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ON DEFORMATION BOUNDARY RIGIDITY AND SPECTRAL RIGIDITY OF RIEMANNIAN SURFACES WITH NO FOCAL POINTS

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Abstract

We prove that a compact two-dimensional Riemannian manifold with strictly convex boundary and with no focal points is deformation boundary rigid. We also show, using similar methods, that two-dimensional Anosov manifolds with no focal points are spectrally rigid.

1. Introduction and statement of the results

Let (M, g) be a compact Riemannian manifold with boundary. The *boundary rigidity problem* consists of determining the Riemannian metric g on M, up to isometries which are the identity on the boundary, by measuring the geodesic distance, d_g, between boundary points. It is easy to see that the answer to this question is no in general. We can find metrics g such that there exists $x_0 \in M$ satisfying $\operatorname{dist}(x_0, \partial M) > \sup_{x,y\in\partial M} d_g(x, y)$. Then we can change the metric g in a neighborhood of x_0 without changing the boundary distance function. A compact Riemannian manifold (M, g) with boundary is called *boundary rigid* if the metric g is uniquely determined by d_g, up to isometries which are the identity on the boundary. There are few classes of metrics which are known to be boundary rigid (see [3], [8], [13], [14]).

In this paper we consider the linearized problem in the two dimensional case. We first state the problem in arbitrary dimension. Let

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g^τ be a one-parameter family of Riemannian metrics on M such that $d_{g^\tau} = d_{g^0},\ -\varepsilon < \tau < \varepsilon$. For a geodesic $\gamma : [a,b] \to M$ of the metric g^0, with $\gamma(a), \gamma(b) \in \partial M$, a straightforward calculation shows that

$$If(\gamma) := \int_a^b f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt = 0, \tag{1.1}$$

where $f = (f_{ij}) = \frac{d}{d\tau} g^\tau|_{\tau=0}$. We remark that the integrand in (1.1) is written in local coordinates. However, it is easy to see that the integrand is independent of the choice of coordinates. The question we address in this paper is the following: Given (M, g), to what extent is a symmetric tensor field f determined by the integrals (1.1) that are known for all geodesics $\gamma : [a,b] \to M$ of the metric g with endpoints on the boundary, $\gamma(a), \gamma(b) \in \partial M$? The operator I defined by (1.1) is called the *ray transform*. We denote by $C^\infty(S^2\tau'_M)$ the space of symmetric tensor two-fields on M. Let $Z(S^2\tau'_M)$ denote the subspace of $C^\infty(S^2\tau'_M)$ consisting of all symmetric tensor two-fields f such that $If(\gamma) = 0$ for every geodesic γ with endpoints on the boundary. The space $Z(S^2\tau'_M)$ is not zero as is seen from the following argument. Let $C^\infty(\tau'_M)$ be the space of covector fields, and

$$\sigma\nabla : C^\infty(\tau'_M) \to C^\infty(S^2\tau'_M) \tag{1.2}$$

be the differential operator defined using coordinates by the equality $(\sigma\nabla v)_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ where ∇ denotes the covariant derivative. In other words, $\sigma\nabla v$ is the symmetric part of the covariant derivative of the field v. A tensor field $f \in C^\infty(S^2\tau'_M)$ is called a *potential field* if it can be represented in the form $f = \sigma\nabla v$ for some $v \in C^\infty(\tau'_M)$ vanishing on the boundary, $v|_{\partial M} = 0$. Let $P(S^2\tau'_M)$ be the space of all potential fields. It is easy to see that

$$P(S^2\tau'_M) \subset Z(S^2\tau'_M). \tag{1.3}$$

Indeed, if $f = \sigma\nabla v$, then the integrand in (1.1) is the total derivative with respect to t, $f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = d(v_i(\gamma(t))\dot\gamma^i(t))/dt$.

The question we address in this paper is the following: For what classes of compact Riemannian manifolds (M, g) with boundary is the inclusion (1.3) in fact equality? In the latter case (M, g) is called a *deformation boundary rigid manifold*. As we have indicated the deformation boundary rigidity problem is a linearization of the boundary rigidity problem. The right-hand side of (1.3) can be considered as the tangent space, at the point g, to the manifold of metrics on M with the same boundary distance function as g; while the left-hand side of (1.3) is the tangent space, at the point g, to the manifold of metrics that are isometric to g via an isometry which is the identity on the boundary. See Chapter 1 of [17] for a detailed discussion of the relationship between these two problems.

We recall some known results regarding the deformation boundary rigidity problem. Equality in (1.3) was first proved [15] for a compact Riemannian manifold of nonpositive sectional curvature with strictly convex boundary. Then this result was generalized [16] by replacing the condition of nonpositivity of the curvature with some weaker curvature condition. In particular, this condition is satisfied for every sufficiently small convex piece of an arbitrary Riemannian manifold. For a simple Riemannian manifold, it is known [18] that the inclusion (1.3) has a finite codimension. A Riemannian manifold (M, g) is called *simple* if it is simply connected, its boundary is strictly convex, and it has no conjugate points. Such a manifold is diffeomorphic to the ball B^n, and every two points of M are joined by a unique geodesic.

The boundary ∂M of a Riemannian manifold is *strictly convex* if the second fundamental form of the boundary is positively definite at every point $x \in \partial M$. A Riemannian manifold (M, g) *has no focal points* if, for every geodesic $\gamma : [a, b] \to M$ and every nonzero Jacobi field $Y(t)$ along γ satisfying the initial condition $Y(a) = 0$, the modulus $|Y(t)|$ is a strictly increasing function on $[a, b]$, i.e., $d|Y(t)|^2/dt > 0$ for $t \in (a, b]$.

In this paper we solve the deformation boundary rigidity problem for Riemannian surfaces with no focal points and with strictly convex boundary. More precisely we have:

Theorem 1.1. *A compact simply connected two-dimensional Riemannian manifold with strictly convex boundary and with no focal points is deformation boundary rigid.*

If a Riemannian manifold has no focal points, then it has no conjugate points. This implies that the manifolds satisfying the hypotheses of Theorem 1.1 are simple and, in particular, are diffeomorphic to the disk B^2.

Now we consider the corresponding question for closed Riemannian manifolds, i.e., compact Riemannian manifolds without boundary. For a symmetric tensor field $f \in C^\infty(S^2\tau'_M)$ on a closed Riemannian manifold

(M, g), we can consider If over a closed geodesic γ. Let $Z(S^2\tau'_M)$ be the space of all fields f such that $If(\gamma) = 0$ for every closed geodesic γ. By $P(S^2\tau'_M)$ we denote the range of the operator (1.2). The elements of $P(S^2\tau'_M)$ are again called potential fields. The inclusion (1.3) holds for every closed Riemannian manifold. The natural question is: for what classes of closed Riemannian manifolds is (1.3) in fact equality?

Of course, the question makes sense only for closed manifolds that have sufficiently many closed geodesics. In the present article we consider the question for Anosov manifolds. A closed Riemannian manifold is said to be an *Anosov manifold* if its geodesic flow is of Anosov type. For such a manifold, the set of closed geodesics is dense in the set of all geodesics. The class of Anosov manifolds contains the class of closed negatively curved manifolds.

For an Anosov manifold of nonpositive sectional curvature, the inclusion (1.3) is equality. This fact is proved in [4] for negatively curved manifolds, but it is only the nonpositivity of the curvature and Anosov type of the geodesic flow that are used in the proof. Without constrains on the curvature, it is proved [5] that the inclusion (1.3) has a finite codimension for an Anosov manifold.

The second main result of the present article is the following:

Theorem 1.2. *The inclusion* (1.3) *is equality for a two-dimensional Anosov manifold with no focal points.*

It is known, [2], [12], that an Anosov manifold has no conjugate points but may have focal points [10].

The second question under consideration is close to the spectral rigidity problem. Let us recall the corresponding definitions. A smooth one-parameter family g^τ $(-\varepsilon < \tau < \varepsilon)$ of metrics on a closed manifold M is said to be a *deformation* of a metric g if $g^0 = g$. Such a family is called an *isospectral deformation* if the spectrum of the Laplace — Beltrami operator Δ^τ of the metric g^τ is independent of τ. A deformation g^τ is called the *trivial deformation* if there exists a family φ^τ of diffeomorphisms of M such that $g^\tau = (\varphi^\tau)^* g$. A manifold (M, g) is called *spectrally rigid* if it does not admit a nontrivial isospectral deformation.

The following result is proved in [9]: an Anosov manifold is spectrally rigid if the inclusion (1.3) is equality. The statement is formulated in [9] for negatively curved manifolds. However, the same proof applies to Anosov manifolds. In view of this result, the left-hand side of (1.3) can be considered as the space of trivial infinitesimal deformations, while the right-hand side is the space of infinitesimal isospectral deformations. In particular, under the hypotheses of Theorem 1.2, (M, g) is a spectrally rigid surface. More precisely we have

Corollary 1.3. *Two-dimensional Anosov manifolds with no focal points are spectrally rigid.*

Theorems 1.1 and 1.2 are shown by the same method which we describe briefly below. First of all, by the same arguments as in [9], [15], we reduce the question to an inverse problem for the kinetic equation

$$Hu(x, \xi) = f_{ij}(x)\xi^i\xi^j \tag{1.4}$$

on the manifold $\Omega M = \{(x, \xi) \mid x \in M, \xi \in T_xM, |\xi| = 1\}$ of unit tangent vectors, where H is the differentiation with respect to the geodesic flow. The claims of the theorems are equivalent to the statement that every solution to the kinetic equation (satisfying the homogeneous boundary condition in the case of Theorem 1.1) is linear in ξ, i.e., $u(x, \xi) = \sum u_i(x)\xi^i$. In [15], [16], [4], [5] the latter statement was proved by decomposing the field f into potential and solenoidal parts and demonstrating that the corresponding boundary value problem has only the trivial solution for a solenoidal field f.

In [19], an integral equation was obtained in the nonlinear boundary rigidity problem whose left-hand side can be considered as a weighted analog I_ρ of the ray transform (1.1). Because of presence of the weight ρ, the operator I_ρ is badly behaved with respect to the decomposition of the field f into potential and solenoidal parts. Another approach to investigation of I_ρ is based on using the fiber-wise Laplacian Δ_f that is naturally defined on the bundle ΩM. In this approach, one has to prove that a solution to the kinetic equation (1.4) is an eigenfunction of Δ_f. In the case of a two-dimensional manifold M, the Laplacian Δ_f coincides with the partial derivative $\partial^2/\partial\theta^2$ with respect to the angle coordinate θ on one-dimensional fibers of ΩM. There is a simple commutator formula for the operators $\partial^2/\partial\theta^2$ and H. Applying the operator $\partial^2/\partial\theta^2$ to equation (1.4) and using the commutator formula, we obtain a quadratic relation for the function $\varphi = u_{\theta\theta} + u$ which plays a key role in the proof. In the case of absence of focal points, the terms of the relation admit estimates that yield to the claims of Theorems 1.1 and 1.2.

In the higher dimensional case, this approach runs into some additional difficulties because of the more complicated form of the commutator formula for the operators Δ_f and H. A similar situation arises in the nonlinear boundary rigidity problem. The commutator formula for Δ_f and I_ρ contains some terms dependent on the weight ρ, and we need some sharp estimates for these terms.

2. Pestov's identity in the two dimensional case

Let (M, g) be a compact oriented two-dimensional Riemannian manifold possibly with boundary. By $\Omega M = \{(x, \xi) \mid x \in M, \xi \in T_x M, |\xi| = 1\}$ we denote the three-dimensional manifold of unit tangent vectors of M. There are three canonically defined vector fields $H, H^\perp$, and $\partial_\theta = \frac{\partial}{\partial\theta}$ on ΩM which are linearly independent at every point. These fields are defined as follows. H is the vector field generating the geodesic flow. The flow generated by the field ∂_θ is the group of rotations of the one-dimensional fibers of the bundle $\Omega M \to M$ which are well defined due to the orientation. In what follows, we use the notation $f_\theta = \partial_\theta f$ for a function $f \in C^\infty(\Omega M)$. Finally, the flow φ^t of the field $H^\perp$ is defined as follows. Given a point $(x, \xi) \in \Omega M$, let $\xi^\perp \in \Omega_x M$ be the unit vector orthogonal to ξ whose direction is chosen with the help of the orientation, and $\gamma_{x,\xi^\perp}$ be the geodesic determined by the initial conditions $\gamma_{x,\xi^\perp}(0) = x$ and $\dot\gamma_{x,\xi^\perp}(0) = \xi^\perp$. Then $\varphi^t(x, \xi) = (\gamma_{x,\xi^\perp}(t), \xi(t))$, where $\xi(t)$ is the result of the parallel transport of the vector ξ along $\gamma_{x,\xi^\perp}$.

The commutator formulas

$$[\partial_\theta, H] = H^\perp, \quad [\partial_\theta, H^\perp] = -H, \quad [H, H^\perp] = K\partial_\theta \tag{2.1}$$

hold where K is the Gaussian curvature. The simplest way of proving these formulas is based on using isothermal coordinates

$$ds^2 = e^{2\mu(x,y)}(dx^2 + dy^2). \tag{2.2}$$

Such a system exists in a neighborhood of every point of M. Given such coordinate system, the local coordinates (x, y, θ) are defined on ΩM, where θ is the angle from ∂_x to the current vector $\xi \in \Omega_{(x,y)} M$. The vector fields H and $H^\perp$ are expressed in these coordinates as follows:

$$H = e^{-\mu}\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + (-\mu_x\sin\theta + \mu_y\cos\theta)\frac{\partial}{\partial\theta}\right), \tag{2.3}$$

$$H^\perp = e^{-\mu}\left(-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} - (\mu_x\cos\theta + \mu_y\sin\theta)\frac{\partial}{\partial\theta}\right). \tag{2.4}$$

By using (2.3), (2.4) and the equality $K = -e^{-2\mu}\Delta\mu$, the formulas (2.1) are proved by straightforward calculations which we omit.

Given an arbitrary function $c \in C^\infty(\Omega M)$, we introduce the operator

$$H_c^\perp = H^\perp + c\partial_\theta.$$

The commutator formulas

$$\begin{aligned} &[\partial_\theta, H] = H_c^\perp - c\partial_\theta, \quad [\partial_\theta, H_c^\perp] = -H + c_\theta\partial_\theta, \\ &[H, H_c^\perp] = -cH_c^\perp + (Hc + c^2 + K)\partial_\theta \end{aligned} \tag{2.5}$$

follow from (2.1).

Let $d\sigma$ be the surface form on M. We use the same notation $d\sigma$ for the 2-form on ΩM which is the pull-back of $d\sigma$ under the projection $\Omega M \to M$. There is also the natural volume form $d\Sigma$ on ΩM. In isothermal coordinates (2.2), these forms are expressed as follows:

$$d\sigma = e^{2\mu} dx \wedge dy, \quad d\Sigma = e^{2\mu} dx \wedge dy \wedge d\theta. \tag{2.6}$$

We denote the operator of inner multiplication by a vector field X by $\iota(X)$. For an arbitrary function $f \in C^\infty(\Omega M)$, the relations

$$\begin{aligned} &d(f d\sigma) = f_\theta d\Sigma, \quad d(f\iota(H)d\Sigma) = Hf \cdot d\Sigma, \\ &d(f\iota(H_c^\perp)d\Sigma) = (H_c^\perp f + c_\theta f)d\Sigma \end{aligned} \tag{2.7}$$

are proved by straightforward calculations in coordinates using (2.3), (2.4) and (2.6).

Lemma 2.1 (Pestov's identity). *For arbitrary real-valued functions $c, \varphi \in C^\infty(\Omega M)$, we have*

$$\begin{aligned} 2\left(H_c^\perp\varphi, \frac{\partial}{\partial\theta}(H\varphi)\right)_{L_2(\Omega M)} &= \|H\varphi\|^2_{L_2(\Omega M)} + \|H_c^\perp\varphi\|^2_{L_2(\Omega M)} \\ &\quad - \int\limits_{\Omega M} (Hc + c^2 + K)\varphi_\theta^2 d\Sigma \\ &\quad + \int\limits_{\partial(\Omega M)} \omega_c(\varphi), \end{aligned} \tag{2.8}$$

where

$$\omega_c(\varphi) = \varphi_\theta H_c^\perp\varphi \cdot \iota(H)d\Sigma - \varphi_\theta H\varphi \cdot \iota(H_c^\perp)d\Sigma. \tag{2.9}$$

The L_2-norm on ΩM is defined by:

$$(f, g)_{L_2(\Omega M)} = \int\limits_{\Omega M} fg\, d\Sigma.$$

This identity can be (and has been) obtained from Pestov's identity with modified horizontal derivative (see Section 8.2 of [17]) by changing coordinates. However the change of coordinates leads to lengthy and complicated calculations. We present the following straightforward proof.

Proof. By using the commutator formulas (2.5), we derive for a function $\varphi \in C^\infty(\Omega M)$

$$\begin{aligned}
&2H_c^\perp\varphi\cdot\frac{\partial}{\partial\theta}(H\varphi)-\frac{\partial}{\partial\theta}(H_c^\perp\varphi\cdot H\varphi)\\
&\qquad = H_c^\perp\varphi\cdot\frac{\partial}{\partial\theta}H\varphi-\frac{\partial}{\partial\theta}H_c^\perp\varphi\cdot H\varphi\\
&\qquad = H_c^\perp\varphi\cdot(H\varphi_\theta+H_c^\perp\varphi-c\varphi_\theta)\\
&\qquad\quad -(H_c^\perp\varphi_\theta-H\varphi+c_\theta\varphi_\theta)\cdot H\varphi\\
&\qquad = (H\varphi)^2+(H_c^\perp\varphi)^2+H_c^\perp\varphi\cdot H\varphi_\theta\\
&\qquad\quad - H_c^\perp\varphi_\theta\cdot H\varphi - c\varphi_\theta H_c^\perp\varphi - c_\theta\varphi_\theta H\varphi\\
&\qquad = (H\varphi)^2+(H_c^\perp\varphi)^2+H(\varphi_\theta H_c^\perp\varphi)-H_c^\perp(\varphi_\theta H\varphi)\\
&\qquad\quad -\varphi_\theta([H,H_c^\perp]\varphi+cH_c^\perp\varphi+c_\theta H\varphi)\\
&\qquad = (H\varphi)^2+(H_c^\perp\varphi)^2-(Hc+c^2+K)\varphi_\theta^2\\
&\qquad\quad + H(\varphi_\theta H_c^\perp\varphi)-H_c^\perp(\varphi_\theta H\varphi)-c_\theta\varphi_\theta H\varphi.
\end{aligned}$$

We have thus proved the equality

$$\begin{aligned}
2H_c^\perp\varphi\cdot\frac{\partial}{\partial\theta}(H\varphi) &= (H\varphi)^2+(H_c^\perp\varphi)^2-(Hc+c^2+K)\varphi_\theta^2\\
&\quad + H(\varphi_\theta H_c^\perp\varphi)-H_c^\perp(\varphi_\theta H\varphi)\\
&\quad + \frac{\partial}{\partial\theta}(H_c^\perp\varphi\cdot H\varphi)-c_\theta\varphi_\theta H\varphi.
\end{aligned}$$

We multiply this equality by $d\Sigma$ and use (2.7) to obtain

$$\begin{aligned}
2H_c^\perp\varphi\cdot\frac{\partial}{\partial\theta}(H\varphi)d\Sigma &= \Big((H\varphi)^2+(H_c^\perp\varphi)^2\Big)d\Sigma-(Hc+c^2+K)\varphi_\theta^2 d\Sigma\\
&\quad + d\Big(\varphi_\theta H_c^\perp\varphi\cdot\iota(H)d\Sigma-\varphi_\theta H\varphi\cdot\iota(H_c^\perp)d\Sigma\\
&\qquad\qquad + H\varphi\cdot H_c^\perp\varphi\cdot d\sigma\Big).
\end{aligned}$$

Integrating this equality over ΩM, we obtain (2.8) because the restriction of $d\sigma$ to $\partial(\Omega M)$ is equal to zero.

3. An identity for a tensor field of second rank

In this section we will prove the following

Lemma 3.1. *Let $c \in C^\infty(\Omega M)$ be an arbitrary real function. If a real function $u \in C^4(\Omega M)$ satisfies the kinetic equation* (1.4)*, then the function*

$$\varphi = u_{\theta\theta} + u$$

satisfies the equality

$$\begin{aligned}\|H\varphi\|^2_{L_2(\Omega M)} + \|H_c^\perp\varphi\|^2_{L_2(\Omega M)} + 4\|H_c^\perp\varphi - c\varphi_\theta/2\|^2_{L_2(\Omega M)} \\ = \int\limits_{\Omega M} (Hc + 2c^2 + K)\varphi_\theta^2 d\Sigma - \int\limits_{\partial(\Omega M)} \omega_c(\varphi),\end{aligned} \tag{3.1}$$

where $\omega_c(\varphi)$ is defined by (2.9).

Proof. In isothermal coordinates (2.2), the kinetic equation (1.4) can be rewritten in the form

$$Hu = \frac{1}{2}f_0 + f_1\cos 2\theta + f_2 \sin 2\theta, \tag{3.2}$$

where the functions

$$f_0 = \frac{1}{2}e^{-\mu}(f_{11} + f_{22}), \quad f_1 = \frac{1}{2}e^{-\mu}(f_{11} - f_{22}), \quad f_2 = e^{-\mu}f_{12}$$

are independent of θ.

Differentiating equation (3.2) with respect to θ and applying the first of the commutator formulas (2.5), we obtain

$$Hu_\theta + H_c^\perp u - cu_\theta = 2(-f_1 \sin 2\theta + f_2 \cos 2\theta). \tag{3.3}$$

We next differentiate (3.3) with respect to θ. By applying (2.5) again, we conclude

$$Hu_{\theta\theta} - 2cu_{\theta\theta} + 2H_c^\perp u_\theta - Hu = -4(f_1\cos 2\theta + f_2 \sin 2\theta). \tag{3.4}$$

We multiply equation (3.2) by 2 and we add the result to (3.4). We get the equality

$$H(u_{\theta\theta} + u) - 2cu_{\theta\theta} + 2H_c^\perp u_\theta = f_0 - 2(f_1 \cos 2\theta + f_2 \sin 2\theta)$$

that can be rewritten in the form

$$H\varphi = -2H_c^\perp u_\theta + 2cu_{\theta\theta} + F \tag{3.5}$$

with

$$F = f_0 - 2(f_1 \cos 2\theta + f_2 \sin 2\theta). \tag{3.6}$$

By (3.5),

$$\frac{\partial}{\partial\theta}(H\varphi) = -2\frac{\partial}{\partial\theta}H_c^\perp u_\theta + 2cu_{\theta\theta\theta} + 2c_\theta u_{\theta\theta} + F_\theta.$$

Using the second of the commutator formulas (2.5), we rewrite the latter equality in the form

$$\frac{\partial}{\partial\theta}(H\varphi) = -2H_c^\perp u_{\theta\theta} + 2Hu_\theta + 2cu_{\theta\theta\theta} + F_\theta. \tag{3.7}$$

By (3.3),

$$Hu_\theta = -H_c^\perp u + cu_\theta + 2(-f_1 \sin 2\theta + f_2 \cos 2\theta).$$

Inserting this expression into (3.7), we obtain

$$\begin{aligned}\frac{\partial}{\partial\theta}(H\varphi) = & -2H_c^\perp u_{\theta\theta} - 2H_c^\perp u + 2cu_\theta + 2cu_{\theta\theta\theta} \\ & + 4(-f_1 \sin 2\theta + f_2 \cos 2\theta) + F_\theta.\end{aligned}$$

By (3.6), the sum of two last terms on the right-hand side of the latter equality is equal to zero, and we arrive at the equation

$$\frac{\partial}{\partial\theta}(H\varphi) = -2H_c^\perp\varphi + 2c\varphi_\theta. \tag{3.8}$$

Although we have proved (3.8) with the help of local coordinates, it is an invariant equation and therefore is valid globally on ΩM.

By (3.8),

$$\begin{aligned}2\left(H_c^\perp\varphi, \frac{\partial}{\partial\theta}(H\varphi)\right)_{L_2(\Omega M)} &= -4\|H_c^\perp\varphi\|^2_{L_2(\Omega M)} + 4(H_c^\perp\varphi, c\varphi_\theta)_{L_2(\Omega M)} \\ &= -4\|H_c^\perp\varphi - c\varphi_\theta/2\|^2_{L_2(\Omega M)} + \int\limits_{\Omega M} c^2\varphi_\theta^2\, d\Sigma.\end{aligned}$$

Inserting this expression into the left-hand side of (2.8), we arrive at (3.1).

4. Proof of Theorem 1.1

First of all we will prove the following

Lemma 4.1. *Under hypotheses of Theorem* 1.1*, there exists a real function* $c \in C^\infty(\Omega M)$ *satisfying the inequality*

$$Hc + 2c^2 + K \leq 0, \tag{4.1}$$

where K *is the Gaussian curvature.*

Proof. Let us fix a unit speed geodesic $\gamma : [0, l] \to M$ with endpoints on the boundary, $\gamma(0), \gamma(l) \in \partial M$. We will first prove that inequality (4.1) has a solution on γ. Putting $x = \gamma(t), \xi = \dot\gamma(t)$ in (4.1), we arrive at the inequality

$$\dot c + 2c^2 + K \leq 0. \tag{4.2}$$

Making the change $c = a/2$, the inequality is transformed into the following one:

$$\dot a + a^2 + 2K \leq 0. \tag{4.3}$$

Since the geodesic γ has no focal points, the Jacobi equation

$$\ddot y + Ky = 0 \tag{4.4}$$

has a positive solution y_1^γ with positive derivative $\dot y_1^\gamma$ on $[0, l]$. Consequently, the function $a_1 = \dot y_1^\gamma / y_1^\gamma$ is positive and satisfies the Riccati equation

$$\dot a_1 + a_1^2 + K = 0. \tag{4.5}$$

Applying the same argument to the geodesic γ with the reversed orientation, we obtain a positive solution y_2^γ to the Jacobi equation (4.4) with negative derivative $\dot y_2^\gamma$ on $[0, l]$. Consequently, the function $a_2 = \dot y_2^\gamma / y_2^\gamma$ is negative and satisfies the Riccati equation

$$\dot a_2 + a_2^2 + K = 0. \tag{4.6}$$

Adding (4.5) and (4.6), we conclude that the function $a = a_1 + a_2$ satisfies the equality

$$\dot a + a^2 + 2K = 2a_1 a_2.$$

Since the functions a_1 and a_2 have different signs, we see that inequality (4.3) is satisfied by a.

We represent ΩM as the union of disjoint curves, the orbits of the geodesic flow, which are geodesics considered as curves in ΩM. We have proved that inequality (4.1) has a solution on every such curve. We have now to choose these solutions in such a way that their union gives us a function that is smooth on the whole of ΩM. To this end we observe that the above-discussed construction of the function c has only the choice of the initial values $y_i^\gamma(0), \dot{y}_i^\gamma(0)$ $(i = 1, 2)$ of the solutions to the Jacobi equation which can be arbitrary. The family of oriented geodesics γ can be parameterized by points of the product $\Omega^1 \times \Omega^1$ of two circles. Choosing smooth functions $y_i^\gamma(0), \dot{y}_i^\gamma(0)$ on $\Omega^1 \times \Omega^1$ we obtain the solution c to inequality (4.1) which is smooth on ΩM. The lemma is proved.

We now start the proof of Theorem 1.1. Given a real tensor field $f \in Z(S^2\tau'_M)$, we define the function $u \in C(\Omega M)$ by

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} f_{ij}(\gamma_{x,\xi}(t))\dot{\gamma}^i_{x,\xi}(t)\dot{\gamma}^j_{x,\xi}(t)\,dt, \tag{4.7}$$

where $\gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is the geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x$, $\dot{\gamma}_{x,\xi}(0) = \xi$ and such that $\gamma_{x,\xi}(\tau_-(x,\xi)) \in \partial M$. The function u satisfies the kinetic equation (1.4) and the homogeneous boundary condition

$$u(x,\xi)|_{x\in\partial M} = 0. \tag{4.8}$$

The function $u(x,\xi)$ depends smoothly on $(x,\xi) \in \Omega M$ except at the points of the set $\Omega(\partial M)$ where some derivatives of u can be infinite. Therefore, some of the integrals considered below are improper and we have to prove their convergence. This verification is performed in the same way as in Section 4.6 of [17], since the singularities of u are due only to the singularities of the low integration limit in (4.7). In order to simplify the presentation, we will not pay attention to these singularities in what follows.

We write down the equality (3.1) for the function $\varphi = u_{\theta\theta} + u$. The boundary integral is equal to zero as is seen from (2.9) and the fact that the function φ_θ vanishes on $\partial(\Omega M)$, the latter fact follows from (4.8).

We thus obtain

$$\|H\varphi\|^2_{L_2(\Omega M)} + \|H_c^\perp\varphi\|^2_{L_2(\Omega M)} + 4\|H_c^\perp\varphi - c\varphi_\theta/2\|^2 = \int\limits_{\Omega M} (Hc + 2c^2 + K)\varphi_\theta^2 d\Sigma.$$

By Lemma 4.1, the right-hand side is nonpositive. Since the left-hand side is nonnegative, this implies that $H\varphi = 0$. The latter equation, together with the homogeneous boundary condition $\varphi|_{\partial(\Omega M)} = 0$, implies that

$$\varphi = u_{\theta\theta} + u = 0.$$

This means that the function u has the form

$$u(x, y, \theta) = \alpha(x, y)\cos\theta + \beta(x, y)\sin\theta \tag{4.9}$$

in isothermal coordinates (2.2).

Substituting the expression (4.9) for u into the kinetic equation (3.2), we see that

$$f_{11} = e^\mu(\alpha_x + \mu_y\beta), \quad f_{12} = \frac{1}{2}e^\mu(\alpha_y + \beta_x - \mu_y\alpha - \mu_x\beta),$$
$$f_{22} = e^\mu(\beta_y + \mu_x\alpha).$$

This equalities are equivalent to the relation $f = \sigma\nabla v$, where v is the covector field with the coordinates $v_1 = e^\mu\alpha$ and $v_2 = e^\mu\beta$. Since v vanishes on ∂M by (4.8), the field f is potential. The theorem is proved.

5. Proof of Theorem 1.2

First of all we will reduce the question to the case of orientable M. Let M be a nonorientable Riemannian surface satisfying the hypotheses of Theorem 1.2, and $\pi : \widetilde{M} \to M$ be the twofold covering with the orientable $\widetilde{M}$. Then $\widetilde{M}$ satisfies also the hypotheses of the theorem. In particular, almost every unit speed geodesic of $\widetilde{M}$ is dense in $\Omega\widetilde{M}$. Let $\eta : \widetilde{M} \to \widetilde{M}$ be the isometry changing two points in every fiber of π. By $\eta^* : C^\infty(S^k\tau'_{\widetilde{M}}) \to C^\infty(S^k\tau'_{\widetilde{M}})$ and $\pi^* : C^\infty(S^k\tau'_M) \to C^\infty(S^k\tau'_{\widetilde{M}})$ we denote the mappings of tensor fields induced by η and π respectively. Since η is an isometry, η^* commutes with the operator $\sigma\nabla$.

Given $f \in Z(S^2\tau'_M)$, the field $\widetilde{f} = \pi^* f$ belongs to $Z(S^2\tau'_{\widetilde{M}})$ and satisfies the relation $\eta^*\widetilde{f} = \widetilde{f}$. Assuming Theorem 1.2 to be valid for $\widetilde{M}$, we find $\widetilde{v} \in C^\infty(\tau'_{\widetilde{M}})$ such that $\widetilde{f} = \sigma\nabla\widetilde{v}$. We have to prove that $\widetilde{v}$ is the lifting of some covector field v on M, i.e., that $\widetilde{v} = \pi^* v$. To this end we should demonstrate that $\eta^*\widetilde{v} = \widetilde{v}$. We have,

$$\sigma\nabla(\eta^*\widetilde{v} - \widetilde{v}) = \eta^*(\sigma\nabla\widetilde{v}) - \sigma\nabla\widetilde{v} = \eta^*\widetilde{f} - \widetilde{f} = 0.$$

We have thus shown that

$$\sigma\nabla(\eta^*\widetilde{v} - \widetilde{v}) = 0.$$

Using Lemma 2.1 of [4], the latter equality implies that

$$\eta^*\widetilde{v} - \widetilde{v} = 0.$$

The following statement is an analog of Lemma 4.1.

Lemma 5.1. *Under the hypotheses of Theorem* 1.2*, there exists a real function $b \in C(\Omega M)$ which is smooth on every orbit of the geodesic flow and satisfies the inequality*

$$Hb + 2b^2 + K \le 0, \tag{5.1}$$

where K is the Gaussian curvature.

Proof. In the proof we essentially repeat the arguments of E. Hopf [11] with slight modifications related to the absence of focal points.

We consider the Jacobi equation (4.4) along a unit speed geodesic $\gamma : \mathbf{R} \to \Omega M$. Absence of conjugate points means that every nontrivial solution to the equation has at most one zero, and any two solutions coincide at most at one point if they are not equal identically. For $a \ne b$, let $y(t; a, b)$ be the solution satisfying the conditions

$$y(a; a, b) = 0, \quad y(b; a, b) = 1. \tag{5.2}$$

These functions satisfy the identity

$$y(t; a, b) = y(\beta; a, b)y(t; \alpha, \beta) + y(\alpha; a, b)y(t; \beta, \alpha). \tag{5.3}$$

Indeed, both sides of the equality are solutions to (4.4) which, by (5.2), coincide at $t = \alpha, \beta$ and therefore are equal identically. In the particular case of $\alpha = a$ and $\beta = b'$, (5.3) gives

$$y(t; a, b) = y(b'; a, b)y(t; a, b'). \tag{5.4}$$

Since γ has no focal points,

$$\dot{y}(t;a,b) > 0 \quad \text{for} \quad a < b \quad \text{and every } t, \tag{5.5}$$

and

$$\begin{aligned} &y(t;a,b) < 0 \quad \text{for} \quad t < a < b, \\ &y(t;a,b) > 0 \quad \text{for} \quad a < b \quad \text{and} \quad t > a. \end{aligned} \tag{5.6}$$

Two solutions $y(t;a,b)$ and $y(t;a',b)$, $a' < a$, coincide at $t = b$ and at no other point. Therefore, by (5.6),

$$y(t;a',b) \leq y(t;a,b) \quad \text{for} \quad a' < a < b \leq t. \tag{5.7}$$

Formulas (5.6) and (5.7) imply existence of the limit

$$y(t;b) = \lim_{a \to -\infty} y(t;a,b) \tag{5.8}$$

for every $t \geq b$.

If α and β in (5.3) are chosen more than b, it becomes evident that limit (5.8) exists for every t, and $y(t;b)$ is a solution to equation (4.4). This implies also that

$$\dot{y}(t;b) = \lim_{a \to -\infty} \dot{y}(t;a,b)$$

for every t. Using (5.2), (5.5) and (5.6) we get that

$$y(b;b) = 1, \quad y(t;b) \geq 0, \quad \dot{y}(t;b) \geq 0$$

for all t. Since $y(t;b)$ is a solution to (4.4), we have the strong inequality $y(t;b) > 0$ for all t.

The function

$$u(t) = \frac{\dot{y}(t;b)}{y(t;b)}$$

is independent of b by (5.4). This function is nonnegative, depends smoothly on t, and satisfies the Riccati equation

$$\dot{u} + u^2 + K = 0.$$

We have thus constructed the function $u(t)$ for every unit speed geodesic $\gamma : \mathbf{R} \to \Omega M$. The value $u(t)$ depends only on the point $\gamma(t)$ but not on the choice of the origin $\gamma(0)$ on γ. In other words, u can be considered as a well-defined function $u(x,\xi)$ on ΩM. As E. Hopf has mentioned in [11] without proof, the function u is continuous on ΩM. In our case continuity of u can be justified as follows. As is established in [7], there is a one-to-one correspondence between the function u and the stable distribution in the case of an Anosov geodesic flow. This means that the stable distribution can be expressed in terms of u and vice-versa. On the other hand, it is well known [1] that the stable distribution of an Anosov flow is continuous. This implies continuity of u.

We have thus found a nonnegative function $u \in C(\Omega M)$ which is smooth on orbits of the geodesic flow and satisfies the Riccati equation

$$Hu + u^2 + K = 0.$$

We define now a new function $a \in C(\Omega M)$ by

$$a(x,\xi) = u(x,\xi) - u(x,-\xi).$$

It satisfies the equation

$$Ha + a^2 + 2K = -2u(x,\xi)u(x,-\xi).$$

Since u is nonnegative, a satisfies the inequality

$$Ha + a^2 + 2K \le 0.$$

Finally, putting $a = 2b$, we arrive at (5.1). The lemma is proved.

We start proving Theorem 1.2. Given a real tensor field $f \in Z(S^2\tau'_M)$, the integral of the function $f_{ij}(x)\xi^i\xi^j \in C^\infty(\Omega M)$ over a closed orbit of the geodesic flow equals zero. Applying the smooth version of the Livčic theorem [6], we obtain a function $u \in C^\infty(\Omega M)$ satisfying the kinetic equation (1.4).

We can assume M to be orientable. Fixing an orientation, the differential operator $\partial/\partial\theta$ is well-defined on ΩM. Our aim is to prove that $\varphi = u_{\theta\theta} + u$ is a constant function.

The statement of Lemma 3.1 looks as follows in the boundaryless case:

$$\|H\varphi\|^2_{L_2(\Omega M)} + \|H_c^\perp\varphi\|^2_{L_2(\Omega M)} + 4\|H_c^\perp\varphi - c\varphi_\theta/2\|^2_{L_2(\Omega M)} = \int\limits_{\Omega M} (Hc + 2c^2 + K)\varphi_\theta^2 d\Sigma. \tag{5.9}$$

Equality (5.9) is valid for an arbitrary function $c \in C^\infty(\Omega M)$. Let now $b \in C(\Omega M)$ be the function constructed in Lemma 5.1. With the help of Lemma 4.1 of [5], we can find a sequence of smooth functions $c_k \in C^\infty(\Omega M)$ $(k = 1, 2, \dots)$ such that c_k and Hc_k converge uniformly on ΩM to b and Hb respectively as $k \to \infty$. Writing down equality (5.9) for $c = c_k$ and passing to the limit in the equality as $k \to \infty$, we obtain the same equality (5.9) with b in place of c. By Lemma 5.1, the right-hand side of the latter equality is nonpositive. Since the left-hand side is nonnegative, the equality implies that $H\varphi = 0$. This means that the function φ is constant on every orbit of the geodesic flow. Since there exists such an orbit dense in ΩM, the function $\varphi = u_{\theta\theta} + u$ is constant on ΩM. This implies that the function u is representable in the form

$$u(x, y, \theta) = u_0 + u_1(x, y)\cos\theta + u_2(x, y)\sin\theta \tag{5.10}$$

in the domain of an isothermal coordinate system, where u_0 is a constant.

The rest of the proof is similar to the arguments at the end of the previous section. Substituting the expression (5.10) for u into the kinetic equation (3.2), we obtain $f = \sigma\nabla v$ for the 1-form $v = e^\mu(u_1dx + u_2dy)$. The latter form is easily seen to be well-defined on the whole of M. The theorem is proved.

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