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# ELLIPTIC SECTORS IN SURFACE THEORY AND THE CARATHÉODORY–LOEWNER CONJECTURES

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### Abstract

We identify a distinguished hyperbolic partial differential equation of importance in the study of principal foliations. We apply Riemann's method to obtain an obstruction to the occurrence of elliptic sectors in principal foliations on surfaces and as a consequence we obtain the first global result on the Carathéodory conjecture.

## 1. Introduction

The principal foliations on a smooth surface in $\mathbb{R}^3$ have been a subject of enduring interest in Differential Geometry and Dynamical Systems from the earliest work of Monge in 1796 through Dupin, Bonnet and Darboux in the $19^{\text{th}}$ century up to the present (see Gutierrez and Sotomayor [5], [6], especially the latter useful survey article). The study of the singularities of the principal foliations, that is the umbilics, is important for other areas of mathematics. Rozoy's beautiful solution [11] of the Lichnerowicz conjecture in General Relativity depends on the vanishing theorem of Smyth and Xavier [13] for umbilics of index $\geq 1$ in smooth surfaces.

The prevailing view is that a principal foliation is somewhat special in the neighborhood of an isolated umbilic in the surface. From the point of view of analysis the special nature of such a singular foliation is completely characterized in the recent paper of Smyth and Xavier ([14], Theorem 2 and Lemma 3). Whether their criterion argues for some topological or diffeomorphism-type implications for umbilics is an

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open question, but very much in line with the interest in the index of umbilics early in this century.

Carathéodory's conjecture asserts that a smooth spherical surface of class $C^3$ immersed in $\mathbb{R}^3$ must have at least two umbilics (see the 1940 paper of Hamburger [7], p. 63). This conjecture appears to have been first made prior to 1928 (see Cohn-Vossen's announcement [4]) and all subsequent studies have been directed toward the (stronger) local Carathéodory conjecture that any isolated umbilic on a piece of a smooth surface must have index $\leq 1$; both the local and global conjectures remain open today. Incidentally (see [14]), the local conjecture is equivalent to Loewner's conjecture [15] that isolated zeroes of $\omega_{\bar{z}\bar{z}}$ (for any real-valued function $\omega$ on $\mathbb{R}^2$) have index $\leq 2$. Until recently there were few results on isolated umbilics on smooth surfaces and a decided emphasis on an approach toward proving that the index of an isolated umbilic in an *analytically* immersed surface is $\leq 1$; all of these assays ([2], [10], [15]) are along the lines of Hamburger's original works [7] and [8], and an elucidation of the gaps in — and counterexamples to — their arguments is contained in Scherbel [12]. All of the results which we present here never need more than a few derivatives.

Our point of view here entertains the possibility that a spherical surface have a lone umbilic — consequently of index 2, by the Euler-Poincaré theorem — and Theorem 1 below addresses the diffeomorphism type of the resulting configuration of principal foliations: it proves the nonexistence of principal foliations locally diffeomorphic to the standard dipole foliation. The standard dipole foliation on $S^2$ may be thought of as the family of circles on $S^2$ obtained by intersecting $S^2$ with a co-axial system of planes with axis a tangent line to $S^2$ (see Figure 1). Smyth and Xavier [14] proved that a principal foliation is never locally diffeomorphic to the standard dipole foliation *under the Gauss map.*

**Theorem 1.** *If $p_0 \in S^2$ is a lone umbilic of an immersion $f\colon S^2 \to \mathbb{R}^3$ of class $C^4$, then the principal foliations can never be $C^2$-diffeomorphic to the standard dipole foliation.*

*Moreover, this result is even local; that is the principal foliations on a neighborhood of an isolated umbilic on a differentiable surface of class $C^4$ can never be locally $C^2$-diffeomorphic to the standard dipole foliation.*

That such a result must hold was conjectured to me by Brian Smyth in 1997. Its very strength suggests that there are broader issues here than the mere numerical value of the index. By Bendixson's index for-

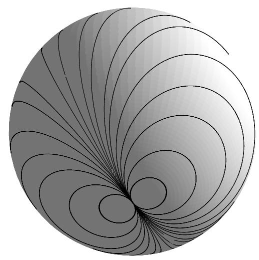

Figure 1: Standard dipole foliation on $S^2$

mula ([1], [9]), $j = 1 + \frac{e-h}{2}$ (where $e$ and $h$ are the numbers of elliptic and hyperbolic sectors, respectively) the salient feature of an isolated umbilic of index $> 1$ — if such exists — would be the existence of an elliptic sector in each of the principal foliations. Thus the study of elliptic sectors in principal foliations is central to the study of the existence of umbilics of index $> 1$. A more ambitious conjecture than either the Carathéodory conjecture or the local Carathéodory conjecture is that a principal foliation cannot contain an elliptic sector. An elliptic sector is a closed simply connected region bounded by a leaf which tends to the singularity in both directions and containing no other singularity (cf [9]). Our main result identifies a fundamental local geometric property of elliptic sectors in a principal foliation.

For any foliation $\mathcal{F}$ of a Riemannian surface with an isolated singularity at $o$, the orthogonal $\mathcal{F}^\perp$ is well-defined and has the same singular set. A *lens* of $\mathcal{F}$ is a closed simply connected region containing $o$ and no other singular point and bounded by an arc of $\mathcal{F}$ and an arc of $\mathcal{F}^\perp$, both exiting a nonsingular point and tending toward $o$ (see §3); an elliptic sector is a continuum of lenses. The reader is reminded in §2 that an equivalence between principal foliations and (plane) Hessian foliations was constructed in [14].

**Main Theorem.** *A $C^2$ singular Hessian foliation cannot have a biconvex lens.*

The biconvexity condition in the theorem amounts to the condition that the orthogonal trajectories are convex at the walls of the lens.

The proof of the Main Theorem in §3 is in the spirit of the work of Smyth and Xavier [14] on the system of wave equations $\omega_{\bar{z}\bar{z}} = g$: It is based on Riemann's method in the theory of hyperbolic partial differential equations. By contrast, our approach identifies one singular hyperbolic partial differential equation which is central to the study. The presence of a lens in the foliation gives simultaneous focusing at $o$ of the characteristics of this partial differential equation and Riemann's method leads to a simple integral identity which must be satisfied by the solution. The curvature assumptions lead to a contradiction to this identity.

The proof of Theorem 1 is given in §5 and is a consequence of our Main Theorem and approximation results for the curvature of foliations.

## 2. Principal foliations and Hessian foliations

A piece of a smooth surface in $\mathbb{R}^3$ may be thought of as a 2-dimensional orientable manifold $M$ together with an immersion $f\colon M \to \mathbb{R}^3$. A choice of unit normal field $\xi$ along $M$ defines a tensor field $A$ of type $(1,1)$ on $M$ by $X\xi = -df(AX)$ for each vector $X$ tangent to $M$. The operator $A$ is symmetric with respect to the induced metric $g = f^*\langle\,,\,\rangle$, where $\langle\,,\,\rangle$ is the euclidean metric on $\mathbb{R}^3$. The points where $A$ is a multiple of the identity are called *umbilics* of the immersion $f$. The complement of the set of umbilics is foliated by a pair of foliations (orthogonal with respect to $g$) given by the eigendirections of $A$ at each point; these are the *principal foliations* of the immersion $f\colon M \to \mathbb{R}^3$.

Let $p_0$ be an isolated umbilic and $C$ a simple closed curve enclosing $p_0$ and no other umbilics and completely contained in a simply connected local coordinate neighborhood of $p_0$. We may continuously choose an oriented unit tangent to the foliation as we describe one counterclockwise circuit of $C$ and measure the total counterclockwise variation $T$ of the tangent. Then $j = \frac{1}{2\pi}T \in \frac{1}{2}\mathbb{Z}$ is independent of the choice of $C$ and called the *index* of the umbilic. The foliation is orientable in a neighborhood of $p_0$ if and only if $j$ is an integer.

Let $p_0$ be an isolated umbilic of $f$. From the definition of an umbilic it is clear that $K(p_0) \geq 0$, where $K$ denotes Gaussian curvature. If $K(p_0) = 0$, then the map $\tilde{f}\colon = \iota \circ f\colon M \to \mathbb{R}^3$, where $\iota$ is the inversion in the unit sphere about $o \in \mathbb{R}^3$, is an immersion (after we arrange the origin so that $f(M) \not\ni o$) with *exactly* the same principal foliations on $M$ as $f$. If we move $f(M)$ so that the origin $o \in \mathbb{R}^3$ does not lie in the tangent plane $df_{p_0}(T_{p_0}M)$, then $\tilde{K}(p_0) > 0$ (see Smyth and Xavier [14]). Thus, there is no loss of generality in taking $K(p_0) > 0$, so that the Gauss map $\xi\colon M \to S^2$ is a diffeomorphism of a neighborhood $U$ of $p_0$

in $M$ onto a neighborhood $V$ of $\xi(p_0)$ in $S^2$. After a change of frame in $\mathbb{R}^3$ we may assume that $\xi(p_0)$ is the south pole of $S^2$. Then, under the stereographic projection $\Pi$ from the north pole onto the equatorial plane, the coordinate $z = x+iy$ on a neighborhood $\Omega$ of the origin $o \in \mathbb{C}$ gives local coordinates $B = \xi^{-1} \circ \Pi^{-1} \colon \Omega \to U$ on the neighborhood $U$ of $p_0 \in M$. These are called *Bonnet coordinates* on $M$ around the umbilic $p_0$.

The function $\omega \colon \Omega \to \mathbb{R}$ defined by

$$\omega(z, \bar{z}) = (1 + z\bar{z})\big(\langle f, \xi \rangle \circ B\big)(z, \bar{z})$$

is called the *Bonnet function*. By [3] or [14], $p_0$ being an isolated umbilic of $f$ implies $o$ is an isolated zero of $\omega_{\bar{z}\bar{z}}$ and in this case $\omega_{\bar{z}\bar{z}} = |\omega_{\bar{z}\bar{z}}|\,\zeta^2$ on $\Omega_0 := \Omega \setminus \{o\}$, where $\zeta$ is a local unit vector field (written as a complex number) representing the Gauss image of either of the two principal foliations on $\Omega_0$. Note that $\omega$ has the same differentiability class as the immersion $f$. If $f$ is $C^r$ $(r \geq 2)$ then $B$ is $C^{r-1}$ and from $\omega = (1+z\bar{z})\langle f \circ B, \Pi^{-1} \rangle$ and $\omega_z = \bar{z}\langle f \circ B, \Pi^{-1} \rangle + (1+z\bar{z})\langle f \circ B, (\Pi^{-1})_z \rangle$ it follows that $\omega$ is in fact $C^r$ (cf [6]).

A continuous 1-dimensional foliation $\mathcal{F}$ of a neighborhood $\Omega$ of $o \in \mathbb{R}^2$ with an isolated singularity at $o$ is called a *singular Hessian foliation* (Smyth and Xavier [14]) if there exists a $C^2$ real-valued function $\omega$ on $\Omega$ whose Hessian operator

$$\mathrm{Hess}[\omega] = \begin{bmatrix} \omega_{xx} & \omega_{xy} \\ \omega_{xy} & \omega_{yy} \end{bmatrix}$$

has the following properties:

1. The operator $\mathrm{Hess}[\omega]$ is proportional to $I$ only at $o \in \Omega$.
2. $\mathcal{F}_p$ is an eigenspace of the operator $\mathrm{Hess}[\omega]$ at each $p \in \Omega_0$.

The Hessian foliation $\mathcal{F}$ above is said to be of class $C^k$ if $\omega \in C^{k+2}(\Omega)$.

Beginning with a continuous foliation $\mathcal{F}$ on a punctured neighborhood $\Omega_0$ of $o \in \mathbb{R}^2$ we may represent it locally on $\Omega_0$ by a continuous unit vector field $\zeta$ (written as a complex number). Then $\zeta^2$ is a well-defined unit vector field on $\Omega_0$, and the foliation $\mathcal{F}$ is a singular Hessian foliation if and only if there exists a function $\omega \in C^2(\Omega)$ such that $o$ is an isolated zero of $\omega_{\bar{z}\bar{z}}$ and

$$\omega_{\bar{z}\bar{z}} = |\omega_{\bar{z}\bar{z}}|\,\zeta^2, \qquad \text{on } \Omega_0.$$

Note that the property of a foliation $\mathcal{F}$ being Hessian is a geometric property in the sense that any orthogonal transformation $L$ of the plane maps $\mathcal{F}$ onto a foliation $L(\mathcal{F})$ which is again a singular Hessian foliation.

Thus, in a Bonnet coordinate neighborhood of an isolated umbilic, the principal foliations are singular Hessian foliations (of the Bonnet function) and conversely — as first proved by Smyth and Xavier [14] — given a function $\omega\colon \Omega \to \mathbb{R}$ on a neighborhood of $o \in \mathbb{R}^2$, there exists an immersion $f\colon \Omega \to \mathbb{R}^3$ such that the principal foliations of $f$ coincide with the Hessian foliations of $\omega$.

Thus there is a simple equivalence between principal foliations in surface theory and Hessian foliations in the plane.

## 3. A geometric obstruction to elliptic sectors in Hessian foliations

Let $\mathcal{F}$ be a foliation of a neighborhood $\Omega$ of $o \in \mathbb{R}^2$ with a singularity at $o$. If there exists a closed simply connected region in $\Omega$ bounded by an arc of $\mathcal{F}$ and an arc of $\mathcal{F}^\perp$ both emanating from some nonsingular point $z_0 \in \Omega$ and meeting at $o$, then we call this region a *lens* of $\mathcal{F}$ and we denote it $L(z_0)$. A foliation need not have a lens; but an elliptic sector has a continuum of lenses. There are also foliations without elliptic sectors which nonetheless have a continuum of lenses.

Let $L(z_0)$ be a lens of some foliation of class $C^2$. We orient its boundary counterclockwise denoting the first arc $C^+(z_0)$ as we leave $o$ and the second arc $C^-(z_0)$. Let us re-label the corresponding foliations by $\mathcal{F}^+$ and $\mathcal{F}^-$. They are then oriented on an open neighborhood $\Omega'_{z_0}$ of $L(z_0)\backslash\{o\}$ in $\Omega_0 = \Omega\backslash\{o\}$ and their curvatures $k^+$ and $k^-$, respectively, are well-defined continuous functions on $\Omega'_{z_0}$. We say that the lens $L(z_0)$ is *biconvex* if $k^- \geq 0$ on $C^+(z_0)$ and $k^+ \geq 0$ on $C^-(z_0)$ (see Figure 2).

It is important to notice that if an isolated singularity of a foliation $\mathcal{F}$ has an elliptic sector then $\mathcal{F}$ and its orthogonal $\mathcal{F}^\perp$ automatically determine a continuum of lenses. In particular, by Bendixson's formula, lenses are *always* present if the index is $> 1$.

The proof of the Main Theorem uses Riemann's method for linear second order hyperbolic equations as presented in [14] and we will explain this first, postponing the proof until later in this section.

Let

$$\mathcal{H}[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Mu_x + Nu_y$$

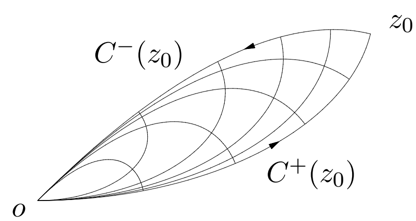


Figure 2: A biconvex lens

be a linear hyperbolic operator of class $C^2$ defined on some domain $\Omega \subset \mathbb{R}^2$. This means that $A, B$ and $C$ are of class $C^2$, $M$ and $N$ are of class $C^1$ and $B^2 - AC > 0$ throughout $\Omega$. Let $\delta = \sqrt{B^2 - AC}$. The *characteristics* of $\mathcal{H}$ are the foliations defined by

$$\pm\delta\, dx = -B\, dx + A\, dy \quad (\text{or } \pm\delta\, dy = -C\, dx + B\, dy),$$

and are denoted accordingly by $C^\pm$. The *adjoint* $\mathcal{H}^*$ of $\mathcal{H}$ is defined by

$$\mathcal{H}^*[v] = (Av)_{xx} + 2(Bv)_{xy} + (Cv)_{yy} - (Mv)_x - (Nv)_y.$$

One also defines the 1-form

$$\sigma[u] = u_x(-B\, dx + A\, dy) + u_y(-C\, dx + B\, dy),$$

and the 1-form

$$\tau = (B_x + C_y - N)\, dx - (A_x + B_y - M)\, dy,$$

which satisfy the identity

$$\Big(\mathcal{H}[u] - u\mathcal{H}^*[1]\Big) dx \wedge dy = d(\sigma[u] + u\tau).$$

Let $Pz_0Q$ be a positively-oriented curvilinear triangle (which together with its interior lies in $\Omega$) such that $Pz_0$ and $z_0Q$ are, respectively, parts of a $C^+$ characteristic and a $C^-$ characteristic, and $\Gamma$ is a simple arc transversal to both $C^\pm$ characteristics at each of its points, as shown in Figure 3.

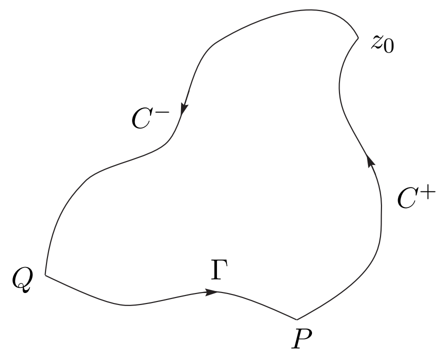


Figure 3: A curvilinear triangle $Pz_0Q$

If $[Pz_0Q]$ denotes the closed domain bounded by $Pz_0Q$ then by Stokes' Theorem

$$\begin{aligned}\iint\limits_{[Pz_0Q]} (\mathcal{H}[u] - u\mathcal{H}^*[1])\, dx\, dy &= \oint\limits_{Pz_0Q} (\sigma[u] + u\tau)\\ &= \int\limits_{Pz_0} (\sigma[u] + u\tau) + \int\limits_{z_0Q} (\sigma[u] + u\tau)\\ &\quad + \int\limits_{\Gamma} (\sigma[u] + u\tau).\end{aligned}$$

From $\sigma[u] = \pm\delta\, du$ along $C^\pm$ one obtains

$$\sigma[u] + u\tau = \pm d(\delta u) + u(\tau \mp d\,\delta) \text{ along } C^\pm$$

and

$$\begin{aligned}2(\delta u)(z_0) = &-\int\limits_{Pz_0} u(\tau - d\delta) - \int\limits_{z_0Q} u(\tau + d\delta)\\ &+ (\delta u)(P) + (\delta u)(Q) - \int\limits_{\Gamma} (\sigma[u] + u\tau)\\ &+ \iint\limits_{[Pz_0Q]} (\mathcal{H}[u] - u\mathcal{H}^*[1])\, dx\, dy.\end{aligned} \tag{1}$$

We also note the identity

$$\mathcal{H}^*[1]\, dx \wedge dy = -d\tau. \tag{2}$$

**Proof of Main Theorem.** Let $L(z_0)$ be a biconvex lens of a singular Hessian foliation of class $C^2$, whose singularity we may take to be at $o$. Thus, on an open neighborhood $\Omega'_{z_0}$ of $L(z_0)\backslash\{o\}$ in $\mathbb{R}^2\backslash\{o\}$, there is defined a $C^2$ unit vector field $F\colon \Omega'_{z_0} \to S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$ which is tangent to $\mathcal{F}^+$, and there exists a function $\omega\colon \Omega \to \mathbb{R}$ of class $C^4$ defined on an open neighborhood $\Omega$ of $o$ containing $\Omega'_{z_0}$, such that

$$\omega_{\bar{z}\bar{z}} = \rho F^2 \tag{3}$$

on $\Omega'_{z_0}$, where $\rho = |\omega_{\bar{z}\bar{z}}| \in C^2(\Omega_0) \cap C^0(\Omega; \mathbb{R}_+)$ vanishes only at $o$.

Differentiating twice with respect to $z$ we have

$$\mathrm{Im}\{(\rho F^2)_{zz}\} = 0 \tag{4}$$

on $\Omega'_{z_0}$. If we define the operator $\mathcal{H}$ by

$$\begin{aligned}\mathcal{H}[\rho] = \frac{F^2 - \bar{F}^2}{2i}\rho_{xx} - 2\frac{F^2 + \bar{F}^2}{2}\rho_{xy} - \frac{F^2 - \bar{F}^2}{2i}\rho_{yy}\\ -4i(FF_z - \bar{F}\bar{F}_{\bar{z}})\rho_x - 4(FF_z + \bar{F}\bar{F}_{\bar{z}})\rho_y,\end{aligned} \tag{5}$$

then (4) becomes

$$\mathcal{H}[\rho] + 4\rho\mathrm{Im}\{(F^2)_{zz}\} = 0. \tag{6}$$

It is easily checked that the operator $\mathcal{H}$ is hyperbolic on $\Omega_0$ with $\delta = |F|^2$ and its $C^\pm$-characteristics are given by

$$\frac{dy}{dx} = \frac{\mathrm{Im}\{F\}}{\mathrm{Re}\{F\}} \quad \text{for } C^+ \qquad \text{and} \qquad \frac{dy}{dx} = \frac{\mathrm{Im}\{iF\}}{\mathrm{Re}\{iF\}} \quad \text{for } C^-.$$

Thus,

$$dz = F\,dt^+ \text{ along } C^+ \qquad \text{and} \qquad dz = iF\,dt^- \text{ along } C^-, \tag{7}$$

where $t^\pm$ is any arclength parameter along $C^\pm$.

By straightforward calculations one also obtains

$$\sigma[\rho] = \rho_z F^2\,d\bar{z} + \rho_{\bar{z}}\bar{F}^2\,dz \tag{8}$$

and

$$\tau = 2\bar{F}\bar{F}_{\bar{z}}\,dz + 2FF_z\,d\bar{z}. \tag{9}$$

Using (2) we obtain

$$\mathcal{H}^*[1] = -4\mathrm{Im}\{(F^2)_{zz}\},$$

which shows that equation (6) is equivalent to

$$\mathcal{H}[\rho] - \rho\mathcal{H}^*[1] = 0. \tag{10}$$

From (9) and (7) we obtain that

$$\begin{cases} \tau - d\delta &= \quad (F\bar{F}\bar{F}_z - \bar{F}^2 F_{\bar{z}} + F\bar{F}\bar{F}_{\bar{z}} - F^2\bar{F}_z)\, dt^+ \quad \text{along } C^+, \\ \tau + d\delta &= -i(F\bar{F}F_z + \bar{F}^2 F_{\bar{z}} - F\bar{F}\bar{F}_{\bar{z}} - F^2\bar{F}_z)\, dt^- \quad \text{along } C^-. \end{cases} \tag{11}$$

The coefficients of $dt^+$ and $dt^-$ in the formulas above are twice the curvatures $k^-$ and $k^+$, respectively. This follows from the general formula of the curvature of a curve in terms of its tangent field viewed as a complex-valued map. To be precise, if $t \longmapsto (x(t), y(t)) \cong x(t) + iy(t) = z(t)$ is a parametrization of a $C^2$ curve then its curvature $k$ is given by the well-known formula

$$k = \frac{x_t y_{tt} - x_{tt} y_t}{({x_t}^2 + {y_t}^2)^{3/2}}$$

and its tangent field is $X = (x_t, y_t) \cong x_t + iy_t$. Using

$$x_{tt} = \partial_x \left( \frac{X + \bar{X}}{2} \right) x_t + \partial_y \left( \frac{X + \bar{X}}{2} \right) y_t,$$

$$y_{tt} = \partial_x \left( \frac{X - \bar{X}}{2i} \right) x_t + \partial_y \left( \frac{X - \bar{X}}{2i} \right) y_t,$$

one obtains, after a straightforward calculation, the formula

$$k = \frac{1}{2i|X|^3}(X\bar{X}X_z + \bar{X}^2 X_{\bar{z}} - X\bar{X}\bar{X}_{\bar{z}} - X^2\bar{X}_z). \tag{12}$$

With (12) above and $|F| = 1$, (11) becomes

$$\begin{cases} \tau - d\delta &= 2k^-\, dt^+ \quad \text{along } C^+, \\ \tau + d\delta &= 2k^+\, dt^- \quad \text{along } C^-. \end{cases} \tag{13}$$

Finally, using (8), (9), and (3) we obtain

$$\sigma[\rho] + \rho\tau = \frac{1}{4} d(\Delta\omega), \tag{14}$$

where $\Delta\omega = 4\omega_{z\bar{z}}$ is the Laplacian of $\omega$.

We are now ready to put the pieces together. We apply (1) using (10), (13), and(14), and obtain that

$$\begin{aligned}\rho(z_0) + \left(\int_P^{z_0} \rho k^- \, dt^+ + \int_{z_0}^{Q} \rho k^+ \, dt^-\right) \\ = \frac{1}{2}[\rho(P) + \rho(Q) + \frac{1}{4}(\Delta\omega)(P) - \frac{1}{4}(\Delta\omega)(Q)]\end{aligned} \tag{15}$$

for all $P \in C^+$ and all $Q \in C^-$.

By the hypotheses, $\rho = |\omega_{\bar{z}\bar{z}}| \geq 0$, $k^+ \geq 0$ on $C^-$ and $k^- \geq 0$ on $C^+$ so the integral term inside the parentheses is positive and nondecreasing as $P$ and $Q$ tend to $o$ (along $C^+$ and $C^-$, respectively). Thus, the left hand side is $\geq \rho(z_0) > 0$. But the right hand side is continuous in $P$ and $Q$, and goes to $\rho(o) = 0$ as $P$ and $Q$ tend to $o$. This contradiction completes the proof of the Main Theorem.

## 4. Curvature of a foliation under diffeomorphisms

In this section we compute two formulas: one for the curvature of the diffeomorphic image of a regular curve in terms of the curvature of the original curve and another one for the curvature of the foliation orthogonal to a given nonsingular foliation. These formulas will be used in §5 to show that the diffeomorphic image of the standard plane dipole foliation always has a biconvex lens.

We begin by recalling that a curve $c\colon (\theta_0, \theta_1) \to \mathbb{R}^2$ is called *regular* of class $C^m$ if $c$ is a differentiable map of class $C^m$ and the tangent vector field $\dot{c} := \dfrac{dc}{d\theta}$ never vanishes. Then the curvature $k_c\colon (\theta_0, \theta_1) \to \mathbb{R}$ of the regular curve $c = (a, b)$ of class $C^2$ is given by the well-known formula

$$k_c = \frac{a_\theta b_{\theta\theta} - b_\theta a_{\theta\theta}}{({a_\theta}^2 + {b_\theta}^2)^{3/2}} \tag{16}$$

where, as usual, the subscripts in the right hand side denote differentiation with respect to the specified variable.

Let $\phi = (\phi^1, \phi^2)\colon U \to \mathbb{R}^2$ be a differentiable map of class $C^2$. For $z \in U$ we define the map $K_\phi(z)\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned}\Big(K_\phi(z)\Big)(X) = \\ \big\langle(\nabla\phi^1)_z, X\big\rangle \mathrm{Hess}_z[\phi^2](X, X) - \big\langle(\nabla\phi^2)_z, X\big\rangle \mathrm{Hess}_z[\phi^1](X, X)\end{aligned}$$

where $\nabla$ denotes the gradient operator and Hess$[f]$ also denotes the symmetric bilinear form naturally induced by the Hessian operator of a real valued function $f$. If $X\colon (\theta_0,\theta_1) \to \mathbb{R}^2$ is a vector field defined along the curve $c$ then $K_\phi(X)\colon (\theta_0,\theta_1) \to \mathbb{R}$ is the map defined by $\Big(K_\phi(X)\Big)(z) = \Big(K_\phi\big(c(\theta)\big)\Big)\big(X(\theta)\big)$. Note that $K_\phi$ is homogeneous of degree 3 in the sense that $K_\phi(\lambda X) = \lambda^3 K_\phi(X)$ for any function $\lambda\colon (\theta_0,\theta_1) \to \mathbb{R}$. We also denote by $J_\phi\colon U \to \mathbb{R}$ the *Jacobian* $J_\phi = \phi^1_x\phi^2_y - \phi^2_x\phi^1_y$ of the map $\phi$.

**Lemma 1.** *Let $c\colon (\theta_0,\theta_1) \to U \subset \mathbb{R}^2$ be a regular curve of class $C^2$ whose image lies in the open set $U$ and let $\phi\colon U \to V \subset \mathbb{R}^2$ be a diffeomorphism of class $C^2$. Then the curvature $k_{\phi\circ c}$ of the curve $\phi\circ c\colon (\theta_0,\theta_1) \to V$ is given by*

$$k_{\phi\circ c} = \left|d\phi\left(\frac{\dot c}{|\dot c|}\right)\right|^{-3}\left[(J_\phi\circ c)k_c + K_\phi\left(\frac{\dot c}{|\dot c|}\right)\right]. \tag{17}$$

*Proof.* Let $\phi\circ c = (\alpha,\beta)$. Then, by a straightforward computation, we have

$$\begin{aligned}
\alpha_\theta =& \langle(\nabla\phi^1)\circ c, \dot c\rangle,\ \beta_\theta = \langle(\nabla\phi^2)\circ c, \dot c\rangle,\\
\alpha_{\theta\theta} =& \langle(\nabla\phi^1)\circ c, \ddot c\rangle + \Big(\mathrm{Hess}[\phi^1]\circ c\Big)(\dot c,\dot c),\ \beta_{\theta\theta}\\
=& \langle(\nabla\phi^2)\circ c, \ddot c\rangle + \Big(\mathrm{Hess}[\phi^2]\circ c\Big)(\dot c,\dot c),
\end{aligned}$$

so that

$$\begin{aligned}
\alpha_\theta\beta_{\theta\theta} - \beta_\theta\alpha_{\theta\theta} =& \Big\langle(\nabla\phi^1)\circ c, \dot c\Big\rangle\Big\langle(\nabla\phi^2)\circ c, \ddot c\Big\rangle\\
&- \Big\langle(\nabla\phi^2)\circ c, \dot c\Big\rangle\Big\langle(\nabla\phi^1)\circ c, \ddot c\Big\rangle\\
&+ K_\phi(\dot c).
\end{aligned}$$

By substituting $\dot c = (a_\theta, b_\theta)$, $\ddot c = (a_{\theta\theta}, b_{\theta\theta})$ and $\nabla\phi^n = (\phi^n_x, \phi^n_y)$, $n \in \{1,2\}$, one obtains easily

$$\alpha_\theta\beta_{\theta\theta} - \beta_\theta\alpha_{\theta\theta} = (J_\phi\circ c)(a_\theta b_{\theta\theta} - b_\theta a_{\theta\theta}) + K_\phi(\dot c)$$

and therefore, by (16) and the homogeneity of $K_\phi$, we obtain that

$$\begin{aligned}
k_{\phi\circ c} =& \frac{(J_\phi\circ c)({a_\theta}^2 + {b_\theta}^2)^{3/2}k_c + K_\phi(\dot c)}{({\alpha_\theta}^2 + {\beta_\theta}^2)^{3/2}}\\
=& \left[\frac{{a_\theta}^2 + {b_\theta}^2}{{\alpha_\theta}^2 + {\beta_\theta}^2}\right]^{3/2}\left[(J_\phi\circ c)k_c + K_\phi\left(\frac{\dot c}{|\dot c|}\right)\right].
\end{aligned}$$

Since ${\alpha_\theta}^2 + {\beta_\theta}^2 = |d\phi(\dot{c})|^2$, formula (17) follows at once. q.e.d.

Let $h\colon (\theta_0, \theta_1) \times (t_0, t_1) \to U \subset \mathbb{R}^2$ be a $t$-family of $\theta$-curves such that $h$ is an orientation preserving diffeomorphism of class $C^m$, $m \geq 2$. Let $\mathcal{F}$ denote the foliation of $U$ by this family of curves; thus $\mathcal{F}$ is generated by the nonvanishing vector field $U \ni h(\theta, t) \longmapsto h_\theta(\theta, t) \in \mathbb{R}^2$. Let $\mathcal{F}^\perp$ be the foliation of $U$ by curves orthogonal (at each of their points) to the leaves of $\mathcal{F}$. Our goal is to obtain an expression for the curvature $k^\perp$ of $\mathcal{F}^\perp$ in terms of $h$. To do this we first construct suitable local parametrizations of $\mathcal{F}^\perp$. Let $(\theta', t') \in (\theta_0, \theta_1) \times (t_0, t_1)$ be fixed but otherwise arbitrarily chosen. We want to obtain a parametrization $h^\perp\colon (t'_0, t'_1) \times (\theta'_0, \theta'_1) \to U$ of $\mathcal{F}^\perp$ by a $\theta$-family of $t$-curves defined in a neighborhood $(t'_0, t'_1) \times (\theta'_0, \theta'_1)$ of $(t', \theta')$. We look for a differentiable function $\Theta\colon (t'_0, t'_1) \times (\theta'_0, \theta'_1) \to (\theta_0, \theta_1)$ with the property that for any $\theta \in (\theta'_0, \theta'_1)$ the correspondence $(t'_0, t'_1) \ni t \longmapsto h\big(\Theta(t, \theta), t\big)$ is a parametrization of the leaf of $\mathcal{F}^\perp$ through $h(\theta, t')$ subject to the initial value condition

$$\Theta(t', \theta) = \theta. \tag{18}$$

If we let $h^\perp\colon (t'_0, t'_1) \times (\theta'_0, \theta'_1) \to U$ be the map defined by

$$h^\perp(t, \theta) = h\big(\Theta(t, \theta), t\big) \tag{19}$$

then the orthogonality condition is

$$\Big\langle h_\theta\big(\Theta(t, \theta), t\big), h^\perp_t(t, \theta) \Big\rangle = 0 \tag{20}$$

for all $(t, \theta) \in (t'_0, t'_1) \times (\theta'_0, \theta'_1)$.

To simplify the notation we will use the following convention: If an expression contains maps pertaining to both foliations $\mathcal{F}$ and $\mathcal{F}^\perp$ then the maps pertaining to $\mathcal{F}^\perp$ (such as $h^\perp$ or $\Theta$) are computed at $(t, \theta)$, the maps pertaining to $\mathcal{F}$ (such as $h$ or $J_h$) are computed at $\big(\Theta(t, \theta), t\big)$ and it should be understood that the formula is valid for all $(t, \theta) \in (t'_0, t'_1) \times (\theta'_0, \theta'_1)$.

Differentiating (19) with respect to $t$ we obtain

$$h^\perp_t = \Theta_t h_\theta + h_t \tag{21}$$

and substituting this in (20) we obtain that $t \longmapsto \Theta(t, \theta)$ satisfies the first order differential equation

$$\Theta_t = -\frac{\langle h_\theta, h_t \rangle}{|h_\theta|^2} \tag{22}$$

with the initial value condition (18). By the theory of first order differential equations, there exists $[t'_0, t'_1] \times [\theta'_0, \theta'_1] \subset (t_0, t_1) \times (\theta_0, \theta_1)$ with $(t', \theta') \in (t'_0, t'_1) \times (\theta'_0, \theta'_1)$ such that (22) and (18) have a unique solution $\Theta$ defined on $[t'_0, t'_1] \times [\theta'_0, \theta'_1]$. Then the map $h^\perp$ defined by (19) satisfies (20) and therefore is a parametrization of $\mathcal{F}^\perp$.

If we now substitute (22) back in (21) we obtain

$$h_t^\perp = h_t - \left\langle h_t, \frac{h_\theta}{|h_\theta|} \right\rangle \frac{h_\theta}{|h_\theta|};$$

that is, $h_t^\perp$ is the component of $h_t$ on $ih_\theta$ (here multiplication of vectors by $i$ means counterclockwise rotation by $\frac{\pi}{2}$). Thus

$$h_t^\perp = \left\langle h_t, \frac{ih_\theta}{|h_\theta|} \right\rangle \frac{ih_\theta}{|h_\theta|}$$

and from the well-known formula

$$\big\langle (v_1, v_2), i(u_1, u_2) \big\rangle = \big\langle (v_1, v_2), (-u_2, u_1) \big\rangle = u_1 v_2 - u_2 v_1 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

we obtain

$$h_t^\perp = \frac{J_h}{|h_\theta|^2}\, ih_\theta, \tag{23}$$

where $J_h$ is the Jacobian of the map $h$.

**Lemma 2.** *The curvature $k^\perp$ of $\mathcal{F}^\perp$ is given by*

$$k^\perp \circ h = \frac{1}{|h_\theta|^3 J_h} \left[ |h_\theta|^2 \partial_\theta J_h - \left( \frac{1}{2} \partial_\theta |h_\theta|^2 \right) J_h \right]. \tag{24}$$

*Proof.* Let $h = (a, b)$ and $h^\perp = (\alpha, \beta)$. Then by (16)

$$k^\perp(h^\perp(t,\theta)) = \frac{{\alpha_t}^2}{\left({\alpha_t}^2 + {\beta_t}^2\right)^{3/2}} \frac{\partial}{\partial t} \left( \frac{\beta_t}{\alpha_t} \right) \Bigg|_{(t,\theta)}$$

and further by (20)

$$k^\perp \circ h^\perp = \frac{{\alpha_t}^2}{\left({\alpha_t}^2 + {\beta_t}^2\right)^{3/2}} \frac{\partial}{\partial t} \left( -\frac{a_\theta}{b_\theta} \right).$$

According to our convention, the function to be differentiated above is

$$\frac{a_\theta}{b_\theta} = \frac{a_\theta\big(\Theta(t,\theta),t\big)}{b_\theta\big(\Theta(t,\theta),t\big)}$$

and therefore

$$k^\perp = \frac{{\alpha_t}^2}{\left({\alpha_t}^2+{\beta_t}^2\right)^{3/2}}\left[-\frac{(a_{\theta\theta}\Theta_t + a_{\theta t})b_\theta - (b_{\theta\theta}\Theta_t + b_{\theta t})a_\theta}{{b_\theta}^2}\right]. \tag{25}$$

By (23)

$${\alpha_t}^2 = \frac{{J_h}^2}{|h_\theta|^4}{b_\theta}^2, \qquad {\alpha_t}^2 + {\beta_t}^2 = \frac{{J_h}^2}{|h_\theta|^2}$$

so that

$$\frac{{\alpha_t}^2}{\left({\alpha_t}^2+{\beta_t}^2\right)^{3/2}}\frac{1}{{b_\theta}^2} = \frac{{b_\theta}^2{J_h}^2/|h_\theta|^4}{{J_h}^3/|h_\theta|^3}\frac{1}{{b_\theta}^2} = \frac{1}{|h_\theta|J_h} \tag{26}$$

(here we have used $J_h > 0$). With a little calculation using (22), formula (25) becomes

(27)
$$k^\perp \circ h = \frac{1}{|h_\theta|^3 J_h}\Big[(b_\theta a_{\theta\theta} - a_\theta b_{\theta\theta})(a_\theta a_t + b_\theta b_t) + |h_\theta|^2(a_\theta b_{\theta t} - b_\theta a_{\theta t})\Big]$$

(we switched from $h^\perp$ to $h$ using (19) since the right hand side is expressed in terms of $h$ only). From $\partial_\theta J_h = a_\theta b_{\theta t} - b_\theta a_{\theta t} + a_{\theta\theta}b_t - b_{\theta\theta}a_t$ we have $a_\theta b_{\theta t} - b_\theta a_{\theta t} = \partial_\theta J_h + b_{\theta\theta}a_t - a_{\theta\theta}b_t$, which we substitute in (27) to obtain

$$\begin{aligned} k^\perp \circ h = \frac{1}{|h_\theta|^3 J_h}\Bigg[&|h_\theta|^2\partial_\theta J_h \\ &+ (b_\theta a_{\theta\theta} - a_\theta b_{\theta\theta})(a_\theta a_t + b_\theta b_t) + ({a_\theta}^2 + {b_\theta}^2)(b_{\theta\theta}a_t - a_{\theta\theta}b_t)\Bigg] \\ = \frac{1}{|h_\theta|^3 J_h}\Bigg[&|h_\theta|^2\partial_\theta J_h - (a_\theta a_{\theta\theta} + b_\theta b_{\theta\theta})(a_\theta b_t - b_\theta a_t)\Bigg], \end{aligned}$$

from which (24) follows at once. q.e.d.

## 5. Local dipoles are not principal foliations

This section is devoted to the proof of Theorem 1 and other immediate applications of the Main Theorem.

Let $f\colon S^2 \to \mathbb{R}^3$ be a $C^4$ immersion of the 2-sphere with only one umbilic at the south pole $\sigma$ and let $\mathcal{P}$ be either one of its principal foliations. The goal of this section is to show that $\mathcal{P}$ cannot be locally $C^2$ diffeomorphic at $\sigma$ to the standard dipole foliation on $S^2$. Let $X$ be a unit vector field on $S^2 \setminus \{\sigma\}$ tangent to $\mathcal{P}$.

Let us then assume, to the contrary, that there exists a diffeomorphism $\chi\colon V_1 \to U_1$ of class $C^2$ between two neighborhoods of $\sigma$ in $S^2$, such that $\chi(\sigma) = \sigma$ and the nowhere vanishing vector field $d\chi(X)$ on $U_1 \setminus \{\sigma\}$ is tangent to the standard dipole foliation. The stereographic projection $\Pi$ from the north pole onto the equatorial plane, maps the standard dipole foliation on $S^2$ onto the *standard plane dipole foliation* $\mathcal{D}$ which is the phase portrait of the vector field $D(z) = z^2$ and consists of circles centered on the imaginary axis and tangent to the real axis. Thus, the diffeomorphism $\Pi \circ \chi\colon V_1 \to U = \Pi(U_1)$ maps the principal foliation $\mathcal{P}$ onto the plane foliation $\mathcal{D}$. On the other hand $\mathcal{P}$ is mapped diffeomorphically through the Bonnet coordinates $B\colon V_1 \to V$, $B(\sigma) = o \in \mathbb{R}^2$, onto a singular Hessian foliation $\mathcal{F}$ (as was shown in §3) determined by the nowhere vanishing vector field $F = dB(X)\colon V \setminus \{o\} \to \mathbb{R}^2$.

Then our initial assumption of the preceeding paragraph that $\mathcal{P}$ is locally diffeomorphic to the standard dipole foliation on $S^2$ leads to the diffeomorphism $\phi\colon = B \circ (\Pi \circ \chi)^{-1}\colon U \to V$ with $\phi(o) = o$ and mapping the standard plane dipole foliation onto the singular Hessian foliation $\mathcal{F}$. It remains to show that the standard plane dipole foliation cannot be locally diffeomorphic to a singular Hessian foliation. The idea of the proof is to show that if $\mathcal{F}$ is locally diffeomeorphic to $\mathcal{D}$, then $\mathcal{F}$ has a biconvex lens and therefore, by the Main Theorem, cannot be a Hessian foliation.

The immersion $f$ being of class $C^4$, the Bonnet function $\omega$ is also of class $C^4$ (see §2) and, since $\phi$ is $C^2$, the foliation $\mathcal{F}$ is $C^2$. Thus the differentiability conditions in the Main Theorem are satisfied. The Hessian property of a foliation being invariant under orthogonal transformations we may assume that $\phi$ is orientation preserving.

Let us begin by considering the map $g\colon (0,\pi) \times (0,\infty) \to \mathbb{R}^2$ defined by

$$g(\theta, t) = \frac{1}{t} \sin\theta(\cos\theta, \sin\theta).$$

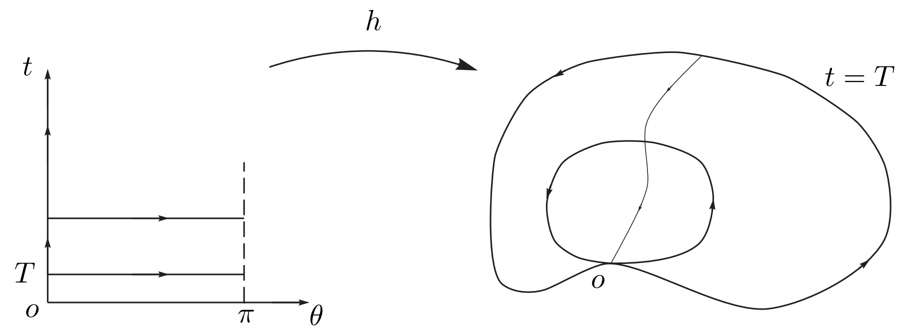


Figure 4: The parametrization of $E_T$

Thus $\theta \longmapsto g(\theta, t)$ is a parametrization of the circle of radius $\frac{1}{2t}$ centered at $(0, \frac{1}{2t})$ and therefore $g$ is a parametrization of the standard plane dipole foliation in the upper half-plane. Clearly, there exists $T > 0$ such that $g\Big((0, \pi) \times [T, \infty)\Big) \subset U$. Then $h = \phi \circ g \colon (0, \pi) \times [T, \infty) \to V$ is a parametrization of an elliptic sector $E_T$ of $\mathcal{F}$. That is $\theta \longmapsto h(\theta, t)$ is a parametrization of a leaf of $\mathcal{F}$ in $E_T$ for each $t \geq T$ and any point $w \neq o$ in $E_T$ is on one of the curves of the $t$-family $\{\theta \longmapsto h(\theta, t)\}_{t \geq T}$; the elliptic sector $E_T$ is clearly bounded by the leaf $\theta \longmapsto h(\theta, T)$ (see Figure 4).

A simple computation shows that the Jacobian $J_g$ of $g$ is given by

$$J_g(\theta, t) = \frac{1}{t^3} \sin^2 \theta$$

and therefore $h$ — whose Jacobian is $J_h = (J_\phi \circ g) J_g$ — establishes an orientation preserving diffeomorphism (of the same class of differentiability as $\phi$) between $(0, \pi) \times [T, \infty)$ and $E_T \setminus \{o\}$. Let $\mathcal{F}^\perp$ be the orthogonal foliation of $\mathcal{F}$. For $\theta \in (0, \pi)$ let $C^-(\theta)$ be the semitrajectory of $\mathcal{F}^\perp$ starting at $h(\theta, T)$. Thus $C^-(\theta)$ enters $E_T$ and approaches $o$. Let $C^+(\theta)$ be the semitrajectory of $\mathcal{F}$ ending at $h(\theta, T)$. Then the closed region $L_\theta$ bounded by $C^+(\theta)$ and $C^-(\theta)$ is a lens of $\mathcal{F}$ with vertex at $h(\theta, T)$ (see Figure 5).

By formula (17) of Lemma 1 the curvature $k$ of $\mathcal{F}$ on $E_T$ is given by

$$k \circ h = \left| d\phi\Big(\frac{g_\theta}{|g_\theta|}\Big) \right|^{-3} \left[ (J_\phi \circ g) k_g + K_\phi \left( \frac{g_\theta}{|g_\theta|} \right) \right] \tag{28}$$

where $k_g(\theta, t) = 2t$ is the curvature of the standard plane dipole foliation. Since $g_\theta / |g_\theta|$ is a unit vector field, the functions $K_\phi \left( g_\theta / |g_\theta| \right)$

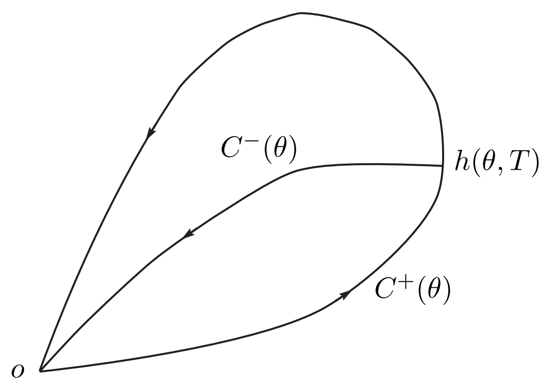


Figure 5: A lens of $\mathcal{F}$

and $\big|d\phi(g_\theta/|g_\theta|)\big|$ are bounded on $(0,\pi)\times[T,\infty)$, the latter being also bounded away from zero since $\phi$ is a diffeomorphism; the function $J_\phi \circ g$ is clearly bounded and positive since $\phi$ is orientation preserving. It is now clear that one can choose $T$ sufficiently large so that

$$k\big(h(\theta,t)\big) > 0 \tag{29}$$

for all $\theta \in (0,\pi)$ and all $t \geq T$.

On the other hand, by (24), the curvature $k^\perp$ of $\mathcal{F}^\perp$ is given by

$$k^\perp \circ h = \frac{1}{|h_\theta|^3}\left[\frac{\partial_\theta J_h}{J_h}|h_\theta|^2 - \left(\frac{1}{2}\partial_\theta |h_\theta|^2\right)\right]. \tag{30}$$

From $h_\theta = d\phi(g_\theta)$, a straightforward computation gives

$$h_{\theta\theta} = \mathrm{Hess}[\phi](g_\theta, g_\theta) + d\phi(g_{\theta\theta})$$

where $\mathrm{Hess}[\phi]\colon U \to L(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ is defined as $\mathrm{Hess}[\phi] = (\mathrm{Hess}[\phi^1], \mathrm{Hess}[\phi^2])$. From

$$g_\theta(\theta,t) = \frac{1}{t}(\cos 2\theta, \sin 2\theta) \qquad \text{and} \qquad g_{\theta\theta}(\theta,t) = \frac{1}{t}(-2\sin 2\theta, 2\cos 2\theta)$$

we obtain $|g_\theta(\theta,t)| = (1/t)$ and $|g_{\theta\theta}(\theta,t)| = (2/t)$ and it is then clear that the function $\theta \longmapsto |h_\theta(\theta,T)|$ is bounded and bounded away from zero on $(0,\pi)$ and that the function $\theta \longmapsto \left(\partial_\theta |h_\theta|^2\right)(\theta,T) = 2\langle h_\theta, h_{\theta\theta}\rangle(\theta,T)$ is bounded.

We will now show that the first term inside the brackets in (30) tends to $+\infty$ as $\theta \longrightarrow 0$. Since $J_g(\theta,t) = (1/t^3)\sin^2\theta$ we have

$$J_h = \frac{1}{t^3}\sin^2\theta\,(J_\phi \circ g)$$

and

$$\partial_\theta J_h = \frac{1}{t^3}\Big[\sin 2\theta\,(J_\phi \circ g) + \sin^2\theta\,\Big(dJ_\phi(g_\theta)\Big)\Big].$$

Clearly

$$\lim_{\theta\searrow 0}\frac{(\partial_\theta J_h)(\theta,T)}{\theta} = \frac{2}{T^3}\,J_\phi(o)$$

and

$$\lim_{\theta\searrow 0}\frac{J_h(\theta,T)}{\theta^2} = \frac{1}{T^3}\,J_\phi(o)$$

so that

$$\lim_{\theta\searrow 0}\theta\,\frac{(\partial_\theta J_h)(\theta,T)}{J_h(\theta,T)} = 2.$$

Thus, for any $M > 0$ there exists $\theta_1 = \theta_1(M) \in (0,\pi)$ such that

$$\frac{(\partial_\theta J_h)(\theta,T)}{J_h(\theta,T)} > M$$

for all $\theta \in (0,\theta_1)$. We have therefore established that there exists $\theta_1 \in (0,\pi)$ such that

$$k^\perp(h(\theta,T)) > 0 \tag{31}$$

for all $\theta \in (0,\theta_1)$.

From (29) and (31) we may now finally conclude that for $\theta \in (0,\theta_1)$ the lens $L_\theta$ with vertex at $h(\theta,T)$ is a biconvex lens of $\mathcal{F}$ and therefore, by the Main Theorem, $\mathcal{F}$ cannot be a singular Hessian foliation. The proof of Theorem 1 is now complete.

**Remark 1.** Let $\mathcal{F}$ be a $C^1$ foliation on $S^2$ with only one singularity at the south pole $\sigma$. By a simple application of the classical theory of dynamical systems ([1], [9]) one obtains that any trajectory $\tau$ of $\mathcal{F}$ must close up at $\sigma$ (that is, both ends of $\tau$ tend to $\sigma$). We say that the trajectories of $\mathcal{F}$ are *nested* if the elliptic sectors within any elliptic sector are ordered by inclusion.

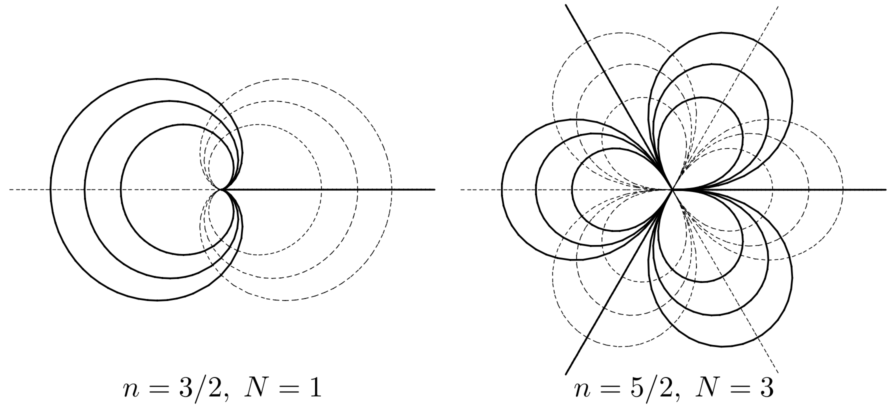


Figure 6: Standard plane $N$-poles.

**Theorem 2.** *Let $f\colon S^2 \to \mathbb{R}^3$ be a smooth immersion of the 2-sphere with only one umbilic at the south pole $\sigma$ and let $\mathcal{P}$ be either one of its principal foliations. If the trajectories of $\mathcal{P}$ are nested then $\mathcal{P}$ is homeomorphic to the standard dipole foliation but can never be diffeomorphic to it.*

The proof of this theorem follows immediately from our Theorem 1 once we show that the principal foliation $\mathcal{P}$ is necessarily homeomorphic to the standard dipole foliation. But this is a simple consequence of the classical topological classification results for smooth foliations on plane or spherical regions (see [1], [9]).

**Remark 2.** For $n \in \frac{1}{2}\mathbb{Z}, n > 1$, let $\mathcal{E}_n$ be the foliation determined by the line field $E_n\colon \mathbb{C}^* \to \mathbb{RP}^1, E_n(z) = l_{z^n}$. Its phase portrait consists of $N = 2n - 2$ elliptic sectors separated by the rays $\theta = m\pi/(n-1)$, $m \in \{0, 1, \ldots, 2n-3\}$. $\mathcal{E}_n$ is the simplest foliation with an isolated singularity at $o$ of index $n$ and is called the *standard plane $N$-pole foliation* (see Figure 6). The orthogonal foliation $\mathcal{E}_n^\perp$ is determined by the line field $E_n^\perp(z) = l_{iz^n}$ and its phase portrait is the phase portrait of $\mathcal{E}_n$ rotated by $-\frac{\pi}{2n-2}$. Using (12) of §3 one obtains at once that the curvatures $k_n$ and $k_n^\perp$ of $\mathcal{E}_n$ and $\mathcal{E}_n^\perp$ respectively are given by

$$k_n(re^{i\theta}) = \frac{n}{r}\sin(n-1)\theta \qquad \text{and} \qquad k_n^\perp(re^{i\theta}) = \frac{n}{r}\cos(n-1)\theta.$$

It is now easy to see that for any $z_0$ with $0 < \arg z_0 < \frac{\pi}{2n-2}$ the lens of $\mathcal{E}_n$ with vertex at $z_0$ is a biconvex lens and so, by the Main Theorem, $\mathcal{E}_n$ cannot be a singular Hessian foliation.

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