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PROOF OF THE RIEMANNIAN PENROSE INEQUALITY USING THE POSITIVE MASS THEOREM

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Abstract

We prove the Riemannian Penrose Conjecture, an important case of a conjecture [41] made by Roger Penrose in 1973, by defining a new flow of metrics. This flow of metrics stays inside the class of asymptotically flat Riemannian 3-manifolds with nonnegative scalar curvature which contain minimal spheres. In particular, if we consider a Riemannian 3-manifold as a totally geodesic submanifold of a space-time in the context of general relativity, then outermost minimal spheres with total area A correspond to apparent horizons of black holes contributing a mass $\sqrt{A/16\pi}$, scalar curvature corresponds to local energy density at each point, and the rate at which the metric becomes flat at infinity corresponds to total mass (also called the ADM mass). The Riemannian Penrose Conjecture then states that the total mass of an asymptotically flat 3-manifold with nonnegative scalar curvature is greater than or equal to the mass contributed by the black holes.

The flow of metrics we define continuously evolves the original 3-metric to a Schwarzschild 3-metric, which represents a spherically symmetric black hole in vacuum. We define the flow such that the area of the minimal spheres (which flow outward) and hence the mass contributed by the black holes in each of the metrics in the flow is constant, and then use the Positive Mass Theorem to show that the total mass of the metrics is nonincreasing. Then since the total mass equals the mass of the black hole in a Schwarzschild metric, the Riemannian Penrose Conjecture follows.

We also refer the reader to the beautiful work of Huisken and Ilmanen [30], who used inverse mean curvature flows of surfaces to prove that the total mass is at least the mass contributed by the largest black hole.

In Sections 1 and 2, we motivate the problem, discuss important quantities like total mass and horizons of black holes, and state the Positive Mass Theorem and the Penrose Conjecture for Riemannian 3-manifolds. In Section 3, we give the proof of the Riemannian Penrose

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Conjecture, with the supporting arguments given in Sections 4 through 13. In Section 14, we apply the techniques used in this paper to define several new quasi-local mass functions which have good monotonicity properties. Finally, in Section 15 we end with a brief discussion of some of the interesting problems which still remain open, and the author thanks the many people who have made important contributions to the ideas in this paper.

1. Introduction

General relativity is a theory of gravity which asserts that matter causes four dimensional space-time to be curved, and that our perception of gravity is a consequence of this curvature. Let $(N^4, \bar{g})$ be the space-time manifold with metric $\bar{g}$ of signature $(-+++)$. Then the central formula of general relativity is Einstein's equation,

$$G = 8\pi T, \tag{1}$$

where T is the stress-energy tensor, $G = \mathrm{Ric}(\bar{g}) - \frac{1}{2}R(\bar{g})\cdot\bar{g}$ is the Einstein curvature tensor, $\mathrm{Ric}(\bar{g})$ is the Ricci curvature tensor, and $R(\bar{g})$ is the scalar curvature of $\bar{g}$. One of the beautiful aspects of general relativity is that this seemingly simple formula explains gravity more accurately than Newtonian physics and experimentally is the best theory of gravity currently known.

However, the nature of the behavior of matter in general relativity is not well understood. It is not even well understood how to define how much energy and momentum exist in a given region, except in special cases. There does exist a well-defined notion of local energy and momentum *density* which is given by the stress-energy tensor which, by Equation (1), can be computed in terms of the curvature of N^4. Also, if we assume that the matter of the space-time manifold N^4 is concentrated in some central region of the universe, then we may consider the idealized situation in which N^4 becomes flatter as we get farther away from this central region. If the curvature of N^4 decays quickly enough and the metric on N^4 is converging to the metric of flat Minkowski space (in some appropriate sense), then N^4 is said to be asymptotically flat (see Definition 21 in Section 13). With these restrictive assumptions it is then possible to define the total ADM mass of the space-time N^4. Interestingly enough, though, the definition of local energy-momentum density, which involves curvature terms of N^4, bears no obvious resemblance to the definition of the total mass of N^4, which is a parameter related to how quickly the metric becomes flat at infinity.

The Penrose Conjecture ([41], [37], [30]) and the Positive Mass Theorem ([44], [45], [46], [47], [52]) can both be thought of as basic attempts at understanding the relationship between the local energy density of a space-time N^4 and the total mass of N^4. In physical terms, the Positive Mass Theorem states that an isolated gravitational system with nonnegative local energy density must have nonnegative total energy. The idea is that nonnegative energy densities must "add up" to something nonnegative. The Penrose Conjecture, on the other hand, states that if an isolated gravitational system with nonnegative local energy density contains black holes contributing a mass m, then the total energy of the system must be at least m.

One curious phenomenon in general relativity which makes the Positive Mass Theorem and the Penrose Conjecture particularly subtle and interesting is the fact that it is possible to construct vacuum space-times which nevertheless have *positive* total mass. Physicists refer to this extra energy as "gravitational energy" which apparently results from the gravitational waves of the space-time. One interpretation of the Positive Mass Theorem then is that in vacuum space-times gravitational energy is always nonnegative.

Hence, we see that the total ADM mass does not always equal the integral of the energy density of the space-time (over a space-like slice) because we expect gravitational waves to make a positive contribution to the total mass. This might lead one to conjecture that the total ADM mass is always greater than the integral of the energy density of the space-time (over a space-like slice), but this is also false. In fact, potential energy contributions between matter (to the extent that this is well-defined by certain examples) tends to make a negative contribution to the total mass. Hence, the relationship between the total mass of a space-time and the distribution of energy and black holes inside the space-time is nontrivial.

Important cases of the Positive Mass Theorem and the Penrose Conjecture can be translated into statements about complete, asymptotically flat Riemannian 3-manifolds (M^3, g) with nonnegative scalar curvature. If we consider (M^3, g, h) as a space-like hypersurface of $(N^4, \bar{g})$ with metric g_{ij} and second fundamental form h_{ij} in N^4, then Equation

(1) implies that

$$\mu = \frac{1}{8\pi} G^{00} = \frac{1}{16\pi} \left[R - \sum_{i,j} h^{ij} h_{ij} + \left(\sum_i h_i^i \right)^2 \right], \tag{2}$$

$$J^i = \frac{1}{8\pi} G^{0i} = \frac{1}{8\pi} \sum_j \nabla_j \left[h^{ij} - \left(\sum_k h_k^k \right) g^{ij} \right], \tag{3}$$

where R is the scalar curvature of the metric g, μ is the local energy density, and J^i is the local current density. The assumption of nonnegative energy density everywhere in N^4, called the dominant energy condition, implies that we must have

$$\mu \geq \left(\sum_i J^i J_i \right)^{\frac{1}{2}} \tag{4}$$

at all points on M^3. Equations (2), (3), and (4) are called the constraint equations for (M^3, g, h) in $(N^4, \bar{g})$. Thus we see that if we restrict our attention to 3-manifolds which have zero second fundamental form h in N^3, the constraint equations are equivalent to the condition that the Riemannian manifold (M^3, g) has nonnegative scalar curvature everywhere.

An asymptotically flat 3-manifold is a Riemannian manifold (M^3, g) which, outside a compact set, is the disjoint union of one or more regions (called ends) diffeomorphic to $(\mathbf{R}^3 \backslash B_1(0), \delta)$, where the metric g in each of these $\mathbf{R}^3$ coordinate charts approaches the standard metric δ on $\mathbf{R}^3$ at infinity (with certain asymptotic decay conditions — see Definition 21, [1], [2]). The Positive Mass Theorem and the Penrose Conjecture are both statements which refer to a particular *chosen end* of (M^3, g). The total mass of (M^3, g), also called the ADM mass [1], is then a parameter related to how fast this chosen end of (M^3, g) becomes flat at infinity. The usual definition of the total mass is given in Equation (225) in Section 13. In addition, we give an alternate definition of total mass in the next section.

The Positive Mass Theorem was first proved by Schoen and Yau [44] in 1979 using minimal surfaces and then by Witten [52] in 1981 using spinors. The Riemannian Positive Mass Theorem is a special case of the Positive Mass Theorem which comes from considering the space-like hypersurfaces which have zero second fundamental form in the spacetime.

The Riemannian Positive Mass Theorem. *Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature and total mass m. Then*

$$m \geq 0, \tag{5}$$

with equality if and only if (M^3, g) is isometric to $\mathbf{R}^3$ with the standard flat metric.

Apparent horizons of black holes in N^4 correspond to outermost minimal surfaces of M^3 if we assume M^3 has zero second fundamental form in N^4. A minimal surface is a surface which has zero mean curvature (and hence is a critical point for the area functional). An outermost minimal surface is a minimal surface which is not contained entirely inside another minimal surface. Again, there is a chosen end of M^3, and "contained entirely inside" is defined with respect to this end. Interestingly, it follows from a stability argument [43] that outermost minimal surfaces are always spheres. There could be more than one outermost sphere, with each minimal sphere corresponding to a different black hole, and we note that outermost minimal spheres never intersect.

As an example, consider the Schwarzschild manifolds $(\mathbf{R}^3 \backslash \{0\}, s)$ where $s_{ij} = (1 + m/2r)^4 \, \delta_{ij}$ and m is a positive constant and equals the total mass of the manifold. This manifold has zero scalar curvature everywhere, is spherically symmetric, and it can be checked that it has an outermost minimal sphere at $r = m/2$.

We define the horizon of (M^3, g) to be the union of all of the outermost minimal spheres in M^3, so that the horizon of a manifold can have multiple connected components. We note that it is usually more common to call each outermost minimal sphere a horizon, so that their union is referred to as "horizons", but it turns out to be more convenient for our purposes to refer to the union of all of the outermost minimal spheres as one object, which we will call the horizon of (M^3, g).

There is a very interesting (but not rigorous) physical motivation to define the mass that a collection of black holes contributes to be $\sqrt{\frac{A}{16\pi}}$, where A is the total surface area of the horizon of (M^3, g) [41]. Then the physical statement that a system with nonnegative energy density containing black holes contributing a mass m must have total mass at least m can be translated into the following geometric statement ([41], [37]), the proof of which is the object of this paper.

The Riemannian Penrose Conjecture. *Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature and total mass m whose outermost minimal spheres have total surface area A. Then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{6}$$

with equality if and only if (M^3, g) is isometric to the Schwarzschild metric $(\mathbf{R}^3\backslash\{0\}, s)$ of mass m outside their respective horizons.

The overview of the proof of this result is given in Section 3. The basic idea of the approach to the problem is to flow the original metric continuously to a Schwarzschild metric (outside the horizon). The particular flow we define has the important property that the area of the horizon stays constant while the total mass of the manifold is nonincreasing. Then since the Schwarzschild metric gives equality in the Penrose inequality, the inequality follows for the original metric.

The first breakthrough on the Riemannian Penrose Conjecture was made by Huisken and Ilmanen who proved the above theorem in the case that the horizon of (M^3, g) has only one connected component [30]. More generally, they proved that $m \geq \sqrt{\frac{A_{\max}}{16\pi}}$, where $A_{\max}$ is the area of the largest connected component of the horizon of (M^3, g). Their proof is as interesting as the result itself. In the seventies, Geroch [18] observed that in a manifold with nonnegative scalar curvature, the Hawking mass of a sphere (but not surfaces with multiple components) was monotone increasing under a $1/H$ flow, where H is the mean curvature of the sphere. Jang and Wald [37] proposed using this to attack the Riemannian Penrose Conjecture by flowing the horizon of the manifold out to infinity. However, it is not hard to concoct situations in which the $1/H$ flow of a sphere develops singularities, preventing the idea from working much of the time. Huisken and Ilmanen's approach then was to define a generalized $1/H$ flow which sometimes "jumps" in order to prevent singularities from developing.

Other contributions have also been made by Herzlich [26] using the Dirac operator which Witten [52] used to prove the Positive Mass Theorem, by Gibbons [19] in the special case of collapsing shells, by Tod [50], by Bartnik [4] for quasi-spherical metrics, and by the author [7] using isoperimetric surfaces. There is also some interesting work of Ludvigsen and Vickers [38] using spinors and Bergqvist [6], both concerning the Penrose inequality for null slices of a space-time.

For more on physical discussions related to the Penrose inequality, gravitational collapse, and cosmic censorship, see also [25], [24], [23], [22], [18], and [13]. Papers less well known to the author but recommended to him by Richard Rennie are [16], [20], [27] on the Positive Mass Theorem and [28], [29], [53] on the Penrose inequality.

2. Definitions and setup

Without loss of generality, we will be able to assume that an asymptotically flat metric (see Definition 21) has an even nicer behavior at infinity in each end because of the following lemma and definition.

Lemma 1 (Schoen, Yau [46]). *Let (M^3, g) be any asymptotically flat metric with nonnegative scalar curvature. Then given any $\epsilon > 0$, there exists a metric g_0 with nonnegative scalar curvature which is harmonically flat at infinity (defined in the next definition) such that*

$$1 - \epsilon \leq \frac{g_0(\vec{v}, \vec{v})}{g(\vec{v}, \vec{v})} \leq 1 + \epsilon \tag{7}$$

for all nonzero vectors $\vec{v}$ in the tangent space at every point in M and

$$|\overline{m}_k - m_k| \leq \epsilon \tag{8}$$

where $\overline{m}_k$ and m_k are respectively the total masses of (M^3, g_0) and (M^3, g) in the kth end.

Notice that because of Equation (7), the percentage difference in areas as well as lengths between the two metrics is arbitrarily small. Hence, since the mass changes arbitrarily little also and since inequality (6) is a closed condition, it follows that the Riemannian Penrose inequality for asymptotically flat manifolds follows from proving the inequality for manifolds which are harmonically flat at infinity.

Definition 1. A Riemannian manifold is defined to be *harmonically flat at infinity* if, outside a compact set, it is the disjoint union of regions (which we will again call ends) with zero scalar curvature which are conformal to $(\mathbf{R}^3 \backslash B_1(0), \delta)$ with the conformal factor approaching a positive constant at infinity in each region.

Now it is fairly easy to define the total mass of an end of a manifold (M^3, g_0) which is harmonically flat at infinity. Define g_{flat} to be a smooth metric on M^3 conformal to g_0 such that in each end of M^3 in the above definition (M^3, g_{flat}) is isometric to $(\mathbf{R}^3 \backslash B_1(0), \delta)$. Define $\mathcal{U}_0(x)$ such that

$$g_0 = \mathcal{U}_0(x)^4 g_{\text{flat}}. \tag{9}$$

Then since (M^3, g_0) has zero scalar curvature in each end, $(\mathbf{R}^3 \backslash B_1(0), \mathcal{U}_0(x)^4 \delta)$ must have zero scalar curvature. This implies that $\mathcal{U}_0(x)$ is harmonic in $(\mathbf{R}^3 \backslash B_1(0), \delta)$ (see Equation (240) in Appendix A). Since $\mathcal{U}_0(x)$ is a harmonic function going to a constant at infinity, we may expand it in terms of spherical harmonics to get

$$\mathcal{U}_0(x) = a + \frac{b}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right), \tag{10}$$

where a and b are constants.

Definition 2. The *total mass* (of an end) of a Riemannian 3-manifold which is harmonically flat at infinity is defined to be $2ab$ in the above equation.

While the constants a and b scale depending on how we represent (M^3, g_0) as the disjoint union of a compact set and ends in Definition 1, it can be checked that $2ab$ does not. Furthermore, this definition agrees with the standard definition of the total mass of an asymptotically flat manifold (defined in Equation (225)) in the case that the manifold is harmonically flat at infinity. We choose to work with this definition because it is more convenient for the calculations we will be doing in this paper.

Now we turn our attention to the definition and properties of horizons. For convenience, we modify the topology of M^3 by compactifying all of the ends of M^3 except for the chosen end by adding the points $\{\infty_k\}$. (However, the metrics will still not be defined on these new points.)

Definition 3. Define $\mathcal{S}$ to be the collection of surfaces which are smooth compact boundaries of open sets in M^3 containing the points $\{\infty_k\}$.

All of the surfaces that we will be dealing with in this paper will be in $\mathcal{S}$. Also, we see that all of the surfaces in $\mathcal{S}$ divide M^3 into two regions, an inside (the open set) and an outside (the complement of the open set). Thus, the notion of one surface in $\mathcal{S}$ (entirely) enclosing another surface in $\mathcal{S}$ is well-defined.

Definition 4. A *horizon* of (M^3, g) is any zero mean curvature surface in $\mathcal{S}$.

A horizon may have multiple connected components. Furthermore, by minimizing area over surfaces in $\mathcal{S}$, a horizon is guaranteed to exist when M^3 has more than one end.

Definition 5. A horizon is defined to be *outermost* if it is not enclosed by another horizon.

We note that when at least one horizon exists, there is always a unique outermost horizon, with respect to the chosen end.

Definition 6. A surface $\Sigma \in \mathcal{S}$ is defined to be (strictly) *outer minimizing* if every other surface $\widetilde{\Sigma} \in \mathcal{S}$ which encloses it has (strictly) greater area.

An outer minimizing surface must have nonnegative mean curvature since otherwise the first variation formula would imply that an outward variation would yield a surface with less area. Also, in the case that Σ is a horizon, it is an outer minimizing horizon if and only if it is not enclosed by a horizon with less area. Interestingly, every component of an outer minimizing horizon must be a 2-sphere. When the horizon is strictly outer minimizing (or outermost), this fact follows from the Gauss-Bonnet formula and a second variation argument ([43], or see Section 8). (Without the strictness assumption, tori become possibilities, but are then ruled out by [12].) We will not use outermost horizons very much in this paper, but point out that outermost horizons are always strictly outer minimizing. Hence, the following theorem, which is the main result of this paper, is a slight generalization of the Riemannian Penrose Conjecture.

Theorem 1. *Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature, total mass m, and an outer minimizing horizon (with one or more components) of total area A. Then*

$$m \geq \sqrt{\frac{A}{16\pi}} \tag{11}$$

with equality if and only if (M^3, g) is isometric to a Schwarzschild manifold outside their respective outermost horizons.

Besides the flat metric on $\mathbf{R}^3$, the Schwarzschild manifolds are the only other complete spherically symmetric 3-manifolds with zero scalar curvature, and as previously mentioned can be described explicitly as $(\mathbf{R}^3\backslash\{0\}, s)$ where

$$s_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}, \tag{12}$$

r is the distance from the origin in $\mathbf{R}^3$, and m is a positive constant and equals the total mass of the manifold. Then since the Schwarzschild manifolds have a single minimal sphere which is the coordinate sphere of radius $m/2$, we can verify they give equality in the above theorem.

3. Overview of the proof

In this section we give the overview of the proof of Theorem 1, which is a slight generalization of the Riemannian Penrose Conjecture. The remainder of the paper is then devoted to proving and finding applications for the claims made in this section.

As discussed in the previous section, without loss of generality for proving the Riemannian Penrose inequality for asymptotically flat manifolds we may restrict our attention to harmonically flat manifolds.

Assumption. From this point on, we will assume that (M^3, g_0) is a complete, smooth, harmonically flat 3-manifold with nonnegative scalar curvature and an outer minimizing horizon Σ_0 (with one or more components) of total area A_0, unless otherwise stated.

We will generalize our results to the asymptotically flat case and handle the case of equality of Theorem 1 in Section 13.

We define a continuous family of conformal metrics $\{g_t\}$ on M^3, where

$$g_t = u_t(x)^4 g_0 \tag{13}$$

and $u_0(x) \equiv 1$. Given the metric g_t, define

$$\Sigma(t) = \text{the outermost minimal area enclosure of } \Sigma_0 \text{ in } (M^3, g_t) \tag{14}$$

where Σ_0 is the original outer minimizing horizon in (M^3, g_0) and we stay inside the collection of surfaces $\mathcal{S}$ defined in the previous section. In the cases in which we are interested, $\Sigma(t)$ will not touch Σ_0, from which it follows that $\Sigma(t)$ is actually a strictly outer minimizing horizon of (M^3, g_t). Then given the horizon $\Sigma(t)$, define $v_t(x)$ such that

$$\text{(15)} \qquad \begin{cases} \Delta_{g_0} v_t(x) & \equiv & 0 & \text{outside } \Sigma(t) \\ v_t(x) & = & 0 & \text{on } \Sigma(t) \\ \lim_{x\to\infty} v_t(x) & = & -e^{-t} & \end{cases}$$

and $v_t(x) \equiv 0$ inside $\Sigma(t)$. Finally, given $v_t(x)$, define

$$\text{(16)} \qquad u_t(x) = 1 + \int_0^t v_s(x)ds$$

so that $u_t(x)$ is continuous in t and has $u_0(x) \equiv 1$.

Theorem 2. *Taken together, Equations* (13), (14), (15) *and* (16) *define a first order o.d.e. in t for $u_t(x)$ which has a solution which is Lipschitz in the t variable, C^1 in the x variable everywhere, and smooth in the x variable outside $\Sigma(t)$. Furthermore, $\Sigma(t)$ is a smooth, strictly outer minimizing horizon in (M^3, g_t) for all $t \geq 0$, and $\Sigma(t_2)$ encloses but does not touch $\Sigma(t_1)$ for all $t_2 > t_1 \geq 0$.*

Since $v_t(x)$ is a superharmonic function in (M^3, g_0), it follows that $u_t(x)$ is superharmonic as well, and from Equation (16) we see that $\lim_{x\to\infty} u_t(x) = e^{-t}$ and consequently that $u_t(x) > 0$ for all x and t. Then since

$$\text{(17)} \qquad R(g_t) = u_t(x)^{-5}(-8\Delta_g + R(g))u_t(x)$$

it follows that (M^3, g_t) is an asymptotically flat manifold with nonnegative scalar curvature.

Even so, it still may not seem like g_t is particularly naturally defined since the rate of change of g_t appears to depend on t and the original metric g_0 in Equation (15). We would prefer a flow where the rate of change of g_t is only a function of g_t (and M^3 and Σ_0 perhaps), and interestingly enough this actually does turn out to be the case. In Appendix A we prove this very important fact and provide a very natural motivation for defining this conformal flow of metrics.

Definition 7. The function $A(t)$ is defined to be the total area of the horizon $\Sigma(t)$ in (M^3, g_t).

Definition 8. The function $m(t)$ is defined to be the total mass of (M^3, g_t) in the chosen end.

The next theorem is the key property of the conformal flow of metrics.

Theorem 3. *The function $A(t)$ is constant in t and $m(t)$ is non-increasing in t, for all $t \geq 0$.*

The fact that $A'(t) = 0$ follows from the fact that to first order the metric is not changing on $\Sigma(t)$ (since $v_t(x) = 0$ there) and from the fact that to first order the area of $\Sigma(t)$ does not change as it moves outward since $\Sigma(t)$ has zero mean curvature in (M^3, g_t). We make this rigorous in Section 5. Hence, the interesting part of Theorem 3 is proving that $m'(t) \leq 0$. We will see that this last statement follows from a nice trick using the Riemannian Positive Mass Theorem.

Another important aspect of this conformal flow of the metric is that outside the horizon $\Sigma(t)$, the manifold (M^3, g_t) becomes more and more spherically symmetric and "approaches" a Schwarzschild manifold $(\mathbf{R}^3 \backslash \{0\}, s)$ in the limit as t goes to ∞. More precisely,

Theorem 4. *For sufficiently large t, there exists a diffeomorphism ϕ_t between (M^3, g_t) outside the horizon $\Sigma(t)$ and a fixed Schwarzschild manifold $(\mathbf{R}^3 \backslash \{0\}, s)$ outside its horizon. Furthermore, for all $\epsilon > 0$, there exists a T such that for all $t > T$, the metrics g_t and $\phi_t^*(s)$ (when determining the lengths of unit vectors of (M^3, g_t)) are within ϵ of each other and the total masses of the two manifolds are within ϵ of each other. Hence,*

$$\lim_{t \to \infty} \frac{m(t)}{\sqrt{A(t)}} = \sqrt{\frac{1}{16\pi}}. \tag{18}$$

Inequality (11) of Theorem 1 then follows from Theorems 2, 3 and 4, for harmonically flat manifolds. In addition, in Section 13 we will see that the case of equality in Theorem 1 follows from the fact that $m'(0) = 0$ if and only if (M^3, g_0) is isometric to a Schwarzschild manifold outside their respective outermost horizons. We will also generalize our results to the asymptotically flat case in that section.

The diagrams in Figures 1 and 2 are meant to help illustrate some of the properties of the conformal flow of the metric. Figure 1 depicts the original metric which has a strictly outer minimizing horizon Σ_0. As t increases, $\Sigma(t)$ moves outwards, but never inwards. In Figure 2, we can observe one of the consequences of the fact that $A(t) = A_0$ is constant in t. Since the metric is not changing inside $\Sigma(t)$, all of the horizons $\Sigma(s)$, $0 \leq s \leq t$ have area A_0 in (M^3, g_t). Hence, inside $\Sigma(t)$, the manifold

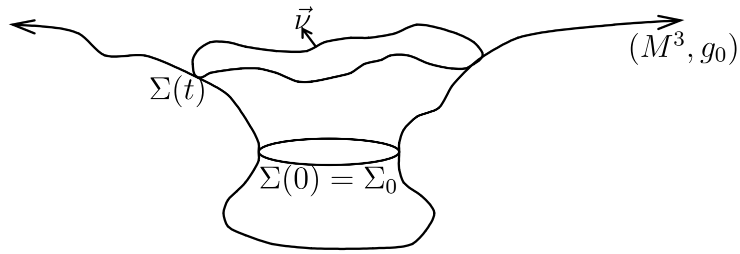


Figure 1.

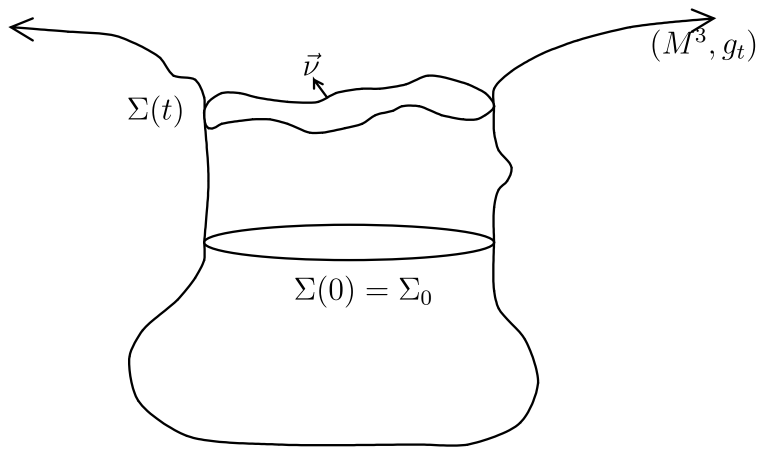


Figure 2.

(M^3, g_t) becomes cylinder-like in the sense that it is laminated by all of the previous horizons which all have the same area A_0 with respect to the metric g_t.

Now let us suppose that the original horizon Σ_0 of (M^3, g) had two components, for example. Then each of the components of the horizon will move outwards as t increases, and at some point before they touch they will suddenly jump outwards to form a horizon with a single component enclosing the previous horizons with two components. Even horizons with only one component will sometimes jump outwards, and it is interesting that this phenomenon of surfaces jumping is also found in the Huisken-Ilmanen approach to the Penrose Conjecture using their generalized $1/H$ flow.

4. Existence and regularity of the flow of metrics $\{g_t\}$

In this section we will prove Theorem 2 which claims that there exists a solution $u_t(x)$ to the o.d.e. in t defined by Equations (13), (14), (15), and (16) with certain regularity properties. To do this, for each $\epsilon \in (0, \frac{1}{2})$ we will define another family of conformal factors $u_t^\epsilon(x)$ which will be easy to prove exists, and then define

$$u_t(x) = \lim_{\epsilon \to 0} u_t^\epsilon(x) \tag{19}$$

which we will then show satisfies the original o.d.e.

Define $\lfloor z \rfloor$ to be the greatest integer less than or equal to z. Define

$$\lfloor z \rfloor_\epsilon = \epsilon \left\lfloor \frac{z}{\epsilon} \right\rfloor \tag{20}$$

which we see is the greatest integer multiple of ϵ less than or equal to z.

Define

$$g_t^\epsilon = u_t^\epsilon(x)^4 g_0 \tag{21}$$

and $u_0^\epsilon(x) \equiv 1$. Given the metric g_t^ϵ, define (for $t \geq 0$)

(22)
$$\Sigma^\epsilon(t) = \begin{cases} \Sigma_0 & \text{if } t = 0 \\ \text{the outermost minimal area enclosure} & \\ \text{of } \Sigma^\epsilon(t-\epsilon) \text{ in } (M^3, g_t^\epsilon) & \text{if } t = k\epsilon,\ k \in \mathbf{Z}^+ \\ \Sigma^\epsilon(\lfloor t \rfloor_\epsilon) & \text{otherwise,} \end{cases}$$

where Σ_0 is the original outer minimizing horizon in (M^3, g_0) and we stay inside the collection of surfaces $\mathcal{S}$ defined in Section 2. Given $\Sigma^\epsilon(t)$, define $v_t^\epsilon(x)$ such that

$$\begin{cases} \Delta_{g_0} v_t^\epsilon(x) & \equiv & 0 & \text{outside } \Sigma^\epsilon(t) \\ v_t^\epsilon(x) & = & 0 & \text{on } \Sigma^\epsilon(t) \\ \lim_{x\to\infty} v_t^\epsilon(x) & = & -(1-\epsilon)^{\lfloor \frac{t}{\epsilon} \rfloor} & \end{cases} \tag{23}$$

and $v_t^\epsilon(x) \equiv 0$ inside $\Sigma^\epsilon(t)$. Finally, given $v_t^\epsilon(x)$, define

$$u_t^\epsilon(x) = 1 + \int_0^t v_s^\epsilon(x)\,ds \tag{24}$$

so that $u_t^\epsilon(x)$ is continuous in t and has $u_0^\epsilon(x) \equiv 1$.

Notice that $\Sigma^\epsilon(t)$ and hence $v_t^\epsilon(x)$ are fixed for $t \in [k\epsilon, (k+1)\epsilon)$. Furthermore, for $t = k\epsilon$, $k \in \mathbf{Z}^+$, $\Sigma^\epsilon(t)$ does not touch $\Sigma^\epsilon(t-\epsilon)$, because it can be shown that $\Sigma^\epsilon(t-\epsilon)$ has negative mean curvature in (M^3, g_t^ϵ) and thus acts as a barrier. Hence, $\Sigma^\epsilon(t)$ is actually a strictly outer minimizing horizon of (M^3, g_t^ϵ) and is smooth since g_t^ϵ is smooth outside $\Sigma^\epsilon(t-\epsilon)$.

From these considerations it follows that a solution $u_t^\epsilon(x)$ to Equations (21), (22), (23), and (24) always exists. We can think of $u_t^\epsilon(x)$ as what results when we approximate the original o.d.e. with a stepping procedure where the step size equals ϵ.

Initially it might seem a little strange that we are requiring

$$\lim_{x\to\infty} v_t^\epsilon(x) = -(1-\epsilon)^{\lfloor \frac{t}{\epsilon} \rfloor}. \tag{25}$$

However, this is done so that

$$\lim_{x\to\infty} v_t^\epsilon(x) = -\lim_{x\to\infty} u_t^\epsilon(x) \tag{26}$$

for t values which are an integral multiple of ϵ. This is necessary to prove that the rate of change of g_t^ϵ is just a function of g_t^ϵ and not of g_0 (when t is an integral multiple of ϵ), and the argument is the same as the one given for the original o.d.e. in Appendix A.

In Corollary 15 of Appendix E, we show that we have upper bounds on the $C^{k,\alpha}$ "norms" of the horizons $\{\Sigma^\epsilon(t)\}$, as defined in Definition 33. Furthermore, these upper bounds do not depend on ϵ, and depend only on T, Σ_0, g_0, and the choice of coordinate charts for M^3. Hence, not only are these surfaces smooth, but any limits of these surfaces, which we will be dealing with later, will also be smooth.

Lemma 2. *The horizon $\Sigma^\epsilon(t_2)$ encloses $\Sigma^\epsilon(t_1)$ for all $t_2 \geq t_1 \geq 0$.*

Proof. Follows from the definition of $\Sigma^\epsilon(t)$ in Equation (22). q.e.d.

Lemma 3. *The horizon $\Sigma^\epsilon(t)$ is the outermost minimal area enclosure of Σ_0 in (M^3, g_t^ϵ) when $t = k\epsilon$, $k \in \mathbf{Z}^+$.*

Proof. The proof is by induction on k. The case when $k = 1$ follows by definition from Equation (22). Now assume that the lemma is true for $t = (k-1)\epsilon$. Then since the metric is not changing inside $\Sigma^\epsilon((k-1)\epsilon)$ for $(k-1)\epsilon \leq t \leq k\epsilon$, the outermost minimal area enclosure of Σ_0 in $(M^3, g_{k\epsilon})$ must be outside $\Sigma^\epsilon((k-1)\epsilon)$. Then the lemma follows for $t = k\epsilon$ from Equation (22) again. q.e.d.

Lemma 4. *The functions $u_t^\epsilon(x)$ are positive, bounded, locally Lipschitz functions (in x and t), with uniform Lipschitz constants independent of ϵ.*

Proof. Since $u_t^\epsilon(x)$ is superharmonic and goes to a positive constant at infinity (which is seen by integrating $v_t^\epsilon(x)$ in Equation (24)), $u_t^\epsilon(x) > 0$. And since $v_t^\epsilon(x) \leq 0$, then from Equation (24) it follows that $u_t^\epsilon(x) \leq 1$. The fact that $u_t^\epsilon(x)$ is Lipschitz in t follows from Equation (24) and the fact that $v_t^\epsilon(x)$ is bounded, and the fact that $u_t^\epsilon(x)$ is Lipschitz in x follows from the fact that $v_t^\epsilon(x)$ is Lipschitz (with Lipschitz constant depending on t) by Corollary 15 of Appendix E. q.e.d.

Corollary 1. *There exists a subsequence $\{\epsilon_i\}$ converging to zero such that*

$$u_t(x) = \lim_{\epsilon_i \to 0} u_t^{\epsilon_i}(x) \tag{27}$$

exists, is locally Lipschitz (in x and t), and the convergence is locally uniform. Hence, we may define the metric

$$g_t = \lim_{\epsilon_i \to 0} g_t^{\epsilon_i} = u_t(x)^4 g_0 \tag{28}$$

for $t \geq 0$ as well.

Proof. Follows from Lemma 4. We will stay inside this subsequence for the remainder of this section. q.e.d.

Definition 9. Define $\{\widetilde{\Sigma}_\gamma(t)\}$ to be the collections of limit surfaces of $\Sigma^{\epsilon_i}(t)$ in the limit as ϵ_i approaches 0.

Initially we define the limits in the measure theoretic sense, which must exist since it is possible to bound the areas of the surfaces $\Sigma^{\epsilon_i}(t)$ from above and below and to show that they are all contained in a compact region. However, we also have bounds on the $C^{k,\alpha}$ "norms" (see Definition 33) of the horizons $\Sigma^\epsilon(t)$ by Corollary 15 in Appendix E. Hence, since the bounds given in Corollary 15 are independent of ϵ, the above limits are also true in the Hausdorff distance sense, and the limit surfaces are all smooth.

Theorem 5. *The limit surface $\widetilde{\Sigma}_{\gamma_2}(t_2)$ encloses $\widetilde{\Sigma}_{\gamma_1}(t_1)$ for all $t_2 > t_1 \geq 0$ and for any γ_1 and γ_2.*

Proof. This theorem would be trivial and would follow directly from Lemma 2 if not for the fact that there can be multiple limit surfaces for $\Sigma^{\epsilon_i}(t)$ as ϵ_i goes to zero. Instead, we have some work to do.

Given any $\delta > 0$, choose $\bar{\epsilon}$ such that

$$|u_t^{\epsilon_i}(x) - u_t(x)| < \delta \quad \text{for all } \epsilon_i < \bar{\epsilon}, \tag{29}$$

which we can do since the convergence in Equation (27) is locally Lipschitz and since the $u_t^{\epsilon}(x)$ are all harmonic outside a compact set. Next, choose $\epsilon_1, \epsilon_2 < \bar{\epsilon}$ such that

$$\mathrm{dis}(\Sigma^{\epsilon_1}(t_1), \widetilde{\Sigma}_{\gamma_1}(t_1)) < \delta \tag{30}$$

$$\mathrm{dis}(\Sigma^{\epsilon_2}(t_2), \widetilde{\Sigma}_{\gamma_2}(t_2)) < \delta \tag{31}$$

which is possible since we have convergence in the Hausdorff distance sense. Note that we define

$$\mathrm{dis}(S, T) = \max\left(\sup_{x\in S}\inf_{y\in T} d(x,y),\ \sup_{x\in T}\inf_{y\in S} d(x,y)\right) \tag{32}$$

where $d(x, y)$ is the usual distance function in (M^3, g_0). Then by Equation (29) and the triangle inequality,

$$|u_t^{\epsilon_1}(x) - u_t^{\epsilon_2}(x)| < 2\delta \tag{33}$$

so that by Equation (24) we have that

$$\left|\int_0^t (v_s^{\epsilon_1}(x) - v_s^{\epsilon_2}(x))\, ds\right| < 2\delta \tag{34}$$

which by the triangle inequality once again implies that

$$\left|\int_{t_1}^{t_2} (v_s^{\epsilon_1}(x) - v_s^{\epsilon_2}(x))\, ds\right| < 4\delta. \tag{35}$$

Since $\Sigma^{\epsilon_2}(t_2)$ encloses $\Sigma^{\epsilon_2}(s)$ for $s < t_2$, it follows from the maximum principle that

$$v_s^{\epsilon_2}(x) \le v_{t_2}^{\epsilon_2}(x)\ (1-\epsilon_2)^{\left(\left\lfloor \frac{s}{\epsilon_2}\right\rfloor - \left\lfloor \frac{t_2}{\epsilon_2}\right\rfloor\right)}. \tag{36}$$

Similarly, since $\Sigma^{\epsilon_1}(t_1)$ is enclosed by $\Sigma^{\epsilon_1}(s)$ for $s > t_1$, it also follows from the maximum principle that

$$v_s^{\epsilon_1}(x) \ge v_{t_1}^{\epsilon_1}(x)\ (1-\epsilon_1)^{\left(\left\lfloor \frac{s}{\epsilon_1}\right\rfloor - \left\lfloor \frac{t_1}{\epsilon_1}\right\rfloor\right)}. \tag{37}$$

Now suppose that $\widetilde{\Sigma}_{\gamma_2}(t_2)$ did not enclose $\widetilde{\Sigma}_{\gamma_1}(t_1)$ for some $t_2 > t_1 \ge 0$. Then choose x_0 strictly inside $\widetilde{\Sigma}_{\gamma_1}(t_1)$ and strictly outside $\widetilde{\Sigma}_{\gamma_2}(t_2)$, and choose any positive $\delta < \frac{1}{2}\max\Big(\mathrm{dis}\Big(x_0, \widetilde{\Sigma}_{\gamma_1}(t_1)\Big),\ \mathrm{dis}\Big(x_0, \widetilde{\Sigma}_{\gamma_2}(t_2)\Big)\Big)$. Then we must have

$$v^{\epsilon_2}_{t_2}(x_0) < 0 \quad \text{and} \quad v^{\epsilon_1}_{t_1}(x_0) = 0. \tag{38}$$

Then combining Equations (35), (36), (37) and (38) yields

$$-v^{\epsilon_2}_{t_2}(x_0) \int_{t_1}^{t_2} (1-\epsilon_2)^{\left(\left\lfloor \frac{s}{\epsilon_2}\right\rfloor - \left\lfloor \frac{t_2}{\epsilon_2}\right\rfloor\right)}\, ds < 4\delta. \tag{39}$$

Let $\lim_{\epsilon_2 \to 0} v^{\epsilon_2}_{t_2}(x_0) = -\alpha$, which must be negative by the maximum principle since $\Sigma^{\epsilon_2}(t_2)$ is approaching $\widetilde{\Sigma}_{\gamma_2}(t_2)$ smoothly. Then taking the limit of inequality (39) as δ and ϵ_2 both go to zero yields

$$\alpha \int_{t_1}^{t_2} e^{(t_2 - s)}\, ds \leq 0, \tag{40}$$

a contradiction since t_2 is strictly greater than t_1. Hence, $\widetilde{\Sigma}_{\gamma_2}(t_2)$ must enclose $\widetilde{\Sigma}_{\gamma_1}(t_1)$ for all $t_2 > t_1 \geq 0$, proving the theorem. q.e.d.

In the next section, we will prove:

Theorem 6. *Let A_0 be the area of the original outer minimizing horizon Σ_0 with respect to the original metric g_0. Then*

$$|\widetilde{\Sigma}_\gamma(t)|_{g_t} = A_0 \tag{41}$$

for all $t \geq 0$, where $|\cdot|_{g_t}$ denotes area with respect to the metric g_t.

Corollary 2. *In addition,*

$$|\widetilde{\Sigma}_\gamma(t_1)|_{g_{t_2}} = A_0 \tag{42}$$

for all $t_2 \geq t_1 \geq 0$.

Proof. This statement follows from the previous two theorems and the fact that $v^{\epsilon_i}_t(x)$ is defined to be zero inside $\Sigma^{\epsilon_i}(t)$ so that $g_{t_1} = g_{t_2}$ inside $\Sigma(t_2)$ for $t_2 \geq t_1 \geq 0$. q.e.d.

Definition 10. Define $\Sigma(t)$ to be the outermost minimal area enclosure of the original horizon Σ_0 in (M^3, g_t).

We note that the outermost minimal area enclosure of a smooth region is well-defined in that it always exists and is unique [5].

Lemma 5. *It is also true that*

$$|\Sigma(t)|_{g_t} = A_0. \tag{43}$$

for all $t \geq 0$.

Proof. Since by Lemma 3 $\Sigma^{\epsilon_i}(t)$ is the outermost minimal area enclosure of Σ_0 in $(M^3, g_t^{\epsilon_i})$, the result follows from Theorem 6 and the fact that $u_t^{\epsilon_i}(x)$ is locally uniformly Lipschitz and is converging to $u_t(x)$ locally uniformly. This is also discussed in the next section. q.e.d.

Lemma 6. *The surface* $\Sigma(t_2)$ *encloses* $\widetilde{\Sigma}_\gamma(t_1)$ *for all* γ *and* $t_2 > t_1 \geq 0$.

Proof. By Lemma 5, the minimal area enclosures of Σ_0 in (M^3, g_{t_2}) have area A_0. But by Corollary 2, $\widetilde{\Sigma}_\gamma(t_1)$ has area A_0 in (M^3, g_{t_2}). Hence, since $\Sigma(t_2)$ is the outermost minimal area enclosure by definition, it follows that $\Sigma(t_2)$ encloses $\widetilde{\Sigma}_\gamma(t_1)$. q.e.d.

Lemma 7. *The surface* $\widetilde{\Sigma}_\gamma(t_2)$ *encloses* $\Sigma(t_1)$ *for all* γ *and* $t_2 > t_1 \geq 0$.

Proof. Suppose $\widetilde{\Sigma}_\gamma(t_2)$ did not (entirely) enclose $\Sigma(t_1)$ for some $t_2 > t_1 \geq 0$. Then we choose a subsequence $\{\epsilon_i'\} \subset \{\epsilon_i\}$ converging to zero such that $\Sigma^{\epsilon_i'}(t_2)$ is converging to $\widetilde{\Sigma}_\gamma(t_2)$ in the Hausdorff distance sense.

On the other hand, by Lemma 5 we have

$$\lim_{\epsilon_i' \to 0} |\Sigma(t_1)|_{g_{t_1}^{\epsilon_i'}} = |\Sigma(t_1)|_{g_{t_1}} = A_0. \tag{44}$$

However, for a given ϵ_i', the metric is shrinking outside of $\Sigma^{\epsilon_i'}(t_2)$ for $t_1 \leq t \leq t_2$ by a uniform amount which can be made independent of ϵ_i', and follows from Lemma 2 and Equations (23) and (24). Hence,

$$\lim_{\epsilon_i' \to 0} |\Sigma(t_1)|_{g_{t_2}^{\epsilon_i'}} = |\Sigma(t_1)|_{g_{t_2}} < A_0, \tag{45}$$

which by Definition 10 violates Lemma 5. q.e.d.

Corollary 3. *The surface* $\Sigma(t_2)$ *encloses* $\Sigma(t_1)$ *for all* $t_2 > t_1 \geq 0$.

Proof. Follows directly from the two previous lemmas. q.e.d.

Definition 11. Define

$$\Sigma^+(t) = \lim_{s\to t^+} \Sigma(s), \qquad \widetilde{\Sigma}^+(t) = \lim_{s\to t^+} \widetilde{\Sigma}_\gamma(s), \tag{46}$$
$$\Sigma^-(t) = \lim_{s\to t^-} \Sigma(s), \qquad \widetilde{\Sigma}^-(t) = \lim_{s\to t^-} \widetilde{\Sigma}_\gamma(s),$$

where we define $\Sigma^-(0) = \Sigma_0 = \widetilde{\Sigma}^-(0)$.

We note that by the inclusion properties of Theorem 5 and Corollary 3 that these limits are always unique.

Definition 12. Define the jump times J to be the set of all $t \geq 0$ with $\Sigma^+(t) \neq \Sigma^-(t)$.

Theorem 7. *We have the inclusion property that for all $t_2 > t_1 \geq 0$, the surfaces $\Sigma(t_2), \widetilde{\Sigma}_\gamma(t_2), \Sigma^+(t_2), \widetilde{\Sigma}^+(t_2), \Sigma^-(t_2), \widetilde{\Sigma}^-(t_2)$ all enclose the surfaces $\Sigma(t_1), \widetilde{\Sigma}_\gamma(t_1), \Sigma^+(t_1), \widetilde{\Sigma}^+(t_1), \Sigma^-(t_1), \widetilde{\Sigma}^-(t_1)$. Also, for $t \geq 0$*

$$\Sigma(t) = \Sigma^+(t) = \widetilde{\Sigma}^+(t) \quad \textit{and encloses} \quad \Sigma^-(t) = \widetilde{\Sigma}^-(t), \tag{47}$$

and all five surfaces are smooth with area A_0 in (M^3, g_t). Furthermore, except for $t \in J$, all five surfaces in Equation (47) are equal, $\widetilde{\Sigma}_\gamma(t)$ is single valued, and $\Sigma(t) = \widetilde{\Sigma}_\gamma(t)$. In addition, the set J is countable.

Proof. The inclusion property follows directly from Lemmas 6 and 7. These two lemmas also prove that $\Sigma^+(t)$ encloses $\widetilde{\Sigma}^+(t)$ and that $\widetilde{\Sigma}^+(t)$ encloses $\Sigma^+(t)$, proving that they are equal. Similarly it follows that $\Sigma^-(t) = \widetilde{\Sigma}^-(t)$, and these two lemmas also imply that $\Sigma^+(t) = \widetilde{\Sigma}^+(t)$ encloses $\Sigma^-(t) = \widetilde{\Sigma}^-(t)$.

By Corollary 1 and Theorem 6, all five surfaces have area A_0 in (M^3, g_t). By Corollary 3, $\Sigma(t)$ is enclosed by $\Sigma^+(t)$. On the other hand, $\Sigma^+(t)$ has area A_0 and $\Sigma(t)$ encloses all other minimal area enclosures of Σ_0, so $\Sigma(t)$ encloses $\Sigma^+(t)$. Hence, $\Sigma(t) = \Sigma^+(t)$.

By the definition of J, all five surfaces are equal for $t \notin J$. Since each $\widetilde{\Sigma}_\gamma(t)$ is sandwiched between the surfaces $\widetilde{\Sigma}^-(t)$ and $\widetilde{\Sigma}^+(t)$ which are equal, it follows that $\widetilde{\Sigma}_\gamma(t)$ is a single limit surface and equals $\widetilde{\Sigma}^-(t) = \widetilde{\Sigma}^+(t)$ for these $t \notin J$, from which it follows that $\Sigma(t) = \widetilde{\Sigma}_\gamma(t)$.

Define $\Delta V(t)$ to equal the volume enclosed by $\Sigma^+(t)$ but not by $\Sigma^-(t)$. Then $\sum_{t\in J\cap[0,T]} \Delta V(t)$ is finite for all $T > 0$ since it is less than or equal to the volume enclosed by $\Sigma(T+1)$ but not by Σ_0 which is finite. Also, $\Delta V(t) > 0$ for $t \in J$ since by Corollary 15 of Appendix E these surfaces are all uniformly smooth. Hence, J is countable. q.e.d.

Definition 13. Given the horizon $\Sigma(t)$, define $v_t(x)$ such that

$$\left\{\begin{array}{rcll} \Delta_{g_0} v_t(x) & \equiv & 0 & \text{outside } \Sigma(t) \\ v_t(x) & = & 0 & \text{on } \Sigma(t) \\ \lim_{x\to\infty} v_t(x) & = & -e^{-t} & \end{array}\right. \tag{48}$$

and $v_t(x) \equiv 0$ inside $\Sigma(t)$.

Lemma 8. *Using the definition of $v_t(x)$ given above and the definition of $u_t(x)$ given in Corollary* 1, *we have that*

$$u_t(x) = 1 + \int_0^t v_s(x) ds. \tag{49}$$

Proof. By Theorem 7, $\lim_{\epsilon_i \to 0} v_s^{\epsilon_i}(x) = v_s(x)$ for almost every value of $s \geq 0$. Then using the inclusion property of Theorem 7 it follows that we can pass the limit into the integral in Equation (24), proving the lemma. q.e.d.

Lemma 9. *The surfaces $\{\Sigma(t)\}$ do not touch for different values of $t \geq 0$. Furthermore, the metric g_t and the conformal factor $u_t(x)$ are C^1 in x, and $\Sigma(t_1)$, $\Sigma^+(t_1)$, and $\Sigma^-(t_1)$ are outer minimizing horizons with area A_0 in (M^3, g_{t_2}) for $t_2 \geq t_1 \geq 0$.*

Proof. First we note that by Equation (23) and Corollary 15 in Appendix E that there exists a locally uniform constant c such that $v_t^\epsilon(x) - cf(x)$ is convex, where $f(x)$ is any locally defined smooth function with all of its second derivatives greater than or equal to one. Hence, by Equation (24), the same statement is true for $u_t^\epsilon(x)$, and then also for $u_t(x)$ after taking the limit. Hence, for all $x \in M$ the directional derivatives of $u_t(x)$ exist in all directions and

$$\nabla_{\vec{w}} u_t(x) \leq -\nabla_{-\vec{w}} u_t(x). \tag{50}$$

The mean curvature of a surface Σ in (M^3, g_t) is

$$H = u_t(x)^{-2} H_0 + 4u_t(x)^{-3} \nabla_{\vec{\nu}} u_t(x) \tag{51}$$

where H_0 is the mean curvature and $\vec{\nu}$ is the outward pointing unit normal vector of Σ in (M^3, g_0). Hence, by Equation (50), the mean curvature of Σ on the outside is always less than or equal to the mean curvature of Σ on the inside (but still using an outward pointing unit normal vector). Since Σ_0 has zero mean curvature in (M^3, g_0) and since

$u_t(x) = 1$ on Σ_0 and $u_t(x) \leq 1$ everywhere else, it follows that the mean curvature of Σ_0 (on the outside) is nonpositive in (M^3, g_t) and thus acts as a barrier for $\Sigma(t)$.

Suppose $\Sigma(t)$ touched Σ_0 at x_0 for some $t > 0$. Since $\Sigma(t)$ is outside Σ_0, it must have nonpositive mean curvature in (M^3, g_0). Then by Equation (49) we have $\nabla_{\vec{\nu}} u_t(x_0) < 0$, where $\vec{\nu}$ is the outward pointing unit normal vector to $\Sigma(t)$ and Σ_0 at x_0 in (M^3, g_0). Hence, by Equation (51) it follows that $\Sigma(t)$ has negative mean curvature on the outside in (M^3, g_t). This is a contradiction, since by the first variation formula and the pseudo-convexity of $u_t(x)$ we could then flow $\Sigma(t)$ out and decrease its area, contradicting the fact that it is defined to be a minimal area enclosure of Σ_0 in (M^3, g_t). Hence, $\Sigma(t)$ does not touch Σ_0.

Then since the mean curvature of $\Sigma(t)$ on the outside is less than or equal to the mean curvature on the inside, it follows that both mean curvatures for $\Sigma(t)$ in (M^3, g_t) must be zero. Otherwise, it would be possible to do a variation of $\Sigma(t)$ which decreased its area in (M^3, g_t). Similarly, by Corollary 2 and Theorem 7, $|\Sigma(t_1)|_{g_{t_2}} = A_0$ too, so that by the same first variation argument the mean curvatures of $\Sigma(t_1)$ on the outside and inside in (M^3, g_2) must both also be zero, for $t_2 \geq t_1 \geq 0$. Hence, $\Sigma(t_1)$ is a horizon in (M^3, g_{t_2}).

Hence, $\Sigma(t_1)$ acts as a barrier for $\Sigma(t_2)$ for $t_2 > t_1 \geq 0$. As discussed in Appendix A, the o.d.e. for $u_t(x)$ is actually translation invariant in t since the rate of change of g_t is a function of g_t and not of t. Thus, the argument proving that $\Sigma(t_2)$ does not touch $\Sigma(t_1)$ is essentially the same as the argument two paragraphs above which proved that $\Sigma(t)$ did not touch Σ_0.

To see that $u_t(x)$ is C^1 in x, we first observe that by Equation (49) the directional derivatives in x for $u_t(x)$ equal

$$\nabla_{\vec{w}} u_t(x) = \int_0^t \nabla_{\vec{w}} v_s(x) ds. \tag{52}$$

Furthermore, $v_t(x)$ is smooth everywhere except on $\Sigma(t)$ and has locally uniform bounds on all of its derivatives off of $\Sigma(t)$ because of the regularity of $\Sigma(t)$ coming from Theorem 7 and Corollary 15 of Appendix E. Hence, $u_t(x)$ will be C^1 at $x = x_0$ if and only if the set $I(x_0) = \{t \mid x_0 \in \Sigma(t)\}$ has measure zero in $\mathbf{R}$. But since we just got through proving that the horizons $\Sigma(t)$ do not touch for different values of t, $I(x_0)$ is at most one point, so $u_t(x)$ is C^1 in x. q.e.d.

Theorem 2 then follows from Corollary 1, Definition 10, Corollary 3, Theorem 7, Definition 13, Lemma 8, Lemma 9, and Corollary 15 in Appendix E.

5. Proof that $A(t)$ is a constant

The fact that $A(t)$, defined to be the area of the horizon $\Sigma(t)$ in (M^3, g_t), is constant in t was proven already in Lemma 5 of the previous section. However, this lemma relied entirely on Theorem 6, the proof of which we have postponed until now.

Proof of Theorem 6. We continue with the same notation as in the previous section.

Definition 14. We define $A^\epsilon(t) = |\Sigma^\epsilon(t)|_{g_t^\epsilon}$ and

$$\Delta A^\epsilon(t) = A^\epsilon(t+\epsilon) - A^\epsilon(t), \tag{53}$$

where t is a nonnegative integer multiple of ϵ.

Then if we can prove that for all $T > 0$,

$$\lim_{\epsilon \to 0} A^\epsilon(t) = A_0 \tag{54}$$

for all $t \in [0, T]$, Theorem 6 follows since each limit surface $\widetilde{\Sigma}_\gamma(t)$ is the limit of a $\Sigma^{\epsilon_i}(t)$ for some choice of $\{\epsilon_i\}$ converging to zero, all of the surfaces involved are uniformly smooth (Corollary 15), and $\{g_t^{\epsilon_i}\}$ are all uniformly Lipschitz and are converging uniformly to g_t (Corollary 1).

Our first observation is that since $v_t^\epsilon(x) \le 0$, the metric g_t^ϵ gets smaller pointwise as t increase. Hence, $\Delta A^\epsilon(t)$ is always negative, where we are requiring that $t = k\epsilon$ for some nonnegative integer k. Then since $A^\epsilon(0) = A_0$ by definition, it follows that

$$A^\epsilon(t) \le A_0. \tag{55}$$

Hence, all that we need to prove Equation (54) is a lower bound on $\Delta A^\epsilon(t)/\epsilon$ which goes to zero as ϵ goes to zero for "most" values of t.

For convenience, we first set $t = 0$ and estimate $\Delta A^\epsilon(0)$. This estimate will then be generalizable for all values of t since the flow is independent of the base metric g_0 and t as discussed in Section 4 and described in Appendix A. Since $u_\epsilon^\epsilon(x) = 1 + \epsilon v_0^\epsilon(x)$, it follows that

$$A^\epsilon(\epsilon) = \int_{\Sigma^\epsilon(\epsilon)} \left(1 + \epsilon v_0^\epsilon(x)\right)^4 \, dA_{g_0^\epsilon} \tag{56}$$

where as usual $dA_{g_0^\epsilon}$ is the area form of $\Sigma^\epsilon(\epsilon)$ with respect to g_0^ϵ. Then since $\Sigma^\epsilon(0)$ is outer minimizing in (M^3, g_0^ϵ), we also have

$$A^\epsilon(0) \leq \int_{\Sigma^\epsilon(\epsilon)} 1 \, dA_{g_0^\epsilon}. \tag{57}$$

Hence,

$$\Delta A^\epsilon(0) \geq \int_{\Sigma^\epsilon(\epsilon)} [(1 + \epsilon v_0^\epsilon(x))^4 - 1] \, dA_{g_0^\epsilon} \tag{58}$$

$$\geq 4\epsilon \int_{\Sigma^\epsilon(\epsilon)} \left(v_0^\epsilon(x) - \epsilon^2\right) \, dA_{g_0^\epsilon} \tag{59}$$

which follows from expanding and the fact that $-1 \leq v_0^\epsilon(x) \leq 0$, so that

$$\Delta A^\epsilon(0) \geq 4\epsilon |\Sigma^\epsilon(\epsilon)|_{g_0^\epsilon} \left(\min_{\Sigma^\epsilon(\epsilon)} v_0^\epsilon(x) - \epsilon^2 \right). \tag{60}$$

More generally, from the discussion at the end of Appendix A, we see that if we define

$$\bar{v}_t^\epsilon(x) = v_t^\epsilon(x) / u_t^\epsilon(x), \tag{61}$$

then

$$\Delta A^\epsilon(t) \geq 4\epsilon |\Sigma^\epsilon(t+\epsilon)|_{g_t^\epsilon} \left(\min_{\Sigma^\epsilon(t+\epsilon)} \bar{v}_t^\epsilon(x) - \epsilon^2 \right), \tag{62}$$

where as before t is a nonnegative integral multiple of ϵ. Furthermore,

$$|\Sigma^\epsilon(t+\epsilon)|_{g_t^\epsilon} \leq |\Sigma^\epsilon(t+\epsilon)|_{g_{t+\epsilon}^\epsilon} (1-\epsilon)^{-4} \leq A_0 (1-\epsilon)^{-4}, \tag{63}$$

since g_t^ϵ and $g_{t+\epsilon}^\epsilon$ are conformally related within a factor of $(1-\epsilon)^{-4}$ and by inequality (55). Thus,

$$\Delta A^\epsilon(t) \geq 4\epsilon A_0 (1-\epsilon)^{-4} \left(\min_{\Sigma^\epsilon(t+\epsilon)} \bar{v}_t^\epsilon(x) - \epsilon^2 \right). \tag{64}$$

Definition 15. We define $V^\epsilon(t)$ to be the volume of the region enclosed by $\Sigma^\epsilon(t)$ which is outside the original horizon Σ_0 and define

$$\Delta V^\epsilon(t) = V^\epsilon(t+\epsilon) - V^\epsilon(t), \tag{65}$$

where t is a nonnegative integer multiple of ϵ.

We now require $t \in [0, T]$, for any fixed $T > 0$. Since (M^3, g_0) is asymptotically flat and $u_t^\epsilon(x)$ has uniform upper and lower bounds, there exists a uniform upper bound V_0 for $V^\epsilon(T)$ which is independent of ϵ. Hence, by Lemma 2 it follows that

$$\Delta V^\epsilon(k\epsilon) \leq V_0\sqrt{\epsilon} \tag{66}$$

for $k = 0$ to $\left\lfloor \frac{T}{\epsilon} \right\rfloor - 1$ except at most $\left\lfloor \frac{1}{\sqrt{\epsilon}} \right\rfloor$ values of k.

Furthermore, it is possible to define a function f which is independent of ϵ and only depends on T, Σ_0, g_0, and the uniform regularity bounds on the surfaces $\Sigma^\epsilon(t)$ in Corollary 15 in Appendix E such that

$$\max_{x \in \Sigma^\epsilon((k+1)\epsilon)} \operatorname{dis}(x, \Sigma^\epsilon(k\epsilon)) \leq f(\Delta V^\epsilon(k\epsilon)), \tag{67}$$

where f is a continuous, increasing function which equals zero at zero. Also, from Corollary 15 we have

$$|\nabla v_t^\epsilon(x)|_{g_0} \leq K \tag{68}$$

for $0 \leq t \leq T$, where K is also independent of ϵ. Hence, since $v_{k\epsilon}^\epsilon(x) = 0$ on $\Sigma^\epsilon(k\epsilon)$,

$$\min_{\Sigma^\epsilon((k+1)\epsilon)} v_{k\epsilon}^\epsilon(x) \geq -Kf(V_0\sqrt{\epsilon}), \tag{69}$$

and since $u_t^\epsilon(x) \geq (1-\epsilon)^{\left\lfloor \frac{T}{\epsilon} \right\rfloor}$ for $0 \leq t \leq T$,

$$\min_{\Sigma^\epsilon((k+1)\epsilon)} \bar{v}_{k\epsilon}^\epsilon(x) \geq -Kf(V_0\sqrt{\epsilon})(1-\epsilon)^{-\left\lfloor \frac{T}{\epsilon} \right\rfloor}, \tag{70}$$

for $k = 0$ to $\left\lfloor \frac{T}{\epsilon} \right\rfloor - 1$ except at most $\left\lfloor \frac{1}{\sqrt{\epsilon}} \right\rfloor$ values of k. At these exceptional values of k we will just use the fact that $\bar{v}_{k\epsilon}^\epsilon(x) > -1$.

Hence, from Equation (64)

$$\Delta A^\epsilon(k\epsilon) \geq -4\epsilon A_0(1-\epsilon)^{-4}\left(Kf(V_0\sqrt{\epsilon})(1-\epsilon)^{-\left\lfloor \frac{T}{\epsilon} \right\rfloor} + \epsilon^2\right) \tag{71}$$

for $k = 0$ to $\left\lfloor \frac{T}{\epsilon} \right\rfloor - 1$ except at most $\left\lfloor \frac{1}{\sqrt{\epsilon}} \right\rfloor$ values of k and

$$\Delta A^\epsilon(k\epsilon) \geq -4\epsilon A_0(1-\epsilon)^{-4}\left(1 + \epsilon^2\right) \tag{72}$$

for all values of k including the exceptional ones.

Then from Equation (53) we have that

$$A^\epsilon(n\epsilon) = A_0 + \sum_{k=0}^{n-1} \Delta A^\epsilon(k\epsilon) \tag{73}$$
$$\geq A_0 - 4TA_0(1-\epsilon)^{-4}\left(Kf(V_0\sqrt{\epsilon})(1-\epsilon)^{-\lfloor \frac{T}{\epsilon} \rfloor} + \epsilon^2\right)$$
$$- 4\sqrt{\epsilon}A_0(1-\epsilon)^{-4}\left(1+\epsilon^2\right)$$

for $1 \leq n \leq \lfloor \frac{T}{\epsilon} \rfloor$. Then since f goes to zero at zero, Equation (54) follows from inequalities (55) and (73) for $0 \leq t \leq T$. Since $T > 0$ was arbitrary, this proves Theorem 6. q.e.d.

6. Green's functions at infinity and the Riemannian Positive Mass Theorem

In this section we will prove a few theorems about certain Green's functions on asymptotically flat 3-manifolds with nonnegative scalar curvature which will be needed in the next section to prove that $m(t)$ is nonincreasing in t. However, the theorems in this section, which follow from and generalize the Riemannian Positive Mass Theorem, are also of independent interest.

The results of this section are closely related to the beautiful ideas used by Bunting and Masood-ul-Alam in [11] to prove the non-existence of multiple black holes in asymptotically flat, static, vacuum space-times. Hence, while Theorems 8 and 9 do not appear in their paper, these two theorems follow from a natural extension of their techniques.

Definition 16. Given a complete, asymptotically flat manifold $(M^3, \bar{g})$ with multiple asymptotically flat ends (with one chosen end), define

$$\mathcal{E}(\bar{g}) = \inf_{\phi} \left\{ \frac{1}{2\pi} \int_{(M^3,\bar{g})} |\nabla\phi|^2 \, dV \right\} \tag{75}$$

where the infimum is taken over all smooth $\phi(x)$ which go to one in the chosen end and zero in the other ends.

Without loss of generality, we may assume that $(M^3, \bar{g})$ is actually harmonically flat at infinity (as defined in Section 2). Then since such a modification can be done so as to change the metric uniformly pointwise as small as one likes (by Lemma 1), it follows that $\mathcal{E}(\bar{g})$ changes as small as one likes as well. We remind the reader that the total mass of $(M^3, \bar{g})$ also changes arbitrarily little with such a deformation.

From standard theory it follows that the infimum in the above definition is achieved by the Green's function $\phi(x)$ which satisfies

$$\begin{cases} \lim_{x \to \infty_0} \phi(x) &= 1 \\ \Delta\phi &= 0 \\ \lim_{x \to \infty_k} \phi(x) &= 0 \text{ for all } k \neq 0 \end{cases} \tag{76}$$

where ∞_k are the points at infinity of the various asymptotically flat ends and ∞_0 is infinity in the chosen end. Define the level sets of $\phi(x)$ to be

$$\Sigma_l = \{x \mid \phi(x) = l\} \tag{77}$$

for $0 < l < 1$. Then it follows from Sard's Theorem and the smoothness of $\phi(x)$ that $\Sigma(l)$ is a smooth surface for almost every l. Then by the co-area formula it follows that

$$\mathcal{E}(\bar{g}) = \frac{1}{2\pi} \int_0^1 dl \int_{\Sigma_l} |\nabla\phi| \, dA = \frac{1}{2\pi} \int_0^1 dl \int_{\Sigma_l} \frac{d\phi}{d\eta} \, dA \tag{78}$$

since $\nabla\phi$ is orthogonal to the unit normal vector η of Σ_l. But by the divergence theorem (and since $\phi(x)$ is harmonic), $\int_\Sigma \frac{d\phi}{d\eta} \, dA$ is constant for all homologous Σ. Hence,

$$\mathcal{E}(\bar{g}) = \frac{1}{2\pi} \int_\Sigma \frac{d\phi}{d\eta} \, dA \tag{79}$$

where Σ is any surface in M^3 in $\mathcal{S}$ (which is the set of smooth, compact boundaries of open regions which contains the points at infinity $\{\infty_k\}$ in all of the ends except the chosen one). Then since $(M^3, \bar{g})$ is harmonically flat at infinity, we know that in the chosen end $\phi(x) = 1 - c/|x| + \mathcal{O}(1/|x|^2)$, so that from Equation (79) it follows that

$$\phi(x) = 1 - \frac{\mathcal{E}(\bar{g})}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{80}$$

by letting Σ in Equation (79) be a large sphere in the chosen end.

Theorem 8. *Let $(M^3, \bar{g})$ be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature which has multiple asymptotically flat ends and total mass $\overline{m}$ in the chosen end. Then*

$$\overline{m} \geq \mathcal{E}(\bar{g}) \tag{81}$$

with equality if and only if $(M^3, \bar{g})$ has zero scalar curvature and is conformal to $(\mathbf{R}^3, \delta)$ minus a finite number of points.

Proof. Again, without loss of generality we will assume that $(M^3, \bar{g})$ is harmonically flat at infinity, which by definition means that all of the ends are harmonically flat at infinity. In Figure 3, $(M^3, \bar{g})$ has three ends, the chosen end (at the top of the figure) and two other ends.

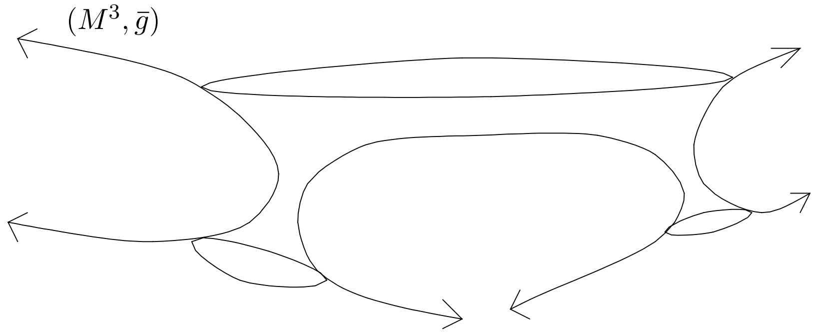


Figure 3.

Then we consider the metric $(M^3, \widetilde{g})$, with $\widetilde{g} = \phi(x)^4 \bar{g}$, as depicted in Figure 4. Since $\phi(x)$ goes to zero (and is bounded above by $C/|x|$) in all of the harmonically flat ends other than the chosen one, the metric $\widetilde{g} = \phi(x)^4 \bar{g}$ in each end is conformal to a punctured ball with the conformal factor being a bounded harmonic function to the fourth power in the punctured ball. Hence, by the removable singularity theorem, this harmonic function can be extended to the whole ball, which proves that the metric $\widetilde{g}$ can be extended smoothly over all of the points at infinity in the compactified ends.

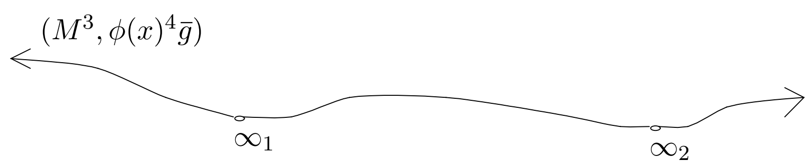


Figure 4.

Furthermore, $(M^3, \widetilde{g})$ has nonnegative scalar curvature since $(M^3, \bar{g})$ has nonnegative scalar curvature and $\phi(x)$ is harmonic with respect to $\bar{g}$ (see Equation (240) in Appendix A). Moreover, $(M^3 \cup \{\infty_k\}, \widetilde{g})$ has nonnegative scalar curvature too since in a neighborhood of each ∞_k the manifold is conformal to a ball, with the conformal factor being a positive harmonic function to the fourth power. Hence, since $(M^3 \cup \{\infty_k\}, \widetilde{g})$ is a complete 3-manifold with nonnegative scalar curvature with a single harmonically flat end, we may apply the Riemannian Positive Mass Theorem to this manifold to conclude that the total mass of this manifold, which we will call $\widetilde{m}$, is nonnegative.

Now we will compute $\widetilde{m}$ in terms of $\overline{m}$ and $\mathcal{E}(\bar{g})$. Since $\bar{g}$ is harmonically flat at infinity, we know that by Definition 2 we have $\bar{g} = \overline{\mathcal{U}}(x)^4 \bar{g}_{\text{flat}}$, where $(M^3, \bar{g}_{\text{flat}})$ is isometric to $(\mathbf{R}^3 \backslash B_r(0), \delta)$ in the harmonically flat end of $(M^3, \bar{g})$,

$$\overline{\mathcal{U}}(x) = 1 + \frac{\overline{m}}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right), \tag{82}$$

and the scale of the harmonically flat coordinate chart $\mathbf{R}^3 \backslash B_r(0)$ has been chosen so that $\overline{\mathcal{U}}(x)$ goes to one at infinity in the chosen end. Furthermore, since $\widetilde{g} = \phi(x)^4 \bar{g}$ is also harmonically flat at infinity (which follows from Equation (239) in Appendix A), we have $\widetilde{g} = \widetilde{\mathcal{U}}(x)^4 \bar{g}_{\text{flat}} = \phi(x)^4 \overline{\mathcal{U}}(x)^4 \bar{g}_{\text{flat}}$ where

$$\widetilde{\mathcal{U}}(x) = 1 + \frac{\widetilde{m}}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{83}$$

Then comparing Equations (80), (82), and (83) with $\widetilde{\mathcal{U}}(x) = \overline{\mathcal{U}}(x)\phi(x)$ yields

$$\widetilde{m} = \overline{m} - \mathcal{E}(\bar{g}) \geq 0 \tag{84}$$

by the Riemannian Positive Mass Theorem, which proves inequality (81) for harmonically flat manifolds. Then since asymptotically flat manifolds can be arbitrarily well approximated by harmonically flat manifolds by Lemma 1, inequality (81) follows for asymptotically flat manifolds as well.

To prove the case of equality, we require a generalization of the case of equality of the Positive Mass Theorem given in [8] as Theorem 5.3. In that paper, we say that a singular manifold has *generalized nonnegative scalar curvature* if it is the limit (in the sense given in [8]) of smooth manifolds with nonnegative scalar curvature.

Note that if $(M^3, \bar{g})$ is only asymptotically flat and not harmonically flat, then we have not shown that $(M^3, \widetilde{g})$ can be extended smoothly over the missing points $\{\infty_k\}$. However, as a possibly singular manifold, it does have generalized nonnegative scalar curvature since it is the limit of smooth manifolds with nonnegative scalar curvature (since $(M^3, \bar{g})$ can be arbitrarily well approximated by harmonically flat manifolds).

Then Theorem 5.3 in [8] states that if $(M^3, \widetilde{g})$ has generalized nonnegative scalar curvature, zero mass, and positive isoperimetric constant, then $(M^3, \widetilde{g})$ is flat (outside the singular set). Hence, it follows that if we have equality in inequality (81), then $\widetilde{g}$ is flat. Hence, $(M^3, \widetilde{g})$ is isometric to $(\mathbf{R}^3, \delta)$ minus a finite number of points, and since the harmonic conformal factor $\phi(x)$ preserves the sign of the scalar curvature by Equation (240), the case of equality of the theorem follows. q.e.d.

Definition 17. Given a complete, asymptotically flat manifold (M^3, g) with horizon $\Sigma \in \mathcal{S}$ (defined in Section 2), define

$$\mathcal{E}(\Sigma, g) = \inf_{\varphi} \left\{ \frac{1}{2\pi} \int_{M^3} |\nabla \varphi|^2 \, dV \right\} \tag{85}$$

where the infimum is taken over all smooth $\varphi(x)$ which go to one at infinity and equal zero on the horizon Σ (and are zero inside Σ). (By definition of $\mathcal{S}$, all of the ends other than the chosen end are contained inside Σ.)

The infimum in the above definition is achieved by the Green's function $\varphi(x)$ which satisfies

$$\begin{cases} \lim_{x \to \infty_0} \varphi(x) &= 1 \\ \Delta \varphi &= 0 \\ \varphi(x) &= 0 \text{ on } \Sigma \end{cases} \tag{86}$$

and as before,

$$\varphi(x) = 1 - \frac{\mathcal{E}(\Sigma, g)}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{87}$$

in the chosen end.

Theorem 9. *Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature with a horizon $\Sigma \in \mathcal{S}$ and total mass m (in the chosen end). Then*

$$m \geq \frac{1}{2} \mathcal{E}(\Sigma, g) \tag{88}$$

with equality if and only if (M^3, g) is a Schwarzschild manifold outside the horizon Σ.

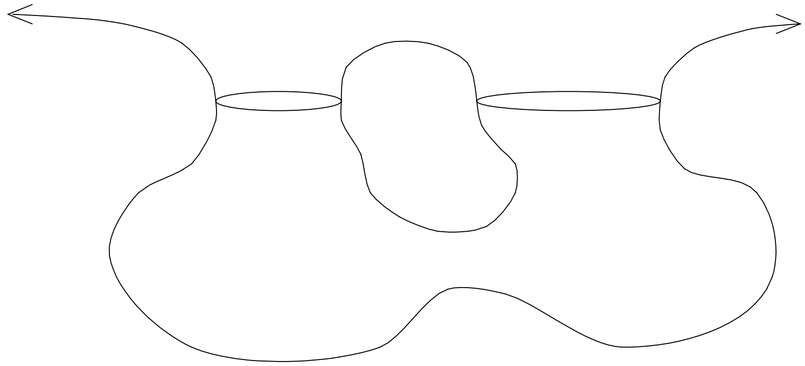


Figure 5.

Proof. Let M^3_Σ be the closed region of M^3 which is outside (or on) Σ. Since $\Sigma \in \mathcal{S}$, (M^3_Σ, g) has only one end, and we recall that Σ could have multiple components. For example, in Figure 5, Σ has two components.

Then the basic idea is to reflect (M^3_Σ, g) through Σ to get a manifold $(\overline{M}^3_\Sigma, \bar{g})$ with two asymptotically flat ends. Then define $\phi(x)$ on $(\overline{M}^3_\Sigma, \bar{g})$ using Equation (76) and $\varphi(x)$ on (M^3_Σ, g) using Equation (86). It follows from symmetry that $\phi(x) = \frac{1}{2}$ on Σ, so that

$$\phi(x) = \frac{1}{2}(\varphi(x) + 1) \tag{89}$$

on (M^3_Σ, g). Then

$$\mathcal{E}(\bar{g}) = \frac{1}{2}\mathcal{E}(\Sigma, g) \tag{90}$$

so that Theorem 9 follows from Theorem 8.

The only technicality is that Theorem 8 applies to smooth manifolds with nonnegative scalar curvature, and $(\overline{M}^3_\Sigma, \bar{g})$ is typically not smooth along Σ, which also makes it unclear how to define the scalar curvature there. However, it happens that because Σ has zero mean curvature, these issues can be resolved.

This idea of reflecting a manifold through its horizon is used by Bunting and Masood-ul-Alam in [11], and the issue of the smoothness of the reflected manifold appears in their paper as well. However, in their setting they have the simpler case in which the horizon not only has zero mean curvature but also has zero second fundamental form. Hence, the reflected manifold is $C^{1,1}$, which apparently is sufficient for their purposes.

However, in our setting we can not assume that the horizon Σ has zero second fundamental form, so that $(\overline{M}^3_\Sigma, \bar{g})$ is only Lipschitz. To solve this problem, given $\delta > 0$ we will define a smooth manifold $(\widetilde{M}^3_{\Sigma,\delta}, \widetilde{g}_\delta)$ with nonnegative scalar curvature which, in the limit as δ approaches zero, approaches $(\overline{M}^3_\Sigma, \bar{g})$ (meaning that there exists a diffeomorphism under which the metrics are arbitrarily uniformly close to each other and the total masses are arbitrarily close). Then by Definition 16 it follows that $\mathcal{E}(\widetilde{g}_\delta)$ is close to $\mathcal{E}(\bar{g})$, from which we will be able to conclude

$$m \approx \widetilde{m}_\delta \geq \mathcal{E}(\widetilde{g}_\delta) \approx \mathcal{E}(\bar{g}) = \frac{1}{2}\mathcal{E}(\Sigma, g), \tag{91}$$

where $\widetilde{m}_\delta$ is the mass of $(\widetilde{M}^3_{\Sigma,\delta}, \widetilde{g}_\delta)$ and the approximations in the above inequality can be made to be arbitrarily accurate by choosing δ small, thereby proving inequality (88).

The first step is to construct the smooth manifolds

$$(\widetilde{M}^3_{\Sigma,\delta}, \bar{g}_\delta) \approx (M^3_\Sigma, g) \cup (\Sigma \times [0, 2\delta], G) \cup (M^3_\Sigma, g), \tag{92}$$

where identifications are made along the boundaries of these three manifolds as drawn below. (To be precise, the second (M^3_Σ, g) in the above union is meant to be a copy of the first (M^3_Σ, g) and therefore distinct.) We will define the metric G such that the metric $\bar{g}_\delta$ is smooth, although it will not have nonnegative scalar curvature. Then we will define

$$\widetilde{g}_\delta = u_\delta(x)^4 \bar{g}_\delta \tag{93}$$

so that $\widetilde{g}_\delta$ is not only smooth but also has nonnegative scalar curvature, and we will show that because of our choice of the metric G, $u_\delta(x)$ approaches one in the limit as δ approaches zero.

We will use the local coordinates (z, t) to describe points on $\Sigma \times [0, 2\delta]$, where $z = (z_1, z_2) \in$ a local coordinate chart for Σ and $t \in [0, 2\delta]$. Then we define $G(\partial_t, \partial_t) = 1$, $G(\partial_t, \partial_{z_1}) = 0$, and $G(\partial_t, \partial_{z_2}) = 0$. Then it follows that $\Sigma \times t$ is obtained by flowing $\Sigma \times 0$ in the unit normal direction for a time t, and that ∂_t is orthogonal to $\Sigma \times t$. Hence, all that remains to fully define the metric G is to define it smoothly on the tangent planes of $\Sigma \times t$ for $0 \leq t \leq 2\delta$.

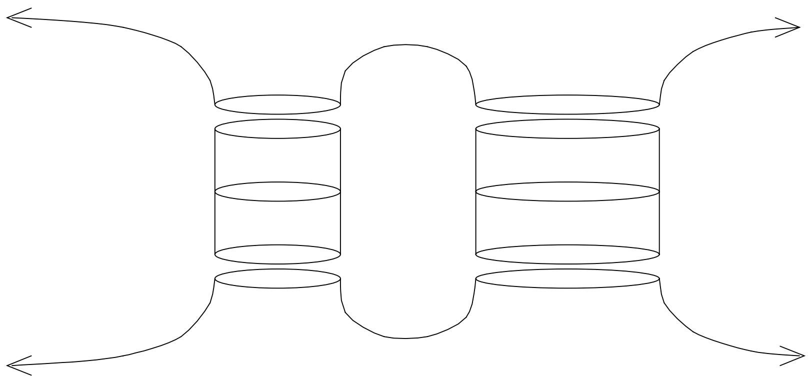


Figure 6.

Let $\overline{G}(z,t)$ be the metric G restricted to $\Sigma \times t$. Then

$$\frac{d}{dt}\overline{G}_{ij}(z,t) = 2\overline{G}_{ik}(z,t)h^k_j(z,t) \tag{94}$$

where $h^k_j(z,t)$ is the second fundamental form of $\Sigma \times t$ in $(\Sigma \times [0,2\delta], G)$ with respect to the normal vector ∂_t. Furthermore, since $\Sigma \times 0$ is identified with $\Sigma \in (M^3_\Sigma, g)$, we can extend the coordinates (z,t) for t slightly less than zero into (M^3_Σ, g), thereby giving us smooth initial data for $\overline{G}_{ij}(z,t)$ and $h^k_j(z,t)$ for $-\epsilon < t \leq 0$, for some positive ϵ.

Now we extend $h^k_j(z,t)$ smoothly for $0 \leq t \leq 2\delta$ in such a way that $h^k_j(z,t)$ is an odd function about $t = \delta$, meaning that $h^k_j(z,t) = -h^k_j(z, 2\delta - t)$. Naturally there are many ways to accomplish this smooth extension.

Then we define $\overline{G}_{ij}(z,t)$ to be the smooth solution to the o.d.e. given in Equation (94) using the initial data for $\overline{G}_{ij}(z,t)$ at $t = 0$. By the oddness of $h^k_j(z,t)$ about $t = \delta$ it follows that $\overline{G}_{ij}(z,t)$ is symmetric about $t = \delta$, that is, $\overline{G}_{ij}(z,t) = \overline{G}_{ij}(z, 2\delta - t)$. Hence, the identification of $\Sigma \times (2\delta)$ with $\Sigma \in$ the second copy of (M^3_Σ, g) is smooth by symmetry. This completes the smooth construction of the metric $(\widetilde{M}^3_{\Sigma,\delta}, \bar{g}_\delta)$.

Now define $H(z,t) = \Sigma_j h^j_j(z,t)$ to be the mean curvature of $\Sigma \times t$ in $(\Sigma \times [0,2\delta], G)$, and let $\dot{H}(z,t) = \frac{d}{dt}H(z,t)$. We note that

$$H(z,0) = 0 = H(z,2\delta) \tag{95}$$

since Σ is a horizon (and hence has zero mean curvature) in (M^3_Σ, g). Let $\alpha = \sup_z \dot{H}(z,0)$ and let $\beta = \sup_z \sum_{jk} h^k_j(z,0) h^j_k(z,0)$, which we note are functions of the metric g on M^3_Σ and are independent of δ. Then we require that the smooth extension we choose for h^k_j satisfies

$$\dot{H}(z,t) \le 2|\alpha| + 1, \tag{96}$$

(which is possible because of Equation (95)) and

$$\sum_{jk} h^k_j(z,t) h^j_k(z,t) \le 2\beta + 1. \tag{97}$$

Then combining Equations (95) and (96) also yields

$$|H(z,t)| \le (2|\alpha| + 1)\delta. \tag{98}$$

These estimates allow us to bound the scalar curvature of $(\widetilde{M}^3_{\Sigma,\delta}, \bar{g}_\delta)$ from below since by the second variation formula and the Gauss equation (see Equations (117) and (118)) we have that

$$R = -2\dot{H} + 2K - |h|^2 - H^2 \tag{99}$$

where $R(z,t)$ is scalar curvature and $K(z,t)$ is the Gauss curvature of $\Sigma \times t$. At this point we realize that we also need a lower bound K_0 for $K(z,t)$ which is independent of δ, which follows from imposing an upper bound on the C^2 norm (in the z variable) of our smooth choice of $h^k_j(z,t)$. Then using this combined with inequalities (96), (97), and (98) we get

$$R(z,t) \ge R_0 \tag{100}$$

where R_0 is independent of δ (for $\delta < 1$).

Now we are ready to define $\widetilde{g}_\delta$ using Equation (93). We already know that $(\widetilde{M}^3_{\Sigma,\delta}, \bar{g}_\delta)$ is smooth and has nonnegative scalar curvature everywhere except possibly in $\Sigma \times [0, 2\delta]$ where it has $R \ge R_0$. If $R_0 \ge 0$, then we just let $u_\delta(x) = 1$ so that $\widetilde{g} = \bar{g}$. Otherwise, we define $u_\delta(x)$ such that

$$(-8\Delta_{\overline{g}} + \mathcal{R}_\delta(x)) u_\delta(x) = 0 \tag{101}$$

and $u_\delta(x)$ goes to one in both asymptotically flat ends, where $\mathcal{R}_\delta(x)$ equals R_0 in $\Sigma\times[0, 2\delta]$, equals zero for x more than a distance δ from $\Sigma\times[0, 2\delta]$, is smooth, and takes values in $[R_0, 0]$ everywhere. Then it follows that for sufficiently small δ, $u_\delta(x)$ is a smooth superharmonic function.

Furthermore, since $\mathcal{R}_\delta$ is zero everywhere except on an open set whose volume is going to zero as δ goes to zero, and since $\mathcal{R}_\delta$ is uniformly bounded from below on this small set, it follows from bounding Green's functions from above that

$$1 \le u_\delta(x) \le 1 + \epsilon(\delta) \tag{102}$$

where ϵ goes to zero as δ approaches zero.

Furthermore, by Equations (100), (93), and (240), $(\widetilde{M}^3_{\Sigma,\delta}, \widetilde{g}_\delta)$ has nonnegative scalar curvature, and since $u_\delta(x)$ and and $(\widetilde{M}^3_{\Sigma,\delta}, \bar{g}_\delta)$ are smooth, so is $(\widetilde{M}^3_{\Sigma,\delta}, \widetilde{g}_\delta)$. In addition, it follows from the construction of $(\widetilde{M}^3_{\Sigma,\delta}, \bar{g}_\delta)$ that there exists a diffeomorphism into $(\overline{M}^3, \bar{g})$ with respect to which the metrics are arbitrarily uniformly close to each other in the limit as δ goes to zero. Hence, by Equation (102), we see that the same statement is true for $(\widetilde{M}^3_{\Sigma,\delta}, \widetilde{g}_\delta)$. Finally, it follows from Equation (101) that $\widetilde{m}_\delta$, the mass of $(\widetilde{M}^3_{\Sigma,\delta}, \widetilde{g}_\delta)$, converges to m, the mass of $(\overline{M}^3_\Sigma, \bar{g})$, in the limit as δ goes to zero. Hence, inequality (91) follows, proving inequality (88).

To prove the case of equality, we note that we can view the above proof in a different way. Since the singular manifold $(\overline{M}^3_\Sigma, \bar{g})$ is the limit of the smooth manifolds $(\widetilde{M}^3_{\Sigma,\delta}, \widetilde{g}_\delta)$ which have nonnegative scalar curvature, it follows that $(\overline{M}^3_\Sigma, \bar{g})$ has generalized nonnegative scalar curvature as defined in [8]. Then if we reexamine the proof of Theorem 8, we see that the theorem, including the case of equality, is also true for singular manifolds like $(\overline{M}^3_\Sigma, \bar{g})$ which have generalized nonnegative scalar curvature (see the discussion at the end of the proof of Theorem 8). Hence, by Equation (90) and Theorem 8, we get equality in inequality (88) if and only if $(\overline{M}^3_\Sigma, \bar{g})$ has zero scalar curvature and is conformal to $(\mathbf{R}^3, \delta)$ minus a finite number of points.

Since $(\overline{M}^3_\Sigma, \bar{g})$ has two ends, it must be conformal to $(\mathbf{R}^3 \backslash \{0\}, \delta)$, and since it has zero scalar curvature, it follows from Equation (240) that it must be a Schwarzschild metric. Hence, in the case of equality for inequality (88), (M^3, g) must be a Schwarzschild manifold outside Σ. q.e.d.

7. Proof that $m(t)$ is nonincreasing

In this section we will finish the proof of Theorem 3 begun in Section 5 by proving that $m(t)$, the total mass of (M^3, g_t), is non-increasing in t. The fact that $m(t)$ is nonincreasing is of course central to the argument presented in this paper for proving the Riemannian Penrose Conjecture and is perhaps the most important property of the conformal flow of metrics $\{g_t\}$.

We begin with a corollary to Lemma 8 and Theorem 7 in Section 4.

Corollary 4. *The left and right hand derivatives $\frac{d}{dt^\pm}$ of $u_t(x)$ exist for all $t > 0$ and are equal except at a countable number of t-values. Furthermore,*

$$\frac{d}{dt^+}\, u_t(x) = v_t^+(x) \tag{103}$$

and

$$\frac{d}{dt^-}\, u_t(x) = v_t^-(x) \tag{104}$$

where $v_t^\pm(x)$ equals zero inside $\Sigma^\pm(t)$ (see Definition 11) and outside $\Sigma^\pm(t)$ is the harmonic function which equals 0 on $\Sigma^\pm(t)$ and goes to $-e^{-t}$ at infinity.

We will use this corollary to compute the left and right hand derivatives of $m(t)$. As proven at the end of Appendix A, the flow of metrics $\{g_t\}$ we are considering has the property that the rate of change of the metric g_t is just a function of g_t and not of t or g_0. Hence, we will just prove that $m'(0) \le 0$, from which it will follow that $m'(t) \le 0$. So without loss of generality, we will assume that the flow begins at some time $-t_0 < 0$, and then compute the left and right hand derivatives of $m(t)$ at $t = 0$.

Also, we remind the reader that we proved that $\Sigma^+(t)$ and $\Sigma^-(t)$ are horizons in (M^3, g_t) in Lemma 9. Furthermore, $v_0^\pm(x)$ is harmonic in (M^3, g_0), equals 0 on $\Sigma^\pm(0)$, and goes to -1 at infinity. Hence, by Equation (87) and Theorem 9 of the previous section,

$$v_0^\pm(x) = -1 + \frac{\mathcal{E}(\Sigma^\pm(0), g_0)}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{105}$$

where

$$m(0) \ge \frac{1}{2}\mathcal{E}(\Sigma^\pm(0), g_0). \tag{106}$$

Now we are ready to compute $m'(t)$. As in Section 2, let $g_0 = \mathcal{U}_0(x)^4 g_{\text{flat}}$, where (M^3, g_{flat}) is isometric to $(\mathbf{R}^3 \backslash B_{r_0}(0), \delta)$ in the harmonically flat end, where we have chosen r_0 and scaled the harmonically flat coordinate chart such that $\mathcal{U}_0(x)$ goes to one at infinity. Then by Definition 2 for the total mass, we have that

$$\mathcal{U}_0(x) = 1 + \frac{m(0)}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{107}$$

We will also let $g_t = \mathcal{U}_t(x)^4 g_{\text{flat}}$ in the harmonically flat end. Then since $g_t = u_t(x)^4 g_0$, it follows that

$$\mathcal{U}_t(x) = u_t(x)\,\mathcal{U}_0(x). \tag{108}$$

Now we define $\alpha(t)$ and $\beta(t)$ such that

$$u_t(x) = \alpha(t) + \frac{\beta(t)}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{109}$$

Then since $u_0(x) \equiv 1$ and $\frac{d}{dt^\pm}\, u_t(x)|_{t=0} = v_0^\pm(x)$, it follows from Equation (105) that

$$\begin{aligned} \alpha(0) &= 1, & \tfrac{d}{dt^\pm}\,\alpha(t)|_{t=0} &= -1, \\ \beta(0) &= 0, & \tfrac{d}{dt^\pm}\,\beta(t)|_{t=0} &= \tfrac{1}{2}\mathcal{E}(\Sigma^\pm(0), g_0). \end{aligned} \tag{110}$$

Thus, by Equation (108),

$$\mathcal{U}_t(x) = \alpha(t) + \frac{1}{|x|}\left(\beta(t) + \frac{m(0)}{2}\alpha(t)\right) + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{111}$$

so that by Definition 2

$$m(t) = 2\alpha(t)\left(\beta(t) + \frac{m(0)}{2}\alpha(t)\right). \tag{112}$$

Hence, by Equation (110)

$$\frac{d}{dt^\pm}\, m(t)|_{t=0} \;=\; \mathcal{E}(\Sigma^\pm(0), g_0) - 2m(0) \;\leq\; 0 \tag{113}$$

by Equation (106). Then since we were able to choose $t = 0$ without loss of generality as previously discussed, we have proven the following theorem.

Theorem 10. *The left and right hand derivatives* $\frac{d}{dt^{\pm}}$ *of* $m(t)$ *exist for all* $t > 0$ *and are equal except at a countable number of* t*-values. Furthermore,*

$$\frac{d}{dt^{\pm}}\, m(t) \le 0 \tag{114}$$

for all $t > 0$ (*and the right hand derivative of* $m(t)$ *at* $t = 0$ *exists and is nonpositive as well*).

Hence, $m'(t)$ exists almost everywhere, and since the left and right hand derivatives of $m(t)$ are all nonpositive, $m(t)$ is nonincreasing. Since we proved that $A(t)$ is constant in Section 5, this completes the proof of Theorem 3.

8. The stability of $\Sigma(t)$

A very interesting and important property of the horizons $\Sigma(t)$ follows from the fact that they are locally stable with respect to outward variations. Since $\Sigma(t)$ is strictly outer minimizing in (M^3, g_t), we know that each component of $\Sigma(t)$ must be stable under outward variations (holding t fixed). In particular, choose any component $\Sigma_i(t)$ of $\Sigma(t)$, and flow it outwards in the unit normal direction at constant speed one. Since $\Sigma(t)$ is smooth, we can do this for some positive amount of time without the surface forming singularities. Let $A(s)$ be the area of the surface after being flowed at speed one for time s. Then since $\Sigma_i(t)$ has zero mean curvature, $A'(0) = 0$. And since $\Sigma(t)$ is strictly outer minimizing, which we recall means that all surfaces in (M^3, g_t) which enclose $\Sigma(t)$ have strictly larger area, we must have $A''(0) \ge 0$.

On the other hand,

$$A'(s) = \int H d\mu \tag{115}$$

where the integral is being taken over the surface resulting from flowing $\Sigma_i(t)$ for time s, H is the mean curvature of the surface, and $d\mu$ is the area form of the surface. Then since $H \equiv 0$ at $s = 0$,

$$A''(0) = \int_{\Sigma_i(t)} \frac{d}{ds}(H) d\mu. \tag{116}$$

We will then use the second variation formula

$$\frac{d}{dt} H = -|h|^2 - \mathrm{Ric}(\vec{\nu}, \vec{\nu}), \tag{117}$$

the Gauss equation,

$$\text{Ric}(\vec{\nu},\vec{\nu}) = \frac{1}{2}R - K + \frac{1}{2}H^2 - \frac{1}{2}|h|^2, \tag{118}$$

and the fact that $|h|^2 = \frac{1}{2}(\lambda_1 - \lambda_2)^2 + \frac{1}{2}H^2$, where h is the second fundamental form of $\Sigma_i(t)$ (so that $H = \text{trace}(h)$), Ric is the Ricci curvature tensor of (M^3, g_t), $\vec{\nu}$ is the outward pointing normal vector to $\Sigma_i(t)$, R is the scalar curvature of (M^3, g_t), K is the Gauss curvature of $\Sigma_i(t)$, and λ_1 and λ_2 are the principal curvatures of $\Sigma_i(t)$, to get

$$A''(0) = \int_{\Sigma_i(t)} -\frac{1}{2}R + K - \frac{1}{4}(\lambda_1 - \lambda_2)^2 \tag{119}$$

since $H = 0$. Hence, since $A''(0) \geq 0$, $R \geq 0$, and $\int_{\Sigma_i(t)} K \leq 4\pi$ by the Gauss-Bonnet formula (actually we have equality in the last inequality since $\Sigma_i(t)$ is a sphere since $\Sigma(t)$ is strictly outer minimizing in (M^3, g_t) - see below), we conclude that

$$\int_{\Sigma_i(t)} (\lambda_1 - \lambda_2)^2 d\mu \leq 16\pi \tag{120}$$

with respect to the metric g_t. However, it happens that the left hand side of inequality (120) is conformally invariant, and g_t is conformal to g_{flat} (defined in Section 2). Thus, inequality (120) is also true with respect to the metric g_{flat}.

Theorem 11. *Each component $\Sigma_i(t)$ of the surface $\Sigma(t)$ (which is a strictly outer minimizing horizon in (M^3, g_t)) is a sphere and satisfies*

$$\int_{\Sigma_i(t)} (\lambda_1 - \lambda_2)^2 d\mu \leq 16\pi \tag{121}$$

with respect to the fixed metric g_{flat}.

We also comment that Equation (119) can be used to prove that each component of a strictly outer minimizing surface Σ in a manifold (M^3, g) with nonnegative scalar curvature is a sphere. First, we choose a superharmonic function $u(x)$ defined outside Σ in (M^3, g) approximately equal to one which has negative outward derivative on Σ. Then the metric $(M^3, u(x)^4 g)$ has *strictly* positive scalar curvature by Equation (240). Also, since Σ now has negative mean curvature in $(M^3, u(x)^4 g)$ by the Neumann condition on $u(x)$, Σ acts a barrier so that the outermost minimal area enclosure of Σ does not touch Σ and hence is a stable horizon with zero mean curvature. Then by Equation (119) and the Gauss-Bonnet formula, each component of this outermost minimal area enclosure of Σ must be a sphere. In the limit as $u(x)$ approaches one, the area of this outermost minimal area enclosure in (M^3, g) must approach the area of Σ. Then since Σ is *strictly* outer minimizing in (M^3, g), these outermost minimal area enclosures must be approaching Σ in the limit as $u(x)$ goes to one, proving that each component of Σ is a sphere.

Unfortunately, this argument doesn't quite work for non-strictly outer minimizing horizons, but Equation (119) does prove that each component of the horizon is either a sphere or a torus. The torus possibility is then ruled out by [12].

We will need Theorem 11 in Sections 10 and 11.

9. The exponential growth rate of Diam$(\Sigma(t))$

In this section we will show that the diameter of $\Sigma(t)$ is growing approximately exponentially with respect to the fixed metric g_0. This fact will be used in Section 10 to prove that $\Sigma(t)$ eventually encloses any bounded set. It will also be used in Section 11 to help bound the behavior of $\Sigma(t)$ for large t.

We recall from Section 2 that we may write $g_0 = \mathcal{U}_0(x)^4 g_{\text{flat}}$, where the harmonically flat end of (M^3, g_{flat}) is isometric to $(\mathbf{R}^3 \backslash B_{r_0}(0), \delta)$, where now we have chosen r_0 such that we can require $\mathcal{U}_0(x)$ to go to one at infinity. Let $S(r)$ denote the sphere of radius r in $\mathbf{R}^3$ centered at zero. Then for $r \geq r_0$, we can define $S(r)$ to be a sphere in (M^3, g_{flat}).

Theorem 12. *Given any $\widetilde{t} \geq 0$, there exists a $t \geq \widetilde{t}$ such that $\Sigma(t)$ is not entirely enclosed by $S(r(t))$ (and $r(t) \geq r_0$), where*

$$r(t) = \left(\frac{A_0}{65\pi}\right)^{1/2} e^{2t}, \tag{122}$$

and $A_0 = A(0)$.

Proof. The proof is a proof by contradiction. We will assume that for some $\widetilde{t} \geq 0$ (with $r(\widetilde{t}) \geq r_0$) that $\Sigma(t)$ is entirely enclosed by $S(r(t))$ for all $t \geq \widetilde{t}$. Then we will use this to prove that $A(t)$ is not constant, contradicting Theorem 3.

In fact, we will show that $|S(r(t))|_{g_t}$ (which is the area of $S(r(t))$ with respect to the metric g_t) is less than $A_0 \equiv A(0)$ for some $t \geq \widetilde{t}$.

Then since $A(t)$ is defined to be the area of $\Sigma(t)$ which is defined to be the outermost minimal area enclosure of Σ_0, this will force $A(t) < A_0$, a contradiction.

As in Section 7, we define $g_t = u_t(x)^4 g_0$ and $g_t = \mathcal{U}_t(x)^4 g_{\text{flat}}$ so that $\mathcal{U}_t(x) = u_t(x)\mathcal{U}_0(x)$. In addition, we define $\mathcal{V}_t(x) = v_t(x)\mathcal{U}_0(x)$ so that

$$\frac{d}{dt}\mathcal{U}_t(x) = \mathcal{V}_t(x) \tag{123}$$

since $\frac{d}{dt}u_t(x) = v_t(x)$. Next we define $\mathcal{Q}_t(x)$ outside $S(r(t))$ in (M^3, g_{flat}) such that

$$\begin{cases} Q_t = & 0 & \text{on } S(r(t)) \\ \Delta_{g_{\text{flat}}} Q_t \equiv & 0 & \text{outside } S(r(t)) \\ Q_t \to & -e^{-t} & \text{at infinity.} \end{cases} \tag{124}$$

so that

$$\mathcal{Q}_t(x) = -e^{-t}\left(1 - \frac{r(t)}{|x|}\right) \tag{125}$$

outside $S(r(t))$.

Then since we are assuming that $\Sigma(t)$ is entirely enclosed by $S(r(t))$, it follows from Equations (15) and (239) that

$$\begin{cases} \mathcal{V}_t \leq & 0 & \text{on } S(r(t)) \\ \Delta_{g_{\text{flat}}} \mathcal{V}_t \equiv & 0 & \text{outside } S(r(t)) \\ \mathcal{V}_t \to & -e^{-t} & \text{at infinity.} \end{cases} \tag{126}$$

Hence, by the maximum principle,

$$\mathcal{V}_t(x) \leq \mathcal{Q}_t(x) \tag{127}$$

for all x outside $S(r(t))$. Consequently,

$$\begin{aligned} \mathcal{U}_t(x) &= \mathcal{U}_{\tilde{t}}(x) + \int_{\tilde{t}}^{t} \mathcal{V}_s(x)\, ds \\ &\leq \mathcal{U}_{\tilde{t}}(x) + \int_{\tilde{t}}^{t} \mathcal{Q}_s(x)\, ds \end{aligned} \tag{128}$$

for x outside $S(r(t))$. Now choose k such that $\mathcal{U}_{\tilde{t}}(x) \leq e^{-\tilde{t}} + \frac{k}{|x|}$ outside $S(r(\tilde{t}))$. Then using this and Equation (125), we get

$$\mathcal{U}_t(x) \leq e^{-t} + \frac{1}{|x|}\left[k + \sqrt{\frac{A_0}{65\pi}}\left(e^t - e^{\tilde{t}}\right)\right] \tag{129}$$

for x outside $S(r(t))$.

Now we define $\mathcal{A}(t)$ to be the area of $S(r(t))$ in (M^3, g_t). Then

$$\begin{aligned}\mathcal{A}(t) &= \int_{S(r(t))} \mathcal{U}_t(x)^4 \, dA_{g_{\text{flat}}} \\ &\leq 4\pi r(t)^2 \left(\sup_{S(r(t))} \mathcal{U}_t(x) \right)^4 \\ &\leq \frac{4}{65} A_0 \left[2 + e^{-t} \left(k\sqrt{\frac{65\pi}{A_0}} - e^{\tilde{t}} \right) \right]^4\end{aligned} \tag{130}$$

by inequality (129), where $dA_{g_{\text{flat}}}$ is the area form of $S(r(t))$ in (M^3, g_{flat}). Hence,

$$\lim_{t\to\infty} \mathcal{A}(t) = \frac{64}{65} A_0, \tag{131}$$

which means that $\mathcal{A}(t) < A_0$ for some $t \geq \tilde{t}$. But by Theorem 3, $|\Sigma(t)|_{g_t} = A_0$ for all $t \geq 0$, and since $\Sigma(t)$ was defined to be a minimal area enclosure of the original horizon Σ_0 in (M^3, g_t), we have a contradiction. q.e.d.

10. Proof that $\Sigma(t)$ eventually encloses any bounded set

In this section we will prove that $\Sigma(t)$ eventually encloses any bounded set in a finite amount of time. In particular, this will allow us to conclude that $\Sigma(t)$ flows into the harmonically flat end of (M^3, g_0) in a finite amount of time, which greatly simplifies the discussion in the next two sections.

Theorem 13. *Given any bounded set $B \subset M^3$, there exists a $t \geq 0$ such that $\Sigma(t)$ encloses B.*

Proof. The proof of this theorem is a proof by contradiction. We will show that if the theorem were false, then we would be able to find a $\bar{t}$ such that $A(\bar{t})$, the area of $\Sigma(\bar{t})$ in $(M^3, g_{\bar{t}})$, was greater than A_0, contradicting Theorem 3.

So suppose there exists a bounded set $B \subset M^3$ such that $\Sigma(t)$ does not (entirely) enclose B for any $t \geq 0$. Then since B is bounded, there exists an $R_1 > r_0$ such that the coordinate sphere $S(R_1)$ (defined in the previous section) encloses B. Hence, $\Sigma(t)$ does not enclose $S(R_1)$ for any t.

Now choose any $R_2 > R_1$. By Theorem 12, there exists a $\bar{t}$ such that $\Sigma(\bar{t})$ is not enclosed by $S(R_2)$. Then it follows that at least one component $\Sigma_i(\bar{t})$ of $\Sigma(\bar{t})$ must satisfy

$$\Sigma_i(\bar{t}) \cap S(R) \neq \varnothing \tag{135}$$

for all $R_1 \leq R \leq R_2$. This follows from the fact that each component of the region inside $\Sigma(\bar{t})$ must intersect the region inside $S(r_0)$ since otherwise the area of $\Sigma(\bar{t})$ could be decreased by simply eliminating that component.

We will show that it is possible to choose R_2 large enough to make the area of $\Sigma_i(\bar{t})$ in $(M^3, g_{\bar{t}})$ as large as we want, which is a contradiction. Since

$$|\Sigma_i(\bar{t})|_{g_{\bar{t}}} = \int_{\Sigma_i(\bar{t})} \mathcal{U}_{\bar{t}}(x)^4 \, dA_{g_{\text{flat}}} \tag{136}$$

where $dA_{g_{\text{flat}}}$ is the area form of $\Sigma_i(\bar{t})$ in (M^3, g_{flat}), we can show that $|\Sigma_i(\bar{t})|_{g_{\bar{t}}}$ is large if we have adequate lower bounds on $\mathcal{U}_{\bar{t}}(x)$ and on the area of $|\Sigma_i(\bar{t})|$ in (M^3, g_{flat}).

To find a lower bound on $\mathcal{U}_{\bar{t}}(x)$ we will use Theorem 20 in Appendix C. Let

$$\mathcal{U}_\infty(x) = \lim_{t\to\infty} \mathcal{U}_t(x) \tag{137}$$

Then since $u_t(x)$ is decreasing in t by Equation (16), so is $\mathcal{U}_t(x)$. Hence,

$$\mathcal{U}_t(x) \geq \mathcal{U}_\infty(x). \tag{138}$$

for all $t \geq 0$. However, to use Theorem 20, we need a superharmonic function defined on $\mathbf{R}^3$, whereas $\mathcal{U}_\infty(x)$ is defined on (M^3, g_{flat}) which is only isometric to $(\mathbf{R}^3 \backslash B_{r_0}(0), \delta)$ in the harmonically flat end. Let $c > 0$ be the minimum value of $\mathcal{U}_\infty(x)$ on $S(r_0)$. and let ϕ be the isometry mapping $(\mathbf{R}^3 \backslash B_{r_0}(0), \delta)$ to the harmonically flat end of (M^3, g_{flat}). Then we can define $\mathcal{U}(x)$ on $\mathbf{R}^3$ as

$$\mathcal{U}(x) = \begin{cases} \min(c/2,\, \mathcal{U}_\infty(\phi(x))), & \text{for } |x| \geq r_0 \\ c/2, & \text{for } |x| < r_0. \end{cases} \tag{139}$$

Since the minimum value of two superharmonic functions is superharmonic, it follows that $\mathcal{U}(x)$ is superharmonic on $(\mathbf{R}^3, \delta)$. Furthermore, since $\mathcal{U}_\infty(x)$ goes to zero at infinity, there exists an $\widetilde{r} \geq r_0$ such that

$$\mathcal{U}(x) = \mathcal{U}_\infty(\phi(x)) \tag{140}$$

for $|x| > \widetilde{r}$. (Without loss of generality, we will assume that we chose R_1 from before so that $R_1 > \widetilde{r}$.)

Next, we observe that

$$|S(R)|_{g_t} = \int_{S(R)} \mathcal{U}_t(x)^4 \, dA_{g_{\text{flat}}} \geq A_0 \tag{141}$$

by Theorem 3 since $\Sigma(t)$ has area A_0 and is a minimal area enclosure of Σ_0 in (M^3, g_t). Hence, taking the limit as t goes to infinity, it follows from Equation (140) that

$$\int_{S_R(0)} \mathcal{U}(x)^4 \, dA \geq A_0 \tag{142}$$

in $(\mathbf{R}^3, \delta)$ for $R > \widetilde{r}$, so that by Theorem 20 in Appendix C

$$\mathcal{U}(x) \geq c\, A_0^{1/4} |x|^{-1/2} \tag{143}$$

for $|x| \geq \widetilde{r}$ and for some $c > 0$. Thus, by Equations (140) and (138)

$$\mathcal{U}_t(x) \geq c\, A_0^{1/4} |x|^{-1/2} \tag{144}$$

for all $t \geq 0$ and for all x in M^3 outside $S(\widetilde{r})$. This lower bound on $\mathcal{U}_t(x)$ is the first of two steps needed to prove that integral in Equation (136) is large.

The second step of the proof is to find the right lower bound on the area of $\Sigma_i(\bar{t})$ in (M^3, g_{flat}). So far all we have is Equation (135) which tells us that the diameter of $\Sigma_i(\bar{t})$ in (M^3, g_{flat}) is at least $R_2 - R_1$. Naturally this is not enough to bound the area from below without some control on the possible geometries of $\Sigma_i(\bar{t})$ in (M^3, g_{flat}).

Fortunately, we do have very good control on the geometry of each component $\Sigma_i(\bar{t})$ of $\Sigma(\bar{t})$ by virtue of Theorem 11. Using the same notation as in the Gauss equation given in Equation (118), we can substitute

$$(\lambda_1 - \lambda_2)^2 = (2R - 4\text{Ric}(\nu, \nu)) - 4K + H^2 \tag{145}$$

into Equation (121) to get

$$\int_{\Sigma_i(\bar{t})} H^2 \, dA_{g_{\text{flat}}} \leq 32\pi + \int_{\Sigma_i(\bar{t})} (4\text{Ric}(\nu, \nu) - 2R) \, dA_{g_{\text{flat}}} \tag{146}$$

where we have used the Gauss-Bonnet theorem and the fact that every component of $\Sigma(\bar{t})$ is a sphere. In addition, $4\text{Ric}(\nu, \nu) - 2R$ is zero in the harmonically flat end of M^3 since g_{flat} is flat.

Now we want to show that the right hand side of Equation (146) is bounded. Let K be the compact set of points outside (or on) the original horizon Σ_0 and inside (or on) $S(r_0)$. Let

$$R_{\max} = \sup_K |4\mathrm{Ric}(\nu, \nu) - 2R| \tag{147}$$

which is finite since (M^3, g_{flat}) is smooth and K is compact. Then since

$$A_0 = |\Sigma(\bar{t})|_{g_{\bar{t}}} \geq |\Sigma_i(\bar{t}) \cap K|_{g_{\bar{t}}} = \int_{\Sigma_i(\bar{t})\cap K} \mathcal{U}_{\bar{t}}(x)^4 \, dA_{g_{\text{flat}}} \tag{148}$$

it follows that

$$|\Sigma_i(\bar{t}) \cap K|_{g_{\text{flat}}} \leq \frac{A_0}{\inf_K \mathcal{U}_{\bar{t}}(x)^4} \leq \frac{A_0}{\inf_K u(x)^4 \mathcal{U}_0(x)^4} \tag{149}$$

where we recall that $\mathcal{U}_{\bar{t}}(x) = u_{\bar{t}}(x)\mathcal{U}_0(x)$ and where $u(x)$ is defined to be the positive harmonic function which equals 1 (and hence also $u_{\bar{t}}(x)$) on the original horizon Σ_0 and goes to zero at infinity and hence is a barrier function for the superharmonic function $u_{\bar{t}}(x)$ in $(M^3, g_{\bar{t}})$. Thus, we have that

$$\int_{\Sigma_i(\bar{t})} H^2 \, dA_{g_{\text{flat}}} \leq 32\pi + \frac{R_{\max} A_0}{\inf_K u(x)^4 \mathcal{U}_0(x)^4} \equiv k. \tag{150}$$

We will need the Willmore functional bound in inequality (150) to use the identity (Equation 16.31 in [21])

$$|\Sigma \cap B_r(x)| \geq \pi r^2 \left(1 - \frac{1}{16\pi} \int_{\Sigma \cap B_r(x)} H^2 \, dA \right) \tag{151}$$

where Σ is any smooth, compact surface which is the boundary of a region in $\mathbf{R}^3$ and everything is with respect to the standard flat metric δ. Since (M^3, g_{flat}) is flat in the harmonically flat end region of M^3, we will be able to use this identity on $\Sigma_i(\bar{t})$.

Since our choice of R_2 could be arbitrarily large, let $R_2 = 3(2^{n^2} - 1)R_1$ where n is a positive integer which may be chosen to be arbitrarily large. Then by Equation (135) we can choose

$$x_k \in \Sigma_i(\bar{t}) \cap S(3(2^k - 1)R_1) \tag{152}$$

for $1 \leq k \leq n^2$. Then if we define $r_k = 2^k R_1$ it follows that the balls $B_{r_k}(x_k)$ are all disjoint. Hence,

$$\int_{\Sigma_i(\bar{t})\cap B_{r_k}(x_k)} H^2 \, dA_{g_{\text{flat}}} \leq \frac{k}{n} \tag{153}$$

except at most n different values of k. Then for these values of k, it follows from Equation (144) that

$$\begin{aligned}(154)\quad \int_{\Sigma_i(\bar{t})\cap B_{r_k}(x_k)} \mathcal{U}_{\bar{t}}(x)^4 \, dA_{g_{\text{flat}}} &\geq c^4 A_0 \int_{\Sigma_i(\bar{t})\cap B_{r_k}(x_k)} |x|^{-2} \, dA_{g_{\text{flat}}} \\ &\geq c^4 A_0 \, |\Sigma_i(\bar{t}) \cap B_{r_k}(x_k)| \, (2^{k+2}R_1)^{-2} \\ &\geq c^4 A_0 \, \pi r_k^2 \left(1 - \frac{k}{16\pi n}\right) (2^{k+2}R_1)^{-2} \\ &= c^4 A_0 \frac{\pi}{16} \left(1 - \frac{k}{16\pi n}\right)\end{aligned}$$

where $|x|$ is defined to be r on $S(r)$ in the harmonically flat end of (M^3, g_{flat}). Hence, we have that

$$\begin{aligned}(155)\quad A_0 = |\Sigma(\bar{t})|_{g_{\bar{t}}} &\geq |\Sigma_i(\bar{t})|_{g_{\bar{t}}} \\ &= \int_{\Sigma_i(\bar{t})} \mathcal{U}_{\bar{t}}(x)^4 \, dA_{g_{\text{flat}}} \\ &\geq \sum_{k=1}^{n^2} \int_{\Sigma_i(\bar{t})\cap B_{r_k}(x_k)} \mathcal{U}_{\bar{t}}(x)^4 \, dA_{g_{\text{flat}}} \\ &\geq c^4 A_0 \frac{\pi}{16} \left(1 - \frac{k}{16\pi n}\right)(n^2 - n)\end{aligned}$$

which is a contradiction since n can be chosen to be arbitrarily large. Hence, given any bounded set $B \subset M^3$, there must exist a $t \geq 0$ such that $\Sigma(t)$ encloses B. q.e.d.

We immediately deduce a very useful corollary. Since $\Sigma(t)$ always flows outwards and must eventually entirely enclose $S(r_0)$ by the above theorem, it follows that after a certain point in time $\Sigma(t)$ is entirely in the harmonically flat end of (M^3, g_{flat}). Furthermore, since $\Sigma(t)$ is defined to be the outermost surface with minimum area in (M^3, g_t) which encloses the original horizon, these $\Sigma(t)$ only have one component since having any additional components would only increase the area of $\Sigma(t)$. Then since it follows from a stability argument that each component is a sphere ([43] or see the end of Section 8), $\Sigma(t)$ is a single sphere. Thus, we get the following corollary to Theorem 13.

Corollary 5. *There exists a $t_0 \geq 0$ such that for all $t \geq t_0$, topologically $\Sigma(t)$ is a single sphere and is in the harmonically flat end of (M^3, g_{flat}) which is isometric to $(\mathbf{R}^3 \backslash B_{r_0}(0), \delta)$.*

We remind the reader that (M^3, g_{flat}) was defined in Section 2 and is conformal to the harmonically flat manifold (M^3, g_0).

11. Bounds on the behavior of $\Sigma(t)$

The main objective of this section is to achieve upper and lower bounds for the diameter of $\Sigma(t)$ in (M^3, g_0) for large t. However, in the process of deriving these bounds we also prove other interesting although not essential bounds on the behavior of $\Sigma(t)$.

As in the previous two sections, $g_0 = \mathcal{U}_0(x)^4 g_{\text{flat}}$, where the harmonically flat end of (M^3, g_{flat}) is isometric to $(\mathbf{R}^3 \backslash B_{r_0}(0), \delta)$, and $\mathcal{U}_0(x)$ goes to one at infinity. From Corollary 5 of the previous section, we see that $\Sigma(t)$ is in the flat region of (M^3, g_{flat}) for $t \geq t_0$ so that the behavior of $\Sigma(t)$ for large t reduces to a problem in $\mathbf{R}^3$.

Furthermore, from Section 9 we know that the diameter of $\Sigma(t)$ is going up roughly as some constant times e^{2t}. Then since $\mathbf{R}^3$ is linear, it makes since and is convenient to rescale distances by the factor e^{-2t} so that in this new rescaled $\mathbf{R}^3$ the diameter of $\Sigma(t)$ is approximately bounded. In fact, the goal of this section is to show that the diameter of $\Sigma(t)$ in this rescaled $\mathbf{R}^3$ is bounded above and below by constants. (From this point on $\Sigma(t)$ will refer to the rescaled $\Sigma(t)$.)

We will use capital letters to denote rescaled quantities. Recall that $g_t = \mathcal{U}_t(x)^4 g_{\text{flat}}$ and $g_t = u_t(x)^4 g_0$ so that $\mathcal{U}_t(x) = u_t(x)\mathcal{U}_0(x)$. Let

$$U_t(x) = e^t \, \mathcal{U}_t(xe^{2t}). \tag{162}$$

so that $U_t(x)$ goes to one at infinity for all t since we arranged $\mathcal{U}_0(x)$ to go to one at infinity and $u_t(x)$ goes to e^{-t} at infinity. Analogously, we define $\mathcal{V}_t(x) = v_t(x)\mathcal{U}_0(x)$ (so that $\frac{d}{dt}\mathcal{U}_t(x) = \mathcal{V}_t(x)$) and we define

$$V_t(x) = e^t \, \mathcal{V}_t(xe^{2t}). \tag{163}$$

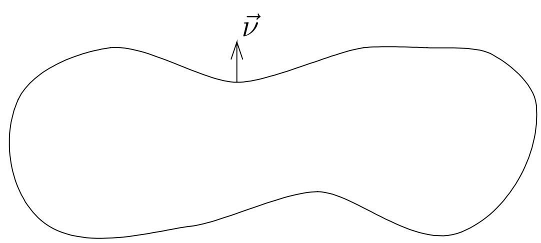
Then we observe that $V_t(x)$ goes to -1 at infinity (since $v_t(x)$ goes to $-e^{-t}$ at infinity) and $V_t(x)$ equals zero on $\Sigma(t)$. Furthermore, differentiating Equation (162) gives us

$$\frac{d}{dt}U_t(x) = V_t(x) + U_t(x) + 2r\frac{\partial}{\partial r}U_t(x) \tag{164}$$

where r is the radial coordinate in $\mathbf{R}^3$.

Lemma 10. *For* $t \geq t_0$ *(as defined in Corollary* 5*), the Riemannian manifold* $(\mathbf{R}^3 \backslash B_{r_0 e^{-2t}}(0), U_t(x)^4 \delta)$ *is isometric to the harmonically flat end of* (M^3, g_t) *and has total mass* $m(t)$*.*

Consequently, $\Sigma(t)$ has zero mean curvature in $(\mathbf{R}^3 \backslash B_{r_0 e^{-2t}}(0), U_t(x)^4 \delta)$, from which it follows that $U_t H + 4 \frac{dU_t}{d\vec{\nu}} = 0$, where H is the mean curvature of $\Sigma(t)$ and $\vec{\nu}$ is the outward pointing unit normal vector of $\Sigma(t)$ in $(\mathbf{R}^3, \delta)$. Also, since (M^3, g_t) has zero scalar curvature outside $\Sigma(t)$ (for $t \geq t_0$ in Corollary 5), it follows from Equation (240) that $U_t(x)$ is harmonic in $(\mathbf{R}^3, \delta)$ outside $\Sigma(t)$. Then from the discussion and definitions in the above paragraphs this in turn implies that $\mathcal{U}_t(x)$ is harmonic, which implies that $\mathcal{V}_t(x)$ is harmonic, which implies that $V_t(x)$ is also harmonic outside $\Sigma(t)$. To summarize:



$\Sigma(t) \subset \mathbf{R}^3$

Figure 7.

$$\text{(165)} \qquad \begin{cases} U_t H + 4 \frac{dU_t}{d\vec{\nu}} = 0 & \text{on } \Sigma(t) \\ \Delta U_t \equiv 0 & \text{outside } \Sigma(t) \\ U_t \to 1 & \text{at infinity} \end{cases}$$

$$\text{(166)} \qquad \begin{cases} V_t = 0 & \text{on } \Sigma(t) \\ \Delta V_t \equiv 0 & \text{outside } \Sigma(t) \\ V_t \to -1 & \text{at infinity.} \end{cases}$$

Equations (164), (165), and (166) characterize the new rescaled first order o.d.e. in t for $U_t(x)$. On the one hand $U_t(x)$ determines $\Sigma(t)$ since $\Sigma(t)$ is the outermost area minimizing horizon of $(\mathbf{R}^3 \backslash B_{r_0 e^{-2t}}(0), U_t(x)^4 \delta)$, and on the other hand $\Sigma(t)$ determines $V_t(x)$ by Equation (166) which determines the first order rate of change of $U_t(x)$ in Equation (164). In the next section we will prove that $U_t(x)$ actually converges to $1+\frac{M}{2r}$ for some positive M in the limit as t goes to infinity in this o.d.e. and that $\Sigma(t)$ converges to a sphere of radius $m/2$. However, first it is necessary to prove that the diameter of $\Sigma(t)$ is bounded, which is what we will do in this section.

In Section 7 we proved that $m(t)$ was nonincreasing. In fact, by closely reexamining Equations (84), (90), and (113) we have that

$$m'(t) = -2\widetilde{m}(t) \leq 0 \tag{167}$$

where $\widetilde{m}(t)$ is the total mass of the manifold $(\overline{M}^3_{\Sigma(t)}, \widetilde{g}_t)$, where $\widetilde{g}_t = \phi(x)^4 \overline{g_t}$, $M^3_{\Sigma(t)}$ is the closed region of M^3 which is outside or on $\Sigma(t)$, $(\overline{M}^3_{\Sigma(t)}, \overline{g_t})$ is the manifold obtained by reflecting $(M^3_{\Sigma(t)}, g_t)$ through $\Sigma(t)$, and $\phi(x)$ is the harmonic function on $(\overline{M}^3_{\Sigma(t)}, \overline{g_t})$ which goes to one in the original end and zero in the other end.

Then since $(M^3_{\Sigma(t)}, g_t)$ is isometric to $(\mathbf{R}^3_{\Sigma(t)}, U_t(x)^4\delta)$ (where $\mathbf{R}^3_{\Sigma(t)}$ is the region in $\mathbf{R}^3$ outside $\Sigma(t)$), $(\overline{M}^3_{\Sigma(t)}, \widetilde{g}_t)$ is isometric to $(\mathbf{R}^3_{\Sigma(t)}, W_t(x)^4\delta)$ where $W_t(x) = \phi(x)U_t(x)$. Furthermore, by Equation (239) it follows that $W_t(x)$ is harmonic in $(\mathbf{R}^3, \delta)$, and since $\phi(x) = \frac{1}{2}$ on $\Sigma(t)$ by symmetry and $U_t(x)$ and $\phi(x)$ both go to one at infinity, we have that

$$\begin{cases} W_t = & \frac{1}{2}U_t & \text{on } \Sigma(t) \\ \Delta W_t \equiv & 0 & \text{outside } \Sigma(t) \\ W_t \to & 1 & \text{at infinity.} \end{cases} \tag{168}$$

Lemma 11. *For $t \geq t_0$, the Riemannian manifold $(\mathbf{R}^3_{\Sigma(t)}, W_t(x)^4\delta)$ is isometric to $(\overline{M}^3_{\Sigma(t)}, \widetilde{g}_t)$ defined above and has total mass $\widetilde{m}(t)$.*

Corollary 6. *For $t \geq t_0$,*

$$m(t) = -\frac{1}{2\pi}\int_{\Sigma(t)} \frac{dU_t}{d\vec{\nu}} \tag{169}$$

and

$$\widetilde{m}(t) = -\frac{1}{2\pi}\int_{\Sigma(t)} \frac{dW_t}{d\vec{\nu}}. \tag{170}$$

Proof. In fact, by the divergence theorem, the above equations are true if we replace $\Sigma(t)$ with any homologous surface containing $\Sigma(t)$

since $U_t(x)$ and $W_t(x)$ are harmonic in $\mathbf{R}^3$. Then the corollary follows from the definition of total mass given in Definition 2 where we consider the above statements with $\Sigma(t)$ replaced by a large sphere at infinity.

q.e.d.

Furthermore, since $U_t(x)$, $V_t(x)$, and $W_t(x)$ are all harmonic functions in $(\mathbf{R}^3, \delta)$ outside $\Sigma(t)$, it follows from their boundary values that

$$V_t(x) = U_t(x) - 2W_t(x) \tag{171}$$

outside $\Sigma(t)$. Then plugging this into Equation (164) we get

$$\frac{d}{dt}U_t(x) = 2(U_t(x) - W_t(x) + r\frac{\partial}{\partial r}U_t(x)) \tag{172}$$

outside $\Sigma(t)$.

The above equation reveals the key idea we will use to study the behavior of the o.d.e. for $U_t(x)$. By Equation (167), it follows that $\widetilde{m}(t)$ must be going to zero for large t since $m(t)$ cannot become less than zero by the Positive Mass Theorem. Also, the Positive Mass Theorem states that there is only one zero mass metric, namely $(\mathbf{R}^3, \delta)$, and in Section 12 we will use this fact to prove that $W_t(x)$ is approaching the constant function one. Then it follows from studying Equation (172) that $U_t(x)$ approaches $1+\frac{M}{2r}$ for some positive M in the limit as t goes to infinity. With a few additional observations this will prove that (M^3, g_t) converges to a Schwarzschild metric outside the horizon $\Sigma(t)$ as claimed in Theorem 4.

In the rest of this section, we will show that the rescaled horizon $\Sigma(t)$ is very well behaved in $\mathbf{R}^3$ as $t \to \infty$. In fact, we will show that both the areas and the diameters of the surfaces $\Sigma(t)$ have uniform upper and lower bounds. Later in Section 12 we will use this to prove that the harmonic functions $U_t(x)$, $V_t(x)$, and $W_t(x)$ also have upper and lower bounds independent of t, which will be needed when we take limits of these harmonic function.

We recall that by Corollary 5, $\Sigma(t)$ has only one component for $t \geq t_0$, and that this component is a sphere. Furthermore, by Equation (146), it follows that the conformal-invariant (and hence scale-invariant) quantity

$$\int_{\Sigma(t)} H^2 d\mu \leq 32\pi, \tag{173}$$

where H is the mean curvature and $d\mu$ is the area form of $\Sigma(t)$ in $(\mathbf{R}^3, \delta)$.

The Willmore functional of a surface in $\mathbf{R}^n$ is defined to be one fourth of the integral of the mean curvature squared over the surface. Surfaces with bounded Willmore functional have been widely studied, and in particular, it was shown by L. Simon in [49] that the ratio of the diameter squared to the area of a surface is bounded both from above and from below by the Willmore functional of the surface. More precisely, for a surface Σ in $\mathbf{R}^3$,

$$\frac{4}{\int_\Sigma H^2 d\mu} \leq \frac{\operatorname{diam}(\Sigma)^2}{|\Sigma|} \leq \frac{C^2}{4}\int_\Sigma H^2 d\mu, \tag{174}$$

where C is some positive constant. Hence, since Theorem 12 tells us that the rescaled $\Sigma(t)$ satisfy

$$\operatorname{diam}(\Sigma(t)) \geq \left(\frac{A_0}{65\pi}\right)^{1/2} \tag{175}$$

for arbitrarily large values of t, we get the following corollary.

Corollary 7. *Given any $\widetilde{t} \geq 0$, there exists a $t \geq \widetilde{t}$ such that*

$$|\Sigma(t)| \geq \frac{A_0}{k}, \tag{176}$$

where $A_0 = A(0)$, $|\Sigma(t)|$ denotes the area of the rescaled $\Sigma(t)$ in $(\mathbf{R}^3, \delta)$, and $k = 8 \cdot 65 C^2 \pi^2$.

Now going back to Corollary 6 and using Equation (171), we get that

$$m(t) - 2\widetilde{m}(t) = -\frac{1}{2\pi}\int_{\Sigma(t)} \frac{dV_t}{d\vec{\nu}}. \tag{177}$$

Then since $\widetilde{m}(t)$ is the total mass of $(\overline{M}^3_{\Sigma(t)}, \widetilde{g}_t)$, by the Positive Mass Theorem it must be positive. Hence,

$$m(t) \geq -\frac{1}{2\pi}\int_{\Sigma(t)} \frac{dV_t}{d\vec{\nu}}. \tag{178}$$

But from Theorem 21 in Appendix D and Equation (173), we have that

$$-\frac{1}{2\pi}\int_{\Sigma(t)} \frac{dV_t}{d\vec{\nu}} \geq (24\pi)^{-1/2}|\Sigma(t)|^{1/2} \tag{179}$$

which proves inequality (180) of the following theorem.

Theorem 14. *For* $t \geq t_0$ *(as defined in Corollary* 5*),*

$$m(t) \geq \left(\frac{|\Sigma(t)|}{24\pi}\right)^{1/2} \tag{180}$$

where $|\Sigma(t)|$ *denotes the area of the rescaled* $\Sigma(t)$ *in* $(\mathbf{R}^3, \delta)$*. Also,*

$$m(t) \geq \left(\frac{A_0}{24\pi k}\right)^{1/2} \tag{181}$$

for all $t \geq 0$*, where again* $A_0 = A(0)$ *and* $k = 8 \cdot 65 C^2 \pi^2$*.*

Inequality (181) then follows from inequality (180), Corollary 7, and the fact that $m(t)$ is nonincreasing. We note that since $m(t)$ and A_0 are respectively the total mass of (M^3, g_t) and the area of the horizon $\Sigma(t)$ in (M^3, g_t), inequality (181) is a weak Penrose inequality for (M^3, g_t). Furthermore, since the area of the horizon $A(t)$ is constant and $m(t)$ is non-increasing, we get this same weak Penrose inequality for the original metric (M^3, g_0).

Now going back to Equation (165), we observe that

$$m(t) = -\frac{1}{2\pi}\int_{\Sigma(t)} \frac{dU_t}{d\vec{\nu}} = \frac{1}{8\pi}\int_{\Sigma(t)} U_t(x) H. \tag{182}$$

Thus, by the Cauchy-Schwarz inequality,

$$m(t) \leq \frac{1}{8\pi}\left(\int_{\Sigma(t)} U_t(x)^2\right)^{1/2}\left(\int_{\Sigma(t)} H^2\right)^{1/2}. \tag{183}$$

Furthermore, since the area of $\Sigma(t)$ in (M^3, g_t) equals A_0, by Lemma 10

$$A_0 = \int_{\Sigma(t)} U_t(x)^4. \tag{184}$$

Thus,

$$\int_{\Sigma(t)} U_t(x)^2 \leq |\Sigma(t)|^{1/2} A_0^{1/2}, \tag{185}$$

so that by Equation (173) we have that

$$m(t) \leq (2\pi)^{-1/2} A_0^{1/4} |\Sigma(t)|^{1/4}, \tag{186}$$

which, when combined with Theorem 14, gives inequality (187) of the following theorem.

Theorem 15. *For* $t \geq t_0$ (*as defined in Corollary* 5),

$$\frac{1}{(12k)^2} \leq \frac{|\Sigma(t)|}{A_0} \leq 12^2 \tag{187}$$

and

$$\frac{1}{8\pi(12k)^2} \leq \frac{diam(\Sigma(t))^2}{A_0} \leq 8\pi(12C)^2 \tag{188}$$

where $A_0 = A(0)$ *and* $k = 8 \cdot 65C^2\pi^2$.

Inequality (188) then follows from inequalities (187), (173), and (174), and is important for Section 12.

12. The limit metric

In this section we will prove that, outside the horizons $\Sigma(t)$, the metrics (M^3, g_t) approach a Schwarzschild metric. More precisely, we will prove Theorem 4 by showing that the rescaled $\Sigma(t)$, defined in the previous section as the original $\Sigma(t)$ rescaled by the factor e^{-2t}, converge to a coordinate sphere of radius $M/2$ in $(\mathbf{R}^3, \delta)$ and that

$$\lim_{t\to\infty} U_t(x) = 1 + \frac{M}{2|x|} \tag{189}$$

for $|x| \geq M/2$, where $M = \lim_{t\to\infty} m(t)$.

The first step is to bound $U_t(x)$ from above. From inequality (188) in Section 11, it follows that the rescaled $\Sigma(t)$ (defined for $t \geq t_0$) stay inside $S_{r_{\max}}(0)$, where $r_{\max} = 12C(8\pi A_0)^{1/2}$. Hence, by the maximum principle and Equation (166),

$$V_t(x) \leq -1 + \frac{r_{\max}}{|x|}. \tag{190}$$

Now choose b such that

$$U_{t_0}(x) \leq 1 + \frac{b}{|x|} \tag{191}$$

and let $c = \max(b, r_{\max})$. Then analyzing Equation (164) allows us to conclude that

$$U_t(x) \leq 1 + \frac{c}{|x|} \tag{192}$$

for all $t \geq t_0$ and all x outside the ball of radius $r_0 e^{-2t}$. Then since $W_t(x) \leq (1 + U_t(x))/2$ by the maximum principle and Equation (168), it follows that

$$W_t(x) \leq 1 + \frac{c}{2|x|} \tag{193}$$

for all $t \geq t_0$ and all x outside $\Sigma(t)$.

Since $W_t(x)$ is harmonic outside $S_{r_{\max}}(0)$ in $(\mathbf{R}^3, \delta)$ and goes to one at infinity, it is completely determined in this region by its values on $S_{r_{\max}+1}(0)$.

Definition 18. For $\alpha \in (0, 1/2)$, we define H_α to be the set of positive harmonic functions $h(x)$ defined outside $S_{r_{\max}}(0)$ in $(\mathbf{R}^3, \delta)$ which go to one at infinity and which satisfy

$$|h(x) - 1| \leq \alpha \tag{194}$$

for all $x \in S_{r_{\max}+1}(0)$.

Now let's put the supremum topology (for x outside $S_{r_{\max}+1}(0)$) on H_α, so that $k(x)$ is in an ϵ neighborhood of $h(x)$ if and only if $|k(x) - h(x)| < \epsilon$ for all $x \in S_{r_{\max}+1}(0)$, for $k(x), h(x) \in H_\alpha$. Then it follows that H_α is a compact space with this topology [21].

Next we define the following very useful continuous functional $\mathcal{F}$ on H_α.

Definition 19. Given $h(x) \in H_\alpha$, let (P^3, k) be the Riemannian manifold isometric to $(\mathbf{R}^3 \backslash \overline{B}_{r_{\max}+2}(0), h(x)^4 \delta)$. Then we define

$$\mathcal{F}(h(x)) = \inf_{\psi(x)} \left\{ \frac{1}{4\pi} \int_{(P^3,k)} |\nabla \psi|^2 \, dV \;\;\middle|\;\; \lim_{x \to \infty} \psi(x) = \psi_0 \right\} \tag{195}$$

where ψ is a spinor, ψ_0 is a fixed constant spinor of norm one defined at infinity, ∇ is the spin connection, and dV is the volume form on P^3 with respect to the metric k.

Furthermore, from standard theory there exists a minimizing spinor for each $h(x)$ which satisfies

$$\begin{cases} \overline{\nabla}^j \nabla_j \psi(x) = 0, & \text{for } x \in P^3 \\ \nu^j \nabla_j \psi = 0, & \text{for } x \in \partial P^3 \\ \lim_{x \to \infty} \psi(x) = \psi_0 \end{cases} \tag{196}$$

where ∇_j is the spin connection in (P^3, k), $\overline{\nabla}^j$ its formal adjoint, and $\vec{\nu}$ is the outward pointing unit normal vector to the boundary of P^3.

Lemma 12. *The functional $\mathcal{F}$ is continuous on H_α.*

Proof. Working with respect to the standard flat metric on $\mathbf{R}^3 \backslash \overline{B}_{r_{\max}+2}(0)$, we can write down explicit formulas for the spin connection derived in [17] for the case of two-component Weyl spinors. Hence, we let $\psi(x) = (\psi^1(x), \psi^2(x))$ be a pair of complex-valued functions and

$$\mathcal{F}(h(x)) = \frac{1}{4\pi} \int_{\mathbf{R}^3 \backslash \overline{B}_{r_{\max}+2}(0)} h(x)^2 \left| \vec{\nabla}\psi + i \left(\frac{\vec{\nabla}h}{h} \times \vec{\sigma} \right) \psi \right|^2 dV \tag{197}$$

where $\times$ is the cross product in $\mathbf{R}^3$, σ^i are the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{198}$$

$\vec{\nabla}$ is now the usual gradient in $\mathbf{R}^3$, and dV is the usual volume form in $\mathbf{R}^3$. Using the fact that (196) is the Euler-Lagrange equation for Equation (197), one can show that $|\psi(x)|$ is uniformly bounded. It also follows that $|\vec{\nabla}\psi(x)| \leq c/r^2$, and since $h(x)$ is harmonic, $|\vec{\nabla}h(x)| \leq c/r^2$ too, for some uniform constant $c > 0$.

Using these facts, we can compute the derivative of $\mathcal{F}(h_t(x))$ with respect to t at $t = 0$. Since the energy functional of the spinors in Equation (195) is strictly convex, it follows that the minimizing spinor varies smoothly for smooth variations of $h(x)$. Furthermore, it follows from Equation (196) that the contribution to the first order rate of change of $\mathcal{F}(h_t(x))$ due to the variation of the minimizing spinor is zero. Hence, from the previous paragraph it follows that the derivative of $\mathcal{F}(h_t(x))$ with respect to t at $t = 0$ is uniformly bounded (with respect to the supremum norm on H_α) in all directions, from which it follows that $\mathcal{F}$ is continuous on H_α. q.e.d.

Definition 20. Define $\mathcal{H}_\alpha$ to be the closure in the topological space H_α of the set of all $h(x)$ such that the corresponding manifold with boundary (P^3, k) defined in the previous definition can be extended to be a complete, smooth, asymptotically flat manifold with nonnegative scalar curvature (with possibly multiple ends but without boundary).

Note that since $\mathcal{H}_\alpha$ is a closed subset of a compact topological space, $\mathcal{H}_\alpha$ is also compact using this same supremum topology.

Using Definition 2, we can define the total mass functional $m(h(x))$ for $h(x) \in H_\alpha$ to be the total mass of (P^3, k) (defined in Definition 19). We note that m is continuous on H_α.

Lemma 13. *Given* $h(x) \in \mathcal{H}_\alpha$,

$$m(h(x)) \geq \mathcal{F}(h(x)). \tag{199}$$

Proof. This lemma follows directly from Witten's proof of the Positive Mass Theorem [52], [40]. In Witten's argument, he showed that the total mass is bounded below by such an integral over the whole manifold for a spinor which satisfies the Dirac equation and goes to a constant spinor with norm one at infinity. Thus, the infimum of the same integral over a smaller region and over more spinors must be smaller than the original integral. This proves the inequality for $h(x)$ which correspond to (P^3, k) which can be extended smoothly. Then the inequality follows for all $h(x) \in \mathcal{H}_\alpha$ since the total mass m and the functional $\mathcal{F}$ are continuous functionals on H_α. q.e.d.

Lemma 14. *For* $h(x) \in H_\alpha$, $\mathcal{F}(h(x)) = 0$ *if and only if* $h(x) \equiv 1$.

Proof. By Equation (196), it follows that the norm of the minimizing spinor is not identically zero, from which it follows that $\mathcal{F}(h(x)) = 0$ implies the existence of a parallel spinor. Hence, since (P^3, k) is asymptotically flat, it must be flat everywhere, which implies that $h(x) \equiv 1$. Conversely, if $h(x) \equiv 1$, choosing $\psi(x) \equiv \psi_0$ proves that $\mathcal{F}(h(x)) = 0$. q.e.d.

Lemma 15. *For all* $\delta > 0$, *there exists an* $\epsilon > 0$ *such that*

$$\mathcal{F}(h(x)) < \epsilon \quad \Rightarrow \quad \sup_{x \in S_{r_{\max}+1}(0)} |h(x) - 1| < \delta \tag{200}$$

for $h(x) \in \mathcal{H}_\alpha$.

Proof. Since $\mathcal{F}$ is a continuous functional on the compact space $\mathcal{H}_\alpha$ and only equals zero if $h(x) \equiv 1$, the lemma follows. q.e.d.

Corollary 8. *For all* $\delta > 0$, *there exists an* $\epsilon \in (0, 2(r_{\max} + 1)\delta)$ *such that*

$$m(h(x)) < \epsilon \quad \Rightarrow \quad \sup_{x \in S_{r_{\max}+1}(0)} |h(x) - 1| < \delta \tag{201}$$

for $h(x) \in \mathcal{H}_\alpha$.

Proof. The fact that there exists an $\epsilon > 0$ follows directly from Lemmas 13 and 15. Then by considering $h(x) = 1 + a/2|x|$ for $a > 0$

which corresponds to a Schwarzschild metric with total mass a, it follows that $\epsilon < 2(r_{\max} + 1)\delta$. q.e.d.

Alternatively, the previous discussion with spinors beginning with Definition 19 and ending with the above corollary could be replaced by quoting Theorem 1.1 of [8] (the proof of which also uses spinors) which proves Corollary 8 as well.

Now we are ready to apply these results to understand the asymptotic behavior of $U_t(x)$ as t goes to infinity.

Theorem 16. *For all $\delta > 0$, there exists a $\bar{t}$ such that for all $t \geq \bar{t}$*

$$\left| U_t(x) - \left(1 + \frac{M}{2|x|}\right) \right| \leq \delta \tag{202}$$

for $|x| \geq r_{\max} + 1$, where $M \equiv \lim_{t\to\infty} m(t) > 0$.

Proof. First, we let

$$\overline{U}_t(x) = U_t(x) - \left(1 + \frac{m(t)}{2|x|}\right) \tag{203}$$

and

$$\overline{W}_t(x) = W_t(x) - \left(1 + \frac{\widetilde{m}(t)}{2|x|}\right) \tag{204}$$

so that by Lemmas 10 and 11 the harmonic functions $\overline{U}_t(x)$ and $\overline{W}_t(x)$ do not have a constant term or a $1/|x|$ term in their expansions using spherical harmonics. Then substituting these two expressions into Equation (172) yields

$$\frac{d}{dt}\overline{U}_t(x) = 2\left(\overline{U}_t(x) - \overline{W}_t(x) + r\frac{\partial}{\partial r}\overline{U}_t(x)\right) \tag{205}$$

by Equation (167).

Also, by Equation (167) and the Positive Mass Theorem, $\int_0^\infty \widetilde{m}(t)\, dt \leq m(0)/2$, so $T_{\text{bad}} = \{t \mid \widetilde{m}(t) \geq \epsilon\}$ has finite measure less than or equal to $m(0)/2\epsilon$. Hence, by Lemma 11 and Corollary 8, for all $\delta > 0$, there exists an $\epsilon > 0$ such that

$$\sup_{x \in S_{r_{\max}+1}(0)} |W_t(x) - 1| < \delta \tag{206}$$

and hence

$$\sup_{x \in S_{r_{\max}+1}(0)} |\overline{W}_t(x)| < 2\delta \tag{207}$$

since

$$\widetilde{m}(t) < \epsilon < 2(r_{\max} + 1)\delta, \tag{208}$$

for all $t \notin T_{\text{bad}}$. Also, for all t we have the uniform bound

$$\sup_{x \in S_{r_{\max}+1}(0)} |\overline{W}_t(x)| < B \tag{209}$$

(where $B = 1 + c/2(r_{\max} + 1)$) by Equations (204) and (193) and since $\widetilde{m}(t) \leq c$ by Equation (193). Then since $\overline{W}_t(x)$ is a harmonic function without a constant term or a $1/|x|$ term, it follows that

$$|\overline{W}_t(x)| \leq k \frac{(r_{\max} + 1)^2}{|x|^2} \begin{cases} 2\delta & \text{for } t \notin T_{\text{bad}} \\ B & \text{for } t \in T_{\text{bad}} \end{cases} \tag{210}$$

for $|x| \geq r_{\max} + 1$ and some positive constant k (which we will not need to compute). Then from analyzing Equation (205) and using the fact that $\overline{U}_t(x)$ does not have a constant term or a $1/|x|$ term and the fact that T_{bad} has finite measure, it follows that we can choose some $\bar{t}$ large enough such that

$$|\overline{U}_t(x)| \leq k \frac{(r_{\max} + 1)^2}{|x|^2} (2.01)\delta \tag{211}$$

for all $t \geq \bar{t}$ and $|x| \geq r_{\max} + 1$. Hence, since $\delta > 0$ was arbitrary and $M = \lim_{t \to \infty} m(t)$ (and is positive by inequality (181)), Theorem 16 follows. q.e.d.

Corollary 9. *For all $\delta > 0$, there exists a $\bar{t}$ such that*

$$\left| V_t(x) - \left(\frac{M}{2|x|} - 1 \right) \right| \leq \delta \tag{212}$$

for $|x| \geq r_{\max} + 1$ and all $t \geq \bar{t}$ except on a set with measure less than δ.

Proof. We recall from Equation (171) that $V_t(x) = U_t(x) - 2W_t(x)$. Hence, since $U_t(x)$ is converging to $1 + \frac{M}{2|x|}$ by Theorem 16 and $W_t(x)$ is converging to 1 (for $t \notin T_{\text{bad}}$) by Equations (204), (207), and (208), then the corollary follows from the fact that T_{bad} has finite measure.

q.e.d.

Corollary 10. *For all $\delta > 0$, there exists a $\bar{t}$ such that for all $t \geq \bar{t}$,*

$$\left| V_t(x) - \left(\frac{M}{2|x|} - 1 \right) \right| \leq \delta \tag{213}$$

and

$$|W_t(x) - 1| \leq \delta \tag{214}$$

for $|x| \geq r_{\max} + 1$.

Proof. Equation (213) follows from the previous corollary, Equation (166), and Theorem 7. Equation (214) then follows from Equation (213), Theorem 16, and Equation (171). q.e.d.

The next two lemmas use Corollary 10 to prove that $\Sigma(t)$ converges to the sphere of radius $M/2$. The main idea is that since $V_t(x)$ converges to $\frac{M}{2|x|} - 1$ and equals zero on $\Sigma(t)$ by definition, then $\Sigma(t)$ must be converging to $S_{M/2}(0)$.

In the rest of this section we will want to take limits of certain sequences of surfaces, and naturally there are several ways to do this. However, in our case, all of the surfaces we are dealing with are boundaries of regions, so it seems most natural to follow [39]. In [39], a very general definition of the measure of the perimeter of a Lebesgue measurable set is given on p. 64. Then on p. 70, it is shown that the space of Lebesgue measurable sets with equally bounded perimeters in a compact region K is compact with respect to the L^1 norm of the characteristic functions of the regions, meaning that any sequence of such sets contains a subsequence such that the characteristic functions of the subsequence converge in $L^1(K)$. Thus, in what follows, we will say that Σ_∞ is a limit of a sequence of surfaces $\{\Sigma(t_i)\}$ if it is the boundary of a Lebesgue measurable set with bounded perimeter whose characteristic function is the L^1 limit of the characteristic functions of a subsequence of the $\{\Sigma(t_i)\}$. We note that (187) implies that the $\{\Sigma(t_i)\}$ have equally bounded perimeters for $t_i \geq t_0$. We also note that since this perimeter function is shown to be lower semicontinuous in [39], Σ_∞ has bounded perimeter. (We also note that we define the boundary of a region to be the set of boundary points such that every open ball around a boundary point contains a positive measure of both the region and the complement of the region.)

From the above considerations, it also turns out that the $\{\Sigma(t_i)\}$ will converge to Σ_∞ in the Hausdorff distance sense as well. This follows from the fact that each $\Sigma(t_i)$ minimizes area in $(\mathbf{R}^3 \backslash B_{r_0 e^{-2t}}(0), U_t(x)^4 \delta)$ and we have uniform upper and lower bounds on the $U_{t_i}(x)$. (This is related to the proof of Lemma 24 in Appendix E.)

Lemma 16. *Let Σ_∞ be any limit of $\Sigma(t)$ in $(\mathbf{R}^3, \delta)$. Then no part of Σ_∞ lies inside $S_{M/2}(0)$.*

Proof. Let R be the open region outside Σ_∞, and consider a sequence of t_i going to infinity such that $\Sigma(t_i)$ is converging to a limit Σ_∞. Since each $\Sigma(t_i)$ is outer minimizing and we have uniform upper and lower bounds on $U_{t_i}(x)$ (and hence on the corresponding metric), it follows that R must be connected. Then $V_{t_i}(x)$ must converge to a harmonic function in R, and by Corollary 10 this limit harmonic function equals $\frac{M}{2|x|} - 1$. But each $V_{t_i}(x) \leq 0$ by definition, so $R \cap B_{M/2}(0) = \varnothing$, proving the lemma. q.e.d.

Lemma 17. *Let Σ_∞ be any limit of $\Sigma(t)$ in $(\mathbf{R}^3, \delta)$. Then no part of Σ_∞ lies outside $S_{M/2}(0)$.*

Proof. Suppose otherwise. Then we can consider a sequence of t_i going to infinity such that $\Sigma(t_i)$ is converging to a limit Σ_∞ which lies at least partially outside $S_{M/2}(0)$. Again, defining the region R as above, $V_{t_i}(x)$ must converge to a harmonic function in R, and by Corollary 10 this limit harmonic function equals $\frac{M}{2|x|} - 1$.

Since $V_{t_i}(x)$ is harmonic, it minimizes its energy among functions with the same boundary data. Thus, since $\Sigma(t_i)$ is contained inside $S_{r_{\max}}(0)$, the energy of $V_{t_i}(x)$ is less than $4\pi r_{\max}$. Now choose $x_0 \in \Sigma_\infty$ which maximizes distance from the origin, and let $|x_0| = M/2 + 2r$ for some $r > 0$. Then by the co-area formula,

(215)

$$\begin{aligned}
4\pi r_{\max} &> \int_{R_i} |\nabla V_{t_i}(x)|^2 \, dV > \int_{R_i \cap B_r(x_0)} |\nabla V_{t_i}(x)|^2 \, dV \\
&= \int_{-1}^{0} dz \int_{L_z \cap B_r(x_0)} |\nabla V_{t_i}(x)| \, dA_z \\
&\geq \int_{-1}^{0} dz \, |L_z \cap B_r(x_0)|^2 \left(\int_{L_z \cap B_r(x_0)} |\nabla V_{t_i}(x)|^{-1} \, dA_z \right)^{-1}
\end{aligned}$$

where R_i is the region outside $\Sigma(t_i)$, dV is the standard volume form on $\mathbf{R}^3$, L_z is the level set on which V_{t_i} equals z, dA_z is the area form of

L_z, and $|\cdot|$ denotes area in $\mathbf{R}^3$. Next, we define

$$\mathcal{V}(z) = \text{ the volume of the region } \{x \in B_r(x_0) \mid V_{t_i}(x) > z\} \tag{216}$$

for $-1 < z \leq 0$. Then from inequality (215),

$$4\pi r_{\max} > \int_{-1}^{0} |L_z \cap B_r(x_0)|^2 \, \mathcal{V}'(z)^{-1} \, dz. \tag{217}$$

Since $V_{t_i}(x)$ equals zero on $\Sigma(t_i)$ and yet is converging to $\frac{M}{2|x|} - 1$ in R, we must have $\mathcal{V}'(z)$ approaching zero and the surfaces $L_z \cap B_r(x_0)$ converging to $\Sigma(t_i) \cap B_r(x_0)$ for $z \in \left(\frac{M}{2|x_0|} - 1, 0\right)$ as t_i goes to infinity. However, since $\Sigma(t_i)$ minimizes area in $(\mathbf{R}^3 \backslash B_{r_0 e^{-2t_i}}(0), U_{t_i}(x)^4 \delta)$ and $U_{t_i}(x)$ is uniformly bounded, it follows that $|L_z \cap B_r(x_0)|$ is not going to zero for $z \in \left(\frac{M}{2|x_0|} - 1, 0\right)$. Hence, the right hand side of inequality (217) is going to infinity as t_i goes to infinity, giving us a contradiction and proving the lemma. q.e.d.

Theorem 17. *The surfaces $\Sigma(t)$ converge to $S_{M/2}(0)$ in the Hausdorff distance sense in the limit as t goes to infinity,*

$$\lim_{t\to\infty} U_t(x) = 1 + \frac{M}{2|x|} \tag{218}$$

for $|x| \geq M/2$, and

$$\lim_{t\to\infty} U_t(x) = \sqrt{\frac{2M}{|x|}} \tag{219}$$

for $0 < |x| \leq M/2$, where as usual $M = \lim_{t\to\infty} m(t) > 0$.

Proof. First we observe that for all $\delta > 0$, there exists a $\bar{t}$ such that

$$\Sigma(t) \subset B_{\frac{M}{2}+\delta}(0) \backslash B_{\frac{M}{2}-\delta}(0) \tag{220}$$

for all $t \geq \bar{t}$. Otherwise, we could choose a sequence of t_i going to infinity such that each $\Sigma(t_i)$ did not lie entirely in the annulus. By previous discussions, a limit Σ_∞ would have to exist, and since this limit is valid in the Hausdorff distance sense as previously discussed, at least part of Σ_∞ would have to lie off of $S_{M/2}(0)$, contradiction at least one of the two previous lemmas. This proves Equation (220).

As a corollary, we get that for all $\delta > 0$, there exists a $\bar{t}$ such that

$$F_{\frac{M}{2}-\delta}(x) \leq V_t(x) \leq F_{\frac{M}{2}+\delta}(x) \tag{221}$$

where we define

$$F_a(x) = \begin{cases} \frac{a}{|x|} - 1 & \text{for } |x| \geq a \\ 0 & \text{for } |x| \leq a \end{cases} \tag{222}$$

for $a > 0$. Then Equations (218) and (219) follow (with uniform convergence on compact subsets of $\mathbf{R}^3 - \{0\}$) from inequality (221) and analyzing the behavior of Equation (164). q.e.d.

Theorem 4 then follows from the above theorem and Corollary 5.

13. Generalization to asymptotically flat manifolds and the case of equality

Up to this point in the paper we have assumed that (M^3, g_0) was harmonically flat at infinity. In particular, Theorems 2, 3, and 4 only apply to harmonically flat manifolds as stated. In this section, we will extend Theorems 2 and 3 and elements of Theorem 4 to asymptotically flat manifolds. This will prove the main theorem, Theorem 1, except for the case of equality, which we will see follows from the case of equality of Theorem 9.

It is worth noting that the main reason for initially considering only harmonically flat manifolds was convenience. Alternatively, we could have ignored harmonically flat manifolds and dealt only with asymptotically flat manifolds. However, this would have complicated some of the arguments unnecessarily, so we chose to delay these considerations until now.

Definition 21. The manifold (M^n, g) is said to be *asymptotically flat* if there is a compact set $K \subset M$ such that $M \backslash K$ is the disjoint union of ends $\{E_k\}$, such that for each end there exists a diffeomorphism $\Phi_k : E_k \to \mathbf{R}^n \backslash B_1(0)$ such that, in the coordinate chart defined by Φ_k,

$$g = \sum_{i,j} g_{ij}(x) dx^i dx^j,$$

where

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-p})$$
$$|x||g_{ij,k}(x)| + |x|^2|g_{ij,kl(x)}| = O(|x|^{-p})$$
$$|R(g)| = O(|x|^{-q})$$

for some $p > \frac{n-2}{2}$ and some $q > n$, where we have used commas to denote partial derivatives in the coordinate chart, and $R(g)$ is the scalar curvature of (M^n, g).

These assumptions on the asymptotic behavior of (M^n, g) at infinity imply the existence of the limit

$$M_{\text{ADM}}(g) = \frac{1}{16\pi} \lim_{\sigma\to\infty} \int_{S_\sigma} \sum_{i,j} (g_{ij,i}\nu_j - g_{ii,j}\nu_j)\, d\mu \tag{225}$$

where S_σ is the coordinate sphere of radius σ, ν is the unit normal to S_σ, and $d\mu$ is the area element of S_σ in the coordinate chart. The quantity M_{ADM} is called the *total mass* of (M^n, g) (see [1], [2], [42], and [46]), and agrees with the definition of total mass for harmonically flat 3-manifolds given in Definition 2.

First we observe that the arguments in the proof of the existence theorem, Theorem 2, did not use harmonic flatness anywhere, so we immediately get existence of the conformal flow of metrics for asymptotically flat manifolds. Similarly, the arguments used in Section 5 to prove that $A(t)$ is constant still hold. Next we reexamine the proof of Theorem 10 which proved that $m(t)$ was nonincreasing. The only modification we need to make is to use the more general definition for the total mass of an asymptotically flat manifold given by Equation (225). It is then straight forward to check that Equation (113) and hence Theorem 10 are still true. Hence:

Theorem 18. *Theorems* 2 *and* 3 *are true for asymptotically flat manifolds as well as harmonically flat manifolds.*

We choose not to extend Theorem 4 to asymptotically flat manifolds, but conjecture that it is still true. Instead, we observe that we must still have

$$m(t) \geq \sqrt{\frac{A(t)}{16\pi}} \tag{226}$$

for asymptotically flat manifolds. Otherwise, given an asymptotically flat counterexample, we could use Lemma 1 to perturb the manifold slightly making it harmonically flat at infinity such that it still violated Equation (226). Then applying the conformal flow of metrics to this harmonically flat manifold would violate Theorems 3 and 4, which is a contradiction. Setting $t = 0$ in inequality (226) then proves the Riemannian Penrose inequality for asymptotically flat manifolds.

The case of equality of Theorem 1 then follows from Equation (113) and Theorem 9. If we have equality in the Riemannian Penrose inequality, then applying the conformal flow of metrics to this initial metric must also give equality in inequality (226) for all $t \geq 0$. Hence, the right hand derivative of $m(t)$ at $t = 0$ equals zero, so by Equation (113),

$$\mathcal{E}(\Sigma^+(0), g_0) = 2m(0). \tag{227}$$

By Definition 11 and Equation (14), $\Sigma^+(0)$ is the outermost minimal area enclosure of Σ_0 in (M^3, g). Furthermore, by the case of equality of Theorem 9, (M^3, g) is a Schwarzschild manifold outside $\Sigma^+(0)$. Hence, $\Sigma^+(0)$ is the outermost horizon of (M^3, g), so (M^3, g) is isometric to a Schwarzschild manifold outside their respective outermost horizons. This completes the proof of Theorem 1 and the Riemannian Penrose inequality.

The reader might also have noticed that none of the arguments in this paper have used anything about the original manifold inside the original horizon Σ_0. Hence, we can generalize Theorem 1 to the following.

Theorem 19. *Let (M^3, g) be a complete, smooth, asymptotically flat* 3*-manifold with boundary which has nonnegative scalar curvature and total mass m. Then if the boundary is an outer minimizing horizon* (*with one or more components*) *of total area A, then*

$$m \geq \sqrt{\frac{A}{16\pi}} \tag{228}$$

with equality if and only if (M^3, g) is isometric to a Schwarzschild manifold outside their respective outermost horizons.

14. New quasi-local mass functions

Given a region in a space-like slice of a space-time, it is natural to ask how much energy and momentum is contained in that region. As described in the introduction, there does exist a well-defined notion of total mass and also of energy-momentum density. However, when the region in question is some finite region which is not the entire manifold or just a single point, it is not very well understood how to define how much energy and momentum is in that region.

Various definitions of "quasi-local" mass exist, such as the Hawking mass, which was used by Huisken and Ilmanen in [30] to prove their Riemannian Penrose inequality, for example. Good definitions of quasi-local mass should satisfy certain reasonable properties [14] such as positivity and some kind of monotonicity, either under a flow or by inclusion. In addition, for a large region, the quasi-local mass function should approach the total mass of the manifold, and for horizons with area A it is thought that the mass should be $\sqrt{A/16\pi}$.

Let (M^3, g) be a complete asymptotically flat manifold with nonnegative scalar curvature and total mass m. Let Σ be any surface in M^3 which is in the class of surfaces $\mathcal{S}$ defined in Section 2.

Definition 22. Suppose $u(x)$ is a positive harmonic function in (M^3, g) outside Σ going to a constant at infinity scaled such that $(M^3, u(x)^4 g)$ has the same total mass as (M^3, g).

Then if Σ is an outer minimizing horizon with area A in $(M^3, u(x)^4 g)$, we define the quasi-local mass of Σ in (M^3, g) to be

$$m_g(\Sigma) = \sqrt{\frac{A}{16\pi}}. \tag{229}$$

Definition 23. We define S to be the subset of $\mathcal{S}$ of surfaces Σ for which such a conformal factor $u(x)$ exists, and we note that (by Equation (239) mostly) this conformal factor is unique for each Σ when it exists.

As usual Σ could have multiple components. It is also interesting that

$$m_g(\widetilde{\Sigma}) = m_{u(x)^4 g}(\widetilde{\Sigma}) \tag{230}$$

for all surfaces $\widetilde{\Sigma} \in S$ where the conformal factor $u(x)$ is any harmonic function in (M^3, g) defined outside $\widetilde{\Sigma}$ which goes to a constant at infinity scaled such that $(M^3, u(x)^4 g)$ has the same total mass as (M^3, g).

Lemma 18. *The quasi-local mass function $m_g(\Sigma)$ defined for (M^3, g) is nondecreasing for the family of surfaces $\Sigma(t)$ defined by Equa-*

tion (14). *That is,* $m_g(\Sigma(t))$ *is nondecreasing in* t. *Furthermore,*

$$m_g(\Sigma(0)) = \sqrt{\frac{A}{16\pi}} \tag{231}$$

where A *is the area of the original outer minimizing horizon* Σ_0 *in* (M^3, g), *and*

$$\lim_{t\to\infty} m_g(\Sigma(t)) = m, \tag{232}$$

the total mass of (M^3, g).

Proof. We consider the conformal flow of metrics g_t beginning with (M^3, g) as discussed throughout this paper and in the previous section for asymptotically flat manifolds. Then we note that by Equation (14), $\Sigma(t)$ is an outer minimizing horizon in $(M^3, u_t(x)^4 g)$ with area $A(t)$. Hence, $u_t(x)$ satisfies the conditions in Definition 22 except that it is not scaled to have the correct mass. Hence, since mass has units of length, it follows that

$$m_g(\Sigma(t)) = \frac{m}{m(t)}\sqrt{\frac{A(t)}{16\pi}} \tag{233}$$

where again m is the total mass of (M^3, g). Then the lemma follows from Theorems 3 and 4 and the fact that $m(0) = m$. q.e.d.

There is a trick which allows us to extend the definition of this quasi-local mass function to all surfaces in $\mathcal{S}$.

Definition 24. Define

$$\widetilde{m}_g(\Sigma) = \sup\left\{ m_g(\widetilde{\Sigma}) \;\mid\; \Sigma \text{ (entirely) encloses } \widetilde{\Sigma} \in S \right\} \tag{234}$$

where Σ is any surface in $\mathcal{S}$.

It follows trivially that $\widetilde{m}_g(\Sigma)$ is monotone with respect to enclosure, which is a desirable property for quasi-local mass functions to have since larger regions should contain more mass in the nonnegative energy density setting which we are in. We also note that this same construction can be used with the Hawking mass to make it monotone with respect to enclosure too, where the original Hawking mass should only be defined to exist for surfaces which are their own outermost minimal area enclosures, as motivated by the results of Huisken and Ilmanen in [30].

We also notice that the existence of the Penrose inequality allows us to define another new quasi-local mass function which is similar in nature to the Bartnik mass [3], [30]. In fact, whereas the Bartnik mass could also be called an outer quasi-local mass function, it makes sense to call the new quasi-local mass function defined below the inner quasi-local mass function, which is clear from the definition. We have recently learned that Huisken and Ilmanen have also considered something similar to the following definition.

Definition 25. Given a surface $\Sigma \in \mathcal{S}$ in (M^3, g), consider all other asymptotically flat, complete, Riemannian manifolds $(\widetilde{M}^3, \widetilde{g})$ with nonnegative scalar curvature which are isometric to (M^3, g) outside Σ. Then we define

$$m_{\text{inner}}(\Sigma) = \sup \sqrt{\frac{\widetilde{A}}{16\pi}} \tag{235}$$

where $\widetilde{A}$ is the infimum of the areas of all of the surfaces in $(\widetilde{M}^3, \widetilde{g})$ in $\widetilde{\mathcal{S}}$.

We note here that $\widetilde{\mathcal{S}}$ is defined the same way as $\mathcal{S}$ in Definition 3 and that the surface in $\widetilde{\mathcal{S}}$ with minimum area may have multiple components. We also note that for $\widetilde{A}$ to be nonzero that $(\widetilde{M}^3, \widetilde{g})$ must have more than one asymptotically flat end.

Lemma 19. *Let (M^3, g) be an asymptotically flat, complete, Riemannian manifold with nonnegative scalar curvature, and let $\Sigma_1, \Sigma_2 \in \mathcal{S}$ such that Σ_2 (entirely) encloses Σ_1. Then*

$$m \geq m_{\text{inner}}(\Sigma_2) \geq m_{\text{inner}}(\Sigma_1) \tag{236}$$

where m is the total mass of (M^3, g).

Proof. Follows directly from the Penrose inequality and Definition 25. q.e.d.

Also, if Σ is outer minimizing (see Definition 6), then

$$m_{\text{outer}}(\Sigma) \geq m_{\text{inner}}(\Sigma) \tag{237}$$

where $m_{\text{outer}}(\Sigma)$ is basically the Bartnik mass [3] except that we only consider extensions of the metric in which Σ continues to be outer minimizing. The proof of this inequality and related discussions will be included in a paper on quasi-local mass which is currently in progress.

15. Open problems and acknowledgments

Even though the original Penrose Conjecture concerned only three dimensional, space-like slices of a space-time, it is easy to generalize the conjecture to higher dimensions using the same motivation as the three-dimensional case. In this paper we have restricted our attention to proving the three dimensional case of the Riemannian Penrose Conjecture, which is perhaps the most physically interesting dimension. However, the techniques presented here generalize to higher dimensions. The author is currently working on a paper to treat dimensions 4,5,6,7. Dimensions 8 and higher present additional technical challenges because the horizons, which are minimal hypersurfaces, can have co-dimension 7 singularities.

The Positive Mass Theorem is also technically still open in dimensions 8 and higher for manifolds which are not spin. The Witten proof [52] only works for spin manifolds (but in any dimension) and the Schoen-Yau proof [46] (which uses minimal hypersurface techniques) has the same technical difficulties that the author has encountered in dimension 8 and higher. It seems plausible that these technical difficulties might be able to be overcome so that the Schoen-Yau proof would work in all dimensions. This is a good problem to study.

Another important problem is to prove the Penrose Conjecture for arbitrary space-like slices (as opposed to totally-geodesic slices treated here) of a space-time as described in Section 1. There seem to be several natural ideas to try in this regard. First, one could attempt to solve a variant of Jang's equation which was used to extend the Riemannian Positive Mass Theorem to the general case [46]. Otherwise, one could try to modify the flow of metrics defined in this paper to define a flow of the Cauchy data (M^3, g, h) which has good monotonicity properties. Both of these approaches are speculative but could yield interesting results.

We also note that in this paper we did not ever prove the uniqueness of the conformal flow of metrics defined by Equations (13), (14), (15), and (16), although we conjecture that this is true. It would also be interesting to understand the relationship between the Hawking mass used in the Huisken-Ilmanen paper and the quasi-local mass function m_g defined in the previous section. In the spherically symmetric case, m_g is bounded below by the Hawking mass on the spherically symmetric spheres since the Hawking mass equals the Bartnik mass in this case (outside the outermost horizon). It is unclear if such a relationship generally holds.

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A. The harmonic conformal class of a metric

In this appendix we define a new equivalence class and partial ordering of conformal metrics. This provides a natural motivation for studying conformal flows of metrics to try to prove the Riemannian Penrose inequality.

Let

$$g_2 = u(x)^{\frac{4}{n-2}} g_1 \tag{238}$$

where g_2 and g_1 are metrics on an n-dimensional manifold M^n, $n \geq 3$. Then we get the surprisingly simple identity that

$$\Delta_{g_1}(u\phi) = u^{\frac{n+2}{n-2}} \Delta_{g_2}(\phi) + \phi \Delta_{g_1}(u) \tag{239}$$

for any smooth function ϕ.

This motivates us to define the following relation.

Definition 26. Define

$$g_2 \sim g_1$$

if and only if Equation (238) is satisfied with $\Delta_{g_1}(u) = 0$ and $u(x) > 0$.

Then from Equation (239) we get the following lemma.

Lemma 20. *The relation $\sim$ is reflexive, symmetric, and transitive, and hence is an equivalence relation.*

Thus, we can define the following equivalence class of metrics.

Definition 27. Define

$$[g]_H = \{\bar{g} \mid \bar{g} \sim g\}$$

to be the *harmonic conformal class* of the metric g.

Of course, this definition is most interesting when (M^n, g) has non-constant positive harmonic functions, which happens for example when (M^n, g) has a boundary.

Also, we can modify the relation $\sim$ to get another relation $\succeq$.

Definition 28. Define

$$g_2 \succeq g_1$$

if and only if Equation (238) is satisfied with $-\Delta_{g_1}(u) \geq 0$ and $u(x) > 0$.

Then from Equation (239) we get the following lemma.

Lemma 21. *The relation $\succeq$ is reflexive and transitive, and hence is a partial ordering.*

Since $\succeq$ is defined in terms of superharmonic functions, we will call it the superharmonic partial ordering of metrics on M^n. Then it is natural to define the following set of metrics.

Definition 29. Define

$$[g]_S = \{\bar{g} \mid \bar{g} \succeq g\}.$$

This set of metrics has the property that if $\bar{g} \in [g]_S$, then $[\bar{g}]_S \subset [g]_S$

Also, the scalar curvature transforms nicely under a conformal change of the metric. In fact, assuming Equation (238) again,

$$R(g_2) = u(x)^{-(\frac{n+2}{n-2})} \left(-c_n \Delta_{g_1} + R(g_1)\right) u(x) \tag{240}$$

where $c_n = \frac{4(n-1)}{n-2}$ [42]. This gives us the following lemma.

Lemma 22. *The sign of the scalar curvature is preserved pointwise by $\sim$. That is, if $g_2 \sim g_1$, then $sgn(R(g_2)(x)) = sgn(R(g_1)(x))$ for all $x \in M^n$.*

Also, if $g_2 \succeq g_1$, and g_1 has non-negative scalar curvature, then g_2 has non-negative scalar curvature.

Hence, the harmonic conformal equivalence relation $\sim$ and the superharmonic partial ordering $\succeq$ are useful for studying questions about scalar curvature. In particular, these notions are useful for studying the Riemannian Penrose inequality which concerns asymptotically flat 3-manifolds (M^3, g) with non-negative scalar curvature. Given such a manifold, define $m(g)$ to be the total mass of (M^3, g) and $A(g)$ to be the area of the outermost horizon (which could have multiple components) of (M^3, g). Define $P(g) = \frac{m(g)}{\sqrt{A(g)}}$ to be the Penrose quotient of (M^3, g). Then an interesting question is to ask which metric in $[g]_S$ minimizes $P(g)$.

This paper can be viewed as an answer to the above question. We showed that there exists a conformal flow of metrics (starting with g_0) for which the Penrose quotient was non-increasing, and in fact this conformal flow stays inside $[g_0]_S$. Furthermore, $g_{t_2} \in [g_{t_1}]_S$ for all $t_2 \geq t_1 \geq 0$. We showed that no matter which metric we start with, the metric converges to a Schwarzschild metric outside its horizon. Hence, the minimum value of $P(g)$ in $[g]_S$ is achieved in the limit by metrics converging to a Schwarzschild metric (outside their respective horizons).

In the case that the g is harmonically flat at infinity, a Schwarzschild metric (outside the horizon) is contained in $[g]_S$. More generally, given any asymptotically flat manifold (M^3, g), we can use $\mathbf{R}^3 \backslash B_r(0)$ as a coordinate chart for the asymptotically flat end of (M^3, g) which we are interested in, where the metric g_{ij} approaches δ_{ij} at infinity in this coordinate chart. Then we can consider the conformal metric

$$g_C = \left(1 + \frac{C}{|x|}\right)^4 g \tag{241}$$

in this end. In the limit as C goes to infinity, the horizon will approach the coordinate sphere of radius C. Then outside this horizon in the limit as C goes to infinity, the function $(1 + \frac{C}{|x|})$ will be close to a superharmonic function on (M^3, g) and the metric g_C will approach a Schwarzschild metric (since the metric g is approaching the standard metric on $\mathbf{R}^3$). Hence, the Penrose quotient of g_C will approach $(16\pi)^{-1/2}$, which is the Penrose quotient of a Schwarzschild metric.

As a final note, we prove that the first order o.d.e. for $\{g_t\}$ defined in Equations (13),(14), (15), and (16) is naturally defined in the sense that the rate of change of g_t is a function only of g_t and not of g_0 or t. To see this, given any solution $g_t = u_t(x)^4 g_0$ to Equations (13),(14), (15), and (16), choose any $s > 0$ and define $\bar{u}_t(x) = u_t(x)/u_s(x)$ so that

$$g_t = \bar{u}_t(x)^4 g_s \tag{242}$$

and $\bar{u}_s(x) \equiv 1$. Then define $\bar{v}_t(x)$ such that

$$\begin{cases} \Delta_{g_s}\bar{v}_t(x) & \equiv & 0 & \text{outside } \Sigma(t) \\ \bar{v}_t(x) & = & 0 & \text{on } \Sigma(t) \\ \lim_{x\to\infty}\bar{v}_t(x) & = & -e^{-(t-s)} & \end{cases} \tag{243}$$

and $\bar{v}_t(x) \equiv 0$ inside $\Sigma(t)$. Then what we want to show is

$$\bar{u}_t(x) = 1 + \int_s^t \bar{v}_r(x)dr \tag{244}$$

To prove the above equation, we observe that from Equations (239), (243), and (15) it follows that

$$v_t(x) = \bar{v}_t(x)\, u_s(x) \tag{245}$$

since $\lim_{x\to\infty} u_s(x) = e^{-s}$. Hence, since

$$u_t(x) = u_s(x) + \int_s^t v_r(x)dr \tag{246}$$

by Equation (16), dividing through by $u_s(x)$ yields Equation (244) as desired. Thus, we see that the rate of change of $g_t(x)$ at $t = s$ is a function of $\bar{v}_s(x)$ which in turn is just a function of $g_s(x)$ and the horizon $\Sigma(s)$. Hence, to understand properties of the flow we need only analyze the behavior of the flow for t close to zero, since any metric in the flow may be chosen to be the base metric. This point is used many times throughout the paper.

B. An example solution to the conformal flow of metrics

In this section we give the simplest example of a solution to the first order o.d.e. conformal flow of metrics defined by Equations (13), (14), (15), and (16). The initial metric in this example is the three dimensional, space-like Schwarzschild metric which represents a single, non-rotating black hole in vacuum. The Schwarzschild metrics are also very natural from a geometric standpoint as well since they are spherically symmetric and have zero scalar curvature.

Since the flow does not change the metric inside the horizon, we will define this metric to have its horizon as a boundary, which is always allowable. Then (M^3, g_0) will be defined to be isometric to $(\mathbf{R}^3 - B_{m/2}(0), \mathcal{U}_0(x)^4\delta_{ij})$, where

$$\mathcal{U}_0(x) = 1 + \frac{m}{2r} \tag{247}$$

where r is the distance from the origin in $(\mathbf{R}^3, \delta_{ij})$ and m is a positive constant equal to the mass of the black hole.

Next we define (M^3, g_t) to be isometric to $(\mathbf{R}^3 - B_{m/2}(0), \mathcal{U}_t(x)^4\delta_{ij})$, where

$$\mathcal{U}_t(x) = \begin{cases} e^{-t} + \frac{m}{2r}e^t, & \text{for } r \geq \frac{m}{2}e^{2t} \\ \sqrt{\frac{2m}{r}}, & \text{for } r < \frac{m}{2}e^{2t}. \end{cases} \tag{248}$$

We note that on this metric the outermost horizon (and also the outermost minimal area enclosure of the original horizon) is the coordinate sphere given by $r = \frac{m}{2}e^{2t}$, so by Equation (14) we define this horizon to be $\Sigma(t)$.

Next we recall from Equation (13) that $g_t = u_t(x)^4 g_0$. Hence, $u_t(x) = \mathcal{U}_t(x)/\mathcal{U}_0(x)$. Furthermore, by Equation (16) we must have $v_t(x) = \frac{d}{dt}u_t(x)$, so

$$v_t(x) = \frac{1}{\mathcal{U}_0(x)} \begin{cases} -e^{-t} + \frac{m}{2r}e^t, & \text{for } r \geq \frac{m}{2}e^{2t} \\ 0, & \text{for } r < \frac{m}{2}e^{2t}. \end{cases} \tag{249}$$

By Equation (239), $v_t(x)$ is harmonic on (M^3, g_0) outside $\Sigma(t)$ since $a + b/r$ is harmonic in $(\mathbf{R}^3, \delta_{ij})$. Then since $v_t(x)$ goes to $-e^{-t}$ at infinity, is continuous, and equals zero inside $\Sigma(t)$, it follows that Equation (15) is satisfied. Hence, (M^3, g_t) is a solution to the first order conformal flow of metrics defined by Equations (13), (14), (15), and (16).

This example is a good example to keep in mind when considering the main theorems of this paper. For example, we notice that by Definition 2 the total mass $m(t)$ of (M^3, g_t) equals m and hence is non-increasing as claimed in Section 7, and the area $A(t)$ of the horizon $\Sigma(t)$ in (M^3, g_t) is constant as claimed in Section 5. Also, we see that the diameter of $\Sigma(t)$ is growing exponentially as claimed in Theorem 12 and contains any given bounded set in a finite amount of time as claimed by Theorem 13.

Finally, we note that for all $t \geq 0$ in this example, (M^3, g_t) is isometric to a Schwarzschild metric of total mass m outside their respective horizons. Hence, even though the metric is shrinking pointwise, it is not changing at all outside its horizon, *after a reparametrization of the metric*. It is in this sense that Theorem 4 states that no matter what the initial metric is, it eventually converges to a Schwarzschild metric outside its horizon.

C. A nonlinear property of superharmonic functions in $\mathbf{R}^3$

In this appendix we present a nonlinear property of superharmonic functions in $\mathbf{R}^3$ which we needed in Section 10. However, this result is of independent interest and, as is proven in [9], directly implies the Riemannian Penrose inequality with suboptimal constant for manifolds which are conformal to $\mathbf{R}^3$. Furthermore, the following theorem can be generalized to higher dimensions (for certain powers which unfortunately are not applicable to the Riemannian Penrose inequality in higher dimensions) and is fully discussed and proven in [9].

Theorem 20. *There exists a constant $c > 0$ such that if $u(x)$ is any positive, continuous, superharmonic function in R^3 satisfying*

$$\int_{S_r(0)} u(x)^4 \, dA \geq a \tag{250}$$

for all $r > r_0$, then

$$u(x) \geq c\, a^{1/4} |x|^{-1/2} \tag{251}$$

for $|x| \geq r_0$.

Discussion of proof. We refer the reader to [9] (which is a joint work with Kevin Iga) for the details of the proof. In that paper we show that without loss of generality we may assume that $u(x)$ goes to zero at infinity. Then the next important step is a symmetrization argument to argue that without loss of generality we may also assume that the support of Δu is on the x-axis. Then it follows that

$$u(\vec{x}) = \int_0^\infty \frac{d\mu(t)}{|\vec{x} - (t, 0, 0)|} \tag{252}$$

where $\mu(t)$ is a positive measure on $[0, \infty)$. The remainder of the proof then involves converting inequality (250) to a lower bound on an integral expression of $d\mu(t)$ which is then used to prove inequality (251). q.e.d.

D. Lower bound for the capacity of a surface in $\mathbf{R}^3$ with bounded Willmore functional

In this appendix we consider surfaces Σ which are smooth, compact boundaries of open sets in $\mathbf{R}^3$ and which have bounded Willmore functional. That is, we will assume that

$$\int_\Sigma H^2 d\mu \leq w, \tag{253}$$

where H is the mean curvature (equal to the trace of the second fundamental form) and $d\mu$ is the area form of Σ.

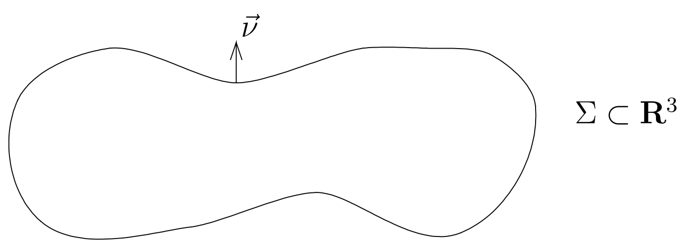

Figure 8.

Define the potential function f of Σ to be the constant function 1 inside Σ and the function satisfying

$$\begin{cases} f(x) = & 1 \quad \text{on } \Sigma \\ \Delta f \equiv & 0 \quad \text{outside } \Sigma \\ f \to & 0 \quad \text{at infinity} \end{cases} \tag{254}$$

outside Σ. Expanding in terms of spherical harmonics, we see that

$$f(x) = \frac{a}{r} + O\left(\frac{1}{r^2}\right). \tag{255}$$

We define a to be the capacity of both $f(x)$ and Σ. Also, since f is nonnegative, a is always nonnegative. Furthermore, we notice that

$$a = \lim_{r\to\infty} -\frac{1}{4\pi}\int_{S_r(0)} \frac{df}{dr} d\mu, \tag{256}$$

where $S_r(0)$ is the sphere of radius r centered around zero. But since $f(x)$ is harmonic, it follows from the divergence theorem that if we perform the above integral over any surface enclosing Σ, we will get the same result. Thus,

$$a = -\frac{1}{4\pi}\int_\Sigma \frac{df}{d\vec{\nu}}d\mu, \tag{257}$$

where $\vec{\nu}$ is the outward pointing unit normal vector to Σ. The goal of this appendix is to find a lower bound for a in terms of the area of Σ and the Willmore bound w.

Since f is harmonic off of Σ, the support of the distribution Δf is on Σ. In fact

$$\Delta f(\phi) = \int_{\mathbf{R}^3} f\Delta\phi = \int_\Sigma \phi\frac{df}{d\vec{\nu}}d\mu \tag{258}$$

where $\frac{df}{d\vec{\nu}}$ is defined to be the outward directional derivative of f (which does not equal the inward directional derivative which is zero). Then since f equals Δf convolved with the Green's function $-\frac{1}{4\pi|x|}$, we get that

$$f(y) = -\frac{1}{4\pi}\int_\Sigma \frac{df}{d\vec{\nu}}\cdot\frac{1}{|y-x|}dx, \tag{259}$$

where dx is the area form of Σ with respect to the variable x. In a moment dy will be the area form of Σ with respect to the variable y. Then using the fact that $f(y)=1$ on Σ, we find that

$$|\Sigma| = \int_\Sigma f(y)dy = -\frac{1}{4\pi}\int_\Sigma \frac{df}{d\vec{\nu}}\cdot\left(\int_\Sigma \frac{1}{|y-x|}dy\right)dx. \tag{260}$$

The next step is to find an upper bound for $\int_\Sigma \frac{1}{|y-x|}dy$. To do this, we need the following lemma which follows from Equation 16.34 in [21] (when R is chosen to go to infinity). We note that the definition of mean curvature in [21] is half that of ours.

Lemma 23. *Given a surface Σ which is a smooth, compact boundary of an open set in $\mathbf{R}^3$,*

$$|\Sigma\cap B_r(x)| \le \left(\frac{3}{4}\int_\Sigma H^2 d\mu\right)r^2 \tag{261}$$

for all $r>0$ and all $x\in\Sigma$.

Hence, in our case,

$$|\Sigma \cap B_r(x)| \le \left(\frac{3}{4}w\right) r^2, \tag{262}$$

for any $x \in \Sigma$. Then since $\frac{1}{|y-x|}$ is maximized when x and y are closest together, we get that $\int_\Sigma \frac{1}{|y-x|}dy$ is maximized subject to the constraint in Equation (262) when $|\Sigma \cap B_r(x)| = (\frac{3}{4}w)r^2$, which gives us

$$\int_\Sigma \frac{1}{|y-x|}dy \le \int_0^{\sqrt{\frac{4|\Sigma|}{3w}}} \frac{1}{r}\cdot 2\left(\frac{3}{4}w\right) r dr = (3w)^{1/2}|\Sigma|^{1/2}. \tag{263}$$

Thus, from Equation (260),

$$|\Sigma| \le -\frac{1}{4\pi}\int_\Sigma \frac{df}{d\vec{\nu}}dx \cdot (3w)^{1/2}|\Sigma|^{1/2}, \tag{264}$$

so that we have

$$a = -\frac{1}{4\pi}\int_\Sigma \frac{df}{d\vec{\nu}}d\mu \ge (3w)^{-1/2}|\Sigma|^{1/2}. \tag{265}$$

Thus, in summary, we have the following theorem.

Theorem 21. *Let Σ be a smooth, compact boundary of an open set in $\mathbf{R}^3$ with $\int_\Sigma H^2 d\mu \le w$. Let $f(x)$ be the harmonic function equal to one on Σ and going to zero at infinity. Then the capacity*

$$a \equiv -\frac{1}{4\pi}\int_\Sigma \frac{df}{d\vec{\nu}}d\mu \ge (3w)^{-1/2}|\Sigma|^{1/2}. \tag{266}$$

We need this lower bound on the capacity of $\Sigma(t)$ in Section 11.

E. Regularity of the horizons $\Sigma^\epsilon(t)$

In this appendix we compute upper bounds on the $C^{k,\alpha}$ "norms" (see Definition 33) of the surfaces $\Sigma^\epsilon(t)$, for some $\alpha \in (0,1)$. We will also compute upper bounds on the $C^{k,\alpha}$ norms of the metrics g_t^ϵ outside $\Sigma^\epsilon(t)$. These bounds are independent of ϵ, allowing us to conclude in Section 4 that that the horizons $\Sigma(t)$ are smooth and the metrics g_t are smooth outside $\Sigma(t)$.

The critical step will be to achieve a uniform (independent of ϵ) bound on the $C^{1,\alpha}$ norms of the surfaces $\Sigma^\epsilon(t)$ in the coordinate charts, for $t \in [0, T]$. Then by Equations (23) and (24), we will also have a uniform bound on the $C^{1,\alpha}$ norms of the metrics $g_t^\epsilon = u_t^\epsilon(x)^4 g_0$ outside $\Sigma^\epsilon(t)$. Schauder estimates applied to the minimal surface equation will then give us a uniform bound on the $C^{2,\alpha}$ norms of the surfaces $\Sigma^\epsilon(t)$. Repeating this bootstrapping argument yields the desired uniform bounds on the $C^{k,\alpha}$ norms of the surfaces $\Sigma^\epsilon(t)$ and the metric g_t^ϵ outside $\Sigma^\epsilon(t)$.

We note that it follows from the definition of $\Sigma^\epsilon(t)$ given in Equation (22) (and from Equation (23)) that $\Sigma^\epsilon(t)$ globally minimizes area among surfaces in $\mathcal{S}$ (defined in Section 2) which contain the original horizon Σ_0. Then as in Section 6, let us define $M^3_{\Sigma_0}$ to be the closed region of M^3 which is outside (or on) Σ_0, and define $(\overline{M}^3_{\Sigma_0}, \bar{g}_0)$ to be two distinct copies of $(M^3_{\Sigma_0}, g_0)$ identified along Σ_0. Hence, $(\overline{M}^3_{\Sigma_0}, \bar{g}_0)$ has a reflection symmetry which keeps Σ_0 fixed.

With out loss of generality, let us now replace (M^3, g_0) with $(\overline{M}^3_{\Sigma_0}, \bar{g}_0)$ for the remainder of this section. We can do this since the portion of (M^3, g_0) which is inside Σ_0 does not affect the conformal flow of metrics defined in Equations (21), (22), (23), and (24). It is true that this operation means that the new (M^3, g_0) will not be smooth along Σ_0, but this turns out not to be important. Furthermore, the big advantage (which we leave to the reader to check) is that now $\Sigma^\epsilon(t)$ globally minimizes area among *all* surfaces in $\mathcal{S}$, which is a fact we will use in the next lemma. However, because of this construction, all of the constants we will be defining will depend on Σ_0, although this is not a problem.

We will continue with the same notation as defined in Section 4. Since the initial manifold (M^3, g_0) is smooth (except possibly on Σ_0 because of the above construction, which we will ignore) and harmonically flat at infinity, it may be covered by a finite number of smooth coordinate charts $\{\mathcal{C}_i\}$, where each harmonically flat end (of which there can only be a finite number) has $\mathbf{R}^3 \backslash B_1(0)$ as an asymptotically flat coordinate chart and all of the other coordinate charts are different copies of $B_1(0) \subset \mathbf{R}^3$. For the rest of this appendix we will work inside the coordinate charts $\{\mathcal{C}_i\}$ which have the standard $\mathbf{R}^3$ metric.

Definition 30. Let $\text{dis}_i(x, S)$ be the infimum (which could equal infinity) of the lengths of all of the paths in $\mathcal{C}_i$ between the point x and the set S with respect to the coordinate chart metric in $\mathcal{C}_i$. Also, let $\text{dis}_{g_0}(x, S)$ be the infimum of the lengths of all of the paths in M^3 between the point x and the set S with respect to the metric g_0.

Definition 31. Let $X^\epsilon(t)$ be the three dimensional open region inside $\Sigma^\epsilon(t)$ so that

$$\Sigma^\epsilon(t) = \partial X^\epsilon(t). \tag{267}$$

The region $X^\epsilon(t)$ always exists since $\Sigma^\epsilon(t) \in \mathcal{S}$.

Lemma 24. *There exists a constant $c_1 > 0$ depending only on T, Σ_0, g_0, and the choice of coordinate charts $\{\mathcal{C}_i\}$ such that for all i and all $x_0 \in \Sigma^\epsilon(t) \cap \mathcal{C}_i$,*

$$|X^\epsilon(t) \cap S_r(x_0)|_{\mathcal{C}_i} \geq c_1 r^2 \tag{268}$$

for all $r \in (0, dis_i(x_0, \partial\mathcal{C}_i))$ and $t \in [0,T]$.

Proof. Let $S_r(x_0)$ and $B_r(x_0)$ respectively be the coordinate 2-sphere and closed coordinate 3-ball of radii r centered at $x_0 \in \Sigma^\epsilon(t) \cap \mathcal{C}_i$. Define

$$A(r) = |\Sigma^\epsilon(t) \cap B_r(x_0)|_{\mathcal{C}_i}, \tag{269}$$

$$L(r) = |\Sigma^\epsilon(t) \cap S_r(x_0)|_{\mathcal{C}_i}, \tag{270}$$

$$\overline{A}(r) = |X^\epsilon(t) \cap S_r(x_0)|_{\mathcal{C}_i}. \tag{271}$$

Let us also define $f(a) = l$ for $a \in [0, 4\pi]$, where l is the minimum length required to enclose a region of area a in the unit sphere S^2. Hence, for small a, $f(a) \approx \sqrt{4\pi a}$.

Finally, we let γ_1 be the infimum of the smallest eigenvalue and let γ_2 be the supremum of the largest eigenvalue of the metric g_0 over every point in all of the coordinate charts $\{\mathcal{C}_i\}$, and then define $\gamma = \gamma_1/\gamma_2$. Since the coordinate charts are smooth and g_0 approaches the standard metric δ_{ij} in the ends of the noncompact coordinate charts, $\gamma > 0$. Then since

$$1 \geq u_t^\epsilon(x) \geq (1-\epsilon)^{\lfloor \frac{t}{\epsilon} \rfloor} \approx e^{-t} \tag{272}$$

by the maximum principle and Equations (25) and (26), it then follows that the corresponding ratio of eigenvalues for the metric g_t^ϵ in the coordinate charts $\{\mathcal{C}_i\}$ is at least γ_T, where

$$\gamma_T = (1-\epsilon)^{\lfloor \frac{4T}{\epsilon} \rfloor} \gamma > 0, \tag{273}$$

$t \in [0,T]$, and as usual we are requiring t to be an integral multiple of ϵ. In the limit as ϵ goes to zero, we note that γ_T approaches $e^{-4T}\gamma$, which is positive.

Now we are ready to begin the actual proof of the lemma. Since $\Sigma^\epsilon(t)$ is globally area minimizing among surfaces in $\mathcal{S}$ with respect to the metric g_t^ϵ, we know that it has area in (M^3, g_t^ϵ) less than or equal to the areas of the two comparison surfaces $\partial(X^\epsilon(t)\backslash B_r(x_0))$ and $\partial(X^\epsilon(t)\cup B_r(x_0))$. Putting this into terms of the above definitions, this yields

$$\gamma_T A(r) \le \overline{A}(r) \le 4\pi r^2 - \gamma_T A(r). \tag{274}$$

Hence, it follows from the above equation that

$$f\left(\frac{\overline{A}(r)}{r^2}\right) \ge f\left(\frac{\gamma_T A(r)}{r^2}\right) \tag{275}$$

since $f(a) = f(4\pi - a)$ and is monotone increasing from 0 to 2π. Furthermore, it follows from the definition of f that

$$L(r) \ge f\left(\frac{\overline{A}(r)}{r^2}\right) r. \tag{276}$$

Hence, since from multivariable calculus we have that $A'(r) \ge L(r)$, we deduce that

$$A'(r) \ge f\left(\frac{\gamma_T A(r)}{r^2}\right) r. \tag{277}$$

It is then straightforward to show that inequalities (274) and (277) imply inequality (268) with $c_1 = 4\gamma_T^2/\pi$, proving the lemma. q.e.d.

Corollary 11. *Let $w(x)$ be any nonnegative harmonic function in (M^3, g_0) defined outside $\Sigma^\epsilon(t)$ which equals zero on $\Sigma^\epsilon(t)$. Define $w(x)$ to be identically zero inside $\Sigma^\epsilon(t)$. Then there exists a constant $c_2 \in (0,1)$ depending only on T, Σ_0, g_0, and $\{\mathcal{C}_i\}$ such that for all i and all $x_0 \in \Sigma^\epsilon(t) \cap \mathcal{C}_i$,*

$$\sup_{S_{r/2}(x_0)} w(y) \le c_2 \sup_{S_r(x_0)} w(y) \tag{278}$$

for all $r \in (0, \mathrm{dis}_i(x_0, \partial\mathcal{C}_i))$ and $t \in [0,T]$.

Proof. The function $w(x)$ is subharmonic in (M^3, g_0), and hence is bounded above by the harmonic function $h(x)$ defined in the coordinate ball $B_r(x_0)$ with Dirichlet boundary data $h(x) = \sup_{S_r(x_0)} w(y)$ on $S_r(x_0)\backslash X^\epsilon(t)$ and $h(x) = 0$ on $S_r(x_0) \cap X^\epsilon(t)$. The corollary then follows by estimating $h(x)$ on $S_{r/2}(x_0)$ using the Poisson kernel and Lemma 24. q.e.d.

Corollary 12. *There exist constants* c_3*,* c_4*, and* $\beta \in (0,1)$ *depending only on* T*,* Σ_0*,* g_0*, and* $\{\mathcal{C}_i\}$ *such that*

$$|v_t^\epsilon(x)| \le c_3 \mathrm{dis}_{g_0}(x, X^\epsilon(t))^\beta \tag{279}$$

and for all $y \in \mathcal{C}_i$ *with* $\mathrm{dis}_{g_0}(x,y) \le \mathrm{dis}_{g_0}(x, X^\epsilon(t))$

$$|v_t^\epsilon(x) - v_t^\epsilon(y)| \le c_4 \mathrm{dis}_{g_0}(x,y)^\beta \tag{280}$$

for all $t \in [0,T]$

Proof. First we note that since distances with respect to the coordinate chart metrics and g_0 are within a bounded factor of each other, we only need to prove each of the above inequalities in each coordinate chart.

Inequality (279) follows from the definition of $v_t^\epsilon(x)$ given in Equation (23) and from recursively applying Corollary 11. Inequality (279) then implies the interior gradient estimate

$$|\nabla v_t^\epsilon(x)| \le c_5 \, \mathrm{dis}_{g_0}(x, X^\epsilon(t))^{\beta-1}, \tag{281}$$

which, when integrated along the straight path connecting x and y, implies inequality (280). q.e.d.

Corollary 13. *Let* $c_6 = 2\max(c_3, c_4)$*. Then*

$$|v_t^\epsilon(x) - v_t^\epsilon(y)| \le c_6 \mathrm{dis}_{g_0}(x,y)^\beta. \tag{282}$$

for all x *and* y *and for* $t \in [0,T]$*.*

Proof. Let $r_x = \mathrm{dis}_{g_0}(x, X^\epsilon(t))$ and $r_y = \mathrm{dis}_{g_0}(y, X^\epsilon(t))$, and let B now denote geodesic balls in (M^3, g_0). Then we consider two cases.

Case 1. Suppose $B_{r_x}(x) \cap B_{r_y}(y) = \varnothing$. Then inequality (282) follows from inequality (279) and the triangle inequality.

Case 2. Suppose $B_{r_x}(x) \cap B_{r_y}(y) \neq \varnothing$. Then choose $z \in B_{r_x}(x) \cap B_{r_y}(y)$ which is on the length minimizing geodesic connecting the points x and y. Then inequality (282) follows from the triangle inequality and inequality (280) applied to the points x and z and to the points y and z. q.e.d.

Definition 32. Let $w(x)$ be any C^k function defined on (M^3, g_0). Then we define the following norm and seminorms (denoted by brackets), all of which depend on our choice of coordinate charts $\{\mathcal{C}_i\}$, $\alpha \in (0,1)$, and $k = 0, 1, 2, \dots$

$$[w]_{k;\Omega} = \sup_i \sup_{x \in \mathcal{C}_i} \sup_{|\gamma|=k} |D^\gamma w(x)|, \tag{283}$$

$$[w]_{k,\alpha;\Omega} = \sup_i \sup_{x \neq y} \sup_{|\gamma|=k} \left\{ \frac{|D^\gamma w(x) - D^\gamma w(y)|}{|x-y|^\alpha} \;\middle|\; x, y \in \mathcal{C}_i \right\}, \tag{284}$$

and

$$||w||_{C^{k,\alpha}(\Omega)} = [w]_{k,\alpha;\Omega} + \sum_{j=1}^{k} [w]_{j;\Omega}, \tag{285}$$

where we also require that $x, y \in \Omega \subset M^3$ in the above equations.

Hence, from Corollary 13 and Equation (24), it follows that

$$||u_t^\epsilon||_{C^{0,\beta}(M^3)} \leq c_7 \tag{286}$$

for $t \in [0, T]$, where c_7 depends only on T, Σ_0, g_0, and $\{\mathcal{C}_i\}$.

Definition 33. Let S be any smooth surface in (M^3, g_0) which is the boundary of a region. Let η be any vector field defined on all of (M^3, g_0) such that on S it equals the outward pointing unit normal vector of S. Let $\eta = (\eta_1, \eta_2, \eta_3)$ be the pull back of η on each coordinate chart. We note that Definition 32 can be used for vector valued functions as well as real valued functions. Then abusing notation slightly (since the following is not a norm), we define

$$||S||_{C^{k,\alpha}} = \inf_\eta ||\eta||_{C^{k-1,\alpha}(S)} \tag{287}$$

for $\alpha \in (0,1)$ and $k = 1, 2, \dots$

The next part of the proof is to use inequality (286) and the fact that $\Sigma^\epsilon(t)$ minimizes area in $(M^3, g_t^\epsilon = u_t^\epsilon(x)^4 g_0)$ to conclude that the $\Sigma^\epsilon(t)$ are uniformly $C^{1,\beta/4}$ surfaces. By this we mean that we will find an upper bound on $||\Sigma^\epsilon(t)||_{C^{1,\beta/4}}$ which is independent of ϵ. Conveniently, the main theorem we need, Theorem 22, was essentially proved already by De Giorgi [15] to understand the regularity of codimension one minimal surfaces in $\mathbf{R}^n$, and is summarized in [39]. However, the application of this theorem to this setting is quite interesting, so we summarize the arguments below.

Definition 34. Let X and Y be regions in (M^3, g_0) (and hence also in the coordinate charts $\{\mathcal{C}_i\}$) with smooth boundaries of finite area. Then we define

$$\psi_{g_0}(X, x, r) = \\ |\partial X \cap B_r(x)|_{g_0} - \inf\left\{|\partial Y \cap B_r(x)|_{g_0} \mid Y = X \text{ outside } B_r(x)\right\}$$

for $x \in \mathcal{C}_i$ and $r \in (0, \mathrm{dis}_i(x, \partial\mathcal{C}_i))$, where $B_r(x)$ is the closed ball of radius r in the coordinate chart $\mathcal{C}_i$.

Hence, the function $\psi_{g_0}(X, x, r)$ can be thought of as the excess area of ∂X in (M^3, g_0) in the coordinate ball $B_r(x)$. We note that the above definition for $\psi_{g_0}(X, x, r)$ in the coordinate chart $\mathcal{C}_i$ equals $\psi(X, x, r)$ defined in [39] in the special case that g_0 equals the coordinate chart metric.

Definition 35. Let X be a region in (M^3, g_0) which has a smooth boundary of finite area. Then following [39], we define X to be $(K, \lambda)_{g_0}$-*minimal* in $\{\mathcal{C}_i\}$ if and only if

$$\psi_{g_0}(X, x, r) \leq K r^{2+\lambda} \tag{288}$$

for all i, $x \in \mathcal{C}_i$, and $r \in (0, \mathrm{dis}_i(x, \partial\mathcal{C}_i))$.

Again, we note that a region which is $(K, \lambda)_{g_0}$-minimal in the coordinate chart $\mathcal{C}_i$ as defined above is (K, λ)-minimal as defined in [39] in the special case that g_0 equals the coordinate chart metric. The usefulness of the above definition can be seen in the next lemma and theorem.

Lemma 25. *Suppose X is a region in (M^3, g_0) which has a smooth boundary of finite area and which is $(K, \lambda)_{g_0}$-minimal in $\{\mathcal{C}_i\}$. We note that the metric g_0 and the coordinate charts $\{\mathcal{C}_i\}$ are assumed to be smooth. Then for all $\epsilon > 0$, there exists a $\delta > 0$ (depending only on K, λ, g_0, and $\{\mathcal{C}_i\}$) such that for all i and $x_0 \in \partial X \cap \mathcal{C}_i$,*

$$\omega_i(X, x_0, r) \equiv |D\chi_X|(B_r(x_0)) - |D\chi_X(B_r(x_0))| < \epsilon\, r^2 \tag{289}$$

for all $r \in (0, \min(\delta, \mathrm{dis}_i(x_0, \partial\mathcal{C}_i))]$.

Proof. We note that we are adopting the notation of [39], so that χ_X is the characteristic function of the region X and $D\chi_X$ is the distributional derivative (with respect to the coordinate chart) of that characteristic function. Hence, ω_i equals zero for regions with flat boundaries in the coordinate chart $\mathcal{C}_i$, and in general can be thought of as a way of measuring how far a boundary is from being flat inside the coordinate ball $B_r(x_0)$.

For convenience, we will also assume that g_0 equals the standard $\mathbf{R}^3$ metric in each of the finite number of coordinate charts. Then since g_0 is smooth and the proof uses a blow up argument, it is easy to adapt the proof we give here to the general case.

We proceed with a proof by contradiction. Suppose that for some $\epsilon > 0$, there existed a counterexample region X_δ for all $\delta > 0$ such that

$$\omega_i(X_\delta, x_0, r_\delta) \geq \epsilon\, r_\delta^2 \tag{290}$$

for some i and $x_0 \in \partial X_\delta \cap \mathcal{C}_i$ and for some $r_\delta \in (0, \min(\delta, \mathrm{dis}_i(x_0, \partial\mathcal{C}_i)]$. We may as well think of each region X_δ being in the same $\mathbf{R}^3$, and for convenience we translate each region by $-x_0$ (which depends on δ) so that $0 \in \partial X_\delta$. Let $\overline{X}_\delta$ be X_δ rescaled by a factor of $1/r_\delta$. Then we have that

$$\omega(\overline{X}_\delta, 0, 1) \geq \epsilon, \tag{291}$$

where ω is defined for regions in $\mathbf{R}^3$. Furthermore, $\overline{X}_\delta$ is $(Kr_\delta^\lambda, \lambda)$-minimal in $B_{1/r_\delta}(0) \subset \mathbf{R}^3$.

Let $\overline{X}$ be a limit region of $\{\overline{X}_\delta\}$, in the sense that the characteristic function of $\overline{X}$ is the L^1 limit of the characteristic functions of $\{\overline{X}_{\delta_i}\}$, for some sequence of $\{\delta_i\}$ converging to zero. The fact that such a limit region exists is proven on p. 70 of [39], and relies on the fact that the areas of the boundaries of $\{\overline{X}_\delta\}$ are uniformly bounded above, which follows from Theorem 1 of Section 2.5.3 of [39]. Furthermore, by the lower semicontinuity of the area functional as defined in [39], it follows that $\overline{X}$ has minimal boundary. Since the only minimizing boundaries in R^3 are planes, $\partial\overline{X}$ must be a plane going through the origin, which we may as well assume is the x-y plane in $\mathbf{R}^3$ after a suitable rotation.

Let π be the projection of $\mathbf{R}^3$ to the x-y axis, and let $\vec{\nu}$ be the outward pointing normal vector of $\partial\overline{X}_{\delta_i}$. Then

$$\lim_{\delta_i \to 0} \left| \int_{\partial\overline{X}_{\delta_i} \cap B_1(0)} \nu_z \right| = \lim_{\delta_i \to 0} |\pi(\partial\overline{X}_{\delta_i} \cap B_1(0))| = \lim_{\delta_i \to 0} |\partial\overline{X}_{\delta_i} \cap B_1(0)| \tag{292}$$

since $\{\overline{X}_\delta\} \to \overline{X}$ and the areas converge as well since each $\overline{X}_\delta$ is $(Kr_\delta^\lambda, \lambda)$-minimal. The above equation then implies that $\lim_{\delta_i \to 0} \omega(\overline{X}_{\delta_i}, 0, 1) = 0$, contradicting inequality (291) and proving the lemma. q.e.d.

Theorem 22. *Suppose X is a region in M^3 which has a smooth boundary of finite area and which is $(K,\lambda)_{g_0}$-minimal in $\{\mathcal{C}_i\}$. Then ∂X is a $C^{1,\lambda/4}$ surface and*

$$||\partial X||_{C^{1,\lambda/4}} \leq \bar{k}, \tag{293}$$

where $\bar{k}$ depends only on K, λ, g_0, and $\{\mathcal{C}_i\}$.

Proof. We restrict our attention to each coordinate chart $\mathcal{C}_i$ one at a time, and first consider the case that g_0 equals the coordinate chart metric. In this case, the theorem follows directly from the proof of Theorem 1 of Section 2.5.4 of [39] and Lemma 25. Then it is then a somewhat long but straightforward task to adapt the relevant theorems of [39] to verify that Theorem 1 of Section 2.5.4 of [39] is still true if we replace the standard metric on $\mathbf{R}^3$ with any smooth fixed metric g_0.

q.e.d.

Lemma 26. *The region $X^\epsilon(t)$, which we recall is defined to be the region inside $\Sigma^\epsilon(t)$, is a $(K,\beta)_{g_0}$-minimal set in $\{\mathcal{C}_i\}$, for $t \in [0,T]$, where K depends only on T, Σ_0, g_0, and $\{\mathcal{C}_i\}$.*

Proof. We need to estimate

$$\begin{aligned}(294)\quad &\psi_{g_0}(X^\epsilon(t),x,r) = \\ &|\partial X^\epsilon(t)\cap B_r(x)|_{g_0} - \inf\left\{|\partial Y \cap B_r(x)|_{g_0} \mid Y = X^\epsilon(t) \text{ outside } B_r(x)\right\}\end{aligned}$$

from above. Well, since $\partial X^\epsilon(t)$ has minimal area in (M^3, g_t^ϵ), we have that

$$|\partial Y \cap B_r(x)|_{g_t^\epsilon} \geq |\partial X^\epsilon(t)\cap B_r(x)|_{g_t^\epsilon}, \tag{295}$$

which gives us

$$|\partial Y \cap B_r(x)|_{g_0} \geq |\partial X^\epsilon(t)\cap B_r(x)|_{g_0}\left(\frac{u_{\min}}{u_{\max}}\right)^4 \tag{296}$$

where $u_{\min}$ and $u_{\max}$ are respectively the minimum and maximum values of $u_t^\epsilon(x)$ on the closed coordinate ball $B_r(x)$. Hence,

$$\psi_{g_0}(X^\epsilon(t),x,r) \leq |\partial X^\epsilon(t)\cap B_r(x)|_{g_0}\left[1-\left(\frac{u_{\min}}{u_{\max}}\right)^4\right]. \tag{297}$$

The lemma then follows from Equations (272), (274), and (286). q.e.d.

Corollary 14. *The surface $\Sigma^\epsilon(t)$ is a $C^{1,\beta/4}$ surface, and*

$$||\Sigma^\epsilon(t)||_{C^{1,\beta/4}} \leq \bar{c}_1, \tag{298}$$

for $t \in [0,T]$, where $\bar{c}_1$ depends only on T, Σ_0, g_0, and $\{\mathcal{C}_i\}$.

Proof. Follows directly from Theorem 22 and Lemma 26. q.e.d.

The remainder of the arguments presented in this appendix are mostly standard applications of [21]. For convenience, we now let $\alpha = \beta/4$.

Lemma 27. *For $k \geq 1$,*

$$||\Sigma^\epsilon(s)||_{C^{k,\alpha}} \leq \bar{c}_k \text{ for all } s \in [0,T] \quad \Rightarrow \quad ||u_t^\epsilon(x)||_{C^{k,\alpha}(M^3 \setminus X^\epsilon(t))} \leq \bar{\bar{c}}_k \tag{299}$$

for $t \in [0,T]$, where $\bar{\bar{c}}_k$ depends only on $\bar{c}_k$, T, Σ_0, g_0, and $\{\mathcal{C}_i\}$.

Proof. By the hypothesis of the lemma, the definition of $v_t^\epsilon(x)$ given in Equation (23), and standard theorems found in [21] and [51] (for $k = 1$), we get an upper bound on $||v_s^\epsilon(x)||_{C^{k,\alpha}(M^3 \setminus X^\epsilon(t))}$. The lemma then follows from Equation (24). q.e.d.

Lemma 28. *For $k \geq 1$ and $t \in [0,T]$ an integer multiple of ϵ,*

$$||u_t^\epsilon(x)||_{C^{k,\alpha}(M^3 \setminus X^\epsilon(t))} \leq \bar{\bar{c}}_k, \quad \Rightarrow \quad ||\Sigma^\epsilon(t)||_{C^{k+1,\alpha}} \leq \bar{c}_{k+1} \tag{300}$$

where $\bar{c}_{k+1}$ depends only on $\bar{\bar{c}}_k$, T, Σ_0, g_0, and $\{\mathcal{C}_i\}$.

Proof. Since $\Sigma^\epsilon(t)$ minimizes area in (M^3, g_t) and since it can be viewed as a graph of a function of two variables over a uniformly large domain by Corollary 14, we can apply Schauder estimates to the minimal surface equation to prove the lemma.

Let the graph of $x_3 = f(x_1, x_2)$ represent the minimal surface $\Sigma^\epsilon(t)$ in the coordinate chart $\mathcal{C}_i \subset \mathbf{R}^3$ with metric $g_{ij}(x_1, x_2, x_3)$ given by the metric $g_t^\epsilon = u_t^\epsilon(x)^4 g_0$. Then we note that the minimal surface equation in this setting is

$$a^{ij}(x) D_{ij} f(x) = p(x) \tag{301}$$

where

$$a^{ij} = G^{ij} - \frac{G^{i\alpha} N_\alpha G^{j\beta} N_\beta}{N^t G N}, \tag{302}$$

$$p = \frac{1}{2} N^t G_3 N + \left(\frac{D_i(G^{\alpha\beta}) N_\alpha N_\beta G^{i\lambda}}{2 N^t G N} - D_i(G^{i\lambda}) \right) N_\lambda, \tag{303}$$

where

$$G^{\alpha\beta} = g_{\alpha+1,\beta+1}g_{\alpha+2,\beta+2} - g_{\alpha+1,\beta+2}g_{\alpha+2,\beta+1} \tag{304}$$

is the determinant of the $\alpha\beta$ minor matrix and the subscript addition in the above equation is modulo 3,

$$G_3^{\alpha\beta} = \frac{\partial}{\partial x_3} G^{\alpha\beta}, \tag{305}$$

and

$$N = (D_1 f, D_2 f, -1), \tag{306}$$

where $x = (x_1, x_2)$ and naturally $x_3 = f(x_1, x_2)$ in the above equations, and we are using the convention that Latin indices are summed from 1 to 2 while Greek indices are summed from 1 to 3.

We also note that since

$$\Lambda_1 |v|^2 \leq g_{\alpha\beta} v^\alpha v^\beta \leq \Lambda_2 |v|^2 \tag{307}$$

for some positive Λ_1 and Λ_2 which depend on t by Equation (272), and since

$$|Df| \leq B \tag{308}$$

over a uniformly sized domain by Corollary 14, then it can be shown that

$$a^{ij} v_i v_j \geq \lambda |v|^2, \tag{309}$$

where

$$\lambda = \frac{\Lambda_1^8}{4\Lambda_2^6 (1 + B^2)^2}. \tag{310}$$

Hence, since the a_{ij} are uniformly positive definite and $a^{ij}(x)$ and $p(x)$ involve only first order derivatives of $f(x)$ and the metric g_{ij}, the lemma follows from Corollary 14 and bootstrapping with Schauder estimates applied to Equation (301). q.e.d.

Then combining Corollary 14 with Lemmas 27 and 28, we get the main result in this appendix.

Corollary 15. *For $t \in [0, T]$, for some $\alpha \in (0, 1/4)$, and for $k \geq 1$, we have uniform bounds on the $C^{k,\alpha}$ norms of $u_t^\epsilon(x)$ outside $\Sigma^\epsilon(t)$ and on the surfaces $\Sigma^\epsilon(t)$, as defined in Definitions* 32 *and* 33. *Furthermore, these bounds depend only on k, T, Σ_0, g_0, and $\{\mathcal{C}_i\}$, and are independent of ϵ.*

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