2\sigma} \overline{u_j} \int_{\gamma_j} t g \Phi_3 = -\sum_{j=1}^{2\sigma} \overline{u_j} \int_{\gamma_j} \tau = 0,$$

which is absurd. This contradiction proves the claim. q.e.d.

Using the above claim we have the existence of $\{t_1, \ldots, t_{4\sigma}\} \subset H\left(\overline{\Omega}\right)$ such that $\det(F(t_1), \ldots, F(t_{4\sigma})) \neq 0$. Up to changing $t_i \leftrightarrow t_i/x$, $x > 0$ large enough, we can assume that

$$\left|\exp\left(\sum_{i=1}^{4\sigma} x_i t_i(z)\right) - 1\right| < 1/(2\alpha), \tag{7}$$

$\forall (x_1, \ldots, x_{4\sigma}) \in \mathbb{R}^{4\sigma},\ |x_i| < 1,\ i = 1, \ldots, 4\sigma, \quad \forall z \in \overline{\Omega}$.

Given $n \in \mathbb{N}$, we apply Runge's theorem and obtain a holomorphic function $t_0^n : \overline{\Omega} \to \mathbb{C}$ satisfying:

- $|t_0^n - n| < 1/n$ in K_1.
- $|t_0^n| < 1/n$ in K_2.

For $\Theta = (\lambda_0, \ldots, \lambda_{4\sigma}) \in \mathbb{R}^{4\sigma+1}$, we define

$$h^{\Theta,n}(z) \stackrel{\text{def}}{=} \exp\left[\lambda_0\, t_0^n(z) + \sum_{j=1}^{4\sigma} \lambda_j\ t_j(z)\right], \quad \forall z \in \overline{\Omega}.$$

Label $g^{\Theta,n} = g/h^{\Theta,n}$ and $\Phi_3^{\Theta,n} = \Phi_3$. As $\left\{t^n_{0|_{K_2}}\right\}_{n\in\mathbb{N}}$ is uniformly bounded, then, up to a subsequence, we have $\left\{t^n_{0|_{K_2}}\right\} \to t_0^\infty \equiv 0$, uniformly on K_2. We also define on K_2 the following Weierstrass data $g^{\Theta,\infty} = g/h^{\Theta,\infty}$, $\Phi_3^{\Theta,\infty} = \Phi_3$, where

$$h^{\Theta,\infty}(z) \stackrel{\text{def}}{=} \exp\left[\sum_{j=1}^{4\sigma} \lambda_j\ t_j(z)\right], \quad \forall z \in K_2.$$

The period problems of all these Weierstrass representations are not solved, except for the third coordinates.

Therefore, we have to deal with the periods of $\Phi_j^{\Theta,n}$, $j = 1, 2$. To do this, we define the map $\mathcal{P}_n : \mathbb{R}^{4\sigma+1} \to \mathbb{R}^{4\sigma}$, $n \in \mathbb{N} \cup \{\infty\}$;

$$\mathcal{P}_n(\Theta) = \left(\operatorname{Re}\left[\int_{\gamma_j} \Phi_1^{\Theta,n}\right]_{j=1,\ldots,2\sigma}, \operatorname{Re}\left[\int_{\gamma_j} \Phi_2^{\Theta,n}\right]_{j=1,\ldots,2\sigma}\right).$$

Since the immersion X is well defined, then one has $\mathcal{P}_n(0, \overset{(4\sigma+1)}{\ldots}, 0) = 0$, $\forall n \in \mathbb{N} \cup \{\infty\}$. Moreover, it is not hard to check that

$$\mathrm{Jac}_{\lambda_1,\ldots,\lambda_{4\sigma}}(\mathcal{P}_n)(0, \overset{(4\sigma+1)}{\ldots}, 0) = \det(F(t_1), \ldots, F(t_{2\sigma})) \neq 0, \quad \forall n \in \mathbb{N} \cup \{\infty\}.$$

So, we can find $\epsilon > 0$ and $1 > r > 0$ such that

- $(\mathrm{Jac}_{\lambda_1,\ldots,\lambda_{4\sigma}}(\mathcal{P}_\infty))|_{[-\epsilon,\epsilon]\times\overline{B}(0,r)} \neq 0$;
- the map $\mathcal{P}_\infty(0,\cdot)|_{\overline{B}(0,r)}$ is injective,

where $\overline{B}(0,r) = \{\Lambda \in \mathbb{R}^{4\sigma} \;/\; \|\Lambda\| \leq r\}$.

As $\{t_0^n\}_{n\in\mathbb{N}}$ uniformly converges to $t_0^\infty \equiv 0$ on K_2 and $\delta_i = z(\gamma_i)$ is contained in K_2, $i = 1, \ldots, 2\sigma$, then it is not hard to see that $\{\mathrm{Jac}_{\lambda_1,\ldots,\lambda_{4\sigma}}(\mathcal{P}_n)\}_{n\in\mathbb{N}}$ uniformly converges to $\mathrm{Jac}_{\lambda_1,\ldots,\lambda_{4\sigma}}(\mathcal{P}_\infty)$ on $[-\epsilon,\epsilon] \times \overline{B}(0,r)$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\mathrm{Jac}_{\lambda_1,\ldots,\lambda_{4\sigma}}(\mathcal{P}_n)(\lambda_0, \Lambda) \neq 0, \;\; \forall(\lambda_0, \Lambda) \in [-\epsilon, \epsilon] \times \overline{B}(0,r), \;\; n \geq n_0.$$

At this point we can apply the Implicit Function Theorem to the map $\mathcal{P}_n$ at $(0, \overset{(4\sigma+1)}{\ldots}, 0) \in [-\epsilon,\epsilon] \times \overline{B}(0,r)$, in order to get a smooth function $L_n : I_n \to \mathbb{R}^{4\sigma}$, satisfying $\mathcal{P}_n(\lambda_0, L_n(\lambda_0)) = 0$, $\forall \lambda_0 \in I_n$, where I_n is an open interval containing 0. We can also assume that I_n is maximal, in the sense that L_n can not be regularly extended beyond I_n.

Label ϵ_n as the supremum of the connected component of $L_n^{-1}(\overline{B}(0,r)) \cap [0,\epsilon]$ that constains $\lambda_0 = 0$. Our next step consists of seeing that $\epsilon_n \in I_n$. Take a sequence $\{\lambda_0^k\}_{k\in\mathbb{N}} \nearrow \epsilon_n$. As $\{L_n(\lambda_0^k)\} \subset \overline{B}(0,r)$, then we can assume, up to a subsequence, that $\{L_n(\lambda_0^k)\}_{k\in\mathbb{N}}$ converges to an element $\Lambda_n \in \overline{B}(0,r)$. Taking into account that $\mathrm{Jac}_{\lambda_1,\ldots,\lambda_{4\sigma}}(\mathcal{P}_n)(\epsilon_n, \Lambda_n) \neq 0$, the local unicity of the curve $(\lambda_0, L_n(\lambda_0))$ around the point (ϵ_n, Λ_n), and that I_n is maximal, we deduce that $\epsilon_n \in I_n$. Therefore, either $\epsilon_n = \epsilon$, or $L_n(\epsilon_n) = \Lambda_n \in \partial(B(0,r))$.

We are going to see that $\epsilon_0 \overset{\text{def}}{=} \liminf\{\epsilon_n\} > 0$. Otherwise, there is a subsequence $\{\epsilon_n\} \to 0$. Without loss of generality, $\epsilon_n < \epsilon$, $\forall n \in \mathbb{N}$, and so $\Lambda_n \in \partial(B(0,r))$, $\forall n \in \mathbb{N}$. Up to a subsequence, $\{\Lambda_n\} \to \Lambda_\infty \in \partial(B(0,r))$. The fact $\mathcal{P}_\infty(0,0) = \mathcal{P}_\infty(0,\Lambda_\infty) = 0$ contradicts the injectivity of $\mathcal{P}_\infty(0,\cdot)$ in $\overline{B}(0,r)$.

We have proved the following assertion:

Claim 4. *There exist $\epsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that the function $L_n : [0,\epsilon_0] \to \overline{B}(0,r)$ is well defined, $\forall n \geq n_0$.*

Label $(\lambda_1^n, \ldots, \lambda_{4\sigma}^n) = L_n(\epsilon_0)$. From (7) we have $|\exp[\sum_{j=1}^{4\sigma} \lambda_j^n t_j] - 1| < 1/(2\alpha)$ on $\overline{\Omega}$. Hence, if $n \geq n_0$ is large enough, the function:

$$h(z) \stackrel{\text{def}}{=} \exp\left[\epsilon_0\, t_0^n(z) + \sum_{j=1}^{4\sigma} \lambda_j^n\, t_j(z)\right]$$

satisfies Statements 1 and 2 in Lemma 1. As the period function $\mathcal{P}_n$ vanishes at $\Theta_n = (\epsilon_0, \lambda_1^n, \ldots, \lambda_{4\sigma}^n)$, then the minimal immersion $\widetilde{F}$ associated to the Weierstrass data $g^{\Theta_n,n}$, $\Phi_3^{\Theta_n,n} = \Phi_3$ is well defined. This proves Statement 3 in the lemma.

5. Proof of Lemma 2

Consider P, the polygon given in the statement of Lemma 2. In a first step, we are going to follow [4] to describe a labyrinth on $\mathrm{Int}(P)$ depending on P and a positive integer N. Later, we use Lemma 1 following Nadirashvili's ideas [5].

Let ℓ be the number of sides of P. Throughout this section, N will be a positive multiple of ℓ.

Remark 1. Along the proof of the lemma, a set of real positive constants $\{c_i,\ i = 1, \ldots, 12\}$ depending on X, P, r, ε, and s will appear. It is important to note that the choice of these constants does not depend on the integer N.

Let $\zeta_0 > 0$ small enough so that P^{ζ_0} is well defined and $\overline{\mathrm{Int}(P^{\varepsilon})} \subset \mathrm{Int}(P^{\zeta_0})$. From now on, we will only consider $N \in \mathbb{N}$ such that $2/N < \zeta_0$. Let c_1 be a lower bound for the length of the sides of polygon P^{ζ} for all $\zeta \leq \zeta_0$. Let $v_1, \ldots, v_{2N}$ be a set of points in the polygon P (containing the vertices of P) that divide each side of P into $\frac{2N}{\ell}$ equal parts. We can transfer this partition to the polygon $P^{2/N}$: $v_1', \ldots, v_{2N}'$ (see Figure 2). We define the following sets:

- $L_i =$ the segment that joins v_i and v_i', $i = 1, \ldots, 2N$.
- $P_i = P^{i/N^3}$, $i = 0, \ldots, 2N^2$.
- $\mathcal{A} = \bigcup_{i=0}^{N^2-1} \overline{\mathrm{Int}(P_{2i}) \setminus \mathrm{Int}(P_{2i+1})}$ and
  $\widetilde{\mathcal{A}} = \bigcup_{i=1}^{N^2} \overline{\mathrm{Int}(P_{2i-1}) \setminus \mathrm{Int}(P_{2i})}$.
- $\mathcal{R} = \bigcup_{i=0}^{2N^2} P_i$.

- $\mathcal{B} = \bigcup_{i=1}^{N} L_{2i}$ and $\widetilde{\mathcal{B}} = \bigcup_{i=0}^{N-1} L_{2i+1}$.
- $L = \mathcal{B} \cap \mathcal{A}$, $\widetilde{L} = \widetilde{\mathcal{B}} \cap \widetilde{\mathcal{A}}$, and $H = \mathcal{R} \cup L \cup \widetilde{L}$.
- $\Omega_N = \left\{ z \in \operatorname{Int}(P_0) \setminus \operatorname{Int}(P_{2N^2}) : \operatorname{dist}_{ds_0,\mathbb{C}}(z, H) \geq \frac{1}{4N^3} \right\}$, where ds_0 is the Euclidean metric on $\mathbb{C}$.

We define ω_i as the union of the segment L_i and those connected components of Ω_N that have nonempty intersection with L_i for $i = 1, \ldots, 2N$. Finally, we label $\varpi_i = \{z \in \mathbb{C} : \operatorname{dist}_{ds_0,\mathbb{C}}(z, \omega_i) < \delta(N)\}$, where $i = 1, \ldots, 2N$, and $\delta(N) > 0$ is chosen in such a way that the sets $\overline{\varpi_i}$ $(i = 1, \ldots, N)$ are pairwise disjoint (see Figure 3). We denote ϖ_i^1 and ϖ_i^2 as the two connected component of $z^{-1}(\varpi_i)$.

The aim of all this construction is to guarantee the following claims for an N large enough.

Claim A *There is a constant c_2 such that $diam_{ds}(\varpi_i^j) \leq c_2/N$, where ds^2 is the Riemannian metric $\|dz/w\|^2$ on M.*

To see this, observe that $\operatorname{diam}_{ds_0}(\varpi_i) \leq \frac{const}{N}$. As we can find a positive constant c_3 such that

$$\frac{1}{c_3} \left\| \frac{dz}{w} \right\| \leq \|dz\| \leq c_3 \left\| \frac{dz}{w} \right\| \quad \text{in } \overline{D} \setminus D^\varepsilon \tag{8}$$

and we have $z^{-1}(\varpi_i) \subset D \setminus D^\varepsilon$ for all $i = 1, \ldots, 2N$, the claim holds.

Claim B *If $\lambda^2(z) ds^2$ is a conformal metric in $\overline{D}$ and $\Upsilon \in \mathbb{R}^+$ satisfies*

$$\lambda(z) \geq \begin{cases} \Upsilon & \textit{in } \operatorname{Int} P, \\ \Upsilon\, N^4 & \textit{in } \Omega_N, \end{cases}$$

and if α is a curve in D connecting $\partial(D^\varepsilon)$ and $\partial(D)$, then the length of α with this metric is greater than $c_4\,\Upsilon N$, where c_4 is a positive constant not depending on Υ.

In order to prove Claim B, if we denote $(z \circ \alpha)_i$ as the piece of $z \circ \alpha$ connecting P_{2i} with P_{2i+2}, for $i = 0, \ldots, N^2 - 1$, then either the Euclidean length of $(z \circ \alpha)_i$ is greater than $\frac{c_1 \ell}{2N}$ or the Euclidean length of $(z \circ \alpha)_i \cap \Omega_N$ is greater than $\frac{1}{2N^3}$. These facts and inequalities (8) give us the existence of the constant c_4.

Now, our purpose is to construct (for N large enough) a sequence of conformal minimal immersions F_i, $i = 0, 1, \ldots, 2N$, in $\overline{D}$, $F_0 = X$, such that:

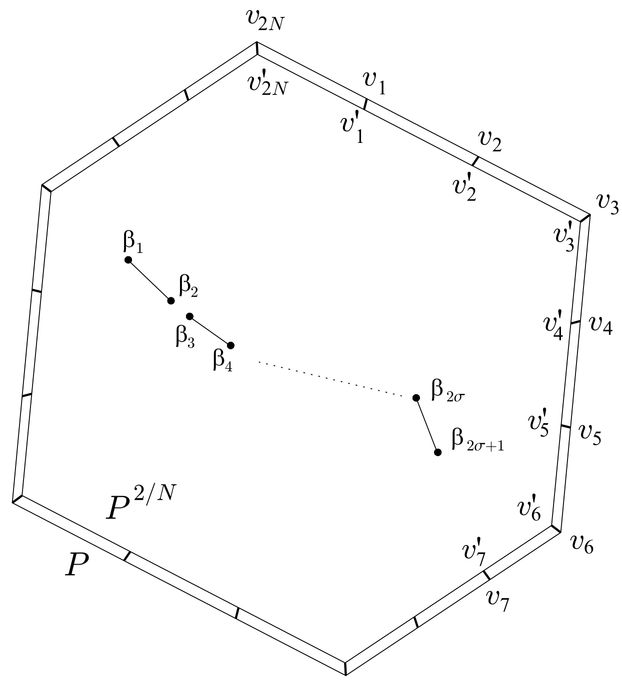

Figure 2: The polygons P and $P^{2/N}$.

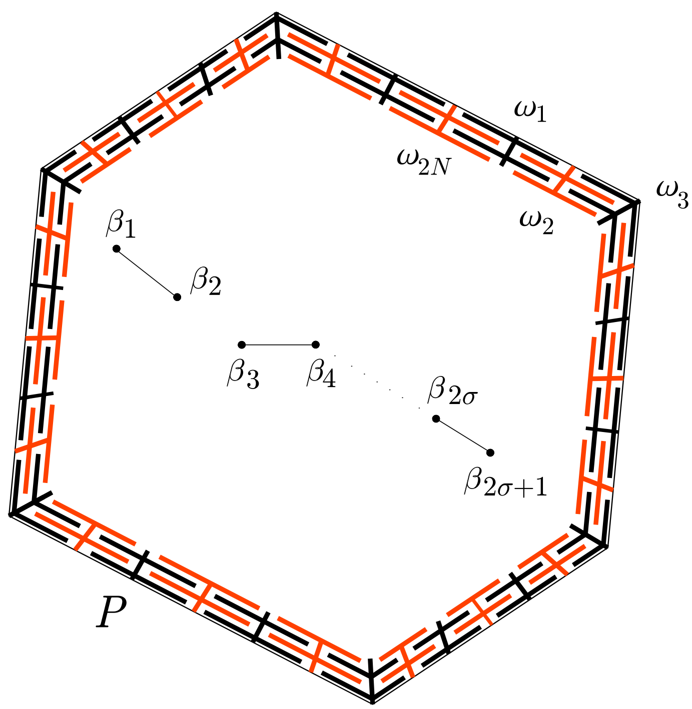

Figure 3: Distribution of the sets ϖ_i^j.

(P1$_i$) $F_i(p) = \mathrm{Re}\left(\int_{p_0}^p \Phi^i\right)$, where $A^*(\Phi^i) = -\Phi^i$, i.e., $\Phi^i = (\varphi_1^i(z), \varphi_2^i(z), \varphi_3^i(z))\frac{dz}{w}$.

(P2$_i$) $\|\varphi^i(z) - \varphi^{i-1}(z)\| \leq 1/N^2$ for all $z \in \mathrm{Int}(P) \setminus \varpi_i$.

(P3$_i$) $\|\varphi^i(z)\| \geq N^{7/2}$ for all $z \in \omega_i$.

(P4$_i$) $\|\varphi^i(z)\| \geq 1/\sqrt{N}$ for all $z \in \varpi_i$.

(P5$_i$) $\mathrm{dist}_{(ds_1,\mathbb{S}^2)}(\mathcal{G}_i(z), \mathcal{G}_{i-1}(z)) < \frac{1}{N\sqrt{N}}$ for all $z \in \mathrm{Int}(P) \setminus \varpi_i$, where ds_1 is the usual Riemannian metric in $\mathbb{S}^2$ and $\mathcal{G}_i$ represents the Gauss map of the immersion F_i.

(P6$_i$) there exists a orthogonal frame $S_i = \{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ and a real constant $c_5 > 0$ such that:

(P6.1$_i$) If $p \in z^{-1}(\overline{\varpi_i})$ and $\|F_{i-1}(p)\| \geq 1/\sqrt{N}$, then

$$\|((F_{i-1}(p))_1, (F_{i-1}(p))_2)\| < \frac{c_5}{\sqrt{N}} \|F_{i-1}(p)\|.$$

(P6.2$_i$) $(F_i(p))_3 = (F_{i-1}(p))_3$ for all $p \in \overline{D}$.

Here, $(\cdot)_k$ is the k-th coordinate function with respect to $\{e_1, e_2, e_3\}$.

Suppose that we have $F_0, \ldots, F_{j-1}$ verifying the claims (P1$_i$), …, (P6$_i$), $i = 1, \ldots, j-1$. Then, for an N large enough, there are positive constants $c_6, \ldots, c_9$ such that the following statments hold:

(L1) $\|\varphi^{j-1}\| \leq c_6$ in $\mathrm{Int}(P) \setminus \bigcup_{k=1}^{j-1} \varpi_k$.
This follows easily from (P2$_l$) for $l = 1, \ldots, j-1$.

(L2) $\|\varphi^{j-1}\| \geq c_7$ in $\mathrm{Int}(P) \setminus \bigcup_{k=1}^{j-1} \varpi_k$.
To obtain this property, it suffices to apply (P2$_l$) for $l = 1, \ldots, j-1$ once again.

(L3) The diameter in $\mathbb{R}^3$ of $F_{j-1}(\varpi_j^l)$ is less than c_8/N, $l = 1, 2$.

This is a consequence of (L1), the bound of $\mathrm{diam}_{ds}(\varpi_j^l)$ in Claim A, and equality (1).

(L4) The diameter in $\mathbb{S}^2$ of $\mathcal{G}_{j-1}(z^{-1}(\varpi_j))$ is less than $c_9/\sqrt{N}$.

Indeed, since $\operatorname{diam}_{ds_0}(\varpi_j) \leq \frac{\text{const}}{N}$, we have a bound of diameter of $\mathcal{G}_0(z^{-1}(\varpi_j))$. From successive applications of (P5$_l$) we have that (L4) holds.

We shall now construct F_j. We look for a set of orthogonal coordinates $S_j = \{e_1, e_2, e_3\}$ in $\mathbb{R}^3$ and a constant $c_{10} > 0$ such that:

(D1) If $p \in z^{-1}(\varpi_j)$ and $\|F_{j-1}(p)\| \geq \frac{1}{\sqrt{N}}$, then $\min\{\angle(e_3, F_{j-1}(p)), \angle(-e_3, F_{j-1}(p))\} \leq \frac{c_{10}}{\sqrt{N}}$.

(D2) $\angle(\pm e_3, \mathcal{G}_{j-1}(z)) \geq \nu/\sqrt{N}$ for all $z \in \varpi_j$.

Here, $\angle(a,b) \in [0,\pi[$ is the angle formed by a and b in $\mathbb{R}^3$ and ν is a constant satisfying $\nu > 1/c_7$. Given $q \in \mathbb{S}^2$, we denote

$$\operatorname{Con}(q,r) = \{x \in \mathbb{S}^2 : \angle(x,q) \leq r\}.$$

Let $g_0 \in \mathcal{G}_{j-1}(\varpi_j)$. Taking (L4) into account, the condition (D2) holds if e_3 is chosen in $\mathbb{S}^2 \setminus \mathcal{C}$, where

$$\mathcal{C} = \operatorname{Con}\left(g_0, \frac{c_9+\nu}{\sqrt{N}}\right) \cup \operatorname{Con}\left(-g_0, \frac{c_9+\nu}{\sqrt{N}}\right).$$

The next step is to find $e_3 \in \mathbb{S}^2 \setminus \mathcal{C}$ satisfying (D1) for a suitable $c_{10} > 0$.

To do this, we define

$$F = \left\{p/\|p\| : p \in F_{j-1}(\varpi_j^1) \text{ and } \|p\| \geq \frac{1}{\sqrt{N}}\right\}.$$

Let q a point in F. Taking into account (L3), we have that $F \subset \operatorname{Con}\left(q, \frac{2c_8}{\sqrt{N}}\right)$. Choose c_{10} such that $2(c_9+\nu+1+c_8) < c_{10}$, and consider $e_3 \in (\mathbb{S}^2 \setminus \mathcal{C}) \cap \operatorname{Con}(q, \frac{c_9+\nu+1}{\sqrt{N}})$. To check property (D1), we take $p \in \varpi_j^1$ verifying $\|F_{j-1}(p)\| \geq 1/\sqrt{N}$, then a straightforward computation leads to

$$\angle(e_3,p) \leq \angle(e_3,q) + \angle(q,p) \leq \frac{2(c_9+\nu+1)}{\sqrt{N}} + \frac{2c_8}{\sqrt{N}} < \frac{c_{10}}{\sqrt{N}}.$$

Thank to $F_{j-1} \circ A = -F_{j-1}$, we have $\angle(-e_3,p) < \frac{c_{10}}{\sqrt{N}}$ for all $p \in \varpi_j^2$, $(\varpi_j^2 = A(\varpi_j^1))$.

Finally, we take e_1, e_2 such that $S_j = \{e_1, e_2, e_3\}$ is a set of orthogonal coordinates in $\mathbb{R}^3$.

Let (Φ_3^{j-1}, g^{j-1}) be the Weierstrass data of the immersion F_{j-1} in the coordinate system S_j. Let h_α be the function given by Lemma 1, for $K_1 = \omega_j$, $K_2 = \overline{\text{Int}(P)} \setminus \varpi_j$ and α large enough in terms of N. We define $\Phi_3^j = \Phi_3^{j-1}$ and $g^j = g^{j-1}/h_\alpha$. Lemma 1 also tell us that the Weierstrass data Φ^j has no real periods. Therefore, the minimal immersion F_j is well-defined and its expression in the set of coordinates S_j is

$$F_j(p) = \text{Re}\left(\int_{p_0}^{p} \varphi^j(z)\,\frac{dz}{w}\right).$$

We shall now see that F_j satisfies the properties (P1$_j$),...,(P6$_j$). (Note that claims (P1$_j$),...,(P6$_j$) do not depend on changes of coordinates in $\mathbb{R}^3$). Claim (P1$_j$) easily holds.

Note that $h_\alpha \to 1$ (resp. $h_\alpha \to \infty$) uniformly on K_2 (resp. on K_1), as $\alpha \to \infty$. Then (P2$_j$), (P3$_j$), and (P5$_j$) easily hold for α large enough.

To verify (P4$_j$), one uses (D2) and obtains:

$$\frac{\sin(\nu/\sqrt{N})}{1+\cos(\nu/\sqrt{N})} \le |g^{j-1}| \le \frac{\sin(\nu/\sqrt{N})}{1-\cos(\nu/\sqrt{N})} \qquad \text{in } \varpi_j,$$

and so, taking (L2) into account one has:

$$\begin{aligned} \|\varphi^j\| \ge |\varphi_3^j| = |\varphi_3^{j-1}| &\ge \sqrt{2}\|\varphi^{j-1}\| \frac{|g^{j-1}|}{1+|g^{j-1}|^2} \\ &\ge c_7 \sin\left(\tfrac{\nu}{\sqrt{N}}\right) \ge \tfrac{1}{\sqrt{N}} \qquad \text{in } \varpi_j \end{aligned}$$

for N large enough, which proves (P4$_j$).

Using (D1), we get (P6.1$_j$) for $c_5 = c_{10}$. To obtain (P6.2$_j$), use that $\Phi_3^{j-1} = \Phi_3^j$ in the frame S_j.

Hence, we have constructed the immersions $F_0, F_1, \dots, F_{2N}$ satisfying claims (P1$_j$),...,(P6$_j$) for $j = 1, \dots, 2N$.

Lemma 2 is a consequence of the following proposition.

Proposition 1. *If N is large enough, then F_{2N} satisfies:*

(i) $2\,s < \text{dist}_{(F_{2N},\overline{D})}(\partial(D), \partial(D^\epsilon))$.

(ii) *There is a constant $c_{11} > 0$ such that $\|F_j(p) - F_{j-1}(p)\| \le \frac{c_{11}}{N^2}$ in $D \setminus z^{-1}(\varpi_j)$.*

(iii) $\|F_{2N} - X\| \leq \frac{2c_{11}}{N}$ *in* $D \setminus \bigcup_{j=1}^{2N} \left(z^{-1}(\varpi_j)\right)$.

(iv) *There is a polygon* $\widetilde{P}$ *satisfying:*

(iv.1) $\overline{\mathrm{Int}(P^\varepsilon)} \subset \mathrm{Int}(\widetilde{P}) \subset \overline{\mathrm{Int}(\widetilde{P})} \subset \mathrm{Int}(P)$.

(iv.2) $s < \mathrm{dist}_{(F_{2N},\overline{\widetilde{D}})}(p, \partial(D^\varepsilon)) < 2\,s$, $\forall p \in \partial(\widetilde{D})$, *where* $\widetilde{D} = z^{-1}\left(\mathrm{Int}(\widetilde{P})\right)$.

(iv.3) $F_{2N}(\widetilde{D}) \subset B_R$, *where* $R = \sqrt{r^2 + (2s)^2} + \varepsilon$.

Proof. If $\lambda_{F_{2N}}^2(z)\|\frac{dz}{w}\|^2$ is the conformal metric induced on $\overline{D}$ by the immersion F_{2N}, then Property (L2) implies

$$\lambda_{F_{2N}}(z) = \frac{\|\varphi^{2N}(z)\|}{\sqrt{2}} \geq \frac{c_7}{\sqrt{2}} > \frac{1}{2\sqrt{N}} \qquad \text{in } \mathrm{Int}(P) \setminus \bigcup_{k=1}^{2N} \varpi_k, \tag{9}$$

for N large enough. Taking into account (P4$_j$) and (P2$_i$) for $i = j + 1, \ldots, 2N$, we have

$$\begin{aligned} \lambda_{F_{2N}}(z) &\geq \frac{\|\varphi^j(z)\| - \|\varphi^{2N}(z) - \varphi^j(z)\|}{\sqrt{2}} \\ &\geq \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{N}} - \frac{2}{N}\right) \geq \frac{1}{2\sqrt{N}} \qquad \text{in each } \varpi_j. \end{aligned} \tag{10}$$

From (P3$_j$) and (P2$_i$) for $i = j + 1, \ldots, 2N$, we obtain

$$\begin{aligned} \lambda_{F_{2N}}(z) &\geq \frac{\|\varphi^j(z)\| - \|\varphi^{2N}(z) - \varphi^j(z)\|}{\sqrt{2}} \\ &\geq \frac{1}{\sqrt{2}}\left(N^{7/2} - \frac{2}{N}\right) \geq \frac{1}{2\sqrt{N}}N^4 \qquad \text{in each } \omega_j. \end{aligned} \tag{11}$$

Using inequalities (9), (10), and (11) together with Claim B, for $\Upsilon = 1/(2\sqrt{N})$, we conclude the proof of assertion (i).

Now we shall prove (ii). Note that the set ϖ_j depends on N, and label $\Xi_N^j = \overline{D} - z^{-1}(\varpi_j)$. It is not hard to see that there exists c_{11} depending only on D such that

$$\sup\left\{\mathrm{dist}_{(ds,\Xi_N^j)}(p_0, p) \;:\; N \in \mathbb{N},\ j \in \{1, \ldots, 2N\},\ p \in \Xi_N^j\right\} \leq c_{11}.$$

Therefore, for all $p \in \Xi_N^j$, there exists a curve α_p in Ξ_N^j, from p_0 to p satisfying length$(\alpha_p, ds) < c_{11}$. Using the former, we obtain

$$\|F_j(p) - F_{j-1}(p)\| = \left\| \mathrm{Re} \int_{\alpha_p} (\varphi^j(z) - \varphi^{j-1}(z)) \frac{dz}{w} \right\| \leq c_{11} \frac{1}{N^2},$$

which proves assertion (ii). From (ii), it is not hard to deduce (iii).

We will construct the polygon $\widetilde{P}$. Let

$$\mathcal{S} = \left\{ p \in D \setminus \overline{D^\varepsilon} : s < \mathrm{dist}_{(F_{2N}, \overline{D})}(p, \partial(D^\varepsilon)) < 2\, s \right\}.$$

Note that $\mathcal{S}$ is a nonempty open subset of $D \setminus \overline{D^\varepsilon}$. As a consequence of (i), we deduce that $z(\mathcal{S})$ contains a Jordan curve, Γ verifying $\overline{\mathrm{Int}(P^\epsilon)} \subset \mathrm{Int}(\Gamma)$. Then we can aproximate Γ by a polygon $\widetilde{P} \subset z(S)$ satisfying statments (iv.1) and (iv.2).

Finally, we prove assertion (iv.3). Thanks to the Maximum Principle, we only need to check that $F_{2N}(\partial(\widetilde{D})) \subset B_R$. Take $p \in \partial(\widetilde{D})$. If $p \in D \setminus \bigcup_{j=1}^{2N} z^{-1}(\varpi_j)$, we have

$$\|F_{2N}(p)\| \leq \|F_{2N}(p) - X(p)\| + \|X(p)\| \leq \frac{2c_{11}}{N} + r \leq R.$$

Suppose now $p \in z^{-1}(\varpi_j)$, $j \in \{1, \ldots, 2N\}$. From (iv.2), it is possible to find a curve $\gamma : [0,1] \to D$ such that $\gamma(0) \in \partial(D^\varepsilon), \gamma(1) = p$, and length$(\gamma, ds_{F_{2N}}) \leq 2\, s$. We define:

$$\bar{t} = \sup\{t \in [0,1] : \gamma(t) \in \partial(z^{-1}(\varpi_j))\}, \qquad \overline{p} = \gamma(\bar{t}).$$

Let γ_1 be the piece of γ from $\overline{p}$ to p.

To continue, we need to demonstrate:

$$\|F_j(\overline{p}) - F_j(p)\| \leq 4\frac{c_{11}}{N} + 2s. \tag{12}$$

Indeed,

$$\begin{aligned} &\|F_j(\overline{p}) - F_j(p)\| \\ &\leq \|F_j(\overline{p}) - F_{2N}(\overline{p})\| + \|F_{2N}(\overline{p}) - F_{2N}(p)\| + \|F_{2N}(p) - F_j(p)\| \end{aligned}$$

using (ii), we have

$$\begin{aligned} &\leq 2\, \frac{2\, c_{11}}{N} + \|F_{2N}(\overline{p}) - F_{2N}(p)\| \\ &\leq 4\, \frac{c_{11}}{N} + \mathrm{length}(\gamma_1, ds_{F_{2N}}) \leq 4\, \frac{c_{11}}{N} + 2\, s. \end{aligned}$$

At this point, we distinguish two cases.

- *Case* 1: $\|F_{j-1}(\overline{p})\| < 1/\sqrt{N}$. Then

$$\begin{aligned}\|F_{2N}(p)\| &\leq \|F_{2N}(p) - F_j(p)\| + \|F_j(p) + F_j(\overline{p})\| \\ &\quad + \|F_j(\overline{p}) - F_{j-1}(\overline{p})\| + \|F_{j-1}(\overline{p})\| \\ &\leq \frac{2c_{11}}{N} + 4\frac{c_{11}}{N} + 2s + \frac{c_{11}}{N^2} + \frac{1}{\sqrt{N}} \leq R\end{aligned}$$

  for an N large enough.

- *Case* 2: $\|F_{j-1}(\overline{p})\| > 1/\sqrt{N}$. In this case, from (P6.2_j) we have, in the frame S_j,

$$\begin{aligned}|(F_j(p))_3| &= |(F_{j-1}(p))_3| \\ &\leq |(F_{j-1}(p))_3 - (X(p))_3| + |(X(p))_3| \\ &\leq \frac{2c_{11}}{N} + r.\end{aligned}$$

  Using inequality (12), the fact that $\overline{p} \in D \setminus z^{-1}(\varpi_j)$, assertion (ii), and property (P6.1_j), one has

$$\begin{aligned}&\|((F_j(p))_1, (F_j(p))_2)\| \\ &\leq \|((F_j(p))_1, (F_j(p))_2) - ((F_j(\overline{p}))_1, (F_j(\overline{p}))_2)\| \\ &\quad + \|((F_j(\overline{p}))_1, (F_j(\overline{p}))_2) - ((F_{j-1}(\overline{p}))_1, (F_{j-1}(\overline{p}))_2)\| \\ &\quad + \|((F_{j-1}(\overline{p}))_1, (F_{j-1}(\overline{p}))_2)\| \\ &\leq 4\frac{c_{11}}{N} + 2s + \frac{c_{11}}{N^2} + \frac{c_5}{\sqrt{N}}\|F_{j-1}(\overline{p})\| \\ &\leq 4\frac{c_{11}}{N} + 2s + \frac{c_{11}}{N^2} + \frac{c_5}{\sqrt{N}}\left(\frac{2c_{11}}{N} + r\right) \\ &\leq 2s + \frac{c_{12}}{\sqrt{N}},\end{aligned}$$

  where $c_{12} = 5\,c_{11} + c_5(2c_{11} + r)$. By Pythagoras' theorem,

$$\begin{aligned}\|F_{2N}(p)\| &\leq \|F_{2N}(p) - F_j(p)\| + \|F_j(p)\| \\ &\leq \frac{2c_{11}}{N} + \sqrt{|(F_j(p))_3|^2 + \|((F_j(p))_1, (F_j(p))_2)\|^2} \\ &< \sqrt{r^2 + (2s)^2} + \varepsilon = R\end{aligned}$$

  for an N large enough.

q.e.d.

In order to finish the proof of the lemma, we define Y as $Y = F_{2N}$. It is straightforward to check that Y verifies all the claims in Lemma 2.

Remark 2. Nadirashvili's construction provides an example of a complete bounded minimal surface with *strictly negative Gauss curvature*, which gives a counter example to a conjecture by Hadamard. The method of construction that we use here never provides surfaces with strictly negative curvature. This is due to the fact that our Weierstrass representations always satisfy $A^*(\Phi) = -\Phi$. In particular, the Gauss map g only depends on z, and so it has branch points at $\{(\beta_1, 0), \dots, (\beta_{2\sigma+1}, 0)\}$.

The authors think that a method of obtaining examples with $K < 0$ and nontrivial topology should be very different to that shown in this article. We would like to thank the referee for pointing out this possibility.

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