

## CONES EMBEDDED IN HYPERBOLIC MANIFOLDS

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**Abstract**

We show that the existence of a maximal embedded tube in a hyperbolic  $n$ -manifold implies the existence of a certain conical region. One application is to establish a lower bound on the volume of the region outside the tube, thereby improving estimates on volume and estimates on lengths of geodesics in small volume hyperbolic 3-manifolds. We also provide new bounds on the injectivity radius and diameter of an  $n$ -manifold.

**1. Introduction**

Lately, there has been much interest in tubes which embed in hyperbolic 3-manifolds. Among the results addressing either the consequences of or existence of such tubes are [3]–[8]. Many of these results establish properties of the smallest volume hyperbolic orientable 3-manifold. The orientability condition is included since one of the main results [5] requires it to prove the existence of a tube of radius  $\frac{\log 3}{2}$ . Hence we shall take the definition of a manifold to include orientability. Specifically, Gabai, Meyerhoff, and Thurston [5] prove:

**Theorem 1.1** ([5]). *Every closed orientable hyperbolic 3-manifold except Vol3 contains an embedded tube of radius at least  $0.52959\dots$  about its shortest geodesic. If the shortest geodesic has length at most  $1.0595\dots$ , there is a tube of radius at least  $\frac{\log 3}{2}$  about it.*

We prove that if there is a maximal embedded tube of radius  $r$ , then there is another region  $W$ , defined later, which also embeds. This region is basically the union of two cones, whose shapes are determined by  $r$ . In addition, we prove that a certain portion of the region  $W$  lies outside of the tube of radius  $r$ .

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In each of [6], [8], and [4], lower bounds on tube volume were used to provide lower bounds on manifold volume. As we provide a lower bound on the volume outside the tube, these earlier results may be augmented. In particular, this author showed [8] that any orientable hyperbolic 3-manifold has volume at least 0.276, which may now be improved to 0.28 and Gabai, Meyerhoff and Milley [4] showed that the smallest volume orientable hyperbolic 3-manifold contains no geodesic of length less than 0.069, which we may now improve to 0.09.

The techniques we use are not specific to three dimensions. Thus, we develop the result in arbitrary dimensions, even though most of our applications are specifically 3-dimensional. We do provide a few simple  $n$ -dimensional applications. Using the relative simplicity of the region  $W$ , we are able to determine the radius of the largest ball which fits inside  $W$ . This ball then embeds in the manifold. This allows us to place a lower bound on the radius of the largest ball embedded in a hyperbolic manifold.

Finally, we locate a point in  $W$  for which we can determine the distance to the geodesic. This then allows us to place a lower bound on the diameter of the manifold.

## 2. Establishing an Embedding

Let  $M^n$  be a hyperbolic  $n$ -manifold with fundamental group  $\Gamma$ . It is known that if  $M$  is closed then the shortest geodesic in  $M$  does not intersect itself and that there is thus an embedded tube about this geodesic. Even if  $M$  is not closed, there is often an embedded tube about some geodesic. Choosing some such geodesic, let  $r$  be the radius of the maximal embedded tube. We now change our viewpoint to that of the universal cover of  $M$ ,  $\mathbb{H}^n$ . The group  $\Gamma$  can be considered as a subgroup of  $\text{Isom}^+(\mathbb{H}^n)$ . The geodesic in  $M$  may be lifted to a line in  $\mathbb{H}^n$  and the maximal tube may be lifted to the set of points within  $r$  of this line. The  $\Gamma$  action will mean that there are many choices of how to perform this lift. Let  $T_1$  be some such lift of the maximal embedded tube and let  $T_2$  be another such lift which intersects  $T_1$  in a single point. We will denote the lines at the core of  $T_1$  and  $T_2$  as  $l_1$  and  $l_2$ , respectively. As  $T_1$  and  $T_2$  intersect at a single point,  $l_1$  and  $l_2$  are at a distance of  $2r$  from one another. Let  $q_1$  and  $q_2$  be the points at which their common perpendicular intersects  $l_1$  and  $l_2$ , respectively. Let  $B_i$  be a ball of radius  $r$  about  $q_i$ . We construct a set which we shall

show packs  $\mathbb{H}^3$  under the action of  $\Gamma$ .

**Definition 2.1.** Let  $X_1$  be the union of all line segments having one endpoint at  $q_1$  and the other endpoint in  $B_2$ . Let  $X_2$  be the union of all line segments having one endpoint at  $q_2$  and the other endpoint in  $B_1$ . Define the set  $W = X_1 \cap X_2$ .

The set  $W$  will be two identical cones which have been attached along their bases. To be specific, we are defining a cone in  $\mathbb{H}^n$  to be formed by an  $n - 1$  dimensional ball and a line segment, called the altitude, with one endpoint lying at the center of the ball such that the line segment is perpendicular to the  $n - 1$  dimensional hyperplane containing the ball. The set of points in the cone is the union of all line segments joining the other end of the altitude to the ball. This is a higher dimensional version of a right circular cone.

We first prove a simple result based on this.

**Lemma 2.2.** *If  $p \in W$ , then  $\min_{i \in \{1,2\}} \text{dist}(p, q_i) < 2r$ .*

*Proof.* The points which are farthest from both  $q_1$  and  $q_2$  will be the points on the boundary of the base of the cones. Thus, it suffices to let  $p$  be one of these points. Then the ray  $q_1\vec{p}$  will intersect  $B_2$  tangentially. Taking this ray, the segment  $\overline{q_1q_2}$ , and the perpendicular dropped from  $q_2$  to  $q_1\vec{p}$ , we form a right triangle. The hypotenuse will be the segment  $\overline{q_1q_2}$  so will have length  $2r$ . The point  $p$  lies on one of the legs of the triangle. Thus  $\text{dist}(p, q_1) < 2r$ . q.e.d.

We wish to show that translates of  $W$  under the  $\Gamma$  action will pack  $\mathbb{H}^3$ . Most of the work lies in proving the following:

**Proposition 2.3.** *Let  $B_3$  be a ball of radius  $r$  in  $\mathbb{H}^n$  whose interior is disjoint from  $B_1$  and  $B_2$ . Then for any point  $p \in W$ , we have that  $\text{dist}(p, B_3) \geq \max_{i \in \{1,2\}} \text{dist}(p, B_i)$ . Equality holds only if  $p \in \partial W$ .*

*Proof.* Let  $q_3$  be the center of  $B_3$ . By Lemma 2.2,  $p \neq q_3$  and, in fact, there is a positive lower bound on the possible distance from  $p$  to  $q_3$ . Without loss of generality, we may assume that  $q_3$  is as close to  $p$  as allowed. Certainly, if  $B_3$  intersects neither  $B_1$  nor  $B_2$ , then it can be moved closer to  $p$ . So we may assume that  $B_3$  is adjacent to at least one of  $B_1$  and  $B_2$ .

Suppose for the moment that  $B_3$  intersects  $B_1$ , but not  $B_2$ . Then, unless  $q_1, q_3$ , and  $p$  are colinear, we may move  $B_3$  closer to  $p$ . So we take this as an additional assumption. If  $q_1$  lies between  $p$  and  $q_3$ , then

clearly  $p$  is closer to  $q_1$  and  $q_2$  than to  $q_3$ . Since  $p$  is within  $2r$  of  $q_1$ ,  $q_3$  cannot lie between  $p$  and  $q_1$ . This leaves only the possibility that  $p$  lies between  $q_1$  and  $q_3$ . However, as was shown in the proof of Lemma 2.2, there is a point of  $B_2$  within  $2r$  of  $q_1$  along the ray  $q_1\vec{p}$ . This point would be in the interior of  $B_3$ , a contradiction. The case in which  $B_3$  intersects  $B_2$  but not  $B_1$  is dealt with similarly.

Hence, we may assume that  $B_3$  intersects both  $B_1$  and  $B_2$ . Since  $B_1, B_2$ , and  $W$  have a rotational symmetry about  $\overline{q_1q_2}$ , there will be an  $n - 2$  sphere of possible locations for  $q_3$ . Along this sphere, the closest point to  $p$  will be one that lies in the plane  $\Pi$  containing  $q_1, q_2$ , and  $p$ . We have now reduced the situation to a two dimensional problem, illustrated by the following diagram.

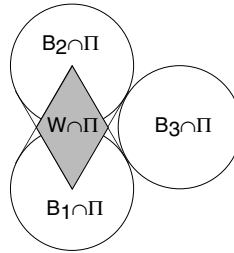


Figure 1.

It is obvious in this situation that  $\text{dist}(p, q_3) \geq \max_{i \in \{1,2\}} \text{dist}(p, q_i)$ . It is also easy to see that equality holds only when  $p$  is equidistant from  $q_3$  and either  $q_1$  or  $q_2$  and hence lies on  $\partial W$ . q.e.d.

We now prove a slightly stronger result.

**Proposition 2.4.** *Let  $T_3$  be a tube of radius  $r$  in  $\mathbb{H}^n$  whose interior is disjoint from  $T_1$  and  $T_2$ . Then for any  $p \in W$ , we have that  $\text{dist}(p, T_3) \geq \max_{i \in \{1,2\}} \text{dist}(p, T_i)$ . Equality holds only if  $p \in \partial W$*

*Proof.* Let  $l_3$  be the axis of  $T_3$  and let  $q_3$  be the point of  $l_3$  which is closest to  $p$ . Taking  $B_3$  to be the ball of radius  $r$  about  $q_3$ , we have that  $B_i \subset T_i$ . The interior of  $B_3$  will then be disjoint from  $B_1$  and  $B_2$ . Since  $\text{dist}(p, T_i) = \text{dist}(p, l_i) - r \leq \text{dist}(p, q_i) - r$ , it would be sufficient to prove that  $\text{dist}(p, l_3) = \text{dist}(p, q_3) \geq \max_{i \in \{1,2\}} \text{dist}(p, q_i)$ . However, this is the result of Proposition 2.3. q.e.d.

We are now ready to produce a packing of  $\mathbb{H}^3$ .

**Theorem 2.5.** *The various translates of  $W$  by elements of  $\Gamma$  have no intersections, except at boundary points.*

*Proof.* A point  $p$  in the interior of  $W$  is closer to  $T_1$  and  $T_2$  than to any other  $\Gamma$  translate of  $T_1$ . If for some  $\gamma \in \Gamma$ ,  $\gamma(p) \in \text{int } W$  then  $\gamma(p)$  is closer to  $T_1$  and  $T_2$  than to any other  $\Gamma$  translate of  $T_1$ . However, since  $\gamma$  is an isometry,  $\gamma(p)$  is closer to  $\gamma(T_1)$  and  $\gamma(T_2)$  than to any other  $\Gamma$  translate of  $T_1$ . Hence,  $\gamma$  carries  $T_1$  and  $T_2$  to, in some order,  $T_1$  and  $T_2$ . This implies that  $T_1 \cap T_2$ , which is a single point, is fixed by  $\gamma$ . The only element of  $\Gamma$  which has fixed points is the identity. q.e.d.

A simple consequence of this is:

**Corollary 2.6.** *The interior of  $W$  projects injectively to  $M$ . The portion of  $W$  lying outside  $T_1 \cup T_2$  projects to a set which does not intersect the projection of  $T_1$ .*

*Proof.* That the interior of  $W$  projects injectively is obvious. To prove the remaining statement, let  $p$  be a point lying in  $W \setminus (T_1 \cup T_2)$ . If  $p$  were inside some  $\Gamma$  translate of  $T_1$ , it would be closer to that tube than to  $T_1$  and  $T_2$ , which contradicts Proposition 2.4. q.e.d.

### 3. Hyperbolic Trigonometry and Integration

Many of our applications will require complicated 3-dimensional computations. Rather than interrupt the flow of thought when we develop these applications, we will get the computations out of the way now, under the guise of computing the volumes of the 3-dimensional version of  $W$  and a specific region within  $W$ .

As was mentioned earlier,  $W$  is the union of two identical cones. Our first step will be to determine the exact shape of the cones. Let  $C$  denote the cone.

**Proposition 3.1.** *The altitude of  $C$  has length  $r$ . The vertex angle  $\alpha$  is  $\sin^{-1} \frac{\sinh r}{\sinh 2r}$  and the slant height is  $\sinh^{-1} \frac{2 \sinh r}{\sqrt{3}}$ .*

*Proof.* It is obvious that the altitude of  $C$  is  $r$ . To determine the vertex angle, we consider that the side of the cone (when extended) will be tangent to a ball of radius  $r$  whose center is at a distance  $2r$  from the vertex of  $C$ . This provides a right triangle whose hypotenuse is  $2r$  and with one of the angles congruent to the vertex angle of  $C$ . The leg opposite the vertex angle is a radius of the ball so is of length  $r$ .

The Law of Sines then determines the vertex angle. To determine the slant height, we consider a right triangle whose hypotenuse is the slant height. One angle is the vertex angle and the adjacent side has length  $r$ . Using the identity  $\cos(\text{angle}) = \tanh(\text{adj}) \coth(\text{hyp})$ , we compute the slant height to be  $\tanh^{-1} \frac{\tanh r}{\cos \sin^{-1} \frac{\sinh r}{\sinh 2r}} = \sinh^{-1} \frac{2 \sinh r}{\sqrt{3}}$ . q.e.d.

We note that the preceding result does not depend on dimension. At this point we start to consider only the 3-dimensional case. We now derive a formula for the volume of a cone.

**Proposition 3.2.** *A right circular cone of altitude  $r$  and vertex angle  $\alpha$  has volume  $\pi \cos \alpha \tanh^{-1} \frac{\tanh r}{\cos \alpha} - \pi r$ .*

*Proof.* We will represent the volume as a triple integral in spherical coordinates  $(\rho, \phi, \theta)$  where  $\rho$  is the distance to the origin,  $\phi$  is the angle from some fixed line, and  $\theta$  is an angle measured in a plane perpendicular to the fixed line. The necessary volume element is  $\sin \phi \sinh^2 \rho \, d\rho \, d\phi \, d\theta$ .

We place the cone so the vertex lies at the origin and the altitude lies along the  $\phi = 0$  direction. In an identical computation to the one performed in the preceding proof, we see that  $\rho$  varies between 0 and  $\tanh^{-1} \frac{\tanh r}{\cos \phi}$ . So the volume  $V_C(r)$  is

$$\begin{aligned} V_C(r) &= \int_0^{2\pi} \int_0^\alpha \int_0^{\tanh^{-1} \frac{\tanh r}{\cos \phi}} \sin \phi \sinh^2 \rho \, d\rho \, d\phi \, d\theta \\ &= \pi \int_0^\alpha \int_0^{\tanh^{-1} \frac{\tanh r}{\cos \phi}} \sin \phi (\cosh 2\rho - 1) \, d\rho \, d\phi \\ &= \pi \int_0^\alpha \frac{\sin \phi \cos \phi \tanh r}{\cos^2 \phi - \tanh^2 r} - \sin \phi \tanh^{-1} \frac{\tanh r}{\cos \phi} \, d\phi. \end{aligned}$$

At this point, we perform the substitution  $u = \cos \phi$  and integrate the second half by parts.

$$\begin{aligned} V_C(r) &= \pi \int_{\cos \alpha}^1 \frac{u \tanh r \, du}{u^2 - \tanh^2 r} - \pi \int_{\cos \alpha}^1 \tanh^{-1} \frac{\tanh r}{u} \, du \\ &= \pi \int_{\cos \alpha}^1 \frac{u \tanh r \, du}{u^2 - \tanh^2 r} \\ &\quad - \pi \left( u \tanh^{-1} \frac{\tanh r}{u} \Big|_{\cos \alpha}^1 + \int_{\cos \alpha}^1 \frac{u}{1 - \frac{\tanh^2 r}{u^2}} \cdot \frac{\tanh r}{u^2} \, du \right) \\ &= \pi \cos \alpha \tanh^{-1} \frac{\tanh r}{\cos \alpha} - \pi r. \end{aligned}$$

q.e.d.

We may now compute the volume of  $W$ .

**Proposition 3.3.** *The volume of  $W$  is*

$$2V_C(r) = \pi\sqrt{4 - \operatorname{sech}^2 r} \sinh^{-1} \frac{2 \sinh r}{\sqrt{3}} - 2\pi r.$$

*Proof.* This is a simple consequence of plugging the relevant vertex angle and altitude into our cone volume formula and then simplifying. q.e.d.

One of our main interests will be determining the volume of the region of  $W$  which lies in neither  $T_1$  nor  $T_2$ . As the actual value is difficult to compute, we determine a lower bound. We do this by determining an upper bound on the volume of  $W \cap T_1$ . First, we give an intractable, but exact, formula for the volume.

**Proposition 3.4.** *The volume of  $W \cap T_1$  is*

$$V(W \cap T_1) = \int_0^{2\pi} \int_0^\alpha \int_0^{\sinh^{-1} \frac{\sinh r}{\sqrt{1 - \cos^2 \theta \sin^2 \phi}}} \sin \phi \sinh^2 \rho \, d\rho \, d\phi \, d\theta.$$

*Proof.* Again we work in spherical coordinates. The axis of  $T_1$  is perpendicular to the altitude of the cones in  $W$ . Placing the cone as before, we may take the axis of  $T_1$  to lie in the  $\phi = \frac{\pi}{2}$  plane in the direction of  $\theta = 0$  (and thus also  $\theta = \pi$ ). The bounds on  $\theta$  and  $\phi$  are easy to establish. Also, the lower bound on  $\rho$  is clearly 0. To determine the upper bound, we will have to do a small amount of work. First we determine the angle  $\beta$  between the axis of  $T_1$  and the line segment joining the origin to the point with spherical coordinates  $(\rho, \phi, \theta)$ . Of course,  $\rho$  will not affect this angle. As is readily seen from the Poincaré disk model, we could just as well perform this computation in Euclidean space. As the spherical coordinates  $(\rho, \phi, \theta)$  correspond to the Cartesian coordinates  $(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi)$ , the angle between this vector and the  $x$ -axis is given by  $\cos \beta = \cos \theta \sin \phi$ . Returning to hyperbolic geometry, we are now able to compute an upper bound on  $\rho$ . We need to know how far one can travel along a line at an angle of  $\beta$  from the axis of  $T_1$ , before one is at a distance of  $r$  from this axis. A quick application of the Law of Sines shows that  $\sinh \rho \leq \sinh r \csc \beta$ . q.e.d.

This expression is difficult to deal with so we make an approximation to simplify matters.

**Proposition 3.5.** *The volume of  $W \cap T_1$  is less than*

$$V_T(r) = \pi \sinh r \left( \csc \tan^{-1} \frac{\cos \alpha}{\sinh r} - \csc \tan^{-1} \operatorname{csch} r \right) - \pi r + \pi \cos \alpha \sinh^{-1} \frac{\sinh r}{\cos \alpha}.$$

*Proof.* Since  $1 - \cos^2 \theta \sin^2 \phi \geq 1 - \sin^2 \phi = \cos^2 \phi$ , we may say that

$$\begin{aligned} V(W \cap T_1) &\leq \int_0^{2\pi} \int_0^\alpha \int_0^{\sinh^{-1} \frac{\sinh r}{\cos \phi}} \sin \phi \sinh^2 \rho \, d\rho \, d\phi \, d\theta \\ &= \pi \int_0^\alpha \int_0^{\sinh^{-1} \frac{\sinh r}{\cos \phi}} \sin \phi (\cosh 2\rho - 1) \, d\rho \, d\phi \\ &= \pi \int_0^\alpha \sin \phi \left( \frac{\sinh r}{\cos \phi} \sqrt{1 + \frac{\sinh^2 r}{\cos^2 \phi}} - \sinh^{-1} \frac{\sinh r}{\cos \phi} \right) d\phi. \end{aligned}$$

At this point, we make the substitution  $u = \cos \phi$  and then integrate the second half by parts.

$$\begin{aligned} V(W \cap T_1) &\leq \pi \int_{\cos \alpha}^1 \frac{\sinh r}{u} \sqrt{1 + \frac{\sinh^2 r}{u^2}} \, du - \pi \int_{\cos \alpha}^1 \sinh^{-1} \frac{\sinh r}{u} \, du \\ &= \pi \int_{\cos \alpha}^1 \frac{\sinh r}{u} \sqrt{1 + \frac{\sinh^2 r}{u^2}} \, du \\ &\quad - \pi \left( u \sinh^{-1} \frac{\sinh r}{u} \Big|_{\cos \alpha}^1 + \int_{\cos \alpha}^1 u \cdot \frac{\frac{\sinh r}{u^2}}{\sqrt{1 + \frac{\sinh^2 r}{u^2}}} \, du \right) \\ &= \pi \int_{\cos \alpha}^1 \frac{\sinh^3 r}{u^2 \sqrt{u^2 + \sinh^2 r}} \, du - \pi r + \pi \cos \alpha \sinh^{-1} \frac{\sinh r}{\cos \alpha}. \end{aligned}$$

Making the substitution  $u = \sinh r \tan t$  allows us to complete the integration.

$$\begin{aligned} V(W \cap T_1) &\leq \pi \int_{\tan^{-1} \frac{\cos \alpha}{\sinh r}}^{\tan^{-1} \operatorname{csch} r} \frac{\sinh r \cos t}{\sin^2 t} \, dt - \pi r + \pi \cos \alpha \sinh^{-1} \frac{\sinh r}{\cos \alpha} \\ &= \pi \sinh r \left( \csc \tan^{-1} \frac{\cos \alpha}{\sinh r} - \csc \tan^{-1} \operatorname{csch} r \right) - \pi r \\ &\quad + \pi \cos \alpha \sinh^{-1} \frac{\sinh r}{\cos \alpha}. \end{aligned}$$

q.e.d.



This now allows us to compute a lower bound on the volume of the region of a manifold lying outside a tube of radius  $r$ .

**Theorem 3.6.** *Let  $M$  be an orientable hyperbolic 3-manifold containing an embedded tube of radius  $r$  about one of its geodesics. Then the region of  $M$  lying outside of this tube has volume at least  $2(V_C(r) - V_T(r))$ .*

*Proof.* In the case in which the embedded tube is of maximal radius, we have already shown that  $W \setminus (T_1 \cup T_2)$  projects injectively to  $M$  and does not intersect the tube. Thus, we need only consider the case in which the tube is not of maximal radius. Expand the tube until it is of maximal radius  $R$ . Then again we know that there is a region of volume  $2(V_C(R) - V_T(R))$  lying outside the maximal tube. All we need to note are that this region is, of course, outside of the original tube and that  $2(V_C(r) - V_T(r))$  is an increasing function of  $r$ . q.e.d.

We perform one last calculation to determine the limiting behavior.

**Proposition 3.7.**  $\lim_{r \rightarrow \infty} 2(V_C(r) - V_T(r)) = \pi(\log \frac{4}{3} - \frac{1}{4})$ .

*Proof.* Using the definitions of the functions involved and the fact that  $\lim_{x \rightarrow \infty} (\sinh^{-1} x - \log(2x)) = 0$ , we have that

$$\begin{aligned} & \lim_{r \rightarrow \infty} 2(V_C(r) - V_T(r)) \\ &= \lim_{r \rightarrow \infty} \left[ \pi \sqrt{4 - \operatorname{sech}^2 r} \sinh^{-1} \frac{2 \sinh r}{\sqrt{3}} \right. \\ & \quad \left. - 2\pi \sinh r \left( \frac{\sqrt{\sinh^2 r + \cos^2 \alpha}}{\cos \alpha} - \cosh r \right) - 2\pi \cos \alpha \sinh^{-1} \frac{\sinh r}{\cos \alpha} \right] \\ &= \lim_{r \rightarrow \infty} \left[ \pi \sqrt{4 - \operatorname{sech}^2 r} \left( r + \log \frac{2}{\sqrt{3}} \right) - 2\pi \cos \alpha (r - \log \cos \alpha) \right. \\ & \quad \left. - 2\pi \frac{\sinh r}{\cos \alpha} \left( \frac{\sinh^2 r + \cos^2 \alpha - \cosh^2 r \cos^2 \alpha}{\sqrt{\sinh^2 r + \cos^2 \alpha + \cosh r \cos \alpha}} \right) \right]. \end{aligned}$$

Earlier, we computed  $\sin \alpha$ . From this, it is easy to determine that

$\cos \alpha = \frac{1}{2}\sqrt{4 - \operatorname{sech}^2 r}$ . Continuing our computation,

$$\begin{aligned} & \lim_{r \rightarrow \infty} 2(V_C(r) - V_T(r)) \\ &= 2\pi \log \frac{2}{\sqrt{3}} - 2\pi \lim_{r \rightarrow \infty} \frac{\sinh^2 r - \sinh^2 r \cos^2 \alpha}{\sqrt{1 + \frac{\cos^2 \alpha}{\sinh^2 r}} + \coth r \cos \alpha} \\ &= 2\pi \log \frac{2}{\sqrt{3}} - 2\pi \lim_{r \rightarrow \infty} \frac{\sinh^2 r \sin^2 \alpha}{2} \\ &= 2\pi \log \frac{2}{\sqrt{3}} - \frac{\pi}{4} = \pi \left( \log \frac{4}{3} - \frac{1}{4} \right). \end{aligned}$$

q.e.d.

#### 4. Applications

We start by looking for embedded balls in hyperbolic manifolds.

**Proposition 4.1.** *If a hyperbolic manifold contains a tube of radius  $r$  about a geodesic, then it also contains an embedded ball of radius  $\sinh^{-1}(\frac{1}{2} \tanh r)$ .*

*Proof.* Let  $r'$  be the radius of the maximal tube about this geodesic. The shortest distance from the center of  $W$  to the boundary of  $W$  will be achieved by dropping a perpendicular from the center to the boundary. This will form a right triangle with hypotenuse  $r'$ , one angle equal to  $\sin^{-1} \frac{\sinh r'}{\sinh 2r'}$  and the opposite leg the desired ball radius. It is easy to see that the radius of the ball is  $\sinh^{-1} \frac{\sinh^2 r'}{\sinh 2r'} = \sinh^{-1}(\frac{1}{2} \tanh r')$ , which is clearly increasing in  $r'$  and hence is at least  $\sinh^{-1}(\frac{1}{2} \tanh r)$ . q.e.d.

This result allows us to prove two corollaries, one 3-dimensional and the other  $n$ -dimensional.

**Corollary 4.2.** *Every closed orientable hyperbolic 3-manifold contains a ball of radius at least  $\sinh^{-1} \frac{1}{4} = 0.24746 \dots$ .*

*Proof.* If the manifold contains a tube of radius  $\frac{\log 3}{2}$ , then it must contain a ball of radius at least  $\sinh^{-1} \frac{1}{4}$ . Thus, we need only note that [5] produces a tube of radius  $\frac{\log 3}{2}$  except when the manifold is Vol3 or when the shortest geodesic has length at least  $1.0595 \dots$ . Vol3 is known to contain a ball of radius  $0.527$ . On the other hand, if the shortest geodesic has length at least  $1.0595 \dots$ , then every point has a ball of radius  $0.529 \dots$  about it. q.e.d.

For the  $n$ -dimensional result, we first cite a theorem of Cao and Waterman [2].

**Theorem 4.3.** *Let  $f$  be a geodesic in a complete hyperbolic  $n$ -manifold, with  $k = \lfloor \frac{n-1}{2} \rfloor$  and  $\sigma(p) = \int_0^{\frac{\pi}{2}} \sin^p x dx$ . If the length  $l$  of  $f$ , is at most  $\frac{2\sigma(k+1)}{\pi^k} \left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$ , then there exists an embedded solid tube around  $f$  whose radius  $r$  satisfies*

$$\sinh^2 r = \frac{1}{4} \left( \frac{\pi^k l}{2\sigma(k+1)} \right)^{-\frac{2}{k+1}} \left( 1 - 4 \left( \frac{\pi^k l}{2\sigma(k+1)} \right)^{\frac{2}{k+1}} \right)^{\frac{1}{2}} - \frac{1}{2}.$$

Further, it is easy to see that  $r$  is a decreasing function of  $l$ . This gives:

**Corollary 4.4.** *If  $M^n$  is a closed hyperbolic  $n$ -manifold, then either the shortest geodesic has length at least  $\frac{2\sigma(k+1)}{\pi^k} \left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$  or there is an embedded ball of radius 0.37. As a consequence, every closed hyperbolic  $n$ -manifold contains a ball of radius  $\frac{\sigma(k+1)}{\pi^k} \left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$ .*

*Proof.* If the shortest geodesic has length  $l < \frac{2\sigma(k+1)}{\pi^k} \left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$  then there is a tube whose radius  $r$  satisfies  $\sinh^2 r = \frac{1}{\sqrt{2}-1} \sqrt{2 - \sqrt{2}} - \frac{1}{2} = 1.3477\dots$ . This leads to a ball of radius at least 0.37\dots

If the shortest geodesic has length at least  $\frac{2\sigma(k+1)}{\pi^k} \left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$  then there must be a ball of radius  $\frac{\sigma(k+1)}{\pi^k} \left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$  about every point in  $M$ . Since this amount is smaller than 0.37 we get the desired lower bound on ball radius. q.e.d.

We may state a corollary to this corollary:

**Corollary 4.5.** *If  $M^n$  is a closed hyperbolic  $n$ -manifold then*

$$\text{Vol}(M) \geq V_n(1) \left( \frac{\sigma(k+1)}{\pi^k} \left( \frac{\sqrt{2}-1}{4} \right)^{\frac{k+1}{2}} \right)^n$$

where  $V_n(1)$  is the volume of an  $n$ -dimensional Euclidean ball of radius 1.

*Proof.* We have proved that there is an embedded ball of radius at least  $\frac{\sigma(k+1)}{\pi^k} \left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$ . The volume of a hyperbolic ball of a given

radius is greater than the volume of the corresponding Euclidean ball of the same radius. Thus we may use the Euclidean volume formula as a lower bound. q.e.d.

In addition to embedded balls, we can consider the diameter of a manifold.

**Proposition 4.6.** *If a hyperbolic manifold  $M$  contains an embedded tube of radius  $r$  about one of its geodesics then*

$$\text{diam}(M) \geq \sinh^{-1} \left( \frac{2 \sinh r}{\sqrt{3}} \sqrt{1 - \frac{\sinh^2 r}{\sinh^2 2r}} \right).$$

*Proof.* As we have shown,  $W$  embeds in  $M$ . We have also shown that the points of  $W$  are closer to  $T_1$  or  $T_2$  than any other  $\pi_1(M)$  translate. Let us consider the points on the boundary of the bases of the cones in  $W$ . Among these points, the ones which are closest to  $l_1$  will lie in the plane containing both  $l_1$  and the common perpendicular to  $l_1$  and  $l_2$ . Working within this plane, we have a point at a distance of  $\sinh^{-1} \frac{2 \sinh r}{\sqrt{3}}$  from  $q_1$  and such that the angle of declination from  $l_1$  is  $\frac{\pi}{2} - \sin^{-1} \frac{\sinh r}{\sinh 2r}$ . Thus, we may use the Law of Sines to see that the distance from this point to  $l_1$  is  $\sinh^{-1} \left( \frac{2 \sinh r}{\sqrt{3}} \sqrt{1 - \frac{\sinh^2 r}{\sinh^2 2r}} \right)$ . The point could not be any closer to  $l_2$  and thus  $l_1$  is closer to the point than any other  $\Gamma$  translate. q.e.d.

From this point on, we restrict our attention to the 3-dimensional case. We first provide a very small improvement of an earlier result of this author [8].

**Proposition 4.7.** *Every closed orientable hyperbolic 3-manifold has volume at least 0.28.*

*Proof.* The exceptional cases to the existence of a  $\frac{\log 3}{2}$  tube all have volume at least 1.01. In [8], it is shown that a tube of radius at least  $\frac{\log 3}{2}$  in an orientable hyperbolic 3-manifold has volume at least 0.27666. Because of Theorem 3.6, we know that the region outside of the tube has volume at least 0.00485. q.e.d.

We now improve a recent result of Gabai, Meyerhoff, and Milley [4]. First, we state the relevant theorems from their paper:

**Theorem 4.8 ([4]).** *If a maximal tube in a complete orientable hyperbolic 3-manifold has length  $l$  and radius  $r > 0.2014$  and a value  $\rho$*

is chosen at most 0.298 then  $l$  is at least as large as the smaller of

$$\frac{\sqrt{3} \cosh(2r)}{2\pi \sinh(2r)} \left( \cosh^{-1} \left( \frac{\sinh^2(2r) + \cosh(2r + \rho)}{\cosh^2(2r)} \right) \right)^2$$

and

$$\frac{1}{\cosh(2r) \sinh(2r)} \left( \frac{R_{2r,0}^2}{4} + \left( R_{2r,\rho} - \frac{R_{2r,0}}{2} \right)^2 \right)$$

where

$$R_{t,\rho} = \sqrt{\sinh(2r) \cosh(2r) \coth(2r + \rho)} \cdot \cosh^{-1} \left( \frac{\sinh(2r) \sinh(2r + \rho) + \cosh(t)}{\cosh(2r) \cosh(2r + \rho)} \right).$$

For their purposes, the value  $\rho = 0.298$  was optimal. They then established various properties of these functions with  $\rho = 0.298$ . We will choose  $\rho = 0.293$ . Rather than reproduce their efforts, we say simply that with one exception, the exact same arguments would work for this new value of  $\rho$ . The exception is that the second function is decreasing so long as  $r \geq 0.6$  whereas in [4] it is shown that this function is invertible whenever its value is less than 0.11014.

**Theorem 4.9.** *The shortest geodesic in the smallest volume orientable hyperbolic 3-manifold has length at least 0.09.*

*Proof.* Suppose that there is a geodesic of length at most 0.09. In [4], it is shown that if a geodesic has length less than 0.10438, then there is a tube of radius at least 1.02 about it. Of course, this applies to our geodesic.

By multiplying the length estimates in Theorem 4.8 by  $\pi \sinh^2 r$  we obtain lower bounds on tube volume. Thus, we can say that with  $\rho = 0.293$ , a tube of radius at least 1.332 has volume at least 0.87906. By Theorem 3.6, the volume outside this same tube is at least 0.06368. Thus the manifold has volume at least 0.94274 which is greater than the volume of the Weeks manifold. Thus,  $1.02 \leq r \leq 1.332$ .

Again using Theorem 4.8, we can say that  $l \geq 0.09009$ , a contradiction. q.e.d.

It is also possible to improve estimates regarding noncompact manifolds. We would like to thank Peter Shalen for suggesting the following application.

**Proposition 4.10.** *If an orientable noncompact hyperbolic 3-manifold  $M$  has betti number at least 4 then  $\text{Vol}(M) \geq \pi(\log \frac{4}{3} + \frac{3}{4})$ .*

*Proof.* In [1], it is established that there is a sequence of manifolds  $M_n$  such that  $\pi_1(M_n)$  converges geometrically to  $\pi_1(M)$ . Further,  $\text{Vol}(M) > \text{Vol}(M_n)$  and the limit as  $n$  goes to infinity of  $l_n$ , the length of the shortest geodesic in  $M_n$  is 0. It is also shown that  $M_n$  contains an embedded tube of volume at least  $V(l_n)$  where  $V$  is a specific function they develop. All that we need to know about  $V$  is that  $\lim_{l \rightarrow 0} V(l) = \pi$ . As  $l_n$  goes to 0, the radius of the maximal embedded tube goes to  $\infty$ . This information is used to establish  $\pi$  as a lower bound on volume.

In  $M_n$ , by Theorem 3.6, we can establish a lower bound on the volume outside of the maximal tube. As  $n \rightarrow \infty$ , this lower bound will approach  $\lim_{r \rightarrow \infty} 2(V_C(r) - V_T(r)) = \pi(\log \frac{4}{3} - \frac{1}{4})$ . Thus, we may say that  $\text{Vol}(M) = \lim_{n \rightarrow \infty} \text{Vol}(M_n) \geq \pi + \pi(\log \frac{4}{3} - \frac{1}{4}) = \pi(\log \frac{4}{3} + \frac{3}{4})$ . q.e.d.

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