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PRESCRIBING SCALAR CURVATURE ON S^N, PART 1: APRIORI ESTIMATES

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Abstract

In this paper, we describe great details of the bubbling behavior for a sequence of solutions w_i of

$$Lw_i + R_i w_i^{\frac{n+2}{n-2}} = 0 \quad \text{on} \quad S^n,$$

where L is the conformal Laplacian operator of (S^n, g_0) and $R_i = n(n-2) + t_i\hat{R}$, $\hat{R} \in C^1(S^n)$. As $t_i \downarrow 0$, we prove among other things the location of blowup points, the spherical Harnack inequality near each blowup point and the asymptotic formulas for the interaction of different blowup points. This is the first step toward computing the topological degree for the nonlinear PDE.

1. Introduction

This is the first of a series of papers to study the problem of prescribing scalar curvature on S^n, the n-dimensional sphere with $n \geq 3$. Let g_0 be the metric on S^n induced from the flat metric of $\mathbb{R}^{n+1}$, and R be a given C^1 positive function on S^n. We are interested in the question whether there exists a metric g conformal to g_0 such that R is the scalar curvature of g. Set $g = c_n w^{\frac{4}{n-2}} g_0$ for a suitable positive constant c_n. Then the question above is equivalent to finding a smooth positive solution of

$$(1.1) \qquad Lw + Rw^{\frac{n+2}{n-2}} = 0 \quad \text{on} \quad S^n,$$

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where $L = \Delta_{g_0} - \dfrac{n(n-2)}{4}$ is the conformal Laplacian operator of (S^n, g_0). In general, the same question can be studied in any Riemannian manifold. For a compact Riemannian manifold and a constant R, this problem is called the Yamabe problem, which was solved in early 80s through the works by Trudinger [22], Aubin [1] and Schoen [19]. For a historic account, we refer the readers to Lee and Parker [14] and references therein. For the last three decades, Equation (1.1) has been continuing to be one of major subjects in nonlinear elliptic PDEs. For recent developments, see [1], [2], [3], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15], [16], [17], [18], [19], [20], [21] and the references therein.

In [5], Chang-Gursky-Yang considered Equation (1.1) when $n = 3$ and R is a positive Morse function on S^3. Under some nondegenerate conditions on the critical points of R, Chang-Gursky-Yang were able to obtain the apriori bound for positive solutions of Equation (1.1). Furthermore, they computed the Leray-Schauder degree d for Equation (1.1) by the following formula

$$d = -\left(1 + \sum_{p \in \Gamma^-} (-1)^{\mathrm{ind}(p)}\right), \tag{1.2}$$

where $\Gamma^- = \{p \in S^3 \mid p \text{ is a critical point of } R \text{ satisfying } \Delta_{g_0} R(p) < 0\}$ and $\mathrm{ind}(p)$ is the Morse index of the Hessian of R at p. When the right-hand side of (1.2) is assumed to be nonzero, the existence of positive solutions to Equation (1.1) was previously obtained by Bahri-Coron [3] and Schoen-Zhang [21]. However, the degree-counting formula (1.2) provides us more information about Equation (1.1). Particularly, it tells us when the concentration phenomenon for solutions of (1.1) could occur. Li [16] proved the apriori bound for Equation (1.1) on S^4 and derived the formula for the Leray-Schauder degree by adding the effect of the interaction of multiple blow-up points. In this series of papers, we will generalize the results of [5] and [16] on S^3 and S^4 to higher dimensions.

As in our previous works [8], [9], it is more convenient for us to study (1.1) in $\mathbb{R}^n$. Without loss of generality, we may assume that the north pole of S^n is *not* a critical point of R. By using the stereographic projection π from S^n to $\mathbb{R}^n$, we set $u(x) = 2^{\frac{n-2}{2}}(1+|x|^2)^{\frac{2-n}{2}} w(\pi^{-1}(x))$

for $x \in \mathbb{R}^n$. Then $u(x)$ satisfies

$$(1.3) \qquad \begin{cases} \Delta u(x) + K(x) u^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbb{R}^n, \\ u(x) = O(|x|^{2-n}) & \text{at } \infty, \end{cases}$$

where $K(x) = R(\pi^{-1}(x))$ for $x \in \mathbb{R}^n$.

When $K(x)$ is a constant, solutions of (1.1) can be classified completely. See [13] and [4]. For nonconstant $R(x)$, it is well-known that existence of solutions depends on K in a very subtle way. So, throughout the paper and [10], we always assume $0 < a \le K(x) \le b$ and $K(x)$ has a finite set of critical points $\{q_1, \dots, q_N\}$. Near each q_j, by Taylor's expansion, $K(x)$ can be written as

$$K(x) = K(q_j) + Q_j(x - q_j) + R_j(x),$$

where $Q_j(x)$ is a C^1 homogeneous function of degree $\beta_j > 1$, i.e., $Q_j(\lambda x) = \lambda^{\beta_j} Q_j(x)$ for $\lambda > 0$ and R_j satisfies

$$\lim_{x \to q_j} |x - q_j|^{-\beta_j} R_j(x) = \lim_{x \to q_j} |x - q_j|^{1-\beta_j} | \bigtriangledown R_j|(x) = 0.$$

Here, β_j is not necessarily an integer. Of course, if $K(x) \in C^\infty$, then β_j must be an integer.

(K0) $\quad | \bigtriangledown Q_j(x)| \ge c_1 |x|^{\beta_j - 1}$ for some $c_1 > 0$.

Let $U_1(x) = (1 + |x|^2)^{-\frac{n-2}{2}}$.

(K1) At each critical point q_j, according to β_j, K satisfies one of the following conditions (i), (ii) and (iii):

(i) If $\beta_j < n$, Q_j satisfies

$$(1.4) \qquad \begin{pmatrix} \int_{\mathbb{R}^n} \bigtriangledown Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x) dx \\ \int_{\mathbb{R}^n} Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x) dx \end{pmatrix} \ne \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for any $\xi \in \mathbb{R}^n$.

(ii) If $\beta_j = n$, then

$$(1.5) \qquad \int_{S^{n-1}} Q_j(x) d\sigma \ne 0$$

provided that there exists a vector $\xi \in \mathbb{R}^n$ satisfying

$$(1.6) \qquad \int_{\mathbb{R}^n} \bigtriangledown Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(y) dy = 0.$$

(iii) If $\beta_j > n$,

$$\int_{\mathbb{R}^n} \langle x - q_j, \bigtriangledown K\rangle |x - q_j|^{-2n} dx \neq 0. \tag{1.7}$$

We note that all integrals in (1.4)–(1.7) are $L^1(\mathbb{R}^n)$. In [5], [9] and [16], we knew that only part of critical points of K might be blowup points for certain solutions. Denote by Γ^- those critical points of K. More precisely:

Definition 1.1. Assume that K satisfies (K0). We say $q_j \in \Gamma^-$ if and only if K satisfies one of the following conditions (i), (ii) and (iii) at q_j according to β_j:

(i) If $\beta_j < n$, there exists $\xi \in \mathbb{R}^n$ such that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} \bigtriangledown Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x) dx = 0 \quad \text{and} \\ \displaystyle\int_{\mathbb{R}^n} Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x) dx < 0. \end{cases} \tag{1.8}$$

(ii) If $\beta_j = n$, there exists $\xi \in \mathbb{R}^n$ satisfying

$$\begin{cases} \displaystyle\int \bigtriangledown Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x) dx = 0 \quad \text{and} \\ \displaystyle\int_{S^{n-1}} Q_j(x) d\sigma < 0. \end{cases} \tag{1.9}$$

(iii) If $\beta_j > n$,

$$\int_{\mathbb{R}^n} \langle x - q_j, \bigtriangledown K\rangle |x - q_j|^{-2n} dx < 0. \tag{1.10}$$

Clearly, the notion $q_j \in \Gamma^-$ and conditions (K0)–(K1) are invariant under the conformal transformations.We list several examples of Q to explain conditions (K0) and (K1).

Example 1.2.

1. $Q(y) = \sum_{j=1}^n a_j y_j^2$. Clearly $a_j \neq 0$ for all j iff (K0) holds. It is easy to see that $\xi = 0$ is the only vector satisfying $\int_{\mathbb{R}^n} \bigtriangledown Q(y + \xi) U_1^{\frac{2n}{n-2}}(y) dy = 0$ and $\int_{\mathbb{R}^n} Q(y) U_1^{\frac{2n}{n-2}}(y) dy = c_n \sum_{j=1}^n a_j$ for some positive constant c_n. Thus, (K0) and (K1) hold for a Morse function R on S^n satisfying $\Delta R(q) \neq 0$ for any critical point q of R. And $q \in \Gamma^-$ iff $\Delta R(q) < 0$.

2. $Q(y) = \sum_{j=1}^n a_j y_j^3, a_j \neq 0$, for $j = 1, 2, \dots, n$. Clearly, no $\xi \in \mathbb{R}^n$ satisfies $\int_{\mathbb{R}^n} \bigtriangledown Q(y+\xi) U_1^{\frac{2n}{n-2}}(y)dy = 0$.

3. $Q(y) = y_1^3 - \lambda y_1 \sum_{j=2}^n y_j^2$. For $\lambda > \frac{3}{n-2}$, $Q(y)$ satisfies (K0) and (1.4). In fact, there are exactly two solutions $\xi = \pm\xi_0$ of $\int_{\mathbb{R}^n} \bigtriangledown Q(y+\xi) U_1^{\frac{2n}{n-2}}(y)dy = 0$, where $\xi_0 = (\xi_{0,1}, 0, \dots, 0)$ for some $\xi_{0,1} > 0$. Direct computations show
$$\int Q(y+\xi_0) U_1^{\frac{2n}{n-2}}(y)dy = -\int Q(y-\xi_0) U_1^{\frac{2n}{n-2}}(y)dy < 0.$$

The main purpose of our work is to show that homogeneous functions $Q_j(x)$ for $q_j \in \Gamma^-$ completely determine the structure of solutions of (1.1). Conditions (K0) and (K1) are already enough for our purpose. However, in order to make our presentation transparent here, each Q_j at $q_j \in \Gamma^-$ is assumed to satisfy

(K2) For each $q_j \in \Gamma^-$ with $\beta_j < n$, assume that

$$\int_{\mathbb{R}^n} Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x)dx < 0 \quad \text{whenever} \quad \int_{\mathbb{R}^n} \bigtriangledown Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x)dx = 0. \tag{1.11}$$

To state our main theorem, we introduce the notion Λ^-. Assume (K0) and (K1). Let Λ^- be a collection of subsets of Γ^- such that a subset A of Γ^- is an element in Λ^- if and only if A satisfies the following conditions.

1. The number of the elements in $A \geq 2$.

2. For any two elements $q_j \neq q_k$ in A, the exponents β_j and β_k satisfies
$$\frac{1}{\beta_j^*} + \frac{1}{\beta_k^*} > \frac{2}{n-2},$$
where

$$\beta_j^* = \min(\beta_j, n). \tag{1.12}$$

Now we can state a special case of the Main Theorem we are going to prove in this paper and the subsequent one [10].

Theorem 1.1. *Assume that K satisfies* (K0) *and* (K1) *such that β_j of Q_j at each critical point q_j in Γ^- satisfies $\beta_j > \dfrac{n-2}{2}$. In addition, we assume*

$$\frac{1}{\beta_j^*} + \frac{1}{\beta_k^*} \neq \frac{2}{n-2} \tag{1.13}$$

for $q_j \neq q_k \in \Gamma^-$. Then there exists a constant $c > 0$ such that for any solution w of (1.1)*, we have*

$$c^{-1} \leq w(y) \leq c \text{ for } y \in S^n. \tag{1.14}$$

Let d denote the Leray-Schauder degree for the nonlinear map $w + L^{-1}(Rw^{\frac{n+2}{n-2}})$ on $C^{2,\alpha}(S^n)$ with $0 < \alpha < 1$. Moreover, if (K2) *holds additionally, then d satisfies*

$$d = -\left[1 + \sum_{j\in\Gamma^-} (-1)^{n+1} \deg F_j + \sum_{A\in\Lambda^-} \prod_{k\in A} ((-1)^{n+1} \deg F_k)\right], \tag{1.15}$$

where $\deg F_j$ denotes the standard topological degree of the mapping $F_j(x) = \triangledown Q_j(x)$ from S^{n-1} to $\mathbb{R}^n \backslash \{0\}$, and Γ^- and Λ^- are defined as above.

We remark that the assumption $\beta_j > \frac{n-2}{2}$ in Theorem 1.1 is an also necessary condition for the existence of apriori bounds for solutions of Equation (1.1). In [11], we constructed blowing up solutions of (1.1) for some K satisfying (K0) and (K1) with $\beta_j < \frac{n-2}{2}$. To establish the apriori bound (1.14), the first step is to understand the details of blowing-up behavior of a sequence of solutions w_i near each blow-up point. In [8], [9] for a sequence of *local solutions* u_i of

$$\Delta u_i + K_i(x) u_i^{\frac{n+2}{n-2}} = 0 \text{ in } B_2 = \{x \mid |x| < 2\} \tag{1.16}$$

where 0 is assumed the only blowup point, we have completely classified types of concentrations of u_i according to the flatness β of Q at the blowup point 0. In particular, if $\frac{n-2}{2} < \beta < n$ then

$$u_i(x) \sim M_i^{-\gamma} \tag{1.17}$$

in any compact set of $\overline{B}_1 \backslash \{0\}$, where

$$\gamma = \begin{cases} \frac{2\beta}{n-2} - 1 & \text{if } \beta < n-2 \\ 1 & \text{if } \beta \geq n-2, \end{cases}$$

and M_i is the maximum of u_i in $\overline{B}_1$. Hereafter, the notation $a_i \sim b_i$ for two sequences of positive numbers denotes that the ratio a_i/b_i is bounded above and below by two positive constants independent of i. Thus, $u_i(x) \downarrow 0$ in $C^2_{\mathrm{loc}}(\overline{B}_1 \backslash \{0\})$. The result (1.17) is important when *global* solutions u_i of Equation (1.3) are considered, because those local maxima must satisfy certain rules according to (1.17). Together with the Pohozaev identity, we must have $\frac{1}{\beta_j^*} + \frac{1}{\beta_k^*} = \frac{2}{n-2}$ for some blowup points q_j and q_k. The apriori bound (1.14) then follows from this. We will give a complete proof of this result in Section 10 of the paper. When $n-2 < \beta_j < n$ for any critical point q_j, the apriori bound was obtained previously in [15].

The degree counting formula (1.15) is more difficult to prove. Usually, there are two ways to establish the Leray-Schauder degree. One is to approach the nonlinear term in Equation (1.1) by subcritical exponents. Another one is to deform the curvature function R, e.g., replace R in Equation (1.1) by $R_t = 1 + t(R-1)$ for $0 \le t \le 1$. For the latter case, if one can show for any $\varepsilon > 0$, solutions of (1.1) with R replaced by R_t are uniformly bounded for $\varepsilon \le t \le 1$, then the Leray-Schauder degree is the same for each $t \ne 0$. Thus, for our purpose, it suffices to compute the Leray-Schauder degree for small $t > 0$. In the situation when t is small enough, the degree theory developed by Chang-Yang [6] can be applied very well. But, Chang-Yang was only able to prove the degree counting formulas (1.2) for the class of Morse functions. More seriously, as we will see, the degree formula in [6] did not count all possible solutions. Roughly speaking, their results only covered the case when solutions of (1.1) possess at most one blow-up point as t tends to zero. Later in this paper, we will prove that under assumptions (K0) and (K1), if a sequence of solutions w_i of (1.1) with R_{t_i} as the scalar curvature blows up as $t_i \to 0$, then the number of blow-up points must be greater than one. Therefore, solutions obtained in [6] only consist of bounded solutions as $t \to 0$. We also remark that if the degree β_j for each $q_j \in \Gamma^-$ is no less than $n-2$, then any sequence of solutions of (1.1) with R replaced by R_{t_i} remains uniformly bounded as $t_i \to 0$. In this case, Λ^- is an empty set and the degree-counting formula (1.15) reduces to $d = -[1 + \sum_{j\in\Gamma^-}(-1)^{n+1}\deg F_j]$. When R is a Morse function on S^3, this is the degree counting (1.2).

In this paper, we consider a sequence of solutions u_i of (1.3) with curvature functions K_i set by

$$K_i(x) = n(n-2) + t_i \hat{K}(x), \tag{1.18}$$

where we assume $t_i \to 0$. Here, $\hat{K}$ is a C^1 function satisfying the nondegenerate conditions (K0)–(K1). Solutions u_i are always assumed to blow up at some points of $\mathbb{R}^n$. The main purpose of this article is to study blowup behavior of u_i near a blowup point and to study the effect due to the interaction between different blowup points. This is the first step for computing the degree-counting formula. Based on these, we will construct all possible blowup solutions of (1.1) as $t_i \downarrow 0$ in [10] and then we are able to compute the "local degree" for each blowup solution. In [10], we will give a complete proof of the degree formula. From the analytic point of view, the main difference between this paper and [9] are: First, we consider the degenerate case $\lim_{i\to\infty} K_i = \mathit{constant}$ here, which can not be covered by the results for nondegenerate $\lim_{i\to\infty} K_i$ in [9]. Second, we allow the number β_j defined in (K0) to be greater than or equal to n in this paper, while we assume $1 < \beta_j \le n-2$ in [9]. Third, we also consider the interaction between different blow-up points here, while we mainly study local behavior near a blow-up point in [9].

The first interesting question concerning a sequence of blowup solutions is to find the location of blowing up points. A general result states that if K_i converges to K in C^1, then any blowup point must be a critical point (see [21], [16], [8]). Obviously, this result could not be of any help for our present situation because the limit function of K_i is identically a constant. Nevertheless, by using more delicate estimates than the nondegenerate case, we are still able to prove the following.

Theorem 1.2. *Suppose $\hat{K}$ satisfies* (K0) *and u_i is a sequence of solutions of* (1.3) *with $K = K_i$ given in* (1.18)*. Then $\triangledown \hat{K}(q) = 0$ for any blowup point q of u_i.*

Throughout the paper, we let $\{q_1, \dots, q_m\}$ be the set of blowup points for $\{u_i\}$, and β_j be the degree of Q_j of $\hat{K}$ at q_j. To analyze the blowup behavior of u_i more accurately, the important step is to show the isolatedness of blowup points, that is, to prove the spherical Harnack inequality (1.19):

$$\max_{|x-q_j|=r} u_i(x) \le c \min_{|x-q_j|=r} u_i(x) \quad \text{for } 0 \le r \le r_0. \tag{1.19}$$

For nondegenerate case, the spherical Harnack inequality (1.19) was proved even for local solutions. See [8], [9] of the reference. For the degenerate case, we do not know whether the spherical Harnack inequality holds or not for local solutions. In Section 4, we study the situation when it fails. Due to the analysis there and the effect of interactions of different blowup points, nevertheless, the spherical Harnack inequaltiy is proved for *global solutions.*

Theorem 1.3. *Suppose that* $\hat{K}$ *satisfies* (K0) *and* (K1). *Assume* $\beta_j \geq \dfrac{2(n-2)}{n}$ *for each* $q_j \in \Gamma^-$. *Then any blowup point is isolated. Furthermore, if* $\beta_j < n+1$ *at a blowup point* q_j, *then* u_i *satisfies*

$$u_i(x) \leq c\,|x-q_j|^{-\frac{n-2}{2}} \tag{1.20}$$

for $|x-q_j| \leq \delta_0$ *with some positive constants* δ_0 *and* c.

By the theory of elliptic equations and the scaling property of Equation (1.3), inequality (1.20) implies (1.19). Hence, we also call (1.20) the spherical Harnack inequality. We note that in Theorem 1.3, (K1) is required only for those q_j where $\beta_j < n-2$.

For each blowup point q_j, we let $M_{i,j}$ and $q_{i,j}$ denote the local maximum and a local maximum point of u_i near q_j, that is,

$$M_{i,j} = u_i(q_{i,j}) = \max_{|x-q_j|\leq\delta_0} u_i(x), \tag{1.21}$$

where δ_0 is a small positive number such that the distance of q_j and q_k are greater than $2\delta_0$. The following theorem is concerned with the asymptotic relations of $M_{i,j}$ for different blowup points. Let l denote the nonnegative positive integer such that $q_1, \ldots, q_l$ are simple blowup points and $q_{l+1}, \ldots, q_m$ are not simple blowup points. For the notion of simple blowup points, we refer the reader to [8], [9] or Section 2 of this paper.

Theorem 1.4. *Assume that* $\hat{K}$ *satisfies* (K0) *and* (K1) *and assume* β *of* $Q > \frac{n-2}{2}$ *at any* $q \in \Gamma^-$. *Let* $\{q_j\}_{j=1}^m$ *be the set of blowup points for* u_i, *and* $M_{i,j}, q_{i,j}$ *and* l *be defined as above. Then* $m \geq 2$, $l \geq 1$ *and* $\beta_1 = \ldots = \beta_l > \beta_j$ *for* $l+1 \leq j \leq m$. *Furthemore, the following conclusions hold:*

(i) *We have* $q_j \in \Gamma^-$ *for* $1 \leq j \leq m$ *and there exists a constant* $c > 0$ *such that*

$$|q_{i,j} - q_j| \leq c \begin{cases} M_{i,j}^{-\frac{2}{n-2}} & \text{if } \beta_j < n+1, \\ M_{i,j}^{-\frac{2}{n-2}} (\log M_{i,j})^{\frac{1}{n}} & \text{if } \beta_j = n+1, \\ M_{i,j}^{-\frac{2}{n-2}\frac{n}{\beta_j - 1}} & \text{if } \beta_j > n+1. \end{cases} \tag{1.22}$$

Moreover, the limit vector $\xi = \lim_{i\to+\infty} M_{i,j}^{\frac{2}{n-2}}(q_{i,j} - q_j)$ *satisfies* (1.8) *if* $\beta_j < n$, *and satisfies* (1.6) *if* $n \le \beta_j < n+1$

(ii) *Assume that* $l = 1$. *We index* q_j *according to the ordering of* $\beta_j : \beta_1 > \beta_2 = \ldots = \beta_{l_1} > \beta_{l_1+1} \ge \ldots \ge \beta_m$ *for some positive integer* l_1. *Then*

$$\frac{1}{\beta_1^*} + \frac{1}{\beta_2} > \frac{2}{n-2}, \tag{1.23}$$

$M_{i,j}$ *satisfies*

(1.24)
$$\left.\begin{array}{ll} t_i M_{i,1}^{-\frac{2\beta_1^*}{n-2}} & \text{if } \beta_1 \ne n \\ t_i M_{i,1}^{-\frac{2n}{n-2}} \log M_{i,1} & \text{if } \beta_1 = n \end{array}\right\} = (1+o(1)) \sum_{j=2}^{l_1} \eta_{1,j} M_{i,j}^{-1} M_{i,1}^{-1},$$

and

$$t_i M_{i,j}^{\frac{-2\beta_j}{n-2}} = (1+o(1))\eta_{j,1} M_{i,j}^{-1} M_{i,1}^{-1} \quad \text{for } 2 \le j \le m, \tag{1.25}$$

where

$$\eta_{j,k} = \frac{n(n-2)|S^{n-1}||q_j - q_k|^{-n+2}}{|b_j|}, \tag{1.26}$$

and

$$b_j = \begin{cases} \beta_j \int_{\mathbb{R}^n} Q_j(x+\xi) U_1^{\frac{2n}{n-2}}(x)dx & \\ \text{with } \ \xi = \lim_{i\to+\infty} M_{i,j}(q_{i,j} - q_j) & \text{if } \beta_j < n \\ n \int_{S^{n-1}} Q_j(x) d\sigma & \text{if } \beta_j = n \\ \int_{\mathbb{R}^n} \langle x - q_j, \nabla \hat{K} \rangle |x - q_j|^{-2n} dx & \text{if } \beta_j > n \end{cases} \tag{1.27}$$

(iii) *Assume* $l \ge 2$. *Then* $\beta_1 = \ldots = \beta_l < n-2$ *and* $M_{i,j}$ *satisfies*

$$t_i M_{i,j}^{-\frac{2\beta_1}{n-2}} = (1+o(1)) \sum_{k=1, k\ne j}^{l} \eta_{j,k} M_{i,j}^{-1} M_{i,k}^{-1} \quad \text{for } 1 \le j \le l, \tag{1.28}$$

and

$$t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} = (1+o(1)) \sum_{k=1}^{l} \eta_{j,k} M_{i,k}^{-1} M_{i,j}^{-1} \quad \text{for } l+1 \le j \le m. \tag{1.29}$$

Theorem 1.4 gives us rather complete information about blowup solutions, that is, the local maxima of blowup solutions must satisfy the necessary conditions (1.24) and (1.25), or (1.28) and (1.29). Conversely, in [10] we will construct such blowup solutions satisfying these relations and compute the contribution of these solutions to the Leray-Schauder degree of Equation (1.1). We note that the third term of the right hand side of (1.15) corresponds to the effect of multiple blowup points.

The paper is organized as follows: In Section 2–Section 9, we consider the degenerate case for Equation (1.3), that is, $K_i(x) = n(n-2) + t_i\hat{K}(x)$ with $t_i \downarrow 0$. In Section 2, main results for local solutions are stated and their proofs are given in the subsequent sections. There are two main issues in Section 2. The first one is the quantity L_i, which is associated with each "good" local maximum point of solutions. The quantity L_i is introduced in Sections 2 and will play an important role because it decides how large of the range where u_i behaves "simply". We will give its proof in Sections 3 and this is the major step where the method of moving planes is applied. Another important issue in Section 2 is the spherical Harnack inequality (1.20). We will see that when the flatness $\beta \geq \frac{n-2}{2}$, the spherical Harnack inequality always holds. See Theorem 2.4. The case $\beta < \frac{n-2}{2}$ is the difficult one for our analysis, even when the Harnack inequality holds. In the general principle, we can obtain the local bubbling informations through the Pohozaev identities. However, we have to compute each term in the identity very accurately and the Harnack inequality itself is not enough for us to achieve this goal. We need a sharper estimate for the error term of the solution and the approximation bubbles. This is a very delicate analysis because in general the solutions might lose the energy more than one bubble. In Section 5, we show that a method of ODE surprisingly gives us fine estimates when the spherical Harnack inequality is validated. Together with suitably chosen comparision functions, we complete the proof of our desired estimate in Sections 5. See Theorem 2.7. This is one of two difficult jobs in the paper. These estimates for the error term are required in the proof of Lemma 7.1 in Section 7. Lemma 7.1 exactly tells us how, through the Pohozaev identities, the local informations can be put together to obtain more global one. Section 4 will deal with the situation when the spherical Harnack inequaltiy (2.19) fails. Here, we employ a technique of Schoen to localize blowup points. Combined with the method of moving planes developed in Section 3, this provides a clear picture for the case when the Harnack inequality does not hold. Based on the analysis in Section 4 and Lemma 7.1, Theorem 1.3 and

Theorem 1.4 are proved in Section 8 and Section 9, respectively. We will prove Theorem 1.2 in Section 6 as a direct consequence of results in Section 2. Finally, we will prove the apriori bound of Theorem 1.1 in Section 10.

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2. Estimates for local solutions

For the convenience of the reader, we briefly review some of previous results from [8] and [9], which would be useful later. Let u_i be a solution of

$$\Delta u_i + K_i(x) u_i^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

where Ω is an open set in $\mathbb{R}^n$. Let x_0 be a blowup point. Following Schoen's idea, a blowup point x_0 is called *simple* if there exists a constant $c > 0$ and a sequence of local maximum points x_i of u_i such that

$$x_0 = \lim_{i \to +\infty} x_i, \tag{2.2}$$

and

$$u_i(x_i + x) \le c\, U_{\lambda_i}(x) \quad \text{for } |x| \le r_0, \tag{2.3}$$

where $r_0 > 0$ is independent of i, $\lambda_i = u_i(x_i)^{-\frac{2}{n-2}}$ tends to zero as $i \to +\infty$ and

$$U_\lambda(x) = \left(\frac{\lambda}{\lambda^2 + |x|^2}\right)^{\frac{n-2}{2}} \quad \text{for} \quad x \in \mathbb{R}^n. \tag{2.4}$$

For any $\lambda > 0$, by elementary calculation, $U_\lambda(x)$ satisfies

$$\Delta U_\lambda + n(n-2) U_\lambda^{\frac{n+2}{n-2}}(x) = 0 \quad \text{in} \quad \mathbb{R}^n.$$

We note that the definition of a simple blowup point is different from the original one given by Schoen. However, it is not difficult to prove that these two definitions are equivalent.

Instead of (2.3), the inequality

$$u_i(x_i + x) \le c\, u_i(x_i)^{-1}|x|^{-n+2} \tag{2.5}$$

is often used when x_0 is a simple blowup point. Also, by (2.4), we have

$$U_\lambda(x) \le (2|x|)^{-\frac{n-2}{2}} \quad \text{for} \quad x \ne 0, \tag{2.6}$$

which implies that if x_0 is a simple blowup point, then

$$u_i(x_i + x) \le c\, |x|^{-\frac{n-2}{2}} \quad \text{for} \quad |x| \le r_0. \tag{2.7}$$

A blowup point x_0 is called *isolated* if (2.7) holds for some c and $r_0 > 0$. It is easy to see a simple blowup point must be isolated. The inequality (2.7) is important because it implies that the Harnack inequality holds for each sphere with center x_i, i.e., there exists a positive constant $c > 0$ such that

$$\max_{|x-x_i|=r} u_i(x) \le c \min_{|x-x_i|=r} u_i(x) \tag{2.8}$$

for $0 \le r \le r_0$.

Suppose that x_0 is a blowup point of u_i. Theorem 1.3 in [8] states that x_0 *is a simple blowup point if* $K_i(x) \to K(x)$ *in* C^1 *and* K_i *satisfies for some constant* c *either* (i) $|\bigtriangledown K_i(x)| \le c$ if $n = 3$ *or* (ii)

$$|\bigtriangledown^j K_i(x)| \le c|\bigtriangledown K_i(x)|^{\frac{\beta-j}{\beta-1}} \tag{2.9}$$

if $n \ge 4$ *in a neighborhood of* x_0 *for* $1 \le j \le \beta = n-2$. Also see [15] for the same conclusion when global solutions are considered. We make some remarks here. First, if $K_j = n(n-2) + t_i\hat{K}$ with $\hat{K}$ satisfying (2.9), then (2.9) holds for K_i also with the same constant c. Thus Theorem 1.3 in [8] can apply to our case. Second, if $\hat{K}$ is smooth and $|\bigtriangledown \hat{K}(x_0)| \ge c > 0$, then obviously condition (2.9) holds for K_i also. Actually, from the first step of the proof of Theorem 1.3 in [8], the smoothness assumption of $\hat{K}$ can be removed if x_0 is not a critical point of $\hat{K}$. Even when x_0 is a critical point, it is not necessary to assume that $\hat{K}$ is smooth. In this case, condition (2.9) can be replaced by

$$\begin{gathered} c_1|x-x_0|^{\beta-1} \le |\bigtriangledown \hat{K}(x)| \le c_2|x-x_0|^{\beta-1} \\ \text{in a neighborhood of } x_0 \text{ for some constants} \\ c_2 > c_1 \text{ and } \beta > 1. \end{gathered} \tag{2.10}$$

Thus, Theorem 1.3 of [8] can be restated as follows:

Theorem A. *Let u_i be a solutions of* (2.1) *with $K_i = n(n-2)+t_i\hat{K}$ and $x_0 \in \Omega$ be a blowup point of u_i. Assume that either x_0 is not a critical point of $\hat{K}$ or x_0 is a critical point of $\hat{K}$ and $\hat{K}$ satisfies* (2.10) *for some $\beta \geq n-2$. Then x_0 is a simple blowup point.*

Obviously, if x_0 is a simple blow-up point, then there are no blowup points in a small neighborhood of x_0. If we further assume that $\hat{K}$ has a discrete set of critical points in Ω, then by Theorem A, u_i has a discrete set of blowup points at most. Hence, throughout Section 2 to Section 5, we always assume that u_i is a solution of

$$\tag{2.11} \begin{cases} \Delta u_i + K_i(x)u_i^{\frac{n+2}{n-2}}(x) = 0 \quad \text{on } \overline{B}_2\backslash\{0\}, \\ u_i(x) \ \text{ is uniformly bounded in any compact} \\ \qquad \text{set of } \overline{B}_2\backslash\{0\}, \end{cases}$$

where $B_2 = \{x : |x| < 2\}$, and $K_i(x) = n(n-2)+t_i\hat{K}$ where $\hat{K}$ satisfies (2.10) with $x_0 = 0$ for $x \in \overline{B}_2$ and some $\beta \geq 1$. Here, solutions u_i is assumed to blow up at 0. Let $\hat{M}_i$ denote the maximum of u_i and x_i be a maximum point of u_i, i.e.,

$$\tag{2.12} \hat{M}_i = u_i(x_i) = \max_{|x|\leq 2} u_i(x) \to +\infty$$

as $i \to +\infty$. Clearly $x_i \to 0$. If $\beta = 1$ or $\beta \geq n-2$, by Theorem A, (2.3) holds for some constant $c > 0$. When $1 < \beta < n-2$, the situation is more complicated as shown in [9].

A solution u_i may have local maximum points beside x_i. Let z_i be any local maximum point of u_i with $u_i(z_i) \to +\infty$. Then by assumption (2.11), $\lim_{i\to\infty} z_i = 0$. Let $v_i(y)$ be the scaled function defined by

$$\tag{2.13} v_i(y) = M_i^{-1}u_i(z_i + M_i^{-\frac{2}{n-2}}y) \quad \text{with} \quad M_i = u_i(z_i).$$

Obviously, $v_i(y)$ is well-defined for $|y| \leq M_i^{\frac{2}{n-2}}$ when i is large. In the paper, we will always reduce the arguments to the situation when

(2.14) $v_i(y)$ is uniformly bounded in any compact set of $\mathbb{R}^n$, that is, for any $\varepsilon > 0$, there exists a sequence of $R_i \to +\infty$ such that

$$|v_i(y) - U_1(y)| \leq \varepsilon U_1(y) \quad \text{for} \quad |y| \leq R_i.$$

In this case, by passing to a subsequence, $v_i(y)$ converges to $U_1(y)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$, where $U_1(y)$ is given in (2.4) with $\lambda = 1$.

For such "good" local maximum point z_i, we set

$$L_i(z_i) = \min\left[(t_i^{-1} u_i(z_i)^{\frac{2}{n-2}} |z_i|^{1-\beta})^{\frac{1}{n-2}}, (t_i^{-1} u_i(z_i)^{\frac{2\hat{\beta}}{n-2}})^{\frac{1}{n-2}}\right], \tag{2.15}$$

where $\hat{\beta} = \beta$ if $\beta < n$ and $\hat{\beta}$ is any positive number in $(n-1, n)$ if $\beta \geq n$. One of the main themes for local solutions is to know if the scaled vector $M_i^{\frac{2}{n-2}} z_i$ is bounded. This is closely related to the quantity $L_i(z_i)$. To see this, let us assume $\beta < n$ for simplicity. In this case, if $\lim\limits_{i\to+\infty} u_i(z_i)|z_i|^{\frac{n-2}{2}} = +\infty$, then

$$u_i(z_i)^{\frac{2}{n-2}} |z_i|^{1-\beta} = (u_i(z_i)^{\frac{2}{n-2}} |z_i|)^{1-\beta} u_i(z_i)^{\frac{2\beta}{n-2}} = o(1) u_i(z_i)^{\frac{2\beta}{n-2}}$$

and

$$L_i(z_i) = (t_i^{-1} u_i(z_i)^{\frac{2}{n-2}} |z_i|^{1-\beta})^{\frac{1}{n-2}}.$$

On the other hand, if

$$\lim_{i\to+\infty} u_i(z_i)|z_i|^{\frac{n-2}{2}} < +\infty,$$

then it is easy to see

$$L_i(z_i) \sim (t_i^{-1} u_i(z_i)^{\frac{2\beta}{n-2}})^{\frac{1}{n-2}}.$$

The quantity $L_i(z_i)$ plays an important role for us to understand the bubbling profile of u_i. Our first result concerns with $L_i(x_i)$ and the simple blowup at 0. We recall x_i is a maximum point of u_i and $\hat{M}_i = u_i(x_i)$ is the maximum of u_i. See (2.12).

Theorem 2.1. *Suppose u_i is a solution of* (2.11) *and $\hat{K}$ satisfies* (2.10) *for some $\beta \geq 1$. Assume* (1.4) *in addition if $\beta < n-2$. Then after passing to a subsequence,* 0 *is a simple blow-up point if and only if there exists a constant $c > 0$ independent of i such that*

$$\hat{M}_i^{\frac{2}{n-2}} \leq c\, L_i(x_i)$$

for all i.

An interesting case is when the ratio $\hat{M}_i^{-\frac{2}{n-2}} L(x_i)$ tends to $+\infty$ as $i \to +\infty$. If u_i is a *global* solution of (1.3), by applying the method of moving planes, we can prove that 0 is the only simple blowup point. See (6.8).

On the other hand, when the ratio $\hat{M}_i^{-\frac{2}{n-2}} L(x_i)$ is bounded, we have the following result.

Theorem 2.2. *Let u_i and $\hat{K}$ satisfy the assumptions of Theorem* 2.1 *and let x_i, $\hat{M}_i$ and $L_i(x_i)$ be defined in* (2.12) *and* (2.15)*, respectively. Suppose that there is $c > 0$ such that*

$$L_i(x_i) \leq c\ \hat{M}_i^{\frac{2}{n-2}},$$

then $\hat{M}_i |x_i|^{\frac{n-2}{2}}$ is bounded and $\beta < n-2$. Furthermore, if assume in addition that $\hat{K}$ satisfies (K0) *with Q being the homogeneous function and $\lim_{i\to+\infty} \xi_i = \xi$ exists with $\xi_i = \hat{M}_i^{\frac{2}{n-2}} x_i$, then ξ satisfies*

$$\int_{\mathbb{R}^n} \nabla Q(x+\xi) U_1^{\frac{2n}{n-2}}(x)dx = 0. \tag{2.16}$$

The following consequence of Theorem 2.2 is important when we come to determine the position of blowup points for *global solutions* of (1.3).

Corollary 2.3. *Let u_i and K_i satisfy the assumptions of Theorem* 2.1*. Assume that either $\nabla \hat{K}(0) \neq 0$ or $\nabla \hat{K}(0) = 0$ with $\beta \geq n-2$, then $\lim_{i\to+\infty} L_i(x_i)\hat{M}_i^{-\frac{2}{n-2}} = +\infty$.*

Both proofs of Theorem 2.1 and 2.2 are given in Section 3, where the application of the reflection method are discussed. By Theorem A, the flatness β of $\hat{K}$ at 0 determines the bubbling behavior of u_i. Conventionally, u_i is said to lose the energy of one bubble at 0 if u_i converges to 0 in $C^1_{\text{loc}}(B_2 \setminus \{0\})$ and

$$\lim_{i\to+\infty} \int_{|x|\leq 1} u_i^{\frac{2n}{n-2}}(x)dx = \left(\frac{S_n}{n(n-2)}\right)^{\frac{n}{2}}, \tag{2.17}$$

where S_n is the Sobolev best constant. Clearly, if u_i blows up at 0 simply, then u_i lost one bubble.

Theorem 2.4. *Assume that $\hat{K}$ satisfies* (K0) *and* (K1) *at* 0 *with $\frac{n-2}{2} \leq \beta$, and u_i is a solution satisfying* (2.11)*. Then u_i loses the energy* *of only one bubble at* 0. *Suppose in addition that* $\overline{\lim\limits_{i\to+\infty}} L_i(x_i)\hat{M}_i^{-\frac{2}{n-2}} < +\infty$. *Then there exists a constant* $c > 0$ *such that*

$$u_i(x) \le c\, |x|^{\frac{2-n}{2}} \quad \textit{for } |x| \le 1. \tag{2.18}$$

Set $\xi_i = \hat{M}_i^{\frac{2}{n-2}} x_i$. *Then after passing to a subsequence, the limit* $\xi = \lim\limits_{i\to+\infty} \xi_i$ *satisfies* (1.8).

When $\beta < \frac{n-2}{2}$, it is possible that (2.18) does not hold and it is also possible that u_i loses energy of more than one bubble even (2.18) holds. We first consider the case when inequality (2.18) does not hold. There are two alternatives in this case.

Theorem 2.5. *Assume that* $\hat{K}$ *satisfies* (K0) *and* (K1) *at* 0, *and* u_i *is a solution of* (2.11). *Suppose*

$$\lim_{i\to+\infty} \sup(u_i(x)|x|^{\frac{n-2}{2}}) = +\infty. \tag{2.19}$$

Then one of the followings holds:

(i) *The origin is a simple blowup point and consequently, an isolated blowup point. More precisely, we have*

$$\begin{cases} u_i(x_i + x) \le c\, U_{\lambda_i}(x) \quad \textit{for } |x| \le 1, \textit{ and} \\ \lim\limits_{i\to+\infty} \hat{M}_i|x_i|^{\frac{n-2}{2}} = +\infty, \end{cases} \tag{2.20}$$

where $\lambda_i = \hat{M}_i^{-\frac{2}{n-2}}$.

(ii) *The origin is not a simple blowup point and is not an isolated blowup point. In this case, we have* $\beta < \dfrac{n-2}{2}$ *and there exists a local maximum point* z_i *of* u_i *satisfying*

$$u_i(z_i)|z_i|^{\frac{n-2}{2}} \to \infty \textit{ and } L_i(z_i)u_i(z_i)^{-\frac{n-2}{2}} \to \infty \textit{ as } i \to +\infty \tag{2.21}$$

such that for any $\delta > 0$, $u_i(x)$ *is a simple blowup with center* z_i *for* $x \notin B(0, \delta|z_i|)$, *i.e.,*

$$u_i(x) \le c\, U_{\lambda_i}(x - z_i) \tag{2.22}$$

for $|x| \geq \delta|z_i|$, where $\lambda_i = u_i(z_i)^{-\frac{2}{n-2}}$. Also, for $x \notin B(z_i, \delta|z_i|)$, we have

$$u_i(x)|x|^{\frac{n-2}{2}} \leq c \tag{2.23}$$

with $c = c(\delta)$ independent of i. Moreover, $u_i(z_i) = o(1)\hat{M}_i$, where $o(1)$ tends to 0 as $i \to +\infty$ and $\hat{M}_i = \max_{|x|\leq 2} u_i(x)$.

Remark 2.6. Two consequences follow from Theorem 2.5. First, since (2.22) implies

$$\min_{|x|=1} u_i(x) \sim u_i(z_i)^{-1}, \tag{2.24}$$

the spherical Harnack inequality (2.18) holds if $u_i(x) \geq c > 0$ on $\overline{B}_2$ for some c independent of i. Second, by (2.21),

$$\lim_{i\to+\infty} L_i(z_i)u_i(z_i)^{-\frac{n-2}{2}} = +\infty.$$

We will see later that this implies if u_i is a sequence of global solutions, then the number of the type of blowup points described in (ii) of Theorem 2.5 is at most one. See (6.8). By using this fact, we then are able to apply Lemma 7.1 to get rid of the blowup point of the type of behavior in case (ii) of Theorem 2.5. This is indeed Theorem 1.3.

When u_i converges to zero in $C^1_{\text{loc}}(\overline{B}_2\backslash\{0\})$, we say u_i loses energy of more than one bubble near 0 if

$$\lim_{i\to+\infty} \int_{|x|\leq 1} u_i^{\frac{2n}{n-2}}(x)dx > \left(\frac{S_n}{n(n-2)}\right)^{\frac{n}{2}}. \tag{2.25}$$

In this case, we have $\beta < \frac{n-2}{2}$ by Theorem A and Theorem 2.4. It is easy to see the blowup described in (ii) of Theorem 2.5 belongs to this case. Actually, when $\beta < \frac{n-2}{2}$, it is possible for u_i to lose infinite energy. See [11] for the existence for such solutions.

To estimate u_i more accurately when it satisfies (2.18) and loses energy of more than one bubble, let

$$\bar{u}_i(r) = \frac{1}{|\partial B_r|}\int_{|x|=r} u_i d\sigma \tag{2.26}$$

be the spherical average of u_i, and

$$w_i(s) = \bar{u}_i(r)r^{\frac{n-2}{2}} \quad \text{with} \quad r = e^s. \tag{2.27}$$

Obviously, $w_i(s)$ is well-defined for $s \leq 0$. Since 0 is a blowup point, w_i has at least one maximum point. Let $s_i \leq 0$ be the local maximum point of w_i, which is nearest to zero. Set

$$M_i = e^{-(\frac{n-2}{2})s_i}, \tag{2.28}$$

$$L_i = \left(t_i^{-1} M_i^{\frac{2\beta}{n-2}}\right)^{\frac{1}{n-2}}, \tag{2.29}$$

$$R_i = L_i^\gamma, \gamma = \frac{1}{1-\frac{2\beta}{n-2}}, \quad \text{and} \tag{2.30}$$

$$\widetilde{u}_i = M_i^{-1} u_i\left(M_i^{\frac{-2}{n-2}} x\right). \tag{2.31}$$

Then we have the following estimates:

Theorem 2.7. *Suppose that $\hat{K}$ satisfies* (K0) *and* (K1) *at* 0 *with* $1 < \beta < \dfrac{n-2}{2}$, *and u_i is a solution of* (2.11) *which converges uniformly to zero in any compact set of $\overline{B}_2 \backslash \{0\}$ and satisfies* (2.18) *and* (2.25). *Define w_i, s_i, M_i, L_i, R_i and $\widetilde{u}_i$ as above. Then $\lim_{i\to+\infty} M_i = +\infty$ and there are $c > 0$, $a_i \to 1$, $z_i \in \mathbb{R}^n$ and $\lambda_i > 0$ such that the following hold:*

(i) $\lim_{i\to\infty} \lambda_i = \lambda$ *and* $\lim_{i\to+\infty} z_i = z$, *where λ and z satisfy*

$$1 = \lambda^2 + |z|^2. \tag{2.32}$$

Set $\xi = \sqrt{\lambda} z$. Then ξ satisfies (1.8).

(ii) $\widetilde{u}_i$ *satisfies*

$$|\widetilde{u}_i(x)| \leq c\,|x|^{-\frac{n-2}{2}} \quad \text{for } |x| \leq R_i^{-2}, \text{ and} \tag{2.33}$$

(2.34)
$$\begin{aligned}
&|\widetilde{u}_i(x) - a_i U_{\lambda_i}(x - z_i)| \\
&\leq c\,(L_i^{-n+2} + R_i^{-n+2}|x|^{-n+2} + \max_{|y|=M_i^{\frac{2}{n-2}}} |\widetilde{u}_i(y) - a_i U_{\lambda_i}(y - z_i)|),
\end{aligned}$$

for $R_i^{-2} \leq |x| \leq M_i^{\frac{2}{n-2}}$.

Remark 2.8. If $L_iM_i^{-\frac{2}{n-2}} \le c$ for some constant $c > 0$, then from the proof of Theorem 2.7, we will see that $\widetilde{u}_i(y) \le c_1 L_i^{-n+2}$ for some constant c_1 when $|y| = M_i^{\frac{2}{n-2}}$. Thus, the third term in the right hand side of (2.34) can be absorbed by L_i^{-n+2} when $L_iM_i^{-\frac{2}{n-2}} \le c$.

To extend the notion of simple blowup to cover the case when u_i loses energy of more than one bubble, we modify (2.3) as follows. Let $B_r(y)$ denote $\{x : |x - y| < r\}$.

Definition 2.9. Assume 0 is a blowup point. The blowup point 0 is called simple-like if there exist $c > 0$, $r_0 > 0$, a sequence of numbers $\{\lambda_i\}$, a sequence of points $\{z_i\}$ and a sequence of balls $\{B_{r_i}(y_i)\}$ such that $\lim_{i\to\infty} \lambda_i = 0$, $\lim_{i\to\infty} z_i = \lim_{i\to\infty} y_i = 0$, $\lim_{i\to\infty} r_i\lambda_i^{-1} = 0$, and

$$u_i(x + z_i) \le c\, U_{\lambda_i}(x) \text{ on } B_{r_0}(0) \setminus B_{r_i}(y_i).$$

According to the definition, it is not difficult to see that there are exactly three types of simple-like blowup point: simple blowup, the blowup described in (ii) of Theorem 2.5, and the blowup in Theorem 2.7 when $L_i \ge cM_i^{\frac{2}{n-2}}$ for some constant $c > 0$. On the other hand, if 0 is non-simple-like, then by Theorem 2.5, inequality (2.18) holds and 0 must be isolated.

Remark 2.10. When the assumption (K1) is concerned in the theorems of this section, (K1) is required only when $\beta < n - 2$.

3. Applications of the method of moving planes

In this section, we will collect some well-known results and prove some lemmas which will be used in the proofs of the theorems in Section 2. In the proofs, we often assume there is a sequence of local maximum points z_i of u_i such that the scaled function v_i in (2.13) satisfies (2.14). By applying the method of moving planes, we can improve the result of (2.14). When K_i satisfies if the nondegenerate conditions (K0) and (K1) with $1 < \beta \le n - 2$, we proved that $u_i(z_i + x)$ could be bounded by $c\, U_{\lambda_i}(x)$ with $\lambda_i = u_i(z_i)^{-\frac{2}{n-2}}$ for $|x| \le L_iM_i^{-\frac{2}{n-2}}$. See Lemma 3.1 in [9]. Actually the proof there can apply to the degenerate case. In the following, we give a brief sketch of the proof for the convenience of readers. In fact, Lemma 3.1 below deals with the case more general than the one considered in [9], namely, u_i is allowed to have very large values, compared with $u_i(z_i)$, in some small region. Let $d(B,0)$ denote the distance from the origin to a ball B.

Lemma 3.1. *Suppose that u_i is a solution of* (2.11)*, z_i is a local maximum point of u_i and v_i is given as in* (2.13)*. Let B be a closed ball in $\mathbb{R}^n$ with $d(B,0)>0$ and ε be a positive* (*small*) *number. Suppose that there is a sequence of $R_i \to +\infty$ as $i \to +\infty$ such that*

$$|v_i(y) - U_1(y)| \le \varepsilon U_1(y)$$

for $|y| \le R_i$ and $y \notin B$. Then there exists $\delta = \delta(\varepsilon, d(B,0)) > 0$ such that

$$\min_{|y|\le r} v_i(y) \le (1+2\epsilon)U_1(r) \tag{3.1}$$

for $0 \le r \le L_i^*(\delta)$, where $L_i^*(\delta) = \min(\delta L_i(z_i), M_i^{\frac{2}{n-2}})$.

Proof. When B is an empty set and $1 \le \beta \le n-2$, this is Lemma 3.1 in [9]. Thus, we only sketch the proof below. For the details, we refer the interested readers to [9].

Let $e_1 = (1,0,\cdots,0)$ and $\tau = d(B,0)$. We may assume the center of B is $r_0 e_1$ for some $r_0 > \tau$. Let

$$\begin{aligned}
F(x) &= \frac{\tau^2 x}{|x|^2} + \tau e_1, \\
\bar v_i(x) &= \Big(\frac{\tau}{|x|}\Big)^{n-2} v_i\big(\frac{\tau^2 x}{|x|^2} + \tau e_1\big), \\
\bar U_1(x) &= \Big(\frac{\tau}{|x|}\Big)^{n-2} U_1\big(\frac{\tau^2 x}{|x|^2} + \tau e_1\big).
\end{aligned} \tag{3.2}$$

By a straighforward calculation, we have

$$\bar U_1(x) = \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}},$$

where $\lambda = \dfrac{\tau^2}{\tau^2+1}$ and $x_0 = -\dfrac{\tau^3 e_1}{\tau^2+1}$. Also we have $F^{-1}(B) = \{x : x = F^{-1}(y), y \in B\} \subset \{(x_1, x_2, \cdots, x_n) : x_1 > 0\}, d(F(B),0) > 0$ and $\bar v_i$ satisfies

$$\triangle \bar v_i + \overline{K}_i(x) \bar v_i^{\frac{n+2}{n-2}} = 0$$

for $x \notin F^{-1}(B)$, where $\overline{K}_i(x) = K_i(z_i + M_i^{-\frac{2}{n-2}} F(x))$.

Now assume that the conclusion of Lemma 3.1 does not hold. Then by passing to a subsequence, there is a sequence of positive number r_i such that $r_i \le L_i^*(\delta)$ and

$$\min_{|y|\le r_i} v_i(y) \ge (1+2\epsilon)U_1(r_i), \tag{3.3}$$

where $\delta = \delta(\varepsilon)$ will be chosen later. By the assumptions, it is easy to see $r_i \ge R_i \to +\infty$ as $i \to +\infty$. Since by (3.2), $\bar{v}_i(x)$ uniformly converges to $\overline{U}_1(x)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$, $\bar{v}_i$ has a local maximum at some point q_i near x_0. Now we are going to apply the method of moving planes to obtain a contradiction.

For any $\lambda < 0$, let $\Sigma_\lambda = \{x \mid x_1 > \lambda\}$, $T_\lambda = \{x \mid x_1 = \lambda\}$ and x^λ denote the reflection point of x with respect to T_λ. We also let $\Sigma'_\lambda = \Sigma_\lambda \cap \{x \mid |x| \ge \tau^2(r_i-\tau)^{-1}\}$. In the following, we will choose a number λ_0 satisfying $-|x_0| < \lambda_0 < -\dfrac{|x_0|}{2}$ and show that for $\lambda \le \lambda_0$, there exists $i_0 = i_0(\lambda_0)$ such that

$$\bar{v}_i(x^\lambda) \le \bar{v}_i(x) \tag{3.4}$$

for $x \in \Sigma'_\lambda$, $\lambda \le \lambda_0$ and $i \ge i_0$. This yields a contradiction to the fact that $\bar{v}_i$ has a local maximum near x_0. Note that the local maximum point q_i tends to x_0 as $i \to \infty$.

Let $w_\lambda(x) = \bar{v}_i(x) - \bar{v}_i(x^\lambda)$. Then w_λ satisfies

$$\triangle w_\lambda + b_\lambda(x) w_\lambda(x) = Q_\lambda(x) \qquad \text{in } \Sigma'_\lambda, \tag{3.5}$$

where

$$\begin{cases} b_\lambda(x) = \overline{K}_i(x)\dfrac{\big(\bar{v}_i(x)^{\frac{n+2}{n-2}} - (\bar{v}_i(x^\lambda)^{\frac{n+2}{n-2}}\big)}{\bar{v}_i(x) - \bar{v}_i(x^\lambda)} \\ Q_\lambda(x) = \big(\overline{K}_i(x^\lambda) - \overline{K}_i(x)\big)\,\bar{v}_i(x^\lambda)^{\frac{n+2}{n-2}}. \end{cases}$$

By (3.2) and (3.3), we have for $|x| = \tau^2(r_i-\tau)^{-1}$,

$$\bar{v}_i(x) \ge \big(\frac{r_i-\tau}{\tau}\big)^{n-2} \min_{|y|\le r_i} v_i \ge (1+\varepsilon)\overline{U}_1(0) \tag{3.6}$$

for i large. On the other hand, $\bar{v}_i(x^{-|x_0|})$ converges to $\overline{U}_1(0^{-|x_0|}) = \overline{U}_1(0)$ uniformly for $|x| = \tau^2 r_i^{-1}$, where $x^{-|x_0|}$ and $0^{-|x_0|}$ are the reflection points of x and 0 with respect to the hyperplane $T_{-|x_0|}$. Hence there exists $-|x_0| < \lambda_0 < \dfrac{-|x_0|}{2}$ such that

$$\bar{v}_i(x^\lambda) \le (1+\frac{\varepsilon}{2})\overline{U}_1(0)$$

for $|x| = \tau^2(r_i - \tau)^{-1}, \lambda \le \lambda_0$ and large i. Together with (3.6), it implies for $|x| = \tau^2(r_i - \tau)^{-1}$,

$$w_\lambda(x) \ge \frac{\varepsilon}{2}\overline{U}_1(0)$$

for $\lambda \le \lambda_0$ and large i. In the following, we fix this λ_0. Then there is a small c_0 such that

$$w_\lambda(x) \ge \frac{\varepsilon}{2}\overline{U}(0) \ge c_0 r_i^{-n+2} G^\lambda(x, 0) \tag{3.7}$$

holds for $|x| = \tau^2(r_i - \tau)^{-1}, \lambda \le \lambda_0$ and large i, where $G^\lambda(x, y)$ is

$$G^\lambda(x, y) = c_n\Big(\frac{1}{|y - x|^{n-2}} - \frac{1}{|y^\lambda - x|^{n-2}}\Big),$$

the Green function of $-\triangle$ on $\Sigma_\lambda = \{x : x_1 > \lambda\}$.

If $\lambda_1 < 0$ and $|\lambda_1|$ is large, then we have

$$w_\lambda(x) \ge \frac{c_0}{2} r_i^{-n+2} G^\lambda(x, 0) \tag{3.8}$$

for $\lambda \le \lambda_1, x \in \Sigma'_\lambda$ and large i. For the details, see [9].

For $\lambda > \lambda_1$, let $Q^+_\lambda = \max(0, Q_\lambda)$, $L_i = L_i(z_i)$ and

$$h_\lambda(x) = aL_i^{-n+2} G^\lambda(x, 0) - \int_{\Sigma'_\lambda} G^\lambda(x, \eta) Q^+_\lambda(\eta)\, d\eta, \tag{3.9}$$

where a is a positive number to be chosen later. Obviously, h_λ satisfies

$$\triangle h_\lambda = Q^+_\lambda \ge Q_\lambda \qquad \text{in } \Sigma'_\lambda.$$

For $\lambda \le \lambda_0$ and $\eta \in \Sigma_\lambda$, since $|\eta^\lambda| \ge |\eta|$ and $|\eta^\lambda| \ge |\lambda_0| \ge \frac{|x_0|}{2} > 0$, one has by (3.2)

$$|\bar{v}_i(\eta^\lambda)| \le c_1(1 + |\eta^\lambda|)^{-(n-2)}.$$

Here, we use $F^{-1}(B) \subset \Sigma_\lambda$ also. For $\eta \in \Sigma'_\lambda$, we have

$$|\eta| \ge \tau^2(r_i - \tau)^{-1} \ge \frac{\tau^2}{2} L^*_i(\delta) \ge \frac{\tau^2}{2} M_i^{-\frac{2}{n-2}}.$$

To estimate the integral term in (3.9), we note

$$Q^+_\lambda(\eta) \le c_2(1+|\eta^\lambda|)^{-(n+2)} \big|K_i(z_i+M_i^{-\frac{2}{n-2}}F(\eta^\lambda)) - K_i(z_i+M_i^{-\frac{2}{n-2}}F(\eta))\big|.$$

By (2.10), when $\eta \in \Sigma'_\lambda$,

(3.10)
$$\begin{aligned}
&|K_i(z_i + M_i^{-\frac{2}{n-2}} F(\eta)) - K_i(z_i)| \\
&\qquad \le c t_i M_i^{-\frac{2}{n-2}} |F(\eta)| \left\{ |z_i|^{\beta-1} + M_i^{-\frac{2(\hat\beta-1)}{n-2}} |F(\eta)|^{\hat\beta-1} \right\} \\
&\qquad \le c_3 t_i M_i^{-\frac{2}{n-2}} (1 + |\eta|^{-1}) \left\{ |z_i|^{\beta-1} + M_i^{-\frac{2(\hat\beta-1)}{n-2}} (1 + |\eta|^{1-\hat\beta}) \right\} \\
&\qquad \le c_4 L_i^{2-n} (1 + |\eta|^{-\hat\beta}),
\end{aligned}$$

where $|\eta| \ge \frac{\tau^2}{2} M_i^{-\frac{2}{n-2}}$ is used and $\hat\beta$ is the number in (2.15). Thus, we have

$$Q^+_\lambda(\eta) \le c_5 L_i^{-n+2} (1 + |\eta|^{-\hat\beta})(1 + |\eta^\lambda|)^{-(n+2)}. \tag{3.11}$$

By (3.11), following the computation in the proof of Lemma 3.1 in [9], we obtain

$$\int_{\Sigma'_\lambda} G^\lambda(x,\eta) Q^+_\lambda(\eta) d\eta \le c_6 L_i^{-n+2} G^\lambda(x,0) \tag{3.12}$$

for $x \in \Sigma'_\lambda$, where c_6 is a constant depending on the constants in (2.10), τ and n only.

Set $a = 2c_6$ in (3.9). Then

$$0 < \frac{a}{2}[L(z_i)]^{-n+2} G^\lambda(x,0) \le h_\lambda(x) \le a[L(z_i)]^{-n+2} G^\lambda(x,0). \tag{3.13}$$

Recall that $r_i \le \delta L_i(z_i)$. Choose δ to be sufficiently small such that $c_0 \delta^{-n+2} \ge 2a$. Then by (3.7) and (3.8), for i large,

$$w_\lambda(x) > h_\lambda(x)$$

holds for $x \in \Sigma'_\lambda$ if $\lambda = \lambda_1$, and holds for $|x| = \tau^2(r_i - \tau)^{-1}$ and $\lambda \le \lambda_0$. It follows that h_λ satisfies the assumptions of Lemma 2.1 in [9] with $\lambda_1 \le \lambda \le \lambda_0$ when i is large. Applying Lemma 2.1 in [9], $w_\lambda(x) > h_\lambda(x) > 0$ for $x \in \Sigma'_\lambda$ and $\lambda \le \lambda_0$. Hence, (3.4) is proved, and then the proof of Lemma 3.1 is finished. q.e.d.

Note that if u_i is a global solution defined in the whole space $\mathbb{R}^n$, then we can choose

$$L_i^*(\delta) = \min(L_i^*(\delta), \lambda M_i^{\frac{2}{n-2}})$$

for any $\lambda > 0$. Inequality (3.1) is very useful when the Harnack inequality holds for v_i on each sphere $|y| = r$. Actually, under some extra condition on u_i, we can derive the spherical Harnack inequality from (3.1) itself by using the Green representation formula. We will explain this in Lemma 3.4, which tells us how to derive the Harnack inequality. Before that, we have to state two well-known lemmas. For their proofs, see [9].

Lemma 3.2. *Suppose $\phi(x)$ satisfies*

$$\triangle\phi(x) + n(n+2)U_1^{\frac{4}{n-2}}\phi(x) = 0 \qquad \text{in } \mathbb{R}^n$$

with $\phi(x) \to 0$ as $|y| \to \infty$. Then $\phi(x)$ can be written as

$$\phi(x) = c_0\psi_0(x) + \sum_{j=1}^{n} c_j\psi_j(x)$$

for some $c_j \in \mathbb{R}$, $j = 0, 1, \dots, n$, where $\psi_j(x) = \dfrac{\partial U_1}{\partial x_j}$ for $1 \le j \le n$ and $\psi_0(x) = \dfrac{n-2}{2}U_1 + x \cdot \nabla U_1$.

Lemma 3.3. *Suppose that u is a positive smooth solution of*

$$\triangle u + K(x)u^{\frac{n+2}{n-2}} = 0 \text{ in } B_r,$$

where $|K(x)| \le b$. Then there exists a small $\epsilon_o > 0$, depending on b and n only, such that if $||u||_{L^{\frac{2n}{n-2}}} \le \epsilon_o$, then the Harnack inequality

$$u(x) \le c\ u(y)$$

holds for $|x|, |y| \le r/4$, where $c > 0$ depends on b and n only.

In Lemma 3.4, we consider a more general setting, which is needed later. Assume that $0 < a \le K(x) \le b$, u is a solution of

$$\triangle u + K(x)u^{\frac{n+2}{n-2}} = 0, u > 0 \text{ for } |x| \le l_0, \tag{3.14}$$

and U is the solution of

$$\begin{cases} \triangle U + K_0 U^{\frac{n+2}{n-2}} = 0, U > 0 \text{ in } \mathbb{R}^n, \\ U(0) = \max\limits_{\mathbb{R}^n} U = 1, \end{cases} \tag{3.15}$$

where K_0 is a positive constant. Let $B_r = \{x : |x| < r\}$.

Lemma 3.4. *Let u, U and l_0 be as above. Suppose $0 < \sigma < 1$, $R \le \frac{l_0}{8}$, and $E \subseteq B_{R/2}$ such that*

$$|u(x) - U(x)| \le \sigma U(x) \tag{3.16}$$

for $x \in B_R \backslash E$,

$$\int_{|x|\le R} |K(x) - K_0| U^{\frac{n+2}{n-2}} dx \le \sigma, \tag{3.17}$$

$$\int_E U^{\frac{n+2}{n-2}} dx < \sigma, \quad \text{and} \tag{3.18}$$

$$\min_{|x|=l} u(x) \le (1+\sigma) U(r) \tag{3.19}$$

for some $l \in [R, \frac{l_0}{4}]$. Then there is a constant c_1 depending on n and b only such that

$$\int_{R\le|x|\le l} u^{\frac{n+2}{n-2}} dx \le c_1 (R^{-2} + \sigma + (\frac{l}{l_0})^{n-2}), \tag{3.20}$$

Furthermore, if

$$u(x) \le c_2 \ (R^{-2} + \sigma + (\frac{l}{l_0})^{n-2})^{-1}, \quad \text{and} \tag{3.21}$$

$$\min_{|x|=r} u(x) \le c_3 \ U(r) \tag{3.22}$$

for $R \le r \le l$ where $c_2 = c_2(n, a, b)$ is a small positive constant and $c_3 > 0$, then

$$u(x) \le c_4 U(x) \tag{3.23}$$

for $|x| \le \frac{l}{2}$ and $x \notin E$, where c_4 depends on c_2 and c_3.

Proof. For $r > 0$, let $B_r = \{x : |x| < r\}$. Let $G(x, \eta)$ be the Green function of the Laplacian operator $-\triangle$ on the ball B_{l_0} with zero boundary value. Let x_0 be a point satisfy $|x_0| = l$ and $u(x_0) = \min_{|x|\leq l} u(x)$. By the Green identity and (3.19),

$$(1+\sigma)U(x_0) \geq u(x_0) \geq \int_{B_{l_0}} G(x_0,\eta)K(\eta)u^{\frac{n+2}{n-2}}(\eta)d\eta, \tag{3.24}$$

and

$$\begin{aligned} U(x_0) &= \int_{B_{l_0}} G(x_0,\eta)K_0 U^{\frac{n+2}{n-2}}\,d\eta + U(l_0) \\ &\leq \int_{B_{l_0}} G(x_0,\eta)K_0 U^{\frac{n+2}{n-2}}\,d\eta + U(l_0) \end{aligned} \tag{3.25}$$

Hence there is c_n depending on n only such that

$$\begin{aligned} &a\ c_n \int_{\frac{R}{2}\leq\eta\leq\frac{l_0}{2}} (l+|\eta|)^{-n+2} u^{\frac{n+2}{n-2}}\,d\eta \\ &\quad\leq\ u(x_0) - \int_{B_{\frac{R}{2}}\setminus E} G(x_0,\eta)K(\eta)u^{\frac{n+2}{n-2}}\,d\eta \\ &\quad\leq\ (1+\sigma)U(x_0) - \int_{B_{\frac{R}{2}}\setminus E} G(x_0,\eta)K(\eta)u^{\frac{n+2}{n-2}}\,d\eta. \end{aligned} \tag{3.26}$$

By the assumptions (3.16) and (3.17), there is c_4 depending on n and b only such that

$$\begin{aligned} &\int_{B_{\frac{R}{2}}\setminus E} G(x_0,\eta)K(\eta)u^{\frac{n+2}{n-2}}\,d\eta \\ &\quad\geq \int_{B_{\frac{R}{2}}\setminus E} G(x_0,\eta)\big\{K_0 U^{\frac{n+2}{n-2}} + K(\eta)(u^{\frac{n+2}{n-2}} - U^{\frac{n+2}{n-2}}) \\ &\qquad\qquad - |K(\eta)-K_0|U^{\frac{n+2}{n-2}}\big\}\,d\eta \\ &\quad\geq \int_{B_{\frac{R}{2}}} G(x_0,\eta)K_0 U^{\frac{n+2}{n-2}}\,d\eta - c_4 l^{-n+2}\sigma. \end{aligned}$$

Together with (3.25), it leads to

$$\begin{aligned}
a\ c_n & \int_{\frac{R}{2}\le\eta\le\frac{l_0}{2}} (1+l^{-1}|x|)^{-n+2}u^{\frac{n+2}{n-2}}\,d\eta \\
& \le\ l^{n-2}\left[\int_{B_{l_0}\setminus B_{\frac{R}{2}}} G(x_0,\eta)K_0U^{\frac{n+2}{n-2}}\,d\eta + c_4 l^{-n+2}\sigma\right] \\
& \quad + l^{n-2}\left[\sigma U(x_0)+U(l_0)\right] \\
& \le\ c_5\ (\sigma+R^{-2}+(\frac{l}{l_0})^{n-2}),
\end{aligned}$$

where c_5 depends on n and b only. Obviously the inequality (3.20) follows immediately.

Let ϵ_0 be the number in Lemma 3.3 and c_2 be a small number such that

$$c_2c_5(c_na)^{-1} < \epsilon_0.$$

If $u(x) \le c_2(R^{-2}+\sigma+(\frac{l}{l_0})^{n-2})^{-1}$ for $\frac{R}{2} \le |x| \le l$, then

$$\int_{\frac{R}{2}\le|\eta|\le l} u^{\frac{2n}{n-2}}\,d\eta < \int_{\frac{R}{2}\le|\eta|\le l} u^{\frac{n+2}{n-2}}d\eta\left(\max_{\frac{R}{2}\le|\eta|\le l} u\right) < \epsilon_0.$$

By Lemma 3.3, the Harnack inequality holds for u on $\{x : |x| = r\}$ with $R \le r \le \dfrac{l}{2}$. The inequality (3.23) then follows from it and (3.22) for $R \le r \le \frac{l}{2}$. Together with (3.16), (3.23) holds for all $|x| \le \frac{l}{2}$ and $x \notin E$.

q.e.d.

Let z_i be a local maximum point and v_i be the scaled solution in (2.13) such that (2.14) holds and $U_i(y)$ be the solution of (3.15) with $K_0 = K_i(z_i)$. In the next step, we are going to estimate the difference between v_i and $U_i(y)$. By (2.14), for any $\varepsilon > 0$, we have a sequence of $R_i \to +\infty$ such that

$$|v_i(y) - U_i(y)| \le \varepsilon U_i(y) \quad \text{for} \quad |y| \le R_i.$$

By Lemma 3.1, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$\min_{|y|=r} v_i(y) \le (1+2\varepsilon)U_i(r) \tag{3.27}$$

for $0 \le r \le L_i^*(\delta_0)$. Then Lemma 3.4 yields the following important result.

Lemma 3.5. *Let v_i and U_i be described as above. Suppose that there is a sequence of positive number $l_i \le L_i^*(\delta_0)$ such that*

$$v_i(y) \le \bar{c}_1 \quad \textit{for} \quad |y| \le l_i. \tag{3.28}$$

Then there exists a small $d > 0$ such that

$$v_i(y) \le \bar{c}_2 U_i(y), \quad \textit{and} \tag{3.29}$$

$$|v_i(y) - U_i(y)| \le \bar{c}_2 r_i^{-n+2} \tag{3.30}$$

for $|y| \le r_i = dl_i$ where d is a constant depending on n only. Furthermore, let $\widetilde{Q}_i(y) = K_i(z_i) - K_i(z_i + M_i^{-\frac{2}{n-2}} y)$. Then for $r \le r_i$,

$$\Big| \int_{|y|\le r} \widetilde{Q}(y) U_i^{\frac{n+2}{n-2}}(y) \psi_0(y) dy \Big| \le c_1 r^{-n+2}, \tag{3.31}$$

and

$$\Big| \int_{|y|\le r} \widetilde{Q}(y) U_i^{\frac{n+2}{n-2}}(y) \psi_j(y) dy \Big| \le c_1 r^{-n+1} \tag{3.32}$$

for $1 \le j \le n$, where $\psi_j(x)$ are given in Lemma 3.2.

Proof. Without loss of generality, we might assume $R_i << l_i$. Otherwise, (3.29)–(3.30) hold automatically. By Lemma 3.1, (3.27) holds for $0 \le r \le l_i$. Since $K_i = n(n-2) + t_i \hat{K}$, we have

$$\int_{|x|\le R_i} |\widetilde{K}_i(x) - K_i(z_i)| U^{\frac{n+2}{n-2}}(x) dx \le \bar{c}\ t_i \le \varepsilon,$$

for t_i small, where $\widetilde{K}_i(x) = K_i\Big(z_i + M_i^{-\frac{2}{n-2}} x\Big)$. Thus, v_i satisfies assumptions (3.16) $\sim$ (3.19) with an empty set E, $R = R_i$, $l = dl_i$ and $l_0 = M_i^{\frac{2}{n-2}}$. Let d be small such that

$$\bar{c}_1 (R_i^{-2} + \varepsilon + d^{n-2}) < c_2,$$

where c_2 is the constant in (3.21). Then by (3.28), we have

$$v_i(y) \le c_2 (R_i^{-2} + \varepsilon + d^{n-2})^{-1} \quad \text{for} \quad |y| \le l_i.$$

Then (3.29) follows immediately from Lemma 3.4. The inequality (3.30) can be proved by the same argument as in Lemma 3.3 of [9]. Hence, we omit the proof here.

To Prove (3.31) and (3.32), we let $w_i = v_i(y) - U_i(y)$. Then w_i satisfies

$$\Delta w_i + \widetilde{b}_i(y) w_i(y) = \widetilde{Q}_i(y) U_i^{\frac{n+2}{n-2}}(y), \tag{3.33}$$

where

$$\begin{cases} \widetilde{b}_i(y) = \widetilde{K}_i(y) \left(\dfrac{v_i^{\frac{n+2}{n-2}} - U_i^{\frac{n+2}{n-2}}}{v_i - U_i} \right), \\ \widetilde{K}_i(y) = K_i\Big(z_i + M_i^{-\frac{2}{n-2}} y\Big), \quad \text{and} \\ \widetilde{Q}_i(y) = K_i(z_i) - \widetilde{K}_i(y). \end{cases} \tag{3.34}$$

Multiplying (3.33) by ψ_j, one has

$$\begin{aligned} \int_{|y| \le r} w_i (\Delta \psi_j + \widetilde{b}_i \psi_j) dy + \int_{|y|=r} \left(\psi_j \frac{\partial w_i}{\partial \nu} - w_i \frac{\partial \psi_j}{\partial \nu} \right) d\sigma& \\ = \int_{|y| \le r} \widetilde{Q}_i U_i^{\frac{n+2}{n-2}} \psi_j dy& \end{aligned} \tag{3.35}$$

for $0 \le j \le n$. Let $r_i = d l_i$. By (3.30), we have for $|y| \le r_i$,

$$|v_i(y) - U_i(y)| \le \bar{c}_2 r_i^{2-n}. \tag{3.36}$$

To estimate the first term of (3.35), we recall

$$\Delta \psi_j + \frac{n+2}{n-2} K_i(z_i) U_i^{\frac{4}{n-2}} \psi_j = 0,$$

and then

$$\begin{aligned} w_i(\Delta \psi_j + \widetilde{b}_i \psi_j) =& (\widetilde{K}_i(y) - K_i(z_i)) \Big(v_i^{\frac{n+2}{n-2}} - U_i^{\frac{n+2}{n-2}} \Big) \psi_j \\ &+ K_i(z_i) \Big(v_i^{\frac{n+2}{n-2}} - U_i^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} U_i^{\frac{4}{n-2}} w_i \Big) \psi_j. \end{aligned}$$

Hence for $j = 0$, we have as in (3.10)

$$\begin{aligned} &|w_i(\Delta \psi_0 + \widetilde{b}_i \psi_0)| \\ &\le c \Big\{ r_i^{2-n} L_i(z_i)^{-n+2} (1+|y|)^{\hat{\beta}-n-2} + r_i^{2(2-n)} (1+|y|)^{-4} \Big\} \\ &\le 2c \; r_i^{2(2-n)} (1+|y|)^{-2}, \end{aligned} \tag{3.37}$$

where $|\psi_0(y)| \le c(1+|y|)^{2-n}$ and $\hat{\beta} < n$ are used. Similarly, by

$$|\psi_j(y)| \le c(1+|y|)^{1-n} \text{ for } 1 \le j \le n,$$

we have

$$|w_i(\Delta\psi_j + \widetilde{b}_i\psi_j)| \le c\, r_i^{2(2-n)}(1+|y|)^{-3}. \tag{3.38}$$

By applying (3.37) and (3.38), we have

$$\Big|\int_{B_r} w_i(\Delta\psi_j + \widetilde{b}_i\psi_j)dy\Big| = O(r^{2-n})$$

for $j = 0$, and

$$\Big|\int_{B_r} w_i(\Delta\psi_j + \widetilde{b}_i\psi_j)dy\Big| = O(r^{1-n})$$

for $1 \le j \le n$. When $|y| = r$, we have

$$|\bigtriangledown v_i(y)| \le c|y|^{-1}v_i(y) = O(|y|^{1-n})$$

by the gradient estimate. Therefore, the boundary term of (3.35) is bounded by $O(r^{2-n})$ for $j = 0$ and is bounded by $O(r^{1-n})$ for $1 \le j \le n$. Both (3.31) and (3.32) then follow from (3.35). q.e.d.

Proof of Theorem 2.2. We prove Theorem 2.2 by contradiction. Suppose $\lim\limits_{i\to+\infty} \hat{M}_i\, |x_i|^{\frac{n-2}{2}} = +\infty$. If $\beta \ge n-2$, by the definition (2.15) and the assumption that $L_i(x_i)\hat{M}_i^{-\frac{2}{n-2}}$ is bounded, we have

$$L_i(x_i) = \left(t_i^{-1}\hat{M}_i^{\frac{2}{n-2}}|x_i|^{1-\beta}\right)^{\frac{1}{n-2}}. \tag{3.39}$$

If $1 \le \beta < n-2$, then

$$\begin{aligned} t_i^{-1}\hat{M}_i^{\frac{2}{n-2}}|x_i|^{1-\beta} &= t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}}\left(\hat{M}_i^{\frac{2}{n-2}}|x_i|\right)^{1-\beta} \\ &\le t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}}, \end{aligned}$$

which implies (3.39) also.

Let $v_i(y)$ be defined as (2.13) with $z_i = x_i$. Obviously, $v_i(y) \le 1$ for $|y| \le \hat{M}_i^{\frac{2}{n-2}}$. By Lemma 3.5, there exists a $\delta_2 > 0$ such that (3.29)–(3.32) hold with dl_i replaced by $\delta_2 L_i(x_i)$. Recall the quantity $\widetilde{Q}_i$ in

Lemma 3.5. We may assume $\lim_{i\to+\infty} \frac{\nabla \hat{K}(x_i)}{|\nabla \hat{K}(x_i)|} = e_1 = (1,0,\dots,0)$. Then

$$
\begin{aligned}
-\widetilde{Q}_i &= K_i\Big(x_i + \hat{M}_i^{\frac{-2}{n-2}} y\Big) - K_i(x_i) \\
&= t_i \hat{M}_i^{\frac{-2}{n-2}} (\nabla \hat{K}(x_i), y) + c(\delta,i) t_i M_i^{-\frac{2}{n-2}} |\nabla \hat{K}(x_i)||y| \\
&= t_i \hat{M}_i^{\frac{-2}{n-2}} |\nabla \hat{K}(x_i)| y_1 + c(\delta,i) t_i M_i^{-\frac{2}{n-2}} |\nabla \hat{K}(x_i)||y|
\end{aligned}
\tag{3.40}
$$

for $|y| \le \delta \hat{M}_i^{\frac{2}{n-2}} |x_i|$, where $c(\delta,i)$ could be arbitrarily small if i is large and δ is small. Therefore, we can choose δ small enough so that

$$
\begin{aligned}
\int_{|y|\le r_i} (-\widetilde{Q}_i) U_i^{\frac{n+2}{n-2}}(y)\psi_1(y)dy &\ge c\ t_i \hat{M}_i^{-\frac{2}{n-2}} |x_i|^{\beta-1} \\
&= c\ (L_i(x_i))^{2-n}
\end{aligned}
\tag{3.41}
$$

for some $c > 0$ where $r_i = \delta \hat{M}_i^{\frac{2}{n-2}} |x_i|$. For the simplicity of notations, we let $l_i = \delta_2 L_i(x_i)$. If $r_i \ge l_i$, then by (3.41), we have

$$
\int_{|y|\le l_i} (-\widetilde{Q}_i) U_i^{\frac{n+2}{n-2}}(y)|\psi_1(y)|dy \ge c_1 (L_i(x_i))^{2-n}. \tag{3.42}
$$

If $l_i \ge r_i$, as in (3.10), we have

$$
\begin{aligned}
&\int_{r_i\le|y|\le l_i} |\widetilde{Q}_i| U_i^{\frac{n+2}{n-2}}(y)\psi_1(y)dy \\
&\quad \le c \int_{r_i\le|y|\le l_i} \Big(L_i(x_i)^{-n+2}|y|^{-2n} + t_i \hat{M}_i^{-\frac{2\hat{\beta}}{n-2}} |y|^{-n-1}\Big) dy \\
&\quad = o(1) L_i(x_i)^{2-n}.
\end{aligned}
$$

Together with (3.41), it implies that (3.42) holds also in the case of $l_i \ge r_i$.

On the other hand, by (3.32), we have

$$
\left| \int_{|y|\le l_i} \widetilde{Q}_i U_i^{\frac{n+2}{n-2}} \psi_1 dy \right| \le c_1 L_i(x_i)^{-n+1}.
$$

This contradicts (3.42). Hence we conclude $\hat{M}_i^{\frac{2}{n-2}} |x_i|$ is bounded.

Suppose $\beta \geq n-2$. Since $\hat{M}_i^{\frac{2}{n-2}}|x_i|$ is bounded, we have

$$\begin{aligned}\hat{M}_i^{\frac{2}{n-2}}|x_i|^{1-\beta} &= \Big(\hat{M}_i^{\frac{2}{n-2}}|x_i|\Big)^{1-\beta}\hat{M}_i^{\frac{2\beta}{n-2}}\\ &\geq c_1\hat{M}_i^2.\end{aligned}$$

Hence,

$$\lim_{i\to+\infty} L_i^{n-2}(x_i)\hat{M}_i^{-2} \geq c_1 \lim_{i\to+\infty} t_i^{-1} = +\infty,$$

which yields a contradiction to our assumptions. Thus, $\beta < n-2$ must hold.

To prove (2.16), we let $w_i(y) = l_i^{n-2}(v_i(y) - U_i(y))$ where $l_i = \delta_2 L_i(x_i)$. Then w_i satisfies (3.33) with $\widetilde{Q}_i(y)$ replaced $l_i^{n-2}\widetilde{Q}_i$ in the right hand side. By (K0),

$$\begin{aligned}\widetilde{Q}_i(y) &= K_i(x_i) - K_i\Big(x_i + \hat{M}_i^{\frac{2}{n-2}}y\Big)\\ &= -t_i\left[Q\Big(x_i + \hat{M}_i^{-\frac{2}{n-2}}y\Big) + R\Big(x_i + \hat{M}_i^{-\frac{2}{n-2}}y\Big)\right]\\ &\quad + (K_i(x_i) - K_i(0))\\ &= -t_i\hat{M}_i^{-\frac{2\beta}{n-2}}\Big[Q(\xi_i + y) + o(1)(|y|^\beta + 1)\Big]\\ &\quad + (K_i(x_i) - K_i(0)),\end{aligned} \tag{3.43}$$

where $\xi_i = \hat{M}_i^{\frac{2}{n-2}}x_i$. By (3.30) of Lemma 3.5, $w_i(y)$ is uniformly bounded in $\mathbb{R}^n$. After passing to a subsequence, we may assume that $w_i(y)$ converges to $w(y)$ in $C^2_{\text{loc}}(\mathbb{R}^n)$. Since $\beta < n-2$ and $\hat{M}_i^{\frac{2}{n-2}}|x_i|$ is bounded, we have $L_i^{-n+2} \sim t_i\hat{M}_i^{-\frac{2\beta}{n-2}}$. We may assume

$$c = \lim_{i\to\infty} t_i l_i^{n-2}\hat{M}_i^{\frac{-2\beta}{n-2}} > 0$$

exists. Multiplying both sides of (3.33) by $\psi_j = \frac{\partial U_i}{\partial y_j}$, we have by integration by parts,

$$\begin{aligned}\int_{B_{l_i}} l_i^{n-2}\widetilde{Q}_i(y)U_i^{\frac{n+2}{n-2}}\psi_j(y)dy = &\int_{B_{l_i}} w_i(\Delta\psi_j + \widetilde{b}_i(y)\psi_j)dy\\ &+ \int_{\partial B_{l_i}}\Big(\psi_j\frac{\partial w_i}{\partial\nu} - w_i\frac{\partial\psi_j}{\partial\nu}\Big)d\sigma.\end{aligned}$$

By (3.30), the boundary term $= O(l_i^{-1}) \to 0$ as $i \to +\infty$, and

$$\begin{aligned}|\Delta\psi_j + \widetilde{b}_i(y)\psi_j| &\le |\widetilde{b}_i(y)\psi_j(y)| + (n+2)n\Big|U_1^{\frac{4}{n-2}}(y)\psi_j(y)\Big| \\ &\le c(1+|y|)^{-(n+2)}.\end{aligned}$$

Thus, by Lebseque's convergence theorem, the right hand side converges to

$$\int_{\mathbb{R}^n} w\Big(\Delta\psi_j + n(n+2)U_1^{\frac{4}{n-2}}\psi_j\Big)dy = 0.$$

Together with (3.43), it implies

$$\begin{aligned}0 &= \lim_{i\to+\infty}\int_{B_{l_i}} l_i^{n-2}\widetilde{Q}_i(y)U_i^{\frac{n+2}{n-2}}\psi_j(y)dy \\ &= c\lim_{i\to+\infty}\int_{B_{l_i}} Q(\xi_i+y)U_1^{\frac{n+2}{n-2}}(y)\frac{\partial U_1(y)}{\partial y_j}dy \\ &= \frac{(n-2)c}{2n}\int_{\mathbb{R}^n} Q(\xi+y)\frac{\partial}{\partial y_j}U_1^{\frac{2n}{n-2}}(y)dy \\ &= \frac{-(n-2)c}{2n}\int_{\mathbb{R}^n}\frac{\partial}{\partial y_j}Q(\xi+y)U_1^{\frac{2n}{n-2}}(y)dy,\end{aligned}$$

where U_1 is defined in (1.4). Here, we have used the fact that $\psi_j(y)$ is odd in y_j, and

$$\int_{B_{l_i}}(K_i(x_i)-K_i(0))\psi_j(y)U_i^{\frac{n+2}{n-2}}(y)dy = 0.$$

The proof of Theorem 2.2 is complete. q.e.d.

Proof of Theorem 2.1. Note that in Section 8, (2.16) is also proved when $\beta < n+1$. This holds only for global solutions. See Lemma 8.1. Let x_i and $\hat{M}_i$ be the maximum point and the maximum of u_i defined in (2.12). We first prove the "if" part. Assume there is a constant $c > 0$ such that

$$L_i(x_i) \ge c\hat{M}_i^{\frac{2}{n-2}}. \tag{3.44}$$

Let $v_i(y)$ be the scaled solution defined in (2.13) with $z_i = x_i$. Obviously, $v_i(y) \le 1$ for $|y| \le \hat{M}_i^{\frac{2}{n-2}}$. By Lemma 3.1, Lemma 3.5 and (3.44), there exists a small positive number $\delta > 0$ such that $v_i(y) \le c\ U_1(y)$ for $|y| \le \delta\hat{M}_i^{\frac{2}{n-2}}$ and for some $c > 0$. Therefore, 0 is a simple blow-up point.

To prove the "only if" part, we assume

$$\lim_{i\to+\infty} L_i(x_i)\hat{M}_i^{-\frac{2}{n-2}} = 0. \tag{3.45}$$

Suppose that 0 is a simple blowup point. Then there exists positive constants c and $\delta_0 < 1$ such that

$$v_i(y) \le c\, U_1(y) \tag{3.46}$$

for $|y| \le \delta_0 \hat{M}_i^{\frac{2}{n-2}}$. Following the notations of Lemma 3.5, we let $w_i(y) = v_i(y) - U_i(y)$ and $\psi_0(y) = \frac{n-2}{2}U_i(y) + y \cdot \bigtriangledown U_i(y)$. By the gradient estimate, we have by (3.46), $|\bigtriangledown v_i(y)| = O(|y|^{-n+1})$ for $|y| \ge 1$. Thus,

$$\int_{|x|=\hat{r}_i} \Big(\psi_0 \frac{\partial w_i}{\partial \nu} - w_i \frac{\partial \psi_0}{\partial \nu}\Big) d\sigma = O(\hat{r}_i^{-n+2}) = O(\hat{M}_i^{-2}), \tag{3.47}$$

where $\hat{r}_i = \delta_0 \hat{M}_i^{\frac{2}{n-2}}$. To estimate the first term of (3.35), we have by Lemma 3.5

$$\begin{aligned}\int_{B_{\hat{r}_i}} w_i(\Delta\psi_0 + \widetilde{b}_i\psi_0)dy = \int_{B_{r_i}} & w_i(\Delta\psi_0 + \widetilde{b}_i\psi_0)dy \\ &+ \int_{B_{\hat{r}_i}\setminus B_{r_i}} w_i(\Delta\psi_0 + \widetilde{b}_i\psi_0)dy,\end{aligned}$$

where $r_i = \delta_0 L_i(x_i)$. By Theorem 2.2, we have $1 \le \beta < n-2$. Similar to (3.37), we have by the fact $\beta < n-2$ that

$$|w_i(\Delta\psi_0 + \widetilde{b}_i\psi_0)| \le c\, r_i^{2(n-2)}(1+|y|)^{-4}$$

for $1 \le r \le r_i$. Hence

$$\left|\int_{B_{r_i}} w_i(\Delta\psi_0 + \widetilde{b}_i\psi_0)dy\right| = O(r_i^{-n+1}).$$

We note that Lemma 3.5 is crucial in the estimate above. By applying $|v_i(y)| + |U_i(y)| \le c|y|^{-n+2}$ and $|\psi_0(y)| \le c|y|^{-n+2}$ for $r_i \le |y| \le \hat{r}_i$,

$$\left|\int_{B_{\hat{r}_i}\setminus B_{r_i}} w_i(\Delta\psi_0 + \widetilde{b}_i\psi_0)dy\right| = O(r_i^{-n+1}).$$

Together with these two estimates, we have

$$\left|\int_{B_{\hat{r}_i}} w_i(\Delta\psi_0 + \widetilde{b}_i\psi_0)dy\right| = O(r_i^{-n+1}). \tag{3.48}$$

By Theorem 2.2, $\xi_i = \hat{M}_i^{\frac{2}{n-2}} x_i$ is bounded. We may assume $\xi = \lim_{i\to+\infty} \xi_i$. Then ξ satisfies

$$\int_{\mathbb{R}^n} \bigtriangledown Q(y+\xi)U_1^{\frac{2n}{n-2}}(y)dy = 0. \tag{3.49}$$

Also, the right hand side of (3.35) converges to

$$\begin{aligned}\lim_{i\to+\infty} t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}} &\left(\int_{B_{\hat{r}_i}} \widetilde{Q}_i U_i^{\frac{n+2}{n-2}}\psi_0(y)dy\right) \\ &= -\int_{\mathbb{R}^n} Q(y+\xi)U_1^{\frac{n+2}{n-2}}(y)\psi_0(y)dy.\end{aligned} \tag{3.50}$$

Recall that $\psi_0(y) = \frac{n-2}{2}U_1(y) + y \bigtriangledown U_1(y)$. From integration by parts, (3.49) and $y \cdot \bigtriangledown Q(y) = \beta Q(y)$, we have

$$\begin{aligned}-\int_{\mathbb{R}^n} & Q(y+\xi)U_1^{\frac{n+2}{n-2}}(y)\psi_0(y)dy \\ &= \frac{n-2}{2n}\int_{\mathbb{R}^n} y\cdot\bigtriangledown Q(y+\xi)U_1^{\frac{2n}{n-2}}(y)dy \\ &= \frac{\beta(n-2)}{2n}\int_{\mathbb{R}^n} Q(y+\xi)U_1^{\frac{2n}{n-2}}(y)dy \neq 0.\end{aligned} \tag{3.51}$$

The last term does not vanish due to (K1).

Recall $L_i(x_i)^{n-2} \sim t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}}$. Putting (3.50), (3.51) and (3.48) together these estimates, we have

$$\begin{aligned}L_i(x_i)^{2-n} &\le c\left|\int_{B_{\hat{r}_i}} \widetilde{Q}_i U_i^{\frac{n+2}{n-2}}\psi_0(y)dy\right| \\ &\le c\,(L_i(x_i)^{1-n} + \hat{M}_i^{-2}),\end{aligned}$$

which yields a contradiction to (3.45). Therefore, the proof of Theorem 2.1 is complete. q.e.d.

4. The method of localizing blow-up points

In this section, we will employ the method of localization of blow-up points to prove Theorem 2.4 and Theorem 2.5. This technique was due to R. Schoen. In the previous work [9], we have used this method to prove the isolatedness of blow-up points. For other applications of this method, see [17], [18]. We begin with the following lemma.

Lemma 4.1. *Let δ, σ and ε be small positive numbers and $R > 1$. Then there exist positive constants $R = R(\delta, \sigma)$ and $C_0 = C_0(\delta, \sigma, R, \varepsilon)$ independent of i such that the following statements hold:*

(i) *If $u_i(y_0)|y_0|^{\frac{n-2}{2}} \geq C_0$, then there exists a local maximum point $z \in B(y_0, 2\delta|y_0|)$ of u_i such that*

$$u_i(y_0) \leq u_i(z) \tag{4.1}$$

and the rescaled function

$$v_i(y) = u_i(z)^{-1} u_i(u_i(z)^{-\frac{2}{n-2}} y + z)$$

satisfies

$$\begin{cases} \text{the origin } 0 \text{ is the only local maximum point of } v_i \\ \text{in } B(0, 4R), \text{ and } |v_i - U_1|_{C^2(B(0,4R))} \leq \sigma(4R)^{2-n}. \end{cases} \tag{4.2}$$

(ii) *Let $\{z_j^i\}_{j=1}^{s_i}$ denote all local maximum points of u_i in the ball $\overline{B}_1$ which satisfy $u_i(z_j^i)|z_j^i|^{\frac{n-2}{2}} \geq C_0$ and (4.2) with $z = z_j^i$. Assume $u_i(z_1^i) \geq u_i(z_2^i) \cdots \geq u_i(z_{s_i}^i)$. Then*

(a) *$u_i(y) \leq 2C_0|y|^{-\frac{n-2}{2}}$ for $y \notin \Omega_i$ where $\Omega_i = \cup_j B(z_j^i, 2\delta|z_j^i|)$. Furthermore,*

$$|z_j^i - z_k^i| \geq 4Ru_i(z_j^i)^{-\frac{2}{n-2}}$$

for $j \neq k$.

(b) *$u_i(x) \leq 2u_i(z_j^i)$ holds for $x \in B(z_j^i, 2\delta|z_j^i|)$ and*

$$|z_j^i| \leq \varepsilon|z_k^i| \quad \text{for} \quad j < k \leq s_i. \tag{4.3}$$

Lemma 4.1 can be proved by the blow-up method of Schoen and the method of moving planes, Lemma 3.1. See Lemma 4.1–Lemma 4.4 in [9]. In fact, we can prove more in Lemma 4.2 below. In the following, z_j^i is indexed by the ordering $u_i(z_1^i) \geq \ldots \geq u_i(z_{s_i}^i)$.

Lemma 4.2. *Let $\{z_j^i\}_{j=1}^{s_i}$ be the local maximum points in Lemma* 4.1 *and $\delta > 0$ be a small number. Then we have the following statements if the positive constant C_0 in Lemma* 4.1 *is large enough.*

(i) *The inequality*

$$L_i(z_j^i) \geq (\delta u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|)^{\frac{n-1}{n-2}}$$

holds for $1 \leq j \leq s_i$.

(ii) *Let*

$$L_i^*(z_j^i) = \min(L_i(z_j^i), u_i(z_j^i)^{\frac{2}{n-2}})$$

and

$$D_j^i = \{y : |y - z_j^i| \leq c L_i^*(z_j^i) u_i(z_j^i)^{-\frac{2}{n-2}}\}$$

with c small. Then

$$z_k^i \notin D_j^i$$

when $k > j$.

Proof. We follow notations in Section 3. Let v_i be defined in (2.13) with $z_i = z_j^i$ and U_i be the solution to (3.15) with $K_0 = K_i(z_i)$.

We may assume C_0 is very large. If $1 < \beta < n$, by (2.15) and $u_i(z_j^i)|z_j^i|^{\frac{n-2}{2}} \geq C_0$, we have

$$u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|^{1-\beta} = (u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|)^{1-\beta} u_i(z_j^i)^{\frac{2\beta}{n-2}} < u_i(z_j^i)^{\frac{2\beta}{n-2}}$$

and

$$L_i(z_j^i) = (t_i^{-1} u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|^{1-\beta})^{\frac{1}{n-2}}. \tag{4.4}$$

If $\beta \geq n$ and $L_i(z_j^i) \neq (t_i^{-1} u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|^{1-\beta})^{\frac{1}{n-2}}$, then by (2.15),

$$L_i(z_j^i) = \left(t_i^{-1} u_i(z_j^i)^{\frac{2\hat{\beta}}{n-2}}\right)^{\frac{1}{n-2}} \geq (u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|)^{\frac{n-1}{n-2}}$$

for large i since $\hat{\beta} > n - 1$, that is, (i) holds in this case. Hence in order to prove (i), we may assume $L_i(z_j^i) = (t_i^{-1} u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|^{1-\beta})^{\frac{1}{n-2}}$ and $1 < \beta < n$.

Let δ be small enough, $M_i = u_i(z_j^i)$, $L_i = L_i(z_j^i)$ and

$$r_i = \delta \min(L_i, M_i^{\frac{2}{n-2}} |z_j^i|).$$

. Then by (b) of part (ii) in Lemma 4.1,

$$v_i(y) \le 2 \quad \text{for} \quad |y| \le r_i.$$

By Lemma 3.1, Lemma 3.4 and Lemma 3.5, we have

$$v_i(x) \le c\, U_i(x)$$

and

$$|v_i(x) - U_i(x)| \le c\, r_i^{-2+n}$$

for $|x| \le r_i$, where c is a constant independent of δ and i. For the sake of simplicity, δ always denotes a small positive number, but could change from line to line. Assume $\nabla \hat{K}(z_j^i)$ is in the direction $e_1 = (1, 0, \cdots, 0)$. Let $\psi_1 = \dfrac{\partial U_i}{\partial y_1}$. By (3.32) of Lemma 3.5,

$$\left| \int_{|x| \le r_i} \widetilde{Q}_i U_i^{\frac{n+2}{n-2}} \psi_1 \, dx \right| \le c_1 r_i^{-n+1}. \tag{4.5}$$

By (3.40), we have

$$-\widetilde{Q}_i(y) = t_i M_i^{-\frac{2}{n-2}} | \bigtriangledown \hat{K}(z_j^i)| (y_1 + o(1)|y|)$$

for $|y| \le r_i$, where $o(1)$ could be arbitrarily small if δ is small. Since $\psi_1(y) y_1 \ge 0$, we have

$$\left| \int_{B_{r_i}} \widetilde{Q}_i U_i^{\frac{n+2}{n-2}} \psi_1 dy \right| \ge c_2 t_i M_i^{-\frac{2}{n-2}} |z_j^i|^{\beta - 1} \tag{4.6}$$

for some $c_2 > 0$. Since we assume $L_i(z_j^i) = (t_i^{-1} u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|^{1-\beta})^{\frac{1}{n-2}}$, it follows from (4.5) and (4.6) that

$$L_i^{-n+2} \le c_3 r_i^{-n+1}. \tag{4.7}$$

Since $r_i \le L_i$, C_0 is large and $L_i \to +\infty$ as $i \to +\infty$, we conclude r_i / L_i is small from (4.7). Thus $r_i = \delta u_i(z_j^i)^{\frac{2}{n-2}} |z_j^i|$. Since both c_1 and c_2 are independent of δ and i, part (i) of Lemma 4.2 follows from (4.7) if δ is chosen to be small enough.

We prove (ii) by contradiction. Assume that after passing to a sequence, there exists $j_i < k_i$ such that $z^i_{k_i} \in D^i_{j_i}$ and both $u_i(z^i_{j_i})|z^i_{j_i}|^{\frac{n-2}{2}}$ and $u_i(z^i_{k_i})|z^i_{k_i}|^{\frac{n-2}{2}}$ tend to $+\infty$. For simplicity of notations, we let $z_i = z^i_{j_i}$ and $w_i = z^i_{k_i}$. Recall that u_i satisfies

$$u_i(w_i) \le u_i(z_i). \tag{4.8}$$

Let $M_i = u_i(z_i)$ and $v_i(y)$ be the solution in (2.13) scaled with respect to the local maximum point z_i. Since $M_i^{\frac{2}{n-2}}|z_i| \to +\infty$ and (4.2) holds, we have for any $\sigma > 0$, by Lemma 3.1

$$\min_{|y|\le r} v_i(y) \le (1+2\sigma)U_1(r) \tag{4.9}$$

if i is large and $0 \le r \le 4d_0L^*_i(z_i)$ with some $d_0 = d_0(\sigma) > 0$. Let $l_i = d_0L^*_i(z_i)$. Applying Lemma 3.5 with an empty set E, $l_0 = M_i^{\frac{2}{n-2}}$ and $l = l_i$, there is a constant c_1 independent of σ and i

$$\int_{R\le|y|\le l_i} v_i^{\frac{n+2}{n-2}}(y)dy \le c_1\sigma, \tag{4.10}$$

provided that $d_0 < \sigma^{\frac12}$ and $R \ge \sigma^{-\frac12}$.

Set

$$B_i = \{x \mid |x-w_i| \le u_i(w_i)^{-\frac{2}{n-2}}\}$$

and

$$\hat{B}_i = \{y \mid M_i^{-\frac{2}{n-2}}(y+z_i) \in B_i\}.$$

By (ii) of Lemma 4.1 and (4.8),

$$4R \le u_i(w_i)^{\frac{2}{n-2}}|z_i-w_i| \le M_i^{\frac{2}{n-2}}|z_i-w_i| \le cL^*_i(z_i)$$

because $w_i \in D_i$. By (ii) of Lemma 4.1, we have $|z_i| = o(1)|w_i|$ and

$$\begin{aligned} M_i^{\frac{2}{n-2}}u_i(w_i)^{-\frac{2}{n-2}} &<< M_i^{\frac{2}{n-2}}|w_i| \\ &= (1+o(1))M_i^{\frac{2}{n-2}}|z_i-w_i| \le cL^*_i(z_i). \end{aligned}$$

Thus, $B_i \subseteq 2D_i$.

Since $u_i(x) \le u_i(w_i) \le u_i(z_i)$ for $x \in B_i$, we have $v_i(y) \le 1$ for $y \in \hat{B}_i$. Since by Lemma 4.1, 0 is the unique local maximum of $v_i(y)$

for $|y| \leq 4R$, we have $\hat{B}_i \subseteq \{y \mid R \leq |y| \leq l_i\}$ if the constant c in D^i_j is small. Again by (i) of Lemma 4.1, we have for some constant $c_2 > 0$,

$$\begin{aligned} 0 < c_2 &\leq \int_{B_i} u_i^{\frac{2n}{n-2}}(x)dx = \int_{\hat{B}_i} v_i^{\frac{2n}{n-2}} dy \leq \int_{\hat{B}_i} v_i^{\frac{n+2}{n-2}}(y)dy \\ &\leq \int_{R \leq |y| \leq l_i} v_i^{\frac{n+2}{n-2}}(y)dy \leq c_1\sigma, \end{aligned}$$

which yields a contradiction if σ is small enough. Therefore, (ii) is proved. q.e.d.

Proof of Theorem 2.4. Let $L_i = L_i(x_i)$ and $\hat{M}_i = u_i(x_i)$. Suppose that $L_i\hat{M}_i^{-\frac{2}{n-2}} \to +\infty$, then by Theorem 2.1, 0 is a simple blowup point and u_i loses the energy of one bubble at 0. Therefore, we suppose that $\lim_{i\to+\infty} L_i\hat{M}_i^{-\frac{2}{n-2}} < +\infty$.

By Theorem 2.2, $\hat{M}_i^{\frac{2}{n-2}}|x_i|^{\frac{n-2}{2}}$ is bounded, $\beta < n-2$ and $\xi = \lim_{i\to+\infty} \hat{M}_i^{\frac{2}{n-2}} x_i$ satisfies (2.16). From the definition (2.15) of L_i, we have $L_i(x_i) \sim \left(t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}}\right)^{\frac{1}{n-2}}$. Applying Lemma 3.1 and Lemma 3.5, u_i satisfies

$$c_1\hat{M}_i^{-1}|x|^{2-n} \leq u_i(x) \leq c_2\hat{M}_i^{-1}|x|^{2-n} \tag{4.11}$$

for

$$\hat{M}_i^{-\frac{2}{n-2}} \leq |x| \leq \delta(t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}-2})^{\frac{1}{n-2}}$$

with a small $\delta > 0$. Let $r_i = \delta\left(t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}-2}\right)^{\frac{1}{n-2}}$. Then, we have

$$\min_{|x|=r_i} u_i(x) \sim t_i\hat{M}_i^{1-\frac{2\beta}{n-2}}. \tag{4.12}$$

Now suppose

$$\lim_{i\to+\infty} \sup_{\overline{B}_2}\left(u_i(x)|x|^{\frac{n-2}{2}}\right) = +\infty.$$

Let $z_i = z_1^i$, where z_1^i is the local maximum point in Lemma 4.2. Let $M_i = u_i(z_i)$. Since $M_i^{\frac{2}{n-2}}|z_i| \geq C_0$ is very large, we have

$$L_i(z_i) \leq \left(t_i^{-1}M_i^{\frac{2\beta}{n-2}}\right)^{\frac{1}{n-2}}, \tag{4.13}$$

and by (i) of Lemma 4.2,

$$M_i^{\frac{2}{n-2}}|z_i| << L_i(z_i). \tag{4.14}$$

Since $u_i(x)$ is a positive superharmonic function, there exists a small constant $c > 0$ such that

$$u_i(z_i + x) \geq c\ M_i^{-1}(|x|^{2-n} - (3/2)^{2-n}) \tag{4.15}$$

for $M_i^{-\frac{2}{n-2}} \leq |x| \leq \frac{3}{2}$. In particular, we have

$$\min_{|x-z_i|\leq\min(\hat{r}_i,1)} u_i(x) \geq cM_i^{-1}\hat{r}_i^{2-n} \geq ct_iM_i^{1-\frac{2\beta}{n-2}}, \tag{4.16}$$

where $\hat{r}_i = \left(t_i^{-1}M_i^{\frac{2\beta}{n-2}-2}\right)^{\frac{1}{n-2}}$. Since $u_i(x)$ has the only maximum point x_i in the region $\{x \mid |x| \leq r_i\}$, we have by (4.14)

$$r_i \leq |z_i| << \hat{r}_i,$$

namely, the ball $B_{r_i}(0)$ is contained inside of the ball $B(z_i, \hat{r}_i)$. Hence, if $\hat{r}_i$ is bounded, by (4.12), (4.16) and the maximum principle, we have

$$\begin{aligned} t_i\hat{M}_i^{1-\frac{2\beta}{n-2}} \sim \min_{|x|=r_i} u_i &\geq \min_{|x-z_i|\leq\hat{r}_i} u_i \\ &\geq ct_iM_i^{1-\frac{2\beta}{n-2}}. \end{aligned}$$

First we consider the case when $\beta > \frac{n-2}{2}$. Since $\beta > \frac{n-2}{2}$ and $\hat{M}_i$ is the maximum of u_i, it implies $\hat{M}_i \sim M_i$. Hence, the function $v_i(y)$ rescaled with respect to the center z_i satisfies

$$v_i(y) \leq c$$

for some constant $c > 0$ and $|y| \leq M_i^{\frac{2}{n-2}}$. Thus, $v_i(y) \sim U_1(y)$ for $|y| \leq \delta L_i(z_i)$ by Lemma 3.4. Particularly, we have

$$\hat{M}_i \sim M_iU_1(|z_i|M_i^{\frac{2}{n-2}}) = M_i(M_i|z_i|^{\frac{n-2}{2}})^{-2} = o(1)M_i,$$

which obviously yields a contradiction.

For the case $\beta = \frac{n-2}{2}$, we have by (4.14), (4.15) and the maximum principle,

$$\begin{aligned} t_i = t_i \hat{M}_i^{1-\frac{2\beta}{n-2}} &\sim \min_{|x|=r_i} u_i \\ &\geq \min_{|x-z_i|\leq 2|z_i|} u_i \geq c M_i^{-1} |z_i|^{2-n}, \end{aligned}$$

which implies

$$\hat{r}_i = (t_i^{-1} M_i^{-1})^{\frac{1}{n-2}} \leq c|z_i|.$$

But by (4.14), $|z_i| << \hat{r}_i$ for large i. Thus, we obtain a contradiction and then (2.18) is proved.

Once that (2.18) is established, (1.8) follows from Lemma 5.2 of Section 5. Also, from (2.18), the energy outside the region, where u_i is not simple, tends to zero. Therefore (2.17) is obtained, and then Theorem 2.4 is proved. q.e.d.

Proof of Theorem 2.5. Suppose that u_i satisfies

$$\lim_{i\to+\infty} \sup_{\overline{B}_2} (u_i(x)|x|^{\frac{n-2}{2}}) = +\infty.$$

Assume that 0 is not a simple blowup point. Then $\beta < n-2$ by Corollary 2.3. Let δ, R, C_0 and the local maximum points $\{z_j^i\}_{j=1}^{s_i}$ of u_i satisfy the assumptions of Lemma 4.1 and Lemma 4.2. We will prove $s_i = 1$ for i large.

Let $z_i = z_1^i$, $L_i = L_i(z_i)$, $M_i = u_i(z_i)$ and $v_i(y)$ be the scaled function defined in (2.13). We claim

$$\lim_{i\to+\infty} L_i M_i^{-\frac{2}{n-2}} = +\infty. \tag{4.17}$$

We prove (4.17) by contradiction. Suppose

$$\lim_{i\to+\infty} L_i M_i^{-\frac{2}{n-2}} < +\infty.$$

Then for any small number $\sigma > 0$, by Lemma 3.1 and Lemma 3.5, there is a small positive number $d_0 = d_0(\sigma)$ such that

$$\min_{|y|=r} v_i(y) \leq (1+\sigma) U_1(r) \tag{4.18}$$

for $0 \leq r \leq d_0 L_i$, and

$$\int_{R\leq|x|\leq d_0 L_i} v_i^{\frac{n+2}{n-2}}(y)dy \leq c_1(\sigma + R^{-2} + (\frac{d_0 L_i}{L_i})^{n-2}) \equiv c_1 \widetilde{\sigma}, \tag{4.19}$$

where R is very large and c_1 is a positive constant independent of σ and i. Note that $L_i^*(d_0) = \min(d_0 L_i, M_i^{\frac{2}{n-2}}) = d_0 L_i$ due to the assumption $L_i \le c\, M_i^{\frac{2}{n-2}}$.

Let Ω_i be the set in Lemma 4.1. Let σ be a small positive number, which will be chosen later. For $|x| \ge \delta |z_i|$ and $z \notin \Omega_i$, we have by Lemma 4.1,

$$u_i(x)|x|^{\frac{n-2}{2}} \le 2C_0 \le 2M_i |z_i|^{\frac{n-2}{2}}$$

for i large, which implies that

$$u_i(x) \le c\, M_i$$

for some $c = c(\delta) > 0$. If $x \in \Omega_i$, then for some j,

$$u_i(x) \le 2u_i(z_j^i) \le 2u_i(z_i).$$

Hence, there is $c_1 = c_1(\delta) > 0$ such that

$$u_i(x) \le c_1 M_i \tag{4.20}$$

for $|x| \ge \delta |z_i|$.

If σ and d_0 are small and R is large, then by (4.19) and (4.20), Lemma 3.5 can be applied to obtain the Harnack inequality for $v_i(y)$ on each sphere $|y| = r \le d_0 L_i$ if the annulus $\{y \mid \frac{r}{2} \le |y| \le 2r\}$ does not intersect with the set $\left\{ y \mid \left| y + M_i^{\frac{2}{n-2}} z_i \right| \le \delta M_i^{\frac{2}{n-2}} |z_i| \right\}$. In particular,

$$v_i(y) \le c\, U_i(y), \tag{4.21}$$

holds for $2M_i^{\frac{2}{n-2}} |z_i| \le |y| \le d_0 L_i$, where c is a constant independent of i and δ. Let

$$r_i = d_0 L_i M_i^{-\frac{2}{n-2}}. \tag{4.22}$$

Going back to the function u_i, (4.21) implies

$$u_i(z_i + x) + |x| |\bigtriangledown u(z_i + x)| \le c M_i^{-1} |x|^{2-n} \tag{4.23}$$

for $2\delta |z_i| \le |x| \le r_i$.

Let $e_i = |\bigtriangledown \hat{K}(z_i)|^{-1} \bigtriangledown \hat{K}(z_i)$ and $e = \lim_{i\to+\infty} e_i$. Applying the Pohozaev identity,

$$\begin{aligned}\frac{n-2}{2n}&\int_{B(z_i,r_i)} \langle e, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}} dx \\ &= \int_{\partial B(z_i,r_i)} \Big[\langle e, \bigtriangledown u_i\rangle \frac{\partial u_i}{\partial \nu} - \langle e,\nu\rangle \frac{|\bigtriangledown u_i|^2}{2} \\ &\qquad + \frac{n-2}{2n}\langle e,\nu\rangle K_i u_i^{\frac{2n}{n-2}}\Big] d\sigma.\end{aligned} \tag{4.24}$$

By (4.23), the right hand side of (4.24) is dominated by $r_i^{-n+1}M_i^{-2}$. To find a lower bound, we decompose $B(z_i,r_i)$ into four parts: $A_1 = \{x \mid |x-z_i| \le M_i^{-\frac{2}{n-2}}R_0\}$, $A_2 = \{x \mid |x| \le 3\delta|z_i|\}$, $A_3 = \{x \mid 3\delta|z_i| \le |x| \le 2|z_i|, |z_i - x| \ge M_i^{-\frac{2}{n-2}}R_0\}$ and $A_4 = \{x \mid 2|z_i| \le |x| \le r_i\}$, where R_0 is a positive number.

For $x \in A_2$, we have by Lemma 4.1

$$u_i(x) \le 2C_0|x|^{-\frac{n-2}{2}}.$$

Then

$$\int_{A_2} |\bigtriangledown K_i| u_i^{\frac{2n}{n-2}}(x)dx \le c_2(\delta|z_i|)^{\beta-1}t_i. \tag{4.25}$$

For $x \in A_3$, we have

$$\int_{A_3} |\bigtriangledown K_i| u_i^{\frac{2n}{n-2}}(x)dx \le c\ t_i|z_i|^{\beta-1}\int_{A_3} u_i^{\frac{2n}{n-2}}(x)dx.$$

By (4.19) and $v_i(y) \le c_1(\delta)$, we have

$$\begin{aligned}\int_{A_3} |\bigtriangledown K_i| u_i^{\frac{2n}{n-2}}dx &\le c\ t_i|z_i|^{\beta-1}\int_{R_0\le|y|\le d_0L_i} v_i^{\frac{2n}{n-2}}(y)dy \\ &\le c\ t_i|z_i|^{\beta-1}(c_2(\delta)\widetilde{\sigma} + R_0^{-n}),\end{aligned} \tag{4.26}$$

where the estimate,

$$\int_{R_0\le|y|\le R} v_i^{\frac{2n}{n-2}}(y)dy \le c\int_{R_0\le|y|\le R}|y|^{-2n}dy \le cR_0^{-n}$$

is used.

For $x \in A_4$, we apply (4.21)

$$u_i(x) \le c M_i^{-1} |x|^{2-n}.$$

Hence

$$\begin{aligned} &\int_{A_4} |\bigtriangledown K_i| u_i^{\frac{2n}{n-2}}(x)dx \\ &\quad \le c\ t_i M_i^{-\frac{2n}{n-2}} \int_{A_4} |x|^{-2n+\beta-1} dx \\ &\quad \le c\ t_i M_i^{-\frac{2n}{n-2}} |z_i|^{-(n+1)+\beta} \\ &\quad = c\ (t_i|z_i|^{\beta-1}) \Big(M_i |z_i|^{\frac{n-2}{2}} \Big)^{\frac{-2n}{n-2}}. \end{aligned} \tag{4.27}$$

For $x \in A_1$, we have a positive $c_0 > 0$ such that

$$\int_{A_1} \langle e, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x)dx \ge c_0(t_i|z_i|^{\beta-1}). \tag{4.28}$$

If we chose σ, d_0 to be small and R_0 to be large, then by (4.25) $\sim$ (4.28), the left hand side of (4.24) has

$$\int_{B(z_i,r_i)} \langle e, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}} dx \ge (c_0/2) t_i |z_i|^{\beta-1} \tag{4.29}$$

when i is large. Combining the estimates of both sides of (4.24), one has

$$t_i|z_i|^{\beta-1} \le c\ r_i^{-n+1} M_i^{-2} = c_1 L_i^{-n+1} M_i^{\frac{2}{n-2}},$$

namely,

$$L_i(z_i)^{-n+2} \le c_1 L_i(z_i)^{-n+1},$$

which obviously yields a contradiction. Hence (4.17) is proved.

If $s_i > 1$, then by (4.17), $L_i^*(z_1^i) u_i(z_1^i)^{-\frac{2}{n-2}} \ge 1$ with $L_i^*(z_1^i) = \min(L_i(z_i), u_i(z_i)^{\frac{2}{n-2}})$ defined in Lemma 4.2. Since $z_2^i \notin D_1^i$, we have $|z_2^i| \ge c_2$ for some $c_2 > 0$. On the other hand, by (2.11) and the Harnack inequality, we have u_i converges to 0 uniformly on any compact subset of $B_1 \setminus \{0\}$. Thus,

$$u_i(z_2^i) \le \max_{|x|=c_2} u_i(x) \to 0 \quad \text{as } i \to +\infty,$$

which yields a contradiction again. Therefore, $s_i = 1$. We note that $x_i \ne z_i$ because 0 is not a simple blowup point. The other conclusions of Theorem 2.5 follow from (4.17) and the lemmas in Section 3. Hence, the proof of Theorem 2.5 completely finished. q.e.d.

5. An ODE approach

In this sectin, we consider a sequence of solution u_i of (2.11) such that

$$\sup_{|x|\le 1}(u_i(x)|x|^{\frac{n-2}{2}}) \le c_1 \quad \text{and } u_i(x) \text{ converges to 0 in } C^2_{\mathrm{loc}}(\overline{B}_1\backslash\{0\}). \tag{5.1}$$

From (5.1) and the theory of elliptic equations, it is easy to see

$$\max_{|x|=r} u_i(x) \le c \min_{|x|=r} u_i(x)$$

for $0 \le r \le \frac{1}{2}$ and some $c > 0$ depending on c_1 only. Let $\bar{u}_i(r), w_i(s), s_i$, M_i and L_i are defined in (2.26) $\sim$ (2.31), respectively. By (5.1), $w_i(s) \le c_1$ for $s \le 0$. Throughout this section, we set

$$R_i = L_i^{\gamma} \quad \text{and} \quad \gamma = \frac{1}{1 - \frac{2\beta}{n-2}} \tag{5.2}$$

By a straightforward computation, w_i satisfies

$$w_i'' - \Big(\frac{n-2}{2}\Big)^2 w_i + \overline{K}_i(s) w_i^{\frac{n+2}{n-2}} = 0 \text{ for } s \le 0, \tag{5.3}$$

where

$$\overline{K}_i(s) = |\partial B_{e^s}(0)|^{-1} w_i^{-\frac{n+2}{n-2}}(s) \int_{|x|=e^s} K_i(x) \Big(u(x)|x|^{\frac{n-2}{2}}\Big)^{\frac{n+2}{n-2}} d\sigma$$

and $B_{e^s}(0)$ is the ball with radius e^s and center 0. Since we assume K_i is bounded between two positive constants, by (5.1), there are $\hat{a}$ and $\hat{b}$ such that $\overline{K}_i(s)$ satisfies

$$0 < \hat{a} \le \overline{K}_i(s) \le \hat{b}. \tag{5.4}$$

From (5.3) and (5.4), there is a constant $c_2 > 0$ such that if s is a local maximum point of w_i, then

$$w_i(s) \ge c_2 > 0. \tag{5.5}$$

In particular, we have $w_i(s_i) \ge c_2 > 0$. Since $u_i(x)$ converges to zero in $C^2_{\mathrm{loc}}(\overline{B}_1\backslash\{0\})$, $s_i \to -\infty$ as $i \to +\infty$. Thus, we have by (5.5),

$$\lim_{i\to+\infty} M_i = +\infty. \tag{5.6}$$

We can obtain some basic estimates for w_i as in the following. For the proof, see [9]. Let w_i be denoted by w.

Lemma 5.1. *There is a small number $\epsilon_0 > 0$ and large M such that the following statements hold:*

(i) *Suppose that $w(s)$ is nonincreasing in (s_o, s_1) with $w(s_o) \le \epsilon_0$. Then there exists a constant c depending on $\hat{a}$ and $\hat{b}$ only such that*

$$s_1 - s_o \le \frac{2}{n-2} \log \frac{w(s_o)}{w(s_1)} + c \tag{5.7}$$

holds. Futhermore, if s_1 is a local minimum point of w, then

$$s_1 - s_o \ge \frac{2}{n-2} \log \frac{w(s_o)}{w(s_1)}. \tag{5.8}$$

(ii) *Suppose that $w(s)$ is nondecreasing in (s_1, s_2) with $w(s_2) \le \epsilon_0$. Then there exists a constant c depending on $\hat{a}$ and $\hat{b}$ only such that*

$$s_2 - s_1 \le \frac{2}{n-2} \log \frac{w(s_2)}{w(s_1)} + c \tag{5.9}$$

holds. Futhermore, if s_1 is a local minimum point of w, then

$$s_2 - s_1 \ge \frac{2}{n-2} \log \frac{w(s_2)}{w(s_1)}. \tag{5.10}$$

Proof Theorem 2.7. The proof of Theorem 2.7 is very long. So, we devide it into two steps. The first step is to estimate u_i via Lemma 5.1, and the second step can refine the estimate further by using comparison functions. First, we want to prove

Step 1. There is a constant c such that

$$u_i(x) \le c\ (t_i M_i^{-1})^{\gamma} |x|^{-n+2} \tag{5.11}$$

for $R_i^{-2} M_i^{-\frac{2}{n-2}} \le |x| \le R_i^{-1} M_i^{-\frac{2}{n-2}}$, and $\gamma = (1 - \frac{2\beta}{n-2})^{-1}$,

$$u_i(x) \le c\ M_i \tag{5.12}$$

for $R_i^{-1} M_i^{-\frac{2}{n-2}} \le |x| \le M_i^{-\frac{2}{n-2}}$,

$$u_i(x) \le c M_i^{-1} |x|^{-n+2} \quad \text{for} \quad M_i^{-\frac{2}{n-2}} \le |x| \le 1 \tag{5.13}$$

if $L_i M_i^{-\frac{2}{n-2}} \geq c_1 > 0$, and

$$\text{(5.14)} \qquad u_i(x) \begin{cases} \leq c\, M_i^{-1}|x|^{-n+2} & \text{for } M_i^{-\frac{2}{n-2}} \leq |x| \leq L_i M_i^{-\frac{2}{n-2}} \\ \leq c\, M_i^{-1} L_i^{-n+2} & \text{for } L_i M_i^{-\frac{2}{n-2}} \leq |x| \leq 1, \end{cases}$$

provided that $\lim_{i\to+\infty} L_i M_i^{\frac{-2}{n-2}} = 0$.

Recall $w_i(s) = \bar{u}_i(r) r^{\frac{n-2}{2}}$ with $s = \log r \leq 0$. Let $\hat{s}_i$ be a local maximum point of w_i. By (5.5), $w_i(\hat{s}_i) \geq c > 0$. Set $\hat{u}_i(x) = \hat{r}_i^{\frac{n-2}{2}} u_i(\hat{r}_i x)$ with $\hat{r}_i = e^{\hat{s}_i}$. Then $\hat{u}_i(x) \leq c|x|^{\frac{2-n}{2}}$ for $0 \leq |x| \leq \hat{r}_i^{-1}$. By passing to a subsequence, $\hat{u}_i(x)$ converges to $\hat{U}(x)$ in $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. In Lemma 5.2 (below), we will show that $\hat{U}(x) = [\hat{\lambda}(\hat{\lambda}^2 + |x - \hat{q}|^2)^{-1}]^{\frac{n-2}{2}}$ for some $\hat{\lambda} > 0$ and $\hat{q} \in \mathbb{R}^n$. A direct computations show that $\overline{\hat{U}(r)} r^{\frac{n-2}{2}}$ has a unique critical point at $r = \sqrt{\hat{\lambda}^2 + |q|^2}$, which is also nondegenerate. From here, we deduce that for each large i, $w_i(s)$ has a sequence of local maximum point $s_{j,i}$ and local minimum point $\underline{s}_{j,i}$ for $j = 1, 2, \ldots, N(i)$. Such that the following holds:

$$\text{(5.15)} \qquad \begin{aligned} &s_{j,i} < \underline{s}_{j,i} < s_{j+1,i} \ \text{ with } s_{N(i)+1,i} = s_i, w(s) \text{ is decreasing} \\ &\text{for } s \in (s_{j,i}, \underline{s}_{j,i}) \text{ and } w(s) \text{ is increasing for } s \in (\underline{s}_{j,i}, s_{j+1,i}) \end{aligned}$$

for $1 \leq j \leq N(i)$. Furthermore, $w(\underline{s}_{j,i}) \to 0$ as $i \to +\infty$ for $j = 1, 2, \ldots, N(i)$, and,

$$\text{(5.16)} \qquad \begin{aligned} &s_{j+1,i} - \underline{s}_{j,i} \text{ and } \underline{s}_{j,i} - s_{j,i} \to +\infty \text{ as } i \to +\infty \\ &\text{for any } j = 1, 2, \ldots, N(i). \text{ Consequently, } M_{j,i}/M_{j+1,i} \to 0 \\ &\text{as } i \to +\infty \text{ for } y \in \{1, 2, \ldots, N(i)\}. \end{aligned}$$

Note that $N(i) \geq 1$ due to the assumption that u_i loses the energy of more than one bubble. For $j = 1, 2, \ldots, N(i)$, we set $\hat{u}_i(x) = r_{j,i}^{\frac{n-2}{2}} u_i(r_{j,i} x)$ with $r_{j,i} = e^{s_{j,i}}$ and $\hat{U}$ to be the limit of $\hat{u}_i$ in $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. Then we have

Lemma 5.2. *Let $\hat{U}$ be described as above. Then*

$$\text{(5.17)} \qquad \hat{U}(x) = \left(\frac{\hat{\lambda}}{\hat{\lambda}^2 + |x - \hat{q}|^2} \right)^{\frac{n-2}{2}},$$

where

$$1 = \hat{\lambda}^2 + |\hat{q}|^2. \tag{5.18}$$

Furthermore, if set $\xi_0 = \sqrt{\hat{\lambda}}\hat{q}$, *then* ξ_0 *satisfies*

$$\int_{\mathbb{R}^n} \bigtriangledown Q(\xi_0 + y) U_1^{\frac{2n}{n-2}}(y)dy = 0 \quad \textit{and} \tag{5.19}$$

$$\int_{\mathbb{R}^n} Q(\xi_0 + y) U_1^{\frac{2n}{n-2}}(y)dy < 0. \tag{5.20}$$

The proof of Lemma 5.2 will be given at the end of this section. Now, we go back to the proof of Step 1. By the remark above, we denote $\hat{s}_i$ and $\underline{s}_i$ to be the local maximum point $\bar{s}_{N(i),i}$ and local minimum point $\underline{s}_{N(i),i}$, respectively. Since $w_i(\underline{s}_i) \to 0$ as $i \to +\infty$, there are $\hat{s}_i < a_i < \underline{s}_i < b_i < s_i$ such that $w_i(a_i) = w_i(b_i) = \epsilon_0$, where ϵ_0 is the small positive number in Lemma 5.1. By a simple scaling argument,

$$s_i - b_i \leq c_3 = c_3\ (\epsilon_0) \tag{5.21}$$

for some constant c_3 independent of i. By Lemma 5.1,

$$\begin{aligned} \frac{2}{n-2} \log \frac{\epsilon_0}{w_i(\underline{s}_i)} &\leq \underline{s}_i - a_i, b_i - \underline{s}_i \\ &\leq \frac{2}{n-2} \log \frac{\epsilon_0}{w_i(\underline{s}_i)} + c. \end{aligned} \tag{5.22}$$

To obtain some estimate for $\underline{s}_i - a_i$ and $b_i - \underline{s}_i$, we need to find upper and lower bounds for $w_i(\underline{s}_i)$. First, we show that

$$(\min_{|x|=r_i} u_i)^{-1} \max_{|x|=r_i} u_i \to 1 \quad \text{and} \quad r_i = e^{\underline{s}_i}. \tag{5.23}$$

uniformly as $i \to \infty$. To see it, let $\hat{x}_i$ be any sequence of points with $|\hat{x}_i| = r_i$. Let $h_i(\eta) = u_i(\hat{x}_i)^{-1} u_i(r_i\eta)$. Since $w_i(\underline{s}_i) \to 0$ as $i \to +\infty$, after passing to a subsequence, $h_i(\eta)$ converges to $h(\eta)$ in $C^2_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$ and satisfies

$$\Delta h(\eta) = 0 \ \text{ in } \ \mathbb{R}^n \backslash \{0\}. \tag{5.24}$$

Let $\bar{h}(r)$ be the spherical average of h. Since $\underline{s}_i$ is a local minimum point of w_i, we have

$$\frac{d}{dr}(\bar{h}(r)r^{\frac{n-2}{2}}) = 0 \quad \text{at} \quad r = 1. \tag{5.25}$$

By the Liouville Theorem, (5.25) implies

$$\begin{cases} h(\eta) = a|\eta|^{2-n} + b, \\ a = b > 0 \end{cases} \tag{5.26}$$

Clearly, from it we obtain (5.23) and

$$|\bigtriangledown u_i|(x) = -\overline{u}_i'(r_i)(1 + o(1)) \tag{5.27}$$

for $|x| = r_i$ as $i \to +\infty$. By (5.23) and (5.27), the Pohozaev identity implies

$$\begin{aligned} P(r_i, u_i) =& |S^{n-1}|\Big\{\frac{1}{2}w_i'^2(\underline{s}_i) - \frac{1}{2}\Big(\frac{n-2}{2}\Big)^2 w_i^2(\underline{s}_i)\Big\} \\ &+ o(1)(w_i'^2(\underline{s}_i) + w_i^2(\underline{s}_i)), \end{aligned} \tag{5.28}$$

where

$$P(r; u_i) = \frac{n-2}{2n}\int_{|x|\le r}(x\cdot \bigtriangledown K_i)u_i^{\frac{2n}{n-2}}(x)dx. \tag{5.29}$$

Hence,

$$\begin{aligned} (1 + o(1))w_i^2(\underline{s}_i) =& - c_n P_n(r_i, u_i) \\ \le& c_n\Big\{\int_{e^{a_i}\le|x|\le r_i}|x||\bigtriangledown K(x)|u_i^{\frac{2n}{n-2}}\,dx \\ &+ \int_{|x|\le e^{a_i}}|x||\bigtriangledown K(x)|u_i^{\frac{2n}{n-2}}\,dx\Big\} \\ \equiv& I_1 + I_2. \end{aligned} \tag{5.30}$$

Since $|\bigtriangledown K_i(x)| \le c|x|^{\beta-1}$, by (5.1),

$$|I_2| \le c\ t_i \exp(\beta a_i).$$

By Lemma 5.1, we have for $a_i \le s \le \underline{s}_i$

$$c\ w_i(\underline{s}_i)\exp\Big[\frac{n-2}{2}(\underline{s}_i - s)\Big] \le w_i(s) \le w_i(\underline{s}_i)\exp\Big[\frac{n-2}{2}(\underline{s}_i - s)\Big].$$

Therefore

$$
\begin{aligned}
|I_1| &\le c\ t_i w_i^{\frac{2n}{n-2}}(\underline{s}_i)\exp(n\underline{s}_i)\int_{a_i}^{\underline{s}_i}\exp[(-n+\beta)s]\,ds \\
&\le c\ t_i w_i^{\frac{2n}{n-2}}(\underline{s}_i)\exp(n\underline{s}_i)\exp[(-n+\beta)a_i].
\end{aligned}
$$

By Lemma 5.1 again,

$$
\begin{aligned}
w_i(a_i)\exp\left(\frac{(n-2)(a_i-\underline{s}_i)}{2}\right) \le& w_i(\underline{s}_i) \\
\le& c\ w_i(a_i)\exp\left(\frac{(n-2)(a_i-\underline{s}_i)}{2}\right).
\end{aligned}
$$

These estimates imply

$$
|I_1| \le c\ t_i \epsilon_0^{\frac{2n}{n-2}}\exp(\beta a_i). \tag{5.31}
$$

Hence, we obtain

$$
w_i(\underline{s}_i) \le c\ t_i^{\frac{1}{2}}\exp\left(\frac{\beta a_i}{2}\right). \tag{5.32}
$$

Together with (5.22), it implies

$$
\underline{s}_i - a_i \ge \frac{1}{n-2}(-\log t_i - \beta a_i) - c(\varepsilon_0). \tag{5.33}
$$

To obtain a lower bound for $w_i(\underline{s}_i)$, we recall $\hat{s}_i < a_i < s_i$ to be the next local maximum point of w_i. Set

$$
\hat{u}_i(y) = \hat{M}_i^{-1} u_i(\hat{M}_i^{-\frac{2}{n-2}} y), \tag{5.34}
$$

where $\hat{M}_i = \exp(-\frac{n-2}{2}\hat{s}_i)$. By Lemma 5.2, by passing to a subsequence, $\hat{u}_i(y)$ converges to $\hat{U}(y)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ with

$$
\hat{U}(y) = \left(\frac{\hat{\lambda}}{\hat{\lambda}+|y-\hat{q}|^2}\right)^{\frac{n-2}{2}}.
$$

Let $\hat{r}_i = \delta \exp \hat{s}_i$ for a small $\delta > 0$. By (5.28) and (5.31),

(5.35)
$$\begin{aligned} w_i^2(\underline{s}_i) \geq & \left| \int_{\hat{r}_i \leq |x| \leq e^{a_i}} \langle x, \bigtriangledown K_i(x) \rangle u_i^{\frac{2n}{n-2}}(x) dx \right| \\ & - c \left\{ t_i \int_{|x| \leq \hat{r}_i} |x|^\beta u_i^{\frac{2n}{n-2}}(x) dx + t_i \int_{e^{a_i} \leq |x| \leq r_i} |x|^\beta u_i^{\frac{2n}{n-2}}(x) dx \right\} \\ \geq & c_n t_i \left\{ \left| \int_{\hat{r}_i \leq |x| \leq e^{a_i}} \langle x, \bigtriangledown \hat{K} \rangle \hat{u}_i^{\frac{2n}{n-2}} dx \right| - \hat{r}_i^\beta - \varepsilon_0^{\frac{2n}{n-2}} \exp(\beta a_i) \right\}. \end{aligned}$$

Since $w_i(a_i) = \epsilon_0$, by the scaling property of $\hat{U}(y)$, we have

$$\exp(a_i - \hat{s}_i) \sim \varepsilon_0^{-\frac{2}{n-2}} >> 1.$$

By the scaling (5.34),

$$\begin{aligned} & \left| \int_{\hat{r}_i \leq |x| \leq e^{a_i}} \langle x, \bigtriangledown K_i(x) \rangle u_i^{\frac{2n}{n-2}}(x) dx \right| \\ & \quad = \beta t_i \hat{M}_i^{-\frac{2\beta}{n-2}} \left| \int_{\delta \leq |y| \leq \exp(a_i - \hat{s}_i)} Q(y) \hat{u}_i(y)^{\frac{2n}{n-2}} dy \right| \\ & \quad = \beta t_i \hat{M}_i^{-\frac{2\beta}{n-2}} \left(\int_{\mathbb{R}^n} -Q(y) \hat{U}(y) dy \right) (1 + o(1)), \end{aligned} \tag{5.36}$$

where $o(1)$ is small provided that both δ and ϵ_0 be small. Thus, by (5.20), (5.35) yields

$$w_i(\underline{s}_i) \geq c_1 \; t_i^{1/2} \exp(\beta \hat{s}_i / 2) \geq c_2(\epsilon_0) t_i^{1/2} \exp(\beta a_i / 2) \tag{5.37}$$

for some $c_2(\epsilon_0) > 0$.

By (5.22),(5.32) and (5.37),

$$\begin{cases} 2(\underline{s}_i - a_i) \leq b_i - a_i \\ |\underline{s}_i - (1 - \frac{\beta}{n-2}) a_i + \frac{1}{n-2} \log t_i| \leq c(\epsilon_0) \\ |b_i - (1 - \frac{2\beta}{n-2}) a_i + \frac{2}{n-2} \log t_i| \leq c(\epsilon_0). \end{cases} \tag{5.38}$$

for some constant $c(\epsilon_0) > 0$. Hence we have

$$\begin{aligned} a_i \leq & \Big(1 - \frac{2\beta}{n-2}\Big)^{-1} \Big[s_i + \frac{2}{n-2} \log t_i \Big] + c \\ \leq & \log [R_i^{-2} M_i^{-\frac{2}{n-2}}] + c, \end{aligned} \tag{5.39}$$

and

$$\begin{aligned} \underline{s}_i \leq& \frac{1}{2}(b_i - a_i) + a_i + c \\ \leq& \frac{(1-\frac{\beta}{n-2})}{(1-\frac{2\beta}{n-2})} s_i + \frac{1}{(n-2)(1-\frac{2\beta}{n-2})} \log t_i + c \\ \leq& \log[R_i^{-1} M_i^{-\frac{2}{n-2}}] + c, \end{aligned} \tag{5.40}$$

where R_i is defined in (5.2). These estimates together with Lemma 5.1 and (5.22) imply

$$\begin{aligned} w_i(s) &\leq w_i(\underline{s}_i) \exp\Big[\frac{n-2}{2}(\underline{s}_i - s)\Big] \\ &\leq c_1\ t_i^{\frac{1}{2}} \exp\Big(-\frac{n-2}{2}s\Big) \exp\Big[\frac{n-2}{2}\underline{s}_i + \frac{1}{2}\beta a_i\Big] \\ &\leq c(\epsilon_0) \exp\Big(-\frac{n-2}{2}s\Big) \exp\left\{\frac{\frac{n-2}{2}b_i + \log t_i}{1-\frac{2\beta}{n-2}}\right\} \end{aligned} \tag{5.41}$$

for $a_i \leq s \leq \underline{s}_i$, and

$$\begin{aligned} w_i(\underline{s}_i) \exp\Big[\frac{n-2}{2}(s - \underline{s}_i)\Big] \leq& w_i(s) \\ \leq& w_i(\underline{s}_i) \exp\Big[\frac{n-2}{2}(s - \underline{s}_i)\Big] \end{aligned} \tag{5.42}$$

for $\underline{s}_i \leq s \leq b_i$. Using (5.41), it follows

$$\begin{aligned} u_i(x) \leq& c(\epsilon_0) \exp\left\{\frac{\frac{n-2}{2}s_i + \log t_i}{1-\frac{2\beta}{n-2}}\right\} |x|^{-n+2} \\ =& c(\epsilon_0)(t_i M_i^{-1})^{\frac{1}{1-\frac{2\beta}{n-2}}} |x|^{-n+2} \end{aligned} \tag{5.43}$$

for $\exp(a_i) \leq |x| \leq \exp(\underline{s}_i)$, and by Lemma 5.1,

$$c w_i(\underline{s}_i) \exp\Big[-\frac{n-2}{2}\underline{s}_i\Big] \leq u_i(x) \leq c_1(\epsilon_0)\ w_i(\underline{s}_i) \exp\Big[-\frac{n-2}{2}\underline{s}_i\Big] \tag{5.44}$$

for $\exp(\underline{s}_i) \leq |x| \leq \exp(s_i)$ and some $c_1(\epsilon_0)$. Since $u_i(x) \sim \exp(-\frac{n-2}{2}s_i)$ for $|x| = \exp(s_i)$, (5.44) leads to

$$u_i(x) \sim \exp\Big(-\frac{n-2}{2}s_i\Big) \sim M_i \tag{5.45}$$

for $\exp(\underline{s}_i) \le |x| \le M_i^{-\frac{2}{n-2}}$. Now (5.39), (5.40), (5.44) and (5.45) imply

$$u_i(x) \le c\,(t_i M_i^{-1})^{\frac{1}{1-\frac{2\beta}{n-2}}} |x|^{-n+2} \tag{5.46}$$

for $R_i^{-2} M_i^{-\frac{2}{n-2}} \le |x| \le R_i^{-1} M_i^{-\frac{2}{n-2}}$, and

$$u_i(x) \le c M_i \tag{5.47}$$

for $R_i^{-1} M_i^{-\frac{2}{n-2}} \le |x| \le M_i^{-\frac{2}{n-2}} \sim e^{s_i}$.

Finally, we want to estimate $u_i(x)$ for $|x| \ge M_i^{\frac{-2}{n-2}}$. Set s_i^* to be a local minimum point of $w_i(s)$ in $(s_i, 0)$ if there is one. Otherwise $s_i^* = 0$. we claim

$$\begin{aligned} &s_i^* \to 0 \ \text{ if and only if } \quad L_i M_i^{-\frac{2}{n-2}} \to 0 \ \text{ and } \ i \to +\infty. \\ &\text{Moreover, if } s_i^* \to 0, \text{ then } e^{s_i^*} \sim L_i M_i^{\frac{-2}{n-2}}. \end{aligned} \tag{5.48}$$

First suppose $L_i M_i^{\frac{-2}{n-2}} \to 0$ and $s_i^* \ge c > 0$. Set

$$\widetilde{u}_i(y) = M_i^{-1} u_i(M_i^{-\frac{2}{n-2}} y).$$

By Lemma 5.1,

$$\widetilde{u}_i(y) \le c\,|y|^{2-n} \ \text{ for } \ 1 \le |y| \le M_i^{\frac{2}{n-2}}, \tag{5.49}$$

because $s_i^* \ge c > 0$. The scaled $\widetilde{u}_i(y)$ converges to

$$U(y) = [\lambda(\lambda^2 + |y-q|^2)]^{\frac{2-n}{2}}$$

for $\lambda > 0$ and $q \in \mathbb{R}^n$. Then by Remark 5.3 (below), we have

$$\int_{\mathbb{R}^n} \triangledown Q(y) U^{\frac{2n}{n-2}}(y) dy = 0. \tag{5.50}$$

Note $\widetilde{u}_i$ satisfies $\Delta \widetilde{u}_i + \widetilde{K}_i(y) \widetilde{u}_i^{\frac{n+2}{n-2}} = 0$ and

$$\widetilde{K}_i(y) = K_i(M_i^{-\frac{2}{n-2}} y).$$

Clearly,

$$y \cdot \triangledown \widetilde{K}_i(y) = t_i M_i^{-\frac{2\beta}{n-2}} [Q(y) + O(|y|^{\beta-1})]$$

and $L_i^{n-2} = t_i^{-1} M_i^{\frac{2\beta}{n-2}}$. Thus, the Pohozave identity yields

$$\begin{aligned}
&\frac{(n-2)\beta}{2n} \int_{\mathbb{R}^n} Q(y) U^{\frac{2n}{n-2}}(y)dy \\
&\quad = \lim_{i\to+\infty} \frac{n-2}{2n} L_i^{n-2} \int_{|y|\le M_i^{\frac{2}{n-2}}} \langle y, \nabla \widetilde{K}_i \rangle \widetilde{u}_i^{\frac{2n}{n-2}}(y)dy \\
&\quad = \lim_{i\to+\infty} L_i^{n-2} \int_{|y|=M_i^{\frac{2}{n-2}}} \left(\frac{n-2}{2} \widetilde{u}_i \frac{\partial \widetilde{u}_i}{\partial r} + \left| \frac{\partial \widetilde{u}_i}{\partial r} \right|^2 r \right. \\
&\qquad\qquad \left. - \frac{1}{2} | \nabla \widetilde{u}_i|^2 r + \frac{n-2}{2n} \widetilde{K}_i(y) \widetilde{u}_i^{\frac{2n}{n-2}} r \right) d\sigma \to 0,
\end{aligned} \tag{5.51}$$

because the boundary term $= O(M_i^{-2})$. By (5.50) and (K2),

$$\int Q(y) U^{\frac{2}{n-2}}(y)dy \neq 0.$$

Thus, (5.51) yields a contradiction.

Conversely, we assume $s_i^* \to 0$. Then as the second inequality in (5.38), we have

$$s_i^* = \left(1 - \frac{\beta}{n-2}\right) s_i - \frac{1}{n-2} \log t_i + O(1), \tag{5.52}$$

which yields

$$e^{s_i^*} \sim L_i M_i^{-\frac{2}{n-2}}, \tag{5.53}$$

and it implies $L_i M_i^{-\frac{2}{n-2}} \to 0$ as $i \to +\infty$. Hence, (5.48) is proved. Clearly, (5.13) and (5.14) follows from Lemma 5.1 and (5.48). Therefore, we have proved Step 1.

Step 2. Recall that $\widetilde{u}_i(x) = M_i^{-1} u_i(M_i^{-\frac{2}{n-2}} x)$. After passing to a subsequence, $\widetilde{u}_i(x)$ converges to $U(x-q)$ in $C^2_{\text{loc}}(\mathbb{R}^n \backslash \{0\})$ with

$$U(x) = \left(\frac{\lambda}{\lambda^2 + |x|^2} \right)^{\frac{n-2}{2}}. \tag{5.54}$$

Now we can estimate the difference of $\widetilde{u}_i$ and $U(x-q)$ more precisely if we rescale $U(x-q)$ and translate the position of its maximum point suitably.

If $q \neq 0$, then there is a local maximum point q_i of $\widetilde{u}_i$ with $\lim_{i\to\infty} q_i = q$. For suitable $a_i \to 1$ and $\lambda_i \to 1$, we let the function $U_i(x) = a_i\lambda_i^{-\frac{n-2}{2}} U(\lambda_i^{-1}(x - q_i)) > 0$ satisfy

$$\begin{cases} \triangle U_i + \widetilde{K}_i(0)U_i^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \\ U_i(q_i) = \max_{\mathbb{R}^n} U_i = \widetilde{u}_i(q_i) \end{cases} \tag{5.55}$$

with $\widetilde{K}_i(x) = K_i(M_i^{-\frac{2}{n-2}}x)$, and

$$\nabla(\widetilde{u}_i(q_i) - U_i(q_i)) = 0. \tag{5.56}$$

Note that λ_i and a_i are uniquely determined because they satisfy

$$\begin{cases} a_i\lambda_i^{-\frac{n-2}{2}} U(0) = \widetilde{u}_i(q_i) \quad \text{and} \\ \widetilde{K}_i(0) = a_i^{-\frac{4}{n-2}} n(n-2). \end{cases} \tag{5.57}$$

If $q = 0$, let $\delta_o > 0$ be a small number which is independent of i and will be chosen later. Then there is $q_i = q_i(\delta_0)$ such that

$$\lim_{i\to\infty} q_i = 0, \quad \text{and}$$

$$\int_{|x-q_i|=\delta_o} (x - q_i)\widetilde{u}_i \, ds = 0, \tag{5.58}$$

since $\widetilde{u}_i(x)$ converges to $U(x)$ in $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$. For suitable $a_i \to 1$ and $\lambda_i \to \lambda$, we may let the function $U_i = a_i\lambda_i^{-\frac{n-2}{2}} U_1(\lambda_i^{-1}(x - q_i)) > 0$ satisfy

$$\begin{cases} \triangle U_i + \widetilde{K}_i(0)U_i^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \\ \displaystyle\int_{|x-q_i|=\delta_o} U_i \, ds = \int_{|x-q_i|=\delta_o} \widetilde{u}_i \, ds, \\ \displaystyle\int_{|x-q_i|=\delta_o} (x - q_i)U_i \, ds = 0. \end{cases} \tag{5.59}$$

Set U_i as above, let $g_i(x) = \widetilde{u}_i(x) - U_i(x)$. Then g_i satisfies

$$\triangle g_i + b(x)g_i = \widetilde{Q}(x)U_i^{\frac{n+2}{n-2}},$$

where

$$\begin{aligned} b(x) =& \widetilde{K}_i(x) \frac{\widetilde{u}_i^{\frac{n+2}{n-2}} - U_i^{\frac{n+2}{n-2}}}{\widetilde{u}_i - U_i}, \\ \widetilde{Q}(x) =& \widetilde{K}_i(0) - \widetilde{K}_i(x). \end{aligned}$$

Let $f_i(x)$ be defined as follows.

$$\begin{aligned} f_i(x) &= |x|^{-\frac{n-2}{2}} \quad \text{for } 0 \le |x| \le R_i^{-2}, \\ f_i(x) &= \{L_i^{-n+2} + R_i^{-n+2}|x|^{-n+2} + \max_{|y|=M_i^{\frac{2}{n-2}}} |\widetilde{u}_i(y) - U_i(y)|\} \\ &\qquad \text{for } R_i^{-2} \le |x| \le M_i^{\frac{2}{n-2}}, \end{aligned}$$

and

$$N_i = \max_{|x| \le M_i^{\frac{2}{n-2}}} f_i^{-1}(x)|g_i(x)|.$$

Let x_i be a point satisfy $|x_i| \le M_i^{\frac{2}{n-2}}$ and satisfy $N_i = f_i^{-1}(x_i)|g_i(x_i)|$. To prove part (ii), it suffices to show $\sup_{i \ge 1} N_i < \infty$.

Assume that N_i is unbounded. Without loss of generality, we may assume $\lim_{i \to \infty} N_i = +\infty$. Let $r_i = \min(L_i, M_i^{\frac{2}{n-2}})$. By (5.1), (5.11), (5.12), (5.13) and (5.14), we can see that $\widetilde{u}_i$ satisfies

$$\widetilde{u}_i(x) \le c\,|x|^{-\frac{n-2}{2}} \quad \text{for} \quad |x| \le M_i^{\frac{2}{n-2}}, \tag{5.60}$$

$$\begin{aligned} \widetilde{u}_i(x) &\le c(t_i M_i^{-1})^{\gamma} M_i |x|^{-n+2} \\ &= c\, R_i^{-n+2}|x|^{-n+2} \quad \text{for} \quad R_i^{-2} \le |x| \le R_i^{-1}, \end{aligned} \tag{5.61}$$

$$\widetilde{u}_i(x) \le c\, U(x), \quad \text{for } R_i^{-1} \le |x| \le r_i, \text{ and} \tag{5.62}$$

(5.63)

If $L_i M_i^{-\frac{2}{n-2}}$ is bounded, then $\widetilde{u}_i(x) \le c\, L_i^{-n+2}$ for $r_i \le |x| \le M_i^{\frac{2}{n-2}}$.

We note that if $L_i M_i^{-\frac{2}{n-2}}$ is unbounded, then $w_i(s)$ has no local minimum for $s_i \le s \le 0$. Thus, $r_i = M_i^{\frac{2}{n-2}}$ and by (5.13), we have for $|x| = r_i$,

$$\widetilde{u}_i(x) \sim M_i^{-2} >> L_i^{-n+2}, \tag{5.64}$$

i.e., (5.63) does not hold in this case.

Since N_i is unbounded, we have by (5.60), (5.61) and (5.63),

$$r_i \geq |x_i| \geq R_i^{-1}. \tag{5.65}$$

By Green's identity, we have for $r_i \geq |x| \geq R_i^{-1}$

$$\begin{aligned} g_i(x) = &\int_{|\eta|\leq r_i} G(x,\eta)(b(\eta)g_i - \widetilde{Q}(\eta)U_i^{\frac{n+2}{n-2}})\, d\eta \\ &- \int_{|\eta|=r_i} \frac{\partial G(x,\eta)}{\partial \nu} g_i(\eta)\, ds \end{aligned} \tag{5.66}$$

and

$$\begin{aligned} \nabla g_i(x) = &\int_{|\eta|\leq r_i} \nabla_x G(x,\eta)(b(\eta)g_i - \widetilde{Q}(\eta)U_i^{\frac{n+2}{n-2}})\, d\eta \\ &- \int_{|\eta|=r_i} \frac{\partial \nabla_x G(x,\eta)}{\partial \nu} g_i(\eta)\, ds, \end{aligned} \tag{5.67}$$

where $G(x,\eta)$ is the Green function of $-\triangle$ on $\{x : |x| \leq r_i\}$. Since we assume $\dfrac{n-2}{2} \geq \beta > 1$, it implies $n \geq 4$. By the inequality $g_i \leq N_i f_i$ and $G(x,\eta) \leq c_n |x-\eta|^{2-n}$, we have the following estimates for $R_i^{-1} \leq |x| \leq r_i$. Their proofs are elementary and are omitted here. By (5.60), we have

$$|b(\eta)g_i(\eta)| \leq c\ |\eta|^{-\frac{n+2}{2}} \quad \text{for } |\eta| \leq R_i^{-2}.$$

Hence

$$\int_{|\eta|\leq R_i^{-1}} G(x,\eta)b(\eta)g_i\, d\eta = O\big(R_i^{-n+2}|x|^{-n+2}\big). \tag{5.68}$$

By (5.62) and (5.63), we have

$$\widetilde{u}_i(x) \leq c\ U(x) \quad \text{for} \quad R_i^{-1} \leq |x| \leq r_i,$$

which implies

$$|b(\eta)| \leq c(1+|\eta|)^{-4} \quad \text{for} \quad R_i^{-1} \leq |\eta| \leq r_i.$$

Hence

(5.69)

$$\begin{aligned}
&\int_{R_i^{-1}\le|\eta|\le r_i} G(x,\eta)b(\eta)g_i\,d\eta \\
&\qquad = O\big[\int_{R_i^{-1}\le|\eta|\le r_i} G(x,\eta)U_i^{\frac{4}{n-2}}N_i f_i\,d\eta\big] \\
&\qquad = N_i\,O\big[R_i^{-n+2}\begin{cases} |x|^{-n+4}(1+|x|)^{-2} & n>4 \\ \log(2+|x|^{-1})(1+|x|)^{-2} & n=4 \end{cases} \\
&\qquad\qquad + (1+|x|)^{-2}(L_i^{-n+2} + \max_{|\eta|=M_i^{\frac{2}{n-2}}} |g_i(\eta)|)\big] \\
&\qquad = O(1)(1+|x|)^{-2}N_i f_i(x).
\end{aligned}$$

Note that for (5.69), we have used $R_i^{-n+2}|x|^{-n+2} = o(1)L_i^{-n+2}$ for $|x|\ge 1$,

$$\begin{aligned}
&\int_{|\eta|\le r_i} G(x,\eta)\widetilde{Q}(\eta)U_i^{\frac{n+2}{n-2}}\,d\eta \\
&= O\left[\frac{1}{(1+|x|)^{n-2}} + \frac{1}{(1+|x|)^{n-\beta}}\right]L_i^{-n+2} \\
&= o(1)N_i f_i(x),
\end{aligned} \tag{5.70}$$

where

$$\begin{aligned}
|\widetilde{Q}(\eta)| &= \left|K_i(0) - K_i(M_i^{-\frac{2}{n-2}}\eta)\right| \\
&\le c\; t_i M_i^{-\frac{2\beta}{n-2}}|\eta|^\beta \\
&= c\; L_i^{-n+2}|\eta|^\beta.
\end{aligned}$$

$$\int_{|\eta|=r_i} \frac{\partial G(x,\eta)}{\partial\nu} g_i(\eta)\,ds = O\big[\max_{|\eta|=r_i}|g_i(\eta)|\big]. \tag{5.71}$$

From (5.62) and (5.63), there is $\hat{c}>0$ such that

$$\max_{|\eta|=r_i}|g_i(\eta)| \le \hat{c}\min_{|x|\le M_i^{\frac{2}{n-2}}} f_i(x).$$

Putting these estimates together, we obtain

$$\begin{aligned}
g_i(x) =& O\big[(1+|x|)^{-2}N_i f_i(x) + \max_{|\eta|=r_i}|g_i(\eta)|\big] \\
=& O\big[(1+|x|)^{-2} + o(1)\big]N_i f_i(x)
\end{aligned} \tag{5.72}$$

for $r_i \ge |x| \ge R_i^{-1}$. Similarly, we have the following estimates for derivatives:

$$\int_{|\eta|\le R_i^{-1}} \nabla_x G(x,\eta) b(\eta) g_i \, d\eta = O\big(R_i^{-n+2}|x|^{-n+1}\big), \tag{5.73}$$

(5.74)

$$\begin{aligned}\int_{R_i^{-1}\le|\eta|\le r_i} & \nabla_x G(x,\eta) b(\eta) g_i \, d\eta \\ &= O\big[\int_{R_i^{-1}\le|\eta|\le r_i} \nabla_x G(x,\eta) U_i^{\frac{4}{n-2}} N_i f_i \, d\eta\big] \\ &= N_i\, O\big[R_i^{-n+2}|x|^{-n+3}(1+|x|)^{-2} \\ &\qquad + (1+|x|)^{-3}(L_i^{-n+2} + \max_{|\eta|=M_i^{\frac{2}{n-2}}} |g_i(\eta)|)\big] \\ &= O(1)(1+|x|)^{-2} N_i f_i(x),\end{aligned}$$

$$\begin{aligned}\int_{|\eta|\le r_i} & \nabla_x G(x,\eta) \widetilde{Q}(\eta) U_i^{\frac{n+2}{n-2}} \, d\eta \\ &= O(1)[\log(2+|x|)(1+|x|)^{-n+1} \\ &\qquad + (1+|x|)^{-n-1+\beta}] L_i^{-n+2},\end{aligned} \tag{5.75}$$

$$\int_{|\eta|=r_i} \frac{\partial \nabla_x G(x,\eta)}{\partial \nu} g_i(\eta) \, ds = O\big[r_i^{-1} \max_{|\eta|=r_i} |g_i(\eta)|\big] \tag{5.76}$$

for $r_i \ge |x| \ge R_i^{-1}$. It follows from these estimates

$$\begin{aligned}\nabla g_i(x) =& O\big[R_i^{-n+2}|x|^{-n+1} + (1+|x|)^{-2} N_i f_i(x) \\ & \qquad + r_i^{-1} \max_{|\eta|=r_i} |g_i(\eta)|\big] \\ =& O\big[R_i^{-n+2}|x|^{-n+1} + ((1+|x|)^{-2} + o(1)) N_i f_i(x)\big]\end{aligned} \tag{5.77}$$

for $r_i \ge |x| \ge R_i^{-1}$.

Let $x = x_i$ in (5.72). We obtain

$$N_i f_i(x_i) = |g_i(x_i)| \le c[(1+|x_i|)^{-2} + o(1)] N_i f_i(x_i)$$

for some c independent of i. Hence x_i must be bounded and

$$|x_i| \le c_1$$

for some c_1 independent of i.

Since $R_i^{-1} \le |x_i| \le c_1$, we have

$$f_i(x_i) = \left\{ L_i^{-n+2} + R_i^{-n+2}|x_i|^{-n+2} + \max_{|y|=M_i^{\frac{2}{n-2}}} |\widetilde{u}_i(y) - U_i(y)| \right\}.$$

Note that $L_i << R_i$. For any $r > 0$, if $|x| \ge r$, then

$$\frac{f_i(x)}{f_i(x_i)} \le 2 + \left(\frac{L_i}{R_i}\right)^{n-2} r^{-n+2}. \tag{5.78}$$

By (5.72) and (5.78), $|g_i(x_i)|^{-1} g_i(x)$ satisfies for $|x| \ge r > 0$,

$$\begin{aligned} \frac{|g_i(x)|}{|g_i(x_i)|} &\le c\ (1+|x|)^{-2} \frac{f_i(x)}{f_i(x_i)} \\ &\le c\ \left[(1+|x|)^{-2} + \left(\frac{L_i}{R_i}\right)^{n-2} r^{-n+2} \right]. \end{aligned}$$

After passing to a subsequence, the sequence $g_i(x_i)^{-1} g_i(x)$ converges in $C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ to a function ϕ which satisfies

$$\left\{ \begin{array}{c} \triangle\phi + n(n+2)U^{\frac{4}{n-2}}\phi = 0 \text{ in } \mathbb{R}^n \setminus \{0\}, \\ |\phi| \le c\ (1+|x|)^{-2}, \end{array} \right. \tag{5.79}$$

where U is given in (5.54). Since $\phi(x)$ is bounded, by the regularity of elliptic equations, ϕ satisfies (5.79) in $\mathbb{R}^n$. Now we show that $\phi \not\equiv 0$. Since x_i is bounded, without loss of generality, we may assume $x_i \to x_0$. If $x_0 \ne 0$, then $\phi(x_0) = 1$. Obviously, $\phi(x) \not\equiv 0$ in $\mathbb{R}^n$. Now we assume $x_0 = 0$. Let δ_1 be a small positive number. For $y_i = \delta_1 |x_i|^{-1} x_i$, we have by (5.77) and the fact $|x_i| > R_i^{-1}$ that

$$\begin{aligned} |g_i(y_i) - g_i(x_i)| &\le \int_{|x_i|}^{|y_i|} |\nabla g_i(s|x_i|^{-1}x_i)| ds \\ &\le c(R_i^{-n+2}|x_i|^{-n+2} + \delta_1 N_i f_i(x_i)) \\ &\le \frac{1}{2} N_i f_i(x_i) \quad \le \quad \frac{1}{2}|g_i(x_i)| \end{aligned}$$

if N_i is large and δ_1 is small. This implies

$$|g_i(x_i)^{-1} g(y_i)| \ge \frac{1}{2}$$

for large i and consequently,

$$\min_{|x|=\delta_1} |\phi(x)| \geq \frac{1}{2}.$$

We conclude that $\phi \not\equiv 0$.

By Lemma 3.2,

$$\phi = \sum \gamma_j \psi_j$$

with $\psi_0 = \frac{n-2}{2} U + (x-q) \cdot \nabla U(x-q)$ and $\psi_j = \frac{\partial U}{\partial x_j}, 1 \leq j \leq n$. By (5.56) and (5.59), we have either $q \neq 0, \phi(q) = 0$ and $\nabla \phi(q) = 0$ or $q = 0$, $\int_{|x|=\delta_o} \phi \, ds = 0$ and $\int_{|x|=\delta_0} x_j \phi ds = 0$, $1 \leq j \leq n$, which implies $\gamma_j = 0$ for $0 \leq j \leq n$. We obtain a contradiction. Hence N_i must be bounded. The proof of Theorem 2.7 is complete. q.e.d.

Proof of Lemma 5.2. We follow the notations in the proof of Theorem 2.7. Recall that $\hat{u}_i(y)$ converges to $\hat{U}(y)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$, where $\hat{U}$ satisfies (5.16). By the Pohozaev identity

$$\frac{n-2}{2n} \int_{|x|\leq 1} \langle x, \nabla \hat{K}_i \rangle \hat{u}_i^{\frac{2n}{n-2}} \, dx = P(1, \hat{u}_i), \tag{5.80}$$

where

$$\hat{K}_i(x) = K_i(\hat{M}_i^{-\frac{2}{n-2}} x), \hat{M}_i = e^{-\frac{n-2}{2}\hat{s}_i},$$

and

$$P(r, \hat{u}_i) = \int_{|x|=r} \left(\frac{n-2}{2} \hat{u}_i \frac{\partial \hat{u}_i}{\partial \nu} - \frac{1}{2} r |\nabla \hat{u}_i|^2 \right. \\ \left. + r \left| \frac{\partial \hat{u}_i}{\partial \nu} \right|^2 + \frac{n-2}{2n} r \hat{K}_i \hat{u}_i^{\frac{2n}{n-2}} \right) d\sigma.$$

Since $\hat{u}_i(x) \leq c|x|^{-\frac{n-2}{2}}$, the left hand side of (5.80) tends to 0 as $i \to \infty$, which implies

$$P(1, U) = \lim_{i \to \infty} P(1, \widetilde{u}_i) = 0.$$

Since $P(r, u) \equiv$ constant < 0 for any singular solution u of (5.16), $\hat{U}$ is smooth at 0. Hence

$$\hat{U}(y) = \left(\frac{\hat{\lambda}}{\hat{\lambda} + |y - \hat{q}|^2} \right)^{\frac{n-2}{2}}.$$

Since
$$\frac{d}{dr}\hat{u}_i(r)r^{\frac{n-2}{2}}|_{r=1}=0,$$
we have
$$\frac{d}{dr}\overline{\hat{U}}(r)r^{\frac{n-2}{2}}|_{r=1}=0.$$
By a straightforward computation, we have
$$\begin{aligned}\frac{d}{dr}\overline{\hat{U}}(r)r^{\frac{n-2}{2}}&=\frac{d}{dr}\frac{1}{|S^{n-1}|}\int_{S^{n-1}}\frac{(r\hat{\lambda})^{\frac{n-2}{2}}d\sigma}{(\hat{\lambda}^2+|ry-\hat{q}|^2)^{\frac{n-2}{2}}}\\&=\frac{(n-2)\hat{\lambda}^{\frac{n-2}{2}}r^{\frac{n-4}{2}}}{2|S^{n-1}|}\int_{S^{n-1}}\frac{(\hat{\lambda}^2+|\hat{q}|^2-r^2)d\sigma}{(\hat{\lambda}^2+|ry-\hat{q}|^2)^{\frac{n}{2}}}.\end{aligned}$$
Thus, $r_0=\sqrt{\hat{\lambda}^2+|\hat{q}|^2}$ is the only critical point of $\overline{\hat{U}}(r)r^{\frac{n-2}{2}}$ and
$$\frac{d^2}{dr^2}(\overline{\hat{U}}(r)r^{\frac{n-2}{2}})\ |_{r_0}<0.$$
(5.18) follows readily.

We want to prove that $\hat{U}(y)$ satisfies
$$\int_{\mathbb{R}^n}\triangledown Q(y)\hat{U}(y)^{\frac{2n}{n-2}}dy=0, \tag{5.81}$$
and
$$\int_{\mathbb{R}^n}Q(y)\hat{U}(y)^{\frac{2n}{n-2}}dy\leq 0. \tag{5.82}$$
By a simple scaling argument, we have (5.22), i.e.,
$$a_i-\hat{s}_i\leq c\ (\varepsilon_0).$$
Hence, by (5.33),
$$\underline{s}_i-\hat{s}_i>\underline{s}_i-a_i\geq\frac{1}{n-2}(-\log t_i-\beta\hat{s}_i)-c. \tag{5.83}$$
Recall that
$$\hat{M}_i=\exp\left(-\frac{n-2}{2}\hat{s}_i\right)\quad\text{and}\quad\hat{L}_i=\left(t_i^{-1}\hat{M}_i^{\frac{2\beta}{n-2}}\right)^{\frac{1}{n-2}}.$$

By (5.83),

$$r_i \equiv e^{\underline{s}_i} \hat{M}_i^{\frac{2}{n-2}} \geq c \left(t_i^{-1} \hat{M}_i^{\frac{2\beta}{n-2}} \right)^{\frac{1}{n-2}} = c \hat{L}_i^{n-2}.$$

Applying Lemma 5.1, we have

$$\hat{u}_i(y) \leq c\, |y|^{2-n} \tag{5.84}$$

for $1 \leq |y| \leq e^{\underline{s}_i} \hat{M}_i^{\frac{2}{n-2}} = r_i$. Since $\hat{u}_i$ satisfies

$$\Delta \hat{u}_i + \hat{K}_i \hat{u}_i^{\frac{n+2}{n-2}} = 0 \quad \text{for} \quad |y| \leq \hat{M}_i^{\frac{2}{n-2}},$$

where

$$\hat{K}_i(y) = K_i(\hat{M}_i^{-\frac{2}{n-2}} y).$$

Let e_j, $1 \leq j \leq n$, be the standard orthorgonal base for $\mathbb{R}^n$. Applying Pohozaev's identities, we have

$$\begin{aligned} &\frac{n-2}{2n} \int_{B(0,r_i)} \langle e_j, \bigtriangledown \hat{K}_i \rangle \hat{u}_i^{\frac{2n}{n-2}}(y) dy \\ &\quad = \int_{\partial B(0,r_i)} \langle e_j, \bigtriangledown \hat{u}_i \rangle \frac{\partial \hat{u}_i}{\partial \nu} - \langle e_j, \nu \rangle \frac{|\bigtriangledown \hat{u}_i|^2}{2} \\ &\qquad + \frac{n-2}{2n} \langle e_j, \nu \rangle \hat{K}_i \hat{u}_i^{\frac{2n}{n-2}} d\sigma \\ &\quad = O(r_i^{-n+1}), \end{aligned} \tag{5.85}$$

by (5.85) and the gradient estimate. From (5.28), we have

$$\begin{aligned} &\frac{n-2}{2n} \int_{B(0,r_i)} \langle y, \bigtriangledown \hat{K}_i \rangle \hat{u}_i^{\frac{2n}{n-2}}(y) dy \\ &\quad = -\frac{|S^{n-1}|}{2} w_i^2(\underline{s}_i)(1 + o(1)). \end{aligned} \tag{5.86}$$

Since $t_i \hat{M}_i^{-\frac{2\beta}{n-2}} = \hat{L}_i^{-n+2}$ and

$$\bigtriangledown \hat{K}_i(y) = t_i \hat{M}_i^{-\frac{2\beta}{n-2}} (\bigtriangledown Q(y) + o(1)|y|^{\beta-1}) \quad \text{for} \quad |y| \leq \hat{M}_i^{\frac{2}{n-2}},$$

(5.84) and (5.85) yield

$$\begin{aligned} \lim_{i\to+\infty} &\left| \hat{L}_i^{n-2} \int_{B(0,r_i)} t_i \hat{M}_i^{-\frac{2\beta}{n-2}} \left(\left(\frac{\partial Q(y)}{\partial y_j} \right) + o(1)|y|^{\beta-1} \right) \hat{u}_j^{\frac{2n}{n-2}}(y) dy \right| \\ &= \left| \int_{\mathbb{R}^n} \frac{\partial Q}{\partial y_j} \hat{U}^{\frac{2n}{n-2}}(y) dy \right| \leq c\, \hat{L}_i^{-1} \to 0 \end{aligned}$$

as $i \to +\infty$, which is (5.81).

To prove (5.82), we note

$$(y, \bigtriangledown \hat{K}_i(y)) = t_i \hat{M}_i^{-\frac{2\beta}{n-2}} (\beta Q(y) + o(1)|y|^\beta).$$

Thus, (5.86) yields

$$\begin{aligned}
\beta \int_{\mathbb{R}^n} Q(y) \hat{U}^{\frac{2n}{n-2}}(y) dy &= \lim_{i\to+\infty} \left(\hat{L}_i^{n-2} \int_{B(0,r_i)} (y, \bigtriangledown \hat{K}_i) \hat{u}_i^{\frac{2n}{n-2}}(y) dy \right) \\
&= -\frac{n(n-2)|S^{n-1}|}{4} \lim_{i\to+\infty} (\hat{L}_i^{n-2} w_i^2(\underline{s}_i)) \\
&\le 0,
\end{aligned}$$

which is (5.82). The proof of Lemma 5.2 is complete. q.e.d.

Remark 5.3. The proof of (5.19) holds also for $\widetilde{u}_i$ of (5.49), when $L_i M_i^{-\frac{2}{n-2}} \to 0$. Because the left hand side of (5.85) $= \frac{n-2}{2n} t_i M_i^{-\frac{2\beta}{n-2}} \times$

$$\left(\int_{\delta \le |y| \le M_i^{\frac{2}{n-2}}} \langle e_j, \bigtriangledown Q(y) \rangle U^{\frac{2n}{n-2}}(y) dy + O(1) \int_{|y|\le\delta} |y|^{\beta-1-n} dy \right),$$

(5.85) yields

$$\left| \int_{\mathbb{R}^n} \bigtriangledown Q(y) U^{\frac{2n}{n-2}}(y) dy \right| \le c\ L_i^{n-2} (M_i^{-\frac{2}{n-2}})^{(n-1)} \to 0$$

as $n \to +\infty$, which is (5.50).

Remark 5.4. If $L_i M_i^{-\frac{2}{n-2}} \ge c > 0$ for some constant $c > 0$, then (5.13) yields $u_i(x) \le c\ M^{-1}|x|^{2-n}$ for $M_i^{-\frac{2}{n-2}} \le |x| \le 1$. By passing to a subsequence, $M_i u_i(x)$ converges to a positive harmonic function $h(x)$ in $C^2_{\text{loc}}(B_2 \backslash \{0\})$. We claim

(5.87) If $\lim_{i\to+\infty} L_i M_i^{-\frac{2}{n-2}} = +\infty$, then $h(x) = \dfrac{a}{|x|^{n-2}} + O(|x|)$ near 0 for some $a > 0$.

Let $h(x) = \frac{a}{|x|^{n-2}} + b + O(|x|)$ for $a > 0$ and $b \in \mathbb{R}$. By applying the

Pohozaev identity, we have

$$\begin{aligned}
&\frac{n-2}{2n}\int_{|x|\le 1}\langle x, \bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx \\
(5.88)\qquad &= \int_{|x|=1}\left(\frac{n-2}{2}\frac{\partial u_i}{\partial r}u + r\left|\frac{\partial u_i}{\partial r}\right|^2\right. \\
&\qquad\left. -\frac{1}{2}|\bigtriangledown u_i|^2 r - \frac{n-2}{2n}K_i(x)u_i^{\frac{2n}{n-2}}\right)d\sigma
\end{aligned}$$

By scaling, it is easy to see the left hand side of (5.88)

$$= \frac{(n-2)\beta}{2n}L_i^{2-n}\left(\int_{\mathbb{R}^n} Q(y)U^{\frac{2n}{n-2}}(y)dy + o(1)\right),$$

and the right hand side $= -c_n ab M_i^{-2}(1+o(1))$. Since

$$\int_{\mathbb{R}^n} \bigtriangledown Q(y)U^{\frac{2n}{n-2}}(y)dy = 0$$

by (5.50), (K1) yields $\int_{\mathbb{R}^n} Q(y)U^{\frac{2n}{n-2}}(y)dy \ne 0$. Hence, if

$$\lim_{i\to+\infty} L_i^{n-2}M_i^{-2} = +\infty,$$

then $ab = 0$, i.e., $b = 0$. Thus, the claim (5.87) is proved.

6. Preliminary results of global solutions

From now on, $u_i(x)$ is considered to be a solution of (1.3) defined in the whole $\mathbb{R}^n$. Theorem 1.2 implies that after passing to a subsequence, $\{u_i\}$ blows up only at finite points. We will prove this later and for the proof of Theorem 1.2, we assume first that $\{\hat{q}_j\}_{j=1}^m$ is the set of blowup points for $\{u_i\}$ with $m \ge 1$, and $u_i \to 0$ on any compact subset of $\mathbb{R}^n\backslash\{\hat{q}_1, \dots, \hat{q}_m\}$. Let $l \le m$ be the nonnegative integer such that $\hat{q}_1, \dots, \hat{q}_l$ are simple-like blowup points and $\hat{q}_{l+1}, \dots, \hat{q}_m$ are non-simple-like blowup points. For the definition of simple-like blowup points, see the end of Section 2. If there are no simple-like blowup points, we let $l = 0$.

For each blowup point $\hat{q}_j$, we define the local maximum $M_{i,j}$ and the local maximum point in the following ways. Let δ_0 be a small positive number such that the distance $d(\hat{q}_j, \hat{q}_k)$ from $\hat{q}_j$ to $\hat{q}_k$ is greater than

$2\delta_0$. If u_i loses energy of one bubble near $\hat{q}_j$, that is, if (2.17) holds, then $M_{i,j}$ and $\hat{q}_{i,j}$ are defined by

$$M_{i,j} = u_i(\hat{q}_{i,j}) = \max_{|\hat{q}_j - x| \le \delta_0} u_i(x). \tag{6.1}$$

Let $L_{i,j} = L_i(\hat{q}_{i,j})$ be the number defined in (2.15). If u_i loses energy of more than one bubble at $\hat{q}_j$, then there are two cases. The first one is described in (ii) of Theorem 2.5. In this case, $\hat{q}_{i,j}$ denotes the local maximum point z_i in the statement of (ii) of Theorem 2.5, and $M_{i,j} = u_i(\hat{q}_{i,j})$. Note that in this case, $\hat{q}_j$ is a simple-like blowup point,

$$\lim_{i\to+\infty} L_{i,j} M_{i,j}^{-\frac{2}{n-2}} = +\infty, \quad \text{and} \quad \lim_{i\to+\infty}\left(|\hat{q}_{i,j} - \hat{q}_j| M_{i,j}^{\frac{2}{n-2}}\right) = +\infty \tag{6.2}$$

by Theorem 2.5. The second case is described in Theorem 2.7. In this case, $M_{i,j}$ and $L_{i,j}$ are defined as in (2.28) and (2.29), and $\hat{q}_{i,j}$ is defined to be $\hat{q}_j + M_{i,j}^{-\frac{2}{n-2}} z_i$, where z_i is in the statement of Theorem 2.7.

By Theorem 2.1, Theorem 2.5, and Theorem 2.7 and the remark after Definition 2.9, $\hat{q}_j$ is a simple-like blowup point if and only if

$$L_{i,j} M_{i,j}^{-\frac{2}{n-2}} \ge c > 0. \tag{6.3}$$

Also, for $j \le l$, we have

$$\min_{|x-\hat{q}_j| \le \delta_0} u_i(x) \sim M_{i,j}^{-1}. \tag{6.4}$$

For $l+1 \le j \le m$, we have then

$$\min_{|x-\hat{q}_j| \le \delta_0} u_i(x) \sim L_{i,j}^{2-n} M_{i,j} \tag{6.5}$$

and

$$u_i(x) \le c|x - \hat{q}_j|^{-\frac{n-2}{2}} \tag{6.6}$$

for $|x - \hat{q}_j| \le \delta_0$ since they are non-simple-like blowup points.

One important situation is that for some j,

$$\lim_{i\to+\infty} L_{i,j} M_{i,j}^{-\frac{2}{n-2}} = +\infty \tag{6.7}$$

occurs. We claim

(6.8) *If* (6.7) *holds for some* j, *then* $\hat{q}_j$ *is the only simple-like blowup point, that is,* $\quad l = 1$.

Proof of (6.8). If $u_i(x)$ satisfies the assumption of Theorem 2.7 at q_j, then $m_i \sim M_{i,j}^{-1}$ by (6.4). Set $h(x)$ to be the limit of $m_i^{-1}u_i(x)$. Since $\lim_{i\to+\infty} L_{i,j}M_{i,j}^{-\frac{2}{n-2}} = +\infty$, (5.87) yields that $h(x) = \frac{a}{|x-q_j|^{n-2}}$ for some $a > 0$. By Lemma 6.1 (below), q_j is the only simple-like blowup point. So, we might assume either q_j is a simple blowup point or q_j is the one described by (ii) of Theorem 2.5. We note that for both cases, by letting an empty set E, $R = R_i, l = \delta L_{i,j}$ and $l_0 = +\infty$, Lemma 3.5 yields

$$\int_{R_i\le|x|\le L^*+_{i,j}(\delta)} v_i^{\frac{n+2}{n-2}}(y)dy \le c_1(R_i^{-2}+\varepsilon),$$

where $v_i(y) = M_{i,j}^{-1}u_i(q_{i,j}+M_{i,j}^{-\frac{2}{n-2}})$, R_i is given in (2.14), and $L^*_{i,j}(\delta) = \min(\delta L_{i,j},\ \lambda M_{i,j}^{\frac{2}{n-2}})$ for some fixed $\delta > 0$. Now let x_0 be another simple-like blowup point, i.e., either x_0 is a simple blowup point or the one in case (ii) of Theorem 2.5. Say $x_0 = q_1 \ne q_j$. In any case, there is a small neighborhood ω of q_0 such that $\min_\omega u_i(x) \sim M_{i,1}^{-1}$. Clearly, $\min_\omega u_i(x) \sim \min_{|x-q_j|\le 1} u_i(x)$. Hence $M_{i,j} \sim M_{i,1}$. Let $\omega_i^* = \{y \mid q_{i,j} + M_{i,j}^{-\frac{2}{n-2}}y \in \omega\}$. Then, $v_i(y) \le c$ for $y \in \omega^*$. Since $\lim_{i\to+\infty} L_{i,j}M_{i,j}^{-\frac{2}{n-2}} = +\infty$, $L_{i,j} >> |q_j - q_1|M_{i,j}^{\frac{2}{n-2}}$ for large i. Therefore, by choosing $\lambda \ge 2|q_j - q_1|$, we have $L^*_{i,j}(\delta) = \lambda M_{i,j}^{\frac{2}{n-2}}$, and

$$\begin{aligned}
0 < c_2 &\le \int_\omega u_i^{\frac{2n}{n-2}}(x)dx \\
&= \int_{\omega_i^*} v_i^{\frac{2n}{n-2}}dy \le c\int_{\omega_i^*} v_i^{\frac{n+2}{n-2}}(y)dy \\
&\le c\int_{R_i\le|x|\le L^*_{i,j}(\delta)} v_i^{\frac{n+2}{n-2}}(y)dy \\
&\le c\ c_1(R_i^{-2}+\varepsilon).
\end{aligned}$$

Clearly, this yields a contradiction. Then (6.8) is proved.

One important consequence of (6.8) is that if $l = 1$ and $j \ge 2$ or if $l \ge 2$ and $j \ge 1$, the inequality (6.6) always holds near $\hat{q}_j$. From it, we have $L_{i,j}^{n-2} \sim t_i^{-1}M_{i,j}^{\frac{2\beta_j}{n-2}}$ which follows definition of $L_{i,j}$. To show (6.6) holds in these cases, it suffices for us to consider the case $l \ge 2$. By

(6.8), $\lim_{i\to\infty} L_{i,j} M_{i,j}^{-\frac{2}{n-2}} < \infty$ for all j. If (6.6) does not hold near $\hat{q}_j$, then Theorem 2.5 and (2.21) imply that $\hat{q}_j$ is a simply blowup point. However, Theorem 2.2 implies $|\hat{q}_{i,j} - \hat{q}_j| M_{i,j}^{\frac{2}{n-2}} \le c$. Together with the fact that $\hat{q}_j$ is a simply blowup point, (6.6) holds at $\hat{q}_j$. Then it yields a contradiction again. Hence we prove the claim.

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Recall that $\{q_1, \cdots, q_N\}$ are the critical points of $\hat{K}$. Let q be a blowup point of $\{u_i\}$. We want to prove $\bigtriangledown \hat{K}(q) = 0$. We may assume $q \ne \infty$. Now suppose that q is not a critical point of $\hat{K}$. Then by Corollary 2.3 and (6.8), we conclude that after passing to a subsequence, q is the only simple-like blowup point. Therefore, $\bigtriangledown \hat{K}(\hat{q}) = 0$ for any other blowup point $\hat{q} \ne q$, and it implies there are at most finite blowup points $\{\hat{q}_1, \cdots, \hat{q}_m\}$ which are contained in $\{q_1, \cdots, q_N\} \cup \{q\}$. Also by the Harnack inequality, $u_i \to 0$ uniformly on any compact subset of $\mathbb{R}^n \setminus \{\hat{q}_1, \cdots, \hat{q}_m\}$.

Let $M_{i,j}$ and $\hat{q}_{i,j}$ be defined as above. We may assume $\hat{q}_1 = q$. Then

$$u_i(x) \le c\, M_{i,1}^{-1} |x - \hat{q}_1|^{2-n} \tag{6.9}$$

for $x \notin \bigcup_{j\ge 2}^{m} B(\hat{q}_j, \delta_0)$, and by (6.6),

$$u_i(x) \le c\, |x - \hat{q}_j|^{-\frac{n-2}{2}} \tag{6.10}$$

holds for $|x - \hat{q}_j| \le \delta_0$ and $j \ge 2$. Let $e_1 = (1, 0, \cdots, 0)$ and $\Omega_i = \mathbb{R}^n \setminus \bigcup_{j=1}^{m} B(\hat{q}_j, \delta_0)$. We may assume $e_1 = \dfrac{\bigtriangledown \hat{K}(\hat{q}_1)}{|\bigtriangledown \hat{K}(\hat{q}_1)|}$. By the Pohozaev identity,

$$\begin{aligned}
\int_{B(\hat{q}_1,\delta_0)} \frac{\partial K_i(x)}{\partial x_1} u_i^{\frac{2n}{n-2}}(x)dx &= -\int_{\mathbb{R}^n\setminus B(\hat{q}_1,\delta_0)} \frac{\partial K_i}{\partial x_1} u_i^{\frac{2n}{n-2}} dx \\
&\le \sum_{j=2}^{m} \int_{B(\hat{q}_j,\delta_0)} |\bigtriangledown K_i| u_i^{\frac{2n}{n-2}}(x)dx \\
&\quad + \int_{\Omega_i} |\bigtriangledown K_i| u_i^{\frac{2n}{n-2}} dx \\
&\le c\, t_i \left\{ \sum_{j=2}^{m} \delta_0^{\beta_j - 1} + M_{i,1}^{-\frac{2n}{n-2}} \right\},
\end{aligned} \tag{6.11}$$

where inequalities (6.9) and (6.10) are used.

On the other hand, since $\hat{q}_1$ is a blowup point,

$$\int_{B(\hat{q}_1,\delta_0)} u_i^{\frac{2n}{n-2}}(x)dx \geq c_n > 0 \tag{6.12}$$

for some constant $c_n > 0$. Since $\beta_j > 1$ for $j \geq 2$, (6.11) implies

$$c_n\ t_i \leq \int_{B(\hat{q}_1,\delta_0)} \frac{\partial K_i}{\partial x_1} u_i^{\frac{2n}{n-2}}(x)dx \leq c\ t_i \left\{ \sum_{j=2}^{m} \delta_0^{\beta_j - 1} + M_{i,1}^{-\frac{2n}{n-2}} \right\},$$

which obviously yields a contradiction when δ_0 is small. The proof is finished q.e.d.

From now on, by passing to a subsequence, we may assume the blowup points are $\{q_1, \cdots, q_m\} \subset \{q_1, \cdots, q_N\}$ and $u_i \to 0$ uniformly on any compact subset of $\mathbb{R}^n \setminus \{q_1, \cdots, q_m\}$. Let $l \leq m$ be the non-negative integer such that $q_1, \ldots, q_l$ are simple-like blowup points and $q_{l+1}, \ldots, q_m$ are non-simple-like blowup points. Set

$$m_i = \inf_{\mathbb{R}^n} (u_i(x)(1 + |x|)^{n-2}). \tag{6.13}$$

Since $u_i(x) \to 0$ for $x \notin \{q_1, \ldots, q_m\}$, $m_i \to 0$ as $i \to +\infty$. Let

$$h_i(x) = m_i^{-1} u_i(x) \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then $h_i(x)$ is bounded in $C^2_{\text{loc}}(\mathbb{R}^n \backslash \{q_1, \ldots, q_m\})$. After passing to a subsequence, $h_i(x)$ converges to $h(x)$ in $C^2_{\text{loc}}(\mathbb{R}^n \backslash \{q_1, \ldots, q_m\})$. Since $m_i \to 0$, $h(x)$ satisfies

$$\begin{cases} \Delta h(x) = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \{q_1, \ldots, q_m\}, \\ h(x) > 0. \end{cases}$$

By the Liouville Theorem, we have

$$h(x) = \sum_{j=1}^{m} \frac{\mu_j}{|x - q_j|}, \tag{6.14}$$

where $\mu_j \geq 0$ and $\displaystyle\sum_{j=1}^{m} \mu_j \neq 0$.

Lemma 6.1. $\mu_j > 0$ *if and only if* q_j *is a simple-like blowup point.*

Proof. Let q_j be a simple-like blowup point. By (6.4),

$$m_i \sim M_{i,j}^{-1}.$$

Thus,

$$\begin{aligned}\frac{1}{m_i}\int_{|x-q_j|\le\delta_0} & K_i(x)u_i^{\frac{n+2}{n-2}}(x)dx \\ & \ge c_1\ M_{i,j}\int_{|x-q_j|\le\delta_0} u_i^{\frac{n+2}{n-2}}dx \ge c_2 > 0.\end{aligned}$$

It implies $\mu_j > 0$.

Conversely, if q_j is not a simple-like blowup point, then by (6.6) and (6.5),

$$u_i(x) \le \begin{cases} |x-q_j|^{-\frac{n-2}{2}} & \text{for } |x| \le M_i^{-\frac{2}{n-2}} \\ M_i^{-1}|x-q_j|^{-n+2} & \text{for } M_i^{-\frac{2}{n-2}} \le |x-q_j| \le L_i M_i^{-\frac{2}{n-2}} \\ L_i^{2-n}M_i & \text{for } L_i M_i^{-\frac{2}{n-2}} \le |x-q_j| \le \delta_0, \end{cases} \tag{6.15}$$

where for the simplicity of notations, M_i and L_i denote $M_{i,j}$ and $L_{i,j}$, respectively. Hence

$$m_i \sim L_i^{2-n}M_i. \tag{6.16}$$

Applying (6.16), a straightforward computation shows

$$\frac{1}{m_i}\int_{|x-q_j|\le\delta_0} K_i(x)u_i^{\frac{n+2}{n-2}}(x)dx \le \frac{c}{m_i}\left\{M_i^{-1} + m_i^{\frac{n+2}{n-2}}\right\} \to 0.$$

Here we have used $m_iM_i \sim L_i^{2-n}M_i^2 \to +\infty$ as $i \to +\infty$ by (6.2). Therefore, $\mu_j = 0$. q.e.d.

From Lemma 6.1, we immediately have $l \ge 1$. The next lemma tell us that there are some constraints for a collection of critical points to be a set of blowup points.

Lemma 6.2.

(i) *If $l \ge 2$, then we have $\beta_j > \frac{n-2}{2}$ for all j, or $\beta_j = \frac{n-2}{2}$ for all j, or $\beta_j < \frac{n-2}{2}$ for all j. Moreover, $\beta_1 = \beta_2 = \ldots = \beta_l$ always holds, $\beta_1 > \beta_j$ for $j \ge l+1$ if $\beta_j > \frac{n-2}{2}$ for all j, and $\beta_1 < \beta_j$ for $j \ge l+1$ if $\beta_j < \frac{n-2}{2}$ for all j.*

(ii) *If* $l = 1$, *then we have* $\beta_j > \frac{n-2}{2}$ *for* $2 \le j \le m$, *or* $\beta_j < \frac{n-2}{2}$ *for* $2 \le j \le m$, *or* $\beta_j = \frac{n-2}{2}$ *for* $2 \le j \le m$. *Furthermore, if* $\beta_1 \le \frac{n-2}{2}$, *then* $\beta_j < \frac{n-2}{2}$ *for* $2 \le j \le m$. *If* $\beta_j > \dfrac{n-2}{2}$ *for* $2 \le j \le m$, *then* $\beta_1 > \beta_j$ *for* $j \ge 2$.

Proof. We prove (i) first. Since $l \ge 2$, by (6.8), the inequality (6.6) holds near any q_j and $L_{i,j}^{2-n} \sim t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}$. By (6.8) again and the fact $q_1, \dots, q_l$ are simple-like blowup points, we also have $L_{i,j} \sim M_{i,j}^{\frac{2}{n-2}}$ for $1 \le j \le l$. Thus, $M_{i,j}$ satisfies

$$L_{i,j}^{2-n} \sim t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} \quad \text{for } 1 \le j \le m, \tag{6.17}$$

$$m_i \sim L_{i,j}^{2-n} M_{i,j} \quad \text{for } \ 1 \le j \le m, \tag{6.18}$$

and by Lemma 6.1

$$M_{i,j} = o(1) M_{i,k} \quad \text{for } 1 \le j \le l \text{ and } k \ge l+1. \tag{6.19}$$

By (6.17) and (6.18), for $j \ne k$,

$$M_{i,j}^{1-\frac{2\beta_j}{n-2}} \sim M_{i,k}^{1-\frac{2\beta_k}{n-2}}, \tag{6.20}$$

which implies that there are only three possibilities: $\beta_j > \frac{n-2}{2}$ for all j, or $\beta_j = \frac{n-2}{2}$ all j, or $\beta_j < \frac{n-2}{2}$ for all j. Since $M_{i,j} \sim M_{i,k}$ if $1 \le j, k \le l$, by (6.20), we have $\beta_j = \beta_k$. Again by (6.20) and (6.19), we obtain the inequalities: $\beta_1 > \beta_j$ for $j \ge l+1$ if $\beta_1 > \frac{n-2}{2}$, or $\beta_1 < \beta_j$ for $j \ge l+1$ if $\beta_1 < \frac{n-2}{2}$.

To prove (ii), we note that by (6.15), (6.17) and (6.18) holds for $2 \le j \le m$. Thus, (6.20) holds for $j \ne k \ge 2$, and then we have $\beta_j > \frac{n-2}{2}$ for all $j \ge 2$, or $\beta_j = \frac{n-2}{2}$ for all $j \ge 2$, or $\beta_j < \frac{n-2}{2}$ for all $j \ge 2$.

By (6.19) and

$$m_i \sim M_{i,1}^{-1} >> L_{i,1}^{2-n} M_{i,1} \ge t_i M_{i,1}^{1-\frac{2\beta_1}{n-2}},$$

we have for $j \ge 2$,

$$M_{i,j}^{1-\frac{2\beta_j}{n-2}} >> M_{i,1}^{1-\frac{2\beta_1}{n-2}}.$$

Hence, if $\beta_1 \le \frac{n-2}{2}$, we have $\beta_j < \frac{n-2}{2}$ for all $j \ge 2$. If $\beta_j \ge \frac{n-2}{2}$ for $j \ge 2$, then $\beta_1 > \frac{n-2}{2}$ and for $j \ge 2$

$$\begin{aligned}\left(\frac{2\beta_j}{n-2} - 1\right)\log M_{i,1} &<< \left(\frac{2\beta_j}{n-2} - 1\right)\log M_{i,j}\\ &<< \left(\frac{2\beta_1}{n-2} - 1\right)\log M_{i,1},\end{aligned}$$

which implies $\beta_1 > \beta_j$. q.e.d.

7. Estimates for the Pohozaev identity

As in Section 6, let $q_1, \dots, q_l$ denote all the simple-like blowup points, and let $q_{l+1}, \dots, q_m$ denote the non-simple-like blowup points. Also, let $M_{i,j}$, $q_{i,j}$ and $L_{i,j}$ be defined as in Section 6. Recall $m_i^{-1} \sim M_{i,1}$. Hereafter, $h(x)$ denotes the limit of $M_{i,1}u_i(x)$. By Lemma 6.1,

$$h(x) = \sum_{j=1}^{l} \frac{\mu_j}{|x - q_j|^{n-2}},$$

where $\mu_j > 0$. For $1 \le j \le l$, the regular part of h at q_j is denoted by

$$h_j(x) = \sum_{k=1, k\ne j}^{l} \frac{\mu_k}{|x - q_k|^{n-2}}.$$

The Pohozaev identity plays an important role when we come to study the interaction of different blowup points. Therefore, we have to compute the terms appearing in the Pohozaev identity very precisely. For example, we consider the case when q_j is not a simple-like blowup point. Then $h(x)$ of Section 6 is smooth at q_j. By a direct computation, the Pohozaev identity leads to

$$\begin{aligned}&\frac{n-2}{2n}\int_{|x-q_j|\le\delta_0} \langle x - q_j, \bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}\,dx\\ &\quad = \int_{|x-q_j|=\delta_0} \left(\frac{n-2}{2}u_i\frac{\partial u_i}{\partial\nu} - \frac{1}{2}\delta_0|\bigtriangledown u_i|^2\right.\\ &\qquad\qquad \left. +\delta_0|\frac{\partial u_i}{\partial \nu}|^2 + \frac{n-2}{2n}\delta_0 K_i u_i^{\frac{2n}{n-2}}\right) d\sigma\\ &\quad = o(1)M_{i,1}^{-2},\end{aligned}$$

because $h(x) = \lim_{i\to+\infty} M_{i,1}u_i$ is smooth at q_j. However, it does not show any information about $M_{i,j}$. The following lemma improves the estimate.

Lemma 7.1. *Suppose $\beta_j \geq \frac{2(n-2)}{n}$ for all j. Then the following hold:*

(1) *For $m \geq j \geq l+1$, we have*

$$\begin{aligned}\frac{n-2}{2n}\int_{|x-q_j|\leq\delta_0} \nabla K_i(x)u_i^{\frac{2n}{n-2}}(x)dx \\ = -(1+o(1)+c_1(\delta_0))(n-2)|S^{n-1}|\nabla h(q_j)M_{i,1}^{-1}M_{i,j}^{-1} \\ + o\left(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}\right) + O\left(\delta_0^{n-1}t_iM_{i,1}^{\frac{-2n}{n-2}}\right),\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}\frac{n-2}{2n}\int_{|x-q_j|\leq\delta_0} \langle x-q_j, \nabla K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx \\ = -(1+o(1)+c_2(\delta_0))\frac{(n-2)^2}{2}|S^{n-1}|h(q_j)M_{i,1}^{-1}M_{i,j}^{-1} \\ + o\left(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}\right) + O\left(\delta_0^{n-1}t_iM_{i,1}^{-\frac{2n}{n-2}}\right),\end{aligned} \tag{7.2}$$

where $o(1)\to 0$ as $i\to+\infty$, $c_1(\delta)$ and $c_2(\delta)\to 0$ as $\delta\to 0$.

(2) *If $l\geq 2$, then $1\leq j\leq l$,*

$$\begin{aligned}\frac{n-2}{2n}\int_{|x-q_j|\leq\delta_0} \nabla K_i(x)u_i^{\frac{2n}{n-2}}dx \\ = -(1+o(1)+c_1(\delta))(n-2)|S^{n-1}|\nabla h_j(q_j)M_{i,1}^{-1}M_{i,j}^{-1} \\ + o\left(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}\right) + O\left(\delta_0^{n-1}t_iM_{i,1}^{-\frac{2n}{n-2}}\right),\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}
&\frac{n-2}{2n}\int_{|x-q_j|\le\delta_0}\langle x-q_j,\bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}\,dx\\
&\qquad= -(1+o(1)+c_2(\delta))\frac{(n-2)^2}{2}|S^{n-1}|h_j(q_j)M_{i,1}^{-1}M_{i,j}^{-1}\\
&\qquad\quad+o\left(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}\right)+O\left(\delta_0^{n-1}t_iM_{i,1}^{-\frac{2n}{n-2}}\right).
\end{aligned}\tag{7.4}$$

Proof of Lemma 7.1. For each q_j considered here, $u_i(x)$ satisfies

$$\begin{cases} u_i(x)\le c\,|x-q_j|^{-\frac{n-2}{2}} \quad\text{for}\quad |x-q_j|\le\delta_0\\ L_{i,j}^{2-n}\sim m_iM_{i,j}^{-1}\sim t_iM_{i,j}^{-\frac{2\beta_j}{n-2}},\quad\text{and}\\ \beta_j<n-2,\end{cases}\tag{7.5}$$

due to (6.8) and Corollary 1.3, where m_i is the minimum of u_i in (6.13). We separate our argument into two cases which require different estimates. Case (I) is when u_i loses energy of one bubble only and Case (II) is when u_i loses energy of more than one bubble.

For Case (I), let

$$\widetilde{u}_i(x)=M_{i,j}^{-1}u_i(q_{i,j}+M_{i,j}^{-\frac{2}{n-2}}x).\tag{7.6}$$

Then by Lemma 3.5 and (7.5), we have

$$|\widetilde{u}_i(x)-U_i(x)|\le c\,L_{i,j}^{-n+2}\quad\text{for}\quad |x|\le\delta_0M_{i,j}^{\frac{2}{n-2}}.\tag{7.7}$$

(Note that in this case, $q_{i,j}$ is the local maximum given in (2.12)), where U_i is the solution of

$$\Delta U_i+K_i(q_{i,j})U_i^{\frac{n+2}{n-2}}=0\quad\text{in}\quad\mathbb{R}^n\tag{7.8}$$

with $U_i(0)=\max\limits_{\mathbb{R}^n}U_1(x)=1$.

For Case (II), we can apply Theorem 2.7 to estimate the difference between $\widetilde{u}_i$ and $a_iU_{\lambda_i}$. In this case, $\beta_j<\dfrac{n-2}{2}$ always.

In the following, let

$$L_{i,j}^*=\min(L_{i,j},\delta_0M_{i,j}^{\frac{2}{n-2}})\quad\text{and}\quad l_i=\delta_0M_{i,j}^{\frac{2}{n-2}}$$

for the simplicity of notations. Let U_i denote the solution of (7.8) for Case (I) and denote $a_i U_{\lambda_i}$ for Case (II). Set $g_i(x) = \widetilde{u}_i(x) - U_i(x)$. Then g_i satisfies

$$\triangle g_i + pK_i(q_{i,j})U_i^{p-1} g_i = (K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p + H_1, \tag{7.9}$$

where $p = \dfrac{n+2}{n-2}$, $\widetilde{K}_i(x) = K_i(q_{i,j} + M_{i,j}^{-\frac{2}{n-2}} x)$, and

$$H_1(x) = K_i(q_{i,j})[U_i^p - \widetilde{u}_i^p + pU_i^{p-1} g_i]. \tag{7.10}$$

To estimate the term H_1, we consider Case (I) first. By Lemma 3.5, we have

$$|H_1(x)| \le c_1 U_i^{p-2} |g_i|^2 \le c_2 U_i^{p-2} \left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} \right)^2 \tag{7.11}$$

when $|x| \le L_{i,j}^*$, and

$$|H_1(x)| \le c_1 \ (m_i M_{i,j}^{-1})^p \tag{7.12}$$

when $L_{i,j}^* \le |x| \le \delta_0 M_{i,j}^{\frac{2}{n-2}}$.

For Case (II), we apply Theorem 2.7 to obtain

$$\begin{aligned}
&|H_1(x)| \le c|x|^{-\frac{n+2}{2}} \qquad \text{for } |x| \le R_i^{-2} \\
&|H_1(x)| \le cR_i^{-n-2}|x|^{-n-2} \qquad \text{for } R_i^{-2} \le |x| \le R_i^{-1} \\
&|H_1(x)| \le cU_i^{p-2}|g_i|^2 \le c_2 U_i^{p-2}(R_i^{-2n+4}|x|^{-2n+4} + L_{i,j}^{-2n+4}) \\
&\qquad\qquad\qquad\qquad \text{for } R_i^{-1} \le |x| \le L_{i,j}^* \\
&|H_1(x)| \le c\ L_{i,j}^{-(n-2)p} \qquad \text{for } L_{i,j}^* \le |x| \le \delta_0 M_{i,j}^{\frac{2}{n-2}},
\end{aligned} \tag{7.13}$$

where $R_i = L_{i,j}^{\gamma}$ and $\gamma = (1 - \dfrac{2\beta_j}{n-2})^{-1}$.

Let

$$\partial_\lambda U_i = -\frac{n-2}{2} U_i(x) - x \cdot \bigtriangledown U_i$$

and

$$\partial_\lambda \widetilde{u}_i(x) = -\frac{n-2}{2} \widetilde{u}_i(x) - x \cdot \bigtriangledown \widetilde{u}_i(x).$$

Multiplying (7.9) by $\bigtriangledown U_i$, we have

$$\begin{aligned}
&\int_{|x|\le l_i} \nabla U_i(\triangle g_i + pK_i(q_{i,j})U_i^{\frac{4}{n-2}} g_i)\,dx \\
&\qquad = \int_{|x|\le l_i} (K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p \nabla \widetilde{u}_i\,dx \\
&\qquad\quad + \int_{|x|\le l_i} (K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p (\nabla U_i - \nabla \widetilde{u}_i)\,dx \\
&\qquad\quad + \int_{|x|\le l_i} H_1(x)\nabla U_i\,dx \\
&\qquad \equiv\ I + II + III.
\end{aligned} \tag{7.14}$$

Multiplying (7.9) by $\partial_\lambda U_i$, we have

$$\begin{aligned}
&\int_{|x|\le l_i} \partial_\lambda U_i(\triangle g_i + pK_i(q_{i,j})U_i^{\frac{4}{n-2}} g_i)\,dx \\
&\qquad = \int_{|x|\le l_i} (K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p \partial_\lambda \widetilde{u}_i\,dx \\
&\qquad\quad + \int_{|x|\le l_i} (K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p (\partial_\lambda U_i - \partial_\lambda \widetilde{u}_i)\,dx \\
&\qquad\quad + \int_{|x|\le l_i} H_1(x)\partial_\lambda U_i\,dx \\
&\qquad \equiv\ I^a + II^a + III^a.
\end{aligned} \tag{7.15}$$

Let $y = M_{i,j}^{-\frac{2}{n-2}} x$. By integration by parts,

$$\begin{aligned}
I =& \frac{1}{p+1} \int_{|x|\le l_i} \nabla_x \widetilde{K}_i \widetilde{u}_i^{p+1}\,dx \\
&+ O\left(\int_{|x|=l_i} |K_i(q_{i,j}) - \widetilde{K}_i(x)| \widetilde{u}_i^{p+1}\,ds \right) \\
=& \frac{1}{p+1} M_{i,j}^{-\frac{2}{n-2}} \int_{|y|\le \delta_0} \nabla_y K_i u_i^{p+1}\,dy \\
&+ O\big(\delta_0^{n-1} t_i M_{i,j}^{-\frac{2}{n-2}} (m_i)^{\frac{2n}{n-2}}\big),
\end{aligned} \tag{7.16}$$

By scaling, we have

$$
\begin{aligned}
I^a =& \frac{1}{p+1}\int_{|x|\le l_i}\langle x,\nabla_x\widetilde{K}_i\rangle \widetilde{u}_i^{p+1}\,dx \\
&+O\bigg(\int_{|x|=l_i} l_i|K_i(q_{i,j})-\widetilde{K}_i(x)|\widetilde{u}_i^{p+1}\,ds\bigg) \\
=& \frac{1}{p+1}\int_{|y|\le\delta_0}\langle y,\nabla_y K_i\rangle u_i^{p+1}\,dy \\
&+O(\delta_0^{n-1}t_i(m_i)^{\frac{2n}{n-2}}).
\end{aligned}
\tag{7.17}
$$

To estimate the terms II, III, II^a and III^a, we consider Case (I) first. By (7.7) and integration by parts,

(7.18)
$$
\begin{aligned}
|II| \le& c\int_{|x|\le l_i}\{|(\nabla_x\widetilde{K}_i)\widetilde{u}_i^p|+|(K_i(q_{i,j})-\widetilde{K}_i(x))\nabla_x\widetilde{u}_i^p|\}|(\widetilde{u}_i-U_i)|\,dy \\
&+c\int_{|x|=l_i}|(K_i(q_{i,j})-\widetilde{K}_i(x))\widetilde{u}_i^p(\widetilde{u}_i-U_i)|\,ds \\
\le& c\int_{|x|\le L_{i,j}^*}\frac{L_{i,j}^{-2n+4}}{(1+|x|)^{n-\beta_j+3}}\,dx+O(\delta_0^{n-1}t_iM_{i,j}^{-\frac{2}{n-2}}m_i^{\frac{2n}{n-2}}) \\
\le& O\left[L_{i,j}^{-2n+4}\left\{\begin{matrix} 1, & \beta_j<3 \\ \log L_{i,j}, & \beta_j=3 \\ L_{i,j}^{\beta_j-3}, & \beta_j>3\end{matrix}\right\}+\delta_0^{n-1}t_iM_{i,j}^{-\frac{2}{n-2}}m_i^{\frac{2n}{n-2}}\right] \\
=& o(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}})+O(\delta_0^{n-1}t_iM_{i,j}^{-\frac{2}{n-2}}(m_i)^{\frac{2n}{n-2}})
\end{aligned}
$$

as $i\to\infty$. Here we have used the fact $M_{i,j}|q_j-q_{i,j}|^{\frac{n-2}{2}}$ is bounded and the following estimates:

$$
\begin{cases}
\widetilde{u}_i(x)\sim m_iM_{i,j}^{-1} \quad \text{for } |x|\ge L_{i,j}^*, \text{ and} \\
|K_i(q_{i,j})-\widetilde{K}_i(x)|\le c_1\ t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}(1+|x|^{\beta_j}) \\
|\bigtriangledown\widetilde{K}_i(x)|\le c_1\ t_iM_i^{-\frac{2(\beta_j-1)}{n-2}}(1+|x|^{\beta_j-1}).
\end{cases}
\tag{7.19}
$$

Similarly, we have

$$
\begin{aligned}
|II^a| \le & c\int_{|x|\le l_i} |\langle x, \nabla_x[(K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p]\rangle(\widetilde{u}_i - U_i)|\,dy \\
& + c\int_{|x|=l_i} |(K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p(u_i - U_i)|\,ds \\
\le & c\int_{|x|\le L_{i,j}^*} \frac{L_{i,j}^{-2n+4}}{(1+|x|)^{n-\beta_j+2}}\,dx + O(\delta_0^{n-1}t_i m_i^{\frac{2n}{n-2}}) \qquad (7.20)\\
\le & O\left[L_{i,j}^{-2n+4}\begin{Bmatrix} 1, & \beta_j < 2 \\ \log L_{i,j}, & \beta_j = 2 \\ L_{i,j}^{\beta_j-2}, & \beta_j > 2\end{Bmatrix}\right] + O(\delta_0^{n-1}t_i m_i^{\frac{2n}{n-2}}) \\
= & o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}}) + O(\delta_0^{n-1}t_i (m_i)^{\frac{2n}{n-2}})
\end{aligned}
$$

as $i \to \infty$. Here we have used the fact that by (7.5) and $m_i \to 0$, which implies

$$
L_{i,j}^{-1} \sim o(M_{i,j}^{-\frac{1}{n-2}}). \tag{7.21}
$$

For the terms III and III^a, we have by (7.11) and (7.12),

(7.22)

$$
\begin{aligned}
|III| \le & c\int_{|x|\le L_{i,j}^*} \left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}\right)^2 \frac{1}{(1+|x|)^5}\,dx + O(\delta_0^{n-1}t_i M_i^{-\frac{2}{n-2}} m_i^{\frac{2n}{n-2}}) \\
\le & O\left[L_{i,j}^{-2n+4}\begin{Bmatrix} 1, & n < 5 \\ \log L_{i,j}, & n = 5 \\ L_{i,j}^{n-5}, & n > 5\end{Bmatrix} + \delta_0^{n-1}t_i M_i^{-\frac{2}{n-2}} m_i^{\frac{2n}{n-2}}\right] \\
= & o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}}) + O(\delta_0^{n-1}t_i M_i^{-\frac{2}{n-2}} m_i^{\frac{2n}{n-2}}),
\end{aligned}
$$

and

(7.23)

$$
\begin{aligned}
|III^a| \le & c\int_{|x|\le L_{i,j}^*} \left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}\right)^2 \frac{1}{(1+|x|)^4}\,dx + O(\delta_0^{n-1}t_i m_i^{\frac{2n}{n-2}}) \\
\le & O\left[L_{i,j}^{-2n+4}\begin{Bmatrix} 1, & n < 4 \\ \log L_{i,j}, & n = 4 \\ L_{i,j}^{n-4}, & n > 4\end{Bmatrix} + \delta_0^{n-1}t_i m_i^{\frac{2n}{n-2}}\right] \\
= & o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}) + O(\delta_0^{n-1}t_i m_i^{\frac{2n}{n-2}})
\end{aligned}
$$

as $i \to \infty$. Thus for Case (I), from (7.16), (7.18) and (7.22), we obtain

$$\begin{aligned}&\int_{|x|\le l_i} \nabla U_i(\triangle g_i + pK_i(q_{i,j})U_i^{\frac{4}{n-2}} g_i)\,dx\\ &\quad= \frac{1}{p+1} M_{i,j}^{-\frac{2}{n-2}} \int_{|y|\le\delta_1} \nabla_y K_i u_i^{p+1}\,dy + o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}})\\ &\qquad+ O(\delta_0^{n-1} t_i M_{i,j}^{-\frac{2}{n-2}} (m_i)^{\frac{2n}{n-2}})\end{aligned} \tag{7.24}$$

as $i \to \infty$. From (7.17), (7.20) and (7.23), we have

$$\begin{aligned}&\int_{|x|\le l_i} \partial_\lambda U_i(\triangle g_i + pK_i(q_{i,j})U_i^{\frac{4}{n-2}} g_i)\,dx\\ &\quad= \frac{1}{p+1} \int_{|y|\le\delta_0} \langle y, \nabla_y K_i\rangle u_i^{p+1}\,dy + o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}})\\ &\qquad+ O(\delta_0^{n-1} t_i (m_i)^{\frac{2n}{n-2}})\end{aligned} \tag{7.25}$$

as $i \to \infty$.

For Case (II), we have $1 < \beta_j < \dfrac{n-2}{2}$ and $n > 4$. By using (ii) of Theorem 2.7, we decompose II and II^a into three terms respectively.

$$\begin{aligned}II =& \int_{|x|\le R_i^{-1}} + \int_{R_i^{-1}\le|x|\le L_{i,j}^*} + \int_{L_{i,j}^*\le|x|\le l_i}\\ \equiv& II_1 + II_2 + II_3\end{aligned}$$

and

$$\begin{aligned}II^a =& \int_{|x|\le R_i^{-1}} + \int_{R_i^{-1}\le|x|\le L_{i,j}^*} + \int_{L_{i,j}^*\le|x|\le l_i}\\ \equiv& II_1^a + II_2^a + II_3^a\end{aligned}$$

From integration by parts, (7.19), the fact $M_{i,j}|q_j - q_{i,j}|^{\frac{n-2}{2}}$ is bounded and Theorem 2.7,

$$
\begin{aligned}
II_1 =& -\frac{1}{p+1}\int_{|x|\le R_i^{-1}} \nabla_x \widetilde{K}_i \widetilde{u}_i^{p+1}\,dx \\
&+ O\Big[\int_{|x|=R_i^{-1}} t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-\beta_j}\,ds\Big] \\
&+ \int_{|x|\le R_i^{-1}} (K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p \nabla U_i\,dx \\
=& -\frac{1}{p+1}\Big(\int_{|x|\le R_i^{-2}} + \int_{R_i^{-2}\le|x|\le R_i^{-1}}\Big) + O(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-n+1-\beta_j}) \\
&+ O\left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}(1+R_i^{-\beta_j})\int_{|x|\le R_i^{-1}} |x|^{-\frac{n-2}{2}p} M_{i,j}^{-\frac{2}{n-2}}\,dx\right) \\
=& O\left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-2\beta_j+2} + t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-n+1-\beta_j} + t_i M_{i,j}^{-\frac{2\beta_j+2}{n-2}} R_i^{-\frac{n-2}{2}}\right).
\end{aligned}
$$

$$
\begin{aligned}
II_1^a =& \frac{-1}{p+1}\int_{|x|\le R_i^{-1}} \langle x, \nabla_x \widetilde{K}_i\rangle \widetilde{u}_i^{p+1}\,dx \\
&+ O\Big[\int_{|x|=R_i^{-1}} R_i^{-1} t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-\beta_j}\,ds\Big] \\
&+ \int_{|x|\le R_i^{-1}} (K_i(q_{i,j}) - \widetilde{K}_i(x))\widetilde{u}_i^p \partial_\lambda U_i\,dx \\
=& \frac{-1}{p+1}\Big(\int_{|x|\le R_i^{-2}} + \int_{R_i^{-2}\le|x|\le R_i^{-1}}\Big) + O(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-n-\beta_j}) \\
&+ O\left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}(1+R_i^{-\beta_j})\int_{|x|\le R_i^{-1}} |x|^{-\frac{n-2}{2}p}\,dx\right) \\
=& O\left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-2\beta_j} + t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-n-\beta_j} + t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} R_i^{-\frac{n-2}{2}}\right).
\end{aligned}
$$

Recall that $R_i^{-1} = L_{i,j}^{-\gamma} = o\left(M_{i,j}^{\frac{-\frac{1}{n-2}}{1-\frac{2\beta_j}{n-2}}}\right)$. Thus,

$$
\begin{aligned}
II_1 &= o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}}) \\
II_1^a &= o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}})
\end{aligned}
$$

as $i \to \infty$ if $\beta_j \ge \dfrac{2(n-2)}{n}$. Here is the place we need $\beta_j \ge \dfrac{2(n-2)}{n}$.

From integration by parts and Theorem 2.7,

$$
\begin{aligned}
|II_2| &\le c\ t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} \int_{R_i^{-1}\le|x|\le L_{i,j}^*} \frac{|x|^{\beta_j-1}}{(1+|x|)^{n+2}} \\
&\qquad\qquad \cdot (R_i^{-n+2}|x|^{-n+2} + L_{i,j}^{-n+2})\,dx \\
&\le c(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}})^2 \begin{Bmatrix} 1, & \beta_j < 3 \\ \log L_{i,j}, & \beta_j = 3 \\ L_{i,j}^{\beta_j-3}, & \beta_j > 3 \end{Bmatrix} \\
&= o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}}), \quad \text{and}
\end{aligned}
\tag{7.26}
$$

$$
\begin{aligned}
|II_2^a| &\le c\ t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} \int_{R_i^{-1}\le|x|\le L_{i,j}^*} \frac{|x|^{\beta_j}}{(1+|x|)^{n+2}} \\
&\qquad\qquad \cdot (R_i^{-n+2}|x|^{-n+2} + L_{i,j}^{-n+2})\,dx \\
&\le c(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}})^2 \begin{Bmatrix} 1, & \beta_j < 2 \\ \log L_{i,j}, & \beta_j = 2 \\ L_{i,j}^{\beta_j-2}, & \beta_j > 2 \end{Bmatrix} \\
&= o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}}).
\end{aligned}
\tag{7.27}
$$

For II_3 and II_3^a, we have

$$
\begin{aligned}
|II_3| \le& c\ t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} \int_{L_{i,j}^*\le|x|\le l_i} |x|^{\beta_j-1}(L_{i,j}^{-n+2})^{p+1}\,dx \\
=& O[\delta_0^{n+\beta_j-1} t_i M_{i,j}^{-\frac{2}{n-2}} (M_{i,j}L_{i,j}^{-n+2})^{p+1}] \\
=& O[\delta_0^{n-1} t_i M_{i,j}^{-\frac{2}{n-2}} (m_i)^{p+1}], \quad \text{and,}
\end{aligned}
\tag{7.28}
$$

$$
\begin{aligned}
|II_3^a| \le& c\ t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} \int_{L_{i,j}\le|x|\le l_i} (L_{i,j}^{-n+2})^{p+1}\,dx \\
=& O[\delta_0^{n+\beta_j} t_i (M_{i,j}L_{i,j}^{-n+2})^{p+1}] \\
=& O[\delta_0^{n-1} t_i (m_i)^{p+1}].
\end{aligned}
\tag{7.29}
$$

To estimate III, note that $n \geq 4$ and then

$$
\begin{aligned}
|III| \leq & c\Bigg[\int_{|x|\leq R_i^{-2}} |x|^{-\frac{n+2}{2}}\,dx + \int_{R_i^{-2}\leq|x|\leq R_i^{-1}} R_i^{-n-2}|x|^{-n-2}\,dx \\
&\quad + \int_{R_i^{-1}\leq|x|\leq L_{i,j}^*} \frac{1}{(1+|x|)^5}(R_i^{-2n+4}|x|^{-2n+4} + L_{i,j}^{-2n+4})\,dx \\
&\quad + \int_{L_{i,j}^*\leq|x|\leq l_i} \frac{L_{i,j}^{-n-2}}{(1+|x|)^{n-1}}\,dx\Bigg] \\
\leq & c\big[R_i^{-n+2} + R_i^{-n-2}R_i^4 \\
&\quad + \left(R_i^{-2n+4}\begin{cases} R_i^{n-4}, & n>4 \\ \log R_i, & n=4 \end{cases} + L_{i,j}^{-2n+4}\begin{cases} 1, & n<5 \\ \log L_{i,j}, & n=5 \\ L_{i,j}^{n-5}, & n>5 \end{cases}\right) \\
&\quad + L_{i,j}^{-n-2}M_{i,j}^{\frac{2}{n-2}}\big],
\end{aligned}
$$

and

$$
\begin{aligned}
|III^a| \leq & c\Bigg[\int_{|x|\leq R_i^{-2}} |x|^{-\frac{n+2}{2}}\,dx + \int_{R_i^{-2}\leq|x|\leq R_i^{-1}} R_i^{-n-2}|x|^{-n-2}\,dx \\
&\quad + \int_{R_i^{-1}\leq|x|\leq L_{i,j}^*} \frac{1}{(1+|x|)^4}(R_i^{-2n+4}|x|^{-2n+4} + L_{i,j}^{-2n+4})\,dx \\
&\quad + \int_{L_{k,i}^*\leq|x|\leq l_i} \frac{L_{i,j}^{-n-2}}{(1+|x|)^{n-2}}\,dx\Bigg] \\
\leq & c\big[R_i^{-n+2} + R_i^{-n-2}R_i^4 \\
&\quad + \left(R_i^{-2n+4}\begin{cases} R_i^{n-4}, & n>4 \\ \log R_i, & n=4 \end{cases} + L_{i,j}^{-2n+4}\begin{cases} \log L_{i,j}, & n=4 \\ L_{i,j}^{n-4}, & n>4 \end{cases}\right) \\
&\quad + L_{i,j}^{-n-2}M_{i,j}^{\frac{2}{n-2}}\big].
\end{aligned}
$$

Since $R_i = L_{i,j}L_{i,j}^{\frac{2\beta_j}{n-2}/(1-\frac{2\beta_j}{n-2})} \geq L_{i,j}L_{i,j}^{\frac{2\beta_j}{n-2}}$, $L_{i,j}^{-n+2} \sim t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}$ and $L_{i,j}^{-1} = o(M_{i,j}^{-\frac{1}{n-2}})$, we have

$$
R_i^{-n+2} \leq L_{i,j}^{-n+2}L_{i,j}^{-2\beta_j} = o(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2\beta_j}{n-2}}), \quad \text{and}
$$

$$
\begin{aligned}
&R_i^{-2n+4}\max(R_i^{n-4},\log R_i) \le R_i^{-n+2} = o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}}),\\
&L_{i,j}^{-2n+4}\begin{cases} 1, & n<5\\ \log L_{i,j}, & n=5\\ L_{i,j}^{n-5}, & n>5\end{cases} \le L_{i,j}^{-n+2}\begin{cases} L_{i,j}^{-n+2}, & n<5\\ L_{i,j}^{-3}\log L_{i,j}, & n=5\\ L_{i,j}^{-3}, & n>5,\end{cases}\\
&L_{i,j}^{-2n+4}\begin{cases} \log L_{i,j}, & n=4\\ L_{i,j}^{n-4}, & n>4\end{cases} \le L_{i,j}^{-n+2}\begin{cases} L_{i,j}^{-2}\log L_{i,j}, & n=4\\ L_{i,j}^{-2}, & n>4,\end{cases}\\
&L_{i,j}^{-n-2}M_{i,j}^{\frac{2}{n-2}} \le L_{i,j}^{-n+2}o(1)M_{i,j}^{-\frac{4}{n-2}}M_{i,j}^{\frac{2}{n-2}}.
\end{aligned}
$$

Putting these estimates together, we have by $L_{i,j}^{-n+2}\sim t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}$ and $L_{i,j}^{-1}=o(M_{i,j}^{-\frac{1}{n-2}})$,

$$|III| = o(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}-\frac{2}{n-2}}), \quad \text{and} \quad |III^a| = o(t_iM_{i,j}^{-\frac{2\beta_j}{n-2}}) \tag{7.30}$$

From (7.26) $\sim$ (7.30), we obtain (7.24) and (7.25) for Case (II) also.

By Lemma 6.1, after passing to a subsequence of $\{u_i\}$, $M_{i,1}u_i$ converges to $h=\sum_{j=1}^{l}\frac{\mu_j}{|x-q_j|^{n-2}}$. From integration by parts and the facts

$$\triangle(\nabla U_i)+pK_i(q_{i,j})U_i^{p-1}\nabla U_i=0,$$

$$\triangle(\partial_\lambda U_i)+pK_i(q_{i,j})U_i^{p-1}\partial_\lambda U_i=0,$$

the left hand sides of (7.24) and (7.25) are equal to

$$\int_{|x|=\delta_0M_{i,j}^{\frac{2}{n-2}}}\left(\frac{\partial g_i}{\partial r}\nabla U_i - g_i\frac{\partial\nabla U_i}{\partial r}\right)d\sigma \tag{7.31}$$

and

$$\int_{|x|=\delta_0M_{i,j}^{\frac{2}{n-2}}}\left(\frac{\partial g_i}{\partial r}\partial_\lambda U_i - g_i\frac{\partial(\partial_\lambda U_i)}{\partial r}\right)d\sigma, \tag{7.32}$$

respectively.

For $1\le j\le l$, since q_j is a simple-like blowing-up point, we have

$$m_i \sim M_{i,j}^{-1}. \tag{7.33}$$

When $j\ge l+1$, $L_{i,j}M_{i,j}^{-\frac{2}{n-2}}\to 0$ as $i\to\infty$. Thus,

$$M_{i,1}^{-1} >> M_{i,j}^{-1}. \tag{7.34}$$

Now assume $j \geq l+1$. On $\left\{x : |x| = \delta_0 M_{i,j}^{\frac{2}{n-2}}\right\}$,

$$g_i = \widetilde{u}_i + O(M_{i,j}^{-2}) \quad \text{and} \quad \nabla g_i = \nabla \widetilde{u}_i + O(M_{i,j}^{-2-\frac{2}{n-2}}).$$

By (7.34), we have on $\left\{x : |x| = \delta_0 M_{i,j}^{\frac{2}{n-2}}\right\}$,

$$\begin{aligned} g_i(x) &= (1+o(1)) M_{1,i}^{-1} M_{i,j}^{-1} h(q_j + M_{i,j}^{-\frac{2}{n-2}} x), \\ \nabla_x g_i(x) &= (1+o(1)) M_{i,1}^{-1} M_{i,j}^{-1-\frac{2}{n-2}} \nabla_y h(q_j + M_{i,j}^{-\frac{2}{n-2}} x), \end{aligned}$$

where $y = q_j + M_{i,j}^{-\frac{2}{n-2}} x$. We have

$$\begin{aligned} &\int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} \frac{\partial g_i}{\partial r} \nabla U_i \\ &\qquad = -(1+o(1)+c_1(\delta_0)) \frac{n-2}{n} |S^{n-1}| \nabla_y h(q_j) M_{i,1}^{-1} M_{i,j}^{-1-\frac{2}{n-2}}, \\ -&\int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} g_i \frac{\partial \nabla U_i}{\partial r} = -(1+o(1)+c_2(\delta_0)) \\ &\qquad\qquad \times \frac{(n-1)(n-2)}{n} |S^{n-1}| \nabla_y h(q_j) M_{i,1}^{-1} M_{i,j}^{-1-\frac{2}{n-2}}, \\ &\int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} \frac{\partial g_i}{\partial r} \partial_\lambda U_i \\ &\qquad = -(o(1)+c_3(\delta_0)) \frac{n-2}{n} |S^{n-1}| \nabla_y h(q_j) M_{i,1}^{-1} M_{i,j}^{-1}, \\ -&\int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} g_i \frac{\partial \partial_\lambda U_i}{\partial r} \\ &\qquad = -(1+o(1)+c_4(\delta_0)) \frac{(n-2)^2}{2} |S^{n-1}| \nabla_y h(q_j) M_{i,1}^{-1} M_{i,j}^{-1}, \end{aligned}$$

where $c_j(\delta_0) \to 0$ as $\delta_0 \to 0$. Hence as $i \to \infty$

(7.35)
$$\begin{aligned} &- \int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} \left(\frac{\partial g_i}{\partial r} \nabla U_i - g_i \frac{\partial \nabla U_i}{\partial r} \right) d\sigma \\ &\qquad = -(1+o(1)+c(\delta_0))(n-2) |S^{n-1}| \nabla_y h(q_j) M_{i,1}^{-1} M_{i,j}^{-1-\frac{2}{n-2}}, \end{aligned}$$

and

$$\int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} \left(\frac{\partial g_i}{\partial r}\partial_\lambda U_{i,\lambda} - g_i \frac{\partial \partial_\lambda U_{i,\lambda}}{\partial r}\right) d\sigma \tag{7.36}$$
$$= -(1+o(1)+\widetilde{c}(\delta_0))\frac{(n-2)^2}{2}|S^{n-1}|h(q_j)M_{i,1}^{-1}M_{i,j}^{-1},$$

where $c(\delta_0), \widetilde{c}(\delta_0) \to 0$ as $\delta_0 \to 0$. Now from (7.24), (7.25), (7.35) and (7.36), we obtain (7.1) and

$$\begin{aligned}&\frac{n-2}{2n}\int_{|x-q_{i,j}|\le\delta_0} \langle x-q_{i,j}, \bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx \\ &\quad = -(1+o(1)+c_2(\delta_0))\frac{(n-2)^2}{2}|S^{n-1}|h(q_j)M_{i,1}^{-1}M_{i,j}^{-1} \\ &\qquad + o\left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}\right) + O\left(\delta_0^{n-1} t_i M_{i,1}^{-\frac{2n}{n-2}}\right).\end{aligned}$$

By (7.1) and the fact $M_{i,j}|q_j - q_{i,j}|^{\frac{n-2}{2}}$ is bounded,

$$\begin{aligned}&\int_{|x-q_j|\le\delta_0} \langle x-q_j, \bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx \\ &\quad = \int_{|x-q_j|\le\delta_0} \langle x-q_{i,j}, \bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx \\ &\qquad + \int_{|x-q_j|\le\delta_0} \langle q_{i,j}-q_j, \bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx \\ &\quad = \int_{|x-q_{i,j}|\le\delta_0} \langle x-q_{i,j}, \bigtriangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx \\ &\qquad + o(1)M_{i,1}^{-1}M_{i,j}^{-1} + o\left(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}}\right) + o\left(\delta_0^{n-1} t_i M_{i,1}^{-\frac{2n}{n-2}}\right).\end{aligned}$$

We obtain (7.2).

When $l \ge 2$ and $1 \le j \le l$, after passing to a subsequence of $\{u_i\}$, we have $\infty > \lim L_{i,j} M_{i,j}^{-\frac{2}{n-2}} = c > 0$. Therefore on $\left\{x : |x| = \delta_0 M_{i,j}^{\frac{2}{n-2}}\right\}$, we have

$$\begin{aligned}&g_i(x) \sim U_i(x), \\ &g_i(x) = (1+o(1))\left[M_{i,1}^{-1}M_{i,j}^{-1}h(q_j + M_{i,j}^{-\frac{2}{n-2}}x) - \frac{1}{|x|^{n-2}}\right], \\ &\nabla_x g_i(x) = (1+o(1))\left[M_{1,i}^{-1}M_{i,j}^{-1}\nabla_y h(q_j + M_{i,j}^{-\frac{2}{n-2}}x) + \frac{(n-2)x}{|x|^n}\right],\end{aligned}$$

From these estimates, we obtain

(7.37)
$$\int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} \left(\frac{\partial g_i}{\partial r}\nabla U_i - g_i \frac{\partial \nabla U_i}{\partial r}\right) d\sigma$$
$$= -(1+o(1)+c(\delta_0))(n-2)|S^{n-1}|\nabla_y h_j(q_j) M_{i,1}^{-1} M_{i,j}^{-1-\frac{2}{n-2}},$$

and

$$\int_{|x|=\delta_0 M_{i,j}^{\frac{2}{n-2}}} \left(\frac{\partial g_i}{\partial r}\partial_\lambda U_{i,\lambda} - g_i \frac{\partial \partial_\lambda U_{i,\lambda}}{\partial r}\right) d\sigma$$
(7.39)
$$= -(1+o(1)+\widetilde{c}(\delta_0))\frac{(n-2)^2}{2}|S^{n-1}| h_j(q_j) M_{i,1}^{-1} M_{i,j}^{-1},$$

where $h_j = h - \frac{\mu_j}{|x-q_j|^{n-2}}, c(\delta_0), \widetilde{c}(\delta_0) \to 0$ as $\delta_0 \to 0$. Putting these estimates into (7.24) and (7.25), we obtain (7.3) and (7.4). q.e.d.

8. Isolated blowing up

Proof of Theorem 1.3. Suppose that there exists a blowup point q which is not isolated. Then by Theorem 2.1, Corollary 2.3, Theorem 2.4, Theorem 2.5 and (6.8), q is the only simple-like blowup point. Thus $l = 1$, $q = q_1$ and $\beta_1 < \dfrac{n-2}{2}$. By (ii) of Theorem 2.5,

$$u_i(x) \le c|x-q_1|^{-\frac{n-2}{2}} \tag{8.1}$$

for $x \in B_i = \{x \mid |x-q_1| \le \delta |q_1 - q_{i,1}|\}$, where c is independent of δ if $\delta \le \dfrac{1}{2}$, and

$$u_i(x) \le c_1 U_{\lambda_i}(x - q_{i,1}) \tag{8.2}$$

for $x \notin B_i$, where $\lambda_i = u_i(q_{i,1})^{-\dfrac{2}{n-2}}$ and $c_1 = c_1(\delta)$.

In particular, we have

$$m_i \sim M_{i,1}^{-1} = u_i(q_{i,1})^{-1}. \tag{8.3}$$

Now, let $\{q_j\}_{j=2}^m$ be the other blowup points, and $\Omega_i = \displaystyle\bigcup_{j=1}^m B(q_j, \delta_0)$. Then, (8.3) implies

$$u_i(x) \le c M_{i,1}^{-1}(1+|x|)^{2-n}$$

for $x \notin \Omega_i$. By the Pohozaev identity,

$$\int_{\mathbb{R}^n} \langle x - q_1, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x)dx = 0, \quad \text{and} \tag{8.4}$$

$$\int_{\mathbb{R}^n} \langle e_i, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x)dx = 0, \tag{8.5}$$

where $e_i = \dfrac{\bigtriangledown \hat{K}(q_{i,1})}{| \bigtriangledown \hat{K}(q_{i,1})|}$. By (8.1) and (8.2), we have

$$\begin{aligned} &\left| \int_{|x-q_1|\le\delta_0} \langle x - q_1, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x)dx \right| \\ &\quad \le c\ t_i \left\{ \int_{B_i} |x - q_1|^{\beta_1 - n} dx \right. \\ &\qquad\qquad \left. + \int_{B(q_1,\delta_0)\setminus B_i} |x - q_1|^{\beta_1} U_{\lambda_i}^{\frac{2n}{n-2}}(x - q_{i,1})dx \right\} \\ &\quad \le c\ t_i |q_{i,1} - q_1|^{\beta_1}, \end{aligned} \tag{8.6}$$

where $\lim\limits_{i\to+\infty} \left(u(q_{i,1})|q_{i,1} - q_1|^{\frac{n-2}{2}} \right) = +\infty$ is used. As in (4.29), we can obtain the lower bound

$$\int_{B(q_1,\delta_0)} \langle e_i, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x)dx \ge c_2\ t_i |q_{i,1} - q_1|^{\beta_1 - 1}, \tag{8.7}$$

provided that δ is small enought.

On the other hand, we have

$$\left| \int_{\mathbb{R}^n\setminus\Omega_i} \langle x - q_1, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x)dx \right| = O(1) t_i M_{i,1}^{-\frac{2n}{n-2}}, \tag{8.8}$$

$$\left| \int_{\mathbb{R}^n\setminus\Omega_i} \langle e_i, \bigtriangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x)dx \right| = O(1) t_i M_{i,1}^{-\frac{2n}{n-2}}, \tag{8.9}$$

and by (7.2) of Lemma 7.1,

(8.10)
$$
\begin{aligned}
&-\sum_{j=2}^{m}\int_{B(q_j,\delta_0)}\langle x-q_1,\triangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx\\
&\quad=\sum_{j=2}^{m}\Bigg[-\int_{B(q_j,\delta_0)}\langle q_j-q_1,\triangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx\\
&\qquad-\int_{B(q_j,\delta_0)}\langle x-q_j,\triangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx\Bigg]\\
&\quad=(1+\epsilon)(n-2)|S^{n-1}|\sum_{j=2}^{m}\Big(\langle q_j-q_1,\triangledown h(q_j)\rangle\\
&\qquad+\frac{n-2}{2}h(q_j)\Big)\cdot M_{i,1}^{-1}M_{i,j}^{-1}+O(1)t_iM_{i,1}^{-\frac{2n}{n-2}},\\
&\quad=-(1+\epsilon)\frac{(n-2)^2}{2}|S^{n-1}|\sum_{j=2}^{m}h(q_j)M_{i,1}^{-1}M_{i,j}^{-1}+O(1)t_iM_{i,1}^{-\frac{2n}{n-2}}
\end{aligned}
$$

with some small ϵ. By (7.3) and (8.10),

$$
\begin{aligned}
\sum_{j=2}^{m}&\left|\int_{B(q_j,\delta_0)}\langle e_i,\triangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx\right|\\
&\le c\sum_{j=2}^{m}M_{i,1}^{-1}M_{i,j}^{-1}+O(1)t_iM_{i,1}^{-\frac{2n}{n-2}}\\
&\le c_1\sum_{j=2}^{m}\int_{B(q_j,\delta_0)}\langle x-q_1,\triangledown K_i(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx\\
&\quad+O(1)t_iM_{i,1}^{-\frac{2n}{n-2}}.
\end{aligned}
\tag{8.11}
$$

Note that $h(x)=\dfrac{\mu_1}{|x-q_1|^{n-2}}$. Hence, we can use the following identity in (8.10)

$$
\langle q_j-q_1,\triangledown h(q_j)\rangle=-(n-2)h(q_j).
\tag{8.12}
$$

By (8.5) $\sim$ (8.11), we have

$$
\begin{aligned}
& c_3\ t_i|q_{i,1}-q_1|^{\beta_1-1} \\
&\qquad \le \int_{B(q_1,\delta_0)} \langle e_i, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx \\
&\qquad \le c\ \sum_{j=2}^{m} M_{i,1}^{-1}M_{i,j}^{-1} + O(1)t_iM_{i,1}^{-\frac{2n}{n-2}} \\
&\qquad \le c_1\left|\int_{\mathbb{R}^n\setminus B(q_1,\delta_0)} \langle x-q_1, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx\right| \\
&\qquad\quad + O(1)t_iM_{i,1}^{-\frac{2n}{n-2}} \\
&\qquad = c_1\left|\int_{B(q_1,\delta_0)} \langle x-q_1, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx\right| \\
&\qquad\quad + O(1)t_iM_{i,1}^{-\frac{2n}{n-2}} \\
&\qquad \le c_2\left\{t_i|q_{i,1}-q_1|^{\beta_1} + t_iM_{i,1}^{-\frac{2n}{n-2}}\right\}.
\end{aligned}
\tag{8.13}
$$

Therefore,

$$
|q_{i,1}-q_1|^{\beta_1-1} \le c\ M_{i,1}^{-\frac{2n}{n-2}}. \tag{8.14}
$$

Recall that $\beta_1 < \dfrac{n-2}{2}$. Then (8.14) yields a contradiction to the assumption that $\lim_{i\to+\infty}(|q_{i,1}-q_1|M_{i,1}^{\frac{2}{n-2}}) = +\infty$. We have proved that every blowup point must be isolated.

To prove the second part, let us assume that q_j is a blowup point with $\beta_j < n+1$ and $\lim_{i\to+\infty}\sup_{B(q_j,\delta_0)}(u_i(x)|x-q_j|^{\frac{n-2}{2}}) = +\infty$. Since (ii) of Theorem 2.5 is excluded, q_j must be a simple blowup point. Thus, u_i lose the energy of only one bubble at q_j and then, $q_{i,j}$ is the local maximum point defined by (6.1). By the assumptions, we have

$$
\lim_{i\to+\infty}\left(|q_{i,j}-q_j|M_{i,j}^{\frac{2}{n-2}}\right) = +\infty \quad \text{and} \tag{8.15}
$$

$$
u_i(x) \le c\ U_{\lambda_i}(x-q_{i,j}) \text{ for } |x-q_j| \le \delta_0, \tag{8.16}
$$

where $\lambda_i = M_{i,j}^{-\frac{2}{n-2}}$. Applying Theorem 2.2, (8.14) implies

$$
\lim_{i\to+\infty} L_{i,j}M_{i,j}^{-\frac{2}{n-2}} = +\infty.
$$

Hence q_j is the only simple-like blowup point. By repeating the same argument as above, we can reach the same conclusion as (8.14), that is,

$$|q_{i,j} - q_j|^{\beta_j - 1} \leq c\ M_{i,j}^{-\frac{2n}{n-2}}$$

for some constant $c > 0$. Since $\beta_j < n + 1$, the inequality yields a contradiction to (8.15). Hence (1.20) is proved. q.e.d.

Set $q_{i,j}$ to be the local maximum point of u_i defined by (1.21) and $\xi_i = M_{i,j}^{\frac{2}{n-2}}(q_{i,j} - q_j)$. Let ξ be any limit of ξ_i. Then we claim:

Lemma 8.1. ξ *satisfies*

$$\int_{\mathbb{R}^n} \triangledown Q_j(y + \xi) U_1^{\frac{2n}{n-2}}(y) dy = 0. \tag{8.17}$$

Proof. If $L_i(q_{i,j}) M_{i,j}^{-\frac{2}{n-2}}$ is bounded where $M_{i,j} = u_i(q_{i,j})$, then (8.17) is proved by Theorem 2.2. So, we may assume

$$\lim_{i \to +\infty} L_i(q_{i,j}) M_{i,j}^{-\frac{2}{n-2}} = +\infty.$$

Thus, q_j is the only simple-like blowup points. Hence Lemma 7.1 can be applied to all blowup point $q_k, k \neq j$. For the simplicity, we assume $j = 1$. By using (7.2) of Lemma 7.1, (8.4), (8.5), (8.10) and (8.11), we have the same conclusion as (8.13), i.e.,

$$\begin{aligned}
&\left| \int_{B(q_1,\delta_0)} \triangledown K_i(x) u_i^{\frac{2n}{n-2}}(x) dx \right| \\
&\quad \leq c_1 \left| \int_{B(q_1,\delta_0)} \langle x - q_1, \triangledown K_i \rangle u_i^{\frac{2n}{n-2}}(x) dx \right| \\
&\qquad + O(1) t_i M_{i,1}^{-\frac{2n}{n-2}} \\
&\quad \leq c_2\ t_i \left\{ \int_{B(q_1,\delta_0)} |x - q_1|^{\beta_1} u_i^{\frac{2n}{n-2}}(x) dx + M_{i,1}^{\frac{-2n}{n-2}} \right\} \\
&\quad \leq c_3\ t_i \begin{cases} M_{i,1}^{-\frac{2n}{n-2}} \log(M_{i,1}) & \text{if } \beta_1 = n \\ M_{i,1}^{-\frac{2\beta_1^*}{n-2}} & \text{if } \beta_1 \neq n, \end{cases}
\end{aligned} \tag{8.18}$$

where $\beta_1^* = \min(\beta_1, n)$. On the other hand, by the scaling and (1.20), we have

$$\begin{aligned}\int_{B(q_1,\delta_0)} & \bigtriangledown K_i(x) u_i^{\frac{2n}{n-2}}(x)dx \\ &= \left(\int_{\mathbb{R}^n} \bigtriangledown Q_1(y+\xi)U_1^{\frac{2n}{n-2}}(y)dy + o(1)\right) t_i M_{i,1}^{-\frac{2(\beta_1-1)}{n-2}}.\end{aligned} \tag{8.19}$$

Since $\beta_1 - 1 < n$, (8.17) follows from (8.18) and (8.19) readily. q.e.d.

9. Asymptotic behaviors of $M_{i,j}$

Proof of Theorem 1.4. We first prove (1.22). By (1.20) in Theorem 1.3, we only need to consider the case $\beta_j \geq n+1$. Suppose $\beta_j \geq n+1$ and (1.22) does not hold. Then we have $|q_{i,j} - q_j|^{\frac{n-2}{2}} M_{i,j} \to \infty$ as $i \to \infty$. By Theorem 2.2 and (6.7), $j = l = 1$. Let $e_i = \dfrac{\nabla K_i(q_{i,1})}{|\nabla K_i(q_{i,1})|}$. By (7.1) and (7.2),

$$\begin{aligned}\int_{|x-q_1|\leq\delta_0} & \langle x - q_1, \nabla K_i\rangle u_i^{\frac{2n}{n-2}} dx \\ &\geq c_1 \sum_{k=2}^{m} \Big\{ M_{i,1}^{-1} M_{i,k}^{-1} + o(t_i M_{i,k}^{\frac{-2\beta_1}{n-2}}) \Big\} \\ &\quad + O(t_i M_{i,1}^{\frac{-2n}{n-2}}),\end{aligned}$$

for some $c_1 > 0$ and

$$\begin{aligned}\int_{|x-q_1|\leq\delta_0} & \langle e_i, \nabla K_i\rangle u_i^{\frac{2n}{n-2}} dx \\ &= O\Big(\sum_{j=2}^{s} \Big\{ M_{i,1}^{-1} M_{i,j}^{-1} + o(t_i M_{i,k}^{\frac{-2\beta_1}{n-2}}) \Big\}\Big) + O(t_i M_{i,1}^{\frac{-2n}{n-2}})\end{aligned}$$

as $i \to \infty$. On the other hand,

$$\begin{aligned}\int_{|x-q_1|\leq\delta_0} & \langle x - q_1, \nabla K_i\rangle u_i^{\frac{2n}{n-2}} dx \\ &\leq c\, t_i |q_{i,1} - q_1|^{\beta_1} + c\, t_i M_{1,i}^{-\frac{2n}{n-2}}\end{aligned}$$

and

$$
\begin{aligned}
&\int_{|x-q_1|\le\delta_0} \langle e_i, \nabla K_i\rangle u_i^{\frac{2n}{n-2}}\,dx \\
&\qquad \ge ct_i|q_{i,1}-q_1|^{\beta_1-1} - c_1 \begin{cases} t_i M_{i,1}^{-\frac{2n}{n-2}}(-\log|q_{i,1}-q_1|) & \beta_1 = n+1 \\ t_i M_{i,1}^{-\frac{2n}{n-2}} & \beta_1 > n+1 \end{cases} \\
&\qquad \ge ct_i|q_{i,1}-q_1|^{\beta_1-1} - c_1 \begin{cases} t_i M_{i,1}^{-\frac{2n}{n-2}}(\log M_{i,1}) & \beta_1 = n+1 \\ t_i M_{i,1}^{-\frac{2n}{n-2}} & \beta_1 > n+1 \end{cases}
\end{aligned}
$$

for some $c > 0$ and $c_1 > 0$. Putting the estimates above together, we obtain

$$
|q_{i,1}-q_1|^{\beta_1-1} \le c|q_{i,1}-q_1|^{\beta_1} + c \begin{cases} t_i M_{i,1}^{-\frac{2(\beta_1-1)}{n-2}}(\log M_{1,i}) & \beta_1 = n+1 \\ t_i M_{1,i}^{-\frac{2n}{n-2}} & \beta_1 > n+1. \end{cases}
$$

Since $|q_{i,1}-q_1| \to 0$ as $i \to \infty$, we conclude

$$
\begin{aligned}
|q_{i,1}-q_1| &= O(M_{i,1}^{-\frac{2}{n-2}}(\log M_{i,1})^{\frac{1}{n}}) \qquad \text{for } \beta_1 = n+1 \\
|q_{i,1}-q_1| &= O(M_{i,1}^{-\frac{2}{n-2}\frac{n}{\beta_1-1}}) \qquad \text{for } \beta_1 > n+1.
\end{aligned}
$$

From these, we conclude that (1.22) holds.

Now we prove $m \ge 2$. Suppose $m = 1$. Then $q = q_1$ is the only blowup point and it must be simple. If $\beta_1 < n$, then by Theorem 1.3,

$$
|q_{i,1}-q_1| \le c\, M_{i,1}^{-\frac{2}{n-2}}. \tag{9.1}
$$

By the Pohozaev identity,

$$
\begin{aligned}
&\left|\int_{|x-q_1|\le\delta_0} \langle x-q_1, \triangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)\,dx\right| \\
&\qquad = \left|\int_{|x-q_1|>\delta_0} \langle x-q_1, \triangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)\,dx\right| \\
&\qquad \le c_1 t_i M_{i,1}^{-\frac{2n}{n-2}}.
\end{aligned} \tag{9.2}
$$

By scaling and (9.1), it is not difficult to see that the left hand side of

(9.2) is

$$\begin{aligned}&\left|\int_{|x-q_1|\le\delta_0}\langle x-q_1,\bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx\right|\\&\qquad = t_i M_{i,1}^{-\frac{2\beta_1}{n-2}}\left|\int_{\mathbb{R}^n} Q(y+\xi)U_1^{\frac{2n}{n-2}}(y)dy\right|\\&\qquad \ge c_1\ t_i M_{i,1}^{-\frac{2\beta_1}{n-2}}.\end{aligned}$$

for some $c_2 > 0$, where $\xi = \lim_{i\to+\infty} M_{i,1}^{\frac{2}{n-2}}(q_{i,1}-q_1)$. Thus, it yields a contradiction to $\beta_1 < n$.

If $\beta_1 = n$, the left hand side of (9.2) is greater than $c_1 M_{i,1}^{-\frac{2\beta_1}{n-2}}\log M_{i,1}$ for some $c_1 > 0$, which also yields a contradiction. Now we assume $\beta_1 > n$. The Pohozaev identity gives

$$\int_{\mathbb{R}^n}\langle x-q_1,\bigtriangledown\hat{K}(x)\rangle u_i^{\frac{2n}{n-2}}(x)dx = 0. \tag{9.3}$$

Since $M_{i,1}u_i(x)\to\dfrac{\mu_1}{|x-q_1|^{n-2}}$ for some $\mu_1>0$ and

$$M_{i,1}u_i(x)\le\frac{c}{|x-q_{i,1}|^{n-2}}$$

for some constant $c > 0$, by multipling both sides of (9.3) by $M_{i,1}^{\frac{2n}{n-2}}$ and using (1.22), we obtain

$$\int_{\mathbb{R}^n}\langle x-q_1,\bigtriangledown\hat{K}(x)\rangle|x-q_1|^{-2n}dx = 0,$$

a contradiction to our assumptions. Hence $m\ge 2$ is proved. Let $\{q_1,\dots,q_l,q_{l+1},\dots,q_m\}$ be indexed by the ordering $\beta_1=\ldots=\beta_l>\beta_{l+1}=\ldots=\beta_{l_1}>\beta_{l_1+1}\ge\ldots\ge\beta_m$ as in Lemma 6.2. To find the asymptotic behavior of $M_{i,j}$, we consider the case $l = 1$ first. Let $h_i(x) = M_{i,1}u_i(x)$. Then $h_i(x)$ converges to $\mu_1|x-q_1|^{2-n}$ for some $\mu_1 > 0$ by Lemma 6.1. To compute μ_1, we use

$$\begin{aligned}\mu_1(n-2)|S^{n-1}| &= \lim_{i\to+\infty}\left(-\int_{|x-q_1|=\delta_0}\frac{\partial h_i}{\partial\nu}d\sigma\right)\\&= \lim_{i\to+\infty} M_{i,1}\int_{|x-q_1|\le\delta_0} K_i(x)u_i^{\frac{n+2}{n-2}}(x)dx \qquad (9.4)\\&= n(n-2)\int_{\mathbb{R}^n} U_1^{\frac{n+2}{n-2}}(y)dy = (n-2)|S^{n-1}|.\end{aligned}$$

From (9.4), $\mu_1 = 1$, that is,

$$h(x) = \frac{1}{|x-q_1|^{n-2}}. \tag{9.5}$$

In (7.2), since after passing to a subsequence, the left hand side is of order $O(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}})$, we may drop the term $o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}})$ in the right hand side. Let $\Omega = \cup_{j=1}^m B(q_j, \delta_0)$. When $\beta_1 < n$, together with (7.1) and (7.2), the Pohozave identity implies

$$\begin{aligned}
&\frac{n-2}{2n}\int_{B(q_1,\delta_0)} \langle x-q_1, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx \\
&\quad = -\sum_{j=2}^m \frac{n-2}{2n}\left(\int_{B(q_j,\delta_0)} \langle x-q_1, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}dx \right.\\
&\qquad\qquad \left. + \int_{\mathbb{R}^n\setminus\Omega} \langle x-q_1, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}dx\right) \\
&\quad = -(1+o(1)+c_1(d_0))\frac{(n-2)^2}{2}|S^{n-1}| \\
&\qquad \sum_{j=2}^m \left(|q_1-q_j|^{2-n} M_{i,1}^{-1} M_{i,j}^{-1} + o(t_i M_{i,j}^{-\frac{2\beta_j}{n-2}})\right) \\
&\qquad + O(t_i M_{i,1}^{-\frac{2n}{n-2}}),
\end{aligned} \tag{9.6}$$

where (9.5) is used. On the other hand, the left hand side of (9.6) is equal to

$$\beta_1\Big(\frac{n-2}{2n}\Big) t_i M_{i,1}^{-\frac{2\beta_1}{n-2}} \left(\int_{\mathbb{R}^n} Q_1(y+\xi) U_1^{\frac{2n}{n-2}}(y)dy\right)(1+o(1)).$$

Thus, we have

$$t_i M_{i,1}^{-\frac{2\beta_1}{n-2}} = \sum_{j=2}^{l_1} \eta_{1,j} M_{i,1}^{-1} M_{i,j}^{-1}(1+o(1)), \tag{9.7}$$

where

$$\eta_{1,j} = \frac{n(n-2)|S^{n-1}||q_1-q_j|^{-n+2}}{\beta_1|\int Q(y+\xi)U_1^{\frac{2n}{n-2}}(y)dy|}. \tag{9.8}$$

When $\beta_1 = n$, we have

$$t_i M_{i,1}^{-\frac{2n}{n-2}} \log M_{i,1} = \sum_{j=2}^{l_1} \eta_{1,j} M_{i,1}^{-1} M_{i,j}^{-1},$$

where

$$\eta_{1,j} = \frac{(n-2)|S^{n-1}||q_1 - q_j|^{-n+2}}{\left|\int_{S^{n-1}} Q(y) d\sigma\right|} \tag{9.9}$$

by noting that the left hand side of (9.6) will give

$$n\Big(\frac{n-2}{2n}\Big) t_i M_{i,1}^{-\frac{2n}{n-2}} \left(\int_{S^{n-1}} Q(y) d\sigma\right) \log M_{i,1}(1 + o(1)).$$

When $\beta_1 > n$, we have

$$\begin{aligned} &\lim_{i\to+\infty} \left(M_{i,1}^{\frac{2n}{n-2}} \int_{\mathbb{R}^n \setminus \bigcup_{j=2}^m B(q_j,\delta)} \langle x - q_1, \bigtriangledown \hat{K}\rangle u_i^{\frac{2n}{n-2}} dx \right) \\ &\qquad = \int_{\mathbb{R}^n \setminus \bigcup_{j=2}^m B(q_j,\delta)} \langle x - q_1, \bigtriangledown \hat{K}\rangle |x - q_1|^{-2n} dx \end{aligned}$$

for any $\delta > 0$. By letting $\delta \to 0$, we have

$$t_i M_{i,1}^{-\frac{2n}{n-2}} = (1 + o(1)) \sum_{j=1}^{l_1} \eta_{1,j} M_{i,1}^{-1} M_{i,j}^{-1},$$

where

$$\eta_{1,j} = \frac{n(n-2)|S^{n-1}||q_1 - q_j|^{-n+2}}{\left|\int_{\mathbb{R}^n} \langle x - q_1, \bigtriangledown \hat{K}\rangle |x - q_1|^{-2n} dx\right|}. \tag{9.10}$$

Thus, (1.24) is proved.

To prove (1.25), we note $\beta_j < n-2$ for $j \geq 2$. By (1.24), $tM_{1,i}^{-\frac{2n}{n-2}} = O(M_{i,1}^{-1} M_{i,j}^{-1})$. Hence if we let d tend to 0 suitably, (7.2) implies

$$\begin{aligned} &\beta_j \frac{n-2}{2n} t_i M_{i,j}^{-\frac{2\beta_j}{n-2}} \left| \int_{\mathbb{R}^n} Q_j(y + \xi) U_1^{\frac{2n}{n-2}}(y) dy \right| \\ &\qquad = (1 + o(1)) \frac{n-2}{2n} \int_{B(q_j,\delta)} \langle x - q_1, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}} \, dx \\ &\qquad = (1 + o(1)) \frac{(n-2)^2}{2} |S^{n-1}||q_1 - q_j|^{-n+2} M_{i,1}^{-1} M_{i,j}^{-1}, \end{aligned}$$

which is (1.25).

To prove (1.28), it is enough to prove (1.28) for $j=1$. As the proof of (9.5), we have

$$h(x)=\sum_{k=1}^{l}\frac{\mu_k}{|x-q_k|^{n-2}} \tag{9.11}$$

and

$$\mu_1=1. \tag{9.12}$$

Since $M_{i,k}u_i(x)$ also converges to $\widetilde{h}(x)$ where

$$\widetilde{h}(x)=\frac{1}{|x-q_k|^{n-2}}+\sum_{j\neq k}^{l}\frac{\widetilde{\mu}_j}{|x-q_j|^{n-2}},$$

we have

$$\mu_k=\lim_{i\to+\infty}\frac{M_{i,1}}{M_{i,k}}. \tag{9.13}$$

Since $l\geq 2$, we recall that Theorem 2.2 and Lemma 6.2 imply $\beta_j<n-2$ for all j. By (7.4), we have for $j=1$

$$\begin{aligned}
&\beta_1\frac{n-2}{2n}t_iM_{i,1}^{-\frac{2\beta_1}{n-2}}\left(\int_{\mathbb{R}^n}Q_1(y+\xi)U_1^{\frac{2n}{n-2}}(y)dy\right)(1+o(1))\\
&\quad=-\frac{(n-2)^2}{2}|S^{n-1}|h_1(q_1)M_{i,1}^{-1}M_{i,1}^{-1}\\
&\quad=-\frac{(n-2)^2}{2}|S^{n-1}|\left(\sum_{j=2}^{l}\frac{\mu_j}{|q_j-q_1|^{n-2}}M_{i,1}^{-1}\right)M_{i,1}^{-1}\\
&\quad=-\frac{n-2}{2}|S^{n-1}|\sum_{j=2}^{l}\frac{1}{|q_j-q_1|^{n-2}}M_{i,1}^{-1}M_{i,j}^{-1},
\end{aligned}$$

where the last equality comes from (9.13). Clearly, (1.28) follows immediately. Identity (1.29) also follows from (7.2) and (9.13) immediately. Thus, the proof of Theorem 1.4 is complete. q.e.d.

10. Apriori estimates

In this final section, we are going to prove the apriori bound of Theorem 1.1. Here, we consider a sequence of blowing up solutions of

Equation (1.3) with $K = K_i$ more general than the one in previous sections. We assume that K_i converges to a function, say K, in C^1, and for simplicity, assume K_i has the same set of critical points $\{q_1, q_2, \ldots, q_N\}$. Let $Q_{i,j}(y)$ be the homogeneous function in (K0) for K_i at q_j. Assume that K satisfies (K0) $\sim$ (K1) and $Q_{i,j}(y) \to Q_j(y)$ in C^1. Let $\beta_{i,j}$ be the degree of $Q_{i,j}$ and

$$\beta_j = \lim_{i\to+\infty} \beta_{i,j} > \frac{n-2}{2} \tag{10.1}$$

for all j such that $q_j \in \Gamma^-$, where Γ^- is defined in Section 1.

By results of [8], [9], it is known that any blowup point is isolated. Without loss of generality, the point $+\infty$ is assumed not to be a blowup point. Let $\{q_1, \ldots, q_m\}$ be the set of blowup points such that $q_1, \ldots, q_l$ are all simple blowup points and $q_{l+1}, \ldots, q_m$ are non-simple blowup points. Following the same proof of Lemma 6.1 and part (i) of Theorem 1.4, we have $l \geq 1$ and $m \geq 2$. Another important result in [8], [9] is that q_j is *simple if and only if* $\beta_j \geq n-2$. This result follows from Theorem 1.3 of [8], [9] when $\beta_j \neq n-2$. For the case $\beta_j = n-2$, it follows from the following lemma similar to Lemma 7.1.

Lemma 10.1. *For $2 \leq j \leq m$ if $l = 1$ and $1 \leq j \leq m$ if $l \geq 2$, we have*

$$\begin{aligned}
&\frac{n-2}{2n}\int_{|x-q_j|\leq\delta} \bigtriangledown K_i(x) u_i^{\frac{2n}{n-2}}(x)dx \\
&\quad = -(1+o(1)+c_1(\delta)) \\
&\qquad \cdot (n-2)|S^{n-1}| \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{2}} \\
&\qquad \bigtriangledown \widetilde{h}_j(q_j)\hat{M}_{i,1}^{-1}\hat{M}_{i,j}^{-1-\frac{2}{n-2}} \\
&\qquad + o(\hat{M}_{i,j}^{-\frac{2\beta_{i,j}}{n-2}}) + O(\delta^{n-1}\hat{M}_{i,1}^{-\frac{2n}{n-2}}),
\end{aligned} \tag{10.2}$$

$$\begin{aligned}
&\frac{n-2}{2n}\int_{|x-q_j|\leq\delta} \langle x-q_j, \bigtriangledown K_i\rangle u_i^{\frac{2n}{n-2}}(x)dx \\
&\quad = -(1+o(1)+c_2(\delta)) \\
&\qquad \cdot \frac{(n-2)^2}{2}|S^{n-1}| \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{2}} \widetilde{h}_j(q_j)\hat{M}_{i,1}^{-1}\hat{M}_{i,j}^{-1} \\
&\quad + o(\hat{M}_{i,j}^{-\frac{2\beta_{i,j}}{n-2}}) + O(\delta^{n-1}\hat{M}_{i,1}^{-\frac{2n}{n-2}}),
\end{aligned} \tag{10.3}$$

where

$$h(x) = \lim_{i\to+\infty} \hat{M}_{i,1} u_i(x) = \sum_{j=1}^{l} \frac{\mu_j}{|x-q_j|^{n-2}},$$

$\widetilde{h}_j(x) = h(x)$ *if* $j \geq l+1$, *and*

$$\widetilde{h}_j(x) = h(x) - \frac{\mu_j}{|x-q_j|^{n-2}}$$

if $1 \leq j \leq l$ *and* $l \geq 2$.

Here, $\hat{M}_{i,j}$ and $q_{i,j}$ are the local maximum and a local maximum point of u_i near q_j satisfying

$$\hat{M}_{i,j} = u_i(q_{i,j}) = \max_{|x-q_j|\leq\delta_0} u_i(x) \tag{10.4}$$

We can prove Lemma 10.1 by the same argument as in Lemma 7.1, but the proof is simpler because $\beta_j > \dfrac{n-2}{2}$ for all j. The position $q_{i,j}$ also satisfies (1.22) for some constant $c > 0$. When $\beta_j \leq n-2$, it was proved in [9]. When $\beta_j > n-2$, it is a consequence of Lemma 10.1, as shown in the previous sections.

Another important consequence of Lemma 10.1 is the asymptotic behavior of $\hat{M}_{i,j}$ which is similar to Theorem 1.4.

Theorem 10.2. *Assume that K satisfies* (K0) *and* (K1) *and $\beta_j > \frac{n-2}{2}$ for all $q_j \in \Gamma^-$. Let $q_1, \dots, q_l$ are simple blowup points and $q_{l+1}, \dots, q_m$ are not simple blowup points. Set*

$$M_{i,j} = \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{4}} \hat{M}_{i,j}$$

where $\hat{M}_{i,j}$ is the local maximum in (10.4)*. Then $m \geq 2$, $l \geq 1$, $\beta_1 = \ldots = \beta_l > \beta_j$ for $j \geq l+1$, and the following hold:*

(i) *If $l = 1$ and q_j is indexed by the ordering $\beta_1 > \beta_2 = \ldots = \beta_{l_1} >$*

$\beta_{l_1+1} \geq \ldots \geq \beta_m$, *then*

$$\left.\begin{array}{ll} |b_1| \left(\frac{n(n-2)}{K(q_1)}\right)^{\frac{n}{2}} M_{i,1}^{-\frac{2\beta_1^*}{n-2}} & \text{if } \beta_1 \neq n \\ \left(\frac{n(n-2)}{K(q_1)}\right)^{\frac{n}{2}} M_{i,1}^{-\frac{2\beta_1}{n-2}} \log M_{i,1} & \text{if } \beta_1 = n \end{array}\right\}$$

$$= (1+o(1))n(n-2)|S^{n-1}| \sum_{j=2}^{l_1} \left(\frac{n(n-2)}{K(q_1)}\right)^{\frac{n-2}{4}} \cdot \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{4}} |q_1 - q_j|^{-n+2} M_{i,1}^{-1} M_{i,j}^{-1} \tag{10.6}$$

and

$$|b_j| \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n}{2}} M_{i,j}^{-\frac{2\beta_j}{n-2}} = (1+o(1))n(n-2)|S^{n-1}| \left(\frac{n(n-2)}{K(q_1)}\right)^{\frac{n-2}{4}} \cdot \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{4}} M_{i,1}^{-1} M_{i,j}^{-1} \tag{10.7}$$

for $2 \leq j \leq m$.

(ii) *If* $l \geq 2$, *then* $\beta_1 = \ldots = \beta_l = n-2$,

$$|b_j| \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n}{2}} M_{i,j}^{-2} = (1+o(1))n(n-2)|S^{n-1}| \sum_{k=1, k\neq j}^{l} \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{4}} \cdot \left(\frac{n(n-2)}{K(q_k)}\right)^{\frac{n-2}{4}} |q_j - q_k|^{-n+2} M_{i,j}^{-1} M_{i,k}^{-1} \tag{10.8}$$

for $1 \leq j \leq l$ *and,*

$$|b_j| \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n}{2}} M_{i,j}^{-\frac{2\beta_j}{n-2}} = n(n-2)|S^{n-1}|(1+o(1)) \sum_{k=1}^{l} \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{4}} \cdot \frac{n(n-2)}{K(q_k)} |q_j - q_k|^{-n+2} M_{i,j}^{-1} M_{i,k}^{-1} \tag{10.9}$$

for $l+1 \le j \le m$ where b_j is given in (1.27).

Now we are in the position to prove the apriori bound of Theorem 1.1. In fact, we are going to prove the result for more general situations. Let $A = \{q_{k_1}, \dots, q_{k_m}\}$ be a subset of Γ^- where $\beta_{k_1} \ge \beta_{k_2} \ge \dots \ge \beta_{k_m}$. A is called admissible if $m \ge 2$ and one of the following conditions holds:

(i) $n \ne \beta_{k_1} > \beta_{k_2}$ and

$$\frac{1}{\beta^*_{k_1}} + \frac{1}{\beta^*_{k_2}} = \frac{2}{n-2}, \tag{10.10}$$

where $\beta^*_j = \min(\beta_j, n)$.

(ii) There exists an integer $l \ge 2$ such that

$$n-2 = \beta_{k_1} = \beta_{k_2} = \dots = \beta_{k_l} > \beta_{k_{l+1}} \ge \dots \ge \beta_m. \tag{10.11}$$

For an admissible set A of case (i), for simplicity, assume it is $\{q_1, \dots, q_m\}$ with $\beta_1 > \beta_2 = \dots = \beta_{l_1} > \beta_{l_1+1} \ge \dots \ge \beta_m,$, we define $\eta = \eta(A)$ by

$$\begin{aligned} \eta(A) = (n(n-2)|S^{n-1}|)^{\frac{2\beta_1^*}{n-2}} \left(\frac{n(n-2)}{K(q_1)}\right)^{\frac{\beta_1^*-n}{2}} \\ \sum_{j=2}^{l_1} \left(\frac{n(n-2)}{K(q_j)}\right)^{-(1+\frac{(n+2)\beta_1^*}{2(n-2)})} |b_j|^{1-\frac{2\beta_1^*}{n-2}} |q_1 - q_j|^{2-n}. \end{aligned} \tag{10.12}$$

For $A = \{q_1, \dots, q_l, \dots, q_m\}$ of case (ii), we associate with a $l \times l$ matrix $\eta_{ij}(A)$:

$$\eta_{jk}(A) = \begin{cases} |b_j| \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n}{2}} & \text{if } j = k \\ -n(n-2)|S^{n-2}| \left(\frac{n(n-2)}{K(q_j)}\right)^{\frac{n-2}{4}} & \\ \cdot \left(\frac{n(n-2)}{K(q_k)}\right)^{\frac{n-2}{4}} |q_j - q_k|^{-n+2} & \text{if } j \ne k. \end{cases} \tag{10.13}$$

Now we can state our main theorem.

Theorem 10.3. *Assume that K satisfies* (K0) $\sim$ (K1) *with $\beta_j > \frac{n-2}{2}$ for any $q_j \in \Gamma^-$. For any admisible set A, assume $\eta(A) \ne 1$ for* *case* (i) *and the first eigenvalue of* $\eta(A)$ *is not zero for case* (ii). *Then there is a constant* $c > 0$ *such that for any solution* w *of Equation* (1.1),

$$c^{-1} \le w(p) \le c$$

holds for any $p \in S^n$.

Proof. Suppose $u_i(x)$ blows up at some point. Let

$$A = \{q_1, \dots, q_l, \dots, q_m\}$$

be the blowup set of u_i. Two cases are discussed separately.

Case 1. If $l = 1$, by (10.7), we can solve $M_{i,j}^{-1}$ in term of $M_{i,1}^{-1}$ for $2 \le j \le l_1$, and substitute it into (10.6). If $\beta_1 = n$, then the additional term $\log M_{i,1}$ makes two sides of (10.6) unbalanced. Thus, $\beta_1 \neq n$. Also, it is easy to see that the exponent of $M_{i,1}^{-1}$ of the right hand side of (10.6) is equal to $1 + \left(\frac{2\beta_2}{n-2} - 1\right)^{-1}$. Hence, we have

$$\frac{2\beta_1^*}{n-2} = 1 + \frac{1}{\frac{2\beta_2}{n-2} - 1}, \tag{10.14}$$

which implies

$$\frac{1}{\beta_1^*} + \frac{1}{\beta_2} = \frac{2}{n-2}. \tag{10.15}$$

Then A is admissible. Applying equality (10.14) and comparing the coefficients of both sides of (10.6) with each other, we have

$$\eta = 1 + o(1),$$

where η is given by (10.12) with $A = \{q_1, q_2, \dots, q_m\}$.

Case 2. $l \ge 2$. Since $\lim_{i \to +\infty} \frac{M_{i,1}}{M_{i,j}} = \lambda_j > 0$ for $1 \le j \le l$, by (10.8), we have

$$\sum_{k=1}^{l} \eta_{jk} \lambda_k = 0,$$

where η_{jk} is given by (10.13) and $A = \{q_1, \dots, q_l, \dots, q_m\}$. Therefore, the first eigenvalue of (η_{jk}) is equal to 0.

Since both cases yield a contradiction to the assumptions, the apriori bound is established. q.e.d.

We note that the assumptions of Theorem 1.1 imply there exist no admissible subsets of Γ^-. Hence, Theorem 1.1 is special case of Theorem 10.3. The asymptotic formulas (10.6) $\sim$ (10.9) will be very helpful when we come to compute the degree for the nonlinear Equation (1.1).

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