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Pluripotential theory and convex bodies: large deviation principle

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Abstract. We continue the study in [2] in the setting of weighted pluripotential theory arising from polynomials associated to a convex body P in $(\mathbb{R}^+)^d$. Our goal is to establish a large deviation principle in this setting specifying the rate function in terms of $P-$pluripotential-theoretic notions. As an important preliminary step, we first give an existence proof for the solution of a Monge-Ampère equation in an appropriate finite energy class. This is achieved using a variational approach.

1. Introduction

As in [2], we fix a convex body $P \subset (\mathbb{R}^+)^d$ and we define the logarithmic indicator function

$$H_P(z) := \sup_{J \in P} \log |z^J| := \sup_{(j_1,\ldots,j_d) \in P} \log[|z_1|^{j_1} \ldots |z_d|^{j_d}]. \tag{1.1}$$

We assume throughout that

$$\Sigma \subset kP \text{ for some } k \in \mathbb{Z}^+ \tag{1.2}$$

where

$$\Sigma := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \le x_i \le 1, \ \sum_{j=1}^d x_i \le 1\}.$$

Then

$$H_P(z) \ge \frac{1}{k} \max_{j=1,\ldots,d} \log^+ |z_j|$$

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where $\log^+|z_j|=\max[0,\log|z_j|]$. We define

$$L_P=L_P(\mathbb{C}^d):=\{u\in PSH(\mathbb{C}^d):u(z)-H_P(z)=O(1),\ |z|\longrightarrow\infty\},$$

and

$$L_{P,+}=L_{P,+}(\mathbb{C}^d)=\{u\in L_P(\mathbb{C}^d):u(z)\geq H_P(z)+C_u\}.$$

These are generalizations of the classical Lelong classes when $P=\Sigma$. We define the finite-dimensional polynomial spaces

$$Poly(nP):=\{p(z)=\sum_{J\in nP\cap(\mathbb{Z}^+)^d} c_J z^J : c_J\in\mathbb{C}\}$$

for $n=1,2,...$ where $z^J=z_1^{j_1}...z_d^{j_d}$ for $J=(j_1,...,j_d)$. For $p\in Poly(nP)$, $n\geq 1$ we have $\frac{1}{n}\log|p|\in L_P$; also each $u\in L_{P,+}(\mathbb{C}^d)$ is locally bounded in $\mathbb{C}^d$. For $P=\Sigma$, we write $Poly(nP)=\mathcal{P}_n$.

Given a compact set $K\subset\mathbb{C}^d$, one can define various pluripotential-theoretic notions associated to K related to L_P and the polynomial spaces $Poly(nP)$. Our goal in this paper is to prove some probabilistic properties of random point processes on K utilizing these notions and their weighted counterparts. We require an existence proof for the solution of a Monge-Ampère equation in an appropriate finite energy class; this is done in Theorem 2.8 using a variational approach and is of interest on its own. The third section recalls appropriate definitions and properties in $P-$pluripotential theory, mostly following [2]. As in [2], our spaces $Poly(nP)$ do not necessarily arise as holomorphic sections of tensor powers of a line bundle. Subsection 3.3 includes a standard elementary probabilistic result on almost sure convergence of probability measures associated to random arrays on K to a $P-$pluripotential-theoretic equilibrium measure. Section 4 sets up the machinery for the more subtle large deviation principle (LDP), Theorem 5.1, for which we provide two proofs (analogous to those in [9]). As in [9], the first proof was inspired by [6] and the second proof was utilized by Berman in [5]. The reader will find far-reaching applications and interpretations of LDP's in the appropriate settings of holomorphic line bundles over a compact, complex manifold in [5]. In particular, the case where P is a convex integral polytope (vertices in $\mathbb{Z}^d$) which is the moment polytope for a toric manifold (P is Delzant) is covered in [5].

2. Monge-Ampère and $P-$pluripotential theory

2.1. Monge-Ampère equations with prescribed singularity

In this section, (X,ω) is a compact Kähler manifold of dimension d.

2.1.1. Quasi-plurisubharmonic functions

A function $u:X\to\mathbb{R}\cup\{-\infty\}$ is called quasi-plurisubharmonic (quasi-psh) if locally $u=\rho+\varphi$, where φ is plurisubharmonic and ρ is smooth.

We let $PSH(X,\omega)$ denote the set of ω-psh functions, i.e. quasi-psh functions u such that $\omega_u:=\omega+dd^c u\geq 0$ in the sense of currents on X.

Given $u,v\in PSH(X,\omega)$ we say that u is more singular than v (and we write $u\prec v$) if $u\leq v+C$ on X, for some constant C. We say that u has the same singularity as v (and we write $u\simeq v$) if $u\prec v$ and $v\prec u$.

Given $\phi\in PSH(X,\omega)$, we let $PSH(X,\omega,\phi)$ denote the set of ω-psh functions u which are more singular than ϕ.

2.1.2. Nonpluripolar Monge-Ampère measure

For bounded ω-psh functions $u_1,...,u_d$, the Monge-Ampère product $(\omega+dd^c u_1)\wedge...\wedge(\omega+dd^c u_d)$ is well-defined as a positive Radon measure on X (see [14], [3]). For general ω-psh functions $u_1,...,u_d$, the sequence of positive measures

$$\mathbf{1}_{\cap\{u_j>-k\}}(\omega+dd^c\max(u_1,-k))\wedge...\wedge(\omega+dd^c\max(u_d,-k))$$

is non-decreasing in k and the limiting measure, which is called the nonpluripolar product of $\omega_{u_1},...,\omega_{u_d}$, is denoted by

$$\omega_{u_1}\wedge...\wedge\omega_{u_d}.$$

When $u_1=...=u_d=u$ we write $\omega_u^d:=\omega_u\wedge...\wedge\omega_u$. Note that by definition $\int_X\omega_{u_1}\wedge...\wedge\omega_{u_d}\leq\int_X\omega^d$.

It was proved in [20, Theorem 1.2] and [11, Theorem 1.1] that the total mass of nonpluripolar Monge-Ampère products is decreasing with respect to singularity type. More precisely,

Theorem 2.1. *Let $\omega_1,...,\omega_d$ be Kähler forms on X. If $u_j\prec v_j$, $j=1,...,d$, are ω_j-psh functions then*

$$\int_X(\omega_1+dd^c u_1)\wedge...\wedge(\omega_d+dd^c u_d)\leq\int_X(\omega_1+dd^c v_1)\wedge...\wedge(\omega_d+dd^c v_d).$$

As noted above, for a general ω-psh function u we have the estimate $\int_X\omega_u^d\leq\int_X\omega^d$. Following [15] we let $\mathcal{E}(X,\omega)$ denote the set of all ω-psh functions with maximal total mass, i.e.

$$\mathcal{E}(X,\omega):=\left\{u\in PSH(X,\omega):\int_X\omega_u^d=\int_X\omega^d\right\}.$$

Given $\phi\in PSH(X,\omega)$, we define

$$\mathcal{E}(X,\omega,\phi):=\left\{u\in PSH(X,\omega,\phi)\,:\,\int_X\omega_u^d=\int_X\omega_\phi^d\right\}.$$

Proposition 2.2. *Let $\phi\in PSH(X,\omega)$. The following are equivalent*:
(1) $\mathcal{E}(X,\omega,\phi)\cap\mathcal{E}(X,\omega)\neq\varnothing$;
(2) $\phi\in\mathcal{E}(X,\omega)$;
(3) $\mathcal{E}(X,\omega,\phi)\subset\mathcal{E}(X,\omega)$.

Proof. We first prove (1)$\Longrightarrow$(2). If $u\in\mathcal{E}(X,\omega,\phi)\cap\mathcal{E}(X,\omega)$ then $\int_X\omega_u^d=\int_X\omega^d$. On the other hand, since u is more singular than ϕ, Theorem 2.1 ensures that

$$\int_X\omega^d=\int_X\omega_u^d\leq\int_X\omega_\phi^d\leq\int_X\omega^d,$$

hence equality holds, proving that $\phi\in\mathcal{E}(X,\omega)$.

Now we prove (2)$\Longrightarrow$(3). If $\phi\in\mathcal{E}(X,\omega)$ and $u\in\mathcal{E}(X,\omega,\phi)$ then

$$\int_X\omega_u^d=\int_X\omega_\phi^d=\int_X\omega^d,$$

hence $u\in\mathcal{E}(X,\omega)$.

Finally (3)$\Longrightarrow$(1) is obvious. □

Proposition 2.3. *Assume that $\phi_j\in PSH(X,\omega_j)$, $j=1,...,d$ with $\int_X(\omega_j+dd^c\phi_j)^d>0$. If $u_j\in\mathcal{E}(X,\omega_j,\phi_j)$, $j=1,...,d$, then*

$$\int_X(\omega_1+dd^cu_1)\wedge...\wedge(\omega_d+dd^cu_d)=\int_X(\omega_1+dd^c\phi_1)\wedge...\wedge(\omega_d+dd^c\phi_d).$$

Proof. Theorem 2.1 gives one inequality. The other one follows from [11, Proposition 3.1 and Theorem 3.14]. □

2.1.3. Model potentials

For a function $f:X\to\mathbb{R}\cup\{-\infty\}$, we let f^* denote its uppersemicontinuous (usc) regularization, i.e.

$$f^*(x):=\limsup_{X\ni y\to x}f(y).$$

Given $\phi\in PSH(X,\omega)$, following J. Ross and D. Witt Nyström [18], we define

$$P_\omega[\phi]:=\left(\lim_{t\to+\infty}P_\omega(\min(\phi+t,0))\right)^*.$$

Here, for a function f, $P_\omega(f)$ is defined as

$$P_\omega(f) := \left(x \longmapsto \sup\{u(x) : u \in PSH(X,\omega), u \le f\}\right)^* .$$

It was shown in [11, Theorem 3.8] that the nonpluripolar Monge-Ampère measure of $P_\omega[\phi]$ is dominated by Lebesgue measure:

$$(\omega + dd^c P_\omega[\phi])^d \le \mathbf{1}_{\{P_\omega[\phi]=0\}} \omega^d \le \omega^d. \tag{2.1}$$

This fact plays a crucial role in solving the complex Monge-Ampère equation. For the reader's convenience, we note that in the notation of [11] (on the left)

$$P_{[\omega,\phi]}(0) = P_\omega[\phi].$$

Definition 2.4. A function $\phi \in PSH(X,\omega)$ is called a model potential if $\int_X \omega_\phi^d > 0$ and $P_\omega[\phi] = \phi$. A function $u \in PSH(X,\omega)$ has model type singularity if u has the same singularity as $P_\omega[u]$; i.e., $u - P_\omega[u]$ is bounded on X.

There are plenty of model potentials. If $\varphi \in PSH(X,\omega)$ with $\int_X \omega_\varphi^d > 0$ then, by [11, Theorem 3.12], $P_\omega[\varphi]$ is a model potential. In particular, if $\int_X \omega_\varphi^d = \int_X \omega^d$ (i.e. $\varphi \in \mathcal{E}(X,\omega)$) then $P_\omega[\varphi] = 0$.

We will use the following property of model potentials proved in [11, Theorem 3.12]: if ϕ is a model potential then

$$u \in PSH(X,\omega,\phi) \Longrightarrow u - \sup_X u \le \phi. \tag{2.2}$$

In the sequel we always assume that ϕ has *model type singularity* and *small unbounded locus*; i.e., ϕ is locally bounded outside a closed complete pluripolar set, allowing us to use the variational approach of [7] as explained in [11].

2.1.4. The variational approach

We call a measure which puts no mass on pluripolar sets a *nonpluripolar measure.* For a positive nonpluripolar measure μ on X we let L_μ denote the following linear functional on $PSH(X,\omega,\phi)$:

$$L_\mu(u) := \int_X (u - \phi)\, d\mu.$$

For $u \in PSH(X,\omega)$ with $u \simeq \phi$, we define the Monge-Ampère energy

$$\mathbf{E}_\phi(u) := \frac{1}{(d+1)} \sum_{k=0}^{d} \int_X (u-\phi)\omega_u^k \wedge \omega_\phi^{d-k}. \tag{2.3}$$

It was shown in [11, Theorem 4.10] (by adapting the arguments of [7]) that $\mathbf{E}_\phi$ is non-decreasing and concave along affine curves, giving rise to its trivial extension to $PSH(X,\omega,\phi)$.

We define

$$\mathcal{E}^1(X,\omega,\phi):=\{u\in PSH(X,\omega,\phi):\mathbf{E}_\phi(u)>-\infty\}. \tag{2.4}$$

The following criterion was proved in [11, Theorem 4.13]:

Proposition 2.5. *Let $u\in PSH(X,\omega,\phi)$. Then $u\in\mathcal{E}^1(X,\omega,\phi)$ iff $u\in\mathcal{E}(X,\omega,\phi)$ and $\int_X(u-\phi)\omega_u^d>-\infty$.*

Lemma 2.6. *If E is pluripolar then there exists $u\in\mathcal{E}^1(X,\omega,\phi)$ such that $E\subset\{u=-\infty\}$.*

Proof. Without loss of generality we can assume that ϕ is a model potential. Then (2.1) gives $\int_X|\phi|\omega_\phi^d=0$. It follows from [7, Corollary 2.11] that there exists $v\in\mathcal{E}^1(X,\omega,0)$, $v\le 0$, such that $E\subset\{v=-\infty\}$. Set $u:=P_\omega(\min(v,\phi))$. Then $E\subset\{u=-\infty\}$ and we claim that $u\in\mathcal{E}^1(X,\omega,\phi)$. For each $j\in\mathbb{N}$ we set $v_j:=\max(v,-j)$ and $u_j:=P_\omega(\min(v_j,\phi))$. Then u_j decreases to u and $u_j\simeq\phi$. Using [11, Theorem 4.10 and Lemma 4.15] it suffices to check that $\{\int_X|u_j-\phi|\omega_{u_j}^d\}$ is uniformly bounded. It follows from [11, Lemma 3.7] that

$$\begin{aligned}\int_X|u_j-\phi|\omega_{u_j}^d\le\int_X|u_j|\omega_{u_j}^d&\le\int_X|v_j|\omega_{v_j}^d+\int_X|\phi|\omega_\phi^d\\&=\int_X|v_j|\omega_{v_j}^d.\end{aligned}$$

The fact that $\int_X|v_j|\omega_{v_j}^d$ is uniformly bounded follows from [15, Corollary 2.4] since $v\in\mathcal{E}^1(X,\omega,0)$. This concludes the proof. $\square$

Lemma 2.7. *Assume that $\mathcal{E}^1(X,\omega,\phi)\subset L^1(X,\mu)$. Then, for each $C>0$, L_μ is bounded on*

$$E_C:=\{u\in PSH(X,\omega,\phi):\sup_X u\le 0\ \text{and}\ \mathbf{E}_\phi(u)\ge -C\}.$$

Proof. By concavity of $\mathbf{E}_\phi$ the set E_C is convex. We now show that E_C is compact in the $L^1(X,\omega^d)$ topology. Let $\{u_j\}$ be a sequence in E_C. We claim that $\{\sup_X u_j\}$ is bounded. Indeed, by [11, Theorem 4.10]

$$\begin{aligned}\mathbf{E}_\phi(u_j)&\le\int_X(u_j-\phi)\omega_\phi^d\\&\le(\sup_X u_j)\int_X\omega_\phi^d+\int_X(u_j-\sup_X u_j-\phi)\omega_\phi^d.\end{aligned}$$

It follows from (2.2) that $u_j - \sup_X u_j \le P_\omega[\phi] \le \phi + C_0$, where C_0 is a constant. The boundedness of $\{\sup_X u_j\}$ then follows from that of $\{\mathbf{E}_\phi(u_j)\}$ and the above estimate. This proves the claim.

A subsequence of $\{u_j\}$, still denoted by $\{u_j\}$, converges in $L^1(X, \omega^d)$ to $u \in PSH(X, \omega)$ with $\sup_X u \le 0$. Since $u_j - \sup_X u_j \le \phi + C_0$, we have $u - \sup_X u \le \phi + C_0$. This proves that $u \in PSH(X, \omega, \phi)$. The upper semicontinuity of $\mathbf{E}_\phi$ (see [11, Proposition 4.19]) ensures that $\mathbf{E}_\phi(u) \ge -C$, hence $u \in E_C$. This proves that E_C is compact in the $L^1(X, \omega^d)$ topology.

The result then follows from [7, Proposition 3.4]. □

The goal of this section is to prove the following result:

Theorem 2.8. *Assume that μ is a nonpluripolar positive measure on X such that $\mu(X) = \int_X \omega_\phi^d$. The following are equivalent*

(1) *μ has finite energy, i.e., L_μ is finite on $\mathcal{E}^1(X, \omega, \phi)$;*

(2) *there exists $u \in \mathcal{E}^1(X, \omega, \phi)$ such that $\omega_u^d = \mu$;*

(3) *there exists a unique $u \in \mathcal{E}^1(X, \omega, \phi)$ such that*

$$F_\mu(u) = \max_{v \in \mathcal{E}^1(X,\omega,\phi)} F_\mu(v) < +\infty$$

where $F_\mu = \mathbf{E}_\phi - L_\mu$.

Remark 2.9. It was shown in [11, Theorem 4.28] that a unique (normalized) solution u in $\mathcal{E}(X, \omega, \phi)$ always exists (without the finite energy assumption on μ). But that proof does not give a solution in $\mathcal{E}^1(X, \omega, \phi)$. Below, we will follow the proof of [11, Theorem 4.28] and use the finite energy condition, $\mathcal{E}^1(X, \omega, \phi) \subset L^1(X, \mu)$, to prove that u belongs to $\mathcal{E}^1(X, \omega, \phi)$.

Lemma 2.10. *Assume that $\mathcal{E}^1(X, \omega, \phi) \subset L^1(X, \mu)$. Then there exists a positive constant C such that, for all $u \in \mathcal{E}^1(X, \omega, \phi)$ with $\sup_X u = 0$,*

$$L_\mu(u) \ge -C(1 + |\mathbf{E}_\phi(u)|^{1/2}). \tag{2.5}$$

The proof below uses ideas in [7], [15].

Proof. Since ϕ has model type singularity, it follows from [11, Theorem 4.10] that $\mathbf{E}_\phi - \mathbf{E}_{P_\omega[\phi]}$ is bounded. Without loss of generality we can assume in this proof that $\phi = P_\omega[\phi]$. Fix $u \in \mathcal{E}^1(X, \omega, \phi)$ such that $\sup_X u = 0$ and $|\mathbf{E}_\phi(u)| > 1$. Then, by [11, Theorem 3.12], $u \le \phi$. Set $a = |\mathbf{E}_\phi(u)|^{-1/2} \in (0, 1)$, and $v := au + (1-a)\phi \in$

$\mathcal{E}^1(X,\omega,\phi)$. We estimate $\mathbf{E}_\phi(v)$ as follows

$$\begin{aligned}(d+1)\mathbf{E}_\phi(v) &= a\sum_{k=0}^{d}\int_X (u-\phi)\omega_v^k\wedge\omega_\phi^{d-k}\\ &= a\sum_{k=0}^{d}\int_X (u-\phi)(a\omega_u+(1-a)\omega_\phi)^k\wedge\omega_\phi^{d-k}\\ &\geq C(d)a\int_X (u-\phi)\omega_\phi^d + C(d)a^2\sum_{k=0}^{d}\int_X (u-\phi)\omega_u^k\wedge\omega_\phi^d,\end{aligned}$$

where $C(d)$ is a positive constant which only depends on d. It follows from $\phi=P_\omega[\phi]$ and [11, Theorem 3.8] that $\omega_\phi^d\leq\omega^d$ (recall (2.1)). This together with [14, Proposition 2.7] give

$$\int_X (u-\phi)\omega_\phi^d \geq -C_1,$$

for a uniform constant C_1. Therefore,

$$(d+1)\mathbf{E}_\phi(v)\geq -C_1C(d)a + C_2a^2\mathbf{E}_\phi(u)\geq -C_3.$$

It thus follows from Lemma 2.7 that $L_\mu(v)\geq -C_4$ for a uniform constant $C_4>0$. Thus

$$\int_X (u-\phi)\, d\mu \geq -C_4/a,$$

which gives (2.5). □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. Without loss of generality we can assume that ϕ is a model potential. We first prove (1)$\Longrightarrow$(2). We write $\mu=f\nu$, where ν is a non-pluripolar positive measure satisfying, for all Borel subsets $B\subset X$,

$$\nu(B)\leq A\,\mathrm{Cap}_\phi(B),$$

for some positive constant A, and $0\leq f\in L^1(X,\nu)$ (cf., [11, Lemma 4.26]). Here Cap_ϕ is defined as

$$\mathrm{Cap}_\phi(B):=\sup\left\{\int_B \omega_u^d : u\in PSH(X,\omega),\ \phi-1\leq u\leq\phi\right\}.$$

Set, for $k\in\mathbb{N}$, $\mu_k:=c_k\min(f,k)\nu$ where $c_k>0$ is chosen so that $\mu_k(X)=\int_X\omega_\phi^d$; this is needed in order to solve the Monge-Ampère equation in the class $\mathcal{E}^1(X,\omega,\phi)$.

For k large enough, $1\le c_k\le 2$ and $c_k\to 1$ as $k\to+\infty$. It follows from [11, Theorem 4.25] that there exists $u_j\in\mathcal{E}^1(X,\omega,\phi)$, $\sup_X u_j=0$, such that $\omega_{u_j}^d=\mu_j$; by [11, Theorem 3.12], $u_j\le\phi$. A subsequence of $\{u_j\}$ which, by abuse of notation, will be denoted by $\{u_j\}$, converges in $L^1(X,\omega^d)$ to $u\in PSH(X,\omega)$ with $u\le\phi$. Define $v_k:=(\sup_{j\ge k} u_j)^*$. Then $v_k\searrow u$ and $\sup_X v_k=0$. It follows from (2.5) and [11, Theorem 4.10] that

$$\begin{aligned}|\mathbf{E}_\phi(u_j)| &\le \int_X |u_j-\phi|\omega_{u_j}^d \le 2\int_X |u_j-\phi|\,d\mu\\ &\le 2C(1+|\mathbf{E}_\phi(u_j)|^{1/2}).\end{aligned}$$

Therefore $\{|\mathbf{E}_\phi(u_j)|\}$ is bounded, hence so is $\{|\mathbf{E}_\phi(v_j)|\}$ since $\mathbf{E}_\phi$ is non-decreasing. It then follows from [11, Lemma 4.15] that $u\in\mathcal{E}^1(X,\omega,\phi)$.

Now, repeating the arguments of [11, Theorem 4.28] we can show that $\omega_u^d=\mu$, finishing the proof of (1)$\Longrightarrow$(2).

We next prove (2)$\Longrightarrow$(3). Assume that $\mu=\omega_u^d$ for some $u\in\mathcal{E}^1(X,\omega,\phi)$. For all $v\in\mathcal{E}^1(X,\omega,\phi)$, by [11, Theorem 4.10] and Proposition 2.5 we have

$$\begin{aligned}L_\mu(v) &= \int_X (v-\phi)\omega_u^d\\ &= \int_X (v-u)\omega_u^d+\int_X (u-\phi)\omega_u^d\\ &\ge \mathbf{E}_\phi(v)-\mathbf{E}_\phi(u)+\int_X (u-\phi)\omega_u^d > -\infty.\end{aligned}$$

Hence L_μ is finite on $\mathcal{E}^1(X,\omega,\phi)$. Now, for all $v\in\mathcal{E}^1(X,\omega,\phi)$, by [11, Theorem 4.10] we have

$$F_\mu(v)-F_\mu(u)=\mathbf{E}_\phi(v)-\mathbf{E}_\phi(u)-\int_X (v-u)\omega_u^d\le 0.$$

This gives (3). Finally, (3)$\Longrightarrow$(1) is obvious. □

2.2. Monge-Ampère equations on $\mathbb{C}^d$ with prescribed growth

As in the introduction we let P be a convex body contained in $(\mathbb{R}^+)^d$ and fix $r>0$ such that $P\subset r\Sigma$. We assume (1.2); i.e., $\Sigma\subset kP$ for some $k\in\mathbb{Z}^+$. This ensures that H_P in (1.1) is locally bounded on $\mathbb{C}^d$ (and of course $H_P\in L_P^+(\mathbb{C}^d)$). Let $u\in L_P(\mathbb{C}^d)$ and define

$$\tilde{u}(z):=u(z)-\frac{r}{2}\log(1+|z|^2),\ z\in\mathbb{C}^d. \tag{2.6}$$

Consider the projective space $\mathbb{P}^d$ equipped with the Kähler metric $\omega := r\omega_{FS}$, where

$$\omega_{FS} = dd^c \frac{1}{2} \log(1+|z|^2)$$

on $\mathbb{C}^d$. Then $\tilde{u}$ is bounded from above on $\mathbb{C}^d$. It thus can be extended to $\mathbb{P}^d$ as a function in $PSH(\mathbb{P}^d, \omega)$.

For a plurisubharmonic function u on $\mathbb{C}^d$, we let $(dd^c u)^d$ denote its nonpluripolar Monge-Ampère measure; i.e., $(dd^c u)^d$ is the increasing limit of the sequence of measures $\mathbf{1}_{\{u>-k\}}(dd^c \max(u,-k))^d$. Then

$$\omega_{\tilde{u}}^d = (\omega + dd^c \tilde{u})^d = (dd^c u)^d \text{ on } \mathbb{C}^d.$$

If $u \in L_P(\mathbb{C}^d)$ then

$$\int_{\mathbb{C}^d} (dd^c u)^d \leq \int_{\mathbb{C}^d} (dd^c H_P)^d = d! Vol(P) =: \gamma_d = \gamma_d(P)$$

(cf., equation (2.4) in [2]). We define

$$\mathcal{E}_P(\mathbb{C}^d) := \left\{ u \in L_P(\mathbb{C}^d) : \int_{\mathbb{C}^d} (dd^c u)^d = \gamma_d \right\}.$$

By the construction in (2.6) we have that $\widetilde{H}_P \in PSH(\mathbb{P}^d, \omega)$. We define

$$\widetilde{\Phi}_P := P_\omega[\widetilde{H}_P].$$

The key point here, which follows from [12, Theorem 7.2], is that $\widetilde{H}_P$ has model type singularity (recall Definition 2.4) and hence the same singularity as $\widetilde{\Phi}_P$. Defining Φ_P on $\mathbb{C}^d$ using (2.6); i.e., for $z \in \mathbb{C}^d$,

$$\Phi_P(z) = \widetilde{\Phi}_P(z) + \frac{r}{2} \log(1+|z|^2),$$

we thus have $\Phi_P \in L_{P,+}(\mathbb{C}^d)$. The advantage of using Φ_P is that, by (2.1), $(dd^c \Phi_P)^d \leq \omega^d$ on $\mathbb{C}^d$. Note that $L_{P,+}(\mathbb{C}^d) \subset \mathcal{E}_P(\mathbb{C}^d)$. For $u, v \in L_P^+(\mathbb{C}^d)$ we define

$$E_v(u) := \frac{1}{(d+1)} \sum_{j=0}^{d} \int_{\mathbb{C}^d} (u-v)(dd^c u)^j \wedge (dd^c v)^{d-j}. \tag{2.7}$$

The corresponding global energy (see (2.3)) is defined as

$$\mathbf{E}_{\tilde{v}}(\tilde{u}) := \frac{1}{(d+1)} \sum_{j=0}^{d} \int_{\mathbb{P}^d} (\tilde{u}-\tilde{v})(\omega + dd^c \tilde{u})^j \wedge (\omega + dd^c \tilde{v})^{d-j}.$$

Then E_v is non-decreasing and concave along affine curves in $L_{P,+}(\mathbb{C}^d)$. We extend E_v to $L_P(\mathbb{C}^d)$ in an obvious way. Note that E_v may take the value $-\infty$. We define

$$\mathcal{E}_P^1(\mathbb{C}^d) := \{u \in L_P(\mathbb{C}^d) : E_{H_P}(u) > -\infty\}.$$

We observe that in the above definition we can replace E_{H_P} by E_{Φ_P}, since for $u \in L_{P,+}(\mathbb{C}^d)$, by the cocycle property (cf. Proposition 3.3 [2]),

$$E_{H_P}(u) - E_{H_P}(\Phi_P) = E_{\Phi_P}(u).$$

We thus have the following important identification (see (2.4)):

$$u \in \mathcal{E}_P^1(\mathbb{C}^d) \quad \Longleftrightarrow \quad \tilde{u} \in \mathcal{E}^1(\mathbb{P}^d, \omega, \widetilde{\Phi}_P). \tag{2.8}$$

We then have the following local version of Proposition 2.5:

Proposition 2.11. *Let $u \in L_P(\mathbb{C}^d)$. Then $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ iff $u \in \mathcal{E}_P(\mathbb{C}^d)$ and $\int_{\mathbb{C}^d} (u - H_P)(dd^c u)^d > -\infty$. In particular, if $supp(dd^c u)^d$ is compact, $u \in \mathcal{E}_P^1(\mathbb{C}^d)$ iff $\int_{\mathbb{C}^d} (dd^c u)^d = \gamma_d$ and $\int_{\mathbb{C}^d} u(dd^c u)^d > -\infty$.*

Proof. Since $\widetilde{H}_P \simeq \widetilde{\Phi}_P$,

$$\int_{\mathbb{P}^d} (\tilde{u} - \widetilde{H}_P)\omega_{\tilde{u}}^d > -\infty \text{ iff } \int_{\mathbb{P}^d} (\tilde{u} - \widetilde{\Phi}_P)\omega_{\tilde{u}}^d > -\infty$$

where $\tilde{u} \in PSH(\mathbb{P}^d, \omega)$ and u are related by (2.6). Moreover, $\Phi_P \in L_{P,+}(\mathbb{C}^d)$ implies $u \le \Phi_P + C$ so that $\tilde{u} \in PSH(\mathbb{P}^d, \omega, \widetilde{\Phi}_P)$. But

$$\int_{\mathbb{P}^d} (\tilde{u} - \widetilde{H}_P)\omega_{\tilde{u}}^d = \int_{\mathbb{C}^d} (u - H_P)(dd^c u)^d$$

and the result follows from (2.8) by applying Proposition 2.5 to $\tilde{u}$. For the last statement, note that for general $u \in L_P(\mathbb{C}^d)$ we may have $\int_{\mathbb{C}^d} H_P(dd^c u)^d = +\infty$, but if $(dd^c u)^d$ has compact support then $\int_{\mathbb{C}^d} H_P(dd^c u)^d$ is finite. □

Note that Theorem 2.1 and Proposition 2.3 give the following result:

Theorem 2.12. *Let $u_1, ..., u_d$ be functions in $\mathcal{E}_P(\mathbb{C}^d)$. Then*

$$\int_{\mathbb{C}^d} dd^c u_1 \wedge ... \wedge dd^c u_d = \gamma_d.$$

For $u_1, ..., u_n \in L_{P,+}(\mathbb{C}^d)$ Theorem 2.12 was proved in [1, Proposition 2.7].

Having the correspondence (2.8) we can state a local version of Theorem 2.8; this will be used in the sequel. Let $\mathcal{M}_P(\mathbb{C}^d)$ denote the set of all positive Borel measures μ on $\mathbb{C}^d$ with $\mu(\mathbb{C}^d) = d! Vol(P) = \gamma_d$.

Theorem 2.13. *Assume that $\mu\in\mathcal{M}_P(\mathbb{C}^d)$ is a positive nonpluripolar Borel measure. The following are equivalent*

(1) $\mathcal{E}^1_P(\mathbb{C}^d)\subset L^1(\mathbb{C}^d,\mu)$;

(2) *there exists $u\in\mathcal{E}^1_P(\mathbb{C}^d)$ such that $(dd^c u)^d=\mu$;*

(3) *there exists $u\in\mathcal{E}^1_P(\mathbb{C}^d)$ such that*

$$\mathcal{F}_\mu(u)=\max_{v\in\mathcal{E}^1_P(\mathbb{C}^d)}\mathcal{F}_\mu(v)<+\infty.$$

A priori the functional $\mathcal{F}_\mu$ is defined for $u\in\mathcal{E}^1_P(\mathbb{C}^d)$ by

$$\mathcal{F}_{\mu,\Phi_P}(u):=E_{\Phi_P}(u)-\int_{\mathbb{C}^d}(u-\Phi_P)\,d\mu.$$

However, using this notation, since

$$\mathcal{F}_{\mu,\Phi_P}(u)-\mathcal{F}_{\mu,H_P}(u)=\mathcal{F}_{\mu,\Phi_P}(H_P),$$

in statement (3) of Theorem 2.13 we can take either of the two definitions $\mathcal{F}_{\mu,\Phi_P}$ or $\mathcal{F}_{\mu,H_P}$ for $\mathcal{F}_\mu$.

Remark 2.14. If μ has compact support in $\mathbb{C}^d$ then $\int_{\mathbb{C}^d}\Phi_P\,d\mu$ and $\int_{\mathbb{C}^d}H_P\,d\mu$ are finite. Therefore, the functional $\mathcal{F}_\mu$ can be replaced by

$$u\longmapsto E_{H_P}(u)-\int_{\mathbb{C}^d}u\,d\mu.$$

Using the remark, for $\mu\in\mathcal{M}_P(\mathbb{C}^d)$ with compact support, it is natural to define the Legendre-type transform of E_{H_P}:

$$E^*(\mu):=\sup_{u\in\mathcal{E}^1_P(\mathbb{C}^d)}[E_{H_P}(u)-\int_{\mathbb{C}^d}u\,d\mu]. \tag{2.9}$$

This functional, which will appear in the rate function for our LDP, will be given a more concrete interpretation using $P-$pluripotential theory in section 4; cf., equation (4.18).

Finally, for future use, we record the following consequence of Lemma 2.6 and the correspondence (2.8).

Lemma 2.15. *If $E\subset\mathbb{C}^d$ is pluripolar then there exists $u\in\mathcal{E}^1_P(\mathbb{C}^d)$ such that $E\subset\{u=-\infty\}$.*

3. $P-$pluripotential theory notions

Given $E\subset\mathbb{C}^d$, the $P-$extremal function of E is

$$V_{P,E}^*(z):=\limsup_{\zeta\to z} V_{P,E}(\zeta)$$

where

$$V_{P,E}(z):=\sup\{u(z):u\in L_P(\mathbb{C}^d),\ u\leq 0 \text{ on } E\}.$$

For $K\subset\mathbb{C}^d$ compact, $w:K\to\mathbb{R}^+$ is an admissible weight function on K if $w\geq 0$ is an uppersemicontinuous function with $\{z\in K: w(z)>0\}$ nonpluripolar. Setting $Q:=-\log w$, we write $Q\in\mathcal{A}(K)$ and define the *weighted $P-$extremal function*

$$V_{P,K,Q}^*(z):=\limsup_{\zeta\to z} V_{P,K,Q}(\zeta)$$

where

$$V_{P,K,Q}(z):=\sup\{u(z):u\in L_P(\mathbb{C}^d),\ u\leq Q \text{ on } K\}.$$

If $Q=0$ we write $V_{P,K,Q}=V_{P,K}$, consistent with the previous notation. For $P=\Sigma$,

$$V_{\Sigma,K,Q}(z)=V_{K,Q}(z):=\sup\{u(z):u\in L(\mathbb{C}^d),\ u\leq Q \text{ on } K\}$$

is the usual weighed extremal function as in Appendix B of [19].

We write (omitting the dependence on P)

$$\mu_{K,Q}:=(dd^cV_{P,K,Q}^*)^d \text{ and } \mu_K:=(dd^cV_{P,K}^*)^d$$

for the Monge-Ampère measures of $V_{P,K,Q}^*$ and $V_{P,K}^*$ (the latter if K is not pluripolar). Proposition 2.5 of [2] states that

$$supp(\mu_{K,Q})\subset\{z\in K: V_{P,K,Q}^*(z)\geq Q(z)\}$$

and $V_{P,K,Q}^*=Q$ q.e. on $supp(\mu_{K,Q})$, i.e., off of a pluripolar set.

3.1. Energy

We recall some results and definitions from [2]. For $u,v\in L_{P,+}(\mathbb{C}^d)$, we define the *mutual energy*

$$\mathcal{E}(u,v):=\int_{\mathbb{C}^d}(u-v)\sum_{j=0}^d (dd^cu)^j\wedge(dd^cv)^{d-j}.$$

For simplicity, when $v=H_P$, we denote the associated (normalized) energy functional by E:

$$E(u):=E_{H_P}(u)=\frac{1}{d+1}\sum_{j=0}^{d}\int_{\mathbb{C}^d}(u-H_P)dd^cu^j\wedge(dd^cH_P)^{d-j}$$

(recall (2.7)).

For $u,u',v\in L_{P,+}(\mathbb{C}^d)$, and for $0\le t\le 1$, we define

$$f(t):=\mathcal{E}(u+t(u'-u),v),$$

From Proposition 3.1 in [2], $f'(t)$ exists for $0\le t\le 1$ and

$$f'(t)=(d+1)\int_{\mathbb{C}^d}(u'-u)(dd^c(u+t(u'-u)))^d$$

Hence, taking $v=H_P$, we have, for $F(t):=E(u+t(u'-u))$, that

$$F'(t)=\int_{\mathbb{C}^d}(u'-u)(dd^c(u+t(u'-u)))^d.$$

Thus $F'(0)=\int_{\mathbb{C}^d}(u'-u)(dd^cu)^d$ and we write

$$<E'(u),u'-u>:=\int(u'-u)(dd^cu)^d. \tag{3.1}$$

We need some applications of a global domination principle. The following version, sufficient for our purposes, follows from [11], Corollary 3.10 (see also Corollary A.2 of [8]).

Proposition 3.1. *Let $u\in L_P(\mathbb{C}^d)$ and $v\in\mathcal{E}_P(\mathbb{C}^d)$ with $u\le v$ a.e. $(dd^cv)^d$. Then $u\le v$ in $\mathbb{C}^d$.*

This will be used to prove an approximation result, Proposition 3.3, which itself will be essential in the sequel. First we need a lemma.

Lemma 3.2. *Assume that $\varphi\le u,v\le H_P$ are functions in $\mathcal{E}_P^1(\mathbb{C}^d)$. Then for all $t>0$,*

$$\int_{\{u\le H_P-2t\}}(H_P-u)(dd^cv)^d\le 2^{d+1}\int_{\{\varphi\le H_P-t\}}(H_P-\varphi)(dd^c\varphi)^d.$$

In particular, the left hand side converges to 0 as $t\to+\infty$ uniformly in u,v.

Proof. For $s>0$, we have the following inclusions of sets:

$$(u\le H_P-2s)\subset\left(\varphi\le \frac{v+H_P}{2}-s\right)\subset(\varphi\le H_P-s).$$

We first note that the left hand side in the lemma is equal to

$$\begin{aligned}&\int_{\{u\le H_P-2t\}}(H_P-u)(dd^cv)^d\\ &\quad =2t\int_{\{u\le H_P-2t\}}(dd^cv)^d+\int_{2t}^{\infty}\left(\int_{\{u\le H_P-s\}}(dd^cv)^d\right)ds.\end{aligned}\tag{3.2}$$

We claim that, for all $s>0$,

$$\int_{\{u\le H_P-2s\}}(dd^cv)^d\le 2^d\int_{\{\varphi\le H_P-s\}}(dd^c\varphi)^d.\tag{3.3}$$

Indeed, the comparison principle ([11, Corollary 3.6]) and the inclusions of sets above give

$$\begin{aligned}\int_{\{u\le H_P-2s\}}(dd^cv)^d&\le\int_{\{\varphi\le\frac{v+H_P}{2}-s\}}(dd^cv)^d\le 2^d\int_{\{\varphi\le\frac{v+H_P}{2}-s\}}\left(dd^c\frac{v+H_P}{2}\right)^d\\ &\le 2^d\int_{\{\varphi\le\frac{v+H_P}{2}-s\}}(dd^c\varphi)^d\le 2^d\int_{\{\varphi\le H_P-s\}}(dd^c\varphi)^d.\end{aligned}$$

The claim is proved. Using (3.3) and (3.2) we obtain

$$\begin{aligned}&\int_{\{u\le H_P-2t\}}(H_P-u)(dd^cv)^d\\ &\le 2^{d+1}t\int_{\{\varphi\le H_P-t\}}(dd^c\varphi)^d+2^{d+1}\int_t^{+\infty}\left(\int_{\{\varphi\le H_P-s\}}(dd^c\varphi)^d\right)ds\\ &=2^{d+1}\int_{\{\varphi\le H_P-t\}}(H_P-\varphi)(dd^c\varphi)^d. \quad\square\end{aligned}$$

Proposition 3.3. *Let $u\in\mathcal{E}^1_P(\mathbb{C}^d)$ with $(dd^cu)^d=\mu$ having support in a non-pluripolar compact set K so that $\int_K u\,d\mu>-\infty$ from Proposition 2.11. Let $\{Q_j\}$ be a sequence of continuous functions on K decreasing to u on K. Then $u_j:=V^*_{P,K,Q_j}\downarrow u$ on $\mathbb{C}^d$ and $\mu_j:=(dd^cu_j)^d$ is supported in K. In particular, $\mu_j\to\mu=(dd^cu)^d$ weak-*. Moreover,*

$$\lim_{j\to\infty}\int_K Q_j\,d\mu_j=\lim_{j\to\infty}\int_K Q_j\,d\mu=\int_K u\,d\mu>-\infty.\tag{3.4}$$

Proof. We can assume $\{Q_j\}$ are defined and decreasing to u on the closure of a bounded open neighborhood Ω of K. By adding a negative constant we can assume that $Q_1 \le 0$ on Ω. Since $\{Q_j\}$ is decreasing, so is the sequence $\{u_j\}$. Moreover, by [4, Proposition 5.1] $u_j \le Q_j$ on $K \setminus E_j$ where E_j is pluripolar. But u is a competitor in the definition of V_{P,K,Q_j} so that $u \le u_j$ on $\mathbb{C}^d$. Thus $\tilde{u} := \lim_{j\to\infty} u_j \ge u$ everywhere and $\tilde{u} \le u$ on $K \setminus E$, where $E := \cup_j E_j$ is a pluripolar set. Since $(dd^c u)^d$ puts no mass on pluripolar sets,

$$\int_{\{u<\tilde{u}\}} (dd^c u)^d \le \int_{E\cup(\mathbb{C}^d\setminus K)} (dd^c u)^d = 0.$$

It thus follows from Proposition 3.1 that $\tilde{u} \le u$, hence $\tilde{u} = u$ on $\mathbb{C}^d$.

The second equality in (3.4) follows from the monotone convergence theorem. It remains to prove that

$$\lim_{j\to\infty} \int_K (-Q_j)\, d\mu_j = \int_K (-u)\, d\mu.$$

For each k fixed and $j \ge k$ we have

$$\int_K (-Q_j)\, d\mu_j \ge \int_K (-Q_k)\, d\mu_j = \int_\Omega (-Q_k)\, d\mu_j,$$

hence $\liminf_{j\to\infty} \int_K (-Q_j)\, d\mu_j \ge \int_K (-Q_k)\, d\mu$ since Ω is open and μ_j, μ are supported on K. Letting $k \to +\infty$ we arrive at

$$\liminf_{j\to\infty} \int_K (-Q_j)\, d\mu_j \ge \int_K (-u)\, d\mu.$$

It remains to prove that

$$\limsup_{j\to\infty} \int_K (-Q_j)\, d\mu_j \le \int_K (-u)\, d\mu.$$

The sequence $\{u_j\}$ is not necessarily uniformly bounded below on K. However, using the facts that $Q_j \ge u$ and H_P is continuous in $\mathbb{C}^d$, it suffices to prove that

$$\limsup_{j\to\infty} \int_K (H_P - u)(dd^c u_j)^d \le \int_K (H_P - u)(dd^c u)^d. \tag{3.5}$$

To verify (3.5), we use Lemma 3.2.

By adding a negative constant we can assume that $u_j \le H_P$. For a function v and for $t>0$ we define $v^t := \max(v, H_P - t)$. Note that for each t the sequence $\{u_j^t\}$ is locally uniformly bounded below. Define

$$a(t) := 2^{d+1} \int_{\{u \le H_P - t/2\}} (H_P - u)(dd^c u)^d.$$

Since $u \in \mathcal{E}^1_P(\mathbb{C}^d)$, from Proposition 2.11 we have $a(t) \to 0$ as $t \to +\infty$. By Lemma 3.2 we have

$$\sup_{j\geq 1} \int_{\{u \leq H_P - t\}} (H_P - u)(dd^c u_j)^d \leq a(t). \tag{3.6}$$

By the plurifine property of non-pluripolar Monge-Ampère measures [10, Proposition 1.4] and (3.6) we have

$$\begin{aligned} \int_K (H_P - u)(dd^c u_j)^d &\leq \int_{K \cap \{u > H_P - t\}} (H_P - u)(dd^c u_j)^d + a(t) \\ &= \int_{K \cap \{u > H_P - t\}} (H_P - u^t)(dd^c u_j^t)^d + a(t) \\ &\leq \int_K (H_P - u^t)(dd^c u_j^t)^d + a(t). \end{aligned}$$

Since H_P is bounded in Ω, it follows from [16, Theorem 4.26] that the sequence of positive Radon measures $(H_P - u^t)(dd^c u_j^t)^d$ converges weakly on Ω to $(H_P - u^t)(dd^c u^t)^d$. Since K is compact it then follows that

$$\limsup_j \int_K (H_P - u)(dd^c u_j)^d \leq \int_K (H_P - u^t)(dd^c u^t)^d + a(t).$$

We finally let $t \to +\infty$ to conclude the proof in the following manner:

$$\begin{aligned} \int_K (H_P - u^t)(dd^c u^t)^d &\leq \int_{K \cap \{u > H_P - t\}} (H_P - u^t)(dd^c u^t)^d + a(t) \\ &\leq \int_K (H_P - u)(dd^c u)^d + a(t), \end{aligned}$$

where in the first estimate we have used $\{u \leq H_P - t\} = \{u^t \leq H_P - t\}$ and Lemma 3.2 and in the last estimate we use again the plurifine property. □

We now give an alternate description of the Legendre-type transform E^* from (2.9) which will be related to the rate function in a large deviation principle. Given $K \subset \mathbb{C}^d$ compact, we let $\mathcal{M}_P(K)$ denote the space of positive measures on K of total mass γ_d and we let $C(K)$ denote the set of continuous, real-valued functions on K.

Proposition 3.4. *Let K be a nonpluripolar compact set and $\mu \in \mathcal{M}_P(K)$. Then*

$$E^*(\mu) = \sup_{v \in C(K)} [E(V^*_{P,K,v}) - \int_K v \, d\mu].$$

Proof. We first treat the case when $E^*(\mu)=+\infty$. By Theorem 2.13 there exists $u\in\mathcal{E}^1_P(\mathbb{C}^d)$ such that $\int_K u\,d\mu=-\infty$. We take a decreasing sequence $Q_j\in C(K)$ such that $Q_j\downarrow u$ on K and set $u_j:=V^*_{P,K,Q_j}$. Then $\{u_j\}$ are decreasing; since $u\in\mathcal{E}^1_P(\mathbb{C}^d)$ and E is non-decreasing, $\{E(u_j)\}$ is uniformly bounded and we obtain

$$E(V^*_{P,K,Q_j})-\int_K Q_j\,d\mu\longrightarrow+\infty,$$

proving the proposition in this case.

Assume now that $E^*(\mu)<+\infty$. Theorem 2.13 ensures that $\int_{\mathbb{C}^d} u\,d\mu>-\infty$ for all $u\in\mathcal{E}^1_P(\mathbb{C}^d)$. By Lemma 2.15, μ puts no mass on pluripolar sets. From monotonicity of E and the definition of E^* in (2.9) we have

$$E^*(\mu)\ge\sup_{v\in C(K)}[E(V^*_{P,K,v})-\int_K v\,d\mu].$$

Here we have used that

$$V^*_{P,K,v}\le v \text{ q.e. on } K \text{ for } v\in C(K).$$

For the reverse inequality, fix $u\in\mathcal{E}^1_P(\mathbb{C}^d)$. Let $\{Q_j\}$ be a sequence of continuous functions on K decreasing to u on K and set $u_j:=V^*_{P,K,Q_j}$. Given $\varepsilon>0$, we can choose j sufficiently large so that, by monotone convergence,

$$\int_K Q_j\,d\mu\le\int_K u\,d\mu+\varepsilon;$$

and, by monotonicity of E,

$$E(V^*_{P,K,Q_j})\ge E(u).$$

Hence

$$E(V^*_{P,K,Q_j})-\int_K Q_j\,d\mu\ge E(u)-\int_K u\,d\mu-\varepsilon$$

so that

$$\sup_{v\in C(K)}[E(V^*_{P,K,v})-\int_K v\,d\mu]\ge E^*(\mu)$$

and equality holds. □

3.2. Transfinite diameter

Let $d_n = d_n(P)$ denote the dimension of the vector space $Poly(nP)$. We write

$$Poly(nP) = \text{span}\{e_1, ..., e_{d_n}\}$$

where $\{e_j(z) := z^{\alpha(j)}\}_{j=1,\ldots,d_n}$ are the standard basis monomials. Given $\zeta_1, ..., \zeta_{d_n} \in \mathbb{C}^d$, let

$$\begin{aligned} VDM(\zeta_1, ..., \zeta_{d_n}) &:= \det[e_i(\zeta_j)]_{i,j=1,\ldots,d_n} \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \ldots & e_{d_n}(\zeta_{d_n}) \end{bmatrix} \end{aligned} \tag{3.7}$$

and for $K \subset \mathbb{C}^d$ compact let

$$V_n = V_n(K) := \max_{\zeta_1,\ldots,\zeta_{d_n} \in K} |VDM(\zeta_1, ..., \zeta_{d_n})|.$$

It was shown in [2] that

$$\delta(K) := \delta(K, P) := \lim_{n\to\infty} V_n^{1/l_n} \tag{3.8}$$

exists where

$$l_n := \sum_{j=1}^{d_n} \deg(e_j) = \sum_{j=1}^{d_n} |\alpha(j)|$$

is the sum of the degrees of the basis monomials for $Poly(nP)$. We call $\delta(K)$ the $P-$*transfinite diameter* of K. More generally, for w an admissible weight function on K and $\zeta_1, ..., \zeta_{d_n} \in K$, let

$$\begin{aligned} VDM_n^Q(\zeta_1, ..., \zeta_{d_n}) &:= VDM(\zeta_1, ..., \zeta_{d_n}) w(\zeta_1)^n ... w(\zeta_{d_n})^n \\ &= \det \begin{bmatrix} e_1(\zeta_1) & e_1(\zeta_2) & \ldots & e_1(\zeta_{d_n}) \\ \vdots & \vdots & \ddots & \vdots \\ e_{d_n}(\zeta_1) & e_{d_n}(\zeta_2) & \ldots & e_{d_n}(\zeta_{d_n}) \end{bmatrix} \cdot w(\zeta_1)^n ... w(\zeta_{d_n})^n \end{aligned} \tag{3.9}$$

be a *weighted Vandermonde determinant.* Let

$$W_n(K) := \max_{\zeta_1,\ldots,\zeta_{d_n} \in K} |VDM_n^Q(\zeta_1, ..., \zeta_{d_n})|.$$

An *$n-th$ weighted $P-$Fekete set for K and w* is a set of d_n points $\zeta_1, ..., \zeta_{d_n} \in K$ with the property that

$$|VDM_n^Q(\zeta_1, ..., \zeta_{d_n})| = W_n(K).$$

The limit

$$\delta^Q(K):=\delta^Q(K,P):=\lim_{n\to\infty} W_n(K)^{1/l_n}$$

exists and is called the *weighted $P-$transfinite diameter*. The following was proved in [2].

Theorem 3.5. (Asymptotic Weighted $P-$Fekete Measures) *Let $K\subset\mathbb{C}^d$ be compact with admissible weight w. For each n, take points $z_1^{(n)},z_2^{(n)},...,z_{d_n}^{(n)}\in K$ for which*

$$\lim_{n\to\infty}\left[|VDM_n^Q(z_1^{(n)},...,z_{d_n}^{(n)})|\right]^{\frac{1}{l_n}}=\delta^Q(K) \tag{3.10}$$

(*asymptotically weighted $P-$Fekete arrays*) *and let $\mu_n:=\frac{1}{d_n}\sum_{j=1}^{d_n}\delta_{z_j^{(n)}}$. Then*

$$\mu_n\longrightarrow\frac{1}{\gamma_d}\mu_{K,Q}\ weak-*.$$

Another ingredient we will use is a Rumely-type relation between transfinite diameter and energy of $V_{P,K,Q}^*$ from [2].

Theorem 3.6. *Let $K\subset\mathbb{C}^d$ be compact and $w=e^{-Q}$ with $Q\in C(K)$. Then*

$$\log\delta^Q(K)=\frac{-1}{\gamma_d dA}\mathcal{E}(V_{P,K,Q}^*,H_P)=\frac{-(d+1)}{\gamma_d dA}E(V_{P,K,Q}^*). \tag{3.11}$$

Here $A=A(P,d)$ was defined in [2]; we recall the definition. For $P=\Sigma$ so that $Poly(n\Sigma)=\mathcal{P}_n$, we have

$$d_n(\Sigma)=\binom{d+n}{d}=0(n^d/d!)\text{ and }l_n(\Sigma)=\frac{d}{d+1}nd_n(\Sigma).$$

For a convex body $P\subset(\mathbb{R}^+)^d$, define $f_n(d)$ by writing

$$l_n=f_n(d)\frac{nd}{d+1}d_n=f_n(d)\frac{l_n(\Sigma)}{d_n(\Sigma)}d_n.$$

Then the ratio l_n/d_n divided by $l_n(\Sigma)/d_n(\Sigma)$ has a limit; i.e.,

$$\lim_{n\to\infty}f_n(d)=:A=A(P,d). \tag{3.12}$$

3.3. Bernstein-Markov

For $K\subset\mathbb{C}^d$ compact, $w=e^{-Q}$ an admissible weight function on K, and ν a finite measure on K, we say that the triple (K,ν,Q) satisfies a weighted Bernstein-Markov property if for all $p_n\in\mathcal{P}_n$,

$$\|w^n p_n\|_K \le M_n \|w^n p_n\|_{L^2(\nu)} \text{ with } \limsup_{n\to\infty} M_n^{1/n}=1. \tag{3.13}$$

Here, $\|w^n p_n\|_K := \sup_{z\in K} |w(z)^n p_n(z)|$ and

$$\|w^n p_n\|^2_{L^2(\nu)} := \int_K |p_n(z)|^2 w(z)^{2n}\, d\nu(z).$$

Following [1], given $P\subset(\mathbb{R}^+)^d$ a convex body, we say that a finite measure ν with support in a compact set K is a Bernstein-Markov measure for the triple (P,K,Q) if (3.13) holds for all $p_n\in Poly(nP)$.

For any P there exists $A=A(P)>0$ with $Poly(nP)\subset\mathcal{P}_{An}$ for all n. Thus if (K,ν,Q) satisfies a weighted Bernstein-Markov property, then ν is a Bernstein-Markov measure for $(P,K,\widetilde{Q})$ where $\widetilde{Q}=AQ$. In particular, if ν is a *strong Bernstein-Markov measure* for K; i.e., if ν is a weighted Bernstein-Markov measure for any $Q\in C(K)$, then for any such Q, ν is a Bernstein-Markov measure for the triple (P,K,Q). Strong Bernstein-Markov measures exist for any nonpluripolar compact set; cf., Corollary 3.8 of [9]. The paragraph following this corollary gives a sufficient mass-density type condition for a measure to be a strong Bernstein-Markov measure.

Given P, for ν a finite measure on K and $Q\in\mathcal{A}(K)$, define

$$Z_n := Z_n(P,K,Q,\nu) := \int_K \dots \int_K |VDM_n^Q(z_1,\dots,z_{d_n})|^2\, d\nu(z_1)\dots d\nu(z_{d_n}). \tag{3.14}$$

The main consequence of using a Bernstein-Markov measure for (P,K,Q) is the following:

Proposition 3.7. *Let $K\subset\mathbb{C}^d$ be a compact set and let $Q\in\mathcal{A}(K)$. If ν is a Bernstein-Markov measure for (P,K,Q) then*

$$\lim_{n\to\infty} Z_n^{\frac{1}{2l_n}} = \delta^Q(K). \tag{3.15}$$

Proof. That $\limsup_{n\to\infty} Z_n^{\frac{1}{2l_n}} \le \delta^Q(K)$ is clear. Observing from (3.7) and (3.9) that, fixing all variables but z_j,

$$z_j \longrightarrow VDM_n^Q(z_1,\dots,z_j,\dots,z_{d_n}) = w(z_j)^n p_n(z_j)$$

for some $p_n\in Poly(nP)$, to show $\liminf_{n\to\infty} Z_n^{\frac{1}{2l_n}} \ge \delta^Q(K)$ one starts with an $n-$th weighted $P-$Fekete set for K and w and repeatedly applies the weighted Bernstein-Markov property. □

Recall $\mathcal{M}_P(K)$ is the space of positive measures on K with total mass γ_d. With the weak-* topology, this is a separable, complete metrizable space. A neighborhood basis of $\mu\in\mathcal{M}_P(K)$ can be given by sets

$$G(\mu,k,\varepsilon):=\{\sigma\in\mathcal{M}_P(K):|\int_K(\mathrm{Re}z)^\alpha(\mathrm{Im}z)^\beta(d\mu-d\sigma)|<\varepsilon$$

$$\text{for } 0\leq|\alpha|+|\beta|\leq k\} \tag{3.16}$$

where $\mathrm{Re}z=(\mathrm{Re}z_1,...,\mathrm{Re}z_n)$ and $\mathrm{Im}z=(\mathrm{Im}z_1,...,\mathrm{Im}z_n)$.

Given ν as in Proposition 3.7, we define a probability measure $Prob_n$ on K^{d_n} via, for a Borel set $A\subset K^{d_n}$,

$$Prob_n(A):=\frac{1}{Z_n}\cdot\int_A|VDM_n^Q(z_1,...,z_{d_n})|^2\cdot d\nu(z_1)...d\nu(z_{d_n}). \tag{3.17}$$

We immediately obtain the following:

Corollary 3.8. *Let ν be a Bernstein-Markov measure for (P,K,Q). Given $\eta>0$, define*

$$A_{n,\eta}:=\{(z_1,...,z_{d_n})\in K^{d_n}:|VDM_n^Q(z_1,...,z_{d_n})|^2\geq(\delta^Q(K)-\eta)^{2l_n}\}. \tag{3.18}$$

Then there exists $n^*=n^*(\eta)$ such that for all $n>n^*$,

$$Prob_n(K^{d_n}\setminus A_{n,\eta})\leq\left(1-\frac{\eta}{2\delta^Q(K)}\right)^{2l_n}.$$

Remark 3.9. Corollary 3.8 was proved in [9], Corollary 3.2, for ν a probability measure but an obvious modification works for $\nu(K)<\infty$.

Using (3.17), we get an induced probability measure $\mathbf{P}$ on the infinite product space of arrays $\chi:=\{X=\{x_j^{(n)}\}_{n=1,2,...;\ j=1,...,d_n}:x_j^{(n)}\in K\}$:

$$(\chi,\mathbf{P}):=\prod_{n=1}^{\infty}(K^{d_n},Prob_n).$$

Corollary 3.10. *Let ν be a Bernstein-Markov measure for (P,K,Q). For $\mathbf{P}$-a.e. array $X=\{x_j^{(n)}\}\in\chi$,*

$$\nu_n:=\frac{1}{d_n}\sum_{j=1}^{d_n}\delta_{x_j^{(n)}}\longrightarrow\frac{1}{\gamma_d}\mu_{K,Q}\ \textit{weak-*}.$$

Proof. From Theorem 3.5 it suffices to verify for **P**-a.e. array $X=\{x_j^{(n)}\}$

$$\liminf_{n\to\infty}\left(|VDM_n^Q(x_1^{(n)},...,x_{d_n}^{(n)})|\right)^{\frac{1}{l_n}}=\delta^Q(K). \tag{3.19}$$

Given $\eta>0$, the condition that for a given array $X=\{x_j^{(n)}\}$ we have

$$\liminf_{n\to\infty}\left(|VDM_n^Q(x_1^{(n)},...,x_{d_n}^{(n)})|\right)^{\frac{1}{l_n}}\leq\delta^Q(K)-\eta$$

means that $(x_1^{(n)},...,x_{d_n}^{(n)})\in K^{d_n}\setminus A_{n,\eta}$ for infinitely many n. Setting

$$E_n:=\{X\in\chi:(x_1^{(n)},...,x_{d_n}^{(n)})\in K^{d_n}\setminus A_{n,\eta}\},$$

we have

$$\mathbf{P}(E_n)\leq Prob_n(K^{d_n}\setminus A_{n,\eta})\leq(1-\frac{\eta}{2\delta^Q(K)})^{2l_n}$$

and $\sum_{n=1}^{\infty}\mathbf{P}(E_n)<+\infty$. By the Borel-Cantelli lemma,

$$\mathbf{P}(\limsup_{n\to\infty}E_n)=\mathbf{P}(\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}^{\infty}E_k)=0.$$

Thus, with probability one, only finitely many E_n occur, and (3.19) follows. □

The main goal in the rest of the paper is to verify a stronger probabilistic result – a large deviation principle – and to explain this result in $P-$pluripotential-theoretic terms.

4. Relation between E^* and J, J^Q functionals

We define some functionals on $\mathcal{M}_P(K)$ using L^2-type notions which act as a replacement for an energy functional on measures. Then we show these functionals $\overline{J}(\mu)$ and $\underline{J}(\mu)$ defined using a "lim sup" and a "lim inf" coincide (see Definitions 4.1 and 4.2); this is the essence of our first proof of the large deviation principle, Theorem 5.1. Using Proposition 3.4, we relate this functional with E^* from (2.9).

Fix a nonpluripolar compact set K and a strong Bernstein-Markov measure ν on K. For simplicity, we normalize so that ν is a probability measure. Recall then for any $Q\in C(K)$, ν is a Bernstein-Markov measure for the triple (P,K,Q). Given $G\subset\mathcal{M}_P(K)$ open, for each $s=1,2,...$ we set

$$\widetilde{G}_s:=\{\mathbf{a}=(a_1,...,a_s)\in K^s:\frac{\gamma_d}{s}\sum_{j=1}^{s}\delta_{a_j}\in G\}. \tag{4.1}$$

Define, for $n=1,2,...,$

$$J_n(G):=[\int_{\widetilde{G}_{d_n}} |VDM_n(\mathbf{a})|^2\, d\nu(\mathbf{a})]^{1/2l_n}.$$

Definition 4.1. For $\mu\in\mathcal{M}_P(K)$ we define

$$\overline{J}(\mu):=\inf_{G\ni\mu}\overline{J}(G) \text{ where } \overline{J}(G):=\limsup_{n\to\infty} J_n(G);$$
$$\underline{J}(\mu):=\inf_{G\ni\mu}\underline{J}(G) \text{ where } \underline{J}(G):=\liminf_{n\to\infty} J_n(G).$$

The infima are taken over all neighborhoods G of the measure μ in $\mathcal{M}_P(K)$. A priori, $\overline{J},\underline{J}$ depend on ν. These functionals are nonnegative but can take the value zero. Intuitively, we are taking a "limit" of $L^2(\nu)$ averages of discrete, equally weighted approximants $\frac{\gamma_d}{s}\sum_{j=1}^s \delta_{a_j}$ of μ. An "L^∞" version of $\overline{J},\underline{J}$ was introduced in [8] where $J_n(G)$ is replaced by

$$W_n(G):=\sup_{\mathbf{a}\in\widetilde{G}_{d_n}} |VDM_n(\mathbf{a})|^{1/l_n}\geq J_n(G). \tag{4.2}$$

The weighted versions of these functionals are defined for $Q\in\mathcal{A}(K)$ using

$$J_n^Q(G):=[\int_{\widetilde{G}_{d_n}} |VDM_n^Q(\mathbf{a})|^2\, d\nu(\mathbf{a})]^{1/2l_n}. \tag{4.3}$$

Definition 4.2. For $\mu\in\mathcal{M}_P(K)$ we define

$$\overline{J}^Q(\mu):=\inf_{G\ni\mu}\overline{J}^Q(G) \text{ where } \overline{J}^Q(G):=\limsup_{n\to\infty} J_n^Q(G);$$
$$\underline{J}^Q(\mu):=\inf_{G\ni\mu}\underline{J}^Q(G) \text{ where } \underline{J}^Q(G):=\liminf_{n\to\infty} J_n^Q(G).$$

The uppersemicontinuity of $\overline{J},\overline{J}^Q,\underline{J}$ and $\underline{J}^Q$ on $\mathcal{M}_P(K)$ (with the weak-* topology) follows as in Lemma 3.1 of [8]. Set

$$b_d=b_d(P):=\frac{d+1}{Ad\gamma_d}.$$

Proposition 4.3. *Fix* $Q\in C(K)$. *Then*

(1) $\overline{J}^Q(\mu)\leq\delta^Q(K)$;

(2) $\overline{J}(\mu)=\overline{J}^Q(\mu)\cdot(e^{\int_K Q\,d\mu})^{b_d}$;

(3) $\log\overline{J}(\mu)\leq\inf_{v\in C(K)}[\log\delta^v(K)+b_d\int_K v\,d\mu]$;

(4) $\log\overline{J}^Q(\mu)\leq\inf_{v\in C(K)}[\log\delta^v(K)+b_d\int_K v\,d\mu]-b_d\int_K Q\,d\mu$.

Properties (1)-(4) *also hold for the functionals* $\underline{J},\underline{J}^Q$.

Proof. Property (1) follows from

$$J_n^Q(G) \le \sup_{\mathbf{a}\in\widetilde{G}_{d_n}} |VDM_n^Q(\mathbf{a})|^{1/l_n} \le \sup_{\mathbf{a}\in K^{d_n}} |VDM_n^Q(\mathbf{a})|^{1/l_n}.$$

The proofs of Corollary 3.4, Proposition 3.5 and Proposition 3.6 of [8] work mutatis mutandis to verify (2), (3) and (4). The relevant estimation, replacing the corresponding one which is two lines above equation (3.2) in [8], is, given $\varepsilon>0$, for $\mathbf{a}\in\widetilde{G}_{d_n}$,

$$\begin{aligned} |VDM_n^Q(\mathbf{a})| e^{\frac{nd_n}{\gamma_d}(-\varepsilon-\int_K Q\,d\mu)} &\le |VDM_n(\mathbf{a})| \\ &\le |VDM_n^Q(\mathbf{a})| e^{\frac{nd_n}{\gamma_d}(\varepsilon+\int_K Q\,d\mu)}. \end{aligned} \tag{4.4}$$

To see this, we first recall that

$$|VDM_n(\mathbf{a})| = |VDM_n^Q(\mathbf{a})| e^{n\sum_{j=1}^{d_n} Q(a_j)}.$$

For $\mu\in\mathcal{M}_P(K)$, $Q\in C(K)$, $\varepsilon>0$, there exists a neighborhood G of μ in $\mathcal{M}_P(K)$ with

$$-\varepsilon < \int_K Q\,d\mu - \frac{\gamma_d}{d_n}\sum_{j=1}^{d_n} Q(a_j) < \varepsilon$$

for $\mathbf{a}\in\widetilde{G}_{d_n}$. Plugging this double inequality into the previous equality we get (4.4). Moreover, from (3.12),

$$\lim_{n\to\infty} \frac{nd_n}{l_n} = \frac{d+1}{Ad} = b_d\gamma_d \tag{4.5}$$

so that $\frac{nd_n}{\gamma_d} \asymp l_n b_d$ as $n\to\infty$. Taking l_n−th roots in (4.4) accounts for the factor of b_d in (2), (3) and (4). □

Remark 4.4. The corresponding $\underline{W}, \underline{W}^Q, \overline{W}, \overline{W}^Q$ functionals, defined using (4.2), clearly dominate their "J" counterparts; e.g., $\overline{W}^Q \ge \overline{J}^Q$.

Note that formula (3.11) can be rewritten:

$$\log\delta^Q(K) = -b_d E(V_{P,K,Q}^*). \tag{4.6}$$

Thus the upper bound in Proposition 4.3 (3) becomes

$$\log\overline{J}(\mu) \le -b_d \sup_{v\in C(K)} [E(V_{P,K,v}^*) - \int_K v\,d\mu] = -b_d E^*(\mu). \tag{4.7}$$

For the rest of section 4 and section 5, we will always assume $Q\in C(K)$. Theorem 4.5 shows that the inequalities in (3) and (4) are equalities, and that the $\overline{J}, \overline{J}^Q$ functionals coincide with their $\underline{J}, \underline{J}^Q$ counterparts. The key step in the proof of Theorem 4.5 is to verify this for $\overline{J}^v(\mu_{K,v})$ and $\underline{J}^v(\mu_{K,v})$.

Theorem 4.5. *Let $K\subset\mathbb{C}^d$ be a nonpluripolar compact set and let ν satisfy a strong Bernstein-Markov property. Fix $Q\in C(K)$. Then for any $\mu\in\mathcal{M}_P(K)$,*

$$\log \overline{J}(\mu)=\log \underline{J}(\mu)=\inf_{v\in C(K)}[\log \delta^v(K)+b_d\int_K v\,d\mu] \tag{4.8}$$

and

$$\log \overline{J}^Q(\mu)=\log \underline{J}^Q(\mu)=\inf_{v\in C(K)}[\log \delta^v(K)+b_d\int_K v\,d\mu]-b_d\int_K Q\,d\mu. \tag{4.9}$$

Proof. It suffices to prove (4.8) since (4.9) follows from (2) of Proposition 4.3. We have the upper bound

$$\log \overline{J}(\mu)\leq \inf_{v\in C(K)}[\log \delta^v(K)+b_d\int_K v\,d\mu]$$

from (3); for the lower bound, we consider different cases.

Case I: $\mu=\mu_{K,v}$ for some $v\in C(K)$.

We verify that

$$\log \overline{J}(\mu_{K,v})=\log \underline{J}(\mu_{K,v})=\log \delta^v(K)+b_d\int_K v\,d\mu_{K,v} \tag{4.10}$$

which proves (4.8) in this case.

To prove (4.10), we use the definition of $\underline{J}(\mu_{K,v})$ and Corollary 3.8. Fix a neighborhood G of $\mu_{K,v}$. For $\eta>0$, define $A_{n,\eta}$ as in (3.18) with $Q=v$. Set

$$\eta_n:=\max\left(\delta^v(K)-\frac{nZ_n^{1/2l_n}}{n+1},\frac{Z_n^{1/2l_n}}{n+1}\right). \tag{4.11}$$

By Proposition 3.7, $\eta_n\to 0$. We claim that we have the inclusion

$$A_{n,\eta_n}\subset \widetilde{G}_{d_n} \text{ for all } n \text{ large enough.} \tag{4.12}$$

We prove (4.12) by contradiction: if false, there is a sequence $\{n_j\}$ with $n_j\uparrow\infty$ and $x^j=(x_1^j,...,x_{d_{n_j}}^j)\in A_{n_j,\eta_{n_j}}\setminus\widetilde{G}_{d_{n_j}}$. However $\mu_j:=\frac{\gamma_d}{d_{n_j}}\sum_{i=1}^{d_{n_j}}\delta_{x_i^j}\notin G$ for j sufficiently large contradicts Theorem 3.5 since $x^j\in A_{n_j,\eta_j}$ and $\eta_j\downarrow 0$ imply $\mu_j\to\mu_{K,v}$ weak-*.

Next, a direct computation using (4.11) shows that, for all n large enough,

$$Prob_n(K^{d_n}\setminus A_{n,\eta_n})\leq \frac{(\delta^v(K)-\eta_n)^{2l_n}}{Z_n}\leq (\frac{n}{n+1})^{2l_n}\leq \frac{n}{n+1} \tag{4.13}$$

(recall ν is a probability measure). Hence

$$\begin{aligned}&\frac{1}{Z_n}\int_{\widetilde{G}_{d_n}} |VDM_n^v(z_1,...,z_{d_n})|^2\cdot d\nu(z_1)...d\nu(z_{d_n})\\ &\quad\ge \frac{1}{Z_n}\int_{A_{n,\eta_n}} |VDM_n^v(z_1,...,z_{d_n})|^2\cdot d\nu(z_1)...d\nu(z_{d_n})\\ &\quad\ge \frac{1}{n+1}.\end{aligned}$$

Since $P\subset r\Sigma$ and $\Sigma\subset kP$ for some $k\in\mathbb{Z}^+$, $l_n=0(n^{d+1})$ and we have $\frac{1}{2l_n}\log(n+1)\to 0$. Since ν satisfies a strong Bernstein-Markov property and $v\in C(K)$, using Proposition 3.7 and the above estimate we conclude that

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{2l_n}\log\int_{\widetilde{G}_{d_n}} |VDM_n^v(z_1,...,z_{d_n})|^2\, d\nu(z_1)...d\nu(z_{d_n})\\ &\ge \log\delta^v(K).\end{aligned}$$

Taking the infimum over all neighborhoods G of $\mu_{K,v}$ we obtain

$$\log \underline{J}^v(\mu_{K,v})\ge \log\delta^v(K).$$

From (1) Proposition 4.3, $\log \overline{J}^v(\mu_{K,v})\le\log\delta^v(K)$; thus we have

$$\log \underline{J}^v(\mu_{K,v})=\log \overline{J}^v(\mu_{K,v})=\log\delta^v(K). \tag{4.14}$$

Using (2) of Proposition 4.3 with $\mu=\mu_{K,v}$ we obtain (4.10).

Case II: $\mu\in\mathcal{M}_P(K)$ with the property that $E^*(\mu)<\infty$.

From Theorem 2.13 and Proposition 2.11 there exists $u\in L_P(\mathbb{C}^d)$ – indeed, $u\in\mathcal{E}_P^1(\mathbb{C}^d)$ – with $\mu=(dd^cu)^d$ and $\int_K u\,d\mu>-\infty$. However, since u is only usc on K, μ is not necessarily of the form $\mu_{K,v}$ for some $v\in C(K)$. Taking a sequence of continuous functions $\{Q_j\}\subset C(K)$ with $Q_j\downarrow u$ on K, by Proposition 3.3 the weighted extremal functions V^*_{P,K,Q_j} decrease to u on $\mathbb{C}^d$;

$$\mu_j:=(dd^cV^*_{P,K,Q_j})^d\longrightarrow\mu=(dd^cu)^d \text{ weak-}*;$$

and

$$\lim_{j\to\infty}\int_K Q_j\,d\mu_j=\lim_{j\to\infty}\int_K Q_j\,d\mu=\int_K u\,d\mu. \tag{4.15}$$

From the previous case we have

$$\log\overline{J}(\mu_j)=\log\underline{J}(\mu_j)=\log\delta^{Q_j}(K)+b_d\int_K Q_j\,d\mu_j.$$

Using uppersemicontinuity of the functional $\mu\to\underline{J}(\mu)$,

$$\limsup_{j\to\infty}\overline{J}(\mu_j)=\limsup_{j\to\infty}\underline{J}(\mu_j)\le\underline{J}(\mu).$$

Since $Q_j\downarrow u$ on K,

$$\limsup_{j\to\infty}\log\delta^{Q_j}(K)=\lim_{j\to\infty}\log\delta^{Q_j}(K). \tag{4.16}$$

Therefore

$$M:=\lim_{j\to\infty}\log\underline{J}(\mu_j)=\lim_{j\to\infty}\big(\log\delta^{Q_j}(K)+b_d\int_K Q_j\,d\mu_j\big)$$

exists and is less than or equal to $\log\underline{J}(\mu)$. We want to show that

$$\inf_v[\log\delta^v(K)+b_d\int_K v\,d\mu]\le M. \tag{4.17}$$

Given $\varepsilon>0$, by (4.15) for $j\ge j_0(\varepsilon)$,

$$\int_K Q_j\,d\mu_j\ge\int_K Q_j\,d\mu-\varepsilon \text{ and } \log\underline{J}(\mu_j)<M+\varepsilon.$$

Hence for such j,

$$\begin{aligned}\inf_v[\log\delta^v(K)+b_d\int_K v\,d\mu]&\le\log\delta^{Q_j}(K)+b_d\int_K Q_j\,d\mu\\&\le\log\delta^{Q_j}(K)+b_d\int_K Q_j\,d\mu_j+b_d\varepsilon\\&=\log\underline{J}(\mu_j)+b_d\varepsilon<M+(b_d+1)\varepsilon,\end{aligned}$$

yielding (4.17). This finishes the proof in Case II.

Case III: $\mu\in\mathcal{M}(K)$ with the property that $E^*(\mu)=+\infty$.

It follows from Proposition 3.4 and Theorem 3.6 that the right-hand side of (4.8) is $-\infty$, finishing the proof. □

Remark 4.6. From now on, we simply use the notation J, J^Q without the overline or underline. Using Proposition 3.4 and Theorem 3.6, we have

$$\begin{aligned}\log J(\mu)&=\inf_{Q\in C(K)}[\log\delta^Q(K)+b_d\int_K Q\,d\mu]\\&=-\sup_{Q\in C(K)}[-\log\delta^Q(K)-b_d\int_K Q\,d\mu]\\&=-\sup_{Q\in C(K)}[b_dE(V^*_{P,K,Q})-b_d\int_K Q\,d\mu]=-b_d\sup_{Q\in C(K)}[E(V^*_{P,K,Q})-\int_K Q\,d\mu]\end{aligned}$$

(recall (4.6)) which one can compare with

$$E^*(\mu) = \sup_{Q\in C(K)} [E(V^*_{P,K,Q}) - \int_K Q\, d\mu]$$

from Proposition 3.4 to conclude

$$\log J(\mu) = -b_d E^*(\mu). \tag{4.18}$$

In particular, J, J^Q are independent of the choice of strong Bernstein-Markov measure for K.

Following the idea in Proposition 4.3 of [9], we observe the following:

Proposition 4.7. *Let $K\subset\mathbb{C}^d$ be a nonpluripolar compact set and let ν satisfy a strong Bernstein-Markov property. Fix $Q\in C(K)$. The measure $\mu_{K,Q}$ is the unique maximizer of the functional $\mu\to J^Q(\mu)$ over $\mu\in\mathcal{M}_P(K)$; i.e.,*

$$J^Q(\mu_{K,Q}) = \delta^Q(K) \ (\textit{and } J(\mu_K) = \delta(K)). \tag{4.19}$$

Proof. The fact that $\mu_{K,Q}$ maximizes J^Q (and μ_K maximizes J) follows from (4.10), (4.14) and Proposition 4.3.

Assume now that $\mu\in\mathcal{M}_P(K)$ maximizes J^Q. From Remark 4.4 and the definitions of the functionals, for any neighborhood $G\subset\mathcal{M}_P(K)$ of μ,

$$\overline{J}^Q(\mu) \le \overline{W}^Q(\mu) \le \sup\{\limsup_{n\to\infty} |VDM_n^Q(\mathbf{a}^{(n)})|^{1/l_n}\} \le \delta^Q(K)$$

where the supremum is taken over all arrays $\{\mathbf{a}^{(n)}\}_{n=1,2,\ldots}$ of d_n−tuples $\mathbf{a}^{(n)}$ in K whose normalized counting measures $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n}\delta_{a_j^{(n)}}$ lie in G. Since $\overline{J}^Q(\mu) = \delta^Q(K)$ there is an asymptotic weighted Fekete array $\{\mathbf{a}^{(n)}\}$ as in (3.10). Theorem 3.5 yields that $\mu_n := \frac{1}{d_n}\sum_{j=1}^{d_n}\delta_{a_j^{(n)}}$ converges weak-* to $\mu_{K,Q}$, hence $\mu_{K,Q}\in\overline{G}$. Since this is true for each neighborhood $G\subset\mathcal{M}_P(K)$ of μ, we must have $\mu = \mu_{K,Q}$. □

5. Large deviation

As in the previous section, we fix $K\subset\mathbb{C}^d$ a nonpluripolar compact set; $Q\in C(K)$; and a measure ν on K satisfying a strong Bernstein-Markov property. For $x_1,\ldots,x_{d_n}\in K$, we get a discrete measure $\frac{\gamma_d}{d_n}\sum_{j=1}^{d_n}\delta_{x_j}\in\mathcal{M}_P(K)$. Define $j_n : K^{d_n}\to\mathcal{M}_P(K)$ via

$$j_n(x_1,\ldots,x_{d_n}) := \frac{\gamma_d}{d_n}\sum_{j=1}^{d_n}\delta_{x_j}.$$

From (3.17), $\sigma_n := (j_n)_*(Prob_n)$ is a probability measure on $\mathcal{M}_P(K)$: for a Borel set $B \subset \mathcal{M}_P(K)$,

$$\sigma_n(B) = \frac{1}{Z_n} \int_{\widetilde{B}_{d_n}} |VDM_n^Q(x_1, ..., x_{d_n})|^2 \, d\nu(x_1)...d\nu(x_{d_n}) \tag{5.1}$$

where $\widetilde{B}_{d_n} := \{\mathbf{a} = (a_1, ..., a_{d_n}) \in K^{d_n} : \frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} \delta_{a_j} \in B\}$(recall (4.1)). Here, $Z_n := Z_n(P, K, Q, \nu)$. Note that

$$\sigma_n(B)^{1/2l_n} = \frac{1}{Z_n^{1/2l_n}} \cdot J_n^Q(B). \tag{5.2}$$

For future use, suppose we have a function $F : \mathbb{R} \to \mathbb{R}$ and a function $v \in C(K)$. We write, for $\mu \in \mathcal{M}_P(K)$,

$$< v, \mu > := \int_K v \, d\mu$$

and then

(5.3)

$$\int_{\mathcal{M}_P(K)} F(< v, \mu >) d\sigma_n(\mu)$$
$$:= \frac{1}{Z_n} \int_K ... \int_K |VDM_n^Q(x_1, ..., x_{d_n})|^2 F\left(\frac{\gamma_d}{d_n} \sum_{j=1}^{d_n} v(x_j)\right) d\nu(x_1)...d\nu(x_{d_n}).$$

With this notation, we offer two proofs of our LDP, Theorem 5.1. We state the result; define LDP in Definition 5.2; and then proceed with the proofs. This closely follows the exposition in section 5 of [9].

Theorem 5.1. *The sequence* $\{\sigma_n = (j_n)_*(Prob_n)\}$ *of probability measures on* $\mathcal{M}_P(K)$ *satisfies a* **large deviation principle** *with speed* $2l_n$ *and good rate function* $\mathcal{I} := \mathcal{I}_{K,Q}$ *where, for* $\mu \in \mathcal{M}_P(K)$,

$$\mathcal{I}(\mu) := \log J^Q(\mu_{K,Q}) - \log J^Q(\mu).$$

This means that $\mathcal{I} : \mathcal{M}_P(K) \to [0, \infty]$ is a lowersemicontinuous mapping such that the sublevel sets $\{\mu \in \mathcal{M}_P(K) : \mathcal{I}(\mu) \leq \alpha\}$ are compact in the weak-* topology on $\mathcal{M}_P(K)$ for all $\alpha \geq 0$ ($\mathcal{I}$ is "good") satisfying (5.4) and (5.5):

Definition 5.2. The sequence $\{\mu_n\}$ of probability measures on $\mathcal{M}_P(K)$ satisfies a **large deviation principle** (LDP) with good rate function $\mathcal{I}$ and speed $2l_n$ if for all measurable sets $\Gamma \subset \mathcal{M}_P(K)$,

$$-\inf_{\mu \in \Gamma^0} \mathcal{I}(\mu) \leq \liminf_{n \to \infty} \frac{1}{2l_n} \log \mu_n(\Gamma) \text{ and} \tag{5.4}$$

$$\limsup_{n\to\infty}\frac{1}{2l_n}\log\mu_n(\Gamma)\leq -\inf_{\mu\in\overline{\Gamma}}\mathcal{I}(\mu). \tag{5.5}$$

In the setting of $\mathcal{M}_P(K)$, to prove a LDP it suffices to work with a base for the weak-* topology. The following is a special case of a basic general existence result for a LDP given in Theorem 4.1.11 in [13].

Proposition 5.3. *Let $\{\sigma_\varepsilon\}$ be a family of probability measures on $\mathcal{M}_P(K)$. Let $\mathcal{B}$ be a base for the topology of $\mathcal{M}_P(K)$. For $\mu\in\mathcal{M}_P(K)$ let*

$$\mathcal{I}(\mu):=-\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\liminf_{\varepsilon\to 0}\varepsilon\log\sigma_\varepsilon(G)\big).$$

Suppose for all $\mu\in\mathcal{M}_P(K)$,

$$\mathcal{I}(\mu)=-\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\limsup_{\varepsilon\to 0}\varepsilon\log\sigma_\varepsilon(G)\big).$$

Then $\{\sigma_\varepsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\varepsilon$.

There is a converse to Proposition 5.3, Theorem 4.1.18 in [13]. For $\mathcal{M}_P(K)$, it reads as follows:

Proposition 5.4. *Let $\{\sigma_\varepsilon\}$ be a family of probability measures on $\mathcal{M}_P(K)$. Suppose that $\{\sigma_\varepsilon\}$ satisfies a LDP with rate function $\mathcal{I}(\mu)$ and speed $1/\varepsilon$. Then for any base $\mathcal{B}$ for the topology of $\mathcal{M}_P(K)$ and any $\mu\in\mathcal{M}_P(K)$*

$$\begin{aligned}\mathcal{I}(\mu)&:=-\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\liminf_{\varepsilon\to 0}\varepsilon\log\sigma_\varepsilon(G)\big)\\&=-\inf_{\{G\in\mathcal{B}:\mu\in G\}}\big(\limsup_{\varepsilon\to 0}\varepsilon\log\sigma_\varepsilon(G)\big).\end{aligned}$$

Remark 5.5. Assuming Theorem 5.1, this shows that, starting with a strong Bernstein-Markov measure ν and the corresponding sequence of probability measures $\{\sigma_n\}$ on $\mathcal{M}_P(K)$ in (5.1), the existence of an LDP with rate function $\mathcal{I}(\mu)$ and speed $2l_n$ implies that necessarily

$$\mathcal{I}(\mu)=\log J^Q(\mu_{K,Q})-\log J^Q(\mu). \tag{5.6}$$

Uniqueness of the rate function is basic (cf., Lemma 4.1.4 of [13]).

We turn to the first proof of Theorem 5.1, using Theorem 4.5, which gives a pluripotential theoretic description of the rate functional.

Proof. As a base $\mathcal{B}$ for the topology of $\mathcal{M}_P(K)$, we can take the sets from (3.16) or simply all open sets. For $\{\sigma_\varepsilon\}$, we take the sequence of probability measures $\{\sigma_n\}$ on $\mathcal{M}_P(K)$ and we take $\varepsilon=\frac{1}{2l_n}$. For $G\in\mathcal{B}$, from (5.2),

$$\frac{1}{2l_n}\log \sigma_n(G)=\log J_n^Q(G)-\frac{1}{2l_n}\log Z_n.$$

From Proposition 3.7, and (4.14) with $v=Q$,

$$\lim_{n\to\infty}\frac{1}{2l_n}\log Z_n=\log \delta^Q(K)=\log J^Q(\mu_{K,Q});$$

and by Theorem 4.5,

$$\inf_{G\ni\mu}\limsup_{n\to\infty}\log J_n^Q(G)=\inf_{G\ni\mu}\liminf_{n\to\infty}\log J_n^Q(G)=\log J^Q(\mu).$$

Thus by Proposition 5.3 $\{\sigma_n\}$ satisfies an LDP with rate function

$$\mathcal{I}(\mu):=\log J^Q(\mu_{K,Q})-\log J^Q(\mu)$$

and speed $2l_n$. This rate function is good since $\mathcal{M}_P(K)$ is compact. □

Remark 5.6. From Proposition 4.7, $\mu_{K,Q}$ is the unique maximizer of the functional

$$\mu\longrightarrow \log J^Q(\mu)$$

over all $\mu\in\mathcal{M}_P(K)$. Thus

$$\mathcal{I}_{K,Q}(\mu)\geq 0 \text{ with } \mathcal{I}_{K,Q}(\mu)=0 \quad\Longleftrightarrow\quad \mu=\mu_{K,Q}.$$

To summarize, $\mathcal{I}_{K,Q}$ is a good rate function with unique minimizer $\mu_{K,Q}$. Using the relations

$$\log J(\mu)=-b_d\sup_{Q\in C(K)}[E(V^*_{P,K,Q})-\int_K Q\,d\mu]$$
$$J(\mu)=J^Q(\mu)\cdot(e^{\int_K Q\,d\mu})^{b_d},\ \text{and } J^Q(\mu_{K,Q})=\delta^Q(K)$$

(the latter from (4.19)), we have

$$\begin{aligned}\mathcal{I}(\mu)&:=\log\delta^Q(K)-\log J^Q(\mu)\\ &=\log\delta^Q(K)-\log J(\mu)+b_d\int_K Q\,d\mu\\ &=b_d\sup_{Q\in C(K)}[E(V^*_{P,K,Q})-\int_K Q\,d\mu]+\log\delta^Q(K)+b_d\int_K Q\,d\mu\\ &=b_d\sup_{v\in C(K)}[E(V^*_{P,K,v})-\int_K v\,d\mu]-b_d[E(V^*_{P,K,Q})-\int_K Q\,d\mu]\end{aligned}$$

from (4.6).

The second proof of our LDP follows from Corollary 4.6.14 in [13], which is a general version of the Gärtner-Ellis theorem. This approach was originally brought to our attention by S. Boucksom and was also utilized by R. Berman in [5]. We state the version of the [13] result for an appropriate family of probability measures.

Proposition 5.7. *Let $C(K)^*$ be the topological dual of $C(K)$, and let $\{\sigma_\varepsilon\}$ be a family of probability measures on $\mathcal{M}_P(K)\subset C(K)^*$ (equipped with the weak-* topology). Suppose for each $\lambda\in C(K)$, the limit*

$$\Lambda(\lambda):=\lim_{\varepsilon\to 0}\varepsilon\log\int_{C(K)^*}e^{\lambda(x)/\varepsilon}d\sigma_\varepsilon(x)$$

exists as a finite real number and assume Λ is Gâteaux differentiable; i.e., for each $\lambda,\theta\in C(K)$, the function $f(t):=\Lambda(\lambda+t\theta)$ is differentiable at $t=0$. Then $\{\sigma_\varepsilon\}$ satisfies an LDP in $C(K)^*$ with the convex, good rate function Λ^*.

Here

$$\Lambda^*(x):=\sup_{\lambda\in C(K)}\big(<\lambda,x>-\Lambda(\lambda)\big),$$

is the Legendre transform of Λ. The upper bound (5.5) in the LDP holds with rate function Λ^* under the assumption that the limit $\Lambda(\lambda)$ exists and is finite; the Gâteaux differentiability of Λ is needed for the lower bound (5.4). To verify this property in our setting, we must recall a result from [2].

Proposition 5.8. *For $Q\in\mathcal{A}(K)$ and $u\in C(K)$, let*

$$F(t):=E(V^*_{P,K,Q+tu})$$

for $t\in\mathbb{R}$. Then F is differentiable and

$$F'(t)=\int_{\mathbb{C}^d}u(dd^cV^*_{P,K,Q+tu})^d.$$

In [2] it was assumed that $u\in C^2(K)$ but the result is true with the weaker assumption $u\in C(K)$ (cf., Theorem 11.11 in [16] due to Lu and Nguyen [17], see also [11, Proposition 4.20]).

We proceed with the second proof of Theorem 5.1. For simplicity, we normalize so that $\gamma_d=1$ to fit the setting of Proposition 5.7 (so members of $\mathcal{M}_P(K)$ are probability measures).

Proof. We show that for each $v\in C(K)$,

$$\Lambda(v):=\lim_{n\to\infty}\frac{1}{2l_n}\log\int_{C(K)^*}e^{2l_n<v,\mu>}d\sigma_n(\mu)$$

exists as a finite real number. First, since σ_n is a measure on $\mathcal{M}_P(K)$, the integral can be taken over $\mathcal{M}_P(K)$. Consider

$$\frac{1}{2l_n}\log\int_{\mathcal{M}_P(K)} e^{2l_n<v,\mu>}d\sigma_n(\mu).$$

By (5.3), this is equal to

$$\frac{1}{2l_n}\log\frac{1}{Z_n}\cdot\int_{K^{d_n}}|VDM_n^{Q-\frac{l_n}{nd_n}v}(x_1,...,x_{d_n})|^2\,d\nu(x_1)...d\nu(x_{d_n}).$$

From (4.5), with $\gamma_d{=}1$, $\frac{l_n}{nd_n}\to\frac{1}{b_d}$; hence for any $\varepsilon{>}0$,

$$\frac{1}{b_d{+}\varepsilon}v\le\frac{l_n}{nd_n}v\le\frac{1}{b_d{-}\varepsilon}v \text{ on } K$$

for n sufficiently large. Recall that

$$Z_n=\int_{K^{d_n}}|VDM_n^Q(x_1,...,x_{d_n}))|^2\,d\nu(x_1)...d\nu(x_{d_n}).$$

Define

$$\widetilde{Z}_n:=\int_{K^{d_n}}|VDM_n^{Q-v/b_d}(x_1,...,x_{d_n})|^2\,d\nu(x_1)...d\nu(x_{d_n}).$$

Then we have

$$\lim_{n\to\infty}\widetilde{Z}_n^{\frac{1}{2l_n}}=\delta^{Q-v/b_d}(K) \text{ and } \lim_{n\to\infty}Z_n^{\frac{1}{2l_n}}=\delta^Q(K)$$

from (3.15) in Proposition 3.7 and the assumption that $(K,\nu,\widetilde{Q})$ satisfies the weighted Bernstein-Markov property for *all* $\widetilde{Q}{\in}C(K)$. Thus

$$\Lambda(v)=\lim_{n\to\infty}\frac{1}{2l_n}\log\frac{\widetilde{Z}_n}{Z_n}=\log\frac{\delta^{Q-v/b_d}(K)}{\delta^Q(K)}. \tag{5.7}$$

Define now, for $v,v'{\in}C(K)$,

$$f(t):=E(V^*_{P,K,Q-(v+tv')}).$$

Proposition 5.8 shows that Λ is Gâteaux differentiable and Proposition 5.7 gives that Λ^* is a rate function on $C(K)^*$.

Since each σ_n has support in $\mathcal{M}_P(K)$, it follows from (5.4) and (5.5) in Definition 5.2 of an LDP with $\Gamma{\subset}C(K)^*$ that for $\mu{\in}C(K)^*\setminus\mathcal{M}_P(K)$, $\Lambda^*(\mu){=}{+}\infty$. By Lemma 4.1.5 (b) of [13], the restriction of Λ^* to $\mathcal{M}_P(K)$ is a rate function. Since $\mathcal{M}_P(K)$ is compact, it is a good rate function. Being a Legendre transform, Λ^* is convex.

To compute Λ^*, we have, using (5.7) and (3.11),

$$\Lambda^*(\mu) = \sup_{v\in C(K)} \Big(\int_K v\,d\mu - \log \frac{\delta^{Q-v/b_d}(K)}{\delta^Q(K)}\Big)$$
$$= \sup_{v\in C(K)} \Big(\int_K v\,d\mu - b_d[E(V^*_{P,K,Q}) - E(V^*_{P,K,Q-v/b_d}])\Big).$$

Thus

$$\Lambda^*(\mu) + b_d E(V^*_{P,K,Q}) = \sup_{v\in C(K)} \Big(\int_K v\,d\mu + b_d E(V^*_{P,K,Q-v/b_d})\Big)$$
$$= \sup_{u\in C(K)} \Big(b_d E(V^*_{P,K,Q+u}) - b_d \int_K u\,d\mu\Big) \text{ (taking } u = -v/b_d).$$

Rearranging and replacing u in the supremum by $v=u+Q$,

$$\Lambda^*(\mu) = \sup_{u\in C(K)} \Big(b_d E(V^*_{P,K,Q+u}) - b_d \int_K u\,d\mu\Big) - b_d E(V^*_{P,K,Q})$$
$$= b_d\Big[\sup_{v\in C(K)} E(V^*_{P,K,v}) - \int_K v\,d\mu\Big] - b_d\Big[E(V^*_{P,K,Q}) - \int_K Q\,d\mu\Big]$$

which agrees with the formula in Remark 5.6 (since μ is a probability measure). □

Remark 5.9. Thus the rate function can be expressed in several equivalent ways:

$$\mathcal{I}(\mu) = \Lambda^*(\mu) = \log J^Q(\mu_{K,Q}) - \log J^Q(\mu)$$
$$= b_d\Big[\sup_{v\in C(K)} E(V^*_{P,K,v}) - \int_K v\,d\mu\Big] - b_d\Big[E(V^*_{P,K,Q}) - \int_K Q\,d\mu\Big]$$
$$= b_d E^*(\mu) - b_d\Big[E(V^*_{P,K,Q}) - \int_K Q\,d\mu\Big]$$

which generalizes the result equating (5.3), (5.10) and (5.11) in [9] for the case $P=\Sigma$ and $b_d=1$. Note in the last equality we are using the slightly different notion of E^* in (2.9) and Proposition 3.4 than that used in [9].

References

1. Bayraktar, T., Zero distribution of random sparse polynomials, *Michigan Math. J.* **66** (2017), 389–419. MR3657224
2. Bayraktar, T., Bloom, T. and Levenberg, N., Pluripotential theory and convex bodies, *Mat. Sb.* **209** (2018), 67–101. MR3769215

3. Bedford, E. and Taylor, B. A., The Dirichlet problem for a complex Monge-Ampère equation, *Invent. Math.* **37** (1976), 1–44. MR0445006
4. Bedford, E. and Taylor, B. A., A new capacity for plurisubharmonic functions, *Acta Math.* **149** (1982), 1–40. MR0674165
5. Berman, R., Determinantal Point Processes and Fermions on Complex Manifolds: Large Deviations and Bosonization, *Comm. Math. Phys.* **327** (2014), 1–47. MR3177931
6. Berman, R. and Boucksom, S., Growth of balls of holomorphic sections and energy at equilibrium, *Invent. Math.* **181** (2010), 337–394. MR2657428
7. Berman, R., Boucksom, S., Guedj, V. and Zeriahi, A., A variational approach to complex Monge-Ampère equations, *Publ. Math. Inst. Hautes Études Sci.* **117** (2013), 179–245. MR3090260
8. Bloom, T. and Levenberg, N., Pluripotential energy, *Potential Anal.* **36** (2012), 155–176. MR2886457
9. Bloom, T. and Levenberg, N., Pluripotential energy and large deviation, *Indiana Univ. Math. J.* **62** (2013), 523–550. MR3158519
10. Boucksom, S., Eyssidieux, P., Guedj, V. and Zeriahi, A., Monge-Ampère equations in big cohomology classes, *Acta Math.* **205** (2010), 199–262. MR2746347
11. Darvas, T., Di Nezza, E. and Lu, C. H., Monotonicity of nonpluripolar products and complex Monge-Ampère equations with prescribed singularity, *Anal. PDE* **11** (2018), 2049–2087. MR3812864
12. Darvas, T., Di Nezza, E. and Lu, C. H., Log-concavity of volume and complex Monge-Ampère equations with prescribed singularity. `arXiv:1807.00276`.
13. Dembo, A. and Zeitouni, O., *Large deviations techniques and applications*, Jones and Bartlett Publishers, Boston, MA, 1993. MR1202429
14. Guedj, V. and Zeriahi, A., Intrinsic capacities on compact Kähler manifolds, *J. Geom. Anal.* **15** (2005), 607–639. MR2203165
15. Guedj, V. and Zeriahi, A., The weighted Monge-Ampère energy of quasiplurisubharmonic functions, *J. Funct. Anal.* **250** (2007), 442–482. MR2352488
16. Guedj, V. and Zeriahi, A., *Degenerate Complex Monge-Ampère Equations*, European Math. Soc. Tracts in Mathematics **26**, 2017. MR3617346
17. Lu, C. H. and Nguyen, V. D., Degenerate complex Hessian equations on compact Kähler manifolds, *Indiana Univ. Math. J.* **64** (2015), 1721–1745. MR3436233
18. Ross, J. and Nyström, D. W., Analytic test configurations and geodesic rays, *J. Symplectic Geom.* **12** (2014), 125–169. MR3194078
19. Saff, E. and Totik, V., *Logarithmic potentials with external fields*, Springer, Berlin, 1997. MR1485778
20. Nyström, D. W., Monotonicity of nonpluripolar Monge-Ampère masses, *Indiana Univ. Math. J.* **68** (2019) 579–591. MR3951074

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