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Mirror symmetry for very affine hypersurfaces

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1. Introduction

Homological mirror symmetry is a story of two categories radically different in origin. The first is a category of Lagrangians in a symplectic manifold, with morphisms defined by intersection points, corrected by holomorphic disks. The second is a category of locally defined modules over the holomorphic functions on a seemingly unrelated complex variety, with morphisms corrected by considerations of homological algebra. Most articles on the subject concern the ingenious manipulations required to identify one with the other, most often requiring heroic calculations of at least one side of this equivalence.

Our contribution is of a different nature. We wish to explain how in many circumstances — we focus on Calabi–Yau hypersurfaces in toric varieties, though the same methods should apply in the generality of Gross–Siebert toric degenerations — both sides can be cut into matching elementary pieces, known to be homologically mirror, and the total mirror symmetry glued together using foundational results in algebraic and symplectic geometry. More precisely, this cutting and gluing is possible at the limiting point where on the one hand the complex manifold degenerates into a union of toric varieties, while on the other, the symplectic form concentrates along certain divisors, and we consider the category associated with their complement. We will be entirely concerned with homological mirror symmetry *at* this limit point.([1])

At this most degenerate point, the category of coherent sheaves on the union of toric varieties — glued together along toric subvarieties — can be calculated as a colimit of the categories of coherent sheaves on the toric components [GR].

([1]) It is a tautology that matching the limit categories matches their infinitesimal deformations, but it remains to identify the geometric meaning of these deformations in a satisfactory way — we do not touch upon this question here.

Mirror symmetry is well studied for toric varieties themselves. The Hori–Vafa prescription is that the mirror A-model category should be associated with a function $W:(\mathbb{C}^*)^n\to\mathbb{C}$ whose Newton polytope is the convex hull of primitive vectors on the 1-dimensional cones of the fan of the toric variety. Different authors have taken different views on how precisely to associate a category with this geometry, either directly in Lagrangian Floer theory [Ab1], [Ab2], or in microlocal sheaf theory [B], [FLTZ3], [T], [Ku] (the latter being known to be calculate Fukaya categories [NZ], [N1], [GPS3]).

A true believer in mirror symmetry should expect the following facts:

(1) The mirror to the toric boundary — a generic fiber of a generic $W:(\mathbb{C}^*)^n\to\mathbb{C}$ whose Newton polytope is the moment polytope of the toric variety — admits a cover by mirrors of toric varieties, glued along mirrors of toric varieties.

(2) There are geometrically defined functors between these Fukaya categories which are mirror to the pullback and pushforward functors corresponding to the inclusion of toric varieties in toric varieties.

(3) There is a descent result for the Fukaya category showing that it carries covers of the sort in (1) to colimits of categories.

Establishing all of these results would show that the Fukaya category of the general fiber of W is equivalent to the category of coherent sheaves of the corresponding toric variety. The recent works [GPS1]–[GPS3] give the necessary general tools to define the functors in (2) and establish the descent required in (1). In fact, these works, together with [NS], build a bridge between Fukaya categories and microlocal sheaf theory, which we cross in order to appeal to the microlocal sheaf calculations of the toric mirror [Ku]. Here we will establish (1) and the 'mirror' assertions of (2) above, and deduce the following result.

THEOREM 1.1. *Suppose we are given the following data*:
- $\mathbb{T}_{\mathbb{C}}$ *an algebraic torus with character and cocharacter lattices* M *and* $M^\vee$;
- $\Delta^\vee\subset M^\vee$ *an integral polytope containing the origin*;
- Σ *a fan in* $M^\vee\otimes\mathbb{R}$ *giving a star-shaped triangulation of* $\Delta^\vee$.

These determine a smooth toric stack $\mathbf{T}_\Sigma$ *with toric boundary divisor* $\partial\mathbf{T}_\Sigma$.

Then, there exists a Laurent polynomial $W:\mathbb{T}^\vee_{\mathbb{C}}\to\mathbb{C}$ *with Newton polytope* $\Delta^\vee$, *a structure of Liouville manifold on a general fiber* F_W, *and an equivalence*

$$\mathrm{Coh}(\partial\mathbf{T}_\Sigma)\cong\mathrm{Fuk}(F_W)$$

between the dg category of coherent sheaves on the variety $\partial\mathbf{T}_\Sigma$ *and the wrapped Fukaya category of the general fiber* F_W.

We close the introduction with some comments about how we will establish items (1) and (2) above.

Regarding (1): in the microlocal sheaf theoretic works beginning with [FLTZ3], a key role is played by a certain conical Lagrangian subvariety $\Lambda_\Sigma \subset T^*\mathbb{T} \cong \mathbb{T}^\vee_{\mathbb{C}}$. It is straightforward to establish that the boundary of this conical subvariety indeed admits a cover corresponding to the cover of the $\partial\mathbf{T}$ by toric subvarieties. What is needed is to relate the geometry of Λ_Σ to the geometry of the Laurent polynomial W. We show in §6 that the deformation equivalence class of the Liouville sector determined by W admits a representative whose relative skeleton is precisely Λ_Σ.(2) The proof uses Mikhalkin–Viro patchworking to reduce the study of F_W to understanding pairs of pants, whose skeleta have been calculated by Nadler.

Regarding (2): after the geometric results in the previous paragraph, existence of the relevant functors of Fukaya categories can be deduced from [GPS1]. To calculate them, we use [GPS3], [NS] to pass to microlocal sheaf theory, where we must now show that the mirror symmetry established in [Ku] can be made functorial with respect to inclusion of toric boundary divisors. We explain how to do this in §7.

In the following section, we explain in more detail the general strategy of the proof, reviewing relevant ideas from sources mentioned above, and we give the proof of Theorem 1.1, up to the calculations mentioned in the previous two paragraphs, which we defer to the main body of the paper.

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2. Our approach to homological mirror symmetry

2.1. An illustration

Consider the degeneration in which a genus-1 holomorphic curve acquires a node. In the mirror degeneration, a symplectic 2-torus acquires a puncture.

(2) The argument we present in §6 requires a certain hypothesis on Σ (see Definition 6.10). After the present article appeared as a preprint, Peng Zhou showed that similar arguments work in general if one replaces the inner product identification of $\mathbb{T}^\vee_{\mathbb{C}} = T\mathbb{T}^\vee$ and $T^*\mathbb{T}^\vee$ by the Legendre transform for a more general homogenous quadratic function [Z]. We give here our original argument, but then appeal [Z] to establish Theorem 1.1 in the stated generality.

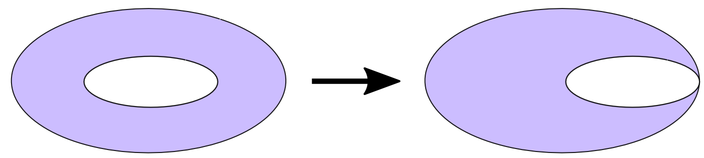

Figure 1. The degeneration of a smooth genus-1 curve to a nodal curve.

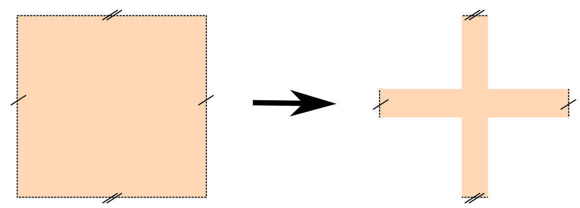

Figure 2. A torus acquiring a puncture as an S^1 fiber approaches infinite radius.

One way to arrive at the view that these two spaces should be mirror is the following "T-duality" account. In general, the spaces on the two sides of mirror symmetry are expected to be dual torus fibrations (in general, with singularities) over the same base, the radii of the fibers on one side being inverse to the radii on the other side. In the present example, on the complex side, we have a torus — a circle bundle over a circle. Under the degeneration, one of the circle fibers is approaching zero radius. Thus, on the symplectic side, we should have a circle bundle over a circle, in which one fiber is approaching infinite radius. A circle of infinite radius is a line — or in other words, the fiber should acquire a puncture.

In the description above, the puncture was just the removal of a point. As we draw only the complement of this point, we are free to imagine the puncture as being larger, as in Figure 2. In our previous description, the fiber containing the puncture was dual to the node. We have expanded the puncture, so in this picture, one should regard the entire horizontal region beneath the puncture as being dual to the node.

On the complex side, we have a singular complex curve; it is natural to take the normalization. This is a smooth curve mapping to the singular curve, and in the case at hand, the map simply identifies points. This is what is indicated in Figure 3. We can describe the symplectic side by a similar gluing. Since the node corresponded to the strip beneath the puncture, the mirror gluing on the A-side involves gluing the two ends of the strip.

The category we associate with this non-compact symplectic manifold is the wrapped

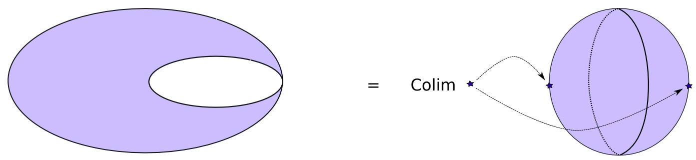

Figure 3. We obtain a nodal curve by gluing smooth pieces.

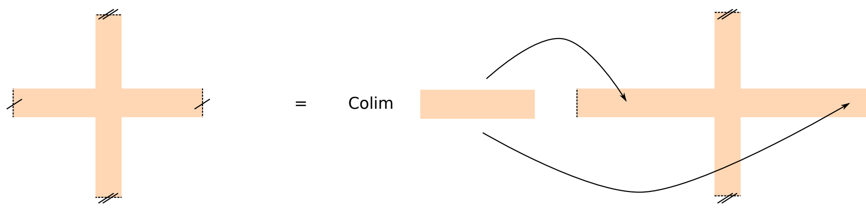

Figure 4. The mirror to the above gluing: a punctured torus is glued together from Liouville sectors.

Fukaya category, which was originally constructed for *Liouville manifolds*, symplectic manifolds with the property that (at least locally near the boundary) there is a primitive for the symplectic form whose dual Liouville vector field is everywhere outward pointing [AS]. In the above gluing, however, the restriction of this Liouville form to the components *does not* have this property: there are boundary components where it is parallel, rather than outward pointing. In particular, the rectangle should be viewed as the cotangent bundle of an interval rather than a disk. That is, the pieces in our gluing are not Liouville manifolds. The appropriate notion is that of *Liouville sector*, which we review in the next subsection. A covariantly functorial Floer theory for these is developed in [GPS1], [GPS2].

We turn now to the question of gluing together a global mirror symmetry from local mirror symmetries. The functor $\mathrm{Coh}(-)$ taking a variety X to its dg category $\mathrm{Coh}(X)$ of coherent sheaves behaves well with respect to the gluings above. Following [GR], we make statements for the Ind-completion $\mathrm{IndCoh}(X)$ of the category $\mathrm{Coh}(X)$; statements for $\mathrm{Coh}(X)$ may be recovered by taking compact objects. We write $\mathrm{IndCoh}^!$ and IndCoh_* for the contravariant and covariant functors from derived stacks to dg categories which carry a stack to its category of Ind-coherent sheaves and carry a map $f\colon X\to Y$ to a pullback $f^!$ or a pushforward f_*, respectively. The key fact [GR, IV.4.A.1.2] is that $\mathrm{IndCoh}^!$ takes pushout squares of affine schemes along closed embeddings to pullback squares of (stable cocomplete) dg categories; by Zariski descent, this holds for schemes more generally. By passing to adjoints, we see that IndCoh_* analogously takes pushouts to pushouts.

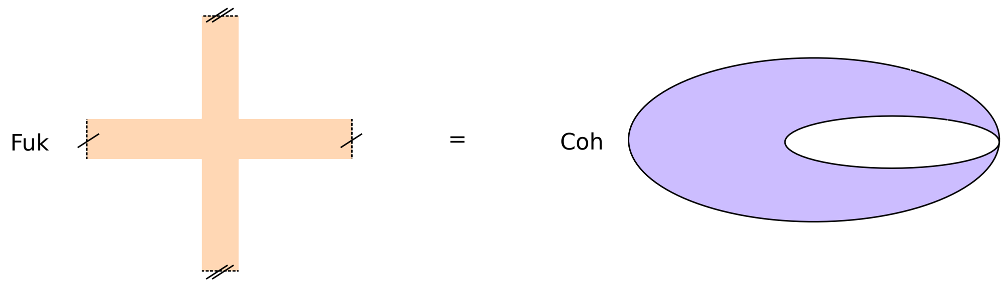

Figure 5. The homological mirror symmetry conjecture for a genus-1 curve at the large volume/complex structure limits.

Homological mirror symmetry as usually stated is an equivalence between coherent sheaves on a given algebraic variety and the Fukaya category of its mirror. What the pictures above suggest is that this should extend to a natural transformation between the functor IndCoh_*, perhaps with respect to some restricted class of maps including normalizations, and a functor IndFuk_*, covariant with respect to some class of maps including those mirror to normalization. It suggests moreover that IndFuk_* should take certain diagrams — those mirror to certain pushouts of varieties — to pushouts of dg or A_∞ categories.

In fact, a covariant functor from a category of Liouville sectors to A_∞ categories has been defined in [GPS1], and shown in [GPS2] to carry diagrams like those illustrated above to pushouts. Given these structural properties, one can establish mirror symmetry by showing that there is an identification, respecting the relevant inclusion functors, of the Fukaya and coherent-sheaf categories of our building blocks.

Remark 2.1. Strictly speaking, this subsection does not illustrate a special case of the statement of Theorem 1.1, because the self-nodal curve is not the boundary of a toric variety. However, an essentially identical argument gives the mirror symmetry between the nodal necklace of three $\mathbb{P}^1$s and a thrice-punctured torus, which is a special case of Theorem 1.1.

This subsection also does not exactly illustrate the proof we will give of Theorem 1.1; instead, as we explain below, we will translate these ideas to the microlocal sheaf setting using [GPS3]. In this setting, we need only cover the skeleton, as we do in Corollary 4.8. A lift of this cover to a sectorial cover in the sense of [GPS2] would yield a proof hewing closer to the above illustration.

2.2. Stops, sectors, skeleta, and partially wrapped Fukaya categories.

For basics on Liouville and Weinstein manifolds, we refer to [CE], [E]. Here we review basic notions of Liouville sectors, stops, and skeleta, and then recall from [GPS1], [GPS2] definitions and results concerning partially wrapped Fukaya categories defined in terms of these geometric structures.

A *Liouville sector* is an exact symplectic manifold-with-boundary $(X, \partial X, \lambda)$ modeled at infinity on the symplectization of a contact manifold-with-boundary $(V, \partial V, \lambda)$, satisfying additional constraints: ∂V should be transverse to a contact vector field, and the characteristic foliation on ∂X should be trivializable as $\partial X = \mathbb{R} \times F$. Such a trivialization makes $(F, \lambda|_F)$ a Liouville manifold. Note that being a Liouville sector is a property of, rather than a structure on, an exact symplectic manifold-with-boundary.

A closed codimension-zero submanifold-with-boundary $Y \subset X$ is a Liouville subsector if (1) each component of ∂Y is either disjoint from or contained in ∂X, and (2) $(Y, \partial Y, \lambda|_{\partial Y})$ is itself a Liouville sector. One can see by inspection that the symplectic manifolds in Figure 4 admit an exact structure making them Liouville sectors, and that the inclusions depicted are inclusions of Liouville sectors.

Another point of view on sectors is obtained by passing to the 'convex completion' $\overline{X}$, which is a Liouville manifold in the usual sense. Up to contractible choices, the data of the sector is equivalent to an embedding $F \subset \partial_\infty \overline{X}$ as a Liouville hypersurface, i.e. some choice of contact form on $\partial_\infty \overline{X}$ restricts to the Liouville form on F. The $(\overline{X}, F)$ form what is termed a *Liouville pair* elsewhere in the literature [Av], [Sy], [E]. The advantage of Liouville sectors over Liouville pairs is that they are better suited to discussions of gluing, in particular because the key notion of Liouville subsector is less natural in the setting of pairs. Basic definitions and constructions relevant to Liouville sectors are found in [GPS1, §2].

We refer to works [AS], [GPS1], [GPS2] for a foundational treatment of partially wrapped Fukaya categories. For our purposes here, we may largely use these works as black boxes. The most general setting for defining partially wrapped Fukaya categories (offered by [GPS2]) takes as input the data of a Liouville sector X and a closed subset in the infinite boundary $\Lambda \subset \partial_\infty X^\circ$. (We recall that a Liouville sector has its actual boundary ∂X, and its ideal contact boundary $\partial_\infty X$; here $\partial_\infty X^\circ$ means the contact boundary minus its intersection with the actual boundary ∂X.) With such a pair is associated a category which we here denote $\mathrm{Fuk}(X, \Lambda)$.([3])

A stopped sector includes in another by enlarging the sector or shrinking the stop: we say $(X', \Lambda') \subset (X, \Lambda)$ if X' is a Liouville subsector of X and $\Lambda' \supset \Lambda \cap X'$. It is shown

([3]) We write Fuk for what is called Perf $\mathcal{W}$ in [GPS1]–[GPS3].

in [GPS1], [GPS2] that in this case there is a functor

$$\mathrm{Fuk}(X', \Lambda') \longrightarrow \mathrm{Fuk}(X, \Lambda).$$

When $X = X'$ we term this functor a "stop removal". These satisfy the natural compatibilities with composition, defining a (strict!) functor from the poset of (stopped) subsectors of (X, Λ) to A_∞ categories.

With a Liouville manifold X is associated the *skeleton* (elsewhere termed *spine* or *core*) $\mathfrak{c}_X$, this being the locus of all points which do not escape to infinity under the Liouville flow. When the Liouville flow is gradient-like and generalized Morse–Smale (such manifolds are said to be Weinstein), the skeleton admits a Whitney stratification by isotropic submanifolds, and the top-dimensional strata admit transverse "cocore" Lagrangian disks. It is this consequence which is relevant for [GPS2], [GPS3],([4]) and some weaker definitions of Weinstein have been proposed which imply it; see e.g. [E]. The above results remain true when the Liouville flow is Morse–Bott, as in the cases studied in this paper.

For a Liouville sector X, one can define the skeleton $\mathfrak{c}_X$ by the same formulation: $\mathfrak{c}_X$ is the locus which does not escape to infinity. However, this definition is only really sensible if the Liouville flow on X is tangent to ∂X along all of ∂X, not just near $\partial_\infty \partial X$. Note [GPS1, Lemma 2.11 and Proposition 2.28] that this can always be arranged after deformation. Evidently, if $X \subset Y$ is an inclusion of Liouville sectors where $\lambda_X = \lambda_Y|_X$ and the Liouville flow on X is tangent to its boundary, then $\mathfrak{c}_X = \mathfrak{c}_Y \cap X$.

We offer also another perspective on the skeleton of a sector. Recall that a sector X is equivalent to the data of a pair $(\overline{X}, F \subset \partial_\infty \overline{X})$. For the pair, it is natural to define the *relative skeleton* $\mathfrak{c}_{\overline{X}, F}$ as the locus of points $\overline{X}$ which do not escape to $\partial_\infty \overline{X} \setminus \mathfrak{c}_F$. Note that this is the union of $\mathfrak{c}_X$ with a $\mathbb{R}$-cone on $\mathfrak{c}_F$. This notion of relative skeleton compares to the skeleton of a sector as follows: it is not difficult, using the techniques of [GPS1, §2] to arrange an inclusion of sectors $X \subset \overline{X}$ such that $\mathfrak{c}_X = \mathfrak{c}_{\overline{X}, F} \cap X$.

From the point of view of Fukaya categories, the significance of skeleta and relative skeleta is in their role in organizing generation results. Indeed, the cocore disks to a Weinstein Morse function provide Lagrangians transverse to each component of the smooth locus of the skeleton, and with any Legendrian point of a stop there is associated a linking disk; according to [GPS2, Theorem 1.10], these generate $\mathrm{Fuk}(X, \Lambda)$ when X is a Weinstein manifold and Λ is mostly Legendrian.

For calculating Fukaya categories, we may always translate back and forth between Liouville sectors and stopped Liouville manifolds, and further we may retract the stop

([4]) Without any assumption beyond isotropicity of the skeleton, one can use the linking disks in $X \times T^*[0,1]$ as replacements for the co-core disks of X; see [GPS2].

to its skeleton. Indeed, per [GPS2, Corollary 2.11], we have equivalences

$$\mathrm{Fuk}(X) \xrightarrow{\sim} \mathrm{Fuk}(\overline{X}, F) \xrightarrow{\sim} \mathrm{Fuk}(\overline{X}, \mathfrak{c}_F).$$

2.3. Notation for tori

Fix lattices M and $M^\vee$. For an abelian group A, we write $M_A := M \otimes_{\mathbb{Z}} A$. We consider the real tori

$$\mathbb{T} = M^\vee_{\mathbb{R}/\mathbb{Z}} = M^\vee_{\mathbb{R}}/M^\vee \quad \text{and} \quad \mathbb{T}^\vee = M_{\mathbb{R}/\mathbb{Z}} = M_{\mathbb{R}}/M,$$

where M and $M^\vee$ are the lattices of characters and cocharacters for $\mathbb{T}$.

We denote the corresponding complex tori by $\mathbb{T}_{\mathbb{C}} = M^\vee_{\mathbb{C}^\times}$ and $\mathbb{T}^\vee_{\mathbb{C}} = M_{\mathbb{C}^\times}$, respectively. The tangent space to a point p in the torus $\mathbb{T}^\vee_{\mathbb{C}}$ is canonically identified with $M_{\mathbb{R}} \times M_{\mathbb{R}}$, and the complex structure on $\mathbb{T}^\vee_{\mathbb{C}}$ is the standard one: $J_p(x,y) = (-y, x)$.

Whenever we wish to do symplectic geometry on a complex torus, we use an inner product $\langle -, - \rangle$ on $M_{\mathbb{R}}$ to obtain an identification with the cotangent bundle:

$$\mathbb{T}^\vee_{\mathbb{C}} = T\mathbb{T}^\vee = \mathbb{T}^\vee \times M_{\mathbb{R}} \xrightarrow[\mathrm{id}_{\mathbb{T}^\vee} \times \langle -,- \rangle]{\sim} \mathbb{T}^\vee \times M^\vee_{\mathbb{R}} = T^*\mathbb{T}^\vee. \tag{2.1}$$

We always regard the latter as an exact symplectic manifold carrying the canonical ("$p\,dq$") Liouville structure.

Under the identification (2.1), the complex structure J from $\mathbb{T}^\vee_{\mathbb{C}}$ and the symplectic structure ω from $T^*\mathbb{T}^\vee$ are compatible, in the sense that $g(-,-) = \omega(-, J-)$ is a (Kähler) metric. Indeed, if we pick a basis $\{m_1, ..., m_n\}$ for M, determining holomorphic coordinates $z_1, ..., z_n$ on $\mathbb{T}^\vee_{\mathbb{C}}$, the metric g may be written in these coordinates as $g = \sum_{i,j=1}^n \langle m_i, m_j \rangle \, dz_i \, d\bar{z}_j$. In particular, under the identification (2.1), complex hypersurfaces of $\mathbb{T}^\vee_{\mathbb{C}}$ become symplectic submanifolds of $T^*\mathbb{T}^\vee$.

2.4. LG model

Partially wrapped Fukaya categories can be used to formulate homological mirror symmetry for Fano varieties. For example, the mirror to $\mathbb{P}^1$ should be somehow associated with the function $W(z) = z + z^{-1}$ on $\mathbb{C}^*$. We interpret this to mean that we should form a Liouville sector from $\mathbb{C}^*$ by deleting the neighborhood of a fiber $W^{-1}(-\infty)$ at infinity. In this special case, any reasonable interpretation of the above description should result in the sector on the left-hand side of Figure 5.

More generally, we would like to obtain a Liouville sector from a function

$$W : (\mathbb{C}^*)^n \longrightarrow \mathbb{C},$$

as such functions were predicted by Hori and Vafa [HV] to provide mirrors to toric varieties. Naïvely, one could attempt to produce a sector from this data as follows: take a half-plane $\mathbb{H}\subset\mathbb{C}$ containing all the critical values (including those associated with critical points at infinity) of W, and take $W^{-1}(\mathbb{H})$ as the sector. Strictly speaking, however, $W^{-1}(\mathbb{H})$ is not generally conical at infinity for the restriction of the most natural Liouville structure on $(\mathbb{C}^*)^n$, so some manipulation of exact structures and use of cutoff functions would be necessary. Similar issues arise in work of Seidel, see e.g. [Se1, §3A] and [Se2, §19B]. Instead, we use the tropical methods of [M], [Ab1] to show the following result.

PROPOSITION 2.2. *Fix a Newton polytope $\Delta^\vee\subset M^\vee$ and regular star subdivision $\mathcal{T}$ induced by some piecewise-linear function α. Consider the function on $M_{\mathbb{C}^*}=\mathbb{T}^\vee_{\mathbb{C}}$:*

$$W(z)=\sum_{n\in\Delta^\vee} t^{-\alpha(n)}z^n.$$

There is a real codimension-2 symplectic submanifold F_Σ of $\mathbb{T}^\vee_{\mathbb{C}}$ such that the following conditions hold:

- [Ab1] *For $t\gg 0$, there is an isotopy of symplectic submanifolds between F_Σ and a general fiber F_W of W;*
- *There is a Liouville subdomain $D\subset\mathbb{T}^\vee_{\mathbb{C}}$, completing to $\mathbb{T}^\vee_{\mathbb{C}}$, such that $\partial D\cap F_\Sigma$ is a Liouville subdomain of F_Σ, completing to F_Σ.*

Note we use (2.1) to define the exact symplectic structure on $\mathbb{T}^\vee_{\mathbb{C}}$.

As indicated, the first item is proven in [Ab1], in a form we recall in Lemma 6.6. The second item follows from our further calculations that the skeleton of F_Σ is contained in the boundary of some subdomain D (Theorem 6.12), and moreover, along the skeleton, F_Σ is nowhere tangent to the Liouville vector field of the ambient $\mathbb{T}^\vee_{\mathbb{C}}$ (Lemma 6.13). Indeed, then we may deform slightly ∂D along the Liouville field in order to contain some neighborhood of the skeleton of F_Σ.

It is the sector associated with the particular pair $(D, F_\Sigma\cap D)$ constructed by our proof of this proposition that is used in this article. In particular in Theorem 1.1, when we assert 'there is a Liouville structure on F_W' we mean that we pull back the Liouville structure mentioned above under the symplectomorphism $F_W\cong F_\Sigma$.

Of course, we expect that any other reasonable construction of such a pair from W will be deformation equivalent to ours, in particular giving the same Fukaya category.

2.5. Sheaves

A prototypical example of a Liouville manifold is the cotangent bundle T^*Q of a closed manifold without boundary; the skeleton for the usual "$p\,dq$" form is the zero section.

If Q had boundary, the cotangent bundle would naturally be a Liouville sector, again with the zero section as skeleton. An open set $U \subset Q$ determines an inclusion of Liouville sectors $T^*U \subset T^*Q$: the stopped boundary of T^*U is the restriction of the cotangent bundle to the boundary of U. Lifting a cover of Q gives a cover of T^*Q by Liouville sectors, whose intersections are again Liouville sectors (with corners). The covariantly functorial [GPS1] assignment $U \mapsto \mathrm{Fuk}(T^*U)$ thus defines a precosheaf of categories on Q.

Suppose we knew this precosheaf were a cosheaf. Then we could compute its global sections from the local data. Indeed, the Fukaya category of the cotangent bundle of a disk is equivalent to the category of chain complexes, so the cosheaf in question would be a locally constant cosheaf of categories. Recall that the ∞-groupoidal version of the Seifert–van Kampen theorem asserts that the fundamental higher groupoid of a space is the global sections of a locally constant cosheaf of spaces with stalk a point. Linearizing this, we see that a locally constant cosheaf of A_∞ categories with stalk the category of chain complexes has global sections (a twisted version of) the category of modules over the algebra of chains on the based loop space of Q. Thus, the Fukaya category of a cotangent bundle is the category of modules over chains on the based loop space. This final statement is originally a result of Abouzaid, by a different argument [Ab3].

Kontsevich's localization conjecture [Ko] asserts the existence of a cosheaf $\mathcal{F}$uk over the skeleton of any Weinstein manifold (e.g. the complement of an ample divisor in a smooth projective variety), whose global sections should recover the wrapped Fukaya category.

A variant, which gives a local-to-global principle without any mention of skeleta, is the main result of [GPS2], which asserts that the Fukaya category satisfies descent with respect to sectorial covers.([5]) This result, together with the well-known calculation of Fukaya categories of disks with stops at the boundary (for a very short calculation, see [GPS2, Example 1.22]), can easily be used to make the discussion of §2.1 above completely rigorous.

In the body of this article we will need a further elaboration of Kontsevich's conjecture, formulated by Nadler [N2] (and further elaborated in [Shen], [NS]), which identifies Kontsevich's conjectural cosheaf of categories on the skeleton with a certain cosheaf of a combinatorial-topological nature which is constructed directly from the microlocal sheaf theory of [KS]. This conjecture is established([6]) in [GPS3], using the theory developed

([5]) To deduce Kontsevich's statement from [GPS2], one would want to know further that appropriate open covers of the skeleton lift to sectorial covers. It is expected that such a lifting is not difficult to construct in general. In the case of relevance to this article, it is likely possible to construct such a cover by hand, though we will not do it here, as we do not invoke this result (instead we use [GPS3]).

([6]) Strictly speaking, this is established in the "stably polarized" case, which includes the examples of interest here.

in [GPS1], [GPS2] and the antimicrolocalization lemma of [NS].

2.6. Remark on toric stacks

To state results in their natural generality, we use the toric stacks of [BCS]. For the purpose of understanding the new ideas in this paper, this can be entirely ignored. Very briefly, toric stacks are smooth Deligne–Mumford stacks associated with the data of a "smooth stacky fan" Σ, which is a simplicial fan together with a choice of integer point along each ray. We term these chosen integer points the "stacky primitives". The coarse moduli space of the toric stack is the toric variety which would ordinarily correspond to the underlying simplicial fan.

Even in the setting of reflexive polytopes, one must in general allow stacks to get the correct category of coherent sheaves for the purposes of mirror symmetry; this is due to the fact that toric varieties do not in general admit crepant resolutions. Of course, if we begin with a smooth fan, no discussion of toric stacks is necessary.

The added generality provided by allowing toric stacks can be seen by the following lemma.

LEMMA 2.3. *Every convex polytope containing the origin is the convex hull of the stacky primitives of a smooth quasi-projective stacky fan.*

Proof. The quasi-projectivity condition is that the triangulation induced by the fan is *regular*, in the sense of being the corner locus of a piecewise-linear function $\alpha\colon \Delta^\vee \to \mathbb{R}$. Let α_0 be the piecewise linear function which is 1 at the origin, and 0 at all facets of the boundary not containing the origin. For each facet of the polytope, τ, choose some α_τ inducing a regular triangulation of τ. Then take the function $\alpha = \alpha_0 + \sum_\tau \varepsilon_\tau \alpha_\tau$ for small ε_τ. (We thank Allen Knutson for this argument.) $\square$

2.7. Proof of Theorem 1.1

Here we give the proof of Theorem 1.1, modulo the results which are the essential mathematical contents of the present article.

Our mirror-symmetric setup is as follows. Let $\Delta^\vee \subset M^\vee_{\mathbb{R}}$ be an integral polytope containing the origin. Choose a regular star-shaped triangulation of $\Delta^\vee$; equivalently, choose a smooth quasi-projective stacky fan $\Sigma \subset M^\vee_{\mathbb{R}}$ whose stacky primitives lie on $\partial\Delta^\vee$ and have convex hull $\Delta^\vee$. This determines a toric stack $\mathbf{T}_\Sigma$ partially compactifying $\mathbb{T}_{\mathbb{C}}$, and we denote its toric boundary by $\partial\mathbf{T}_\Sigma$. (A brief review of relevant algebraic geometry of toric varieties is included in §3.)

$$\begin{tikzcd}
\mathrm{Coh}(\partial\mathbf{T}_\Sigma) \arrow[r] \arrow[d, equal, "\text{Theorem 7.13}"'] & \mathrm{Coh}(\mathbf{T}_\Sigma) \arrow[d, equal, "\text{[Ku]}"] \\
\mu\,\mathrm{sh}_{\partial\mathbb{L}_\Sigma}(\partial\mathbb{L}_\Sigma)^c \arrow[r] \arrow[d, equal, "\text{[GPS3, Theorem 1.4]}"'] & \mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee)^c \arrow[d, equal, "\text{[GPS3, Theorem 1.1]}"] \\
\mathrm{Fuk}(F_\Sigma) \arrow[r] & \mathrm{Fuk}(T^*\mathbb{T}^\vee, F_\Sigma)
\end{tikzcd}$$

Figure 6. The commutative diagram organizing our proof of Theorem 1.1.

We take $W\colon\mathbb{T}^\vee_\mathbb{C}\to\mathbb{C}$ a Laurent polynomial whose Newton polytope is $\Delta^\vee$. (How to choose this polynomial will be discussed further below, though generic choices are isotopic and hence will determine the same categories.)

Finally, we will need a certain conical (singular) Lagrangian $\mathbb{L}_\Sigma\subset T^*\mathbb{T}^\vee$ introduced in [FLTZ3] to study toric mirror symmetry. We recall its definition in §4.

The proof of Theorem 1.1 proceeds by establishing the commutative diagram in Figure 6. Indeed, the theorem follows from the left column (whose notation we have not yet explained in its entirety), together with the fact that F_W is deformation equivalent to a general fiber of W (per Proposition 2.2), and hence has the same Fukaya category. The full diagram gives a functoriality result connecting mirror symmetry for the toric variety and for its boundary. In fact, we will prove even stronger functoriality results on our way to the theorem.

Let us now explain the diagram in detail. We have by now introduced all the geometric players: the real torus $\mathbb{T}$ and its dual real torus $\mathbb{T}^\vee$; the toric variety $\mathbf{T}_\Sigma$ and its boundary $\partial\mathbf{T}_\Sigma$; the [FLTZ3] Lagrangian $\mathbb{L}_\Sigma$ and its Legendrian boundary at infinity $\partial\mathbb{L}_\Sigma$; the Laurent polynomial $W\colon\mathbb{T}^\vee_\mathbb{C}\to\mathbb{C}$, which, under a choice of isomorphism

$$T^*\mathbb{T}^\vee = T\mathbb{T}^\vee = \mathbb{T}^\vee_\mathbb{C},$$

becomes $W\colon T^*\mathbb{T}^\vee\to\mathbb{C}$; and finally F_Σ, the deformation F_W of a general fiber of W.

For an algebraic scheme (or stack) X, we write $\mathrm{Coh}(X)$ for the dg category of complexes of sheaves with coherent cohomology on X, localized at quasi-isomorphisms. The top horizontal arrow is the pushforward.

For Q a manifold and $\Lambda\subset T^*Q$, the notation $\mathrm{Sh}_\Lambda(Q)$ means the category of sheaves whose microsupport is contained in Λ. (When Λ is instead a Legendrian in S^*Q, we use the same notation of sheaves whose microsupport at infinity is contained in Λ.) This notion is introduced and studied in [KS]. Following more modern conventions, and unlike in [KS], by Sh we mean the dg category of all complexes of sheaves localized at the acyclic complexes, rather than the bounded derived category. We write $\mathrm{Sh}(-)^c$ for the subcategory of compact objects, i.e., the "wrapped microlocal sheaves" of [N2].

The particular example of $\mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee)$ is the subject of [FLTZ1], [T], [Ku]. The top-right vertical equality is the main result of [Ku],(7) building on [FLTZ1], [T]. This equality holds for any Σ, without the hypotheses of smoothness or quasi-projectivity.

For $\Lambda \subset T^*Q$ or $\Lambda \subset S^*Q$, the notation $\mu\,\mathrm{sh}_\Lambda$ denotes a certain sheaf of categories on Λ constructed out of the microlocal sheaf theory, called the Kashiwara–Schapira stack. We recall its properties in §7.3.1 below. For formal reasons, taking compact objects in $\mu\,\mathrm{sh}_\Lambda$ gives a *cosheaf* of categories $\mu\,\mathrm{sh}(-)^c$. One of our main results is the following.

For Σ determining a smooth toric stack $\mathbf{T}_\Sigma$, there is an isomorphism

$$\mathrm{Coh}(\partial \mathbf{T}_\Sigma) \cong \mu\,\mathrm{sh}(\partial \mathbb{L}_\Sigma)^c$$

(*Theorem* 7.13) *ensuring that the top square commutes.*

Remark 2.4. In fact such an isomorphism exists without the smoothness hypothesis. We do not show this here, but briefly indicate how one can; see Remarks 7.3 and 7.14.

As the horizontal arrows in the diagram are not fully faithful, the existence of a morphism $\mathrm{Coh}(\partial \mathbf{T}) \to \mu\,\mathrm{sh}(\partial \mathbb{L}_\Sigma)$ making the top square commute *does not* imply that said morphism is an isomorphism. A separate argument is required. We then use the fact (explained in §4.3) that $\partial \mathbb{L}_\Sigma$ has a cover by mirror skeleta to the toric varieties in $\partial \mathbf{T}$, together with the fact that Coh and $\mu\,\mathrm{sh}$ satisfy certain local-to-global principles, to deduce this result. To make this work, we will need to prove a functoriality result ("restriction is mirror to microlocalization") for the isomorphism

$$\mathrm{Coh}(\mathbf{T}) \xrightarrow{\sim} \mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee).$$

This top square is where homological mirror symmetry happens: the sheaf categories are already some kind of interpretation of the A-model (morally: in a rescaling limit under the Liouville flow).

The bottom square compares the microlocal sheaf categories with the Fukaya category(8) The engine for this is the work [GPS3], whose main results we summarize in the following result.(9)

(7) When Σ is not smooth and proper, even the functor is new in [Ku]: the functor described in [B], [FLTZ3], [T] takes values in quasi-coherent sheaves, and it is necessary to lift this functor to take values in ind-coherent sheaves.

(8) This has a purpose aside from merely matching historical formulations: it is the Fukaya category which one knows how to deform away from the large volume limit (by holomorphic disks passing through a compactifying boundary divisor). However, we do not take up the study of this deformation in the present work.

(9) To compare with what is written in [GPS3], note the canonical equivalence

$$\mathrm{Fuk}(T^*Q, \Lambda)^{\mathrm{op}} = \mathrm{Fuk}(T^*Q, -\Lambda),$$

also noted in Remark 1.2 of that reference.

THEOREM 2.5. ([GPS3, Theorems 1.1 and 1.4 and Corollary 7.22]) *Let Q be a real analytic manifold and $\Lambda \subset S^*Q$ an isotropic subanalytic subset. Then there is an equivalence of categories* $\mathrm{Fuk}(T^*Q, -\Lambda) \cong \mathrm{Sh}_\Lambda(Q)^c$. *If in addition $-\Lambda$ is the core of a Liouville hypersurface F which admits homological cocores, then there is a commutative diagram*

$$\begin{array}{ccc} \mu\,\mathrm{sh}_{-\Lambda}(-\Lambda)^c & \longrightarrow & \mathrm{Sh}_{-\Lambda}(Q)^c \\ \| & & \| \\ \mathrm{Fuk}(F) & \longrightarrow & \mathrm{Fuk}(T^*Q, F) \end{array}$$

where the top map is the left adjoint to microlocalization, the bottom map is the [GPS1], [GPS2] *functor associated with a Liouville pair, and the right column is related to the aforementioned equivalence by the canonical*

$$\mathrm{Fuk}(T^*Q, F) \xrightarrow{\sim} \mathrm{Fuk}(T^*Q, -\Lambda).$$

In the case at hand, our $\mathbb{L}_\Sigma$ will be evidently subanalytic. The commutative diagram asserted to exist will match the bottom square, once we establish the following:

> *The construction of F_Σ in Proposition 2.2 may be arranged so that the skeleton of F_Σ is $-\partial\mathbb{L}_\Sigma$ (Theorem 6.12).*

We show this by using Mikhalkin–Viro patchworking [M] to deform the hypersurface in such a way that the calculation of the skeleton localizes to "pairs of pants," where in fact it has already been studied by Nadler [N2]. Our construction will show that $\mathbb{L}_\Sigma$ is the skeleton associated with a Morse–Bott Liouville flow, hence F_Σ admits geometric cocores (and thus homological cocores). More precisely, as noted in the introduction, the argument we present requires a certain hypothesis on Σ (see Definition 6.10); there is now a similar but more general version does not require this hypothesis [Z]; see Remark 6.16.

This completes the proof of Theorem 1.1, modulo the bolded promissory notes. $\square$

2.8. Other related works

We end the introduction by attempting to situate our work in the landscape of homological mirror symmetry.

Our approach has been to pass as quickly as possible to microlocal sheaf theory, and match functorial structures on both sides in order to reduce mirror symmetry to elementary calculations. Previous works in this spirit include [FLTZ1], [Ku], [N2]; the particular approach used in this article is close to what is suggested in [TZ]. The underlying topological spaces of some of the Lagrangian skeleta we construct were studied earlier in [RSTZ].

Note we use the foundational work [GPS1]–[GPS3] rather than [NZ], [N1]; among other reasons, this allows us to make statements regarding the *wrapped* Fukaya category.

Another strategy to approach mirror symmetry is to identify particular Lagrangians, compute their Floer theory, and identify the resulting algebra with some endomorphism algebra on the mirror. We view this as the approach taken to the quartic K3 in [Se3], to toric varieties in [Ab1], [Ab2], and to hypersurfaces in projective space in [Sher1]–[Sher3].

After finding the skeleton and corresponding cover of the hypersurface, we could perhaps have used [Ab1], [Ab2] to complete the proof of mirror symmetry for Calabi–Yau hypersurfaces. However, this would require reworking those arguments in the wrapped setting and establishing the appropriate functoriality with respect to inclusion of toric divisors. In addition, the works [Ab1], [Ab2], as [FLTZ1], [FLTZ3], [T], give only a fully faithful embedding of the coherent sheaf category into the Fukaya category; one would need to prove generation. In any case, the form of the results in [Ku] is better adapted to our uses here.

Finally we note that in [AAK], one finds a mirror proposal for very affine hypersurfaces in terms of a category of singularities; it is a priori different from the category we have found here. The reason for the difference is that the [AAK] mirrors correspond to a maximal subdivision of $\Delta^\vee$, and we have taken a decomposition centered at a single point. One could try and compare algebraically the resulting categories. For that matter, we have provided here many mirrors, depending on the choice of point, and it should be interesting to understand the derived equivalences between them in algebro-geometric terms.

The [AAK] mirrors can also be approached directly by the methods of this paper. The main new difficulty in carrying this out is that the amoebal complements have many bounded components, making it more difficult to find a contact-type hypersurface containing the skeleton. It is, however, possible to use a higher-dimensional version of the inductive argument in [PS]. That proof has two essential ingredients: a gluing result and a way to move around the skeleton to allow further gluings. The gluing result needed is exactly our microlocalization of the theorem of Kuwagaki. We hope to return elsewhere to the question of its interaction with deformations of the skeleton.

3. Toric geometry

We recall here some standard notations and concepts from toric geometry; proofs, details, and further exposition can be found, e.g., in the excellent resources [F], [CLS].

In most of this paper we will be interested in a fixed toric variety $\mathbf{T}$, with dense open torus $\mathbb{T}_{\mathbb{C}}$ whose character and cocharacter lattices are denoted by M and $M^\vee$, respectively. When we must discuss another toric variety $\mathbf{T}'$, we indicate the corresponding characters and cocharacters by $M(\mathbf{T}')$ and $M^\vee(\mathbf{T}')$, respectively. In our review here we confine ourselves to the case of toric varieties; for toric stacks see [BCS].

3.1. Orbits and fans

A toric variety $\mathbf{T}$ is stratified by the finitely many orbits of the torus $\mathbb{T}_\mathbb{C}$. The geometry of this stratification determines a configuration of rational polyhedral cones (the 'fan') in the cocharacter space. We briefly review this correspondence.

For any cocharacter $\eta: \mathbb{G}_m \to \mathbb{T}_\mathbb{C}$, one can ask whether $\lim_{t\to 0}\eta(t)\in\mathbf{T}$, and if so, in which orbit it lies.

This gives a collection of regions in $M^\vee$, and for such a region σ we denote the corresponding orbit by $O(\sigma)$. Each cone σ is readily seen to be closed under addition; in fact, each is the collection of interior integral points inside a rational polyhedral cone $\sigma\subset M^\vee_\mathbb{R}$. This collection of cones is called the *fan* of $\mathbf{T}$. Every face of a cone in the fan is again a cone in the fan.

A character $\chi\in M$ is by definition a map $\mathbb{T}_\mathbb{C}\to\mathbb{G}_m$, but composing with the inclusion $\mathbb{G}_m\to\mathbb{A}^1$ determines a function on $\mathbb{T}_\mathbb{C}$. One can ask whether such a function can be extended to a given torus orbit $O(\sigma)$. Evaluating on 1-parameter subgroups $\eta\in\sigma$, one needs

$$\lim_{t\to 0}\chi(\eta(t)) = \lim_{t\to 0} t^{\langle\chi,\eta\rangle}$$

to be well defined, or in other words that $\langle\chi,\eta\rangle\geqslant 0$. In fact, this condition is also sufficient, and moreover the ring of all functions on $\mathbb{T}$ extending to $O(\sigma)$ is $k[\sigma^\vee]$, where

$$\sigma^\vee = \{\chi\in M : \langle\chi,\sigma\rangle\geqslant 0\}.$$

In other words, if we write $\mathbf{T}_\sigma$ for the locus in $\mathbf{T}$ on which all the $k[\sigma^\vee]$ are well defined, the natural map $\mathbf{T}_\sigma\to\operatorname{Spec} k[\sigma^\vee]$ is an isomorphism.

For cones σ and τ in a fan, the following are equivalent: $\tau\subset\bar\sigma$ if and only if $\sigma^\vee\subset\tau^\vee$ if and only if $O(\sigma)\subset\overline{O(\tau)}$ if and only if $k[\sigma^\vee]\subset k[\tau^\vee]$ if and only if $\mathbf{T}_\tau\subset\mathbf{T}_\sigma$. As sets,

$$\mathbf{T}_\sigma = \coprod_{\tau\subset\bar\sigma} O(\tau) \quad\text{and}\quad \overline{O(\sigma)} = \coprod_{\bar\tau\supset\sigma} O(\tau)$$

Definition 3.1. Let Σ be a fan of cones in $M^\vee_\mathbb{R}$. We denote by $\mathbf{T}_\Sigma$ the toric variety determined as above by the fan Σ.

3.2. Orbit closures

Let $\sigma \subset M^\vee_{\mathbb{R}}$ be a cone of the fan. The corresponding orbit $O(\sigma)$ is acted on trivially by the cocharacters in σ, hence by their span $\mathbb{Z}\sigma$. That is, if we denote by $\mathbb{T}_{\mathbb{C}}/\sigma$ the complex torus $(M^\vee/\mathbb{Z}\sigma)\otimes\mathbb{C}^\times$, then the $\mathbb{T}_{\mathbb{C}}$ action factors through $\mathbb{T}_{\mathbb{C}}/\sigma$. In fact, the resulting action is free, and admits a canonical section inducing an identification $\mathbb{T}_{\mathbb{C}}/\sigma \cong O(\sigma)$. Note in particular that the dimension of the orbit is the codimension of the cone in the fan.

This identification can be extended to the structure of a toric variety on the orbit closure $\overline{O(\sigma)}$. As mentioned above, as a set,

$$\overline{O(\sigma)} = \coprod_{\tau \supset \sigma} O(\tau)$$

The identification of the open torus with $\mathbb{T}_{\mathbb{C}}/\sigma$ induces the following description of the lattice of cocharacters:

$$M^\vee(\overline{O(\sigma)}) \cong M^\vee/\mathbb{Z}\sigma$$

The fan of $\overline{O(\sigma)}$ is obtained from the fan Σ by taking the cones τ such that $\tau \supset \bar{\sigma}$ and projecting them along $M^\vee \to M^\vee/\mathbb{Z}\sigma$.

The orbit closures have the relation $\overline{O}_\sigma \cap \overline{O}_\tau = \overline{O}_{\sigma\wedge\tau}$, where $\sigma\wedge\tau$ is the smallest cone in the fan containing both σ and τ if such a cone exists, and by convention $\overline{O}_{\sigma\wedge\tau} = \varnothing$ if no such cone exists. That is, the association $\sigma \to \overline{O}_\sigma$ is inclusion reversing.

3.3. Fans from triangulations

Let $\Delta^\vee \subset M^\vee_{\mathbb{R}}$ be an integral convex polytope containing zero. We will be interested in stacky fans obtained from star-shaped triangulations of $\Delta^\vee$.

Definition 3.2. A triangulation $\mathcal{T}$ of $\Delta^\vee$ is a *star-shaped triangulation* if every simplex in $\mathcal{T}$ which is not contained in $\partial\Delta^\vee$ has zero as a vertex.

Such a triangulation defines a stacky fan Σ: the stacky primitives of Σ are the 1-dimensional cones in $\mathcal{T}$, and the higher-dimensional cones in Σ are cones on the simplices in $\mathcal{T}$ which are contained in $\partial\Delta^\vee$.

Remark 3.3. Note that not every fan Σ arises in the above fashion. The above construction produces only those fans Σ satisfying the following property: *Let $\Delta^\vee$ be the convex hull of the primitives of Σ. Then every primitive of Σ lies on $\partial\Delta^\vee$.* A more complete discussion of this restriction can be found in §8.

Since the subdivision $\mathcal{T}$ of $\Delta^\vee$ was a triangulation, the fan Σ is necessarily smooth. But we would also like to require that Σ be quasi-projective; recall that this is equivalent to the condition that the triangulation $\mathcal{T}$ be regular.

Definition 3.4. A subdivision $\mathcal{T}$ of $\Delta^\vee$ is *regular* (sometimes also called *coherent*) if it is obtained by projection of finite faces of the overgraph of a convex piecewise linear function $\alpha\colon \Delta^\vee \cap M^\vee \to \mathbb{R}$.

3.4. The toric boundary

In this paper, we are interested in the boundary $\partial\mathbf{T}_\Sigma$ of a toric variety $\mathbf{T}_\sigma$, which by definition is the union of the non-trivial orbit closures:

$$\partial\mathbf{T}_\Sigma = \bigcup_{0\neq\sigma\in\Sigma} \overline{O}_\sigma.$$

In fact, we need a scheme- (or stack-)theoretic version of this statement. Below we always take both each $\overline{O}_\sigma$ and $\partial\mathbf{T}_\Sigma$ with their reduced structure.

LEMMA 3.5. *The diagram of algebraic stacks $\overline{O}_\sigma$ and their inclusions, indexed by the poset of non-zero cones σ in Σ, gives a presentation*

$$\varinjlim_{\sigma\in\Sigma\setminus\{0\}} \overline{O}_\sigma \cong \partial\mathbf{T}_\Sigma \tag{3.1}$$

of the toric boundary $\partial\mathbf{T}_\Sigma$ as a sequence of pushouts along closed embeddings.

Proof. That the colimit in (3.1) can be understood as a sequence of pushouts along closed embeddings is clear, as is the existence of the map $\varinjlim_\sigma \overline{O}_\sigma \to \partial\mathbf{T}_\Sigma$. We must check that this map is an equivalence. As the question is étale local, we may reduce to the case where $\mathbf{T}_\Sigma$ is an affine toric variety, so that $\mathbf{T}_\Sigma = \operatorname{Spec} k[\tau^\vee]$, where τ is the unique maximal cone in Σ.

The ring of functions $\mathcal{O}(\partial\mathbf{T}_\Sigma)$ is the quotient of $k[\tau^\vee]$ by all functions which vanish on all faces; observe that this ideal is generated by the points of the interior of $\tau^\vee$. That is, $\mathcal{O}(\partial\mathbf{T}_\Sigma) = k[\tau^\vee]/k[\operatorname{Int}\tau^\vee]$. Meanwhile, the rings of functions $\mathcal{O}(\overline{O}_\sigma)$ are the further quotients of this by all functions except for those on the facet of $\tau^\vee$ corresponding to σ.

Thus we are interested in whether the map $k[\tau^\vee]/k[\operatorname{Int}\tau^\vee] \to \lim_{\eta<\tau^\vee} k[\eta]$ is an isomorphism, where the η are the faces of $\tau^\vee$. We can study this character by character, i.e., separately at each integer point of $\partial\tau^\vee$. What we must show is that the character-χ part of the limit is 1-dimensional. As pointed out to us by Martin Olsson, this can be seen by observing that the character-χ part of the limit is computing precisely the cohomology of the normal cone to τ at the character χ—and this cone is contractible. $\square$

We will discuss the mirror to this cover in §4.3.

4. The FLTZ skeleton

Here we recall from [FLTZ1], [FLTZ3], [FLTZ4] the conic Lagrangian $\mathbb{L}_\Sigma \subset T^*\mathbb{T}^\vee$.

4.1. Non-stacky definition and examples

With a non-stacky fan Σ, [FLTZ3] associated a conic Lagrangian

$$\mathbb{L}_\Sigma = \bigcup_{\sigma\in\Sigma}(\sigma^\perp)\times(-\sigma) \subset (M_\mathbb{R}^\vee/M^\vee)\times M_\mathbb{R} = T^*\mathbb{T}^\vee.$$

This skeleton is meant to encode the mirror geometry to the toric variety $\mathbf{T}_\Sigma$, and we will term it the mirror skeleton of $\mathbf{T}_\Sigma$.

We draw two examples in Figures 7 and 8. The drawing convention is that the hairs indicate conormal directions along a hypersurface; likewise the circles or angles indicate conormals at a point. Thus, each picture depicts a conical Lagrangian, and the corresponding FLTZ skeleton is the union of this with the zero section.

Example 4.1. (The mirror skeleton of $\mathbb{A}^1$.) Consider the fan in $\mathbb{R}$ whose sole nontrivial cone is spanned by $1\in\mathbb{R}$. We write $\mathbb{L}_1 \subset T^*S^1 = S^1\times\mathbb{R}$ for the corresponding FLTZ skeleton; it is the union of the zero section and half a cotangent fiber at the origin:

$$\mathbb{L}_1 = \{(\theta,0):\theta\in S^1\}\cup\{(0,\xi):-\xi\in\mathbb{R}_{\geqslant 0}\}.$$

Example 4.2. (The mirror skeleton of $\mathbb{A}^n$.) Consider the fan in $\mathbb{R}^n$ consisting of all cones generated by subsets of $e_1,...,e_n$. One easily sees that the corresponding FLTZ skeleton $\mathbb{L}_n \subset T^*T^n$ satisfies $\mathbb{L}_n = (\mathbb{L}_1)^n$.

Another useful description of it is as follows:

$$\mathbb{L}_n = T^*_{(S^1)^n}(S^1)^n \cup \bigcup_{k=1}^{n} \mathbb{L}_{n-1}\times(T^*_{S^1}S^1)_k, \tag{4.1}$$

where by $T^*_{S^1}S^1$ we mean the zero section of T^*S^1, with the subscript k indicating that it is to be inserted in the kth coordinate (with the $k,...,n$ coordinates of $\mathbb{L}_{n-1}$ moved forward one place).

4.2. Stacky definition and example

In [FLTZ4], a 'stacky' version of this construction is given. Note first that we can understand the torus $\mathbb{T}^\vee$ as the Pontrjagin dual of the lattice $M^\vee$:

$$\mathbb{T}^\vee = \widehat{M^\vee} = \mathrm{Hom}(M^\vee, \mathbb{R}/\mathbb{Z}).$$

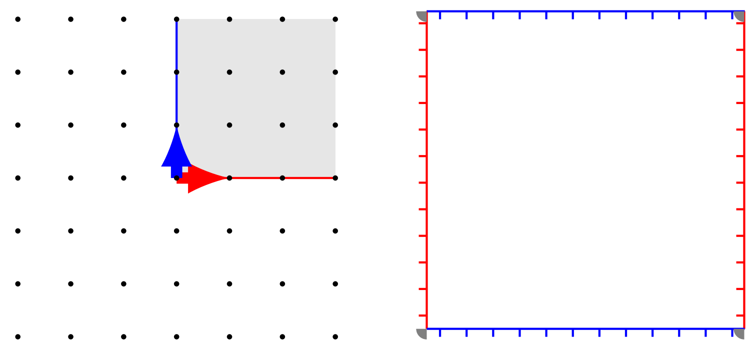

Figure 7. The fan and FLTZ skeleton for $\mathbb{A}^2$.

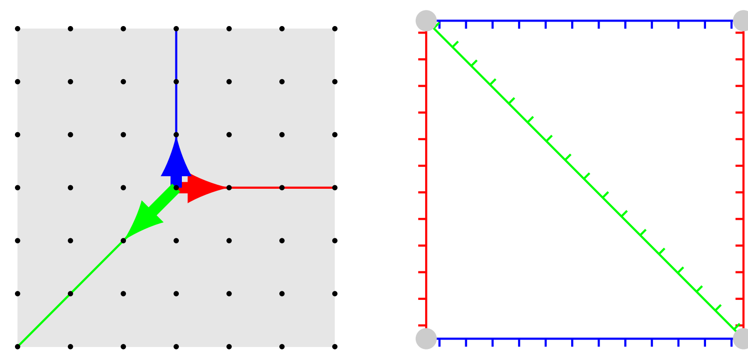

Figure 8. The fan and FLTZ skeleton for $\mathbb{P}^2$.

Now let $\sigma \in \Sigma$ be a cone, corresponding to a face F_σ of the polytope $\Delta^\vee$. If F_σ has vertices $\beta_1, ..., \beta_k$, then we denote by $M_\sigma^\vee$ the quotient

$$M_\sigma^\vee = M^\vee / \langle \beta_1, ..., \beta_k \rangle.$$

This abelian group (which may no longer be free) depends not only on the cone σ, but also the 'stacky' data of the choice of primitive generators β_i of the rays in σ.

Thus the group of homomorphisms $\mathrm{Hom}(M_\sigma^\vee, \mathbb{R}/\mathbb{Z})$, which we will denote by G_σ, is a possibly disconnected subgroup of $\widehat{M^\vee} = \mathbb{T}^\vee$. We write Γ_σ for the group $\pi_0(G_\sigma)$ of components of G_σ. We use these possibly disconnected tori to define $\mathbb{L}_\Sigma$ in the general case.

Definition 4.3. The FLTZ *skeleton* $\mathbb{L}_\Sigma \subset T^*\mathbb{T}^\vee$ is the conic Lagrangian

$$\mathbb{L}_\Sigma = \bigcup_{\sigma \in \Sigma} (G_\sigma \times (-\sigma)).$$

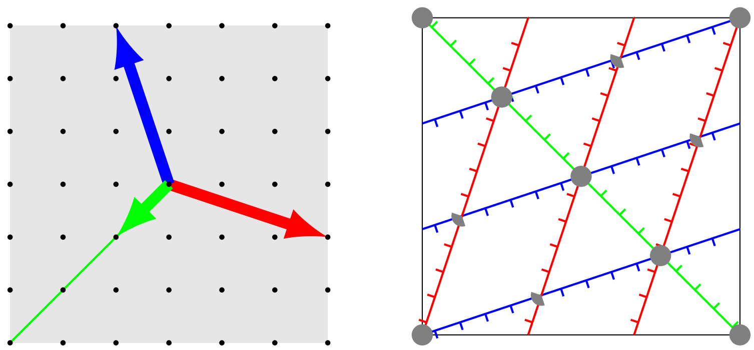

Figure 9. The stacky fan and FLTZ skeleton described in Example 4.5. The "stackiness" of this fan is due to the fact that G_σ is non-zero for each top-dimensional cone σ; on the mirror, this is reflected by the presence of cocircles in $\mathbb{L}_\Sigma^\infty$ (the gray circles in the figure) above points other than zero.

We will denote by $\mathbb{L}_\Sigma^\infty$ or $\partial\mathbb{L}_\Sigma$ the corresponding Legendrian in $T^\infty\mathbb{T}^\vee$: it is the spherical projectivization of $\mathbb{L}_\Sigma\backslash\mathbb{T}^\vee$. When Σ is a non-stacky fan, this reduces to the above definition.

Remark 4.4. The relative skeleton of the Liouville sector associated with the Hori–Vafa superpotential W will be $-\mathbb{L}_\Sigma$ rather than $\mathbb{L}_\Sigma$. This minus sign is a feature: it cancels the need for taking opposite category in the sheaf-Fukaya equivalence of [GPS3].

Example 4.5. Let Σ be the complete fan of cones in $\mathbb{R}^2$ which has three 1-dimensional cones σ_1, σ_2, and σ_3, spanned by the respective vectors $(-1,3)$, $(3,-1)$, and $(-1,-1)$, and three 2-dimensional cones, which we will denote by τ_{ij}, where σ_i and σ_j are the boundaries of τ_{ij}.

The tori $\sigma_i^\perp$ have four points of triple intersection, and the tori $\sigma_1^\perp, \sigma_2^\perp$ have four additional points of intersection. For any τ_{ij}, the group $\Gamma_{\tau_{ij}}$ of discrete translations is equal to the group $\sigma_i^\perp \cap \sigma_j^\perp$, so that for each τ_{ij} and each $p \in \sigma_i^\perp \cap \sigma_j^\perp$, there is an interval in the cosphere fiber $T_p^\infty\mathbb{T}^\vee$ connecting the Legendrian lifts of the tori $\sigma_i^\perp$ and $\sigma_j^\perp$. See Figure 9.

The discrete data is used in the definition of the stacky skeleton to add pieces that will connect the Legendrian lifts of tori $\sigma^\perp$ and $\tau^\perp$ in $\partial\mathbb{L}_\Sigma$ over points where those tori intersect in the base $\mathbb{T}^\vee$.

4.3. Recursive structure

The Legendrian boundary of the FLTZ skeleton admits a structure that will be crucial in our proof of mirror symmetry: it is a union of stabilized FLTZ skeleta for lower-dimensional fans, glued along their own Legendrian boundaries. This is mirror to the fact, described above in §3.4, that the boundary of a toric variety is the union of closures of toric orbits, which are themselves toric varieties, as are their intersections.

Let Σ be a (possibly stacky) fan as above. We have seen that each cone σ in Σ contributes a piece $G_\sigma \times (-\sigma)$ to the FLTZ Lagrangian $\mathbb{L}_\Sigma$. Write

$$\mathbb{L}_\Sigma^\sigma := \bigcup_{\tau \supset \sigma} G_\tau \times (-\tau) \subset \mathbb{L}_\Sigma$$

for the union, over all cones τ in which σ is a face, of these pieces. Observe that we have inclusion maps

$$\mathbb{L}_\Sigma^\tau \hookrightarrow \mathbb{L}_\Sigma^\sigma \tag{4.2}$$

for any inclusion of cones $\tau \supset \sigma$.

For a cone σ, consider the quotient $M_\mathbb{R}^\vee/\langle\sigma\rangle$ of $M_\mathbb{R}^\vee$ by the subspace spanned by σ. In this quotient, consider the reduced fan $\Sigma(\sigma)$ formed by the images of cones containing σ. For a cone τ containing σ, we write τ/σ for the image of τ in this cone. We have seen in §3.2 that this is the fan of the closure in the toric variety $\mathbf{T}_\Sigma$ of the toric orbit $O(\sigma)$. We write $\mathbb{L}_{\Sigma(\sigma)}$ for the FLTZ skeleton of the fan $\Sigma(\sigma)$, which we imagine as living in the cotangent bundle of the possibly disconnected torus T^*G_σ. (In other words, we take a disjoint union of copies of the usual FLTZ Lagrangian for this fan in order to account for the stackiness of σ.)

Example 4.6. Let Σ be the stacky fan of cones in $\mathbb{R}^2 = \mathbb{R}\langle e_1, e_2\rangle$ generated by e_2, $2e_1$, and $-e_1-e_2$, and let σ be the ray generated by $-e_1-e_2$. Then, by choosing $\bar{e}_1$ as a basis vector for the quotient space $\mathbb{R}^2/\langle -e_1-e_2\rangle$, the quotient fan $\Sigma(\sigma)$ may be identified with the stacky fan of cones in $\mathbb{R}$ with generators 2 and -1.

Observe that, for any inclusion of cones $\tau \supset \sigma$, the quotient τ/σ is the cone of conormal directions to G_τ in G_σ. Using the factorization $T^*G_\sigma|_{G_\tau} = T^*G_\tau \times T^*_{G_\tau} G_\sigma$, we can write the restriction to G_τ of the cotangent bundle to G_σ as

$$T^*G_\sigma|_{G_\tau} = T^*G_\tau \times \tau/\sigma. \tag{4.3}$$

Now note that the component of $\mathbb{L}_{\Sigma(\sigma)} \subset T^*G_\sigma$ contributed by τ—the product of the perpendicular torus G_τ with the cone τ/σ—is a product Lagrangian in the factorization (4.3). In other words, we have an inclusion

$$G_\tau \times \tau/\sigma \hookrightarrow \mathbb{L}_{\Sigma(\sigma)}.$$

Moreover, any cone τ' containing τ will also contribute to $\mathbb{L}_{\Sigma(\sigma)}$ a product Lagrangian contained inside (4.3); putting these all together, we get an inclusion of all of $\mathbb{L}_{\Sigma(\tau)}$:

$$\mathbb{L}_{\Sigma(\tau)} \times \tau/\sigma \hookrightarrow \mathbb{L}_{\Sigma(\sigma)}.$$

This induces an inclusion

$$\mathbb{L}_{\Sigma(\tau)} \times \tau = (\mathbb{L}_{\Sigma(\tau)} \times \tau/\sigma) \times \sigma \hookrightarrow \mathbb{L}_{\Sigma(\sigma)} \times \sigma. \tag{4.4}$$

In particular, we may take $\sigma=0$, and hence $\Sigma(\sigma)=\Sigma$. Then the images of the $\mathbb{L}_\tau$ agree with the aforementioned pieces:

LEMMA 4.7. *The image of the map $\mathbb{L}_{\Sigma(\tau)} \times \tau \hookrightarrow \mathbb{L}_\Sigma$ is $\mathbb{L}_\Sigma^\tau$. Moreover, under this identification, the inclusions* (4.2) *and* (4.4) *agree.*

We can rephrase this as a statement about a cover of the Legendrian boundary $\partial\mathbb{L}_\Sigma$ of the FLTZ skeleton $\mathbb{L}_\Sigma$. Let $S_\sigma \subset T^\infty \mathbb{T}^\vee$ denote the boundary of $G_\sigma \times \sigma \subset T^* \mathbb{T}^\vee$.

COROLLARY 4.8. *The Legendrian $\partial\mathbb{L}_\Sigma$ has an open cover by subsets $\Omega_\sigma \subset \partial\mathbb{L}_\Sigma$, anti-indexed by the poset of non-zero cones in the fan Σ, such that $\Omega_\sigma \cong \mathbb{L}_{\Sigma(\sigma)} \times S_\sigma$, with the inclusions among these as described in Lemma 4.7.*

4.4. T-duality description

In the next section we will explain how $\mathbb{L}_\Sigma$ is related to the symplectic geometry of the Hori–Vafa superpotential. Here we informally describe another way to arrive at $\mathbb{L}_\Sigma$, by studying the dual to the moment fibration of the toric variety. This subsection contains no rigorous mathematical statements and nothing in the remainder of the article depends upon it.

Consider the example where $\Sigma \subset \mathbb{R}$ has as cones the loci 0, $[0,\infty)$, and $(-\infty, 0]$, i.e., where Σ is the fan whose toric variety is the projective line $\mathbb{P}^1$. The momentum map gives this space the structure of a circle fibration over an interval whose circle fibers degenerate to zero radius at the ends. The mirror should be again a circle fibration over an interval, this time with fibers degenerating to infinite radius on both ends. Above, we made this precise by declaring that the mirror is the exact symplectic manifold T^*S^1, endowed with the Liouville sectorial structure in which each end of the cylinder has some stopped boundary. Imposing these stops results in a skeleton given by the union of the zero section and the conormal to a point. This is precisely the skeleton $\mathbb{L}_\Sigma$ associated in [FLTZ3] with the fan Σ.

More generally, consider a toric Fano variety $\mathbf{T}_\Sigma$, compactifying a torus $\mathbb{T}$, corresponding to a fan Σ in $M_\mathbb{R}^\vee$. Let $\mathbf{T}_\Sigma \to \Delta \subset M_\mathbb{R}$ be the anticanonical momentum map. The polytope Δ has the property that the cone over its polar dual $\Delta^\vee$ is just Σ.

To find the mirror, we should take the dual torus $\mathbb{T}^\vee$ as a fiber of the dual fibration over the polytope $\Delta^\vee \subset M_\mathbb{R}^\vee$. This polytope will not be used to define another toric variety but rather, under the principle that the T-dual of a collapsing fibration is a blowing up one, we use this polytope to define stopping conditions. Before, the torus spanned by the cocharacters of σ would degenerate to radius zero along the corresponding face; now, we want it to be impossible to go all the way around the dualized version of this torus. Correspondingly, for each cone $\sigma\in\Sigma$, we introduce the stop $\sigma^\perp$ over the face of $\Delta^\vee$ whose cone is σ. The result (up to a sign) is the skeleton $\mathbb{L}_\Sigma$.

Another derivation of $\mathbb{L}_\Sigma$ by this sort of T-duality reasoning can found in [FLTZ3].

5. Pants

5.1. Pants

By an $(n-1)$-dimensional pants, we mean the complement in $(\mathbb{C}^\times)^{n-1}$ of a linear hypersurface transverse to all coordinate subspaces, or equivalently such a linear hypersurface inside $(\mathbb{C}^\times)^n$.

Throughout our discussion of hypersurfaces in $(\mathbb{C}^\times)^n$, we will use the map

$$\begin{aligned}\mathrm{Log}\colon (\mathbb{C}^\times)^n &\longrightarrow \mathbb{R}^n,\\ (z_1,...,z_n) &\longmapsto (\log|z_1|,...,\log|z_n|),\end{aligned}$$

the moment map for the action of the compact torus $(S^1)^n$ on $(\mathbb{C}^\times)^n$.

Definition 5.1. For $n\geqslant 1$, the *standard* $(n-1)$*-dimensional pants* is

$$\mathcal{P}_{n-1}=\{z_1+...+z_n-1=0\}\subset(\mathbb{C}^\times)^n.$$

The *amoeba* of $\mathcal{P}_{n-1}$ is its image $\mathcal{A}_{n-1}:=\mathrm{Log}(\mathcal{P}_{n-1})$ in $\mathbb{R}^n$ under the Log map.

Remark 5.2. The pants $\mathcal{P}_{n-1}$ has an obvious action of the symmetric group Σ_n, but in fact this action extends to an action of the symmetric group Σ_{n+1}. This can be seen by writing $(\mathbb{C}^\times)^n$ as the dense torus in $\mathbb{P}^n$, and hence embedding $\mathcal{P}_{n-1}$ as an open subset of the hypersurface $\overline{\mathcal{P}}_{n-1}$ in $\mathbb{P}^n$ defined by the equation

$$\overline{\mathcal{P}}_{n-1}=\{z_1+...+z_n+z_{n+1}=0\}\subset\mathbb{P}^n.$$

This closed hypersurface has a manifest Σ_{n+1} action which respects the open part $\mathcal{P}_{n-1}$. In our original coordinates, this action is generated from the Σ_n action by the extra generator

$$(z_1,...,z_n)=[z_1:...:-z_{n+1}]\longmapsto[-z_{n+1}:z_1:...:z_n]=\left(\frac{-1}{z_n},\frac{z_1}{z_n},...,\frac{z_{n-1}}{z_n}\right), \tag{5.1}$$

and the Log map becomes equivariant for the Σ_{n+1} action on $\mathbb{R}^n$ obtained by descending the symmetry (5.1) to $\mathbb{R}^n$ in the evident way:

$$(x_1,...,x_n)\longmapsto(-x_n,x_1-x_n,...,x_{n-1}-x_n).$$

Let $\Delta^\vee_{n-1}\subset\mathbb{R}^n$ be the standard n-simplex, i.e., the convex hull of the origin and standard basis vectors $\{e_1,...,e_n\}$. Let Π_{n-1} be the union of positive-codimensional cones in the fan generated by $\{-e_1,...,-e_n,\sum_i e_i\}$. Then, Π_{n-1} is a translate of the dual complex of $\Delta^\vee_{n-1}$, and a deformation retract of the amoeba $\mathcal{A}_{n-1}$. The relationships between $\mathcal{P}_{n-1}$, $\Delta^\vee_{n-1}$, $\mathcal{A}_{n-1}$, and Π_{n-1} are the simplest instances of the general relationship between very affine hypersurfaces and their tropicalizations, as will be recalled in detail in §6.1.

More generally we will consider, for $\ell_1,...,\ell_n\gg 0$, the *translated pants*

$$\mathcal{P}^\ell_{n-1}=\{e^{-\ell_1}z_1+...+e^{-\ell_n}z_n-1=0\}\subset(\mathbb{C}^\times)^n, \tag{5.2}$$

whose amoeba we denote by

$$\mathcal{A}^\ell_{n-1}:=\mathrm{Log}(\mathcal{P}^\ell_{n-1}).$$

This amoeba can be obtained as a translation of $\mathcal{A}_{n-1}$ by the vector $\ell\in\mathbb{R}^n$, which pushes it far into the first orthant.

Because the coefficients are all real, we have the following.

LEMMA 5.3. *The components of $\partial\mathcal{A}^\ell_{n-1}$ are the images of certain components of the real points of $\mathcal{P}^\ell_{n-1}$. In particular, the component of $\partial\mathcal{A}^\ell_{n-1}$ bounding the region containing all sufficiently negative points* (*which corresponds to the vertex zero of the simplex $\Delta^\vee_{n-1}$*) *is the image of the real positive points of $\mathcal{P}^\ell_{n-1}$.*

Proof. That the critical points of $\mathrm{Log}|_{\mathcal{P}^\ell_{n-1}}$ are precisely the real points of $\mathcal{P}^\ell_{n-1}$ is proved in [M, Proposition 4.4]. The critical values of this map certainly include the boundary components of the amoeba, and one can check that the "bottom-left" boundary component contains the image of the real positive points by observing that it contains the real positive point $(e^{\ell_1}/n,...,e^{\ell_n}/n)$. □

We will also want to consider certain other hypersurfaces which are naturally unramified covers of pants, or products of these with copies of $\mathbb{C}^\times$.

Definition 5.4. Let $M^\vee$ be a lattice equipped with a choice of lattice k-simplex $P\subset M^\vee_\mathbb{R}$ containing zero as a vertex, and choose moreover an ordering of the vertices of P. This data determines an injection $T^\vee:\mathbb{Z}^k\to M^\vee$, inducing a dual map of tori $f_T: M_{\mathbb{C}^\times}\to(\mathbb{C}^\times)^k$. We write $\mathcal{P}_P:=f_T^{-1}(\mathcal{P}_{k-1})$ for the variety obtained from the pants $\mathcal{P}_{k-1}$ by pullback along the map f_T, and $\mathcal{A}_P:=\mathrm{Log}(\mathcal{P}_P)$ for its amoeba.

We will refer to the variety $\mathcal{P}_P$, which by construction is an unramified cover of the standard pants $\mathcal{P}_{k-1}$, as the *P-pants.* As in equation (5.2) above, we may also scale the coefficients of $\mathcal{P}_P$ by $e^{-\ell_i}$ in order to obtain the *translated P-pants* $\mathcal{P}^\ell_P$, whose amoeba $\mathcal{A}^\ell_P$ is related to the amoeba $\mathcal{A}_P$ by translation into the first orthant.

Remark 5.5. In our notation $\mathcal{P}_P$, we suppress the additional choice of ordering of the vertices of P: this choice does not affect the variety $\mathcal{P}_P$, since it changes the map f_T by a permutation of coordinates, and $\mathcal{P}_{k-1}$ is permutation invariant. However, this choice will be used implicitly in our labeling of the legs of $\mathcal{P}_P$ in Definition 5.11.

We have the following relationship between amoebae.

LEMMA 5.6. *Let $T\colon M_{\mathbb{R}}\to\mathbb{R}^k$ be the dual of $T^\vee\otimes\mathbb{R}$. Then, $T(\mathcal{A}_T)=\mathcal{A}_{k-1}$.*

Proof. By construction, the diagram

$$\begin{array}{ccc} M_{\mathbb{C}^\times} & \xrightarrow{f_T} & (\mathbb{C}^\times)^k \\ \downarrow{\scriptstyle \mathrm{Log}} & & \downarrow{\scriptstyle \mathrm{Log}} \\ M_{\mathbb{R}} & \xrightarrow[T]{} & \mathbb{R}^k \end{array}$$

commutes, and the map f_T is surjective. ☐

Note that if P is unimodular (e.g., when $T^\vee$ is an inclusion of a coordinate subspace), then there is an isomorphism $\mathcal{P}_P\cong\mathcal{P}_{k-1}\times(\mathbb{C}^\times)^{n-k}$.

5.2. Tailoring

PROPOSITION 5.7. ([M, §6.6], [Ab1, Propositions 4.2 and 4.9]) *Fix $\varepsilon, K\in\mathbb{R}$ with $0<\varepsilon\ll K$. There is a Σ_{n+1}-equivariant symplectic isotopy from $\mathcal{P}_{n-1}$ to a hypersurface $\widetilde{\mathcal{P}}_{n-1}$ with the following properties*:

(1) *On the region*

$$L_1=\{(z_1,...,z_n)\in\widetilde{\mathcal{P}}_{n-1}:\log|z_1|<-K\},$$

there is an equality

$$L_1=\{z_1\in\mathbb{C}^\times:\log|z_1|<-K\}\times\widetilde{\mathcal{P}}_{n-2},$$

and analogous equalities hold on the other n ends of $\widetilde{\mathcal{P}}_{n-1}$.

(2) *Let*

$$L_1^\varepsilon=\{(z_1,...,z_n)\in\mathcal{P}_{n-1}:\log|z_1|<-K+\varepsilon\},$$

and similarly for the other n ends of $\mathcal{P}_{n-1}$. Then, the isotopy is constant outside of

$$\bigcup_{i=1}^{n+1}L_i^\varepsilon.$$

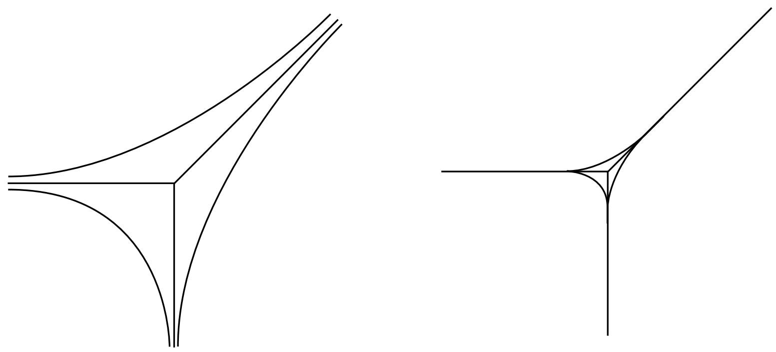

Figure 10. The spine Π_1, included in the amoebae of $\mathcal{P}_1$ and $\widetilde{\mathcal{P}}_1$. (In the second picture, we have also rescaled the base of the Log map to make the situation clearer.)

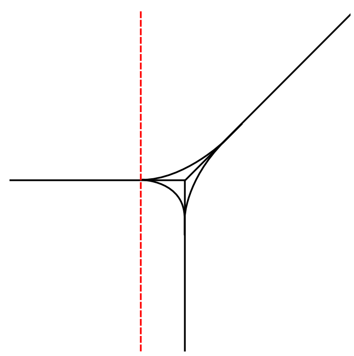

Figure 11. The amoeba of $\widetilde{\mathcal{P}}_1$ together with a red line indicating the region $\log|z_1|=-K$. The leg L_1 of $\widetilde{\mathcal{P}}_1$ is the set of points in $\widetilde{\mathcal{P}}_1$ which project to the area to the left of the red dashed line.

In particular, the amoeba $\tilde{\mathcal{A}}_{n-1}:=\mathrm{Log}(\widetilde{\mathcal{P}}_{n-1})$ *differs from* Π_{n-1} *only in a neighborhood of the singularities of the latter.* (*See Figure* 10 *for the case* $n=2$.)

In Remark 6.7 below, we recall from [Ab1, §4] the construction of this isotopy, in the context of an arbitrary Newton polytope.

Definition 5.8. We call the regions L_i defined above the *legs* of the pants $\widetilde{\mathcal{P}}_{n-1}$.

Definition 5.9. Following [N2], we will call the hypersurface $\widetilde{\mathcal{P}}_{n-1}$ the *tailored pants.* (In [M], it was called the "localized pants".)

We analogously write $\widetilde{\mathcal{P}}^{\ell}_{n-1}$ and $\tilde{\mathcal{A}}^{\ell}_{n-1}$ for the corresponding construction applied to the translated pants $\mathcal{P}^{\ell}_{n-1}$. Likewise, in the situation of Definition 5.4, we have a *tailored* P*-pants* $\widetilde{\mathcal{P}}_P:=f_T^{-1}(\widetilde{\mathcal{P}}_{k-1})$ defined as the preimage of the tailored pants under the map f_T corresponding to a choice of k-simplex $P=T^\vee(\Delta_{k-1})$, and the *translated tailored* P*-pants* $\widetilde{\mathcal{P}}^{\ell}_P$ obtained by rescaling its coefficients.

As in the case of the P-pants $\mathcal{P}_P$, the tailored P-pants $\widetilde{\mathcal{P}}_P$ is an unramified cover of the standard tailored pants $\widetilde{\mathcal{P}}_{n-1}$, so that $\widetilde{\mathcal{P}}_P$ is easy to understand in terms of the tailoring construction we have already discussed. In particular, the analogue of Lemma 5.6 holds for the tailored P-pants.

LEMMA 5.10. $T(\mathrm{Log}(\widetilde{\mathcal{P}}_P)){=}\tilde{\mathcal{A}}_{n-1}$.

The P-pants $\widetilde{\mathcal{P}}_P$ also inherits from $\widetilde{\mathcal{P}}_{n-1}$ an inductive structure on its legs, which we summarize as follows:

Definition 5.11. The ith *leg* L_i of the P-pants $\widetilde{\mathcal{P}}_P$ is the preimage, under the map $\bar{f}_T$, of the ith leg of the standard pants $\widetilde{\mathcal{P}}_{n-1}$. It is isomorphic to $\widetilde{\mathcal{P}}_{F_i}$, where

$$F_i = \mathrm{Conv}(0, v_1, ..., \hat{v}_i, ..., v_k)$$

is the corresponding facet of P. (Note that the labeling of L_i as the ith leg depends on the ordering of vertices of P, which the space $\widetilde{\mathcal{P}}_P$ itself is insensitive to.)

5.3. Skeleta of pants

5.3.1. The skeleton of $\widetilde{\mathcal{P}}^{\ell}_{n-1}$

We now begin to study the Liouville flow of the translated tailored pants $\widetilde{\mathcal{P}}^{\ell}_{n-1}$. Throughout this section, we make the following requirement on the translation parameter ℓ:

$$\ell \gg K,$$

where K is the parameter from the tailoring construction of Proposition 5.7. We equip $\widetilde{\mathcal{P}}^{\ell}_{n-1}$ with the restriction λ of the symplectic primitive from $(\mathbb{C}^*)^n$. This is compatible with the recursive structure from Proposition 5.7 (1).

LEMMA 5.12. *Consider the leg* L_i *of* $\widetilde{\mathcal{P}}^{\ell}_{n-1}$ *for* $1{\leqslant}i{\leqslant}n$. *There is an isomorphism of Liouville manifolds* $L_i{\cong}\widetilde{\mathcal{P}}^{\ell_{\hat{i}}}_{n-2}{\times}\mathrm{Cyl}_i$, *where* $\mathrm{Cyl}{\subset}\mathbb{C}^{\times}$ *is a half-cylinder containing the zero section. The subscript* i *on the second factor indicates that it is placed as the* i*-th coordinate, and we write* $\ell_{\hat{i}}$ *for* $(\ell_1, ..., \hat{\ell}_i, ..., \ell_n)$.

Proof. The product decomposition of each leg follows from Proposition 5.7 (1), and the fact that the $\mathbb{C}^{\times}$ factor in the product contains the unit circle is ensured by the translation by ℓ: due to the assumption $\ell{\gg}K$, this translation shifts the beginning of each of the legs $L_1, ..., L_n$ (under the amoeba projection) into the first orthant of $\mathbb{R}^n$. For an illustration, see Figure 12, where the unit circle in each leg is indicated by a red dot. □

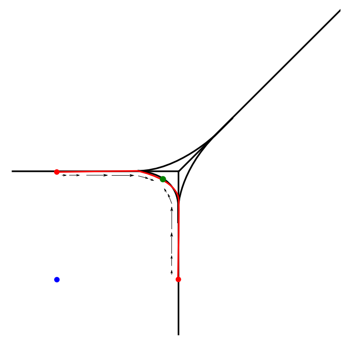

Figure 12. The simplex $\bar{S}_+$, drawn in red on the amoeba of $\widetilde{\mathcal{P}}_1^\ell$, with its barycenter illustrated in green. Note that the vertices of $\bar{S}_+$ are the closest points on their respective legs to the origin (blue). The arrows indicate Liouville flow along $\bar{S}_+$.

COROLLARY 5.13. *The Liouville flow for λ on $\widetilde{\mathcal{P}}_{n-1}^\ell$ is complete*; *i.e.*, $(\widetilde{\mathcal{P}}_{n-1}^\ell, \lambda)$ *is a Liouville manifold.*

Proof. Recall that the product of Liouville manifolds is Liouville. Now, Lemma 5.12 inductively characterizes the Liouville flow in the complement of a compact set. □

Remark 5.14. Because the original $\mathcal{P}_{n-1}$ was algebraic and hence in particular a Stein submanifold of $(\mathbb{C}^\times)^n$, and because the Liouville form on $(\mathbb{C}^\times)^n$ arises from a Kähler potential (namely $|\mathrm{Log}|^2$), it is also the case that the restriction of the ambient Liouville form to $\mathcal{P}_{n-1}$ gives a Liouville structure on $\mathcal{P}_{n-1}$. It is presumably true that the tailoring isotopy (recalled in Remark 6.7 below from [Ab1, §4]) is an isotopy of Liouville manifolds, but we do not prove this here.

Recall that we write $\mathbb{L}_n$ for the FLTZ skeleton mirror to affine n-space, as described in Example 4.2.

THEOREM 5.15. ([N2]) *Let $\partial^0\tilde{\mathcal{A}}_{n-1}^\ell \subset \tilde{\mathcal{A}}_{n-1}^\ell$ be the component of $\partial\tilde{\mathcal{A}}_{n-1}^\ell$ which bounds the region of $\mathbb{R}^n$ containing the all-negative orthant. Let $C = \mathrm{Log}^{-1}(\partial^0\tilde{\mathcal{A}}_{n-1}^\ell) \subset (\mathbb{C}^\times)^n$.*

Then, C is a contact hypersurface, and the skeleton of $\widetilde{\mathcal{P}}_{n-1}^\ell$ is $C \cap (-\mathbb{L}_n)$.

Proof. We proceed by induction on the dimension of the pants, the case $n=1$ being trivial. Many of the ideas of the proof can be seen in the illustration of Figure 12.

Let us consider the legs of $\widetilde{\mathcal{P}}_{n-1}^\ell$. From Lemma 5.12, it is clear that any zero of the Liouville vector field contained in the leg L_i must be contained inside the zero section, i.e., the unit circle, of its $\mathbb{C}_i^\times$ factor; in other words, any zero of the Liouville vector field on L_i must project under the Log map to the ith coordinate hyperplane in $\mathbb{R}^n$. In particular, no vanishing happens on the $(n+1)$-th leg of the pants, since any vanishing must be contained in the hyperplane given by the sum of the coordinate directions, and the translation by ℓ ensures that this hyperplane is disjoint from the leg L_{n+1}.

Moreover, the preimage in $\widetilde{\mathcal{P}}^{\ell}_{n-1}$ of the coordinate hyperplanes in $\mathbb{R}^n$ is entirely contained in the legs, and stable under the Liouville flow. By Lemma 5.12 and the induction hypothesis, the portion of the skeleton contained in L_i is

$$(-\mathbb{L}_{n-1}\times(T^*_{S^1}S^1)_i)\cap C,$$

using the notation of equation (4.1) in Example 4.2. By comparing that equation to the statement of this theorem, we see that our remaining task is to show there is exactly one more component of the skeleton, and to identify it with the intersection of C with the positive real points of $\widetilde{\mathcal{P}}^{\ell}_{n-1}$.

Away from the legs of the pants $\widetilde{\mathcal{P}}^{\ell}_{n-1}$, the map Log is a local submersion everywhere except the real points $R:=\widetilde{\mathcal{P}}^{\ell}_{n-1}\cap\mathbb{R}^n$. Let $z=(z_1,...,z_n)\in R$ be a real point where the Liouville vector field vanishes. The equation of the pants $\sum e^{-\ell_i}z_i=1$ prevents all z_i from being negative; if z_i is positive and z_j is negative, then the function $|\mathrm{Log}|^2$, which is gradient-like for the Liouville vector field, has a differential which pairs positively with the direction $(z_1,...,z_i+\varepsilon,...,z_j-\varepsilon,...,z_n)$ at this point, so that the Liouville vector field cannot vanish there. Thus $z\in R_+=\widetilde{\mathcal{P}}^{\ell}_{n-1}\cap(\mathbb{R}_{>0})^n$. In order that z not lie in the legs, it must be contained in

$$S_+=\{z\in R_+:\mathrm{Log}(z)\in(\mathbb{R}_{>0})^n\}.$$

Recall that Log restricts to a diffeomorphism $R_+\to\partial^0\tilde{\mathcal{A}}^{\ell}_{n-1}$ from R_+ to the inner boundary component of the tailored amoeba.

Since S_+ is contained inside the real points of $\widetilde{\mathcal{P}}^{\ell}_{n-1}$, the Liouville form vanishes on its tangent vectors, so it is preserved by the Liouville vector field. The Liouville flow increases distance to $0\in\mathbb{R}^n$ under the Log projection, and the embedding of $\mathrm{Log}(S_+)$ in $\mathbb{R}^n$ is concave and symmetric under exchange of coordinates. Hence the Liouville field everywhere points along S_+ toward the barycenter of S_+. This barycenter gives the sole remaining zero of the Liouville form, and it contributes its stable cell S_+ to the skeleton. □

Remark 5.16. The closure $\overline{S}_+$ of the region S_+ is an $(n-1)$-simplex, each facet of which is contained in one of the legs, and whose boundary projects to the intersection of the amoeba with the coordinate hyperplanes. The case $n=2$ is depicted in Figure 12.

5.3.2. Skeleta for P-pants

Let $P=\mathrm{Conv}(0,v_1,...,v_k)\subset M^{\vee}_{\mathbb{R}}$ be a simplex. In Definition 5.4, we described the *P-pants* $\widetilde{\mathcal{P}}_P\subset\mathbb{T}^{\vee}_{\mathbb{C}}$ obtained as a cover of the pants $\mathcal{P}_{n-1}$, and in Definition 5.9 we described its tailored translated version $\widetilde{\mathcal{P}}_{n-1}$.

After choosing an inner product on $M_{\mathbb{R}}$ and hence respective symplectic and Liouville forms ω and λ on $\mathbb{T}^{\vee}_{\mathbb{C}}\cong T^*\mathbb{T}^{\vee}$, we can restrict these to the translated tailored P-pants $\widetilde{\mathcal{P}}^{\ell}_P$ to equip this space with the structure of a Liouville manifold. As for the standard pants, we will be interested in computing the Lagrangian skeleton of $(\widetilde{\mathcal{P}}^{\ell}_P, \lambda)$, closely following the calculation in Theorem 5.15.

Let Σ_P be the stacky fan whose primitives are the non-zero vertices of P. As in the statement of Theorem 5.15, let $\partial^0\mathcal{A}^{\ell}_P$ be the component of the amoeba boundary $\partial\mathcal{A}^{\ell}_P$ bounding the "lower-left" orthant of $\mathbb{R}^n$. We will be interested in the contact hypersurface $C_P\subset\mathbb{T}^{\vee}_{\mathbb{C}}$ lying above this boundary:

$$C_P := \{z\in\mathbb{T}^{\vee}_{\mathbb{C}} : \mathrm{Log}(z)\in\mathcal{A}^0_P\}.$$

As in §4, let G_σ be the possibly disconnected torus $\mathrm{Hom}(M^{\vee}_\sigma, \mathbb{R}/\mathbb{Z})$, where $M^{\vee}_\sigma$ is the quotient of $M^{\vee}$ by the vertices of the stacky primitives in σ. This defines a Lagrangian

$$\mathbb{L}_{\Sigma_P} := \bigcup_{\sigma\in\Sigma_P} G_\sigma\times\sigma\subset T^*\mathbb{T}^{\vee};$$

using the inner product, we can treat Σ as a fan of cones in $M_{\mathbb{R}}$, and hence $\mathbb{L}_{\Sigma_P}$ as a subset of $\mathbb{T}^{\vee}_{\mathbb{C}}$.

For $1\leqslant i\leqslant k$, write Σ^i_P for the fan of cones on the $k-1$ vectors $v_1, ..., v_{\hat{i}}, ..., v_k$. As was the case for the standard pants, we find it helpful to rewrite the FLTZ Lagrangian as a union

$$\mathbb{L}_{\Sigma_P} = (G_{\Sigma_P}\times\Sigma_P)\bigcup_{i=1}^{k}\mathbb{L}_{\Sigma^i_P}$$

of one new piece (where we write Σ_P for the big cone in the fan), living in the cotangent fibers over the points G_{Σ_P}, and FLTZ skeleta for lower-dimensional cones of Σ.

LEMMA 5.17. *There is an equality*

$$\Lambda_P = C_P\cap(-\mathbb{L}_{\Sigma_P})$$

between the skeleton Λ_P of $\widetilde{\mathcal{P}}^{\ell}_P$ and the intersection of the contact hypersurface C_P with the negative stacky FLTZ *Lagrangian for Σ_P.*

Proof. The proof of Theorem 5.15 proceeded by induction on dimension, using the fact that each leg L_i of the standard pants was itself (the product of $\mathbb{C}^{\times}$ with) a pants one dimension lower. The proof here follows the same strategy: we need to consider here P-pants for all P (not necessarily top-dimensional), but as before we induct on the dimension of P.

For clarity, we spell out explicitly the base case, when $P=\mathrm{Conv}(0,v)$ is 1-dimensional. In this case, the tailoring construction is unnecessary, since $\mathcal{P}_P^\ell$ is the hypersurface defined (in coordinates $z=(z_1,...,z_n)$) by $\{z_1^{v_1}...z_n^{v_n}=e^\ell\}$, whose amoeba $\mathrm{Log}(\mathcal{P}_P^\ell)$ is the hyperplane

$$\mathcal{A}_P^\ell=\{v_1x_1+...v_nx_n=\ell\}\subset\mathbb{R}^n\cong M_\mathbb{R}.$$

In other words, the hypersurface $\mathcal{P}_P^\ell$ is a disjoint union of copies of $(\mathbb{C}^\times)^{n-1}$, with its symplectic and Liouville form restricted from those of the ambient $(\mathbb{C}^\times)^n$. Hence, its Liouville vector field is given by the gradient of the restriction of the Morse–Bott function $|\mathrm{Log}|^2$. The critical locus of this function is the fiber of $\mathcal{P}_P^\ell$ over the point $p\in\mathcal{A}_P^\ell$ nearest to $0\in M_\mathbb{R}$, which is a manifold of minima for $|\mathrm{Log}|^2|_{\mathcal{P}_P^\ell}$. As v is the normal vector to the hyperplane $\mathcal{A}_P^\ell$, the point p is the point of $\tilde{\mathcal{A}}_P^\ell$ where it intersects the ray defined by v. The fiber over this point is the preimage, under the covering map f_T, of the corresponding fiber of the standard pants: this is the subtorus $G_v\subset\mathbb{T}^\vee$.

We now assume by induction that we have proven the lemma for all P'-pants with $\dim(P')<n$, and we return to the case where $P=\mathrm{Conv}(0,v_1,...v_n)$ is an n-simplex. From this point the proof follows very closely the proof for the standard pants. As in that case, we first investigate the legs $L_1,...,L_n$ of $\widetilde{\mathcal{P}}_P^\ell$. Each of these is itself a P'-pants, for $P'=\mathrm{Conv}(0,v_1,...,v_{\hat{i}},...,v_n)$, and by induction we know that the vanishing of the Liouville vector field on leg L_i contributes to the skeleton of $\widetilde{\mathcal{P}}_P^\ell$ the piece $\mathbb{L}_{\Sigma_P^i}\cap C_P$. It remains for us to determine the vanishing loci of the Liouville vector field on the interior of the pants. (As for the standard pants, it is obvious that no vanishing happens on the final leg.)

We now consider the simplex $\overline{S}_+=\partial^0\tilde{\mathcal{A}}_P\cap\Sigma_P$, where we write Σ_P for the top-dimensional cone in the fan, and we write $p\in\overline{S}_+$ for the point in the interior of $\overline{S}_+$ which is closest to zero. Let S_+ denote the preimage of the interior of this simplex, which is now a disjoint union

$$S_+=\bigsqcup_{i=1}^d S_+^i$$

of $d=\mathrm{vol}(P)$ open simplices S_+^i. Each of these simplices is preserved by the Liouville flow, which flows each simplex to the point lying over p, on which the Liouville field vanishes. Hence, the remaining pieces of the skeleton are the open simplices S_+^i, each of which is mapped diffeomorphically by Log onto the interior of $\overline{S}_+$. As $\overline{S}_+$ is the intersection of $\partial^0\tilde{\mathcal{A}}_P$ with the big cone in Σ_P, and the fiber in $\widetilde{\mathcal{P}}_P^\ell$ over a point in $\overline{S}_+$ is the discrete group G_Σ, this is the desired extra piece in $\mathbb{L}_{\Sigma_P}\cap C_P$.

Finally, if there were any other vanishing of the Liouville form in the interior of the pants, it would have to lie over a critical value of Log. These critical values are just the

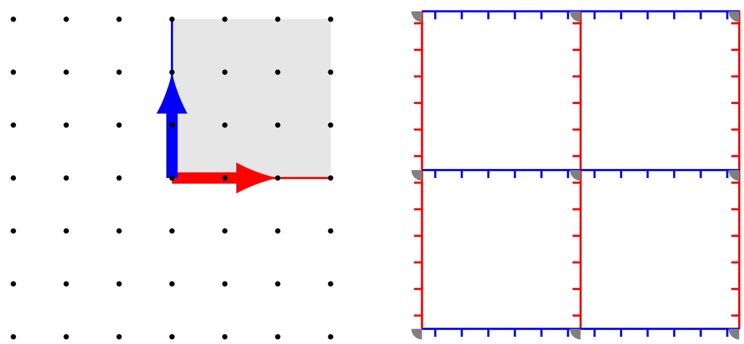

Figure 13. The stacky fan and FLTZ skeleton for $\mathbb{A}^2/(\mathbb{Z}/2\times\mathbb{Z}/2)$.

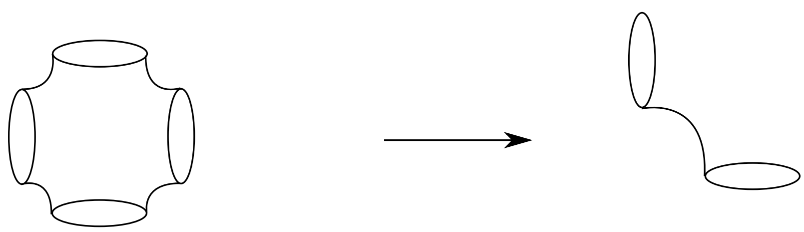

Figure 14. The skeleton $\mathbb{L}_\Delta$ of the pants $\widetilde{\mathcal{P}}_\Delta$ associated with the simplex Δ with vertices $(0,0)$, $(2,0)$, and $(0,2)$, and its 4:1 cover of the skeleton of the standard pants.

preimage (under the cover f_A) of the real points of $\widetilde{\mathcal{P}}_{n-1}$, and we have already seen in the proof of Theorem 5.15 that the Liouville vector field is non-vanishing there. $\square$

A crucial point is that the above result holds in the case of a simplex P with arbitrary volume, obtained as a cover of the standard simplex Δ_{n-1}. For instance, when $n=2$, the P-pants $\widetilde{\mathcal{P}}_P$ may be higher genus.

Example 5.18. Let $\Delta\subset\mathbb{R}^2$ be the simplex with vertices $\{(0,0),(2,0),(0,2)\}$, so that the corresponding stacky fan Σ is a stacky fan for the stack $\mathbb{A}^2/(\mathbb{Z}/2\times\mathbb{Z}/2)$. We draw the stacky fan and FLTZ skeleton in Figure 13. The boundary $\partial\mathbb{A}^2/(\mathbb{Z}/2\times\mathbb{Z}/2)$ matches the mirror skeleton pictured in Figure 14.

6. Patchworking and skeleta

Fix a complex torus $\mathbb{T}^\vee_\mathbb{C}=T\mathbb{T}^\vee$, along with a toric partial compactification $\mathbf{T}_\Sigma$ arising from a (stacky) fan $\Sigma\subset M^\vee_\mathbb{R}$. We write $\Delta^\vee$ for the convex hull of the stacky primitives.

According to [HV], the mirror to $\mathbf{T}_\Sigma$ is the Landau–Ginzburg model associated with a function $W_\Sigma\colon \mathbb{T}^\vee_{\mathbb{C}} \to \mathbb{C}$ whose Newton polytope is $\Delta^\vee$. In addition, the expected mirror to $\partial \mathbf{T}_\Sigma$ is a general fiber F_W of W_Σ.

In this section we will explain how W_Σ determines a Liouville sector (i.e. prove Proposition 2.2) and show that the relative skeleton of this sector is the FLTZ Lagrangian $\mathbb{L}_\Sigma$.

Let us briefly outline the ideas involved. We will study the hyperplane F_W through its *amoeba* ([GKZ]), the projection of F_W to the tangent fiber:

$$\mathcal{A} := \operatorname{Log}(F_W) \subset M_{\mathbb{R}}.$$

The cones of Σ give a triangulation of the polytope $\Delta^\vee$. We choose the Laurent polynomial W_Σ so that its tropicalization Π_Σ is a spine onto which $\mathcal{A}$ retracts. The complex Π_Σ is a piecewise-affine locus dual to the triangulation of $\Delta^\vee$ by the cones of Σ. By assumption, this triangulation is *star-shaped* (all non-boundary simplices share a common vertex zero); the distinguished vertex corresponds to a distinuished component of the complement of the amoeba. We denote the boundary of this region by $\partial^0\mathcal{A} \subset \partial \mathcal{A}$.

Mikhalkin [M] shows how to isotope the hypersurface F_W to another hypersurface F_Σ whose amoeba is "close" to the spine Π_Σ. As we have recalled in §5.2 and §5.3, this isotopy was used by Nadler [N2] to compute the skeleton of the "n-dimensional pants", i.e., the zero locus of the polynomial

$$W_\Sigma = 1 + \sum_{i=1}^{n} z_i.$$

In our more general setting, Mikhalkin's isotopy ensures that the critical points of $\operatorname{Log}|_{F_\Sigma}$ — and in fact the entire skeleton $\mathbb{L}_\Sigma$ — lie above the distinguished boundary component $\partial^0\mathcal{A}$ of the amoeba. The preimage of such a boundary component is precisely a contact type hypersurface. Finally, to each pants in the decomposition of F_Σ we apply the argument from [N2] described in the previous section to obtain the precise form of the skeleton.

6.1. Pants decomposition of F_Σ

In order to construct F_Σ and produce its skeleton, we will follow [N2] in using Mikhalkin's theory of localized hypersurfaces, which we now recall.

6.1.1. Triangulation and dual complex

Recall that we are assuming the fan Σ is smooth and quasi-projective, or equivalently, that the subdivision $\mathcal{T}$ of $\Delta^\vee$ is a *regular* triangulation. By definition, regularity of $\mathcal{T}$

means that $\mathcal{T}$ is the corner locus of a convex piecewise-linear function $\alpha: \Delta^\vee \to \mathbb{R}$. The *Legendre transform* of α is the function

$$\begin{aligned} L_\alpha: M_\mathbb{R} &\longrightarrow \mathbb{R}, \\ m &\longmapsto \max_{n\in\Delta^\vee}(\langle m,n\rangle - \alpha(n)). \end{aligned}$$

Definition 6.1. The *dual complex* for the regular triangulation $\mathcal{T}$ is the polyhedral complex in $M_\mathbb{R}$ obtained as the corner locus of the Legendre transform L_α. We will denote the dual complex for $\mathcal{T}$ by Π_Σ.

Example 6.2. Let $e_1, ..., e_n$ be a basis of $M^\vee$, and let $\Delta^\vee_{\mathrm{std}}$ be the polytope with vertices $0, e_1, ..., e_n$. Then we can define a piecewise-linear function α on $\Delta^\vee_{\mathrm{std}}$ by declaring $\alpha(0)=0$, and $\alpha(e_i)=\alpha_i$ for some $\alpha_i \in \mathbb{R}$. The resulting dual complex Π_{std} is the corner locus of the function $(a_1, ..., a_n) \mapsto \max(0, a_1-\alpha_1, ..., a_n-\alpha_n)$; in other words, it is a translation by $(\alpha_1, ..., \alpha_n)$ of the tropical pants Π_{n-1} defined in §5.1. Note that the function α is not just piecewise-linear but actually linear (and potentially even constant, if we take $\alpha_i=0$ for all i: the break locus of α is necessary to induce a triangulation of $\Delta^\vee$, but the polytope $\Delta^\vee_{\mathrm{std}}$ in this case is already triangulated.

The geometric significance of Π_Σ is the following. Recall that the *amoeba* of a hypersurface in $(\mathbb{C}^\times)^n$ is its image in $\mathbb{R}^n$ under the map

$$\begin{aligned} \mathrm{Log}: (\mathbb{C}^\times)^n &\longrightarrow \mathbb{R}^n, \\ (z_1, ..., z_n) &\longmapsto (\log(|z_1|), ..., \log(|z_n|)). \end{aligned}$$

PROPOSITION 6.3. ([M]) *Let V denote the set of vertices in the triangulation $\mathcal{T}$, and let $H^t=\{f^t=0\}$, where*

$$f^t = \sum_{m\in V} t^{-\alpha(m)} z^m.$$

For $t \gg 0$, the complex $\Pi_\Sigma \cdot \log(t)$ will sit as a spine inside the amoeba of $\mathrm{Log}(H_t)$, and as $t \to \infty$, the rescaled amoebae $\mathrm{Log}(H_t)/\log(t)$ converge (Gromov–Hausdorff) to Π_Σ.

Proof. The basic idea is as follows. Consider a face E of the dual complex Π_Σ, corresponding to a face $E^\vee$ of the triangulation $\mathcal{T}$. Then, the portion of the amoeba lying over the interior of E is a region where the behavior of f_t is dominated by those monomials in f_t corresponding to vertices of $E^\vee$. See [M, §6] for details. □

One says that the complex Π_Σ is the *tropical hypersurface* associated with the Newton polytope $\Delta^\vee$ with regular triangulation $\mathcal{T}$. We term t the 'tropicalization parameter'.

Since the triangulation $\mathcal{T}$ of $\Delta^\vee$ is star-shaped, we have the following result.

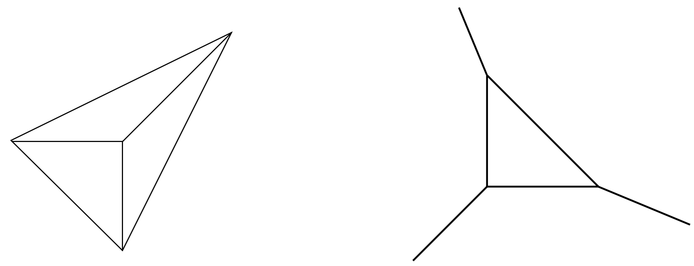

Figure 15. A maximal subdivision of the polygon with vertices $(-1,0)$, $(0,-1)$, and $(1,1)$, and its (unimodular) dual complex Π.

LEMMA 6.4. *Let Π^0_Σ be the component of $M_\mathbb{R}\backslash\Pi_\Sigma$ corresponding to the vertex zero of the triangulation $\mathcal{T}$, and denote its boundary by $\partial^0\Pi_\Sigma$. The polytope Π^0_Σ is a (possibly unbounded) polytope with face poset anti-equivalent to the poset of non-zero cones in the fan Σ.*

The polytope Π^0_Σ will be bounded if and only if the toric variety $\mathbf{T}_\Sigma$ is proper, in which case Π^0_Σ will be the only bounded polytope in $M_\mathbb{R}\backslash\Pi^0_\Sigma$.

6.1.2. Tropical pants

We required the subdivision $\mathcal{T}$ of $\Delta^\vee$ to be a *triangulation*, which means that all of the faces in $\mathcal{T}$ are simplices. This allows us to divide up Π_Σ into pieces we understand.

Definition 6.5. The neighborhood in Π_Σ of any vertex is a *tropical pants*.

These pants will be our basic building blocks in the construction to follow. This has two appealing features: the first is that the complex Π_Σ is obtained by gluing these pants together. Second, a $(k-1)$-face in Π_Σ is the product of $\mathbb{R}^k$ with a $(n-k-1)$-dimensional tropical pants. Hence, the loci along which pants involved in the description of Π_Σ are glued are products of the form (lower-dimensional pants)$\times\mathbb{R}^k$.

6.1.3. Tailoring

We now recall the construction of [M] giving an isotopy from F_W to some F_Σ whose amoeba is closer to the tropical hypersurface Π_Σ. In the case of the pants $\mathcal{P}_{n-1}$, this isotopy was described in Proposition 5.7 above.

It is straightforward to see what F_Σ should be. Suppose two simplices P_1 and P_2 in the triangulation $\mathcal{T}$ share a common face F, so that their respective dual complexes Π_{P_1} and Π_{P_2} overlap in a common subcomplex Π_F, and let U be a neighborhood of the interior of Π_F. Then, the inductive structure of the tailored P-pants ensures that,

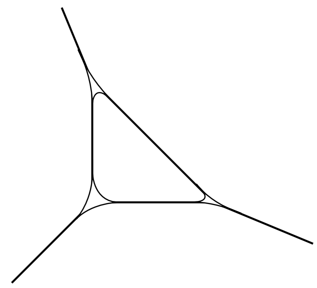

Figure 16. The amoeba of the localization of the hypersurface $xy+1/x+1/y=0$.

above U, the pants $\widetilde{\mathcal{P}}_{P_1}$ and $\widetilde{\mathcal{P}}_{P_2}$ agree: both are equal to the tailored leg $\widetilde{\mathcal{P}}_F$. Thus, we may take the union of all these pants to define F_Σ.

The isotopy can be glued similarly.

LEMMA 6.6. ([M, §6.6], [Ab1, Propositions 4.2 and 4.9]) *There is a Hamiltonian isotopy of symplectic hypersurfaces $F_W \to F_\Sigma$ such that for each face F in the tropical curve $\Pi_\Sigma \subset M_\mathbb{R}$, corresponding to a polytope P in the triangulation $\mathcal{T}$, there is a neighborhood $U_P \subset M_\mathbb{R}$ such that $\operatorname{Log}^{-1}(U_v)$ is equal to the intersection $\widetilde{\mathcal{P}}^\ell_P$ with a large ball in $\mathbb{T}^\vee_\mathbb{C}$.*

Remark 6.7. The symplectic isotopy from [Ab1] is defined as follows: for $t \geqslant 0$ and $0 \leqslant s \leqslant 1$, write $H^{t,s} = \{f^{t,s} = 0\}$,

$$f^{t,s} := \sum_m t^{-\alpha(m)}(1 - s\phi_m(\operatorname{Log}(z)))z^m, \tag{6.1}$$

where the sum is taken over the vertices of the triangulation $\mathcal{T}$ of $\Delta^\vee$, and $\phi_m \in C^\infty(\mathbb{R}^n)$ is a certain function which is 1 in a neighborhood of the component of $M_\mathbb{R} \backslash \Pi_\Sigma$ corresponding to m, and 0 away from that region; as in §6.1.1, taking the tropicalization parameter t large ensures that $\operatorname{Log}(H^{t,s})$ contains $\log(t) \cdot \Pi_\Sigma$ as a spine. Taking the "tailoring parameter" s from 0 to 1 deforms the hypersurface $\{\sum_m t^{-\alpha(m)} z^m\}$ by forcing that, on each region of the amoeba t, any term which does not dominate the behavior of $f^{t,0}$ in that region (as described in the proof of Proposition 6.3) does not contribute at all.

Remark 6.8. As in our definition of the standard pants, our convention in this paper will differ from that in equation (6.1) by our choice to take the sign of the constant coefficient of $f^{t,s}$ to be negative rather than positive. This ensures that the real positive points of $H^{t,s}$ lie over the boundary of the central component of the amoeba complement.

6.2. The skeleton of F_Σ

As in §5.3, by choosing an inner-product on M, we obtain an isomorphism

$$\mathbb{T}^\vee_\mathbb{C} = T\mathbb{T}^\vee \cong T^*\mathbb{T}^\vee,$$

and we restrict the symplectic form ω, and its primitive λ from this space to F_Σ. We will use the pants decomposition of F_Σ to (observe that it is a Liouville manifold and) compute its skeleton, which we denote by Λ_Σ.

However, in order to avoid performing any calculations beyond those described so far, we must adopt a certain technical hypothesis on the fan Σ. This hypothesis was later removed in [Z]; see Remark 6.16 for some discussion.

Definition 6.9. A polytope $P \subset M_\mathbb{R}$ is called *perfectly centered* if for each non-empty face $F \subset P$, the normal cone of F (transported to $M^\vee_\mathbb{R}$ by the inner product $M^\vee_\mathbb{R} \cong M_\mathbb{R}$) has non-empty intersection with the relative interior of F.

As in the proof of Lemma 2.3, we write $\alpha\colon \Delta^\vee \to \mathbb{R}$ for a function inducing the regular triangulation of $\Delta^\vee$ defined by Σ. The complex Π_Σ depends on our choice of α.

Definition 6.10. We will say that a fan Σ is PC if there exists some α as above for which the polytope Π^0_Σ is perfectly centered.

Assume now that the fan Σ is PC.

Remark 6.11. So far, no fan is known to us not to be PC; nor, however, do we know any compelling reason why all fans should be PC.

We will denote the amoeba of F_Σ by

$$\tilde{\mathcal{A}}_\Sigma := \mathrm{Log}(F_\Sigma).$$

Recall that we write Π^0_Σ for the component of $M_\mathbb{R} \setminus \Pi_\Sigma$ dual to the unique zero-dimensional cone in Σ and $\partial^0 \Pi_\Sigma$ for its boundary. Write $\partial^0 \tilde{\mathcal{A}}_\Sigma$ for the corresponding boundary component of the amoeba.

Recall that we write

$$-\mathbb{L}_\Sigma = \bigcup_{0 \neq \sigma \in \Sigma} \sigma^\perp \times \sigma$$

for the (negative) FLTZ skeleton.

THEOREM 6.12. *The skeleton Λ_Σ of F_Σ can be written as the intersection*

$$\Lambda_\Sigma = (\mathbb{T}^\vee \times \partial^0 \tilde{\mathcal{A}}_\Sigma) \cap (-\mathbb{L}_\Sigma).$$

Proof. From our hypothesis that the fan Σ is PC, we may assume that the polytope Π^0_Σ is perfectly centered, so that each non-zero cone σ in Σ intersects its dual face in Π^0_Σ, as in Figure 17. This allows us to define an open cover of F_Σ as follows: for each top-dimensional cone σ in Σ, let V_σ be a neighborhood of the cone σ, thought of as in $M_\mathbb{R}$.

Let $U_\sigma = \mathrm{Log}^{-1}(V_\sigma) \cap F_\Sigma$ be the lift of V_σ to an open subset of F_Σ. Then, U_σ is an open subset in a pants $\widetilde{\mathcal{P}}^\ell_\sigma$. By construction, the image of U_σ in $\widetilde{\mathcal{P}}^\ell_\sigma$ contains the whole skeleton Λ_σ of $\widetilde{\mathcal{P}}^\ell_\sigma$. On the other hand, every zero of $\lambda|_{F_\Sigma}$ is contained in some U_τ, as is its stable manifold. We conclude the skeleton Λ_Σ is equal to the union of the skeleta Λ_τ. $\square$

Note $\mathbb{T}^\vee \times \partial^0 \tilde{\mathcal{A}}_\Sigma$ is transverse to the Liouville flow on $T^*\mathbb{T}^\vee$, hence contact. In addition, we have the following.

LEMMA 6.13. *In a neighborhood of the skeleton Λ_Σ, the hypersurface F_Σ is nowhere tangent to the ambient Liouville vector field of the Weinstein manifold $T^*\mathbb{T}^\vee \cong \mathbb{T}^\vee_\mathbb{C}$.*

Proof. The pants cover of F_Σ allows us to reduce to the case where $F_\Sigma = \widetilde{\mathcal{P}}^\ell_P$ is a P-pants. Now we can proceed by the induction used in the proof of Lemma 5.17.

In the base case, $P = \mathrm{Conv}(0, v)$ is 1-dimensional, and $\widetilde{\mathcal{P}}^\ell_P = \{z_1^{v_1} \dots z_n^{v_n} = e^\ell\}$ is a copy of $(\mathbb{C}^\times)^{n-1}$ projecting by Log to its tropical hypersurface Π_Σ. The Liouville vector field on $\mathbb{T}^\vee_\mathbb{C} = (\mathbb{C}^\times)^n$, under the Log projection, points directly outward from Π_Σ.

Now, suppose that $P = \mathrm{Conv}(0, v_1, \dots, v_n)$ is an n-simplex. We know the result on the legs of $\widetilde{\mathcal{P}}^\ell_P$ by induction, so we only need to prove it in a neighborhood of the big simplex S_+ in the skeleton, which is the preimage under the cover $\widetilde{\mathcal{P}}^\ell_P \to \widetilde{\mathcal{P}}^\ell_{n-1}$ of the positive real points of $\widetilde{\mathcal{P}}^\ell_{n-1}$. But the Liouville vector field on $\mathbb{T}^\vee_\mathbb{C} = (\mathbb{C}^\times)^n$ in coordinates $z_j = e^{\xi_j + i\theta_j}$ is $\sum_j \xi_j \partial_{\xi_j}$, so if $\widetilde{\mathcal{P}}^\ell_P$ had any tangent vectors along S_+ in the direction of the Liouville flow, this would imply that $\widetilde{\mathcal{P}}^\ell_{n-1}$ had positive real points which do not project to the boundary of the amoeba, which is false. $\square$

COROLLARY 6.14. *There is a Liouville domain $D \subset \mathbb{T}^\vee$ completing to $\mathbb{T}^\vee$ and a Liouville domain $F \subset F_\Sigma$ completing to F_Σ, such that $F \subset \partial D$ and the* FLTZ *Lagrangian $-\mathbb{L}_\Sigma$ is a relative skeleton for the pair (D, F).*

Proof. The point is that we may deform $C := \mathbb{T}^\vee \times \partial^0 \tilde{\mathcal{A}}_\Sigma$ transversely to the Liouville flow in such a way as to cause it to contain some neighborhood $U_\Sigma \subset F_\Sigma$ of Λ_Σ.

Indeed, the Liouville flow gives an identification $T^*\mathbb{T}^\vee \setminus \mathbb{T}^\vee \cong C \times \mathbb{R}$, with C included as $C \times \{0\}$. By Lemma 6.13, for some closed manifold $V_\Sigma \subset F_\Sigma$ neighborhood of Λ_Σ, the corresponding projection $V_\Sigma \to C$ is an embedding. Its image is some codimension-1 smooth hypersurface (with boundary) of C, over which we may write V_Σ as the graph of a smooth function. Extend this function arbitrarily to all of C. The graph of the result will be the boundary of our desired D.

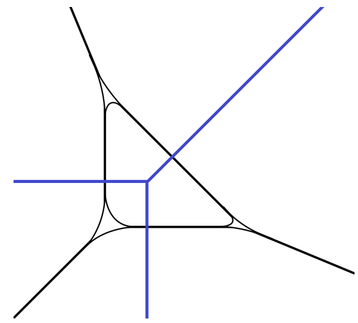

Figure 17. The fan Σ for $\mathbb{P}^2$ superimposed on the amoeba $\mathcal{A}_\Sigma$.

We thank John Pardon for this method of constructing Liouville pairs. $\square$

Example 6.15. Let $\Delta^\vee$ be the polytope with vertices $(1,1)$, $(0,-1)$, and $(-1,0)$, as in Figures 15 and 16. In Figure 17, the fan Σ is drawn superimposed on the amoeba $\mathcal{A}_\Sigma$. A neighborhood of each top-dimensional cone in Σ is a pair of pants which contributes to $\mathbb{L}$ a pair of circles attached by an interval. The circles live over the points where the rays of Σ intersect Π, and the intervals lie over the boundary of the bounded region in the center of the amoeba.

Remark 6.16. Here we have worked under the hypothesis that our fan Σ is PC, i.e., that we can arrange that the polytope $\Pi^0_\Sigma \subset M^\vee_\mathbb{R}$ is perfectly centered. Let us explain how Zhou [Z] lifts this hypothesis. Recall the property that a polytope is perfectly centered depended on a choice of inner product on $M_\mathbb{R}$ and corresponding isomorphism $M_\mathbb{R} \cong M^\vee_\mathbb{R}$. Zhou [Z] considers more generally an arbitrary convex homogenous degree-2 function $\varphi\colon M_\mathbb{R} \to \mathbb{R}$ and corresponding Legendre transform $\Phi_\varphi\colon M_\mathbb{R} \cong M^\vee_\mathbb{R}$. The point is that there always exists φ such that the analogous property of perfectly centered holds with respect to Φ_φ. Then (2.1) is replaced by

$$\mathbb{T}^\vee_\mathbb{C} = T\mathbb{T}^\vee = M_\mathbb{R} \times \mathbb{T}^\vee \xrightarrow{\Phi_\varphi} M^\vee_\mathbb{R} \times \mathbb{T}^\vee = T^*\mathbb{T}^\vee, \tag{6.2}$$

and one always uses the canonical exact symplectic structure on $T^*\mathbb{T}^\vee$. The main result stated in [Z] is the analogue of Theorem 6.12: namely, that the skeleton of $\Phi_\varphi(F_\Sigma)$ with respect to the restriction of the canonical 1-form on $T^*\mathbb{T}^\vee$ is

$$(\mathbb{T}^\vee \times L_\varphi(\partial^0 \tilde{\mathcal{A}}_\Sigma)) \cap (-\mathbb{L}_\Sigma).$$

As noted in [Z, Proposition 2.7], the integral curves of the Liouville vector field on $\mathbb{T}^\vee_\mathbb{C}$ do not depend on φ; since also the Legendre transform preserves convexity, $\Phi_\varphi(\partial^0 \tilde{\mathcal{A}}_\Sigma)$ is a convex neighborhood of the origin, and the proofs of Lemma 6.13 and Corollary 6.14 go through. Finally, note that when interpreting Proposition 2.2 in this more general setup, one should use the symplectic structures on $\mathbb{T}^\vee_\mathbb{C}$ transported from $T^*\mathbb{T}^\vee$ via (6.2) rather than (2.1).

7. Microlocalizing Bondal's correspondence

Recall that we denote by $\mathbb{T}$ a torus with respective character and cocharacter lattices M and $M^\vee$. Fix a (stacky) fan $\Sigma\subset M^\vee_\mathbb{R}$ and the corresponding toric partial compactification $\mathbb{T}_\mathbb{C}\subset\mathbf{T}_\Sigma$.

Bondal [B] described a fully faithful embedding of the category of coherent sheaves on $\mathbf{T}_\Sigma$ into the category of constructible sheaves on the real torus $\mathbb{T}^\vee_\mathbb{R}:=M\otimes\mathbb{R}/\mathbb{Z}$. This was developed further in [FLTZ1], [FLTZ4], [T]; in particular, the constructible sheaves in question were observed to have microsupport contained in $\mathbb{L}_\Sigma$ and conjectured to generate the category of such sheaves. This conjecture was established in [Ku].

We use this equivalence to prove a similarly-flavored equivalence "at infinity", i.e., an equivalence between the category of coherent sheaves on the toric boundary and the category of wrapped microlocal sheaves away from the zero section.

Categories and conventions. We work with dg categories over a fixed ground ring k. This theory can be set up either directly [Ke1], [Ke2], [D], or by specializing the theory of stable $(\infty,1)$-categories of [L1], [L2] as in [GR, I.1.10].

The microlocal sheaf theory of [KS] was originally developed in the setting of the bounded derived category. It is essential for our work here to work with the dg category of unbounded complexes. It is well known to experts that it is straightforward to set up the sheaf theory in this setting (see e.g. [N1, §2.2] or [GPS3, §4.1]) and that, with the use of [Sp], [RS] to deal with some issues around unbounded complexes, all constructions of [KS] may be translated to this setting.

For a manifold Q, we write $\mathrm{Sh}(Q)$ for the unbounded dg derived category of sheaves of k-modules on Q. We impose no restrictions on the stalks; i.e., we write Sh for what in [N2] is called $\mathrm{Sh}^\diamond$ (and similarly for the later $\mu\mathrm{sh}$).

For a conical subset $Z\subset T^*Q$, we write $\mathrm{Sh}_Z(Q)$ for the full subcategory of $\mathrm{Sh}(Q)$ consisting of those sheaves with microsupport in Z. When Z is subanalytic Lagrangian, then this subcategory is compactly generated, and we write $\mathrm{Sh}_Z(Q)^c$ for the subcategory of compact objects. This subcategory is generally larger than the category of sheaves with perfect stalks in $\mathrm{Sh}_Z(Q)$; for instance, when $Z=\varnothing$ it contains the tautological (derived) local system with fiber $C_*(\Omega Q)$. The idea to use compact objects in the unbounded category to model the *wrapped* Fukaya category stopped at Z is due to Nadler [N2]; that it works is now a theorem [GPS3]. The reader is referred to these articles for further discussions of this category.

For X an algebraic variety (or stack), we write
$myQCoh(X)$ for the dg derived category of quasi-coherent sheaves on X in the sense of [GR]; as observed there, the bounded subcategory agrees with the usual usage of this term. It is useful to remember that perfect complexes (bounded complexes of finitely generated projectives) are precisely the compact objects in
$myQCoh(X)$, which can be recovered from $\mathrm{Perf}(X)$ by ind-completion. Similarly, we will write $\mathrm{IndCoh}(X)$ for the Ind-completion of the category $\mathrm{Coh}(X)$ of coherent sheaves on X ([GR]). We can recover the category $\mathrm{Coh}(X)$ by passing to compact objects.

To simplify notation, we write as if Σ is an ordinary (non-stacky) fan. To arrive at the corresponding statements in the stacky case, one need merely remember the data of a finite abelian group Γ_σ for each cone in σ, and correspondingly replace the sets $\{A(\sigma)\}_{\sigma\in\Sigma}$ and $\{B(\sigma)\}_{\sigma\in\Sigma}$ with sets $\{A(\sigma,\chi)\}_{\sigma\in\Sigma,\chi\in\Gamma_\sigma}$ and $\{B(\sigma,\chi)\}_{\sigma\in\Sigma,\chi\in\Gamma_\sigma}$, where the added χ denotes translation in $\mathbb{T}^\vee$ and twists by a character, respectively. See [FLTZ4, §5] for details.

7.1. Bondal's coherent-constructible correspondence

For a cone $\sigma\subset M^\vee$, we write $B(\sigma)$ for the structure sheaf on $\mathrm{Spec}(k[\sigma^\vee])$, or its pushforward to any toric variety whose fan contains the cone σ. On the other hand, we write $A(\sigma)$ for the constructible sheaf on $M\otimes\mathbb{R}/\mathbb{Z}$ obtained by taking the $!$-pushforward of the dualizing (constructible) sheaf on the interior of $\sigma^\vee$. One then makes the following.

Basic calculation. ([B], [FLTZ1], [T]) Let $\mathbf{T}_\Sigma$ be a toric variety with fan Σ, with dense torus $\mathbb{T}_\mathbb{C}$. Let $\sigma,\tau\in\Sigma$ be cones. Then, there are canonical isomorphisms

$$H^*\mathrm{Hom}(A(\sigma),A(\tau))\cong k[\tau^\vee]\cong H^*\mathrm{Hom}(B(\sigma),B(\tau)) \quad \text{if } \sigma\supset\tau,$$

and all other Homs between such objects vanish. This is moreover compatible with the evident composition structure.

We denote full dg subcategories generated by the $A(\sigma)$ and $B(\sigma)$ by

$$A_\Sigma := \{A(\sigma):\sigma\in\Sigma\}\subset \mathrm{Sh}(\mathbb{T}^\vee),$$
$$B_\Sigma := \{B(\sigma):\sigma\in\Sigma\}\subset \mathrm{QCoh}(\mathbf{T}_\Sigma).$$

While the calculation above might seem to imply only the equivalence

$$H^0(A_\Sigma)\cong H^0(B_\Sigma)$$

of *triangulated* categories, we recall the following useful fact.

LEMMA 7.1. *Let $\mathcal{C}_i$ be a collection of dg categories, each of which has all morphisms concentrated in cohomological degree zero. Then, any diagram valued in the $H^0(\mathcal{C}_i)$ lifts canonically to a homotopy coherent diagram in the corresponding $\mathcal{C}_i$.*

Proof. The hypothesis on $\mathcal{C}_i$ implies that the natural maps

$$H^0(\mathcal{C}) \longleftarrow \tau_{\leqslant 0}\mathcal{C} \longrightarrow \mathcal{C}$$

are quasi-isomorphisms. Thus any diagram among the $H^0(\mathcal{C}_i)$ can be lifted to a diagram among the $\mathcal{C}_i$ by composing with this pair of quasi-isomorphisms. □

As the category of quasi-coherent sheaves on a toric variety is generated by the structure sheaves of the affine toric charts, the restriction to the subcategory B_Σ is really no restriction: the morphism $\mathrm{QCoh}(\mathbf{T}_\Sigma) \to \text{Mod-}B_\Sigma$ is an isomorphism.

On the other side, the objects of A_Σ all satisfy the microsupport estimate

$$\mathrm{ss}(A(\tau)) \subset \bigcup_{\sigma \subset \tau} \sigma^\perp \times (-\sigma) \subset \mathbb{T}^\vee \times M_{\mathbb{R}}^\vee = T^*\mathbb{T}^\vee.$$

In particular, writing

$$\mathbb{L}_\Sigma := \bigcup_{\sigma \in \Sigma} \sigma^\perp \times (-\sigma),$$

we have that $A_\sigma \in \mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee)$ for all $\sigma \in \Sigma$. As conjectured by [FLTZ3] and [T], and proven by Kuwagaki [Ku], these objects generate this category.

THEOREM 7.2. ([Ku]) *When $\mathbf{T}_\Sigma$ is a smooth orbifold, the morphism*

$$\mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee) \longrightarrow \text{Mod-}A_\Sigma$$

is an isomorphism.

Remark 7.3. In fact what Kuwakagi proves is that for any, not necessarily smooth, $\mathbf{T}_\Sigma$ there is an isomorphism $\mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee) \cong \mathrm{IndCoh}(\mathbf{T}_\Sigma)$. The above statement follows because in the smooth case, IndCoh=QCoh, which as we mentioned above is generated by the B_σ. We use the above formulation rather than Kuwagaki's more general result, because we will later make calculations with the A_σ and B_σ directly, and we restrict ourselves to the smooth case to avoid e.g. worrying about how to lift the B_σ to IndCoh.

This causes no loss of generality, since it is anyway only in the smooth case that we have been able to identify $\partial\mathbb{L}_\Sigma$ as a relative skeleton.

7.2. Restriction is mirror to microlocalization

Let $\mathbf{T}$ be a toric variety, σ be a cone of the fan $\Sigma(\mathbf{T})$, and $i_\sigma\colon\overline{O(\sigma)}\to\mathbf{T}$ be the inclusion of the orbit closure corresponding to the cone σ. As the orbit closure is itself a toric variety, one can ask what functor of constructible sheaf categories corresponds under Bondal's correspondence to the pullback i_σ^*. We will see that the answer is a sort of microlocalization functor.

7.2.1. Restriction to orbit closures

Recall that the orbit closure $\overline{O(\sigma)}$ carries the structure of a toric variety, with associated cocharacter lattice $M^\vee/\mathbb{Z}\sigma$. For τ a cone containing σ, we write τ/σ for the image of τ in $M^\vee/\mathbb{Z}\sigma$. The map $\tau\to\tau/\sigma$ gives a bijection between cones containing σ and cones in the fan of $\Sigma(\overline{O(\sigma)})$. We will therefore write $\Sigma/\sigma:=\Sigma(\overline{O(\sigma)})$.

Let us recall that

$$\mathbf{T}_\tau=\coprod_{\tau\supset\eta}O(\eta)\quad\text{and}\quad\overline{O(\sigma)}=\coprod_{\eta\supset\sigma}O(\eta),$$

and therefore the intersection of the orbit closure $\overline{O(\sigma)}$ with the affine piece $\mathbf{T}_\tau$ decomposes as

$$\mathbf{T}_\tau\cap\overline{O(\sigma)}=\coprod_{\tau\supset\eta\supset\sigma}O(\eta)=\begin{cases}\overline{O(\sigma)}_{\tau/\sigma}, & \text{if } \tau\supset\sigma,\\ \varnothing, & \text{otherwise.}\end{cases}$$

For $\tau\supset\sigma$, there is a natural identification $(\tau/\sigma)^\vee\cong\tau^\vee\cap\sigma^\perp\subset\tau^\vee$. The corresponding map $k[(\tau/\sigma)^\vee]\hookrightarrow k[\tau^\vee]$ has a unique M-graded left-inverse $k[\tau^\vee]\to k[(\tau/\sigma)^\vee]$, which gives the affine inclusion $\overline{O(\sigma)}_{\tau/\sigma}\hookrightarrow\mathbf{T}_\tau$. We conclude the following result.

LEMMA 7.4. *We have canonical isomorphisms*

$$i_\sigma^*B(\tau)=\begin{cases}B(\tau/\sigma), & \textit{if } \tau\supset\sigma,\\ 0, & \textit{otherwise.}\end{cases}$$

The source or target of the induced map

$$i_\sigma^*\colon H^*\mathrm{Hom}(B(\tau'),B(\tau))\longrightarrow H^*\mathrm{Hom}(B(\tau'/\sigma),B(\tau/\sigma))$$

vanishes unless $\tau'\supset\tau\supset\sigma$, and in this case is canonically identified with the map

$$k[\tau^\vee]\longrightarrow k[(\tau/\sigma)^\vee].$$

7.2.2. Microlocalization

Our description of the mirror to the restriction functor i_σ^* will be given in terms of Sato's microlocalization. We now briefly review this notion; for details see [KS, Chapter 4].

Microlocalization is built from Verdier specialization, and the Fourier–Sato transform. The Verdier specialization along a submanifold $X\subset Y$ carries sheaves on Y to conic sheaves on T_XY, by pushing forward along a deformation to the normal cone. The Fourier–Sato transformation carries conic sheaves on a bundle to conic sheaves on its dual, by convolution with the kernel given by the constant sheaf on the locus $\{(x,x^*):x^*(x)\leqslant 0\}$. Sato's microlocalization is the composition of these, and carries sheaves on Y to conic sheaves on T_X^*Y; we denote it by μ_X.

As usual, write

$$\mathbb{L}_\Sigma=\bigcup_{\sigma\in\Sigma}\sigma^\perp\times(-\sigma)\subset T^*\mathbb{T}^\vee$$

for the [FLTZ3] skeleton mirror to $\mathbf{T}_\Sigma$.

For the orbit closure $\overline{O(\sigma)}$, we denote the corresponding torus

$$\mathbb{T}_\mathbb{C}(\sigma):=\mathbb{T}_\mathbb{C}/(\mathbb{Z}\sigma\otimes\mathbb{G}_m),$$

and the corresponding skeleton $\mathbb{L}_{\Sigma/\sigma}\subset T^*\mathbb{T}(\sigma)^\vee$. Note the canonical identification

$$\mathbb{T}(\sigma)^\vee\cong\sigma^\perp.$$

We compute the following microlocalization.

LEMMA 7.5. *Let* $\pi:\sigma^\perp\times(-\sigma^\circ)\to\sigma^\perp\cong\mathbb{T}(\sigma)^\vee$ *be the projection. Then the morphism*

$$\begin{aligned} m_\sigma:\mathrm{Sh}(\mathbb{T}^\vee)&\longrightarrow\mathrm{Sh}(\sigma^\perp),\\ F&\longmapsto\pi_*((\mu_{\sigma^\perp}F)|_{\sigma^\perp\times(-\sigma^\circ)}).\end{aligned}$$

respects FLTZ *skeleta, i.e. restricts to* $m_\sigma:\mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee)\to\mathrm{Sh}_{\mathbb{L}_{\Sigma/\sigma}}(\sigma^\perp)$. *Moreover, there are canonical isomorphisms*

$$m_\sigma A(\tau)=\begin{cases}A(\tau/\sigma), & \textit{if }\tau\supset\sigma,\\ 0, & \textit{otherwise.}\end{cases}$$

The source or target of the induced map

$$\mu_\sigma:H^*\mathrm{Hom}(A(\tau'),A(\tau))\longrightarrow H^*\mathrm{Hom}(A(\tau'/\sigma),A(\tau/\sigma))$$

vanishes unless $\tau'\supset\tau\supset\sigma$; *in this case the map is canonically identified with*

$$k[\tau^\vee]\longrightarrow k[(\tau/\sigma)^\vee].$$

Proof. As we will we see, the sheaves in question are constant along the fibers of π, which are contractible. Thus, the pushforward π_* does essentially nothing, and we subsequently omit it from the notation.

As $\mathbb{L}_\Sigma$ is the union of the microsupports of the $A(\sigma)$, our argument showing that $m_\sigma A(\sigma)=A(\sigma/\tau)$ will also show that

$$m_\sigma(\mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee)) \subset \mathrm{Sh}_{\mathbb{L}_{\Sigma/\sigma}}(\sigma^\perp).$$

One could also directly use the formula [KS, Theorem 6.4.1] showing that the microsupport of the microlocalization is the specialization of the microsupport to the normal bundle of the conormal bundle.

Suppose that $\sigma \not\subset \tau$. We want to show that $m_\sigma(A_\tau)=0$. In fact, we will show that $\mu_{\sigma^\perp} A(\tau)$ is disjoint from $\sigma^\perp \times(-\sigma^\circ)$, and in fact disjoint from $\mathbb{T}^\vee \times(-\sigma^\circ)$. To do so, recall that for any submanifold $Y \subset X$ and any sheaf F on X, one has the estimate

$$\mathrm{Supp}(\mu_Y F) \subset T^*_Y X \cap \mathrm{ss}(F).$$

As a consequence of [FLTZ3, Proposition 5.1], we have the following inclusion of subsets of $T^*\mathbb{T}^\vee = \mathbb{T}^\vee \times M^\vee_\mathbb{R}$:

$$\mathrm{ss}(A(\tau)) \subset \bigcup_{\eta \subset \tau} \eta^\perp \times(-\eta)$$

In particular, $\mathrm{ss}(A(\tau)) \subset \mathbb{T}^\vee \times(-\tau)$. Meanwhile, σ° is disjoint from τ, unless $\sigma \subset \tau$.

Now, consider $A(\tau)$ with $\sigma \subset \tau$. The specialization of $A(\tau)$ along $\sigma^\perp$ can be understood as follows. Choose a splitting $\mathbb{T}^\vee = \sigma^\perp \times \mathbb{T}'$, where $\mathbb{T}' = \mathrm{Hom}(\mathbb{Z}\sigma, \mathbb{R}/\mathbb{Z})$. Let $\mathbb{T}'_\varepsilon$ be an epsilon ball around the origin of $\mathbb{T}'$. Then the Verdier specialization along $\sigma^\perp$ can be visualized as first restricting to $\sigma^\perp \times \mathbb{T}'_\varepsilon$, and then rescaling the $\mathbb{T}'_\varepsilon$ factor to be very large, in the limit as $\varepsilon \to 0$. In this limit, the $\mathbb{T}'_\varepsilon$ factor can be identified with $\mathrm{Hom}(\mathbb{Z}\sigma, \mathbb{R})$.

Restricting to $\sigma^\perp \times \mathbb{T}'_\varepsilon$ breaks $A(\tau)$ into a direct sum of $\mathbb{N}^k$ pieces, where the $\mathbb{N}^k$ grading counts how many times the cone has wrapped around $(S^1)^k$. Let us call the result $A'(\tau)$.

First, we study the grading zero component, $A'(\tau)_0$. The rescaling limit carries $A'(\tau)_0$ to $(A'(\tau)_0)|_{\sigma^\perp} \boxtimes A_\varepsilon(\sigma)$, where $A_\varepsilon(\sigma)$ is the costandard sheaf on the dual cone to σ inside $\mathrm{Hom}(\mathbb{Z}\sigma, \mathbb{R})$. The Fourier transform (which happens only in the second factor) of $A_\varepsilon(\sigma)$ returns the standard sheaf on $-\sigma$, which restricts to the constant sheaf on $-\sigma^\circ$. On the other hand, $(A'(\tau)_0)|_{\sigma^\perp}$ is readily seen to be $A(\tau/\sigma)$.

For the remaining components, note that since each has already wrapped around at least once in some direction, they are invariant along the line spanned by some extremal ray of the dual cone to σ inside $\mathrm{Hom}(\mathbb{Z}\sigma, \mathbb{R}/\mathbb{Z})$. It follows that their Fourier transform is supported on the face of σ annihilated by that ray; hence, the restriction of such a component to $-\sigma^\circ$ is zero.

Finally, for the statement about Homs, let us recall from [T, Proposition 2.3] their description. Consider the universal cover $\pi\colon M_{\mathbb{R}}\to M_{\mathbb{R}/\mathbb{Z}}=\mathbb{T}^\vee$; as we have mentioned $A(\sigma):=\pi_! k_{(\sigma^\vee)^\circ}$. One calculates

$$\begin{aligned}\operatorname{Hom}(A(\tau'),A(\tau)) &= \operatorname{Hom}(\pi_! k_{(\tau'^\vee)^\circ},\pi_! k_{(\tau^\vee)^\circ})\\ &= \operatorname{Hom}(k_{(\tau'^\vee)^\circ},\pi^! \pi_! k_{(\tau^\vee)^\circ})\\ &= \bigoplus_{m\in M}\operatorname{Hom}(k_{(\tau'^\vee)^\circ},m+k_{(\tau^\vee)^\circ}).\end{aligned}$$

Finally, it is easy to see that

$$\operatorname{Hom}(k_{(\tau'^\vee)^\circ},m+k_{(\tau^\vee)^\circ})=\begin{cases}k, & \text{if } \tau'\supset\tau \text{ and } m\in\tau^\vee,\\ 0, & \text{otherwise.}\end{cases}$$

This is why

$$\operatorname{Hom}(A(\tau'),A(\tau))\cong k[\tau^\vee].$$

Now the point is just that the discussion above is compatible with this calculation of Homs by passage to the universal cover. (E.g. the $\mathbb{N}^k$-grading discussed earlier is just the appropriate part of the M grading appearing in $\bigoplus_{m\in M}$ above.) □

In words: Bondal's correspondence intertwines the pullback i_σ^* with the microlocalization m_σ, at least as far as A_Σ and B_Σ are concerned. By Theorem 7.2 (and noting again Lemma 7.1), this can be extended to the larger categories.

Remark 7.6. In [FLTZ1], [T] a different functoriality statement is established, which however does not apply to the case of an inclusion of a toric divisor. Their result concerns morphisms which, on the A-side, can be described in terms of just sheaves on the base manifold, rather than in terms of microlocalization.

7.3. Microlocal sheaves

7.3.1. The Kashiwara–Schapira stack

Let Q be a manifold. Using the tools of [KS], one can construct a sheaf of categories on T^*Q, the *Kashiwara–Schapiraù stack*, whose global sections recover the usual category of sheaves on Q. To define it, one begins with the presheaf of categories $\mu\,\mathrm{sh}^{\mathrm{pre}}$, whose sections in a small ball U are the quotient category

$$\mu\,\mathrm{sh}^{\mathrm{pre}}(U)=\mathrm{Sh}(Q)/\,\mathrm{Sh}_{T^*Q\setminus U}(Q).$$

For a conical subset $\mathbb{L}\subset T^*Q$, there is a presheaf of full subcategories $\mu\,\mathrm{sh}_{\mathbb{L}}^{\mathrm{pre}}$ on objects whose microsupport near $\mathbb{L}$ is contained in $\mathbb{L}$.

To be precise, we regard $\mu\,\mathrm{sh}^{\mathrm{pre}}$ as a presheaf valued in the ∞-category of all ∞-categories. We denote the corresponding sheaf by $\mu\,\mathrm{sh}$. Because the ∞-category of ∞-categories embeds in the ∞-category of spaces, from the properties of which one can deduce that (1) sheafification preserves stalks and (2) morphisms of sheaves of categories can be checked to be isomorphisms on stalks. Because $\mu\,\mathrm{sh}^{\mathrm{pre}}$ is in fact a presheaf of dg categories, $\mu\,\mathrm{sh}$ carries canonically the structure of a sheaf of dg categories.

Although [KS] never discusses the sheafification of $\mu\,\mathrm{sh}^{\mathrm{pre}}$, the stalks of a bounded version are studied in detail in [KS, Chapter 6] (the $D^b(X;p)$). Morevoer, what are in fact the Hom sheaves of $\mu\,\mathrm{sh}$ are also discussed (under the name $\mu\,\mathrm{hom}$). The sheaf $\mu\,\mathrm{sh}$ is discussed in some detail in [Gu1], [Gu2], [N2], [NS].([10])

The category $\mathrm{Sh}(Q)$ is complete and cocomplete; additionally it is 'presentable'. Let us fix some notation to discuss such properties. We write dg to mean the category whose objects are small stable (aka pre-triangulated) dg categories, and whose morphisms are exact functors. We write DG for the category whose objects are presentable stable dg categories, and whose morphisms are exact functors. There are various not full subcategories of DG characterized by what sort of adjoints the morphisms are. We indicate by ${}^*\mathrm{DG}$ the category in which all morphisms are left adjoints; by ${}^{**}\mathrm{DG}$ the category in which all morphisms are left adjoints of left adjoints; and so on.

Taking adjoints gives equivalences of categories switching the restrictions on adjoints; for instance, ${}^*\mathrm{DG}\cong(\mathrm{DG}^*)^{\mathrm{op}}$, and so on. This turns out to be very useful: as described in [Ga], we can turn colimits into limits. Taking ind-completion and then adjoints gives an embedding $\mathrm{dg}\hookrightarrow{}^{**}\mathrm{DG}\cong({}^*\mathrm{DG}^*)^{\mathrm{op}}$; with the image being the compactly generated categories. Thus a colimit in dg becomes a limit in ${}^*\mathrm{DG}^*$, which we can compute in DG^*. Taking adjoints again and passing to compact objects gives the originally desired colimit.

It is easy to see that $\mu\,\mathrm{sh}^{\mathrm{pre}}$ is valued in ${}^*\mathrm{DG}^*$, but because we sheafified in the ∞-category of ∞-categories (rather than in say DG^*, ${}^*\mathrm{DG}$, ${}^*\mathrm{DG}^*$), it does not follow formally that $\mu\,\mathrm{sh}$ has the corresponding properties; in fact we do not know whether this is the case or not. However, $\mathbb{L}$ is subanalytic Lagrangian, and so the sections of $\mu\,\mathrm{sh}_{\mathbb{L}}^{\mathrm{pre}}$ stabilize in a sequence of contractible open neighborhoods around any given point $p\in\mathbb{L}$.

([10]) The discussion in [Gu2, Chapter 10] gives many details, including an explanation of how the results of [KS] may be translated into the assertions that $D^b(X;p)$ is the stalk of $\mu\,\mathrm{sh}$ and that $\mu\,\mathrm{hom}$ is the sheaf of homs. While the official language of [Gu2] is that of triangulated categories rather than DG categories — an unfortunate choice insofar as triangulated structures do not glue well whereas DG categories do — the constructions of [Gu2] are all compatible with DG enhancement, and the proofs go through unchanged once one has set up the basic sheaf theory in the DG setting. These categorical aspects are discussed explicitly in [N2], [NS].

From this, we can conclude that $\mu\,\mathrm{sh}_{\mathbb{L}}$ and its stalks indeed take values in ${}^*DG^*$.

The sheaf $\mu\,\mathrm{sh}$ can also be regarded as a cosheaf via the equivalence

$$({}^*\mathrm{DG}^*)^{\mathrm{op}} \cong {}^{**}\mathrm{DG}.$$

Again because $\mathbb{L}$ is subanalytic Lagrangian, all sections are compactly generated; we may pass to compact objects in the cosheaf to obtain a cosheaf valued in dg.

7.3.2. Microlocal restriction

We now give some lemmas about how to compute the restriction of $\mu\,\mathrm{sh}$. Let X be a manifold and $Y \subset X$ a submanifold. We write $T^*_Y X \subset T^*X$ for the conormal bundle to Y. Recall from [KS, §3.7 and §4.3] the Sato microlocalization functor:

$$\mu_Y\colon \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(T^*_Y X)^{\mathbb{R}^+}$$

Here, the "$\mathbb{R}^+$" indicates the sheaves are conic, i.e., constant along the cotangent directions.

Note that, locally near $T^*_Y X$, the ambient cotangent bundle T^*X is also the cotangent bundle of $T^*_Y X$. Thus, it is natural to expect an expression for $\mathrm{SS}(\mu_Y F) \subset T^*(T^*_Y X)$ in terms of $\mathrm{SS}(F) \subset T^*X$. These cannot be equal in general, since $\mathrm{SS}(\mu_Y F)$ must be conic in both the cotangent and in the cotangent-to-cotangent directions in $T^*(T^*_Y X)$, whereas $\mathrm{SS}(F) \subset T^*X$ will only be conic in the T^*X cotangent directions. This, however, is the "only" difference.

THEOREM 7.7. ([KS, Theorem 6.4.1]) *$\mathrm{SS}(\mu_Y F) \subset T^*(T^*_Y X)$ is obtained by specializing $\mathrm{SS}(F)$ to the normal cone to $T^*_Y X$.*

For the notion of specialization to the normal cone, see [KS, §4.1]. We will now draw some consquences of this fundamental result.

LEMMA 7.8. *The Sato microlocalization $\mathrm{Sh}(X) \to \mathrm{Sh}^{\mathbb{R}^+}(T^*_Y X)$ factors through the (global sections of) a morphism of sheaves of categories*

$$\mu\,\mathrm{sh}\,|_{T^*_Y X} \longrightarrow \mathrm{Sh}^{\mathbb{R}^+}. \tag{7.1}$$

Here, the right-hand side is the (sheaf of categories of) conic sheaves on $T^*_Y X$.

Moreover, for $\Lambda \subset T^*X$ a conic closed subset, this map restricts to

$$\mu\,\mathrm{sh}_\Lambda|_{T^*_Y X} \longrightarrow \mathrm{Sh}^{\mathbb{R}^+}_{C_{T^*_Y X}(\Lambda)}, \tag{7.2}$$

where $C_{T^*_Y X}(\Lambda)$ is the specialization of Λ to the normal bundle of T^*Y_X, with that normal bundle then identified with $T^*(T^*_Y X)$.

Proof. Because the target is a sheaf of categories, it is enough to construct a map from $\mu\,\mathrm{sh}^{\mathrm{pre}}$. To do this, we should show that for any $\Omega\subset T^*X$, the microlocalization μ_Y induces a functor

$$\mathrm{Sh}(X)/\,\mathrm{Sh}_{T^*X\setminus\Omega}(X)\longrightarrow \mathrm{Sh}(\Omega\cap T^*_YX).$$

In other words, we should show that if F has no microsupport in Ω, then $\mu_Y(F)$ has no support in $\Omega\cap T^*_YX$. This follows from Theorem 7.7, which gives the microsupport of $\mu_Y(F)$, so in particular the support. The final statement characterizing the behavior of microsupports is a direct translation of Theorem 7.7. □

Remark 7.9. Under the identification of $\mu\,\mathrm{hom}$ with the Hom in $\mu\,\mathrm{sh}$, the above functor acts on Hom sheaves as the natural map

$$\mu\,\mathrm{hom}(F,G)|_{T^*_YX}\longrightarrow \mathcal{H}\mathrm{om}(\mu_YF,\mu_YG). \tag{7.3}$$

We are interested conditions on Λ which ensure that (7.2) is an equivalence, and in particular that the map (7.3) is an isomorphism. It is possible to give counterexamples showing some condition is necessary for (7.3) to be an isomorphism [GS] (so in particular, the map (7.1) is not an isomorphism). One known sufficient condition is $\Lambda=T^*_YX$ (see e.g. [Gu2, Lemma 10.2.2 and Proposition 10.2.4], and [KS, Proposition 6.6.1 and Lemma 7.5.2], and more generally around [KS, §7.5], for some precursors). This fact is fundamental to the analysis of $\mu\,\mathrm{sh}$ along a smooth conic Lagrangian.

Here we note that, more generally, it suffices if Λ is in an appropriate sense *already* conic along T^*_YX, so the specialization to the normal cone is an innocent operation.

Lemma 7.10. *Consider $\Lambda\subset T^*X$. Suppose that, in the neighborhood of a point $\xi\in T^*_YX$, the set Λ is contained in the union of conormals* (*in T^*X*) *to strata in a subanalytic* (*or otherwise o-minimal*) *stratification of Y. Then, the map* (7.2)

$$\mu\,\mathrm{sh}_\Lambda|_{T^*_YX}\longrightarrow \mathrm{Sh}^{\mathbb{R}^+}_{C_{T^*_YX}(\Lambda)}$$

is an isomorphism at ξ.

Proof. For morphisms of sheaves valued in the category of categories, whether a map is an isomorphism may be checked on stalks. The stalks of $\mu\,\mathrm{sh}$ are (unbounded versions of) the categories $D^b(X;p)$ studied in [KS, Chapters 6 and 7]; as noted above the results therein continue to hold in the unbounded case.

We may replace X by a tubular neighborhood of Y, and fix a metric so as to identify this neighborhood with the normal bundle to Y. That is, we replace $X=T_YX$. Note that on sheaves conic with respect to the scaling on T_YX, the given functor is just the Fourier–Sato transform, which is an equivalence.

Now the point is just that the hypothesis of the theorem ensures that the sheaves in question are microlocally conic, i.e. isomorphic to a conic on $T_Y X$ sheaf in a neighborhood of $T^*_Y X$. That is because the conormals to strata in Y remain constant under the deformation to the normal cone to $T^*_Y X$. Thus the Fourier transform is (microlocally) an isomorphism on these sheaves. $\square$

Remark 7.11. Evidently, the criterion of Lemma 7.10 holds at every point away from the zero section in the FLTZ skeleton, when Y is taken any subtorus. More generally, it is automatic away from the zero section for sheaves constructible with respect to a piecewise linear stratification, where Y is amongst the strata.

COROLLARY 7.12. *Let $\Sigma \subset M^\vee_{\mathbb{R}}$ be a fan and $\mathbb{L}_\Sigma$ be the corresponding skeleton inside $T^*\mathbb{T}^\vee_{\mathbb{R}}$.*

Let $\sigma \in \Sigma$ be a cone, and let $\pi: \sigma^\perp \times (-\sigma^\circ) \to \sigma^\perp$ be the projection. Then, the functor (from Lemma 7.5)

$$\begin{aligned} m_\sigma: \mathrm{Sh}_{\mathbb{L}_\Sigma}(\mathbb{T}^\vee) &\longrightarrow \mathrm{Sh}_{\mathbb{L}_{\Sigma/\sigma}}(\sigma^\perp) \\ F &\longmapsto \pi_*((\mu_{\sigma^\perp} F)|_{\sigma^\perp \times (-\sigma^\circ)}) \end{aligned}$$

factors canonically through an isomorphism

$$\mu\,\mathrm{sh}_{\mathbb{L}_\Sigma}(\sigma^\perp \times (-\sigma^\circ)) \longrightarrow \mathrm{Sh}_{\mathbb{L}_{\Sigma/\sigma}}(\sigma^\perp).$$

Proof. As we have remarked, sheaves constructible with respect to a piecewise linear stratification necessarily satisfy the hypothesis of Lemma 7.10. $\square$

7.4. At infinity

We are now ready to pass to the boundary on both sides of Bondal's correspondence. On the B-side, this means passing from the toric variety $\mathbf{T}_\Sigma$ to the union of its toric boundary divisors, and on the A-side, this means moving from the relative skeleton $\mathbb{L}_\Sigma$ of the LG model $W: T^*\mathbb{T}^\vee \to \mathbb{C}$ to the complement of the zero section: $\mathbb{L}^\circ_\Sigma := \mathbb{L}_\Sigma \setminus \mathbb{T}^\vee$.

THEOREM 7.13. *For $\mathbf{T}_\Sigma$ smooth, there is an equivalence of categories*

$$\mathrm{Coh}(\partial \mathbf{T}_\Sigma) \cong \mu\,\mathrm{sh}(\partial \mathbb{L}_\Sigma)^c.$$

Proof. To avoid worrying about whether various colimits exist, we will work with the cocomplete categories IndCoh and μsh, and we will return to the above statement at the end by passing to compact objects. This is essentially only a matter of notation.

In [GR, Vol. II, §8.A, Theorem A.1.2], it is stated that if an affine scheme is a pushout of affine schemes along closed embeddings, then its category of ind-coherent sheaves is computed by the corresponding pushout of categories of ind-coherent sheaves.(11) By descent, the same holds for stacks in general.

Per Lemma 3.5 the toric boundary $\partial\mathbf{T}_\Sigma$ can indeed be presented as a sequence of pushouts, of the orbit closures $\overline{O(\sigma)}$, along closed embeddings, so that we may deduce an equivalence of categories

$$\mathrm{IndCoh}(\partial\mathbf{T}_\Sigma)\cong \varinjlim_{\sigma\in\Sigma\setminus\{0\}} \mathrm{IndCoh}(\overline{O(\sigma)}).$$

By Zariski (or étale in the stack case) descent we may trade

$$\mathrm{IndCoh}(\overline{O(\sigma)})\cong \text{Mod-}B_{\widehat{\Sigma}(\overline{O(\sigma)})}.$$

(For a detailed explanation of this isomorphism, see [Ku].)

The coherent-constructible correspondence of [B], [FLTZ3], [T] and Kuwagaki's theorem [Ku], respectively, give the following two equivalences:

$$\varinjlim_{\sigma\in\Sigma\setminus\{0\}} \text{Mod-}B_{\widehat{\Sigma}(\overline{O(\sigma)})}\cong \varinjlim_{\sigma\in\Sigma\setminus\{0\}} \text{Mod-}A_{\widehat{\Sigma}(\overline{O(\sigma)})}\cong \varinjlim_{\sigma\in\Sigma\setminus\{0\}} \mathrm{Sh}_{\mathbb{L}_{\widehat{\Sigma}(\overline{O(\sigma)})}}(\mathbb{T}(\sigma)^\vee).$$

Finally, by taking adjoints to the restriction morphisms we analyzed in Lemma 7.5 and Corollary 7.12, we obtain the following identification:

$$\varinjlim_{\sigma\in\Sigma\setminus\{0\}} \mathrm{Sh}_{\mathbb{L}_{\widehat{\Sigma}(\overline{O(\sigma)})}}(\mathbb{T}(\sigma)^\vee)=\varinjlim_{\sigma\in\Sigma\setminus\{0\}} \mu\,\mathrm{sh}_{\mathbb{L}_\Sigma}(\sigma^\perp\times(-\sigma^\circ)).$$

On the right, the maps are co-restriction functors of wrapped microlocal sheaves, and this colimit is just the one associated with a cover of $\partial\mathbb{L}_\Sigma$. This completes the proof. □

Remark 7.14. The result holds without the smoothness hypothesis, as e.g. can be seen by taking some toric resolution, applying Theorem 7.13, and then matching the semiorthogonal decomposition of the category of the resolution on the B-side with stop removal on the A-side. We content ourselves with the smooth case here because we anyway have only in this case identified $\partial\mathbb{L}_\Sigma$ as a Weinstein skeleton.

(11) Note that a colimit of underived schemes or stacks remains a colimit of the corresponding items viewed as derived objects, since the inclusion of underived geometry into derived geometry is left adjoint to truncation of derived structure.

8. A glimpse in the mirror of birational toric geometry

Since the works [BO], [Ka], it has been understood that birational features of algebraic geometry often have natural interpretations in the derived category of coherent sheaves. Mirror symmetry provides an illuminating perspective on these derived equivalences, which in algebraic geometry seem to be among a discrete set of objects. Remarkably, on the mirror this discretization becomes unnatural, and one can continuously interpolate between the mirrors of derived equivalent varieties. Many other features of birational geometry (e.g., semi-orthogonal decompositions associated with blowups) also have beautiful new geometric interpretations in terms of mirror geometry. For discussions in the context of toric varieties, see [FLTZ2], [CKK], [BDF+].

Here is another result in this direction.

Corollary 8.1. *Let* $W\colon(\mathbb{C}^\times)^n\to\mathbb{C}$ *be a Laurent polynomial with Newton polytope* Δ *and* Σ_1,Σ_2 *a pair of fans obtained as star-shaped triangulations of* Δ*. Then, there is a derived equivalence* $\mathrm{Coh}(\mathbf{T}_{\Sigma_1})\cong\mathrm{Coh}(\mathbf{T}_{\Sigma_2})$.

Proof. Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be the corresponding [FLTZ3] skeleta. We have shown that (a Liouville domain completing to) the general fiber $\partial\mathbf{T}^{mir}$ of W is isotopic both to a domain with skeleton $\partial\mathbb{L}_1$, and to a domain with skeleton $\partial\mathbb{L}_2$. By [GPS2, Corollary 2.9], we have an equivalence of the wrapped Fukaya categories $\mathrm{Fuk}(T^*\mathbb{T}^\vee,\partial\mathbb{L}_1)\cong\mathrm{Fuk}(T^*\mathbb{T}^\vee,\partial\mathbb{L}_2)$. By [GPS3] we may trade this for an equivalence of constructible sheaf categories, and by [Ku] we may trade the latter for the asserted equivalence of coherent sheaf categories. □

What the above argument does not yet give is a formula for the above equivalence. In fact, there are many such derived equivalences, corresponding to monodromies (as we vary the coefficients of f) around the discriminant locus. We will describe these in future work.

8.1. Non-Fano mirror symmetry

It was observed in [AKO], [Ab2] that mirror symmetry for toric varieties requires modification in the case of a non-Fano variety $\mathbf{T}$: the naive interpretation of the Hori–Vafa mirror Landau–Ginzburg model for a non-Fano variety contains $\mathrm{Coh}(\mathbf{T})$ as a full subcategory but can be strictly larger. One procedure to remedy this discrepancy is suggested in [BDF+]. By contrast, [Ku] holds for *all* toric varieties. Here we explain this discrepancy in an example; in future work, we plan to use the same ideas to establish the conjectures of [BDF+].

In the body of this paper, we began with a polytope $\Delta^\vee$ with star-shaped triangulation, and let Σ be the fan given by this star-shaped triangulation. Any fan Σ obtained

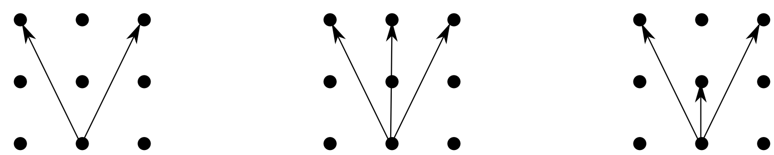

Figure 18. The fans Σ_1, Σ_2, Σ'.

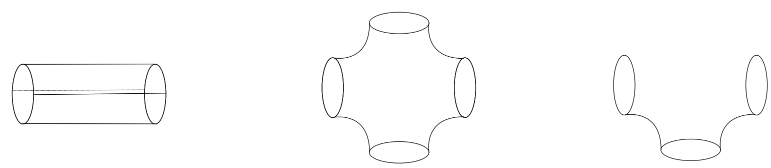

Figure 19. The FLTZ boundary skeleta $\Lambda_1, \Lambda_2, \Lambda'$ for the fans $\Sigma_1, \Sigma_2, \Sigma'$.

in this way has the following property: let $v_1, ..., v_k$ be (stacky) primitives for the rays in Σ, and let $\Delta^\vee$ be the convex hull of the v_i. Then each v_i is on the boundary of $\Delta^\vee$. This excludes fans Σ in which one of the primitives v_i is too short to reach $\partial\Delta^\vee$. In this case, the mirror to $\partial\mathbf{T}_\Sigma$ will not be a hypersurface with Newton polytope Δ, but only a Liouville subdomain of such a hypersurface. The simplest case of this is described in the following example.

Example 8.2. Let Σ_1 be the fan with primitive rays $(-1,2),(1,2)$; Σ_2 be the fan with primitives $(-1,2),(0,2),(1,2)$; Σ' be the fan with primitives $(-1,2),(0,1),(1,2)$; and $\Delta^\vee$ be the polytope obtained as convex hull of the primitives for any of the three fans above. (These convex hulls obviously agree.)

Then, each of Σ_1 and Σ_2 is obtained as a star-shaped triangulation of $\Delta^\vee$; hence, the results of this paper show that the boundaries $\partial\mathbf{T}_{\Sigma_1}$ and $\partial\mathbf{T}_{\Sigma_2}$ are both mirror to a generic hypersurface H with Newton polytope $\Delta^\vee$.

We obtain two different skeleta Λ_1 and Λ_2 of the hypersurface H, corresponding to the respective triangulations Σ_1 and Σ_2, and we know that each of these is the boundary of a stacky FLTZ skeleton; by studying the fans Σ_i, we conclude that Λ_1 consists of two circles connected by four different intervals (since the two rays in Σ_1 share a non-unimodular simplex of area 4), and Λ_2 consists of four circles, cyclically connected by intervals (there being four circles since the middle ray, of length 2, is double-counted by the stacky FLTZ procedure). Each of these is a skeleton for H, which is a quadruply-punctured genus-1 curve.

Let Λ' be the boundary of the [FLTZ3] skeleton for Σ'. Then, Λ' is no longer a skeleton for the hypersurface H, as Λ_1 and Λ_2 are. It resembles the skeleton Λ_2, except that the central ray, now of length 1, is no longer double-counted. This means that Λ'

is obtained from Λ_2 by deleting one of the two double-counted circles along with its two connecting intervals. Hence, Λ' consists of three circles, connected in a row by a pair of intervals. It is the skeleton of a triply-punctured genus-1 curve, a subdomain of the quadruply-punctured curve H.

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