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# The local image problem for complex analytic maps

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**Abstract.** We address the question "when the local image of a map is well defined" and answer it in case of holomorphic map germs with target $(\mathbb{C}^2, 0)$. We prove a criterion for holomorphic map germs $(X, x) \to (Y, y)$ to be locally open, solving a conjecture by Huckleberry in all dimensions.

## 1. Introduction

Let $F:(X,x)\to(Y,y)$ be a non-constant complex analytic map germ between complex analytic set germs of pure dimensions. Consider an embedding $(X,x)\subset(\mathbb{C}^N,0)$ and the intersections of some representative $X$ with arbitrarily small balls $B_\varepsilon\subset\mathbb{C}^N$ centred at 0. We say that $F$ has a well defined *local image* if and only if the germ at $y$ of the set $F(B_\varepsilon\cap X)$ is independent of the small enough $\varepsilon>0$. This extends the notion of *locally open maps*. The map germ $F:(X,x)\to(Y,y)$ is said to be *open at* $y$ (or that it is locally open, see e.g. [Hu]) if for every open neighbourhood $U$ of $x$, $F(U)$ has $y$ as an interior point. Therefore "locally open" trivially implies "well defined local image".

For instance in case of a non-constant holomorphic function germ $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$, the Open Mapping Theorem tells that $f(U)$ is open. Therefore the equality of set germs $(\operatorname{Im} f, 0)=(\mathbb{C},0)$ holds, thus $f$ is locally open, and in particular the function germ $f$ has a well-defined image as a set-germ.

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If the target of a holomorphic map $F$ has higher dimension, then $F$ may still have a well defined local image, for instance if $F$ is a *proper map at* $y$, by Remmert's Proper Mapping Theorem [GR1] and [Re2]. Nevertheless, without the properness, map germs as simple as $G:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, $(x,y)\mapsto(x,xy)$, do not have a well defined local image. In this example, the image by $G$ of any small ball $B_\varepsilon$ centred at the origin is a subanalytic set (but not analytic) which depends on the radius $\varepsilon$ so radically that the set germs $(G(B_\varepsilon),0)$ and $(G(B_{\varepsilon'}),0)$ are different if $\varepsilon\neq\varepsilon'$.

We address the following question:

**The image problem.** *Under what conditions the image of a complex analytic map germ is a well defined set germ?*

We ask the local image to be a set germ, and not necessarily to be analytic. It is another long standing question under what conditions the local image of a holomorphic map is an analytic variety, see for instance [Hu, p. 447] for some comments. If the image is a well defined set germ then it must be a subanalytic set germ (see Note 3.6).

Beyond the case of holomorphic functions germs $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ evoked before, one encounters locally open images, according to Hamm's result [Ha], also in case of complex map germs $F:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ which define an *isolated complete intersection singularity* $(X,0)\subset(\mathbb{C}^n,0)$, where $X$ is the zero locus $Z(F)$ of $F$. Moreover, it turns out that the image by $F$ of the germ of the singular locus $(\operatorname{Sing} F,0)$ is an analytic set germ, and more precisely a hypersurface germ. It then follows, cf. [Ha] and [Lo], that there exists a fibration over the germ at 0 of the complement $\mathbb{C}^p\setminus F(\operatorname{Sing} F)$, and on these bases one was able to further study the topology of the Milnor fibration in relation to algebraic invariants([1]) of the isolated complete intersection $X$. A far-reaching extension of local fibrations to map germs $F$ such that their zero locus $Z(F)$ has *nonisolated singularities* is done in [JT].

While the image problem remains widely open in general (and with almost no hope), we provide here the following classification of holomorphic map germs $(f,g):(\mathbb{C}^n,0)\to(\mathbb{C}^2,0)$:

**Theorem 1.1.** *Let* $(f,g):(\mathbb{C}^n,0)\to(\mathbb{C}^2,0)$ *be a non-constant holomorphic map germ. Then*:

(i) *The image of* $(f,g)$ *is locally open, i.e.* $(\operatorname{Im}(f,g),0)=(\mathbb{C}^2,0)$, *if and only if:*

(a) $\dim Z(f)\cap Z(g)=n-2$, *or*

(b) $\dim Z(f)\cap Z(g)=n-1$ *and* $Z(f)\not\subset Z(g)$, $Z(g)\not\subset Z(f)$, *and* $\operatorname{Im}(f,g)$ *is well defined as a set germ at* 0.

([1]) The well known monograph [Lo], recently re-edited, contains some of the most significant results on this rich topic.

(ii) *In case* $Z(g)\subset Z(f)$ *or* $Z(f)\subset Z(g)$*, the map* $(f,g)$ *has a well defined local image if and only if* $(\mathrm{Im}\,(f,g),0)$ *is an irreducible plane curve germ, and this is equivalent to* $\mathrm{Jac}(f,g)\equiv 0$*.*

All situations described in Theorem 1.1 can be realised, and we discuss some examples in §2.3 and §2.4. The case (b) of Theorem 1.1 leaves us with the problem of how to decide whether the image of $(f,g)$ is well defined as a set germ (and thus it is locally open), which we will solve as follows. We first produce a handy test for the local openness of the image in terms of *gap lines*, Proposition 3.3. Although this is a sufficient condition only, it leads to the notion of *gap curves* which is central in our Theorem 3.5, a fully general result that sounds as follows:

*Let* $F:(X,a)\to(Y,b)$, $\dim X\geq\dim Y\geq 1$ *be a holomorphic map germ between two germs of reduced, locally irreducible complex spaces. Then the image of* $F$ *is open at* $b\in Y$ *if and only if* $F$ *has no gap curve.*

This provides an equivalence criterion for a holomorphic map germ $F:(X,a)\to(Y,b)$ to be locally open, and thus completes the solution to our problem. Moreover, this criterion allows us to answer a much older question, as follows.

About 50 years ago, Huckleberry addressed in [Hu] the problem of locally open maps, which is a particular case of the Image Problem stated above (see the discussion at the beginning of this Introduction). Huckleberry introduced the notion of a *subflat* map (an algebraic condition, see Definition 3.10) and proved that a holomorphic map germ $(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ is open if and only if it is subflat. He conjectured that this characterisation holds for any holomorphic map germ $(X,x)\to(Y,y)$, in arbitrary dimensions.

We derive from Theorem 3.5 the following positive answer to Huckleberry's conjecture [Hu]:

**Theorem 1.2.** *Let* $F:(X,a)\to(Y,b)$, $\dim X\geq\dim Y\geq 1$, *be a holomorphic map germ between two germs of reduced locally irreducible complex spaces, and such that* $\mathrm{Sing}\,F\neq X$.

*Then* $(\mathrm{Im}\,F,b)=(Y,b)$ *if and only if* $F$ *is subflat.*

We end this introduction by a remark on possible further developments. One of the anonymous referee's questions, to whom we thank for the constructive criticism, advice, and for the interesting remarks, has been "why is it difficult to address the image problem in case of a higher dimensional target", as we have written "no hope in the general case". Indeed, the classification provided by Theorem 1.1 does not extend by similar methods. Nevertheless, in the subsequent paper [JT] which aims to finding necessary conditions for the existence of fibrations of map germs, we ought to study the image problem from a different angle, by enlarging it to the discriminant. More precisely, we ask not only that the image of the map germ is a well defined set germ, but also that the image of its singular locus is so, since these are preliminary conditions for the existence of a local fibration. We have shown for instance the following result, which extends in particular Theorem 1.1(i)(a) to a higher dimensional target, cf. [JT, Proposition 2.3]: *Let* $F\colon(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, $n\geq p\geq 2$, *be a holomorphic map germ. If the fibre* $F^{-1}(0)$ *has dimension* $n-p$ *then* $(\operatorname{Im} F,0)=(\mathbb{C}^p,0)$. *If moreover* $\operatorname{Sing} F\cap F^{-1}(0)=\{0\}$ *then the image by* $F$ *of the singular set* $\operatorname{Sing} F$ *is a well-defined set germ.*

## 2. The image of the map germ $(f,g)$

Our proof of Theorem 1.1(i) will use the proof of (ii), therefore we start with the later.

### 2.1. Proof of Theorem 1.1(ii)

If the image of $(f,g)$ is a curve (necessarily irreducible) then it is a germ by definition. In fact, if $\operatorname{Im}(f,g)$ is only included in a complex curve, and $\operatorname{Im}(f,g)\neq\{(0,0)\}$, then $\operatorname{Im}(f,g)$ must be the whole irreducible curve germ, as an application of the Open Mapping Theorem.

If $Z(g)\subset Z(f)$ then $\operatorname{Im}(f,g)$ cannot be $(\mathbb{C}^2,0)$ since the axis $\mathbb{C}\times\{0\}$ is missing from the image. By the next result, if $\operatorname{Im}(f,g)$ is a well defined set germ then it must be an irreducible curve germ.

**Proposition 2.1.** *Let* $(f,g)\colon(\mathbb{C}^n,0)\to(\mathbb{C}^2,0)$ *be non-constant holomorphic map germ.*

*If* $\operatorname{Im}(f,g)$ *is a well defined set germ at the origin, then either* $(\operatorname{Im}(f,g),0)=(\mathbb{C}^2,0)$ *or* $(\operatorname{Im}(f,g),0)=(C,0)$ *where* $(C,0)\subset(\mathbb{C}^2,0)$ *is an irreducible complex curve germ.*

To prove this proposition we need the following:

**Lemma 2.2.** *Suppose that* $X$ *is a complex space and* $F\colon X\to\mathbb{C}^2$ *is a holomorphic map. Then* $\operatorname{Im} F$ *can be written as a disjoint union* $U\sqcup A$, *where* $U$ *is a (possibly empty) open subset of* $\mathbb{C}^2$ *and* $A$ *has* 3*-Hausdorff measure equal to* 0.

*Proof.* We stratify $X$ by the rank of $F$ and write $X=M_2\sqcup M_1\sqcup M_0$, where the $M_j$'s are complex manifolds, such that $F_{|M_j}$ has rank $j$. Note that $M_j$ is not necessarily connected and its connected components may not have the same dimension. Then $F(M_2)$ is open and $F(M_1\cup M_0)$ is a countable union of complex curves or points and hence has 3-Hausdorff measure equal to 0. The subsets $U:=F(M_2)$ and $A:=F(X)\setminus U\subset F(M_1\cup M_0)$ verify our claim. □

*Proof of Proposition 2.1.* Let $B_\varepsilon\subset\mathbb{C}^n$ be an open ball centred at the origin of radius $\varepsilon$ such that $(f,g)$ is holomorphic on $B_\varepsilon$. By our hypothesis, for all $0<\varepsilon'<\varepsilon$ we have the equality of germs $((f,g)(B_\varepsilon),0)=((f,g)(B_{\varepsilon'}),0)$.

We fix $0<\varepsilon'<\varepsilon$ and $r>0$ such that $(f,g)(B_{\varepsilon'})\cap D_r=(f,g)(B_\varepsilon)\cap D_r$ where $D_r\subset\mathbb{C}^2$ denotes the open ball centred at 0 and of radius $r$. Since

$$(f,g)(B_{\varepsilon'})\cap D_r\subset(f,g)(\overline{B_{\varepsilon'}})\cap D_r\subset(f,g)(B_\varepsilon)\cap D_r$$

we get the equality $(f,g)(\overline{B_{\varepsilon'}})\cap D_r=(f,g)(B_\varepsilon)\cap D_r$.

By the above equality, since $K:=(f,g)(\overline{B_{\varepsilon'}})$ is a compact subset of $\mathbb{C}^2$, the image $(f,g)(B_\varepsilon)\cap D_r=K\cap D_r$ is closed in $D_r$.

By Lemma 2.2 we have that $K\cap D_r$ is equal to a disjoint union $U\sqcup A$ of an open subset $U\subset D_r$ and a subset $A\subset D_r$ which has 3-Hausdorff measure equal to zero.

This implies:

$$D_r\setminus A=U\sqcup(D_r\setminus K) \tag{1}$$

which implies in turn that $A$ is closed in $D_r$. Since $A$ has 3-Hausdorff measure equal to zero, it follows that $D_r\setminus A$ is connected, see e.g. [Ch, Proposition 6, p. 347].

We distinguish two cases, where the notation $\operatorname{int}(A)$ is for the interior of the set $A$.

**Case 1.** $0\in\overline{\operatorname{int}((f,g)(B_\varepsilon))}$.

Our assumption implies that $0\in\partial\overline{U}$, hence $U\neq\varnothing$. Then the disjoint union decomposition (1) of the connected set $D_r\setminus A$ into open sets shows that $D_r\setminus K=\varnothing$. Therefore we have the equality $K\cap D_r=D_r$, which shows that the image $(f,g)(\overline{B_{\varepsilon'}})$ contains the ball $D_r$. We conclude that $(\operatorname{Im}(f,g),0)=(\mathbb{C}^2,0)$ in this case.

**Case 2.** $0\notin\overline{\operatorname{int}((f,g)(B_\varepsilon))}$.

By shrinking $B_\varepsilon$ we may assume that $\operatorname{int}(f,g)(B_\varepsilon)=\varnothing$. This implies that all non-empty fibres $(f,g)^{-1}((f,g)(p))$ arbitrarily close to $(f,g)^{-1}(0)$ have pure dimension $n-1$. Indeed, if there is a sequence of points $x_i\to 0$ in the source, such that the local dimension of the fibre $(f,g)^{-1}((f,g)(x_i))$ is $n-2$ then we use the reasoning in the proof of Theorem 1.1(a) which shows $(f,g)(x_i)$ is an interior point of the image for any $i$, and thus $0\in\overline{\operatorname{int}((f,g)(B_\varepsilon))}$, which contradicts the hypothesis.

In order to finish the proof we need the following result, which is also called "Remmert's Rank Theorem" by Łojasiewicz in [Łoj1, Theorem 1, p. 295], see also [Na, Proposition 3, Chapter VII]:

**Proposition 2.3.** ([Re1, Satz 18, p. 30]) *Let $X$ and $Y$ be complex spaces such that $X$ is pure dimensional and $f:X\to Y$ be a holomorphic map. If $r=$*

$\dim_x f^{-1}(f(x))$ *is independent of* $x\in X$ *then any point* $a\in X$ *has a fundamental system of neighbourhoods* $\{U_i\}$ *such that* $f(U_i)$ *is analytic at* $f(a)$, *of dimension* $\dim X-r$.

Since all non-empty fibres of $(f,g)$ have dimension $n-1$, it follows from Proposition 2.3 that there exists a connected neighbourhood $W\subset\mathbb{C}^n$ of 0 such that $(f,g)(W)$ is a complex analytic subset of $\mathbb{C}^2$, thus a complex curve $C$ containing the origin, by Proposition 2.1 and since in our case the image cannot be $(\mathbb{C}^2,0)$. Since $W$ is connected, we get that $C$ is also irreducible, thus we get the equality $(\operatorname{Im}(f,g),0)=(C,0)$.

This ends the proof of Proposition 2.1, and thus of the first equivalence of Theorem 1.1(ii). □

Let us finally prove the second equivalence claimed by Theorem 1.1(ii), namely: $(\operatorname{Im}(f,g),0)$ is a well-defined curve germ $\Leftrightarrow$ $\operatorname{Jac}(f,g)\equiv 0$.

If $\operatorname{Im}(f,g)$ is a curve then the map $(f,g)$ cannot be a submersion at $x$, for any point $x$ in the neighbourhood of 0, hence $\operatorname{Jac}(f,g)\equiv 0$. It remains to prove the converse. The hypothesis $\operatorname{Jac}(f,g)\equiv 0$ implies $\operatorname{rank}_x(f,g)\le 1$, for any $x$ in the neighbourhood of 0. Since $\operatorname{rank}(f,g)\equiv 0$ implies $(f,g)\equiv 0$, we are left with the case $\operatorname{rank}(f,g)\not\equiv 0$.

If $B$ denotes a small enough open ball centred at $0\in\mathbb{C}^n$, then $B_0:=\{x\in B\mid \operatorname{rank}_x(f,g)>0\}$ is an open, connected and dense analytic subset of $B$. As we have just seen above, we actually have $\operatorname{rank}_x(f,g)=1$, $\forall x\in B_0$. By the rank theorem we deduce that $\dim_x(f,g)^{-1}(f(x),g(x))=n-1$ for all $x\in B_0$. The next result on the semi-continuity of the dimension of fibres is useful in order to figure out what happens at points in $B\setminus B_0$, see also [Na, p. 66]:

**Lemma 2.4.** ([Re1, Satz 16]) *Let* $F\colon X\to Y$ *be a holomorphic map between complex spaces. Then every* $a\in X$ *has a neighbourhood* $U\subset X$ *such that*

$$\dim_x F^{-1}(F(x))\le \dim_a F^{-1}(F(a))$$

*for any* $x\in U$.

Since $B_0$ is dense in $B$, the above lemma tells that in our case, for any $x\in B$ one has at least the inequality $\dim_x(f,g)^{-1}(f(x),g(x))\ge n-1$. However, this inequality cannot be strict (i.e. not even at points in $B\setminus B_0$) since the converse inequality $\dim_x(f,g)^{-1}(f(x),g(x))\le n-1$ necessarily holds, as we have shown in the first part of the above proof.

And now, since $\dim_x(f,g)^{-1}(f(x),g(x))=n-1$ for all $x\in B$, our claim follows from Proposition 2.3 applied to the point $0\in B$.

This ends the proof of Theorem 1.1(ii).

## 2.2. Proof of Theorem 1.1(i)

Let us prove the implication "⇒". The dimension $\dim Z(f)\cap Z(g)$ may be either $n-2$ or $n-1$. If this dimension is $n-1$ and $Z(f)\subset Z(g)$ or $Z(g)\subset Z(f)$ then, by Theorem 1.1(ii) proved above, the map germ $(f,g)$ cannot be open at 0. Therefore, if $(\operatorname{Im}(f,g),0)=(\mathbb{C}^2,0)$, then one must have either $\dim Z(f)\cap Z(g)=n-2$, or $\dim Z(f)\cap Z(g)=n-1$ and $Z(f)\not\subset Z(g)$ and $Z(g)\not\subset Z(f)$.

To show the converse implication "⇐", we first consider the case (i)(a). Let $(H,0)\subset(\mathbb{C}^n,0)$ be a general complex 2-plane germ such that 0 is an isolated point of $H\cap(f,g)^{-1}(0)$. It then follows, e.g. by [GR2, Proposition, p. 63], that there exist an open ball $B$ at 0 in $\mathbb{C}^n$ and an open neighbourhood $U$ of the origin in $\mathbb{C}^2$ such that $(f,g)(H\cap B)\subset U$ and the induced map $(f,g)_{|H\cap B}:H\cap B\to U$ is finite. By the Open Mapping Theorem (cf. [GR2, p. 107]), this implies that $(f,g)(H\cap B)$ is open, thus $(\operatorname{Im}(f,g),0)=(\mathbb{C}^2,0)$.

We continue with the proof of the converse implication "⇐" in the case (i)(b). If the image $\operatorname{Im}(f,g)$ is a set germ then by Proposition 2.1 it is either a curve germ or $(\mathbb{C}^2,0)$. Let us show that it cannot be a curve. Indeed, if it is a curve $(C,0)$ different from the axes, then this has a Puiseux parametrisation, say $(h(t),t^\gamma)$, for some holomorphic function $h$ with $\operatorname{ord}_0 h>0$, and some positive integer $\gamma$. Then $g(x)=0$ implies that $t=0$, thus $f(x)=0$, which means $Z(g)\subset Z(f)$, and this contradicts the hypothesis. If the curve $(C,0)$ is one of the axes, then we immediately get the same contradiction (for instance, if the axis is $\mathbb{C}\times\{0\}$, then this implies $Z(f)\subset Z(g)$).

## 2.3. Examples where the image of $(f,g)$ is not a set germ

Let $G:\mathbb{C}^2\to\mathbb{C}^2$, $G(x,y)=(x,xy)$. The global image of this algebraic map is the semi-algebraic set $(\mathbb{C}^2\setminus\{x=0\})\cup\{(0,0)\}$, but since $\operatorname{Jac}G=x$ (thus not identically 0), Theorem 1.1(ii) tells that the image of the map germ $G:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ is not well defined as a set germ. Actually, for the 2-disks $D_t:=\{|x|<t,|y|<t\}$ as basis of open neighbourhoods of 0 for $t>0$, the image $A_t:=G(D_t)$ is the open angle of vertex 0, having the horizontal axis as bisector, and of slope $t$.

Let $F:\mathbb{C}^2\to\mathbb{C}^2$, $F(x,y)=(x(x+y),xy)$. The global image of this algebraic map is $(\mathbb{C}^2\setminus\Delta)\cup\{(0,0)\}$, where $\Delta$ denotes the diagonal line.

Let us see that the image of the map germ $F$ is not a set germ at $(0,0)$. The images of the line segments $x+(1-\alpha)y=0$ inside a small ball $B_\varepsilon$ at the origin are line segments centred at 0 in the target. When $\alpha$ tends to 1, the image segments tend to the diagonal $\Delta$, and their lengths tend to 0. It follows that the images by $F$ of small balls intersected with arbitrarily small balls $D_\delta$ in the target are different, namely $F(B_{\varepsilon_1})\cap D_\delta\neq F(B_{\varepsilon_2})\cap D_\delta$.

The property "the image of $G$ is well defined as a set germ" being invariant under change of coordinates in the source or in the target, we consider the linear change of coordinates $(a,b)\mapsto(a-b,b)$ in the target. The resulting map germ $G$: $(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, $G(x,y)=(x^2,xy)$ has one single *gap line* $\{x=0\}$ (see §3.1), and the image $\operatorname{Im} G$ is not well defined as set germ, by Theorem 1.1(ii), since $\operatorname{Jac}F\not\equiv 0$.

### 2.4. Examples where the image of $(f,g)$ is a set germ

Let $F\colon(\mathbb{C}^3,0)\to(\mathbb{C}^2,0)$, $F(x,y,z)=(xy,xz)$. This satisfies the hypothesis of Theorem 1.1(i)(b). The fibers are not equidimensional, i.e. all fibres are curves except of the one over the origin which contains the plane $\{x=0\}=\operatorname{Sing} F$. However, the image of any open ball $B_\varepsilon\subset\mathbb{C}^3$ centred at 0 contains the open ball $B_{\varepsilon^2}\subset\mathbb{C}^2$ centred at 0, thus the image of the map germ $F$ is a germ and $(\operatorname{Im} F,0)=(\mathbb{C}^2,0)$.

Another example for Theorem 1.1(i)(b) is $F\colon(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, $F(x,y)=(x(y+x^2),y(y+x^2))$. It is not trivial but still not difficult to show that $(\operatorname{Im} F,0)=(\mathbb{C}^2,0)$. In §3.1 we give a sufficient criterion (Proposition 3.3) for the image to be $(\mathbb{C}^2,0)$ which is easily verified by this example, cf. Example 3.7.

## 3. When is the image of a map germ locally open?

We first give a sufficient test for the image of $(f,g)$ to be locally open which is handy (see Example 3.7), and show its limits. Next we prove an "if and only if" criterium in full generality, that is for a holomorphic map germ $F\colon(X,a)\to(Y,b)$.

### 3.1. Gap lines

In order to test the "locally open" possibility in Theorem 1.1(i)(b) we introduce the notion of "gap line", which loosely speaking means a line germ in the target which does not contain points of the image except of $0\in\mathbb{C}^2$.

*Definition* 3.1. Suppose that $\dim Z(f)\cap Z(g)=n-1$ and $Z(f)\not\subset Z(g)$, $Z(g)\not\subset Z(f)$. Let $f=h\hat{f}$, $g=h\hat{g}$, where $h=\gcd(f,g)\in\mathfrak{m}_n$, up to invertible elements in $\mathcal{O}_n$, and such that $\hat{f},\hat{g}\in\mathfrak{m}_n$. We say that $Z(\beta x+\alpha y)\subset\mathbb{C}^2$, for some $[\alpha:\beta]\in\mathbb{P}^1$, is a *gap line* for $(f,g)$ if the analytic set germ $(Z(\beta\hat{f}+\alpha\hat{g}),0)$ is included in the fibre $(f,g)^{-1}(0,0)$.

*Remark* 3.2. For a given map germ $(f,g)$ there are at most finitely many gap lines. Indeed, since $\hat{f}$ and $\hat{g}$ are co-prime, for a gap line we must have the inclusion $Z(\beta\hat{f}+\alpha\hat{g})\subset Z(h)$. As $Z(h)$ has finitely many irreducible components, our conclusion follows.

**Proposition 3.3.** *Let* $f=h\hat{f}$, $g=h\hat{g}$, *where* $h=\gcd(f,g)\in\mathfrak{m}_n$, *and such that* $\hat{f}$ *and* $\hat{g}$ *are not units. If* $\dim Z(h)\cap Z(\beta\hat{f}+\alpha\hat{g})=n-2$ *for any* $[\alpha:\beta]\in\mathbb{P}^1$, *then* $(\mathrm{Im}\,(f,g),0)=(\mathbb{C}^2,0)$.

The above result is not an equivalence, although it looks close to that; see Example 3.9. The equality $(\mathrm{Im}\,(f,g),0)=(\mathbb{C}^2,0)$ implies that there are no gap lines (trivially), but the converse is not true, see Example 3.8.

The proof of Proposition 3.3 will be given after the next §3.2 where we obtain a desired equivalence, and moreover in the most general setting. But there is a price to pay.

### 3.2. The gap variety, and a general criterion for the existence of locally open image

We consider here a holomorphic map germ $F:(X,a)\to(Y,b)$ between two germs of reduced, locally irreducible complex spaces with $\dim X\geq\dim Y\geq 1$.

*Definition* 3.4. Let $(V,b)\subset(Y,b)$ be a complex analytic germ of positive dimension. We say that $(V,b)$ is a *gap variety* for $F$ if the inclusion of analytic set germs $(F^{-1}(V),a)\subset(F^{-1}(b),a)$ holds.

Let us remark that Huckleberry [Hu] had used the same concept under the name "$F$ omits $V$". Our general criterion is the following:

**Theorem 3.5.** *Let* $F:(X,a)\to(Y,b)$, $\dim X\geq\dim Y\geq 1$ *be a holomorphic map germ between two germs of reduced, locally irreducible complex spaces. Then the image of* $F$ *is open at* $b\in Y$ *if and only if* $F$ *has no gap curve.*

In case $\dim_a X<\dim_b Y$, the image of $F$ cannot be open, and Theorem 3.5 tells that $F$ must have gap curves. We do not assume anything about the singular locus of $F$. For instance if $\mathrm{Sing}\,F=X$ then the image of $F$ cannot be dense in the neighbourhood of 0, and, once again, Proposition 3.5 implies that there are gap curves.

Let us remind a useful tool in the realm of subanalytic sets that will be employed in the proof.

*Note* 3.6. The Curve Selection Lemma was first proved for semi-analytic sets, see e.g. [BC] and [Mi]. We shall need it in the *subanalytic* setting. The subanalytic concept has been introduced because the image by a real analytic map of a semi-analytic set is not semi-analytic in general. A subset $X$ of a real analytic manifold $M$ is called *subanalytic* if for each point $x\in M$ there exists a neighbourhood $U\subset X$ of $x$, a real analytic manifold $N$, and a relatively compact semi-analytic subset $A$ of $M\times N$ such that $U$ is the image of $A$ by the projection $M\times N\to M$. This category is closed under taking closures, interiors, complements, finite intersections, finite unions. Moreover, the image of a relatively compact subanalytic set by an analytic map is subanalytic. For more details, see e.g. [BM], [Ga], [Hi] and [Łoj2].

In the setting of subanalytic sets, the Curve Selection Lemma is due to Hironaka [Hi, Proposition 3.9, p. 482]: *Let* $X \subset M$ *be a subanalytic set and let* $b \in \partial X$. *There exists a real analytic function* $\gamma : ]-\varepsilon, \varepsilon[ \to M$ *such that* $\gamma(0) = b$ *and that* $\gamma(]0, \varepsilon[) \subset X$.

### 3.3. Proof of Theorem 3.5.

"$\Rightarrow$" is obvious, so let us show "$\Leftarrow$".

We assume that $F$ has no gap curve, and we want to show that the image of $F$ is open at $b \in Y$. The proof is by contradiction. Suppose that the image of our holomorphic map $F$ is not open at $b \in Y$. Then there exists some open ball $B$ centred at $a$ such that $b \in \partial(Y \setminus F(B))$, where $Y$ denotes here some representative of the analytic set germ $(Y, b)$. Since $Y \setminus F(B)$ is a subanalytic set (as being the complement of the image of a relatively compact analytic set, see Note 3.6 and its references), the above cited Curve Selection Lemma says that there exists a real analytic function $\gamma : ]-\varepsilon, \varepsilon[ \to Y$ such that $\gamma(0) = b$ and that $\gamma(]0, \varepsilon[) \subset Y \setminus F(B)$.

We then consider the complexification at 0 of the image $\gamma(]-\varepsilon, \varepsilon[)$, i.e. the (unique) irreducible complex analytic curve $C$ containing $\gamma(]-\varepsilon', \varepsilon'[)$, for some positive $\varepsilon' \leq \varepsilon$.

We claim that $C$ is a gap curve for $F$. If so, then this yields a contradiction to our assumption, so the proof of Theorem 3.5 is finished.

*Proof of the Claim.* By contradiction, suppose that $C$ is not a gap curve for $F$. Then $(F^{-1}(C), a) \not\subset (F^{-1}(b), a)$. Let $(A, a)$ be an irreducible component of $(F^{-1}(C), a)$ such that $(A, a) \not\subset (F^{-1}(b), a)$. Since the restriction $F_{|A} : (A, a) \to (C, b)$ is not constant and $(C, b)$ is irreducible, we deduce that $F_{|A} : (A, a) \to (C, b)$ is locally open (see the argument below). In particular $F(B)$ will contain a neighbourhood of $b$ in $C$. This contradicts the fact that, by construction we have $\gamma(]0, \varepsilon[) \subset C$ and $\gamma(]0, \varepsilon[) \cap F(B) = \varnothing$. This ends our proof of our claim. □

For the reader's convenience, let us also show that the restriction $F_{|A} : (A, a) \to (C, b)$ is locally open, a fact which has been used in the above proof. Let $(A_1, a)$ be the germ of an irreducible curve such that $(A_1, a) \subset (A, a)$ and $F_{|A_1}$ is not the constant map. It then suffices to show that $F_{|A_1}$ is locally open. By the Riemann Extension Theorem, $F_{|A_1} : (A_1, a) \to (C, b)$ lifts to a holomorphic map between the normalisations,(2) $\widetilde{F}_{|A_1}:(\tilde{A}_1,a)\to(\widetilde{C},\tilde{b})$. By the Open Mapping Theorem, $\widetilde{F}_{|A_1}$ is an open map. Since both $(A_1,a)$ and $(C,b)$ are irreducible, the normalisation maps $(\tilde{A}_1,a)\to(A_1,a)$ and $(\widetilde{C},\tilde{b})\to(C,b)$ are local homeomorphisms. It follows that $F_{|A_1}$: $(A_1,a)\to(C,b)$ is locally open.

### 3.4. Proof of Proposition 3.3.

Let us suppose by contradiction that the image of $(f,g)$ is not open at $(0,0)\in\mathbb{C}^2$, or does not exist as a set germ. Applying Proposition 3.5 to our setting, it follows that $(f,g)$ has a gap curve germ $(C,0)\subset(\mathbb{C}^2,0)$. If $\{\varphi=0\}$ is a local equation for $C$, then by Definition 3.4 we have the inclusion $Z(\varphi(f,g))\subset Z(f)\cap Z(g)=Z(h)\cup(Z(\hat{f})\cap Z(\hat{g}))$. Since $Z(\varphi(f,g))$ has pure dimension $n-1$ and since $\dim Z(\hat{f})\cap Z(\hat{g})=n-2$, we deduce the inclusion $Z(\varphi(f,g))\subset Z(h)$. In local coordinates we may write $\varphi(x,y)=P(x,y)+\sum_{i+j\geq p+1}c_{i,j}x^iy^j$, where $P(x,y)$ is a homogeneous polynomial of degree $p\geq 1$. Therefore $\varphi(f,g))=P(\hat{f},\hat{g})h^p+h^{p+1}\tilde{h}$, for some holomorphic function $\tilde{h}$. Then $Z(\varphi(f,g))\subset Z(h)$ implies that $Z(P(\hat{f},\hat{g})+h\tilde{h})\subset Z(h)$. We then deduce $Z(P(\hat{f},\hat{g})+h\tilde{h})\subset Z(h)\cap Z(P(\hat{f},\hat{g}))$, and in particular $\dim Z(h)\cap Z(P(\hat{f},\hat{g}))=n-1$.

Writing now the homogeneous polynomial $P(x,y)$ as a product of linear factors, we deduce that there exists $[\alpha:\beta]\in\mathbb{P}^1$ such that $\dim Z(h)\cap Z(\beta\hat{f}+\alpha\hat{g})=n-1$. This ends the proof of Proposition 3.3.

The condition "no gap curve" is nice in theory but not easily verifiable in practise, thus proving the locally openness of $F$ usually amounts to direct computations. See Example 3.9 below, preceded by a couple of other examples for the above described situations.

*Example* 3.7. Let $F:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$, $F(x,y)=(x(y+x^2),y(y+x^2))$, which we have already seen before (§2.4). It is trivial to check the criterion of Proposition 3.3, thus we have $(\operatorname{Im}F,0)=(\mathbb{C}^2,0)$.

Let us also remark that by changing coordinates locally $x=u, y=v-u^2$ one gets the map germ $(uv,v(v-u^2))$ which is an example that Huckleberry computed explicitly in [Hu, p. 449].

*Example* 3.8. Let $(f,g):\mathbb{C}^2\to\mathbb{C}^2$, $(f,g)(x,y)=(xy,x^2y^2+y^3)$.
Then $h(x,y)=y$, $\hat{f}(x,y)=x$, $\hat{g}(x,y)=x^2y+y^2$.
If $[\alpha:\beta]\neq[1:0]$ then $Z(\beta\hat{f}+\alpha\hat{g})\cap Z(h)=\{(0,0)\}$ and hence $Z(\beta\hat{f}+\alpha\hat{g})\not\subset Z(h)$.
If $[\alpha:\beta]=[1:0]$ then $Z(\beta\hat{f}+\alpha\hat{g})=Z(y(x^2+y))\not\subset Z(y)$.

---

(2) This is a general fact about the normalisation of complex spaces and holomorphic maps $X_1\to X_2$ for which the pre-image of the non-normal locus of $X_2$ is nowhere dense in $X_1$, see e.g. [GR2, Proposition 8.4.3], whereas in the 1-dimensional case it follows easily from the Riemann Extension Theorem.

We deduce that $(f,g)$ has no gap line. Nevertheless $(f,g)$ has a gap curve since $\mathrm{Im}\, f\cap\{(u,v)\in\mathbb{C}^2 \,|\, v=u^2\}=(0,0)$. By Theorem 3.5 together with Theorem 1.1(i)(b), it then follows that the image of $(f,g)$ is not a well defined set germ.

*Example* 3.9. Let $(f,g)\colon\mathbb{C}^2\to\mathbb{C}^2$, $(f,g)(x,y)=(x(x^4+y),y(x^4+y)^2)$. Note that $h(x,y)=x^4+y$, $\hat{f}(x,y)=x$, $\hat{g}(x,y)=y(x^4+y)$ and hence $\dim Z(h)\cap Z(\hat{g})=n-1$. We claim that $(\mathrm{Im}\,(f,g),0)=(\mathbb{C}^2,0)$.

Let $D_r\subset\mathbb{C}$ denote the closed disk centred at the origin and of radius $r$. We will show that for any $\frac{1}{2}>\varepsilon>0$, there is $r>0$ such that $f(D_\varepsilon\times D_\varepsilon)\supset D_r\times D_r$. We shall actually prove this inclusion in the following by assuming that $r<\varepsilon^{10}$.

We need to show that for any $(a,b)\in D_r\times D_r$ there exists $(x_0,y_0)\in D_\varepsilon\times D_\varepsilon$ such that $(f,g)(x_0,y_0)=(a,b)$. The proof falls into two cases.

**Case 1.** $|a|^2\geq|b|$. *Let* $k:=\frac{b}{a^2}$, *thus* $|k|\leq 1$. *We consider the equation* $x(x^4+k^2x^2)=a$. *This has five complex solutions and their product is* $a$. *It follows that for at least one of them, say* $x_0$, *we have* $|x_0|\leq|a|^{1/5}<r^{1/5}<\varepsilon$. *For* $y_0:=kx_0^2$ *we get* $(x_0,y_0)\in D_\varepsilon\times D_\varepsilon$ *and* $f(x_0,y_0)=(a,b)$.

**Case 2.** $|a|^2\leq|b|$. *Let* $k':=\frac{a^2}{b}$, *thus* $|k'|\leq 1$. *We consider the equation* $y((k')^2y^2+y)^2=b$ *and claim that this equation has at least three solutions in* $D_{\varepsilon^2}\subset D_\varepsilon$. *Since the three solutions of the equation* $y^3-b=0$ *are in* $D_{\varepsilon^2}$, *by Rouche's Theorem it suffices to show that* $|(k')^4y^5+2(k')^2y^4|<|y^3-b|$ *on* $\partial D_{\varepsilon^2}$. *However, if* $|y|=\varepsilon^2$, *we have* $|y^3-b|\geq\varepsilon^6-r>\varepsilon^6-\varepsilon^{10}\geq\frac{15}{16}\varepsilon^6$ *(since* $\varepsilon<1/2$*). On the other hand* $|(k')^4y^5+2(k')^2y^4|<\varepsilon^{10}+2\varepsilon^8<(\frac{1}{16}+\frac{1}{2})\varepsilon^6<\frac{15}{16}\varepsilon^6$, *and therefore we have* $|(k')^4y^5+2(k')^2y^4|<|y^3-b|$ *on* $\partial D_{\varepsilon^2}$ *indeed.*

*We then use such a solution* $y_0$. *For* $x_0:=\frac{a}{(k')^2y_0^2+y_0}$ *we get* $x_0^2=\frac{a^2}{b}y_0=k'y_0$ *and therefore* $|x_0|^2<\varepsilon^2$, *thus* $|x_0|<\varepsilon$. *We also have* $x_0(x_0^4+y_0)=a$ *and* $y_0(x_0^4+y_0)^2=b$, *thus* $(f,g)(x_0,y_0)=(a,b)$.

### 3.5. Locally open image and the "subflat" condition

Huckleberry conjectured in [Hu, p. 461] that if $F$ is not totally singular (i.e. $\mathrm{Sing}\,F=X$ as germs at $a$), then $(\mathrm{Im}\,F,a)=(Y,b)$ if and only if $F$ is *subflat*.

*Definition* 3.10. ([Hu]) If $X$ and $Y$ are reduced, locally irreducible complex spaces and $F\colon X\to Y$ is a holomorphic mapping then $F$ is called *subflat* at $p\in X$ if for every prime ideal $I\subset\mathcal{O}_{Y,F(p)}$ such that $\dim V(I)>0$ we have that $\langle F^*(I)\rangle\cap F^*(\mathcal{O}_{Y,F(p)})=F^*(I)$, where $\langle F^*(I)\rangle$ denotes the ideal generated by $F^*(I)$.

Huckleberry proves his conjecture in case of holomorphic maps $(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$. We are now in position to derive a proof in the general setting:

**Theorem 3.11.** *Let* $F:(X,a)\to(Y,b)$, $\dim X\geq\dim Y\geq 1$, *be a holomorphic map germ*(3) *between two germs of reduced, locally irreducible complex spaces. Then* $(\operatorname{Im} F,b)=(Y,b)$ *if and only if* $F$ *is subflat.*

*Proof.* Huckleberry [Hu, Proposition 3.2] had actually proved in the same setting the following statement, by using essentially the Nullstellensatz:

(*) *a holomorphic* $F$ *such that* $\operatorname{Sing} F\neq X$ *is subflat if and only if* $F$ *has no gap curve.*

The notion of "gap curve" has been given in Definition 3.4. Therefore the proof of Theorem 3.11 reduces, via (*), to the equivalence "$(\operatorname{Im} F,b)=(Y,b)$ *iff* $F$ *has no gap curve*" which is precisely our Theorem 3.5. □

*Remark* 3.12. Given two germs of complex spaces $(X,a)$ and $(Y,b)$, it might happen that there is no holomorphic map germ $F:(X,a)\to(Y,b)$ with $(\operatorname{Im} F,b)=(Y,b)$. In [CJ] one obtains a characterisation of all two-dimensional complex germs $(Y,b)$ for which there exists a holomorphic map $F:(\mathbb{C}^n,0)\to(Y,b)$ such that $(\operatorname{Im} F,b)=(Y,b)$.

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(3) Again, this only makes sense for $F$ which are not totally singular, i.e. $\operatorname{Sing} F\neq X$.

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