

Hitchin characters and geodesic laminations

by

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Introduction

0.1. Background and motivation

For a closed, connected, oriented surface S of genus $g > 1$, the *Hitchin component* $\text{Hit}_n(S)$ is a preferred component of the character variety

$$\mathcal{X}_{\text{PSL}_n(\mathbb{R})}(S) = \{\text{homomorphisms } \varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})\} // \text{PSL}_n(\mathbb{R})$$

consisting of group homomorphisms $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ from the fundamental group $\pi_1(S)$ to the Lie group $\text{PSL}_n(\mathbb{R})$ (equal to the special linear group $\text{SL}_n(\mathbb{R})$ if n is odd, and to $\text{SL}_n(\mathbb{R})/\{\pm \text{Id}\}$ if n is even), where $\text{PSL}_n(\mathbb{R})$ acts on these homomorphisms by conjugation. The quotient should normally be taken in the sense of geometric invariant theory [34], but this subtlety is irrelevant here, as this quotient construction coincides with the usual topological quotient on the Hitchin component. The “character variety” terminology is justified by the fact that, by [37], the class of $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ in $\mathcal{X}_{\text{PSL}_n(\mathbb{R})}(S)$ is determined by the character $\pi_1(S) \rightarrow \mathbb{R}$ associating the trace $\text{Tr } \varrho(\gamma) \in \mathbb{R}$ with $\gamma \in \pi_1(S)$ (defined up to sign when n is even). In particular, we will refer to the elements of $\mathcal{X}_{\text{PSL}_n(\mathbb{R})}(S)$ as *characters*.

When $n=2$, the Lie group $\text{PSL}_2(\mathbb{R})$ is also the orientation-preserving isometry group of the hyperbolic plane \mathbb{H}^2 , and the Hitchin component $\text{Hit}_2(S)$ of $\mathcal{X}_{\text{PSL}_2(\mathbb{R})}(S)$ consists

of all characters represented by injective homomorphisms $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ whose image $\varrho(\pi_1(S))$ is discrete in $\mathrm{PSL}_2(\mathbb{R})$ and for which the natural homotopy equivalence $S \rightarrow \mathbb{H}^2/\varrho(\pi_1(S))$ has degree 1. The Hitchin component $\mathrm{Hit}_2(S)$ is in this case called the *Teichmüller component*, and can also be described as the space of isotopy classes of hyperbolic metrics on S .

When $n > 2$, there is a preferred homomorphism $\mathrm{PSL}_2(\mathbb{R}) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ coming from the unique n -dimensional representation of $\mathrm{SL}_2(\mathbb{R})$ (or, equivalently, from the natural action of $\mathrm{SL}_2(\mathbb{R})$ on the vector space $\mathbb{R}[X, Y]_{n-1} \cong \mathbb{R}^n$ of homogeneous polynomials of degree $n-1$ in two variables). This provides a natural map $\mathcal{X}_{\mathrm{PSL}_2(\mathbb{R})}(S) \rightarrow \mathcal{X}_{\mathrm{PSL}_n(\mathbb{R})}(S)$, and the *Hitchin component* $\mathrm{Hit}_n(S)$ is the component of $\mathcal{X}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ that contains the image of $\mathrm{Hit}_2(S) \subset \mathcal{X}_{\mathrm{PSL}_2(\mathbb{R})}(S)$. The terminology is motivated by the following fundamental result by Hitchin [23], who was the first to single out this component.

THEOREM 0.1. (Hitchin) *The Hitchin component $\mathrm{Hit}_n(S)$ is diffeomorphic to*

$$\mathbb{R}^{2(g-1)(n^2-1)}.$$

A *Hitchin character* is an element of the Hitchin component $\mathrm{Hit}_n(S)$, and a *Hitchin homomorphism* is a homomorphism $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ representing a Hitchin character. We will use the same letter to represent the Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ and the corresponding Hitchin character $\varrho \in \mathrm{Hit}_n(S)$.

About 15 years after [23], Labourie [27] showed that Hitchin homomorphisms satisfy many important geometric and dynamical properties, and in particular are injective with discrete image; see also [18].

Hitchin's construction of the parametrization of $\mathrm{Hit}_n(S)$ given by Theorem 0.1 is based on geometric analysis techniques that provide little information on the geometry of the Hitchin homomorphisms themselves; see [30], [28], [29] for different geometric analytic parametrizations when $n=3$. The current article is devoted to developing another parametrization of the Hitchin component $\mathrm{Hit}_n(S)$ which is much more geometric, and has the additional advantage of being well-behaved with respect to the data of a geodesic lamination. Geodesic laminations were introduced by Thurston in order to develop a continuous calculus for simple closed curves on the surface S , and provide very powerful tools for many topological and geometric problems in dimensions 2 and 3. See §9 and §10 for two simple applications of our parametrization, one to the dynamics of the action of a pseudo-Anosov homomorphism of S on the Hitchin component, and another one to the length functions defined by a Hitchin character on Thurston's space $\mathcal{ML}(S)$ of measured laminations on S .

In fact, our construction is a natural extension of Thurston's parametrization of the Teichmüller component by shearing coordinates [42], [5]. It draws its inspiration from

this classical case where $n=2$, but also from work of Fock–Goncharov [18] on a variant of the Hitchin component where the surface S has punctures, and where these punctures are endowed with additional information. As in the classical case where $n=2$, the situation is conceptually and analytically much more complicated for a closed surface than in the case considered in [18]. The major contributions of the article are to adapt the analysis of [42], [5] to the case where $n \geq 2$. As a consequence, many of the arguments are new even for the case $n=2$: this includes the slithering map of §5.1, the explicit reconstruction of a character from its shearing invariants in §6.2, and the estimates of §8.2 relating growth rates to positivity.

The companion article [9] is devoted to a special case of our parametrization, when the geodesic lamination has only finitely many leaves. The situation is much simpler in that case, and in particular the arguments of [9] tend to be very combinatorial in nature. The current article has a much more analytic flavor. It is also more conceptual, and provides a homological interpretation of some of the invariants and phenomena that were developed in a purely computational way in [9]. And of course the framework of general geodesic laminations, possibly with uncountably many leaves, considered in this article is better suited for applications.

The article [16] was developed, to a large extent, as a first step towards the more general results of the current paper. It investigates all deformations of a Hitchin character $\varrho \in \text{Hit}_n(S)$ that respect its triangle invariants, as discussed in the next section.

0.2. Main results

We can now be more specific. Let λ be a maximal geodesic lamination in S . See §2 for precise definitions. What we need to know here is just that, for an arbitrary auxiliary metric of negative curvature on the surface, λ is decomposed as a union of disjoint geodesic leaves, and that its complement $S \setminus \lambda$ consists of $4(g-1)$ infinite triangles with geodesic boundary. Some maximal geodesic laminations, such as the ones considered in [9], have only a finite number of leaves, but generic examples have uncountably many leaves.

Given a Hitchin character $\varrho \in \text{Hit}_n(S)$, the rich dynamical structure for ϱ discovered by Labourie [27] associates with each triangle component T_j of $S \setminus \lambda$ a triple (E, F, G) of three flags of \mathbb{R}^n , well defined modulo the action of $\text{PGL}_n(\mathbb{R})$. In addition, Fock and Goncharov [18] prove that this flag triple (E, F, G) is positive, in a sense discussed in §1.5, and is determined by $\frac{1}{2}(n-1)(n-2)$ invariants $\tau_{abc}^\varrho(E, F, G) \in \mathbb{R}$. Since $S \setminus \lambda$ has $4(g-1)$ components, these flag triple invariants can be collected into a single *triangle invariant* $\tau^\varrho \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$.

The really new feature introduced in this article describes how to glue these flag triples across the (possibly uncountably many) leaves of the lamination, and simultaneously involves analytic and combinatorial arguments. The analytic part of this analysis is based on the slithering map constructed in §5.1, which is a higher-dimensional analogue of the horocyclic foliation that is at the basis of the case $n=2$ [42], [5]. This slithering map enables us to control the gluing by elements of the homology of a train-track neighborhood U for λ , which we now briefly describe. The precise definition of train-track neighborhoods can be found in §4.2 (and is familiar to experts); at this point, it suffices to say that U is obtained from S by removing $4(g-1)$ disjoint disks, one in each component of $S \setminus \lambda$. In addition, the boundary ∂U is decomposed into a *horizontal boundary* $\partial_h U$ and a *vertical boundary* $\partial_v U$ in such a way that, for each component T of $S \setminus \lambda$, the intersection $T \cap \partial U$ is a hexagon made up of three arc components of $\partial_h U$ running parallel to the sides of T and three arc components of $\partial_v U$ facing the spikes of T .

The geodesic lamination λ has a well-defined 2-fold *orientation cover* $\hat{\lambda}$, whose leaves are continuously oriented, and the covering map $\hat{\lambda} \rightarrow \lambda$ uniquely extends to a 2-fold cover $\hat{U} \rightarrow U$. In particular, $\hat{\lambda}$ is a geodesic lamination in the surface \hat{U} . Also, as the components of $S \setminus \lambda$ are infinite triangles, λ is non-orientable and it follows that $\hat{\lambda}$ and \hat{U} are connected.

Our new invariant for a Hitchin character $\varrho \in \text{Hit}_n(S)$ is a certain *shearing class* $[\sigma^\varrho] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$. This shearing class has the property that $\iota_*([\sigma^\varrho]) = -[\sigma^\varrho]$, for the covering involution ι of the cover $\hat{U} \rightarrow U$ and for the involution $x \mapsto \bar{x}$ of \mathbb{R}^{n-1} that associates $\bar{x} = (x_{n-1}, x_{n-2}, \dots, x_1)$ with $x = (x_1, x_2, \dots, x_{n-1})$. In particular, $[\sigma^\varrho]$ can also be interpreted as a twisted homology class $[\sigma^\varrho] \in H_1(U, \partial_v U; \tilde{\mathbb{R}}^{n-1})$ valued in a suitable coefficient bundle $\tilde{\mathbb{R}}^{n-1}$ over U with fiber \mathbb{R}^{n-1} .

The triangle invariant τ^ϱ and the shearing class $[\sigma^\varrho]$ satisfy two types of constraints. The first constraint is a homological equality.

PROPOSITION 0.2. (Shearing cycle boundary condition) *The homological boundary $\partial[\sigma^\varrho] \in H_0(\partial_v \hat{U}; \mathbb{R}^{n-1})$ of the shearing class $[\sigma^\varrho] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$ of a Hitchin character $\varrho \in \text{Hit}_n(S)$ is completely determined by the triangle invariant $\tau^\varrho \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$, by an explicit linear formula given in Lemma 5.14 and Proposition 5.15.*

The second constraint is a positivity property, proved as Corollary 7.13 in §7.2. Because the leaves of the orientation cover $\hat{\lambda}$ are oriented, a famous construction by Ruelle and Sullivan [38] interprets every transverse measure μ for $\hat{\lambda}$ as a 1-dimensional de Rham current in \hat{U} . In particular, such a transverse measure μ determines a homology class $[\mu] \in H_1(\hat{U}; \mathbb{R})$. See also Proposition 4.2 for a more explicit description of this homology class $[\mu]$.

PROPOSITION 0.3. (Positive intersection condition) *For every transverse measure μ for the orientation cover $\hat{\lambda}$, the algebraic intersection vector $[\mu] \cdot [\sigma^\varrho] \in \mathbb{R}^{n-1}$ of the homology class $[\mu] \in H_1(\hat{U}; \mathbb{R})$ with the shearing class $[\sigma^\varrho] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$ of $\varrho \in \text{Hit}_n(S)$ is positive, in the sense that all its coordinates are positive.*

The shearing cycle boundary and positive intersection conditions restrict the pair $(\tau^\varrho, [\sigma^\varrho])$ to a convex polyhedral cone \mathcal{P} in $\mathbb{R}^{2(g-1)(n-1)(n-2)} \times H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$, with finitely many faces. The main result of the article, proved as Theorem 8.13 in §8.3, shows that these are the only restrictions on the triangle and shearing invariants, and that these provide a parametrization of the Hitchin component $\text{Hit}_n(S)$.

THEOREM 0.4. (Parametrization of the Hitchin component) *The map $\text{Hit}_n(S) \rightarrow \mathcal{P}$, which to a Hitchin character $\varrho \in \text{Hit}_n(S)$ assigns the pair $(\tau^\varrho, [\sigma^\varrho])$ formed by its triangle invariant $\tau^\varrho \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$ and its shearing class $[\sigma^\varrho] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$, is a homeomorphism.*

When $n=2$, all triangle invariants are trivial and the shearing class $[\sigma^\varrho]$ is closed. The above theorem is just a rephrasing of the parametrization of the Teichmüller space $\mathcal{T}(S)$ by the shearing coordinates defined by λ , as developed in [42] and [5].

The shearing cycle boundary condition provides some unexpected constraints on the triangle invariants of Hitchin characters, as well as on their shearing classes. The following two statements are abbreviated expressions of more specific computations given in §8.4. These restrictions are somewhat surprising when one considers the relatively large dimension $2(g-1)(n^2-1)$ of $\text{Hit}_n(S)$.

PROPOSITION 0.5. *An element $\tau \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$ is the triangle invariant τ^ϱ of a Hitchin character $\varrho \in \text{Hit}_n(S)$ if and only if it belongs to a certain explicit subspace of codimension $\lfloor \frac{1}{2}(n-1) \rfloor$ of $\mathbb{R}^{2(g-1)(n-1)(n-2)}$.*

We already saw that shearing cycles have a simple symmetry property under the covering involution ι of \hat{U} , and therefore they belong to the twisted homology space $H_1(U, \partial_v U; \tilde{\mathbb{R}}^{n-1})$, consisting of those $\alpha \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$ such that $\iota_*(\alpha) = -\alpha$. The dimension of this twisted homology space $H_1(U, \partial_v U; \tilde{\mathbb{R}}^{n-1})$ is computed in (the proof of) Proposition 4.7, and is equal to $18(g-1)(n-1)$.

PROPOSITION 0.6. *A twisted relative homology class $[\sigma] \in H_1(U, \partial_v U; \tilde{\mathbb{R}}^{n-1})$ is the shearing class $[\sigma^\varrho]$ of a Hitchin character $\varrho \in \text{Hit}_n(S)$ if and only if it belongs to a certain open convex polyhedral cone, bounded by finitely many faces, in an explicit linear subspace of $H_1(U, \partial_v U; \tilde{\mathbb{R}}^{n-1})$ of codimension $24(g-1)$ if $n > 3$, of codimension $20(g-1)$ if $n=3$, and of codimension $12(g-1)$ if $n=2$.*

At first, the relative homology space $H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}^{n-1})$ of a train-track neighborhood U , and its twisted version $H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1})$, may not appear very natural. In fact, although we decided to privilege this more familiar point of view in this introduction, $H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}^{n-1})$ occurs as a space $\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ of tangent cycles for the orientation cover $\widehat{\lambda}$ relative to its slits, where the slits of $\widehat{\lambda}$ are lifts of the spikes of the complement $S \setminus \lambda$; Proposition 4.5 then provides an isomorphism

$$\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}^{n-1}) \cong H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}^{n-1}).$$

A relative tangent cycle $\alpha \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ assigns a vector $\alpha(k) \in \mathbb{R}^{n-1}$ to each arc k transverse to $\widehat{\lambda}$, in a quasi-additive way: If k is split into two subarcs k_1 and k_2 , then $\alpha(k)$ is equal to the sum of $\alpha(k_1)$, $\alpha(k_2)$ and of a correction factor depending on the slit of $\widehat{\lambda}$ facing the point $k_1 \cap k_2$ along which k was split. In particular, $\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ depends only on the maximal geodesic lamination λ , and not on the train-track neighborhood U . For the above isomorphism, the twisted homology space

$$H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}) \subset H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}^{n-1})$$

corresponds to those $\alpha \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ such that $\alpha(\iota(k)) = \overline{\alpha(k)}$ for every arc k transverse to $\widehat{\lambda}$, and for the covering involution ι ; in particular, $H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1})$ is also independent of the choice of U .

The lack of additivity of a relative tangent cycle $\alpha \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ has a nice expression in terms of the boundary map $\partial: H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}^{n-1}) \rightarrow H_0(\partial_v \widehat{U}; \mathbb{R}^{n-1})$, and is at the basis of the shearing cycle boundary condition of Proposition 0.2. In the classical case where $n=2$, all triangle invariants are trivial, and the shearing cycle boundary condition says that the shearing class $[\sigma^e] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}^{n-1})$ has boundary equal to zero, and in particular that the corresponding tangent cycle $[\sigma^e] \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$ is additive with no correction factors; such objects were called “transverse cocycles” in [7] and [5].

This point of view enables us to shed some light on the positive intersection condition of Proposition 0.3. Given a Hitchin character $\varrho \in \text{Hit}_n(S)$, Labourie [27] shows that for every non-trivial $\gamma \in \pi_1(S)$ the matrix $\varrho(\gamma) \in \text{PSL}_n(\mathbb{R})$ is diagonalizable, and that its eigenvalues $m_a^\varrho(\gamma)$ can be ordered in such a way that $|m_1^\varrho(\gamma)| > |m_2^\varrho(\gamma)| > \dots > |m_n^\varrho(\gamma)|$. If we define $\ell^e(\gamma) \in \mathbb{R}^{n-1}$ by the property that its a th coordinate is

$$\ell_a^e(\gamma) = \log \frac{|m_a^\varrho(\gamma)|}{|m_{a+1}^\varrho(\gamma)|},$$

the second author showed in [15] that this formula admits a continuous linear extension $\ell^e: \mathcal{C}^H(S) \rightarrow \mathbb{R}^{n-1}$ to the space $\mathcal{C}^H(S)$ of Hölder geodesic currents of S , a topological

vector space that contains all conjugacy classes of $\pi_1(S)$ in a natural way; this continuous extension $\ell^e: \mathcal{C}^H(S) \rightarrow \mathbb{R}^{n-1}$ is unique on the subspaces of $\mathcal{C}^H(S)$ that are of interest to us in this paper (see Remark 7.3).

A tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}; \mathbb{R})$, namely a relative tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ with no correction factor, defines a Hölder geodesic current $\alpha \in \mathcal{C}^H(S)$ (see [7]), and we can restrict the length function of [15] to $\ell^e: \mathcal{Z}(\hat{\lambda}; \mathbb{R}) \rightarrow \mathbb{R}^{n-1}$.

The following result, proved as Theorem 7.5 in §7.2, relates the length vector $\ell^e(\alpha) \in \mathbb{R}^{n-1}$ to the shearing class $[\sigma^e] \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1}) \cong H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$.

THEOREM 0.7. (Length and intersection formula) *If $[\sigma^e] \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1}) \cong H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$ is the shearing cycle of a Hitchin character $\varrho \in \text{Hit}_n(S)$, and if $\alpha \in \mathcal{Z}(\hat{\lambda}; \mathbb{R}) \cong H_1(\hat{U}; \mathbb{R})$ is a tangent cycle for the orientation cover $\hat{\lambda}$, then the length vector*

$$\ell^e(\alpha) = [\alpha] \cdot [\sigma^e] \in \mathbb{R}^{n-1}$$

is equal to the algebraic intersection vector of the homology classes $[\alpha] \in H_1(\hat{U}; \mathbb{R})$ and $[\sigma^e] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}^{n-1})$ in the train-track neighborhood \hat{U} of $\hat{\lambda}$.

This is the natural extension of the formula which, in the classical case $n=2$, relates shearing coordinates and lengths of tangent cycles through Thurston's symplectic form [5, §3].

In the special case where the tangent cycle $\alpha \in \mathcal{Z}(\lambda; \mathbb{R})$ is defined by a transverse measure μ for $\hat{\lambda}$, Theorem 0.7 establishes an equivalence between the positive intersection condition of Proposition 0.3 and the property that all coordinates of the vector $\ell^e(\mu)$ are positive. In this version, this positivity property is an immediate consequence of the Anosov property that is central to [27] (see Proposition 7.4).

The article concludes, in §9 and §10, with two brief applications of Theorems 0.4 and 0.7. The first one is concerned with the dynamics of the action of a pseudo-Anosov diffeomorphism $\varphi: S \rightarrow S$ on the Hitchin component $\text{Hit}_n(S)$; applying the parametrization of Theorem 0.4 to the case of a maximal geodesic lamination λ containing the stable lamination of φ shows that the dynamics of the action of φ on $\text{Hit}_n(S)$ are concentrated on submanifolds of $\text{Hit}_n(S)$ of relatively large codimension. The second application considers the restriction of the length function $\ell^e: \mathcal{C}^H(S) \rightarrow \mathbb{R}^{n-1}$ to Thurston's space $\mathcal{ML}(S)$ of measured laminations on S ; a consequence of Theorem 0.7 is that, at each $\alpha \in \mathcal{ML}(S)$, the tangent map $T_\alpha \ell^e: T_\alpha \mathcal{ML}(S) \rightarrow \mathbb{R}^{n-1}$ is linear on each of the (possibly infinitely many) linear faces that naturally decompose $T_\alpha \mathcal{ML}(S)$ for the piecewise linear structure of $\mathcal{ML}(S)$. In particular, $T_\alpha \ell^e$ is linear at those generic $\alpha \in \mathcal{ML}(S)$ where $T_\alpha \mathcal{ML}(S)$ admits a vector space structure (with a single face).

It should be possible to place the results of this article in a broader perspective, by replacing the Lie group $\text{PSL}_n(\mathbb{R})$ with another split real algebraic group G . Indeed,

many properties of the Hitchin component remain valid in this more general context [23], [27], [18], [22]. In this extended framework, our triangle invariant τ^ϱ should then associate with each component of $S \setminus \lambda$ a positive triple in the flag space $B \setminus G$, where B is a Borel subgroup of the Lie group and where $B \setminus G$ here denotes the left quotient. The shearing class would be a relative homology class $[\sigma^\varrho] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathfrak{h})$ valued in the Cartan algebra \mathfrak{h} of G , and equivariant with respect to the covering involution $\iota: \widehat{U} \rightarrow \widehat{U}$ and to the opposition involution of \mathfrak{h} multiplied by -1 . The shearing cycle boundary condition would then state that the boundary $\partial[\sigma^\varrho] \in H_0(\partial_v \widehat{U}; \mathfrak{h})$ is completely determined by the triangle invariant $\tau^\varrho \in (B \setminus G)^{4(g-1)}$, while the positive intersection condition would require that the algebraic intersection vector $[\mu] \cdot [\sigma^\varrho] \in \mathfrak{h}$ belong to the principal Weyl chamber of \mathfrak{h} . There are still a few missing technical steps in the existing literature to complete this program in its full generality (and an additional stumbling block in the limited expertise of the authors), but a large part of the extension to this more general context should mostly be entailed by the use of the right vocabulary.

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1. Generic configurations of flags

Flags in \mathbb{R}^n play a fundamental role in our construction of invariants of Hitchin characters. This section is devoted to certain invariants of finite families of flags, borrowed from [18].

1.1. Flags

A *flag* in \mathbb{R}^n is a family F of nested linear subspaces $F^{(0)} \subset F^{(1)} \subset \dots \subset F^{(n-1)} \subset F^{(n)}$ of \mathbb{R}^n where each $F^{(a)}$ has dimension a .

A pair of flags (E, F) is *generic* if every subspace $E^{(a)}$ of E is transverse to every subspace $F^{(b)}$ of F . This is equivalent to the property that $\mathbb{R}^n = E^{(a)} \oplus F^{(n-a)}$ for every a .

Similarly, a triple of flags (E, F, G) is *generic* if each triple of subspaces $E^{(a)}$, $F^{(b)}$ and $G^{(c)}$, respectively in E , F and G , meets transversely in the sense that

$$\dim(E^{(a)} + F^{(b)} + G^{(c)}) = \min\{a + b + c, n\}.$$

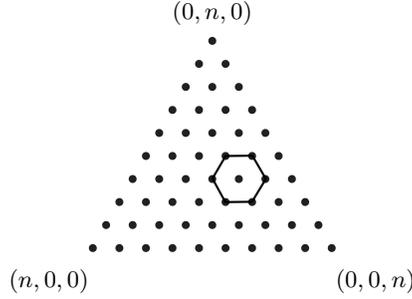


Figure 1. The discrete triangle Θ_n , with a hexagon cycle.

Again, this is equivalent to the property that $\mathbb{R}^n = E^{(a)} \oplus F^{(b)} \oplus G^{(c)}$ for every a, b and c with $a+b+c=n$.

1.2. Wedge-product invariants of generic flag triples

Elementary linear algebra shows that, for any two generic flag pairs (E, F) and (E', F') , there is a linear isomorphism $\mathbb{R}^n \rightarrow \mathbb{R}^n$ sending E to E' and F to F' . However, the same is not true for generic flag triples. Indeed, there is a whole moduli space of generic flag triples modulo the action of $GL_n(\mathbb{R})$, and this moduli space can be parametrized by invariants that we now describe. These invariants, originally introduced by Fock and Goncharov [18], are expressed in terms of the exterior algebra $\Lambda^\bullet(\mathbb{R}^n)$ of \mathbb{R}^n .

Consider the discrete triangle

$$\Theta_n = \{(a, b, c) \in \mathbb{Z}^3 : a+b+c = n \text{ and } a, b, c \geq 0\}$$

represented in Figure 1.

A function $\varphi: \Theta_n \rightarrow \mathbb{Z}$ is *balanced* if, for every a_0, b_0 and c_0 ,

$$\sum_{(a_0, b, c) \in \Theta_n} \varphi(a_0, b, c) = \sum_{(a, b_0, c) \in \Theta_n} \varphi(a, b_0, c) = \sum_{(a, b, c_0) \in \Theta_n} \varphi(a, b, c_0) = 0,$$

namely if the sum of the $\varphi(a, b, c)$ over each line parallel to one side of the triangle Θ_n is equal to zero.

Such a balanced function φ defines an invariant of a generic flag triple (E, F, G) as follows. For each a, b and c between 0 and n , the spaces $\Lambda^a(E^{(a)})$, $\Lambda^b(F^{(b)})$ and $\Lambda^c(G^{(c)})$ are each isomorphic to \mathbb{R} . Choose non-zero elements $e^{(a)} \in \Lambda^a(E^{(a)})$, $f^{(b)} \in \Lambda^b(F^{(b)})$ and $g^{(c)} \in \Lambda^c(G^{(c)})$. We will use the same letters to denote their images $e^{(a)} \in \Lambda^a(\mathbb{R}^n)$, $f^{(b)} \in \Lambda^b(\mathbb{R}^n)$ and $g^{(c)} \in \Lambda^c(\mathbb{R}^n)$. We then define

$$W_\varphi(E, F, G) = \prod_{(a, b, c) \in \Theta_n} (e^{(a)} \wedge f^{(b)} \wedge g^{(c)})^{\varphi(a, b, c)} \in \mathbb{R},$$

where we choose an isomorphism $\Lambda^n(\mathbb{R}^n) \cong \mathbb{R}$ to interpret each term in the product as a real number. The fact that the flag triple is generic guarantees that these numbers are non-zero, while the property that φ is balanced is exactly what is needed to make sure that this product is independent of the choices of the elements $e^{(a)} \in \Lambda^a(E^{(a)})$, $f^{(b)} \in \Lambda^b(F^{(b)})$ and $g^{(c)} \in \Lambda^c(G^{(c)})$ and of the isomorphism $\Lambda^n(\mathbb{R}^n) \cong \mathbb{R}$. We say that W_φ is the *wedge-product invariant* of generic flag triples associated with the balanced function $\varphi: \Theta \rightarrow \mathbb{Z}$.

We now consider a fundamental special case. For $a, b, c \geq 1$ with $a+b+c=n$, namely for a point (a, b, c) in the interior of the triangle Θ_n , the (a, b, c) -*hexagon cycle* is the balanced function $\varphi_{abc}: \Theta_n \rightarrow \mathbb{Z}$ defined by

$$\varphi_{abc} = \delta_{(a+1,b,c-1)} - \delta_{(a-1,b,c+1)} + \delta_{(a,b-1,c+1)} - \delta_{(a,b+1,c-1)} + \delta_{(a-1,b+1,c)} - \delta_{(a+1,b-1,c)},$$

where $\delta_{(a,b,c)}: \Theta_n \rightarrow \mathbb{Z}$ denotes the Kronecker function such that

$$\delta_{(a,b,c)}(a', b', c') = \begin{cases} 1, & \text{if } (a, b, c) = (a', b', c'), \\ 0, & \text{otherwise.} \end{cases}$$

The terminology is explained by the fact that the support of φ_{abc} is a small hexagon in the discrete triangle Θ_n , centered at the point (a, b, c) ; see Figure 1 for the case where $n=9$ and $(a, b, c)=(2, 3, 4)$. The wedge-product invariant associated with the hexagon cycle φ_{abc} is the (a, b, c) -*triple ratio*

$$T_{abc}(E, F, G) = \frac{e^{(a+1)} \wedge f^{(b)} \wedge g^{(c-1)}}{e^{(a-1)} \wedge f^{(b)} \wedge g^{(c+1)}} \frac{e^{(a)} \wedge f^{(b-1)} \wedge g^{(c+1)}}{e^{(a)} \wedge f^{(b+1)} \wedge g^{(c-1)}} \frac{e^{(a-1)} \wedge f^{(b+1)} \wedge g^{(c)}}{e^{(a+1)} \wedge f^{(b-1)} \wedge g^{(c)}}.$$

Note that there is no triple ratio invariant when $n=2$, and exactly one triple ratio $T_{111}(E, F, G)$ when $n=3$. See [18, §9.5] for a reduction of any triple ratio invariant $T_{abc}(E, F, G)$ to the 3-dimensional case, by consideration of the quotient space $\mathbb{R}^n / (E^{(a-1)} + F^{(b-1)} + G^{(c-1)}) \cong \mathbb{R}^3$, and its interpretation as a cross-ratio in the associated projective plane.

Triple ratios have the following properties under permutation of the flags.

LEMMA 1.1.

$$T_{abc}(E, F, G) = T_{bca}(F, G, E) = T_{bac}(F, E, G)^{-1}.$$

Proof. The proof is elementary. □

The natural action of the linear group $GL_n(\mathbb{R})$ on the flag variety $\text{Flag}(\mathbb{R}^n)$ descends to an action of the projective linear group $PGL_n(\mathbb{R})$, quotient of $GL_n(\mathbb{R})$ by its center $(\mathbb{R} \setminus \{0\}) \text{Id}_n$ consisting of all non-zero scalar multiples of the identity matrix. Note that the projective special linear group $PSL_n(\mathbb{R})$ is equal to $PGL_n(\mathbb{R})$ if n is odd, and is an index-2 subgroup of $PGL_n(\mathbb{R})$ otherwise.

PROPOSITION 1.2. *Two generic flag triples (E, F, G) and (E', F', G') are equivalent under the action of $\mathrm{PGL}_n(\mathbb{R})$ if and only if $T_{abc}(E, F, G) = T_{abc}(E', F', G')$ for every $a, b, c \geq 1$ with $a + b + c = n$.*

In addition, for any set of non-zero numbers $t_{abc} \in \mathbb{R} \setminus \{0\}$, there exists a generic flag triple (E, F, G) such that $T_{abc}(E, F, G) = t_{abc}$ for every $a, b, c \geq 1$ with $a + b + c = n$.

Proof. See [18, §9]. □

In particular, the moduli space of generic flag triples (E, F, G) under the action of $\mathrm{PGL}_n(\mathbb{R})$ is homeomorphic to $(\mathbb{R} \setminus \{0\})^{(n-1)(n-2)/2}$.

Corollary 1.4 below partially accounts for the important role played by the triple ratios T_{abc} in Proposition 1.2. We will not really need this property, but it explains why we will always be able to express in terms of triple ratios T_{abc} the various wedge-product invariants that we will encounter in the paper.

LEMMA 1.3. *The hexagon cycles $\{\varphi_{abc} : a, b, c \geq 1 \text{ with } a + b + c = n\}$ form a basis for the free abelian group consisting of all balanced functions $\varphi : \Theta_n \rightarrow \mathbb{Z}$.*

Proof. The proof is elementary, by induction on n . □

COROLLARY 1.4. *Every wedge-product invariant can be uniquely expressed as a product of integer powers of triple ratios.*

Proof. This is an immediate consequence of Lemma 1.3. □

1.3. Quadruple ratios

In addition to triple ratios, the following wedge-product invariants of generic flag triples will play a very important role in this article.

For $a = 1, 2, \dots, n - 1$, the a -th quadruple ratio of the generic flag triple (E, F, G) is the wedge-product invariant

$$Q_a(E, F, G) = \frac{e^{(a-1)} \wedge f^{(n-a)} \wedge g^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge g^{(1)}} \frac{e^{(a)} \wedge f^{(1)} \wedge g^{(n-a-1)}}{e^{(a-1)} \wedge f^{(1)} \wedge g^{(n-a)}} \frac{e^{(a+1)} \wedge f^{(n-a-1)}}{e^{(a+1)} \wedge g^{(n-a-1)}} \frac{e^{(a)} \wedge g^{(n-a)}}{e^{(a)} \wedge f^{(n-a)}}$$

where, as usual, we consider arbitrary non-zero elements $e^{(b)} \in \Lambda^b(E^{(b)})$, $f^{(b)} \in \Lambda^b(F^{(b)})$ and $g^{(b)} \in \Lambda^b(G^{(b)})$, and where the ratios are computed in $\Lambda^n(\mathbb{R}^n) \cong \mathbb{R}$.

Note that $Q_a(E, G, F) = Q_a(E, F, G)^{-1}$, but that this quadruple ratio usually does not behave well under the other permutations of the flags E, F and G , as E plays a special role in $Q_a(E, F, G)$.

For this wedge-product invariant, we can explicitly determine the formula predicted by Corollary 1.4.

LEMMA 1.5. For $a=1, 2, \dots, n-1$,

$$Q_a(E, F, G) = \prod_{b+c=n-a} T_{abc}(E, F, G),$$

where the product is over all integers $b, c \geq 1$ with $b+c=n-a$. In particular,

$$Q_{n-1}(E, F, G) = 1 \quad \text{and} \quad Q_{n-2}(E, F, G) = T_{(n-2)11}(E, F, G).$$

Proof. When computing the right-hand side of the equation, most of the terms $e^{(a')} \wedge f^{(b')} \wedge g^{(c')}$ cancel out and we are left with the eight terms of $Q_a(E, F, G)$. \square

1.4. Double ratios

We now consider quadruples (E, F, G, H) of flags $E, F, G, H \in \text{Flag}(\mathbb{R}^n)$. Such a flag quadruple is *generic* if each quadruple of subspaces $(E^{(a)}, F^{(b)}, G^{(c)}, H^{(d)})$ meets transversely. As usual, we can restrict attention to the cases where $a+b+c+d=n$.

For $1 \leq a \leq n-1$, the a -th *double ratio* of the generic flag quadruple (E, F, G, H) is

$$D_a(E, F, G, H) = -\frac{e^{(a)} \wedge f^{(n-a-1)} \wedge g^{(1)} \quad e^{(a-1)} \wedge f^{(n-a)} \wedge h^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge h^{(1)} \quad e^{(a-1)} \wedge f^{(n-a)} \wedge g^{(1)}},$$

where we choose arbitrary non-zero elements $e^{(a')} \in \Lambda^{a'}(E^{(a')})$, $f^{(b')} \in \Lambda^1(F^{(b')})$, $g^{(1)} \in \Lambda^1(G^{(1)})$ and $h^{(1)} \in \Lambda^1(H^{(1)})$. As usual, $D_a(E, F, G, H)$ is independent of these choices.

The following statement gives a better feeling of what is actually measured by this double ratio.

LEMMA 1.6. For a generic flag quadruple (E, F, G, H) , consider the decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n L_a$, where $L_a = E^{(a)} \cap F^{(n-a+1)}$. For arbitrary non-zero vectors $g \in G^{(1)}$ and $h \in H^{(1)}$, let $g_a, h_a \in L_a$ be the respective projections of g and h to the line L_a parallel to the other lines L_b with $b \neq a$. Then

$$D_a(E, F, G, H) = -\frac{g_{a+1} h_a}{h_{a+1} g_a},$$

where the ratios $g_b/h_b \in \mathbb{R}$ are measured in the lines L_b .

Proof. This results from a simple computation. \square

When $n=2$, $-D_1(E, F, G, H)$ is the familiar cross-ratio of the lines $E^{(1)}, F^{(1)}, G^{(1)}, H^{(1)}$ in \mathbb{R}^2 . See [18, §9.5] for a reduction of any double ratio invariant $D_a(E, F, G, H)$ to this 2-dimensional case, by consideration of the quotient space

$$\frac{\mathbb{R}^n}{E^{(a-1)} + F^{(n-a-1)}} \cong \mathbb{R}^2.$$

Note that $D_a(E, F, G, H)$ does not really depend on the whole flags G and H , but only on the lines $G^{(1)}$ and $H^{(1)}$. The following elementary properties indicate how it behaves under transposition of E and F , or of G and H .

LEMMA 1.7.

$$\begin{aligned} D_a(E, F, H, G) &= D_a(E, F, G, H)^{-1} \\ D_a(F, E, G, H) &= D_{n-a}(E, F, G, H)^{-1}, \\ D_a(E, F, G, K) &= -D_a(E, F, G, H)D_a(E, F, H, K). \end{aligned}$$

Proof. Again, the proof is elementary. □

The minus sign in the definition of $D_a(E, F, G, H)$ is justified by the positivity property of the next section, and in particular by Proposition 1.8.

1.5. Positivity

An ordered family of flags $(E_1, E_2, \dots, E_m) \in \text{Flag}(\mathbb{R}^n)^m$ is *positive* if both the following conditions hold:

- (1) for every distinct i, j and k , and for every $a, b, c \geq 1$ with $a+b+c=n$, the triple ratio $T_{abc}(E_i, E_j, E_k)$ is positive;
- (2) for every distinct i, j, k and l with $i < k < j < l$ or $k < i < l < j$, and for every $1 \leq a \leq n-1$, the double ratio $D_a(E_i, E_j, E_k, E_l)$ is positive.

When $n=2$, the first condition is irrelevant. The second condition just means that, up to reflection, the flags $E_1, E_2, \dots, E_m \in \text{Flag}(\mathbb{R}^2) = \mathbb{R}\mathbb{P}^1$ occur in this order around the projective line $\mathbb{R}\mathbb{P}^1$.

Fock and Goncharov [18, §5] give a much more conceptual definition of positivity, building on earlier work of Lusztig [31], [32]. In particular, they prove the following result.

PROPOSITION 1.8. ([18]) *If the flag m -tuple (E_1, E_2, \dots, E_m) is positive, any flag m -tuple obtained by dihedral permutation of the E_i is also positive.*

Recall that a *dihedral permutation* is either a cyclic permutation, or the composition of the order reversal $(E_1, E_2, \dots, E_m) \mapsto (E_m, E_{m-1}, \dots, E_1)$ with a cyclic permutation.

2. Geodesic laminations

Geodesic laminations are a now very classical tool in 2-dimensional topology and geometry. They occur in many different contexts, for instance when one takes limits of

sequences of simple closed curves. We state here a few basic definitions and facts, and refer to [39], [14], [36], [8] for proofs and background.

To define geodesic laminations, one first chooses a metric m of negative curvature on the surface S .

An m -geodesic lamination is a closed subset $\lambda \subset S$ that can be decomposed as a disjoint union of simple complete m -geodesics, called its *leaves*. Recall that a geodesic is *complete* if it cannot be extended to a longer geodesic, and it is *simple* if it has no transverse self-intersection point. The leaves of a geodesic lamination can be closed or bi-infinite. A geodesic lamination can have finitely many leaves (as in the case considered in [9]), or uncountably many leaves.

An m -geodesic lamination has measure zero, and in fact Hausdorff dimension 1 [1], and its decomposition as a union of leaves is unique. The complement $S \setminus \lambda$ of an m -geodesic lamination λ is a surface of finite topological type, bounded by finitely many leaves of λ . The completion of $S \setminus \lambda$ for the path metric induced by m is a finite area surface with geodesic boundary; it is the union of a compact part and of finitely many spikes homeomorphic to $[0, 1] \times [0, \infty[$, where $\{0, 1\} \times [0, \infty[$ is contained in two leaves of λ . The width of these spikes decreases exponentially in the sense that the parametrization by $[0, 1] \times [0, \infty[$ can be chosen so that its restriction to each $\{x\} \times [0, \infty[$ has speed 1, and so that the length of each arc $[0, 1] \times \{t\}$ decreases exponentially with t .

Because the leaves of λ are disjoint, every point of S has a neighborhood U homeomorphic to $[0, 1] \times [0, 1]$ for which the intersection $U \cap \lambda$ corresponds to $K \times [0, 1]$ for some totally disconnected compact subset $K \subset [0, 1]$; beware that, in general, the homeomorphism cannot be made differentiable, only Hölder bicontinuous.

We will make heavy use of *transverse arcs* for λ . These are arcs differentially immersed in S that are transverse to the leaves of λ . In addition, we require that the endpoints of such a transverse arc be disjoint from λ .

The notion of geodesic lamination is independent of the choice of the negatively curved metric m in the sense that, if m' is another negatively curved metric on S , there is a natural one-to-one correspondence between m -geodesic laminations and m' -geodesic laminations.

A geodesic lamination λ is *maximal* if it is not contained in any other geodesic lamination. This is equivalent to the property that each component of its complement $S \setminus \lambda$ is a triangle, bounded by three infinite leaves of λ and containing three spikes of $S \setminus \lambda$. If the surface S has genus g , an Euler characteristic argument shows that the number of triangle components of the complement $S \setminus \lambda$ of a maximal geodesic lamination is equal to $4(g-1)$.

Every geodesic lamination is contained in a maximal geodesic lamination.

We can think of maximal geodesic laminations as some kind of triangulations of the surface S , where the edges are geodesic and where the vertices have been pushed to infinity. This point of view explains why maximal geodesic laminations are powerful tools for many problems, such as the ones considered in the current article.

3. Triangle invariants

Let $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin homomorphism. We will use a maximal geodesic lamination λ to construct invariants of the corresponding character $\varrho \in \mathrm{Hit}_n(S)$.

3.1. The flag curve

The key to the definition of these invariants is the following construction of Labourie [27].

Let T^1S and $T^1\tilde{S}$ be the unit tangent bundles of the surface S and of its universal cover \tilde{S} , respectively. For convenience, lift the homomorphism $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ to a homomorphism $\varrho': \pi_1(S) \rightarrow \mathrm{SL}_n(\mathbb{R})$. The fact that such a lift exists is classical when $n=2$, and therefore also holds when $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ comes from a discrete representation $\pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$; the existence of the lift in the general case follows by connectedness of the Hitchin component $\mathrm{Hit}_n(S)$, and by homotopy invariance of the obstruction to lift. We can then consider the twisted product

$$T^1S \times_{\varrho'} \mathbb{R}^n = \frac{T^1\tilde{S} \times \mathbb{R}^n}{\pi_1(S)},$$

where the fundamental group $\pi_1(S)$ acts on $T^1\tilde{S}$ by its usual action on the universal cover \tilde{S} , and acts on \mathbb{R}^n by ϱ' . The natural projection $T^1S \times_{\varrho'} \mathbb{R}^n \rightarrow T^1S$ presents $T^1S \times_{\varrho'} \mathbb{R}^n$ as a vector bundle over T^1S with fiber \mathbb{R}^n .

Endow the surface S with an arbitrary metric of negative curvature. This defines a circle at infinity $\partial_\infty\tilde{S}$ for the universal cover \tilde{S} , and a geodesic flow on the unit tangent bundle T^1S . It is well known (see for instance [20], [11] and [19]) that these objects are actually independent of the choice of the negatively curved metric, at least if we do not care about the actual parametrization of the geodesic flow (which is the case here).

The geodesic flow $(g_t)_{t \in \mathbb{R}}$ of T^1S has a natural flat lift to a flow $(G_t)_{t \in \mathbb{R}}$ on the total space $T^1S \times_{\varrho'} \mathbb{R}^n$. The flatness property here just means that the flow $(G_t)_{t \in \mathbb{R}}$ is the projection of the flow $(\tilde{G}_t)_{t \in \mathbb{R}}$ on $T^1\tilde{S} \times \mathbb{R}^n$ that acts by the geodesic flow $(\tilde{g}_t)_{t \in \mathbb{R}}$ of $T^1\tilde{S}$ on the first factor, and by the identity $\mathrm{Id}_{\mathbb{R}^n}$ on the second factor.

Endow each fiber of the vector bundle $T^1S \times_{\varrho'} \mathbb{R}^n \rightarrow T^1S$ with a norm $\|\cdot\|$ depending continuously on the corresponding point of T^1S .

THEOREM 3.1. (Labourie [27, Theorem 4.1]) *If $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ is a Hitchin homomorphism, the vector bundle $T^1S \times_{\varrho'} \mathbb{R}^n \rightarrow T^1S$ admits a unique decomposition as a direct sum $L_1 \oplus L_2 \oplus \dots \oplus L_n$ of n line sub-bundles $L_a \rightarrow T^1S$ such that*

- (1) *each line bundle L_a is invariant under the lift $(G_t)_{t \in \mathbb{R}}$ of the geodesic flow;*
- (2) *for every $a > b$, there exist constants $A_{ab}, B_{ab} > 0$ such that, for every $v_a \in L_a$ and $v_b \in L_b$ in the same fiber of $T^1S \times_{\varrho'} \mathbb{R}^n$ and for every $t \geq 0$,*

$$\frac{\|G_t(v_b)\|}{\|v_b\|} \leq A_{ab} \frac{\|G_t(v_a)\|}{\|v_a\|} e^{-B_{ab}t}.$$

The second property is clearly independent of the choice of the norm $\|\cdot\|$. It is referred to as the *Anosov property* of the Hitchin homomorphism ϱ . This relative property does not say anything about whether the flow $(G_t)_{t \in \mathbb{R}^n}$ expands or contracts the fibers of any individual sub-bundle L_a , but states that, when $a < b$, the flow $(G_t)_{t \in \mathbb{R}^n}$ contracts the fibers of L_b much more than those of L_a . Writing this in a more intrinsic way, this means that $(G_t)_{t \in \mathbb{R}^n}$ induces on the line bundle $\mathrm{Hom}(L_a, L_b)$ a flow that is uniformly contracting when $a > b$.

Lift the sub-bundles L_a of $T^1S \times_{\varrho'} \mathbb{R}^n = (T^1\tilde{S} \times \mathbb{R}^n) / \pi_1(S)$ to sub-bundles \tilde{L}_a of $T^1\tilde{S} \times \mathbb{R}^n$. Because the line sub-bundles L_a are invariant under the lift $(G_t)_{t \in \mathbb{R}}$ of the geodesic flow, the fiber of \tilde{L}_a over $\tilde{x} \in \tilde{S}$ is of the form $\{\tilde{x}\} \times \tilde{L}_a(g)$ for some line $\tilde{L}_a(g) \subset \mathbb{R}^n$ depending only on the orbit g of \tilde{x} for the geodesic flow of $T^1\tilde{S}$.

The line $\tilde{L}_a(g) \subset \mathbb{R}^n$ depends on the orbit g of the geodesic flow of $T^1\tilde{S}$ or, equivalently, on the corresponding oriented geodesic g of \tilde{S} . The Anosov property has the following relatively easy consequence. Define a flag $E(g) \in \mathrm{Flag}(\mathbb{R}^n)$ by the property that $E(g)^{(a)} = \tilde{L}_1(g) \oplus \tilde{L}_2(g) \oplus \dots \oplus \tilde{L}_a(g)$; then $E(g)$ depends only on the positive endpoint of g . More precisely, we have the following result.

PROPOSITION 3.2. (Labourie [27, Theorem 4.1]) *For a Hitchin homomorphism*

$$\varrho: \pi_1(S) \longrightarrow \mathrm{PSL}_n(\mathbb{R}),$$

there exists a unique map $\mathcal{F}_\varrho: \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ such that

- (1) *\mathcal{F}_ϱ is Hölder continuous for the visual metric of $\partial_\infty \tilde{S}$ and for an arbitrary riemannian metric on $\mathrm{Flag}(\mathbb{R}^n)$;*
- (2) *for every oriented geodesic g of \tilde{S} with positive endpoint $\tilde{x}_+ \in \partial_\infty \tilde{S}$, the image $\mathcal{F}_\varrho(\tilde{x}_+)$ is equal to the flag $E(g)$ defined above.*

In addition, \mathcal{F}_ϱ is ϱ -equivariant in the sense that $\mathcal{F}_\varrho(\gamma \tilde{x}) = \varrho(\gamma)(\mathcal{F}_\varrho(\tilde{x}))$ for every $\tilde{x} \in \partial_\infty \tilde{S}$ and $\gamma \in \pi_1(S)$.

By definition, this map $\mathcal{F}_\varrho: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ is the *flag curve* of the Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$. It is independent of the choice of the lift $\varrho': \pi_1(S) \rightarrow \text{SL}_n(\mathbb{R})$ of $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, and of the negatively curved metric on S used to define the geodesic flow of the unit tangent bundle T^1S .

Note that the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g)$, associated by Theorem 3.1 with an oriented geodesic g of \tilde{S} , is easily recovered from the flag curve $\mathcal{F}_\varrho: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ by the property that $\tilde{L}_a(g) = \mathcal{F}_\varrho(\tilde{x}_+)^{(a)} \cap \mathcal{F}_\varrho(\tilde{x}_-)^{(n-a+1)}$ for the positive and negative endpoints $\tilde{x}_+, \tilde{x}_- \in \partial_\infty \tilde{S}$ of g . The main content of Proposition 3.2 is that, for two oriented geodesics g and g' of \tilde{S} that share the same positive endpoint $\tilde{x}_+ \in \partial_\infty \tilde{S}$, the corresponding line decompositions $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g)$ and $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g')$ may be different but define the same linear subspaces $\mathcal{F}_\varrho(\tilde{x}_+)^{(a)} = \bigoplus_{b=1}^a \tilde{L}_b(g) = \bigoplus_{b=1}^a \tilde{L}_b(g')$ for every $a=1, 2, \dots, n$.

See also [21] for a nice exposition of Theorem 3.1 and Proposition 3.2.

The flag curve \mathcal{F}_ϱ has the following important positivity property.

THEOREM 3.3. (Fock–Goncharov [18, Theorem 1.15]) *For every finite set of distinct points $x_1, x_2, \dots, x_k \in \partial_\infty \tilde{S}$ occurring in this order on the circle at infinity $\partial_\infty \tilde{S}$, the flag k -tuple $(\mathcal{F}_\varrho(x_1), \mathcal{F}_\varrho(x_2), \dots, \mathcal{F}_\varrho(x_k))$ is positive in the sense of §1.5.*

3.2. Triangle invariants of Hitchin characters

We now define a first set of invariants for the Hitchin character represented by a homomorphism $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$.

The complement of the maximal geodesic lamination λ consists of finitely many infinite triangles T_1, T_2, \dots, T_m , each with three spikes.

Consider such a triangle component T of $S \setminus \lambda$, and select one of its spikes s . Lift T to an ideal triangle \tilde{T} in the universal cover \tilde{S} , and let \tilde{s} be the spike of \tilde{T} corresponding to s . The spike \tilde{s} uniquely determines a point of the circle at infinity $\partial_\infty \tilde{S}$, which we will also denote by \tilde{s} .

Label the spikes of T as s, s' and s'' in counterclockwise order around T , and let $\tilde{s}, \tilde{s}', \tilde{s}'' \in \partial_\infty \tilde{S}$ be the corresponding points of the circle at infinity. The flag triple $(\mathcal{F}_\varrho(\tilde{s}), \mathcal{F}_\varrho(\tilde{s}'), \mathcal{F}_\varrho(\tilde{s}''))$, associated with $\tilde{s}, \tilde{s}', \tilde{s}'' \in \partial_\infty \tilde{S}$ by the flag curve

$$\mathcal{F}_\varrho: \partial_\infty \tilde{S} \longrightarrow \text{Flag}(\mathbb{R}^n),$$

is positive by Theorem 3.3. We can therefore consider the logarithms

$$\tau_{abc}^\varrho(s) = \log T_{abc}(\mathcal{F}_\varrho(\tilde{s}), \mathcal{F}_\varrho(\tilde{s}'), \mathcal{F}_\varrho(\tilde{s}''))$$

of its triple ratios, defined for every $a, b, c \geq 1$ with $a+b+c=n$. By ϱ -equivariance of the flag curve \mathcal{F}_ϱ , these triple ratio logarithms depend only on the triangle T and on the spike s of T , and not on the choice of the lift \tilde{T} .

The following statement indicates how the invariant $\tau_{abc}^\varrho(s) \in \mathbb{R}$ changes if we choose a different vertex of the triangle T .

LEMMA 3.4. *If s, s' and s'' are the three spikes of the component T of $S \setminus \lambda$, indexed counterclockwise around T , then*

$$\tau_{abc}^\varrho(s) = \tau_{bca}^\varrho(s') = \tau_{cab}^\varrho(s'').$$

Proof. This is an immediate consequence of Lemma 1.1. □

By invariance of triple ratios under the action of $\mathrm{PGL}_n(\mathbb{R})$ on $\mathrm{Flag}(\mathbb{R}^n)$, it is immediate that the triangle invariants $\tau_{abc}^\varrho(s)$ depend only on the character $\varrho \in \mathrm{Hit}_n(S)$, and not on the homomorphism $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ representing it.

Because of Lemma 3.4, we can think of the invariant $\tau_{abc}^\varrho(s)$ as mainly associated with the triangle component T of $S \setminus \lambda$ that has the spike s as a vertex, since choosing a different vertex of T only affects the order in which the indices a, b and c are considered. For this reason, we will refer to the $\tau_{abc}^\varrho(s)$ as the *triangle invariants* of the Hitchin character $\varrho \in \mathrm{Hit}_n(S)$.

Remark 3.5. The companion article [9] uses a clockwise labeling convention for the vertices of a triangle. As a consequence, the triangle invariants of [9] are the opposite of those introduced here.

4. Tangent cycles for a geodesic lamination

The second type of invariants associated with a Hitchin character $\varrho \in \mathrm{Hit}_n(S)$ are more closely tied to the maximal geodesic lamination λ , and have a homological flavor. This section is devoted to the definitions and basic properties of the corresponding objects.

Because of the scope of this article, we are restricting attention to maximal geodesic laminations. However, much of the discussion can be adapted to general geodesic laminations.

4.1. Tangent cycles

Let $\hat{\lambda}$ be the orientation cover of the geodesic lamination λ , consisting of all pairs (x, o) , where $x \in \lambda$ and o is a continuous orientation of the leaves of λ near x . The map $(x, o) \mapsto x$ defines a 2-fold covering map $\hat{\lambda} \rightarrow \lambda$.

Intuitively, a tangent cycle for $\hat{\lambda}$ is a certain real-valued local weight for the leaves of $\hat{\lambda}$, and defines a 1-dimensional de Rham current supported in $\hat{\lambda}$ as in [38]. Tangent cycles were called “transverse cocycles” in [7] and in subsequent papers, with the discrepancy

being explained by Poincaré duality. The change in terminology is motivated by the relative tangent cycles that will be introduced in §4.4.

Let U be a neighborhood of the geodesic lamination λ in S . If U is small enough that it avoids at least one point of each component of $S \setminus \lambda$, the cover $\hat{\lambda} \rightarrow \lambda$ extends to a 2-fold cover $\hat{U} \rightarrow U$ for some surface \hat{U} .

A *tangent cycle* α for the geodesic lamination $\hat{\lambda}$ is the assignment of a number $\alpha(k) \in \mathbb{R}$ to each arc $k \subset \hat{U}$ transverse to $\hat{\lambda}$ such that the the following conditions hold:

(1) α is *finitely additive*, in the sense that $\alpha(k) = \alpha(k_1) + \alpha(k_2)$ whenever the arc k is split into two transverse arcs k_1 and k_2 ;

(2) α is *invariant under homotopy respecting $\hat{\lambda}$* , in the sense that $\alpha(k) = \alpha(k')$ whenever the transverse arcs k and k' are homotopic by a homotopy respecting $\hat{\lambda}$.

To be more precise in the definitions, a *transverse arc* is a closed differentiable arc embedded in the surface \hat{U} , whose boundary is disjoint from $\hat{\lambda}$ and whose interior has transverse intersection with the leaves of $\hat{\lambda}$ that it meets. Such an arc is not assumed to be oriented. Two transverse arcs k and k' are *homotopic respecting $\hat{\lambda}$* if there is a continuous homotopy $H: [0, 1] \times [0, 1] \rightarrow \hat{U}$, restricting to homeomorphisms $[0, 1] \times \{0\} \rightarrow k$ and $[0, 1] \times \{1\} \rightarrow k'$, such that $H^{-1}(\hat{\lambda}) = K \times [0, 1]$ for some closed subset $K \subset [0, 1]$. (Experts will know that in general H cannot be chosen differentiable, but this subtlety is irrelevant here.)

It easily follows from the above two conditions that $\alpha(k) = 0$ for every arc k disjoint from $\hat{\lambda}$. As a consequence, the notion of tangent cycle is independent of the choice of the neighborhood U .

A well-known example of tangent cycle are *transverse measures* for $\hat{\lambda}$. These can be defined as tangent cycles μ such that $\mu(k) \geq 0$ for every transverse arc k . Indeed, this positivity property enhances the finite additivity condition (1) to countable additivity, so that a positive tangent cycle defines on each transverse arc k a measure with support $k \cap \hat{\lambda}$ that is invariant under homotopy respecting $\hat{\lambda}$. See, for instance, [7, Proposition 18].

4.2. Train-track neighborhoods

To determine the space of tangent cycles for the geodesic lamination λ , we will use a very specific type of neighborhood U for λ . Train tracks and train-track neighborhoods are standard tools to work with geodesic laminations. See, for instance, [39] and [36] for an introduction to these notions, with the warning that many slight variations in conventions and terminology occur in the existing literature.

A (trivalent) *train-track neighborhood* for the geodesic lamination λ is a closed neigh-

neighborhood U of λ which can be decomposed as a union of finitely many embedded rectangles R_i such that the following conditions are satisfied:

- (1) the boundary of each rectangle $R_i \cong [0, 1] \times [0, 1]$ is divided into a *horizontal boundary* $\partial_h R_i = [0, 1] \times \{0, 1\}$ and a *vertical boundary* $\partial_v R_i = \{0, 1\} \times [0, 1]$;
- (2) each component of the intersection $R_i \cap R_j$ of two distinct rectangles R_i and R_j is, up to swapping i and j , a component of $\partial_v R_i$ contained in $\partial_v R_j$ and containing one of the endpoints of $\partial_v R_j$;
- (3) each of the four endpoints of $\partial_v R_i$ is contained in some rectangle R_j different from R_i ;
- (4) the leaves of λ are transverse to the arcs $\{x\} \times [0, 1]$ in each rectangle $R_i \cong [0, 1] \times [0, 1]$;
- (5) a fifth condition indicated below is satisfied.

By construction, the boundary ∂U of the train-track neighborhood U naturally splits into two pieces. The *horizontal boundary* $\partial_h U$ is the union of the horizontal boundaries $\partial_h R_i$ of all rectangles R_i . The *vertical boundary* consists of those points of ∂U that are contained in the vertical boundary $\partial_v R_i$ of some rectangle R_i .

We can now state the missing condition.

- (5) no component of $S \setminus U$ is a disk with 0, 1 or 2 components of the vertical boundary $\partial_v U$ in its closure.

In particular, the arcs $\{x\} \times [0, 1]$ of each rectangle $R_i \cong [0, 1] \times [0, 1]$ provide a foliation of U , whose leaves are called the *ties* of the train-track neighborhood. A tie is *generic* if it meets the boundary of U only at its endpoints. Otherwise, it is *singular*.

The origin of the train track terminology should become apparent when U is chosen so that its ties are relatively short. See Figure 2. In particular, a singular tie is also often called a *switch*, and the rectangles R_i are the *branches* of U . The branches of the train-track neighborhood U correspond to the edges of the (1-dimensional) train track associated with U , which explains our frequent use of the letter e to denote a branch.

The definitions are such that a singular tie t is adjacent to three branches R_i , R_j and R_k , in such a way that t is equal to a component of the vertical boundary $\partial_v R_i$, and is also the union of a component of $\partial_v R_j$, of a component of $\partial_v R_k$ and of a component of $\partial_v U$. The rectangles R_j and R_k are not necessarily distinct.

Every geodesic lamination admits a train-track neighborhood; see for instance [39] and [36].

The above definitions make sense for any geodesic lamination. When the geodesic lamination λ is maximal, there is a crucial property of its train-track neighborhoods U that we will use on a regular basis.

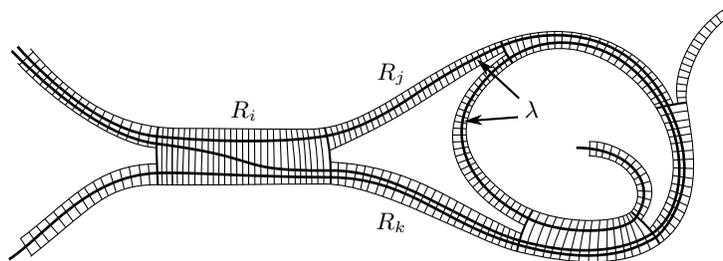


Figure 2. A train-track neighborhood.

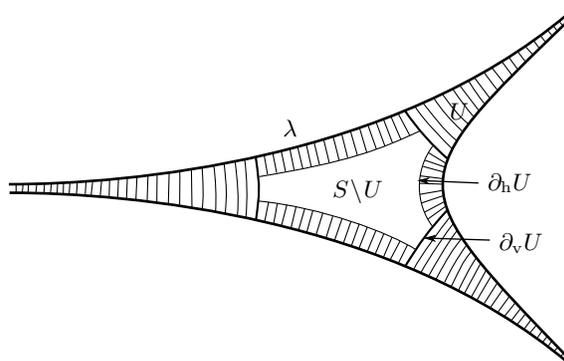


Figure 3. Train-track neighborhoods and maximal geodesic laminations.

PROPOSITION 4.1. *Let U be a train-track neighborhood of the maximal geodesic lamination λ . Then, every component T of the complement $S \setminus \lambda$ contains exactly one component $H = T \setminus U$ of $S \setminus U$; this component H is a hexagon, namely a disk whose boundary is the union of three components of the horizontal boundary $\partial_h U$ and three components of the vertical boundary $\partial_v U$. In addition, the foliation of $T \cap U$ by the ties of U is as indicated in Figure 3.*

Proof. Since λ is assumed to be maximal, its complement consists of infinite triangles. The property is then easily proved by extending the foliation of U by its ties to a foliation of S with saddle-type singularities, and by using an index computation on each component of the complement $S \setminus \lambda$. \square

Incidentally, another index argument applied to the whole surface S shows that the complement $S \setminus U$ consists of $4(g-1)$ hexagons. In particular, this proves that the complement $S \setminus \lambda$ consists of $4(g-1)$ triangles. It also follows that the train-track neighborhood U has $12(g-1)$ switches and $18(g-1)$ branches.

4.3. Homological interpretation of tangent cycles

Train-track neighborhoods provide a convenient tool to perform computations in the vector space $\mathcal{C}(\hat{\lambda}; \mathbb{R})$ consisting of all tangent cycles for the orientation cover $\hat{\lambda}$ of λ .

Let U be a train-track neighborhood of the maximal geodesic lamination λ . Using Proposition 4.1, the orientation cover map $\hat{\lambda} \rightarrow \lambda$ has a unique extension to a cover $\hat{U} \rightarrow U$. Note that $\hat{\lambda}$ is a geodesic lamination in the surface \hat{U} , and that \hat{U} is a train-track neighborhood of $\hat{\lambda}$. Also, each component of $\hat{U} \setminus \hat{\lambda}$ is an annulus bounded on one side by a chain of six leaves of $\hat{\lambda}$, and on the other side by a dodecagon made up of six components of the horizontal boundary $\partial_h \hat{U}$ and six components of the vertical boundary $\partial_v \hat{U}$.

The leaves of the orientation cover $\hat{\lambda}$ are canonically oriented (use the orientation o near the point $(x, o) \in \hat{\lambda}$). This enables us to orient each tie k of \hat{U} from left to right with respect to this canonical orientation of the leaves of $\hat{\lambda}$. Indeed, Proposition 4.1 guarantees that, for each component d of $k \setminus \hat{\lambda}$, the left-to-right orientations of k at the endpoints of d coincide; as a consequence, all leaves of $\hat{\lambda}$ meeting k induce the same left-to-right orientation for k .

PROPOSITION 4.2. *A tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}; \mathbb{R})$ uniquely determines a homology class $[\alpha] \in H_1(\hat{U}; \mathbb{R})$ by the property that*

$$\alpha(k) = [k] \cdot [\alpha]$$

for every generic tie k of the train-track neighborhood \hat{U} , where $[k] \cdot [\alpha]$ is the algebraic intersection number of $[\alpha] \in H_1(\hat{U}; \mathbb{R})$ with the relative homology class $[k] \in H_1(\hat{U}, \partial \hat{U}; \mathbb{R})$ defined by the tie k , endowed with the above left-to-right orientation.

In addition, the rule $\alpha \mapsto [\alpha]$ defines a linear isomorphism $\mathcal{Z}(\hat{\lambda}; \mathbb{R}) \rightarrow H_1(\hat{U}; \mathbb{R})$.

Proof. Because the geodesic lamination λ is maximal, Proposition 4.1 shows that it is tightly carried by the train track U , in the sense that each component of $U \setminus \lambda$ is an annulus. It follows that $\hat{\lambda}$ is tightly carried by \hat{U} . The result is then a consequence of [7, Theorem 11]. □

LEMMA 4.3. *If the surface S has genus g ,*

$$\mathcal{C}(\hat{\lambda}; \mathbb{R}) \cong H_1(\hat{U}; \mathbb{R}) \cong \mathbb{R}^{12g-11}.$$

Proof. Since the complement $S \setminus \lambda$ consists of infinite triangles, the geodesic lamination λ is non-orientable. This implies that $\hat{\lambda}$ is connected, and therefore so is \hat{U} . By definition of the Euler characteristic χ ,

$$\dim H_1(\hat{U}; \mathbb{R}) = -\chi(\hat{U}) + \dim H_0(\hat{U}; \mathbb{R}) = -\chi(\hat{U}) + 1 = -2\chi(U) + 1.$$

We observed that the complement of U in S consists of $4(g-1)$ hexagons. Therefore, $\chi(U) = \chi(S) - 4(g-1) = -6(g-1)$. The result follows. □

4.4. Tangent cycles relative to the slits

We now relax the additivity condition for a tangent cycle.

Let U be a neighborhood of λ in S that avoids at least one point of each component of $S \setminus \lambda$. For instance, U can be a train-track neighborhood of λ . Extend the orientation cover $\hat{\lambda} \rightarrow \lambda$ to a 2-fold cover $\hat{U} \rightarrow U$. The complement $\hat{U} \setminus \hat{\lambda}$ has a certain number of infinite spikes, in fact $24(g-1)$ spikes because the complement $S \setminus \lambda$ consists of $4(g-1)$ infinite triangles and because each spike of $S \setminus \lambda$ lifts to two spikes of $\hat{U} \setminus \hat{\lambda}$. In particular, the spikes of $\hat{U} \setminus \hat{\lambda}$ are really independent of the choice of the neighborhood U . For this reason, we will also refer to the spikes of $\hat{U} \setminus \hat{\lambda}$ as the *slits* of $\hat{\lambda}$.

We need to restrict our attention to a special class of transverse arcs for $\hat{\lambda}$. An arc $k \subset \hat{U}$ is *tightly transverse* to the geodesic lamination $\hat{\lambda}$ if it is transverse to $\hat{\lambda}$, if it has non-empty intersection with $\hat{\lambda}$ and if, for every component d of $k \setminus \hat{\lambda}$, one of the following holds:

- d contains one of the endpoints of k ;
- d separates one of the spikes of $\hat{U} \setminus \hat{\lambda}$ from the rest of $\hat{U} \setminus \hat{\lambda}$.

A fundamental example arises when U is a train-track neighborhood of the maximal geodesic lamination λ , so that its lift \hat{U} is a train-track neighborhood of $\hat{\lambda}$. It then follows from Proposition 4.1 that every tie of \hat{U} is tightly transverse to $\hat{\lambda}$.

The slits of $\hat{\lambda}$, namely the spikes of $\hat{U} \setminus \hat{\lambda}$, come in two types because of the canonical orientation of the leaves of the orientation cover $\hat{\lambda}$: the *positive slits* s where the two leaves of $\hat{\lambda}$ that are adjacent to s are oriented towards s for the canonical orientation of $\hat{\lambda}$, and the *negative slits* where these two leaves are oriented away from s . Define the *sign* ε of the slit s of $\hat{U} \setminus \hat{\lambda}$ by

$$\varepsilon(s) = \begin{cases} +1, & \text{if } s \text{ is positive,} \\ -1, & \text{if } s \text{ is negative.} \end{cases}$$

For mnemonic purposes, it may be useful to observe that our positive/negative terminology is compatible with the boundary orientation coming from orientation of the leaves of $\hat{\lambda}$. A positive slit corresponds to the positive endpoints of the two leaves of $\hat{\lambda}$ that are adjacent to it, whereas a negative slit corresponds to their negative endpoints.

An \mathbb{R} -valued *tangent cycle relative to the slits* for $\hat{\lambda}$ assigns a number $\alpha(k) \in \mathbb{R}$ to each arc $k \subset \hat{U}$ tightly transverse to $\hat{\lambda}$ in such a way that

- α is, as before, *invariant under homotopy respecting $\hat{\lambda}$* in the sense that $\alpha(k) = \alpha(k')$ whenever the transverse arcs k and k' are homotopic by a homotopy that keeps each point of $k \cap \hat{\lambda}$ in the same leaf of $\hat{\lambda}$;
- α is *quasi-additive* in the following sense: there is a number $\partial\alpha(s) \in \mathbb{R}$ associated

with each slit s of $\hat{\lambda}$ such that

$$\alpha(k) = \alpha(k_1) + \alpha(k_2) - \varepsilon(s)\partial\alpha(s)$$

whenever the arc $k \subset \widehat{U}$ is tightly transverse to $\hat{\lambda}$, the arcs k_1 and k_2 are obtained by splitting k at a point $x \in k \setminus \hat{\lambda}$ contained in a component d of $k \setminus \hat{\lambda}$ that is disjoint from ∂k , s is the spike separated from the rest of $\widehat{U} \setminus \hat{\lambda}$ by the component d , and $\varepsilon(s) = \pm 1$ is its sign as defined above.

The function $\partial\alpha: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ is the *boundary* of the relative cycle α . We could have combined $\partial\alpha$ with the sign ε to create a single function $\{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$, but the current convention simplifies the homological interpretation of relative tangent cycles that is given in Proposition 4.5 below. This homological interpretation will also explain the boundary terminology.

We let $\mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ denote the space of tangent cycles relative to the slits for $\hat{\lambda}$.

Using the quasi-additivity property, one easily shows that the notion of tangent cycle relative to the slits is independent of the choice of the neighborhood U of λ .

These relative tangent cycles generalize the tangent cycles of §4.1, as indicated by the following statement.

LEMMA 4.4. *There is a natural correspondence between the set $\mathcal{Z}(\hat{\lambda}; \mathbb{R})$ of tangent cycles for $\hat{\lambda}$ and the set $\{\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) : \partial\alpha = 0\}$ of tangent cycles relative to the slits with boundary zero.*

Proof. A relative tangent cycle with boundary equal to zero is additive. So the only point that requires some discussion is the fact that relative tangent cycles are restricted to arcs tightly transverse to $\hat{\lambda}$, whereas the definition of tangent cycles involves all arcs transverse to $\hat{\lambda}$. In particular, a transverse arc k is allowed to backtrack so that a component d of $k \setminus \hat{\lambda}$ has both endpoints on the same leaf of $\hat{\lambda}$, or to cut a component of $\widehat{U} \setminus \hat{\lambda}$ diagonally so that the endpoints of a component d of $k \setminus \hat{\lambda}$ sit on leaves of $\hat{\lambda}$ that are not asymptotic to each other; both possibilities are forbidden for tightly transverse arcs.

However, every arc k transverse to $\hat{\lambda}$ can be split into the union of finitely many arcs k_1, k_2, \dots, k_l that are tightly transverse to $\hat{\lambda}$. It easily follows that every relative tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ with $\partial\alpha = 0$ uniquely extends to a tangent cycle, by the property that $\alpha(k) = \sum_{i=1}^l \alpha(k_i)$ for every transverse arc k split as above into finitely many tightly transverse arcs k_i . Indeed, the additivity property guarantees that this $\alpha(k)$ does not depend on the decomposition of k into tightly transverse arcs. □

4.5. Homological interpretation of tangent cycles relative to the slits

We now focus on a train-track neighborhood U of the maximal geodesic laminations λ . As before, let $\hat{\lambda}$ be the orientation cover of λ , and extend the covering map $\hat{\lambda} \rightarrow \lambda$ to a cover $\hat{U} \rightarrow U$. The canonical orientation of the leaves of $\hat{\lambda}$ provides a left-to-right orientation for the ties of \hat{U} .

By Proposition 4.1, there is a one-to-one correspondence between the slits of $\hat{\lambda}$ and the components of the vertical boundary $\partial_v \hat{U}$. Indeed, each component c of $\partial_v \hat{U}$ faces a unique spike s of $U \setminus \hat{\lambda}$ (that is, a slit of $\hat{\lambda}$) in the sense that, if k is the singular tie of \hat{U} that contains c and if d is the component of $k \setminus \hat{\lambda}$ that contains c , then d separates s from the rest of $\hat{U} \setminus \hat{\lambda}$; see Figure 3.

For a relative tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$, the boundary $\partial\alpha: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ therefore assigns a real-valued weight to each component of $\partial_v \hat{U}$, and therefore can be interpreted as an element of $H_0(\partial_v \hat{U}; \mathbb{R})$.

PROPOSITION 4.5. *Let U be a train-track neighborhood of the maximal geodesic lamination λ , and let \hat{U} be its lift to a train-track neighborhood of the orientation cover $\hat{\lambda}$. A tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ relative to the slits of $\hat{\lambda}$ uniquely determines a relative homology class $[\alpha] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$ by the property that*

$$\alpha(k) = [k] \cdot [\alpha]$$

for every generic tie k of the train-track neighborhood \hat{U} , where $[k] \cdot [\alpha]$ is the algebraic intersection number of $[\alpha] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$ with the relative homology class

$$[k] \in H_1(\hat{U}, \partial_h \hat{U}; \mathbb{R})$$

defined by the tie k , endowed with the above left-to-right orientation.

In addition, the rule $\alpha \mapsto [\alpha]$ defines a linear isomorphism

$$\mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) \cong H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}),$$

for which the boundary $\partial\alpha: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ of the relative tangent cycle α corresponds to the image of $[\alpha] \in H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$ under the boundary homomorphism

$$\partial: H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R}) \longrightarrow H_0(\partial_v \hat{U}; \mathbb{R}).$$

Proof. We split the proof into a few steps to improve readability.

Step 1. Construct a linear map $\varphi: \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) \rightarrow H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$.

Pick a generic tie k_e in each branch e of \widehat{U} . An easy homological computation (for instance using a Mayer–Vietoris-type argument) shows that, as e ranges over all branches of \widehat{U} , the relative homology classes $[k_e]$ form a basis for $H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$.

If we are given a relative tangent cycle $\alpha \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$, the map $[k_e] \mapsto \alpha(k_e)$ therefore extends to a linear map $H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R}) \rightarrow \mathbb{R}$. By Poincaré–Lefschetz duality and since the boundary $\partial \widehat{U}$ is the union of $\partial_h \widehat{U}$ and $\partial_v \widehat{U}$, there consequently exists a unique class $[\alpha] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$ such that $\alpha(k_e) = [k_e] \cdot [\alpha]$ for every branch e .

An arbitrary generic tie k of \widehat{U} is contained in a branch e . Therefore, $[k] = [k_e]$ in $H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$, and $\alpha(k) = \alpha(k_e)$ by invariance of α under homotopy respecting $\widehat{\lambda}$. This proves that $\alpha(k) = [k] \cdot [\alpha]$ for every generic tie k of \widehat{U} . As a consequence, $[\alpha]$ satisfies the identity indicated in the statement of Proposition 4.5.

This provides a map $\varphi: \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}) \rightarrow H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$, associating the above class $[\alpha] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$ with $\alpha \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$.

Step 2. Construct a linear map $\psi: H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}) \rightarrow \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$.

We first associate a homology class $[k] \in H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$ with each arc k that is tightly transverse to $\widehat{\lambda}$.

A key observation is that the canonical orientation of $\widehat{\lambda}$ specifies a natural orientation for k . Indeed, the definition of tight transversality implies that, if the arc k is tightly transverse to $\widehat{\lambda}$, the canonical orientations of the leaves of $\widehat{\lambda}$ passing through the endpoints of a component d of $k \setminus \widehat{\lambda}$ induce the same left-to-right orientation for k . As a consequence, all leaves of $\widehat{\lambda}$ define the same left-to-right orientation for k . We orient k with this orientation.

We now extend the tightly transverse arc k to an arc $k' \subset \widehat{U}$ with $\partial k' \subset \partial_h \widehat{U}$. There is a natural one-to-one correspondence between the components of the horizontal boundary $\partial_h \widehat{U}$ and the boundary leaves of $\widehat{\lambda}$ (namely those which are in the boundary of $\widehat{U} \setminus \widehat{\lambda}$, counting isolated leaves twice); indeed, Proposition 4.1 shows that all ties originating from a component of $\partial_h \widehat{U}$ leave $\widehat{U} \setminus \widehat{\lambda}$ on the same boundary leaf of $\widehat{\lambda}$. For each component d of $k \setminus \widehat{\lambda}$ containing an endpoint of k , we can extend d to an arc $d' \subset \widehat{U} \setminus \widehat{\lambda}$ going from a boundary leaf l of $\widehat{\lambda}$ to the component of $\partial_h \widehat{U}$ that faces l in the component of $\widehat{U} \setminus \widehat{\lambda}$ containing d , in the homotopy class specified by the arcs in ties of \widehat{U} that connect l to $\partial_h \widehat{U}$. Performing this operation for each of the two components d of $k \setminus \widehat{\lambda}$ that contain an endpoint of k , we have extended k to an oriented arc $k' \supset k$ whose boundary is contained in $\partial_h \widehat{U}$. There are many possible choices for k' but all give the same relative homology class in $H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$, which we denote by $[k]$.

Given a relative homology class $c \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$, we can consider, for every arc k tightly transverse to $\widehat{\lambda}$, the algebraic intersection number

$$\alpha_c(k) = [k] \cdot c \in \mathbb{R}$$

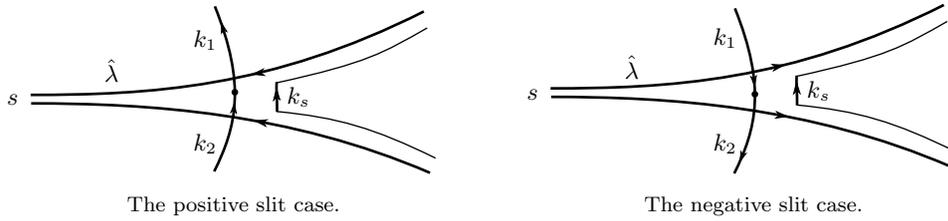


Figure 4. Splitting a tightly transverse arc.

of c with the homology class $[k] \in H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$ associated with k as above. We want to show that this defines a relative tangent cycle $\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$.

The invariance of $\alpha_c(k)$ under homotopy of k respecting $\widehat{\lambda}$ is immediate.

We need to check the quasi-additivity property. Let the arc $k \subset \widehat{U}$ be tightly transverse to $\widehat{\lambda}$, let k_1 and k_2 be obtained by splitting k at a point $x \in k \setminus \widehat{\lambda}$ contained in a component d of $k \setminus \widehat{\lambda}$ that is disjoint from ∂k , and let s be the spike separated from the rest of $\widehat{U} \setminus \widehat{\lambda}$ by the component d . Let k_s be the component of $\partial_v \widehat{U}$ that faces the slit s . Orient k_s by the boundary orientation of $\partial \widehat{U}$.

Then, by the definition of the relative homology classes $[k], [k_1], [k_2] \in H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$, including the left-to-right choice of orientation for k , one has

$$[k] = [k_1] + [k_2] + \varepsilon(s)[k_s] \in H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R}),$$

where $\varepsilon(s) = \pm 1$ is the sign of the slit s . See Figure 4. Taking intersection numbers with $c \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$, it follows that

$$\alpha_c(k) = \alpha_c(k_1) + \alpha_c(k_2) + \varepsilon(s)[k_s] \cdot c.$$

This proves that α_c is a tangent cycle for $\widehat{\lambda}$ relative to its slits, with boundary $\partial \alpha_c$ defined by the property that $\partial \alpha_c(s) = -[k_s] \cdot c$ for every slit s . The minus sign comes from our convention in the definition of quasi-additivity for relative tangent cycles in §4.4.

We define $\psi: H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}) \rightarrow \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$ by the property that $\psi(c) = \alpha_c$ for every $c \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$.

Step 3. For every $c \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$ and every slit s of $\widehat{\lambda}$, $\partial \psi(c)(s) \in \mathbb{R}$ is the weight associated by $\partial c \in H_0(\partial_v \widehat{U}; \mathbb{R})$ with the component k_s of $\partial_v \widehat{U}$ facing s .

This is just a rephrasing of the property that $\partial \alpha_c(s) = -[k_s] \cdot c$.

Step 4. The maps

$$\varphi: \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}) \longrightarrow H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}) \quad \text{and} \quad \psi: H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}) \longrightarrow \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$$

are inverse of each other.

Pick a generic tie k_e in each branch e of \widehat{U} . By construction, the image $c=\varphi(\alpha)$ of $\alpha\in\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$ is defined by the property that $\alpha(k_e)=[k_e]\cdot c$ for every branch e . Conversely, for every $c\in H_1(\widehat{U}, \partial_v\widehat{U}; \mathbb{R})$, $\alpha=\psi(c)$ is characterized by the fact that $\alpha(k)=[k]\cdot c$ for every arc k tightly transverse to $\widehat{\lambda}$.

In particular, $[k_e]\cdot\varphi(\psi(c))=[k_e]\cdot c$ for every branch e , and it follows that $\varphi(\psi(c))=c$ in $H_1(\widehat{U}, \partial_v\widehat{U}; \mathbb{R})$ by Poincaré–Lefschetz duality, since the $[k_e]$ generate $H_1(\widehat{U}, \partial_h\widehat{U}; \mathbb{R})$. This proves that $\varphi\circ\psi$ is equal to the identity.

Conversely, for a relative tangent cycle $\alpha\in\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$, the same argument shows that $\psi(\varphi(\alpha))(k_e)=\alpha(k_e)$ for every branch e of \widehat{U} . For a slit s , let k_s be the component of $\partial_v\widehat{U}$ that faces s , let e be the branch of \widehat{U} that contains k_s , and let e_1 and e_2 be the other two branches that touch k_s . Then, by quasi-additivity of α and $\psi(\varphi(\alpha))\in\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$,

$$\begin{aligned} \varepsilon(s)\partial\alpha(s) &= \alpha(k_{e_1})+\alpha(k_{e_2})-\alpha(k_e) = \psi(\varphi(\alpha))(k_{e_1})+\psi(\varphi(\alpha))(k_{e_2})-\psi(\varphi(\alpha))(k_e) \\ &= \varepsilon(s)\partial\psi(\varphi(\alpha))(s). \end{aligned}$$

This proves that $\psi(\varphi(\alpha))-\alpha$ has boundary zero, and is therefore a tangent cycle by Lemma 4.4. Since $\psi(\varphi(\alpha))(k_e)-\alpha(k_e)=0$ for every branch e of \widehat{U} , it follows from Proposition 4.2 that $\psi(\varphi(\alpha))-\alpha=0$.

This proves that $\psi\circ\varphi$ is the identity, and completes the proof of Proposition 4.5. \square

4.6. Twisted relative tangent cycles

So far, we have considered relative tangent cycles valued in \mathbb{R} . In our analysis of Hitchin characters, we will encounter relative tangent cycles that are valued in \mathbb{R}^{n-1} and behave in a very specific manner with respect to the involution $\iota:\widehat{U}\rightarrow\widehat{U}$ that exchanges the two sheets of the cover $\widehat{U}\rightarrow U$.

More precisely, an \mathbb{R}^{n-1} -valued tangent cycle for $\widehat{\lambda}$ relative to its slits associates a vector $\alpha(k)\in\mathbb{R}^{n-1}$ with each arc k tightly transverse to $\widehat{\lambda}$, in such a way that α is invariant under homotopy respecting $\widehat{\lambda}$ and is quasi-additive with respect to a boundary function $\partial\alpha:\{\text{slits of } \widehat{\lambda}\}\rightarrow\mathbb{R}^{n-1}$.

A twisted tangent cycle for λ relative to its slits and valued in $\widehat{\mathbb{R}}^{n-1}$ is an \mathbb{R}^{n-1} -valued relative tangent cycle α for $\widehat{\lambda}$ such that, for every tightly transverse arc k ,

$$\alpha(\iota(k)) = \overline{\alpha(k)},$$

where $x\mapsto\bar{x}$ is the involution of \mathbb{R}^{n-1} that reverses the order of the coordinates, namely that associates $\bar{x}=(x_{n-1}, x_{n-2}, \dots, x_1)$ to $x=(x_1, x_2, \dots, x_{n-1})\in\mathbb{R}^{n-1}$. Let

$$\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) = \{\alpha\in\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R}^{n-1}) : \alpha(\iota(k)) = \overline{\alpha(k)} \text{ for all } k\}$$

denote the space of these twisted relative tangent cycles.

The terminology and notation is justified by the fact that these twisted relative tangent cycles can be interpreted as tangent cycles for the geodesic lamination λ , relative to the slits of λ , and valued in the twisted coefficient bundle $\widehat{\mathbb{R}}^{n-1} = (\widehat{U} \times \mathbb{R}^{n-1})/\mathbb{Z}_2$ over U , where \mathbb{Z}_2 acts by ι on \widehat{U} and by $x \mapsto \bar{x}$ on \mathbb{R}^{n-1} .

We can similarly define the space of twisted tangent cycles

$$\mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1}) = \{\alpha \in \mathcal{Z}(\hat{\lambda}; \mathbb{R}^{n-1}) : \alpha(\iota(k)) = \overline{\alpha(k)} \text{ for all } k\} = \{\alpha \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) : \partial\alpha = 0\},$$

where the second equality comes from Lemma 4.4.

To compute the dimensions of $\mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1})$ and $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$, we use a version of Propositions 4.2 and 4.5 that gives a homological interpretation of twisted tangent cycles. Because the orientation of arcs is irrelevant for relative tangent cycles but does matter in homology, we need to consider a different coefficient bundle $\widetilde{\mathbb{R}}^{n-1} = (\widehat{U} \times \mathbb{R}^{n-1})/\mathbb{Z}_2$, where \mathbb{Z}_2 still acts by the covering involution ι on \widehat{U} but now acts on \mathbb{R}^{n-1} by $x \mapsto -\bar{x}$.

More precisely, because ι reverses the orientation of $\hat{\lambda}$, the map

$$\varphi: \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) \longrightarrow H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$$

of the proof of Proposition 4.5 conjugates the action of ι on $\mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ to $-\iota_*$, where $\iota_*: H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}) \rightarrow H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$ is the homomorphism induced by ι . Therefore, the tensor product $\varphi \otimes \text{Id}_{\mathbb{R}^{n-1}}$ sends $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ to $\{c \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R}^{n-1}); \iota_*(c) = -\bar{c}\}$, which is naturally identified to $H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1})$. This provides a natural isomorphism $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1})$, which also induces an isomorphism

$$\mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1}) \cong H_1(U; \widetilde{\mathbb{R}}^{n-1}).$$

PROPOSITION 4.6. *The above construction provides natural isomorphisms*

$$\begin{aligned} \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) &\cong H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}), \\ \mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1}) &\cong H_1(U; \widetilde{\mathbb{R}}^{n-1}). \end{aligned}$$

Proof. This follows from the above discussion. □

PROPOSITION 4.7. *The vector spaces $\mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1})$ and $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ have dimensions*

$$\begin{aligned} \dim \mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1}) &= 6(g-1)(n-1) + \lfloor \tfrac{1}{2}(n-1) \rfloor, \\ \dim \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) &= 18(g-1)(n-1), \end{aligned}$$

where $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x .

Proof. Considering Euler characteristics,

$$\chi(U)(n-1) = \dim H_0(U; \widetilde{\mathbb{R}}^{n-1}) - \dim H_1(U; \widetilde{\mathbb{R}}^{n-1}).$$

Since \widehat{U} is connected,

$$H_0(U; \widetilde{\mathbb{R}}^{n-1}) = \{c \in H_0(\widehat{U}; \mathbb{R}^{n-1}) : \iota_*(c) = -\bar{c}\} \cong \{x \in \mathbb{R}^{n-1} : x = -\bar{x}\}$$

has dimension $\lfloor \frac{1}{2}(n-1) \rfloor$. Moreover, because the complement $S \setminus U$ consists of $4(g-1)$ hexagons, $\chi(U) = \chi(S) - 4(g-1) = -6(g-1)$. It follows that

$$\begin{aligned} \dim \mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1}) &= \dim H_1(U; \widetilde{\mathbb{R}}^{n-1}) = -\chi(U)(n-1) + \dim H_0(U; \widetilde{\mathbb{R}}^{n-1}) \\ &= 6(g-1)(n-1) + \lfloor \frac{1}{2}(n-1) \rfloor. \end{aligned}$$

For $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1})$, consider the exact sequence

$$0 \longrightarrow H_1(U; \widetilde{\mathbb{R}}^{n-1}) \longrightarrow H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}) \longrightarrow H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1}) \longrightarrow H_0(U; \widetilde{\mathbb{R}}^{n-1}) \longrightarrow 0.$$

We already observed that $\dim H_0(U; \widetilde{\mathbb{R}}^{n-1}) = \lfloor \frac{1}{2}(n-1) \rfloor$. Since ι respects no component of $\partial_v \widehat{U}$, the twisted homology space $H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1})$ is isomorphic to $H_0(\partial_v U; \mathbb{R}^{n-1})$ and therefore has dimension $12(g-1)(n-1)$, as $\partial_v U$ has $12(g-1)$ components. It follows from the exact sequence above that

$$\begin{aligned} \dim \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) &= \dim H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}) \\ &= \dim H_1(U; \widetilde{\mathbb{R}}^{n-1}) + \dim H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1}) - \dim H_0(U; \widetilde{\mathbb{R}}^{n-1}) \\ &= 18(g-1)(n-1). \end{aligned} \quad \square$$

As suggested by the formula and the fact that U has $18(g-1)$ branches, there is a more elementary (non-homological) computation which deduces the dimension of $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ from Proposition 4.5, but this one is more intrinsic and conceptual.

4.7. Relative tangent cycles from another viewpoint

We give a different description of relative tangent cycles. Compared to the original definition, this presentation does not lend itself as well to the homological interpretation and computations of the previous sections. However, it will be better adapted to the geometric constructions that form the core of this article. It also bypasses the need to consider the orientation cover $\hat{\lambda}$.

In the universal cover \widetilde{S} of S , let \widetilde{U} be the preimage of a train-track neighborhood U of λ .

A relative tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ associates a number $\alpha(T, T') \in \mathbb{R}$ with each ordered pair of distinct components T and T' of $\tilde{S} \setminus \tilde{\lambda}$ as follows. Choose an oriented arc $\tilde{k} \subset \tilde{S}$ that is tightly transverse to $\tilde{\lambda}$ and joins T to T' ; in this preliminary stage, one can for instance take for \tilde{k} any geodesic arc going from T to T' , since every component of $\tilde{S} \setminus \tilde{\lambda}$ is a triangle. Using Proposition 4.1, one can modify \tilde{k} by a homotopy respecting $\tilde{\lambda}$ so that it is contained in \tilde{U} , and is tightly transverse to $\tilde{\lambda}$ in \tilde{U} . Project \tilde{k} to an arc $k \subset U$, which is tightly transverse to λ .

The tightly transverse arc k admits two lifts to the 2-fold cover \hat{U} of U , each oriented so that the canonical orientation of the leaves of the orientation cover $\hat{\lambda}$ points to the left of these arcs at each intersection point. Let $\hat{k} \subset \hat{U}$ be the lift whose orientation projects to the same orientation of k as that of \tilde{k} . By construction, \hat{k} is tightly transverse to $\hat{\lambda}$ in \hat{U} , and we can consider the number $\alpha(\hat{k}) \in \mathbb{R}$ defined by $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$.

In this construction, the arc \tilde{k} is uniquely determined by T and T' up to homotopy respecting $\tilde{\lambda}$ in \tilde{U} , which determines \hat{k} up to homotopy respecting $\hat{\lambda}$ in \hat{U} . It follows that $\alpha(\hat{k})$ depends only on T and T' , and we can define $\alpha(T, T') = \alpha(\hat{k}) \in \mathbb{R}$.

The quasi-additivity property of $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ has a relatively simple translation in this context. Each slit s of λ , namely each spike of $S \setminus \lambda$, lifts to two slits of $\hat{\lambda}$: a positive spike s^+ of $\hat{U} \setminus \hat{\lambda}$ where the leaves of $\hat{\lambda}$ adjacent to s^+ are oriented towards the end of this spike by the canonical orientation of $\hat{\lambda}$, and a negative spike s^- where the adjacent leaves are oriented away from the end of s^- . Define two functions $\partial^+ \alpha, \partial^- \alpha: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}$ by the property that $\partial^+ \alpha(s) = \partial \alpha(s^+)$ and $\partial^- \alpha(s) = \partial \alpha(s^-)$ for every slit s of λ , where $\partial \alpha: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ is the boundary of $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$.

If T, T' and T'' are three components of $\tilde{S} \setminus \tilde{\lambda}$ such that T'' separates T from T' in \tilde{S} , let \tilde{s}'' be the spike of T'' delimited by the two sides of T'' that separate T from T' , and let s'' be the projection of \tilde{s}'' to S . The quasi-additivity of $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ then translates to the property that

$$\alpha(T, T') = \alpha(T, T'') + \alpha(T'', T') - \partial^+ \alpha(s'')$$

if the spike \tilde{s}'' of T'' points to the left as seen from T , and

$$\alpha(T, T') = \alpha(T, T'') + \alpha(T'', T') + \partial^- \alpha(s'')$$

if \tilde{s}'' points to the right as seen from T .

PROPOSITION 4.8. *The above construction provides a one-to-one correspondence between relative tangent cycles $\alpha \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ and maps α associating a number $\alpha(T, T') \in \mathbb{R}$ with each ordered pair of distinct components T and T' of $\tilde{S} \setminus \tilde{\lambda}$ for which there exist two functions $\partial^\pm \alpha: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}$ with the following properties:*

- (1) α is $\pi_1(S)$ -invariant, in the sense that $\alpha(\gamma T, \gamma T') = \alpha(T, T')$ for every $\gamma \in \pi_1(S)$ and every pair of distinct components T and T' of $\tilde{S} \setminus \tilde{\lambda}$;

(2) if T'' separates T from T' in \tilde{S} , if \tilde{s}'' is the spike of T'' delimited by the two sides of T'' that separate T from T' , and if s'' is the slit of λ defined by the projection of \tilde{s}'' to S , then

$$\alpha(T, T') = \alpha(T, T'') + \alpha(T'', T') - \partial^+ \alpha(s'')$$

if \tilde{s}'' points to the left as seen from T , and

$$\alpha(T, T') = \alpha(T, T'') + \alpha(T'', T') + \partial^- \alpha(s'')$$

if \tilde{s}'' points to the right as seen from T .

In addition, the boundary $\partial\alpha: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ is related to the functions

$$\partial^\pm \alpha: \{\text{slits of } \lambda\} \longrightarrow \mathbb{R}$$

by the property that $\partial\alpha(s^\pm) = \partial^\pm \alpha(s)$ for every slit s of λ lifting to a positive slit s^+ and a negative slit s^- of the orientation cover $\hat{\lambda}$.

Proof. This is an immediate consequence of the discussion above. □

Proposition 4.8 has an immediate factor-by-factor extension to relative tangent cycles valued in \mathbb{R}^{n-1} . By restriction to the space of twisted relative tangent cycles $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}^{n-1}}) \subset \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$, this automatically gives the following statement. Recall that $x \mapsto \bar{x}$ denotes the involution of \mathbb{R}^{n-1} that sends $x = (x_1, x_2, \dots, x_{n-1})$ to $\bar{x} = (x_{n-1}, x_{n-2}, \dots, x_1)$.

PROPOSITION 4.9. Proposition 4.8 provides a one-to-one correspondence between twisted relative tangent cycles $\alpha \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}^{n-1}})$ and maps α which associate a vector $\alpha(T, T') \in \mathbb{R}^{n-1}$ with each ordered pair of components T and T' of $\tilde{S} \setminus \tilde{\lambda}$ such that there exists a function $\partial^+ \alpha: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}^{n-1}$ with the following properties:

(1) α is $\pi_1(S)$ -equivariant, in the sense that $\alpha(\gamma T, \gamma T') = \alpha(T, T')$ for each $\gamma \in \pi_1(S)$ and each pair of distinct components T and T' of $\tilde{S} \setminus \tilde{\lambda}$;

(2) if T'' separates T from T' in \tilde{S} , if \tilde{s}'' is the spike of T'' delimited by the two sides of T'' that separate T from T' , and if s'' is the slit of λ defined by the projection of \tilde{s}'' to S , then

$$\alpha(T, T') = \alpha(T, T'') + \alpha(T'', T') - \partial^+ \alpha(s'')$$

if \tilde{s}'' points to the left as seen from T , and

$$\alpha(T, T') = \alpha(T, T'') + \alpha(T'', T') - \overline{\partial^+ \alpha(s'')}$$

if \tilde{s}'' points to the right as seen from T ;

(3) $\alpha(T', T) = \overline{\alpha(T, T')}$ for every pair of distinct components T and T' of $\tilde{S} \setminus \tilde{\lambda}$.

In addition, the boundary $\partial\alpha: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ is related to the function

$$\partial^+ \alpha: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}$$

by the property that $\partial\alpha(s^+) = \partial^+ \alpha(s)$ and $\partial\alpha(s^-) = -\overline{\partial^+ \alpha(s)}$ for every slit s of λ lifting to a positive slit s^+ and a negative slit s^- of the orientation cover $\tilde{\lambda}$.

Proof. This statement immediately follows from Proposition 4.8. □

Note that the function $\partial^- \alpha: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}^{n-1}$ that one would have expected in this case is equal to $\partial^- \alpha = -\overline{\partial^+ \alpha}$ by the third condition of Proposition 4.9. In particular, $\partial\alpha(s^\mp) = -\overline{\partial\alpha(s^\pm)}$ for every $\alpha \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$, when s^+ and s^- are the positive and negative slits of the orientation cover $\hat{\lambda}$ that lift the same slit s of λ .

5. The shearing tangent cycle of a Hitchin character

We will now associate a twisted relative tangent cycle

$$\sigma^\rho \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong H_1(U, \partial_\nu U; \widetilde{\mathbb{R}}^{n-1})$$

with each Hitchin character $\rho \in \text{Hit}_n(S)$. The key ingredient of this construction is the slithering map introduced in the next section.

5.1. Slithering

The slithering construction is a higher-dimensional analogue of the horocyclic foliation defined, in the case where $n=2$, by a hyperbolic metric and a maximal geodesic lamination λ on the surface S ; see [42] and [5] for that case. It only uses the Anosov property of Theorem 3.1 and Proposition 3.2, and is independent of the positivity property of Theorem 3.3. In particular, it quite possibly extends to more general Anosov contexts, such as those of [22].

Consider a Hitchin homomorphism $\rho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, and its associated flag map $\mathcal{F}_\rho: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ as in §3.1.

In the universal cover \tilde{S} , let g be a leaf of the preimage $\tilde{\lambda} \subset \tilde{S}$ of the maximal geodesic lamination $\lambda \subset S$. Choose an arbitrary orientation for g , and let x_+ and x_- be its positive and negative endpoints, respectively. By Theorem 3.3, the flag pair $(\mathcal{F}_\rho(x_+), \mathcal{F}_\rho(x_-))$ is generic. It therefore defines a decomposition of \mathbb{R}^n as the direct sum of the lines $\tilde{L}_a(g) = \mathcal{F}_\rho(x_+)^{(a)} \cap \mathcal{F}_\rho(x_-)^{(n-a+1)}$, as in §3.1.

Note that reversing the orientation of g exchanges x_+ and x_- , and therefore replaces $\tilde{L}_a(g)$ by $\tilde{L}_{n-a+1}(g)$.

Now consider two leaves g and $g' \subset \tilde{\lambda}$. We say that g and g' are *oriented in parallel* if exactly one of the orientations of g and g' coincides with the boundary orientation determined by the component of $S \setminus (g \cup g')$ that separates g from g' .

PROPOSITION 5.1. *There exists a unique family of linear isomorphisms*

$$\Sigma_{gg'}: \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

indexed by all pairs of leaves $g, g' \subset \tilde{\lambda}$, such that the following conditions hold:

- (1) $\Sigma_{gg} = \text{Id}_{\mathbb{R}^n}$, $\Sigma_{g'g} = (\Sigma_{gg'})^{-1}$ and $\Sigma_{gg''} = \Sigma_{gg'} \circ \Sigma_{g'g''}$ when g' separates g from g'' ;
- (2) $\Sigma_{gg'}$ depends locally Hölder continuously on g and g' ; namely, for an arbitrary riemannian metric on the manifold consisting of all geodesics of the universal cover \tilde{S} , the map $(g, g') \mapsto \Sigma_{gg'}$ is Hölder continuous on (the square of) any compact subset of the space of leaves of $\tilde{\lambda}$;
- (3) if g and g' have an endpoint $x \in \partial_\infty \tilde{S}$ in common and are oriented towards x , and if $E = \mathcal{F}_\rho(x) \in \text{Flag}(\mathbb{R}^n)$, then $\Sigma_{gg'}$ sends each line $\tilde{L}_a(g')$ to $\tilde{L}_a(g)$ and its restriction $\tilde{L}_a(g') \rightarrow \tilde{L}_a(g)$ is the composition of the two natural isomorphisms

$$\tilde{L}_a(g') \cong E^{(a)} / E^{(a-1)} \cong \tilde{L}_a(g).$$

In addition, the maps $\Sigma_{gg'}$ satisfy:

- (4) if g and g' are oriented in parallel, $\Sigma_{gg'}$ sends each line $\tilde{L}_a(g')$ to the line $\tilde{L}_a(g)$;
- (5) $\Sigma_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ has determinant 1.

By definition, $\Sigma_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the *slithering map* from the line decomposition

$$\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g')$$

to the line decomposition

$$\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g).$$

We will construct $\Sigma_{gg'}$ by sweeping through all the leaves of $\tilde{\lambda}$ that separate g from g' , and by composition of a (usually infinite) sequence of pivot moves as in condition (3) of Proposition 5.1. The terminology of “slithering” is motivated by the fact that, in general, any small section of this sweep involves both pivot moves to the left and pivot moves to the right.⁽¹⁾

Note that, although the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g)$ depends on an orientation for the leaf g , the slithering map $\Sigma_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is independent of a choice of orientation for g or g' .

⁽¹⁾ In particular, this is unrelated to Thurston’s notion [41], [12] of “slithering” for foliations of 3-dimensional manifolds, beyond the analogy with the movements of a snake.

Proof of Proposition 5.1. We will split the construction of the slithering map of Proposition 5.1 into several steps and lemmas.

Let T be a component of $\tilde{S} \setminus \tilde{\lambda}$ that separates the leaves g and g' of $\tilde{\lambda}$. It is a triangle since the geodesic lamination λ is maximal, and two of its three sides separate g from g' ; among these two sides, let g_T be the one that is closest to g , and g'_T the one closest to g' . Define $\Sigma_T = \Sigma_{g_T g'_T}$ by condition (3) of Proposition 5.1. Namely, if $E_T = \mathcal{F}_\rho(x_T) \in \text{Flag}(\mathbb{R}^n)$ is the image under the flag map $\mathcal{F}_\rho: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ of the common endpoint $x_T \in \partial_\infty \tilde{S}$ of g_T and g'_T , the map $\Sigma_T = \Sigma_{g_T g'_T}$ sends $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g'_T)$ to $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g_T)$ by the property that its restriction $\tilde{L}_a(g'_T) \rightarrow \tilde{L}_a(g_T)$ coincides with the composition of the natural isomorphisms $\tilde{L}_a(g'_T) \cong E_T^{(a)} / E_T^{(a-1)} \cong \tilde{L}_a(g_T)$. Note that, in a basis of \mathbb{R}^n where the a th vector belongs to $\tilde{L}_a(g'_T)$, the map Σ_T is represented by a triangular matrix with all diagonal entries equal to 1. In particular, $\tilde{L}_a(g'_T)$ has determinant 1 and therefore belongs to $\text{SL}_n(\mathbb{R})$.

We will now define

$$\Sigma_{gg'} = \overrightarrow{\prod}_T \Sigma_T$$

as the composition of the maps $\Sigma_T = \Sigma_{g_T g'_T}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ as T ranges over all components of $\tilde{S} \setminus \tilde{\lambda}$ separating g from g' . Of course, there usually are infinitely many maps in this composition, and we also must be careful with the order in which we compose these maps; the arrow over the product symbol is here to remind us that this is an ordered product, if the components T are ordered from g to g' . To make sense of this composition, let $\mathcal{T}_{gg'}$ be the set of components of $\tilde{S} \setminus \tilde{\lambda}$ that separate g from g' . Let $\mathcal{J} = \{T_1, T_2, \dots, T_m\}$ be a finite subset of $\mathcal{T}_{gg'}$, where the indexing is chosen so that each ideal triangle T_j separates g from T_{j+1} . We can then consider the finite composition

$$\Sigma_{\mathcal{J}} = \Sigma_{T_1} \circ \Sigma_{T_2} \circ \dots \circ \Sigma_{T_{m-1}} \circ \Sigma_{T_m} \in \text{SL}_n(\mathbb{R}).$$

We will then show that $\Sigma_{\mathcal{J}}$ converges to some linear map $\Sigma_{gg'} \in \text{SL}_n(\mathbb{R})$ as the finite subset $\mathcal{J} = \{T_1, T_2, \dots, T_m\}$ tends to the whole set $\mathcal{T}_{gg'}$ of those components of $\tilde{S} \setminus \tilde{\lambda}$ which separate g from g' .

The proof of convergence relies on the following estimate. Choose an arc $k \subset \tilde{S}$ that is tightly transverse to the geodesic lamination $\tilde{\lambda}$, and crosses both g and g' ; for instance, we can choose k to be a geodesic arc.

In particular, for every component T of $\tilde{S} \setminus \tilde{\lambda}$ that separates g from g' , $k \cap T$ consists of a single arc.

Endow the space $\text{End}(\mathbb{R}^n)$ of linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ with any of the classical norms $\|\cdot\|$ such that $\|\text{Id}_{\mathbb{R}^n}\| = 1$ and $\|\varphi \circ \psi\| \leq \|\varphi\| \|\psi\|$. Our estimates will also depend on the choice of a negatively curved metric m on S for which the leaves of λ are geodesic.

LEMMA 5.2. *Given an arc k as above, there exist constants A and $\nu > 0$ such that*

$$\|\Sigma_T - \text{Id}_{\mathbb{R}^n}\| \leq A\ell(k \cap T)^\nu$$

for every component T of $\tilde{S} \setminus \tilde{\lambda}$ that separates g from g' , where ℓ denotes the arc length for the auxiliary metric m .

Proof. Let $x_T, y_T, y'_T \in \partial_\infty \tilde{S}$ denote the three vertices of the triangle T , in such a way that x_T and y_T are the endpoints of the side g_T that is closest to g , and x_T and y'_T are the endpoints of the side g'_T closest to g' . Then $\Sigma_T = \Sigma_{g_T g'_T}$ depends only on the two generic flag pairs $(\mathcal{F}_\rho(x_T), \mathcal{F}_\rho(y_T))$ and $(\mathcal{F}_\rho(x_T), \mathcal{F}_\rho(y'_T))$. In fact, Σ_T depends differentiably on these two flag pairs and, as T varies, these pairs stay in a compact subset of the space of generic flag pairs (depending on k and on the continuity of the flag curve \mathcal{F}_ρ). Therefore,

$$\|\Sigma_T - \text{Id}_{\mathbb{R}^n}\| = O(d(\mathcal{F}_\rho(y_T), \mathcal{F}_\rho(y'_T))),$$

where d is an arbitrary riemannian metric on $\text{Flag}(\mathbb{R}^n)$.

Since the flag curve \mathcal{F}_ρ is Hölder continuous (Proposition 3.2),

$$d(\mathcal{F}_\rho(y_T), \mathcal{F}_\rho(y'_T)) = O(d(y_T, y'_T)^\nu)$$

for some Hölder exponent ν , where d now denotes the visual metric of $\partial_\infty \tilde{S}$. The required estimate then follows from an easy geometric argument in the triangle T showing that

$$d(y_T, y'_T) = O(\ell(k \cap T)). \quad \square$$

Note that the constant A depends on the arc k . The Hölder exponent ν depends only on the flag curve \mathcal{F}_ρ .

The second ingredient is a now classical property of geodesic laminations, based on the fundamental property that $S \setminus \lambda$ has only finitely many spikes and that the width of these spikes decay exponentially.

LEMMA 5.3. *As T ranges over all components of $\tilde{S} \setminus \tilde{\lambda}$ separating g from g' , the sum*

$$\sum_{T \in \mathcal{T}_{gg'}} \ell(k \cap T)^\nu$$

is convergent for every $\nu > 0$.

More precisely, there is a function $r: \mathcal{T}_{gg'} \rightarrow \mathbb{N}$ and constants $B, C, B', C' > 0$ such that the following conditions hold:

- (1) $B e^{-Cr(T)} \leq \ell(k \cap T) \leq B' e^{-C'r(T)}$ for every $T \in \mathcal{T}_{gg'}$;
- (2) for every $m \in \mathbb{N}$, the number of triangles $T \in \mathcal{T}_{gg'}$ with $r(T) = m$ is uniformly bounded, independently of m .

Proof. See for instance [5, Lemmas 4 and 5], and compare §8.2. □

We are now ready to show the convergence of the infinite product $\overrightarrow{\prod}_T \Sigma_T$.

Recall that $\mathcal{T}_{gg'}$ denotes the set of components of $\tilde{S} \setminus \tilde{\lambda}$ that separate g from g' and that, for every finite subset $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ of $\mathcal{T}_{gg'}$ where the T_i are ordered from g to g' ,

$$\Sigma_{\mathcal{T}} = \Sigma_{T_1} \circ \Sigma_{T_2} \circ \dots \circ \Sigma_{T_{m-1}} \circ \Sigma_{T_m}.$$

LEMMA 5.4. *As \mathcal{T} ranges over all finite subsets of $\mathcal{T}_{gg'}$, the matrices $\Sigma_{\mathcal{T}}$ remain uniformly bounded.*

Proof. If $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$, Lemma 5.2 shows that $\|\Sigma_{T_i}\| \leq 1 + A \ell(k \cap T_i)^\nu$ for some constants $A, \nu > 0$. Then,

$$\|\Sigma_{\mathcal{T}}\| \leq \prod_{i=1}^m (1 + A \ell(k \cap T_i)^\nu) \leq \prod_{T \in \mathcal{T}_{gg'}} (1 + A \ell(k \cap T)^\nu) < \infty,$$

where the finiteness of the second product follows from Lemma 5.3. □

LEMMA 5.5. *As the finite subset \mathcal{T} tends to $\mathcal{T}_{gg'}$, the limit*

$$\Sigma_{gg'} = \overrightarrow{\prod}_{T \in \mathcal{T}_{gg'}} \Sigma_T = \lim_{\mathcal{T} \rightarrow \mathcal{T}_{gg'}} \Sigma_{\mathcal{T}}$$

exists in $SL_n(\mathbb{R})$.

Proof. Let $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ be a finite subset of $\mathcal{T}_{gg'}$, where the T_i are ordered from g to g' . If $\mathcal{T}' = \mathcal{T} \cup \{T\}$ has one more element $T \in \mathcal{T}_{gg'}$, and if T separates T_i from T_{i+1} , set $\mathcal{T}_1 = \{T_1, T_2, \dots, T_i\}$ and $\mathcal{T}_2 = \{T_{i+1}, T_2, \dots, T_m\}$; then

$$\|\Sigma_{\mathcal{T}'} - \Sigma_{\mathcal{T}}\| = \|\Sigma_{\mathcal{T}_1} \circ (\Sigma_T - \text{Id}_{\mathbb{R}^n}) \circ \Sigma_{\mathcal{T}_2}\| = O(\ell(k \cap T)^\nu)$$

by Lemmas 5.2 and 5.4. Lemma 5.3 then shows that, as \mathcal{T} ranges over all finite subsets of $\mathcal{T}_{gg'}$, the family of maps $\Sigma_{\mathcal{T}} \in SL_n(\mathbb{R})$ satisfies the Cauchy property. The limit therefore exists. □

Having defined the slithering map $\Sigma_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, we now show that it satisfies the properties of Proposition 5.1. We begin with condition (1).

LEMMA 5.6. *For any two leaves g and g' of $\tilde{\lambda}$, one has $\Sigma_{gg} = \text{Id}_{\mathbb{R}^n}$ and $\Sigma_{g'g} = \Sigma_{gg'}^{-1}$. In addition, $\Sigma_{gg''} = \Sigma_{gg'} \circ \Sigma_{g'g''}$ when one of the three leaves g, g' and g'' separates the other two.*

In the missing case where none of g , g' and g'' separates the other two, the relationship between $\Sigma_{gg''}$, $\Sigma_{gg'}$ and $\Sigma_{g'g''}$ is more complicated and involves the triangle invariants of the unique component T of $\tilde{S} \setminus \tilde{\lambda}$ that separates these three leaves, as we will implicitly see in §5.2 (and in particular in the proof of Proposition 5.15).

Proof. The first two properties are immediate from definitions. When g' separates g from g'' , $\mathcal{T}_{gg''}$ is the disjoint union of $\mathcal{T}_{gg'}$ and $\mathcal{T}_{g'g''}$, and the property that $\Sigma_{gg''} = \Sigma_{gg'} \circ \Sigma_{g'g''}$ is again an immediate consequence of the construction. The other two cases follow from this one by an algebraic manipulation. \square

We now turn to condition (2).

LEMMA 5.7. *The slithering map $\Sigma_{gg'}$ provided by Lemma 5.5 depends Hölder continuously on the leaves $g, g' \subset \tilde{\lambda}$ meeting the tightly transverse arc k .*

Proof. If the leaf h is close to g , and if the leaf h' is close to the leaf g' , we can apply Lemma 5.6 to decompose $\Sigma_{hh'}$ as

$$\Sigma_{hh'} = \Sigma_{hg} \circ \Sigma_{gg'} \circ \Sigma_{g'h'}.$$

The argument used in the proof of Lemma 5.5 shows that, for some $\nu > 0$,

$$\|\Sigma_{hg} - \text{Id}_{\mathbb{R}^n}\| = O\left(\sum_{T \in \mathcal{T}_{hg}} \ell(k \cap T)^\nu\right).$$

By Lemma 5.3, the above series is dominated by a geometric series and, using the precise estimate provided by the second half of that statement,

$$\begin{aligned} \sum_{T \in \mathcal{T}_{hg}} \ell(k \cap T)^\nu &= O\left(\max_{T \in \mathcal{T}_{hg}} e^{-\nu C' r(T)}\right) = O\left(\max_{T \in \mathcal{T}_{hg}} \ell(k \cap T)^{\nu'}\right) \\ &= O(\ell(k_{hg})^{\nu'}) = O(d(h, g)^{\nu'}) \end{aligned}$$

for $\nu' = \nu C' / C$ with the constants $C, C' > 0$ of Lemma 5.3, and where k_{hg} is the subarc of k that joins the two points $k \cap g$ and $k \cap h$. Therefore,

$$\|\Sigma_{hg} - \text{Id}_{\mathbb{R}^n}\| = O(d(g, h)^{\nu'}).$$

Similarly,

$$\|\Sigma_{g'h'} - \text{Id}_{\mathbb{R}^n}\| = O(d(g', h')^{\nu'}).$$

Combining these two estimates with the bound provided by Lemma 5.4,

$$\begin{aligned} \|\Sigma_{hh'} - \Sigma_{gg'}\| &\leq \|\Sigma_{hh'} - \Sigma_{gh'}\| + \|\Sigma_{gh'} - \Sigma_{gg'}\| \\ &\leq \|\Sigma_{hg} - \text{Id}_{\mathbb{R}^n}\| \|\Sigma_{gh'}\| + \|\Sigma_{gg'}\| \|\Sigma_{g'h'} - \text{Id}_{\mathbb{R}^n}\| \\ &= O(d(g, h)^{\nu'} + d(g', h')^{\nu'}), \end{aligned}$$

which proves that the map $(g, g') \mapsto \Sigma_{gg'} \in \text{SL}_n(\mathbb{R})$ is Hölder continuous over the square of the space of leaves of $\tilde{\lambda}$ that cross the arc k . \square

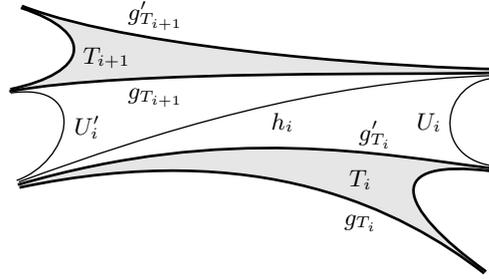


Figure 5. Adding triangles between T_i and T_{i+1} .

Lemma 5.7 proves the local Hölder continuity condition (2) of Proposition 5.1.

If the leaves g and g' share a common endpoint $x \in \partial_\infty \tilde{S}$, then all leaves of $\tilde{\lambda}$ that separate g from g' also have x as an endpoint. In particular, $\Sigma_{gg'}$ is defined as an infinite product of elementary slitherings $\Sigma_T = \Sigma_{g_T g'_T}$ that respect the flag $E = \mathcal{F}_\varrho(x)$ and act as the identity on each line $E^{(a)}/E^{(a-1)}$. It follows that $\Sigma_{gg'}$ satisfies the same property, which proves condition (3) of Proposition 5.1.

LEMMA 5.8. *Suppose that the leaves $g, g' \subset \tilde{\lambda}$ are oriented in parallel. Then, the slithering map $\Sigma_{gg'}$ provided by Lemma 5.5 sends the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g')$ to the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g)$.*

Proof. The strategy is to approximate by a finite lamination the part of $\tilde{\lambda}$ that separates g from g' . The slithering map associated with this finite lamination will send the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g')$ to the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g)$, and approximate the slithering map $\Sigma_{gg'}$. Passing to the limit in the approximation process will conclude the proof.

Let $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ be a finite subset of $\mathcal{T}_{gg'}$, where the T_i are ordered from g to g' . We insert two triangles U_i and U'_i between T_i and T_{i+1} as follows. Recall that g_{T_i} and g'_{T_i} are the two sides of T_i separating g from g' , with g_{T_i} closest to g . Let h_i be the geodesic of \tilde{S} that joins the left-hand side (as seen from g) endpoint of g'_{T_i} to the right-hand side endpoint of $g_{T_{i+1}}$. The two geodesics g'_{T_i} and h_i are two sides of a unique ideal triangle $U_i \subset \tilde{S}$, possibly reduced to a single geodesic when $g'_{T_i} = h_i$. We can similarly consider the ideal triangle U'_i , possibly reduced to a single geodesic, with sides h_i and $g_{T_{i+1}}$. See Figure 5. The same construction with the conventions that $g'_{T_0} = g$ and $g_{T_{m+1}} = g'$ also defines triangles U_0, U'_0, U_m and U'_m .

As before, the triangles U_i and U'_i define an elementary slithering map Σ_{U_i} sending the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(h_i)$ to the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g'_{T_i})$, and another one $\Sigma_{U'_i}$ sending the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g_{T_{i+1}})$ to the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(h_i)$. These slithering maps are equal to the identity when

the corresponding triangles are reduced to geodesics.

Now consider

$$\widehat{\Sigma}_{\mathcal{T}} = (\Sigma_{U_0} \circ \Sigma_{U'_0}) \circ \Sigma_{T_1} \circ (\Sigma_{U_1} \circ \Sigma_{U'_1}) \circ \Sigma_{T_2} \circ (\Sigma_{U_2} \circ \Sigma_{U'_2}) \circ \Sigma_{T_3} \circ \dots \\ \dots \circ \Sigma_{T_{m-1}} \circ (\Sigma_{U_{m-1}} \circ \Sigma_{U'_{m-1}}) \circ \Sigma_{T_m} \circ (\Sigma_{U_m} \circ \Sigma_{U'_m}).$$

By construction, $\widehat{\Sigma}_{\mathcal{T}}$ sends the line decomposition

$$\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g_{T_{m+1}}) = \bigoplus_{a=1}^n \tilde{L}_a(g')$$

to the line decomposition

$$\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g'_{T_0}) = \bigoplus_{a=1}^n \tilde{L}_a(g).$$

To compare $\widehat{\Sigma}_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}}$, choose an arc k tightly transverse to $\tilde{\lambda}$ and meeting both g and g' . Then, Lemma 5.2 provides constants $A, \nu > 0$ such that $\|\Sigma_{U_i} - \text{Id}_{\mathbb{R}^n}\| \leq A\ell(k \cap U_i)^\nu$ and $\|\Sigma_{U'_i} - \text{Id}_{\mathbb{R}^n}\| \leq A\ell(k \cap U'_i)^\nu$.

Since a ν -Hölder continuous function on a compact set is also ν' -Hölder continuous for every $\nu' < \nu$, we can assume that $\nu \leq 1$ without loss of generality. Then, with this condition,

$$\ell(k \cap U_i)^\nu \leq \ell(k \cap (U_i \cup U'_i))^\nu \leq \sum_{T \in \mathcal{T}_{g'_{T_i} g_{T_{i+1}}}} \ell(k \cap T)^\nu,$$

where the sum is over all components T of $\tilde{S} \setminus \tilde{\lambda}$ that separate T_i from T_{i+1} . A similar estimate holds for $\ell(k \cap U'_i)^\nu$. It follows that

$$\|\Sigma_{U_i} \circ \Sigma_{U'_i} - \text{Id}_{\mathbb{R}^n}\| = O\left(\sum_{T \in \mathcal{T}_{g'_{T_i} g_{T_{i+1}}}} \ell(k \cap T)^\nu\right).$$

The arguments used in the proof of Lemmas 5.4 and 5.5 can then be applied to show that

$$\|\widehat{\Sigma}_{\mathcal{T}} - \Sigma_{\mathcal{T}}\| = O\left(\sum_{T \in \mathcal{T}_{gg'} - \mathcal{T}} \ell(k \cap T)^\nu\right).$$

Lemma 5.3 then shows that $\widehat{\Sigma}_{\mathcal{T}}$ and $\Sigma_{\mathcal{T}}$ have the same limit as the finite subset \mathcal{T} tends to $\mathcal{T}_{gg'}$. Therefore, $\widehat{\Sigma}_{\mathcal{T}}$ also converges to the slithering map $\Sigma_{gg'}$ as \mathcal{T} tends to $\mathcal{T}_{gg'}$.

We already observed that each $\widehat{\Sigma}_{\mathcal{T}}$ sends the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g')$ to the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(g)$. Passing to the limit, we conclude that $\Sigma_{gg'}$ has the same property. \square

Lemma 5.8 proves condition (4) of Proposition 5.1.

We already observed in Lemma 5.5 that $\Sigma_{gg'}$ has determinant 1, which is condition (5).

The only property of Proposition 5.1 remaining to prove is the uniqueness of the slithering map.

LEMMA 5.9. *If a family of linear isomorphisms $\Sigma'_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, indexed by all pairs of leaves $g, g' \subset \tilde{\lambda}$, satisfies conditions (1)–(3) of Proposition 5.1, then $\Sigma'_{gg'}$ is equal to the map $\Sigma_{gg'}$ constructed above for every g and g' .*

In particular, conditions (4) and (5) are consequences of conditions (1)–(3).

Proof. As usual, let k be a tightly transverse arc that crosses both g and g' . Let $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ be a finite subset of the set $\mathcal{T}_{gg'}$ of components of $\tilde{S} \setminus \tilde{\lambda}$ that separate g from g' , indexed in such a way that the T_i occur in this order as one goes from g to g' . Let g_{T_i} and g'_{T_i} be the sides of T_i that are closest to g and g' , respectively.

By condition (1),

$$\Sigma'_{gg'} = \Sigma'_{gg_{T_1}} \circ \Sigma'_{g_{T_1}g'_{T_1}} \circ \Sigma'_{g'_{T_1}g_{T_2}} \circ \Sigma'_{g_{T_2}g'_{T_2}} \circ \dots \circ \Sigma'_{g_{T_{m-1}}g'_{T_{m-1}}} \circ \Sigma'_{g'_{T_{m-1}}g_{T_m}} \circ \Sigma'_{g_{T_m}g'_{T_m}} \circ \Sigma'_{g'_{T_m}g'}.$$

Condition (3) implies that $\Sigma'_{g_{T_i}g'_{T_i}} = \Sigma_{g_{T_i}g'_{T_i}} = \Sigma_{T_i}$, so that

$$\Sigma'_{gg'} = \Sigma'_{gg_{T_1}} \circ \Sigma_{T_1} \circ \Sigma'_{g'_{T_1}g_{T_2}} \circ \Sigma_{T_2} \circ \dots \circ \Sigma_{T_{m-1}} \circ \Sigma'_{g'_{T_{m-1}}g_{T_m}} \circ \Sigma_{T_m} \circ \Sigma'_{g'_{T_m}g'}.$$

By condition (2), the map $(h, h') \mapsto \Sigma'_{hh'}$ is Hölder continuous over the space of leaves of $\tilde{\lambda}$ that meet the arc k . As a consequence, there exists a constant $\nu > 0$ such that, for every i ,

$$\|\Sigma'_{g'_{T_i}g_{T_{i+1}}} - \text{Id}_{\mathbb{R}^n}\| = O(d(g'_{T_i}, g_{T_{i+1}})^\nu).$$

Because the leaves g'_{T_i} and $g_{T_{i+1}}$ are disjoint, a classical estimate in negative curvature geometry (see, for instance, [13, §5.2.6]) shows that $d(g'_{T_i}, g_{T_{i+1}})$ is bounded by a constant times the length $\ell(k_{g'_{T_i}g_{T_{i+1}}})$ of the subarc $k_{g'_{T_i}g_{T_{i+1}}} \subset k$ delimited by the points $k \cap g'_{T_i}$ and $k \cap g_{T_{i+1}}$.

The geodesic lamination $\tilde{\lambda}$ has area zero ([39, §8.5], [1]). Its intersection with the transverse arc k therefore has length zero, and

$$\ell(k_{g'_{T_i}g_{T_{i+1}}}) = \sum_{T \in \mathcal{T}_{g'_{T_i}g_{T_{i+1}}}} \ell(k \cap T).$$

Assuming $\nu \leq 1$ without loss of generality, we can combine all these estimates and conclude that

$$\|\Sigma'_{g'_{T_i}g_{T_{i+1}}} - \text{Id}_{\mathbb{R}^n}\| = O\left(\sum_{T \in \mathcal{T}_{g'_{T_i}g_{T_{i+1}}}} \ell(k \cap T)^\nu\right).$$

This also holds for $i=0$ and $i=m$, with the convention that $g'_{T_0}=g$ and $g_{T_{m+1}}=g'$.

From this estimate, we can then use the arguments of the proofs of Lemmas 5.4 and 5.5 to show that

$$\|\Sigma'_{gg'} - \Sigma_{\mathcal{T}}\| = O\left(\sum_{T \in \mathcal{T}_{gg'} - \mathcal{T}} \ell(k \cap T)^\nu\right).$$

By Lemma 5.3, this proves that

$$\Sigma'_{gg'} = \lim_{\mathcal{T} \rightarrow \mathcal{T}_{gg'}} \Sigma_{\mathcal{T}} = \Sigma_{gg'},$$

which concludes the proof of Lemma 5.9. □

This uniqueness property completes the proof of Proposition 5.1. □

Remark 5.10. In Proposition 5.1 (and in Lemma 5.9), the uniqueness property would be false without the hypothesis that the slithering map $\Sigma_{gg'}$ depends locally Hölder continuously (and not just continuously) on the leaves g and g' . This is already true in the classical case where $n=2$. Indeed, let α be a transverse measure for λ that has no atom (which is automatic if λ has no closed leaf). The atom-free hypothesis guarantees that the α -mass $\alpha(g, g')$ of the set of leaves of $\tilde{\lambda}$ separating g from g' depends continuously on g and g' . If g and g' are oriented in parallel in such a way that g' is to the left of g , let $\Sigma'_{gg'}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be obtained by postcomposing the slithering map $\Sigma_{gg'}$ with the linear map that acts by multiplication by $e^{-\alpha(g, g')}$ on the line $\tilde{L}_1(g)$, and by multiplication by $e^{\alpha(g, g')}$ on $\tilde{L}_2(g)$. This new family of maps $\Sigma'_{gg'}$ satisfies conditions (1) and (3)–(5) of Proposition 5.1, the maps $\Sigma'_{gg'}$ depend continuously (but not locally Hölder continuously) on g and g' , and they are of course different from the original family of slithering maps $\Sigma_{gg'}$ if α is non-trivial.

To generalize this construction to all n , let $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ be atom-free transverse measures for λ such that $\alpha_{n-a} = \alpha_a$ for every a (so that in practice we only have $\lfloor \frac{1}{2}n \rfloor$ such α_a). For two leaves g and g' of $\tilde{\lambda}$, define $\beta_1(g, g'), \beta_2(g, g'), \dots, \beta_n(g, g')$ by the properties that $\alpha_a(g, g') = \beta_{a+1}(g, g') - \beta_a(g, g')$ and $\sum_{a=1}^n \beta_a(g, g') = 0$. If g and g' are oriented in parallel in such a way that g' is to the left of g , let $\Sigma'_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be obtained by postcomposing the slithering map $\Sigma_{gg'}$ with the linear map that respects each line $\tilde{L}_a(g)$ and acts by $e^{\beta_a(g, g')}$ on $\tilde{L}_a(g)$. The maps $\Sigma'_{gg'}$ satisfy conditions (1) and (3)–(5) of Proposition 5.1, and depend continuously on g and g' . They clearly differ from the slithering maps $\Sigma_{gg'}$ if at least one of the α_a is non-zero. This construction automatically generalizes to the situation where the α_a are topological differential forms in the sense of [26], in which case it completely describes how the uniqueness can fail if we remove the Hölder condition from Proposition 5.1.

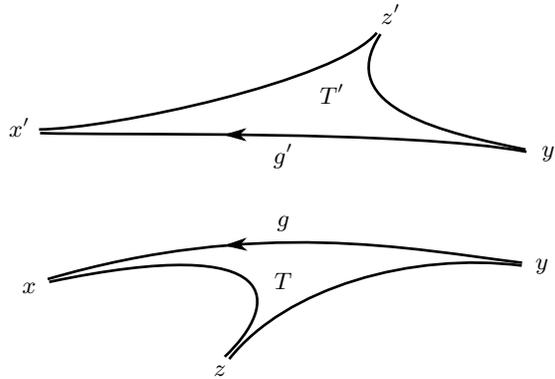


Figure 6. The triangles T and T' and their vertices.

5.2. The shearing cycle

We now use the slithering map to associate with the Hitchin homomorphism

$$\varrho: \pi_1(S) \longrightarrow \mathrm{PSL}_n(\mathbb{R})$$

a certain twisted tangent cycle $\sigma^\varrho \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ relative to the slits of λ . This relative tangent cycle is the *shearing cycle* of the Hitchin homomorphism ϱ .

Unlike §5.1, this construction uses in a fundamental way the positivity property of Theorem 3.3; see Lemma 5.11.

We will use the point of view of §4.7. Let T and T' be two components of $\widetilde{S} \setminus \widetilde{\lambda}$.

Let g be the side of T that is closest to T' , and let g' be the side of T' closest to T . We orient these two leaves of $\widetilde{\lambda}$ to the left as seen from T . In particular, g and g' are oriented in parallel, and the slithering map $\Sigma_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ of Proposition 5.1 sends each line $\widetilde{L}_a(g')$ to the line $\widetilde{L}_a(g)$.

Let $x, y \in \partial_\infty \widetilde{S}$ be the positive and negative endpoints of g , and let z be the third vertex of the ideal triangle T . Similarly, let $x', y' \in \partial_\infty \widetilde{S}$ be the positive and negative endpoints of g' , and let z' be the third vertex of T' . See Figure 6. The flag curve $\mathcal{F}_\varrho: \partial_\infty \widetilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ of Proposition 3.2 now associates six flags

$$\mathcal{F}_\varrho(x), \mathcal{F}_\varrho(y), \mathcal{F}_\varrho(z), \mathcal{F}_\varrho(x'), \mathcal{F}_\varrho(y'), \mathcal{F}_\varrho(z') \in \mathrm{Flag}(\mathbb{R}^n)$$

with these vertices. By our definitions, the slithering map $\Sigma_{gg'}$ sends $\mathcal{F}_\varrho(x')$ to $\mathcal{F}_\varrho(x)$ and $\mathcal{F}_\varrho(y')$ to $\mathcal{F}_\varrho(y)$.

We want to consider the double ratio $D_a(\mathcal{F}_\varrho(x), \mathcal{F}_\varrho(y), \mathcal{F}_\varrho(z), \Sigma_{gg'}(\mathcal{F}_\varrho(z')))$, as in §1.4.

LEMMA 5.11. *The double ratio $D_a(\mathcal{F}_\varrho(x), \mathcal{F}_\varrho(y), \mathcal{F}_\varrho(z), \Sigma_{gg'}(\mathcal{F}_\varrho(z')))$ is finite and positive.*

Proof. When T and T' are adjacent so that $g=g'$, then $\Sigma_{gg'}=\text{Id}_{\mathbb{R}^n}$ and the statement is an immediate consequence of the positivity property of Theorem 3.3. In the general case, however, the appearance of the slithering map $\Sigma_{gg'}$ requires a more elaborate argument. The key ingredient is a deeper consequence of the positivity property, which is that the family of the line bundles $L_a \rightarrow T^1S$ of Theorem 3.1 carries a canonical joint orientation, as we now define.

A *joint orientation* for the line bundles L_a is the choice of an orientation for (the fibers of) each bundle, but determined only up to simultaneous reversal of all orientations. More formally, a joint orientation for the bundles L_a is an orientation for each of the line bundles $L_a \otimes L_{a+1}$.

Actually, we will see that the line bundles L_a admit two equally canonical but opposite joint orientations: the left-hand-side and right-hand-side joint orientations.

To define these joint orientations, focus attention on a point $\tilde{u} \in T^1\tilde{S}$. As in §3.1, consider the line decomposition $\mathbb{R}^n = \bigoplus_{a=1}^n \tilde{L}_a(\tilde{u})$ defined by the fibers over \tilde{u} of the line bundles $\tilde{L}_a \rightarrow T^1\tilde{S}$ lifting the bundles $L_a \rightarrow T^1S$. Then, if $p, q \in \partial_\infty\tilde{S}$ are the positive and negative endpoints of the orbit g of \tilde{u} under the geodesic flow, we have

$$\tilde{L}_a(\tilde{u}) = \mathcal{F}_\rho(p)^{(a)} \cap \mathcal{F}_\rho(q)^{(n-a+1)},$$

by definition of the flag curve \mathcal{F}_ρ in Proposition 3.2. Consider another point $r \in \partial_\infty\tilde{S}$ that is different from p and q , and that sits to the *left* of p as seen from q . By Theorem 3.3, the flag triple $(\mathcal{F}_\rho(p), \mathcal{F}_\rho(q), \mathcal{F}_\rho(r))$ is generic. As a consequence, if v is a non-trivial vector in the line $\mathcal{F}_\rho(r)^{(1)}$, the projection of $v \in \mathbb{R}^n = \bigoplus_{b=1}^n \tilde{L}_b(\tilde{u})$ to the line $\tilde{L}_a(\tilde{u})$ parallel to all $\tilde{L}_b(\tilde{u})$ with $b \neq a$ is non-trivial, and therefore specifies an orientation for $\tilde{L}_a(\tilde{u})$. Replacing v by any other non-trivial vector $v' \in \mathcal{F}_\rho(r)^{(1)}$ determines the same orientation on $\tilde{L}_a(\tilde{u})$ if the ratio v/v' in the line $\mathcal{F}_\rho(r)^{(1)}$ is positive, or reverses all these orientations if $v/v' < 0$. Therefore, the joint orientation of the lines $\tilde{L}_a(\tilde{u})$ is independent of the choice of $v \in \mathcal{F}_\rho(r)^{(1)}$.

To show that the joint orientation of the lines $\tilde{L}_a(\tilde{u})$ is independent of the choice of the point $r \in \partial_\infty\tilde{S}$, consider another point $r' \in \partial_\infty\tilde{S}$ different from p and q , and now located on the *right* of p as seen from q . This point r' similarly defines a joint orientation for the lines $\tilde{L}_a(\tilde{u})$, and we will see that this joint orientation is exactly the opposite of that defined by r (considering joint orientations as orientations of the line bundles $L_a \otimes L_{a+1}$). To prove this, pick non-trivial vectors $v \in \mathcal{F}_\rho(r)^{(1)}$ and $v' \in \mathcal{F}_\rho(r')^{(1)}$. Let v_a and v'_a denote the respective projections of v and v' to the line $\tilde{L}_a(\tilde{u})$ parallel to all $\tilde{L}_b(\tilde{u})$ with $b \neq a$. Since r and r' are in different components of $\partial_\infty\tilde{S} \setminus \{p, q\}$, Lemma 1.6 and the positivity condition of Theorem 3.3 show that

$$0 < D_a(\mathcal{F}_\rho(p), \mathcal{F}_\rho(q), \mathcal{F}_\rho(r), \mathcal{F}_\rho(r')) = -\frac{v_{a+1} v'_a}{v'_{a+1} v_a},$$

where the ratios $v'_b/v_b \in \mathbb{R} \setminus \{0\}$ are computed in the lines $\tilde{L}_b(\tilde{u})$. As a consequence, v and v' induce opposite orientations on the lines $\tilde{L}_a(\tilde{u}) \otimes \tilde{L}_{a+1}(\tilde{u})$. In other words, the joint orientation of the lines $\tilde{L}_a(\tilde{u})$ defined by the point $r' \in \partial_\infty \tilde{S}$ is the opposite of that defined by r . It immediately follows that the joint orientation defined by r is independent of the choice of r in the left-hand-side component of $\partial_\infty \tilde{S} \setminus \{p, q\}$ (as seen from q).

We will refer to the joint orientation defined by r as the *left-hand-side joint orientation* of the lines $\tilde{L}_a(\tilde{u})$, whereas the *right-hand-side joint orientation* will be the one defined by r' . These two joint orientations are opposite of each other.

Let h and h' be two oriented geodesics of \tilde{S} that share the same positive endpoint $p \in \partial_\infty \tilde{S}$, and let $\Sigma_{hh'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the elementary slithering map, sending each line $\tilde{L}_a(h')$ to $\tilde{L}_a(h)$, defined as in Proposition 5.1 (4). The definition of $\Sigma_{hh'}$ through the isomorphisms

$$\tilde{L}_a(h') \cong \mathcal{F}^a(p) / \mathcal{F}^a(p)^{(a-1)} \cong \tilde{L}_a(h)$$

makes it clear that $\Sigma_{hh'}$ sends the left-hand-side joint orientation of the family of lines $\tilde{L}_a(h')$ to the left-hand-side joint orientation of the $\tilde{L}_a(h)$.

We now return to the leaves g and g' of $\tilde{\lambda}$. As in the proof of Lemma 5.8 and with the notation used there, approximate the part of $\tilde{\lambda}$ that separates g and g' by a finite lamination, and the slithering map $\Sigma_{gg'}$ by a finite composition

$$\begin{aligned} \widehat{\Sigma}_{\mathcal{T}} &= (\Sigma_{U_0} \circ \Sigma_{U'_0}) \circ \Sigma_{T_1} \circ (\Sigma_{U_1} \circ \Sigma_{U'_1}) \circ \Sigma_{T_2} \circ (\Sigma_{U_2} \circ \Sigma_{U'_2}) \circ \Sigma_{T_3} \circ \dots \\ &\dots \circ \Sigma_{T_{m-1}} \circ (\Sigma_{U_{m-1}} \circ \Sigma_{U'_{m-1}}) \circ \Sigma_{T_m} \circ (\Sigma_{U_m} \circ \Sigma_{U'_m}) \end{aligned}$$

of elementary slitherings where, for any two consecutive terms, the corresponding triangles T_i and U_i , or U_i and U'_i , or U'_i and T_{i+1} , share a side g_{T_i} , h_i or $g'_{T_{i+1}}$, respectively. By our earlier observation, each of these elementary slitherings respects joint orientations of the appropriate families of lines. It follows that $\widehat{\Sigma}_{\mathcal{T}}$ sends the left-hand-side joint orientation of the lines $\tilde{L}_a(g')$ to the left-hand-side joint orientation of the $\tilde{L}_a(g)$. Passing to the limit as the approximation $\widehat{\Sigma}_{\mathcal{T}}$ tends to $\Sigma_{gg'}$, we conclude that the slithering map $\Sigma_{gg'}$ sends the left-hand-side joint orientation of the lines $\tilde{L}_a(g')$ to the left-hand-side joint orientation of the $\tilde{L}_a(g)$.

We are now ready to determine the sign of the double ratio

$$D_a(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z), \Sigma_{gg'}(\mathcal{F}_\rho(z'))).$$

Pick non-trivial vectors v and v' in the lines $\mathcal{F}_\rho(z)^{(1)}$ and $\mathcal{F}_\rho(z')^{(1)}$, respectively. The left-hand-side joint orientation of the family of lines $\tilde{L}_a(g)$ is defined by the projections v_a of v to $\tilde{L}_a(g)$ parallel to the other lines $\tilde{L}_b(g)$ with $b \neq a$. Similarly, the right-hand-side joint orientation of the lines $\tilde{L}_a(g')$ is defined by the projections v'_a of v' to $\tilde{L}_a(g')$ parallel

to the lines $\tilde{L}_b(g')$ with $b \neq a$. Since we just proved that the slithering map $\Sigma_{gg'}$ respects joint orientations, and since the left- and right-hand-side orientations are opposite of each other, the joint orientation of the $\tilde{L}_a(g)$ by the vectors v_a is opposite to that defined by the vectors $\Sigma_{gg'}(v'_a)$. In other words, all ratios

$$\frac{\Sigma_{gg'}(v'_a)}{v_a} \frac{v_{a+1}}{\Sigma_{gg'}(v'_{a+1})}$$

are negative. By Lemma 1.6,

$$D_a(\mathcal{F}_\varrho(x), \mathcal{F}_\varrho(y), \mathcal{F}_\varrho(z), \Sigma_{gg'}(\mathcal{F}_\varrho(z'))) = -\frac{v_{a+1}}{\Sigma_{gg'}(v'_{a+1})} \frac{\Sigma_{gg'}(v'_a)}{v_a} > 0,$$

which concludes the proof of Lemma 5.11. □

Lemma 5.11 enables us to define the a -th *shearing parameter* of the Hitchin homomorphism ϱ between the components T and T' of $\tilde{S} \setminus \tilde{\lambda}$ as

$$\sigma_a^\varrho(T, T') = \log D_a(\mathcal{F}_\varrho(x), \mathcal{F}_\varrho(y), \mathcal{F}_\varrho(z), \Sigma_{gg'}(\mathcal{F}_\varrho(z'))) \in \mathbb{R}.$$

These shearing parameters are then combined in the *shearing vector*

$$\sigma^\varrho(T, T') = (\sigma_1^\varrho(T, T'), \sigma_2^\varrho(T, T'), \dots, \sigma_{n-1}^\varrho(T, T')) \in \mathbb{R}^{n-1}.$$

We now show that the family of shearing vectors $\sigma^\varrho(T, T')$ defines a relative tangent cycle $\sigma^\varrho \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ for λ valued in the twisted coefficient bundle $\widehat{\mathbb{R}}^n$, as in Proposition 4.9. We begin with the easier part, namely condition (3) of that statement.

LEMMA 5.12. *For any two components T and T' of $\tilde{S} \setminus \tilde{\lambda}$,*

$$\sigma_a^\varrho(T', T) = \sigma_{n-a}^\varrho(T, T').$$

Proof. Using the notation of Figure 6,

$$\begin{aligned} \sigma_a^\varrho(T', T) &= \log D_a(\mathcal{F}_\varrho(y'), \mathcal{F}_\varrho(x'), \mathcal{F}_\varrho(z'), \Sigma_{g'g}(\mathcal{F}_\varrho(z))) \\ &= \log D_{n-a}(\mathcal{F}_\varrho(x'), \mathcal{F}_\varrho(y'), \Sigma_{g'g}(\mathcal{F}_\varrho(z)), \mathcal{F}_\varrho(z')) \\ &= \log D_{n-a}(\Sigma_{g'g}(\mathcal{F}_\varrho(x)), \Sigma_{g'g}(\mathcal{F}_\varrho(y)), \Sigma_{g'g}(\mathcal{F}_\varrho(z)), \mathcal{F}_\varrho(z')) \\ &= \log D_{n-a}(\mathcal{F}_\varrho(x), \mathcal{F}_\varrho(y), \mathcal{F}_\varrho(z), \Sigma_{gg'}(\mathcal{F}_\varrho(z'))) \\ &= \sigma_{n-a}^\varrho(T, T'), \end{aligned}$$

where the second equality is a consequence of the elementary properties of double ratios stated in Lemma 1.7, the third equality comes from the fact that $\Sigma_{g'g}$ sends each line $\tilde{L}_b(g)$ to $\tilde{L}_b(g')$, and the fourth equality follows from the invariance of double ratios under the action of $\Sigma_{gg'} = \Sigma_{g'g}^{-1} \in \text{SL}_n(\mathbb{R})$. □

The quasi-additivity property, namely condition (2) of Proposition 4.9, will involve the quadruple ratios $Q_a(E, F, G)$ of §1.3. The following computation, which connects quadruple ratios to double ratios and slithering maps, is the key to this relationship.

LEMMA 5.13. *Let $x, y, y' \in \partial_\infty \tilde{S}$ be the vertices of a component T of $\tilde{S} \setminus \tilde{\lambda}$, occurring in this order counterclockwise around T . Let g be the side of T going from x to y , and let g' be the side going from x to y' . In particular, the elementary slithering map $\Sigma_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ respects the flag $E = \mathcal{F}_\rho(x) \in \text{Flag}(\mathbb{R}^n)$ and sends $F' = \mathcal{F}_\rho(y')$ to $F = \mathcal{F}_\rho(y)$. Then,*

$$Q_a(E, F, F') = D_a(E, F, F', \Sigma_{gg'}(F))^{-1}.$$

Proof. By definition of the double ratio,

$$D_a(E, F, F', \Sigma_{gg'}(F)) = - \frac{e^{(a)} \wedge f^{(n-a-1)} \wedge f'^{(1)}}{e^{(a)} \wedge f^{(n-a-1)} \wedge \Sigma_{gg'}(f^{(1)})} \frac{e^{(a-1)} \wedge f^{(n-a)} \wedge \Sigma_{gg'}(f^{(1)})}{e^{(a-1)} \wedge f^{(n-a)} \wedge f'^{(1)}}$$

for arbitrary non-zero $e^{(b)} \in \Lambda^b(E^{(b)})$, $f^{(b)} \in \Lambda^b(F^{(b)})$, $f'^{(b)} \in \Lambda^b(F'^{(b)})$.

The elementary slithering map $\Sigma_{gg'}$ respects the flag E and sends F' to F . By condition (3) of Proposition 5.1, it acts trivially on each $\Lambda^b(E^{(b)})$, including $\Lambda^n(E^{(n)}) = \Lambda^n(\mathbb{R}^n)$. If we choose $f'^{(b)} = \Sigma_{gg'}^{-1}(f^{(b)})$, we consequently have that

$$\begin{aligned} e^{(b)} \wedge f^{(n-b-1)} \wedge \Sigma_{gg'}(f^{(1)}) &= \Sigma_{gg'}(e^{(b)}) \wedge \Sigma_{gg'}(f'^{(n-b-1)}) \wedge \Sigma_{gg'}(f^{(1)}) \\ &= \Sigma_{gg'}(e^{(b)} \wedge f'^{(n-b-1)} \wedge f^{(1)}) = e^{(b)} \wedge f'^{(n-b-1)} \wedge f^{(1)} \end{aligned}$$

for every b . Similarly, $e^{(b)} \wedge f^{(n-b)} = e^{(b)} \wedge f'^{(n-b)}$ for every b .

Combining these properties and rearranging terms provides

$$\begin{aligned} D_a(E, F, F', \Sigma_{gg'}(F)) &= - \frac{e^{(a)} \wedge f^{(n-a-1)} \wedge f'^{(1)}}{e^{(a)} \wedge f'^{(n-a-1)} \wedge f^{(1)}} \frac{e^{(a-1)} \wedge f'^{(n-a)} \wedge f^{(1)}}{e^{(a-1)} \wedge f^{(n-a)} \wedge f'^{(1)}} \\ &= \frac{e^{(a)} \wedge f^{(n-a-1)} \wedge f'^{(1)}}{e^{(a-1)} \wedge f^{(n-a)} \wedge f'^{(1)}} \frac{e^{(a-1)} \wedge f^{(1)} \wedge f'^{(n-a)}}{e^{(a)} \wedge f^{(1)} \wedge f'^{(n-a-1)}} \frac{e^{(a+1)} \wedge f'^{(n-a-1)}}{e^{(a+1)} \wedge f^{(n-a-1)}} \frac{e'^{(a)} \wedge f^{(n-a)}}{e^{(a)} \wedge f'^{(n-a)}} \\ &= Q_a(E, H, H')^{-1}. \end{aligned} \quad \square$$

Let s be a slit of λ or, equivalently, a spike of the complement $S \setminus \lambda$. Lift s to a spike of $\tilde{S} \setminus \tilde{\lambda}$, namely to a vertex $x \in \partial_\infty \tilde{S}$ of a triangle component T of $\tilde{S} \setminus \tilde{\lambda}$. Let y and z be the other two vertices of T , indexed so that x, y and z occur in this order counterclockwise around T . The flag curve \mathcal{F}_ρ then determines a positive triple of flags $\mathcal{F}_\rho(x)$, $\mathcal{F}_\rho(y)$ and $\mathcal{F}_\rho(z) \in \text{Flag}(\mathbb{R}^n)$. Considering their quadruple ratios as in §1.3, define

$$\xi_a^\rho(s) = \log Q_a(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z)),$$

which is clearly independent of the lift of the slit s to the universal cover \tilde{S} .

This invariant $\xi_a^\varrho(s)$ can be expressed in terms of the triangle invariants $\tau_{abc}^\varrho(s)$ of ϱ that we introduced in §3.2.

LEMMA 5.14.

$$\xi_a^\varrho(s) = \sum_{b+c=n-a} \tau_{abc}^\varrho(s).$$

Proof. This is an immediate consequence of Lemma 1.5. □

Recall that, by definition, a slit \hat{s} of the orientation cover $\hat{\lambda}$ is positive if the canonical orientation of $\hat{\lambda}$ orients towards \hat{s} the two leaves that are adjacent to \hat{s} , and that \hat{s} is negative when these two leaves are oriented away from \hat{s} .

PROPOSITION 5.15. *The rule $(T, T') \mapsto \sigma_a^\varrho(T, T')$ defines a relative tangent cycle $\sigma_a^\varrho \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$. The boundary $\partial\sigma_a^\varrho: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ is defined by the property that, for every slit \hat{s} of $\hat{\lambda}$ projecting to a slit s of λ ,*

$$\partial\sigma_a^\varrho(\hat{s}) = \begin{cases} \xi_a^\varrho(s), & \text{if } \hat{s} \text{ is a positive slit of } \hat{\lambda}, \\ -\xi_{n-a}^\varrho(s), & \text{if } \hat{s} \text{ is negative.} \end{cases}$$

In the classical case where $n=2$, the boundary $\partial\sigma_1^\varrho$ is trivial, since there are no triangle invariants, and $\sigma_1^\varrho \in \mathcal{Z}(\hat{\lambda}; \mathbb{R})$ is consequently a (non-relative) tangent cycle.

Proof. Using the framework of Proposition 4.8, let T, T' and T'' be three components of $\tilde{S} \setminus \tilde{\lambda}$ such that T'' separates T from T' in \tilde{S} . Let \tilde{s}'' be the spike of T'' delimited by the two sides of T'' that separate T from T' .

We first consider the case where \tilde{s}'' points to the left as seen from T .

Let g be the side of T that is closest to T' and T'' , and let g' be the side of T' that is closest to T and T'' . Let f be the side of T'' that faces T , and let f' be the side of T'' that faces T' . Orient these leaves of $\tilde{\lambda}$ to the left as seen from T . Let $E, F, E', F', E'', H, H' \in \text{Flag}(\mathbb{R}^n)$ be the flags respectively assigned by the flag curve $\mathcal{F}_\varrho: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ to the positive endpoint of g , the negative endpoint of g , the positive endpoint of g' , the negative endpoint of g' , the positive endpoint \tilde{s}'' of f and f' , the negative endpoint of f , and the negative endpoint of f' . Similarly, let $G, G' \in \text{Flag}(\mathbb{R}^n)$ be respectively associated with the vertex of T that is not contained in g , and with the vertex of T' that is not contained in g' . See Figure 7, where the vertices of T, T' and T'' are labelled by the flags assigned to them by the flag curve \mathcal{F}_ϱ .

Then,

$$\sigma_a^\varrho(T, T') = \log D_a(E, F, G, \Sigma_{gg'}(G')) = \log D_a(E'', H, \Sigma_{fg}(G), \Sigma_{fg'}(G')),$$

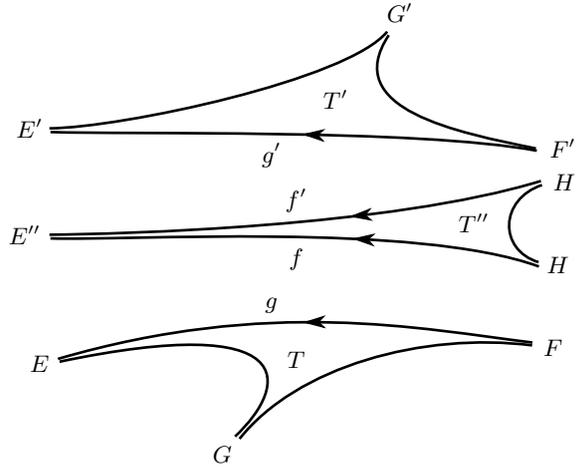


Figure 7. The triangles T , T' and T'' , their sides, and the flags associated with their vertices.

by using the fact that the slithering map Σ_{fg} sends E to E'' and F to H . Similarly,

$$\sigma_a^\theta(T, T'') = \log D_a(E, F, G, \Sigma_{gf}(H')) = \log D_a(E'', H, \Sigma_{fg}(G), H')$$

and

$$\sigma_a^\theta(T'', T') = \log D_a(E'', H', H, \Sigma_{f'g'}(G')) = \log D_a(E'', H, \Sigma_{ff'}(H), \Sigma_{fg'}(G')).$$

Using the elementary properties of double ratios stated in Lemma 1.7, it follows that

$$\sigma_a^\theta(T, T') = \sigma_a^\theta(T, T'') + \sigma_a^\theta(T'', T') + \log D_a(E'', H, H', \Sigma_{ff'}(H)).$$

Lemma 5.13 now shows that

$$\sigma_a^\theta(T, T') = \sigma_a^\theta(T, T'') + \sigma_a^\theta(T'', T') - \log Q_a(E'', H, H') = \sigma_a^\theta(T, T'') + \sigma_a^\theta(T'', T') - \xi_a^\theta(s''),$$

where s'' is the slit of λ that is the projection of the slit \tilde{s}'' of $\tilde{\lambda}$.

This computation holds when \tilde{s}'' points to the left as seen from T . When \tilde{s}'' points to the right, a very similar computation or an application of Lemma 5.12 shows that in this case

$$\sigma_a^\theta(T, T') = \sigma_a^\theta(T, T'') + \sigma_a^\theta(T'', T') - \xi_{n-a}^\theta(s'').$$

Considering these two cases, Proposition 4.8 then shows that the rule

$$(T, T') \mapsto \sigma_a^\theta(T, T')$$

defines a relative tangent cycle $\sigma_a^\theta \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$, whose boundary $\partial \sigma_a^\theta: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}$ is the one described in the statement of Proposition 5.15. This concludes the proof of that proposition. \square

Through Proposition 4.9, the combination of Lemma 5.12 and Proposition 5.15 shows that the relative tangent cycles $\sigma_a^\varrho \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ can be combined to define a relative tangent cycle $\sigma^\varrho \in \mathcal{Z}(\lambda, \text{slits}; \hat{\mathbb{R}}^{n-1})$ valued in the twisted coefficient bundle $\hat{\mathbb{R}}^{n-1}$ introduced in §4.6. This twisted relative tangent cycle is the *shearing cycle* of the Hitchin character $\varrho \in \text{Hit}_n(S)$ with respect to the maximal geodesic lamination λ .

6. Hitchin characters are determined by their invariants

The goal of this section is to show that, if two Hitchin homomorphisms $\varrho, \varrho': \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ have the same triangle invariants and the same shearing cycle, then they represent the same character in the Hitchin component $\text{Hit}_n(S)$.

6.1. Revisiting the slithering map

We give in Proposition 6.2 below a different description of the slithering map $\Sigma_{gg'}$ of §5.1. This new formulation is based on the following simple algebraic trick.

LEMMA 6.1. *Let A_1, A_2, \dots, A_m be elements of a group. Then,*

$$A_1 A_2 \dots A_{m-1} A_m = \hat{A}_m \hat{A}_{m-1} \dots \hat{A}_2 \hat{A}_1$$

where $\hat{A}_i = (A_1 A_2 \dots A_{i-1}) A_i (A_1 A_2 \dots A_{i-1})^{-1}$.

Proof. Observe that $A_1 A_2 \dots A_{m-1} A_m = \hat{A}_m A_1 A_2 \dots A_{m-1}$, and proceed by induction. □

We return to the construction of the slithering map $\Sigma_{gg'}$ in §5.1. Let g and g' be two leaves of the preimage $\tilde{\lambda} \subset \tilde{S}$ of the geodesic lamination λ , and let $\mathcal{T}_{gg'}$ be the set of components of $\tilde{S} \setminus \tilde{\lambda}$ that separate g from g' , where these components are ordered from g to g' . For such a component $T \in \mathcal{T}_{gg'}$, we consider the elementary slithering $\Sigma_T = \Sigma_{g_T g'_T}$ defined by condition (3) of Proposition 5.1, where g_T and g'_T are the two sides of T that are respectively closest to g and g' .

We now consider the infinite product of the maps

$$\hat{\Sigma}_T = \Sigma_{gg_T} \circ \Sigma_T \circ \Sigma_{gg_T}^{-1}.$$

More precisely, let $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ be a finite subset of $\mathcal{T}_{gg'}$, where each T_i separates T_{i+1} from g . We then consider the limit

$$\overleftarrow{\prod}_{T \in \mathcal{T}_{gg'}} \hat{\Sigma}_T = \lim_{\mathcal{T} \rightarrow \mathcal{T}_{gg'}} \hat{\Sigma}_{T_m} \circ \hat{\Sigma}_{T_{m-1}} \circ \dots \circ \hat{\Sigma}_{T_2} \circ \hat{\Sigma}_{T_1}.$$

The reverse arrow on top of the product sign is here to remind us that the composition of the $\widehat{\Sigma}_T$ is taken in the order opposite to the ordering of the elements of $\mathcal{T}_{gg'}$ from g to g' ,

PROPOSITION 6.2. *The slithering map $\Sigma_{gg'}$ is equal to*

$$\Sigma_{gg'} = \overleftarrow{\prod}_{T \in \mathcal{T}_{gg'}} \widehat{\Sigma}_T.$$

Proof. First of all, the fact that the infinite product converges is proved by the estimates of §5.1, using the fact that the Σ_{gg_T} are uniformly bounded (Lemma 5.4) and the estimates on $\Sigma_T - \text{Id}_{\mathbb{R}^n}$ given by Lemmas 5.2 and 5.3.

As usual, let $\mathcal{T} = \{T_1, T_2, \dots, T_m\}$ be a finite subset of $\mathcal{T}_{gg'}$, where each T_i separates T_{i+1} from g . By Lemma 6.1,

$$\overrightarrow{\prod}_{T \in \mathcal{T}} \Sigma_T = \Sigma_{T_1} \circ \Sigma_{T_2} \circ \dots \circ \Sigma_{T_{m-1}} \circ \Sigma_{T_m} = \widehat{\Sigma}_{T_m}^{\mathcal{T}} \circ \widehat{\Sigma}_{T_{m-1}}^{\mathcal{T}} \circ \dots \circ \widehat{\Sigma}_{T_2}^{\mathcal{T}} \circ \widehat{\Sigma}_{T_1}^{\mathcal{T}} = \overleftarrow{\prod}_{T \in \mathcal{T}} \widehat{\Sigma}_T^{\mathcal{T}},$$

where

$$\widehat{\Sigma}_{T_i}^{\mathcal{T}} = (\Sigma_{T_1} \circ \Sigma_{T_2} \circ \dots \circ \Sigma_{T_{i-1}}) \circ \Sigma_{T_i} \circ (\Sigma_{T_1} \circ \Sigma_{T_2} \circ \dots \circ \Sigma_{T_{i-1}})^{-1}.$$

For a fixed T , the map $\widehat{\Sigma}_T^{\mathcal{T}}$ tends to $\widehat{\Sigma}_T = \Sigma_{gg_T} \circ \Sigma_T \circ \Sigma_{gg_T}^{-1}$ as the finite family \mathcal{T} tends to the set $\mathcal{T}_{gg'}$ of all components of $\widetilde{S} \setminus \widetilde{\lambda}$ separating g from g' , by definition of the slithering map. By uniformity in the exponential estimates guaranteeing the convergence of the infinite products, it follows that

$$\Sigma_{gg'} = \lim_{\mathcal{T} \rightarrow \mathcal{T}_{gg'}} \overrightarrow{\prod}_{T \in \mathcal{T}} \Sigma_T = \lim_{\mathcal{T} \rightarrow \mathcal{T}_{gg'}} \overleftarrow{\prod}_{T \in \mathcal{T}} \widehat{\Sigma}_T^{\mathcal{T}} = \lim_{\mathcal{T} \rightarrow \mathcal{T}_{gg'}} \overleftarrow{\prod}_{T \in \mathcal{T}} \widehat{\Sigma}_T = \overleftarrow{\prod}_{T \in \mathcal{T}_{gg'}} \widehat{\Sigma}_T. \quad \square$$

6.2. Reconstructing a Hitchin homomorphism from its invariants

We now show how to reconstruct, up to conjugation by an element of $\text{PSL}_n(\mathbb{R})$, a Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ from its triangle invariants and its shearing cycle.

For this, we first normalize ϱ to avoid having to worry about conjugations. Fix a component T_0 of $\widetilde{S} \setminus \widetilde{\lambda}$, with vertices $x_0, y_0, z_0 \in \partial_\infty \widetilde{S}$. Also, choose a generic flag triple (E_0, F_0, G_0) .

LEMMA 6.3. *The flag triple (E_0, F_0, G_0) can be chosen so that every Hitchin character is represented by a unique homomorphism $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ for which the flag $\mathcal{F}_\varrho(x_0)$ is equal to E_0 , the flag $\mathcal{F}_\varrho(y_0)$ is equal to F_0 , and the line $\mathcal{F}_\varrho(z_0)^{(1)}$ is equal to the line $G_0^{(1)}$.*

Note that the last condition involves only the line $\mathcal{F}_\varrho(z_0)^{(1)}$, not the whole flag $\mathcal{F}_\varrho(z_0)$.

Proof. For a Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$, we know from Theorem 3.3 that the flag triple $(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), \mathcal{F}_\varrho(z_0))$ is positive, and in particular generic. By elementary linear algebra, there consequently exists a unique projective isomorphism $\varphi_\varrho \in \mathrm{PGL}_n(\mathbb{R})$ sending the flag $\mathcal{F}_\varrho(x_0)$ to E_0 , the flag $\mathcal{F}_\varrho(y_0)$ to F_0 , and the line $\mathcal{F}_\varrho(z_0)^{(1)}$ to the line $G_0^{(1)}$. This φ_ϱ depends continuously on ϱ .

If $\varphi_{\varrho_0} \in \mathrm{PGL}_n(\mathbb{R})$ is in $\mathrm{PSL}_n(\mathbb{R})$ for some Hitchin homomorphism ϱ_0 (which is automatic when n is odd), then φ_ϱ belongs to $\mathrm{PSL}_n(\mathbb{R})$ for all ϱ , by connectedness of the space of all Hitchin homomorphisms. Conjugating ϱ by $\varphi_\varrho \in \mathrm{PSL}_n(\mathbb{R})$ replaces the flag curve $\mathcal{F}_\varrho: \partial_\infty \tilde{S} \rightarrow \mathrm{Flag}(\mathbb{R}^n)$ by its composition with the action of φ_ϱ on $\mathrm{Flag}(\mathbb{R}^n)$, and consequently provides a representative of the class of ϱ in $\mathrm{Hit}_n(S)$ with the requested properties. The uniqueness of this representative is an immediate consequence of the fact that the stabilizer of $(E_0, F_0, G_0^{(1)})$ in $\mathrm{PSL}_n(\mathbb{R})$ is trivial.

Otherwise, φ_ϱ is in the complement of $\mathrm{PSL}_n(\mathbb{R})$ in $\mathrm{PGL}_n(\mathbb{R})$ for every Hitchin homomorphism ϱ . Replace the flag triple (E_0, F_0, G_0) by its image under an arbitrary ψ in this complement. This replaces each φ_ϱ by $\psi \circ \varphi_\varrho$, which is now in $\mathrm{PSL}_n(\mathbb{R})$ for every Hitchin homomorphism. We then conclude as in the earlier case. \square

Note that all generic flag triples (E_0, F_0, G_0) can be used in Lemma 6.3 when n is odd, and “half” of them (namely those in one of the two equivalence classes of an appropriate equivalence relation) when n is even.

The following lemma will help in the exposition, by decreasing the number of cases to consider. Let g_0 be the side of T_0 joining x_0 and y_0 , and let h_0 be the side joining x_0 and z_0 .

LEMMA 6.4. *The fundamental group $\pi_1(S)$ is generated by finitely many elements $\gamma \in \pi_1(S)$ whose axes cross both g_0 and h_0 , and send T_0 to a triangle γT_0 contained in the component of $\tilde{S} \setminus T_0$ that is adjacent to g_0 , as in Figure 8.*

Proof. The axes of $\pi_1(S)$ are dense in the space of geodesics of \tilde{S} . Therefore, there exists an element $\gamma_0 \in \pi_1(S)$ whose axis crosses both g_0 and h_0 , and whose attracting fixed point in $\partial_\infty \tilde{S}$ is contained in the closure of the component U of $\tilde{S} \setminus T_0$ delimited by g_0 . In particular, $\gamma_0 T_0$ is contained in U .

Let $\gamma_1, \gamma_2, \dots, \gamma_k$ be a set of generators for $\pi_1(S)$. The ping-pong lemma shows that, for $m_i, n_i > 0$ large enough, the attracting and repelling fixed points of $\gamma'_i = \gamma_0^{m_i} \gamma_i \gamma_0^{n_i}$ are very close to the attracting and repelling fixed points of γ_0 . In particular, the axis of γ'_i crosses both g_0 and h_0 , and $\gamma'_i T_0$ is contained in U .

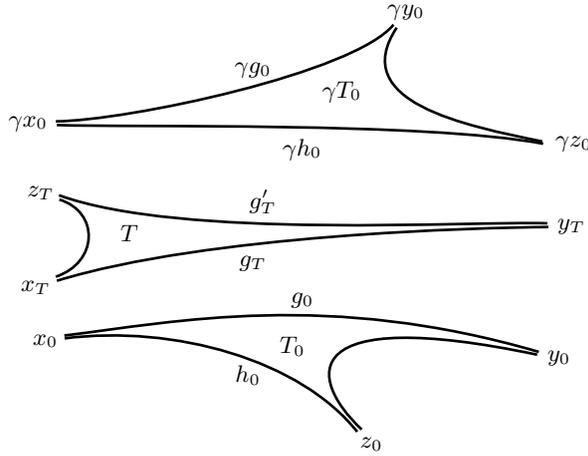


Figure 8. The triangles T_0 , γT_0 and T .

Then, the family of elements $\gamma_0, \gamma'_1, \gamma'_2, \dots, \gamma'_k$ generates $\pi_1(S)$ and has the required properties. \square

For a Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ and an element $\gamma \in \pi_1(S)$, the projective map $\varrho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ sends the flag pair $(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(z_0))$ to $(\mathcal{F}_\varrho(\gamma x_0), \mathcal{F}_\varrho(\gamma z_0))$ by ϱ -equivariance of the flag map \mathcal{F}_ϱ . We would like to compare it to another map sending $(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(z_0))$ to $(\mathcal{F}_\varrho(\gamma x_0), \mathcal{F}_\varrho(\gamma z_0))$, namely the slithering map $\Sigma_{(\gamma h_0)h_0} = \Sigma_{(\gamma h_0)g_0} \circ \Sigma_{g_0 h_0}: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Focusing on the side g_0 of the triangle T_0 instead of h_0 , we conclude that $\Sigma_{g_0(\gamma h_0)} \circ \varrho(\gamma) \circ \Sigma_{h_0 g_0}$ stabilizes the flag pair $(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0))$, and we expect this element of the stabilizer to be related to the shearing vector $\sigma^e(T_0, \gamma T_0) \in \mathbb{R}^{n-1}$.

With this in mind, we introduce a family of elements of the stabilizer of $(E_0, F_0) = (\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0))$, assuming ϱ normalized as in Lemma 6.3. For $t = (t_1, t_2, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$, let u_1, u_2, \dots, u_n be uniquely determined by the properties that $t_a = u_a - u_{a+1}$ and $\sum_{a=1}^n u_a = 0$. Namely,

$$u_a = \frac{1}{n} \sum_{b=1}^{n-1} (n-b)t_b - \sum_{b=1}^{a-1} t_b.$$

Then, let $\Theta_{E_0 F_0}^t: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the element of $\mathrm{SL}_n(\mathbb{R})$ that acts by multiplication by e^{u_a} on each line $E_0^{(a)} \cap F_0^{(n-a+1)}$.

We will also need to replace the slithering map $\Sigma_{g_0 h_0}$ by another map $\Phi_0 \in \mathrm{PGL}_n(\mathbb{R})$ that stabilizes the flag $\mathcal{F}_\varrho(x_0)$, and sends $\mathcal{F}_\varrho(z_0)$ to $\mathcal{F}_\varrho(y_0)$, but is normalized in such a way that

$$D_a(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), \mathcal{F}_\varrho(z_0), \Phi_0(\mathcal{F}_\varrho(y_0))) = 1$$

for every $a \in \{1, 2, \dots, n-1\}$. The existence and uniqueness of the projective map $\Phi_0 \in \mathrm{PGL}_n(\mathbb{R})$ is elementary. By comparison, the slithering map $\Sigma_{g_0 h_0}$ was normalized so that

it acts by the identity on each $\mathcal{F}_\varrho(x_0)^{(a+1)}/\mathcal{F}_\varrho(x_0)^{(a)} \cong \mathbb{R}$. The normalization defining Φ_0 is better adapted to shearing parameters.

LEMMA 6.5. *Let ϱ be normalized as in Lemma 6.3 and $\gamma \in \pi_1(S)$ be as in Lemma 6.4. Then,*

$$\varrho(\gamma) = \Sigma_{g_0(\gamma h_0)}^{-1} \circ \Theta_{E_0 F_0}^{\sigma^\varrho(T_0, \gamma T_0)} \circ \Phi_0 \in \text{PGL}_n(\mathbb{R}),$$

where $\Theta_{E_0 F_0}^t$ and Φ_0 are defined as above, and where $\sigma^\varrho(T_0, \gamma T_0) \in \mathbb{R}^{n-1}$ is the shearing vector of ϱ between T_0 and γT_0 defined in §5.2.

Proof. Note that, since γ satisfies the conclusions of Lemma 6.4, the side of γT_0 that is closest to T_0 is γh_0 , while g_0 is the side of T_0 closest to γT_0 . See Figure 8.

The map $\Sigma_{g_0(\gamma h_0)} \circ \varrho(\gamma)$ sends the flag triple $(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(z_0), \mathcal{F}_\varrho(y_0))$ to

$$(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), G'_0) = (E_0, F_0, G'_0),$$

where $G'_0 = \Sigma_{g_0(\gamma h_0)} \circ \mathcal{F}_\varrho(\gamma y_0)$.

Similarly, $\Theta_{E_0 F_0}^{\sigma^\varrho(T_0, \gamma T_0)} \circ \Phi_0$ sends $(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(z_0), \mathcal{F}_\varrho(y_0))$ to

$$(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), G''_0) = (E_0, F_0, G''_0),$$

where $G''_0 = \Theta_{E_0 F_0}^{\sigma^\varrho(T_0, \gamma T_0)} \circ \Phi_0 \circ \mathcal{F}_\varrho(y_0)$.

We want to compare the lines $G'_0{}^{(1)}$ and $G''_0{}^{(1)}$. The key property is that

$$\begin{aligned} D_a(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), \mathcal{F}_\varrho(z_0), G'_0) &= D_a(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), \mathcal{F}_\varrho(z_0), \Sigma_{g_0(\gamma h_0)} \circ \mathcal{F}_\varrho(\gamma y_0)) \\ &= e^{\sigma_a^\varrho(T_0, \gamma T_0)}, \end{aligned}$$

by definition of the shearing parameter $\sigma_a^\varrho(T_0, \gamma T_0)$.

We have designed the definitions so that

$$\begin{aligned} D_a(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), \mathcal{F}_\varrho(z_0), G''_0) &= D_a(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), \mathcal{F}_\varrho(z_0), \Theta_{E_0 F_0}^{\sigma^\varrho(T_0, \gamma T_0)} \circ \Phi_0(\mathcal{F}_\varrho(y_0))) \\ &= e^{\sigma_a^\varrho(T_0, \gamma T_0)} D_a(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(y_0), \mathcal{F}_\varrho(z_0), \Phi_0(\mathcal{F}_\varrho(y_0))) \\ &= e^{\sigma_a^\varrho(T_0, \gamma T_0)}, \end{aligned}$$

where the second equality comes from Lemma 1.6 and from the definition of $\Theta_{E_0 F_0}^{\sigma^\varrho(T_0, \gamma T_0)}$.

Comparing these two double ratios for every $a=1, 2, \dots, n$, we conclude that the lines $G'_0{}^{(1)}$ and $G''_0{}^{(1)}$ are equal (use for instance Lemma 1.6 again).

Therefore, the two projective maps $\Sigma_{g_0(\gamma h_0)} \circ \varrho(\gamma)$ and $\Theta_{E_0 F_0}^{\sigma^\varrho(T_0, \gamma T_0)} \circ \Phi_0$ send the flag $\mathcal{F}_\varrho(x_0)$ to the same flag $\mathcal{F}_\varrho(x_0)$, send the flag $\mathcal{F}_\varrho(z_0)$ to the same flag $\mathcal{F}_\varrho(y_0)$, and send the line $\mathcal{F}_\varrho(y_0)^{(1)}$ to the same line $G'_0{}^{(1)} = G''_0{}^{(1)}$. By genericity of the flag triple $(\mathcal{F}_\varrho(x_0), \mathcal{F}_\varrho(z_0), \mathcal{F}_\varrho(y_0))$ they consequently coincide, which proves the lemma. \square

In the formula of Lemma 6.5, the term $\Theta_{E_0 F_0}^{\sigma^\ell(T_0, \gamma T_0)}$ depends only on the shearing cycle σ^ℓ . The term Φ_0 is defined by the generic flag triple $(\mathcal{F}_\ell(x_0), \mathcal{F}_\ell(z_0), \mathcal{F}_\ell(y_0))$ and, because of our normalization that $\mathcal{F}_\ell(x_0) = E_0$, $\mathcal{F}_\ell(y_0) = F_0$ and $\mathcal{F}_\ell(z_0)^{(1)} = G_0^{(1)}$, is completely determined by the triangle invariants $\tau_{abc}^\ell(s)$ of the base triangle T_0 . We now turn our attention to the remaining term, the slithering map $\Sigma_{g_0(\gamma h_0)}$.

By Proposition 6.2,

$$\Sigma_{g_0(\gamma h_0)} = \prod_{T \in \mathcal{T}_{g_0(\gamma h_0)}}^{\leftarrow} \widehat{\Sigma}_T,$$

with the notation of that statement.

Consider the contribution $\widehat{\Sigma}_T = \Sigma_{g_0 g_T} \circ \Sigma_T \circ \Sigma_{g_0 g_T}^{-1}$ of a triangle $T \in \mathcal{T}_{g_0(\gamma h_0)}$, separating g_0 from γh_0 .

LEMMA 6.6. *For each triangle $T \in \mathcal{T}_{g_0(\gamma h_0)}$, the map $\widehat{\Sigma}_T \in \text{PSL}_n(\mathbb{R})$ is uniquely determined by the shearing vector $\sigma^\ell(T_0, T) \in \mathbb{R}^{n-1}$ and by the triangle invariants $\tau_{abc}^\ell(s) \in \mathbb{R}$ of the triangle T .*

More precisely,

$$\widehat{\Sigma}_T = \Theta^{\sigma^\ell(T_0, T)} \circ \widehat{\Sigma}'_T \circ \Theta^{-\sigma^\ell(T_0, T)}$$

where $\widehat{\Sigma}'_T$ depends only on the triangle invariants $\tau_{abc}^\ell(s)$ of T .

Proof. Index the vertices of T as x_T, y_T and z_T , in such a way that the side $g_T = y_T x_T$ is the one that is closest to $g_0 = y_0 x_0$, and is oriented in parallel with g_0 . See Figure 8.

As in the definition of the map Φ_0 , there is a unique projective map $\Phi_T \in \text{PGL}_n(\mathbb{R})$ sending $\mathcal{F}_\ell(x_T)$ to $\mathcal{F}_\ell(x_0) = E_0$, $\mathcal{F}_\ell(y_T)$ to $\mathcal{F}_\ell(y_0) = F_0$, and $\mathcal{F}_\ell(z_T)$ to a flag G'_T such that

$$D_a(\mathcal{F}_\ell(x_0), \mathcal{F}_\ell(y_0), \mathcal{F}_\ell(z_0), G'_T) = 1$$

for every $a = 1, 2, \dots, n$.

A fundamental observation is that, although the map Φ_T depends on the flag triple $(\mathcal{F}_\ell(x_T), \mathcal{F}_\ell(y_T), \mathcal{F}_\ell(z_T))$, the flag $G'_T = \Phi_T(\mathcal{F}_\ell(z_T))$ depends only on the orbit of that flag triple under the action of $\text{PGL}_n(\mathbb{R})$ or, equivalently, on the triangle invariants $\tau_{abc}^\ell(s)$ of the triangle T .

The two maps Φ_T and $\Sigma_{g_0 g_T}$ both send $\mathcal{F}_\ell(x_T)$ to $\mathcal{F}_\ell(x_0)$ and $\mathcal{F}_\ell(y_T)$ to $\mathcal{F}_\ell(y_0)$. As in the proof of Lemma 6.5,

$$D_a(\mathcal{F}_\ell(x_0), \mathcal{F}_\ell(y_0), \mathcal{F}_\ell(z_0), \Sigma_{g_0 g_T}(\mathcal{F}_\ell(z_T))) = e^{\sigma_a(T_0, T)}$$

by definition of the shearing parameter $\sigma_a(T_0, T)$, and

$$D_a(\mathcal{F}_\ell(x_0), \mathcal{F}_\ell(y_0), \mathcal{F}_\ell(z_0), \Theta_{E_0 F_0}^{\sigma^\ell(T_0, T)} \circ \Phi_T(\mathcal{F}_\ell(z_T))) = e^{\sigma_a(T_0, T)}$$

by construction. Again as in the proof of Lemma 6.5, it follows that

$$\Sigma_{g_0 g_T} = \Theta_{E_0 F_0}^{\sigma^\ell(T_0, T)} \circ \Phi_T.$$

Noting that $(\Theta_{E_0 F_0}^t)^{-1} = \Theta_{E_0 F_0}^{-t}$, we set

$$\begin{aligned} \widehat{\Sigma}'_T &= \Theta_{E_0 F_0}^{-\sigma^\ell(T_0, T)} \circ \widehat{\Sigma}'_T \circ \Theta_{E_0 F_0}^{\sigma^\ell(T_0, T)} \\ &= \Theta_{E_0 F_0}^{-\sigma^\ell(T_0, T)} \circ \Sigma_{g_0 g_T} \circ \Sigma_T \circ \Sigma_{g_0 g_T}^{-1} \circ \Theta_{E_0 F_0}^{\sigma^\ell(T_0, T)} = \Phi_T \circ \Sigma_T \circ \Phi_T^{-1}. \end{aligned}$$

There are now two cases to distinguish in the exposition, according to whether the side g'_T of T that is closest to γT_0 is equal to $z_T x_T$ or to $y_T z_T$.

Consider the case where T points to the right, namely where g'_T is equal to $y_T z_T$, as in Figure 8. Then, the elementary slithering $\Sigma_T = \Sigma_{g_T g'_T}$ is the unique linear map that fixes the flag $\mathcal{F}_\varrho(y_T)$, acts by the identity on each line $\mathcal{F}_\varrho(y_T)^{(a+1)} / \mathcal{F}_\varrho(y_T)^{(a)}$, and sends the flag $\mathcal{F}_\varrho(z_T)$ to $\mathcal{F}_\varrho(x_T)$. It follows that $\widehat{\Sigma}'_T = \Phi_T \circ \Sigma_T \circ \Phi_T^{-1}$ is the unique linear map that fixes the flag $\Phi_T(\mathcal{F}_\varrho(y_T)) = F_0$, acts as the identity on each line $F_0^{(a+1)} / F_0^{(a)}$, and sends the flag $\Phi_T(\mathcal{F}_\varrho(z_T)) = G'_T$ to $\Phi_T(\mathcal{F}_\varrho(x_T)) = E_0$. Since we already observed that the flag G'_T is uniquely determined by the triangle invariants $\tau_{abc}^\varrho(s)$, so is the map $\widehat{\Sigma}'_T$ and

$$\widehat{\Sigma}'_T = \Theta^{\sigma^\ell(T_0, T)} \circ \widehat{\Sigma}'_T \circ \Theta^{-\sigma^\ell(T_0, T)}$$

depends only on these triangle invariants as well as the shearing vector $\sigma^\ell(T_0, T)$.

The argument is essentially identical in the case where the triangle T points to the left, namely when g'_T is equal to the geodesic $z_T x_T$. In this case, $\widehat{\Sigma}'_T$ is the unique linear map that fixes E_0 , acts as the identity on each line $E_0^{(a+1)} / E_0^{(a)}$, and sends G'_T to F_0 . Again, this proves that $\widehat{\Sigma}'_T$ is uniquely determined by the triangle invariants $\tau_{abc}^\varrho(s)$. \square

LEMMA 6.7. *Let ϱ be normalized as in Lemma 6.3 and $\gamma \in \pi_1(S)$ be as in Lemma 6.4. Then,*

$$\varrho(\gamma) = \left(\prod_{T \in \mathcal{T}_{g_0(\gamma h_0)}} \left(\Theta_{E_0 F_0}^{\sigma^\ell(T_0, T)} \circ \widehat{\Sigma}'_T \circ \Theta_{E_0 F_0}^{-\sigma^\ell(T_0, T)} \right) \right)^{-1} \circ \Theta_{E_0 F_0}^{\sigma^\ell(T_0, \gamma T_0)} \circ \Phi_0$$

in $\text{PGL}_n(\mathbb{R})$, with the definitions introduced above. In particular, the maps $\widehat{\Sigma}'_T$ and Φ_0 depend only on the triangle invariants $\tau_{abc}^\varrho(s)$ of ϱ , while the terms $\Theta_{E_0 F_0}^{\pm\sigma^\ell(T_0, T)}$ are determined by its shearing cycle $\sigma^\ell \in \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$.

Proof. This follows from the combination of Lemmas 6.5 and 6.6, together with the observation that Φ_0 depends only on the triangle invariants τ_{abc}^ϱ . \square

COROLLARY 6.8. *Two Hitchin homomorphisms $\varrho, \varrho' : \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$ that have the same triangle invariants $\sigma_{abc}^\varrho(s) = \sigma_{abc}^{\varrho'}(s)$ and the same shearing cycles $\sigma^\ell = \sigma^{\ell'} \in \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$ are conjugate by an element of $\text{PSL}_n(\mathbb{R})$, and therefore represent the same character in $\text{Hit}_n(S)$.*

Proof. Conjugate ϱ and ϱ' by elements of $\mathrm{PSL}_n(\mathbb{R})$ in order to normalize them as in Lemma 6.3. Then, for every element $\gamma \in \pi_1(S)$ satisfying the conditions of Lemma 6.4, the formula of Lemma 6.7 shows that $\varrho(\gamma) = \varrho'(\gamma)$. Since these γ generate $\pi_1(S)$, this proves that $\varrho = \varrho'$. \square

7. Length functions

Our next goal is to determine which triangle invariants and shearing cycles can be realized as invariants of Hitchin characters. The length functions considered in this section provide one of the constraints that need to be satisfied by these invariants.

7.1. Length functions associated with Hitchin characters

Let $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin homomorphism. Labourie proves in [27] that, for every non-trivial $\gamma \in \pi_1(S)$, the matrix $\varrho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ is diagonalizable and its eigenvalues can be indexed as $\mu_1(\varrho(\gamma)), \mu_2(\varrho(\gamma)), \dots, \mu_n(\varrho(\gamma))$ in such a way that

$$\frac{\mu_a(\varrho(\gamma))}{\mu_{a+1}(\varrho(\gamma))} > 1$$

for every $i=1, 2, \dots, n-1$. (Note that eigenvalues of an element of $\mathrm{PSL}_n(\mathbb{R})$ are only defined up to sign when n is even, but that the quotient between two such eigenvalues makes intrinsic sense.) This property is in fact an easy consequence of Theorem 3.1.

Eigenvalues are invariant under conjugation. This consequently defines $n-1$ functions

$$\ell_a^{\varrho}: \{\text{non-trivial conjugacy classes of } \pi_1(S)\} \longrightarrow \mathbb{R}$$

by the property that

$$\ell_a^{\varrho}(\gamma) = \log \frac{\mu_a(\varrho(\gamma))}{\mu_{a+1}(\varrho(\gamma))} > 0.$$

The same conjugation invariance shows that the length function ℓ_a^{ϱ} depends only on the Hitchin character $\varrho \in \mathrm{Hit}_n(S)$, not on the Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ that represents it.

The set of conjugacy classes of the fundamental group $\pi_1(S)$ is discrete, but these length functions have a natural extension to a continuous space. Indeed, endowing the surface S with an arbitrary negatively curved riemannian metric, a conjugacy class of $\pi_1(S)$ uniquely determines an oriented closed geodesic of S , and therefore a closed orbit of the geodesic flow of the unit tangent bundle T^1S . This closed leaf is endowed with an integer weight $m > 0$ if the conjugacy class is an m -power of a primitive class. Considering

the Dirac transverse measure defined by this closed orbit and this weight, this provides an analytic interpretation of a conjugacy class of $\pi_1(S)$ as a transverse measure for the *geodesic foliation* \mathcal{F}_S of T^1S , whose leaves are the orbits of the geodesic flow.

This defines a completion of the set of conjugacy classes of $\pi_1(S)$ by the space $\mathcal{C}^m(S)$ of all (positive Radon) *transverse measures* for the geodesic foliation \mathcal{F}_S [2], [3], [4], analogous to Thurston’s completion [39], [17], [36] of the set of isotopy classes of simple closed curves in S by the space $\mathcal{ML}(S)$ of measured laminations on S .

For differentiability properties, it is useful to consider more general transverse structures for the geodesic foliation, namely *transverse Hölder distributions* in the sense of [7] and [6]. When the geodesic foliation \mathcal{F}_S is realized by the geodesic flow of a riemannian metric m of constant negative curvature, it inherits from that construction a transverse differential structure, namely a smooth structure on the space of leaves of $\mathcal{F}_S \cap U$ for every sufficiently small open subset of T^1S . A transverse Hölder distribution then is a distribution, in the usual sense, on these local leaf spaces with additional regularity properties that makes the notion independent of the auxiliary riemannian metric m . See [7] and [6] for details.

In particular, a transverse measure is a special type of transverse Hölder distribution. This embeds the set of conjugacy classes of $\pi_1(S)$ in the topological vector space $\mathcal{C}^H(S)$ of all transverse Hölder distributions for the geodesic foliation \mathcal{F}_S . In other words, we now have embeddings

$$\{\text{non-trivial conjugacy classes of } \pi_1(S)\} \subset \mathcal{C}^m(S) \subset \mathcal{C}^H(S).$$

The elements of $\mathcal{C}^m(S)$ and $\mathcal{C}^H(S)$ are respectively called *measure geodesic currents* and *Hölder geodesic currents* for the surface S . See the references mentioned above for a proof that these constructions depend only on the topology of the surface S .

THEOREM 7.1. ([15]) *For each Hitchin character $\varrho \in \text{Hit}_n(S)$ and for each $a=1, 2, \dots, n-1$, the length function*

$$\ell_a^\varrho: \{\text{non-trivial conjugacy classes of } \pi_1(S)\} \longrightarrow \mathbb{R}$$

extends to a continuous linear map $\ell_a^\varrho: \mathcal{C}^H(S) \rightarrow \mathbb{R}$.

Remark 7.2. The reader should beware that the above functions ℓ_a^ϱ are slightly different from those introduced in [15]. Namely, our functions ℓ_a^ϱ would be called $\ell_a^\varrho - \ell_{a+1}^\varrho$ in [15]. Although mathematically equivalent to those of [15], our conventions are better adapted to the framework of the current article, as will be apparent in Proposition 7.4 and Theorem 7.5 below.

Remark 7.3. By linearity and continuity, the extension $\ell_a^g: \mathcal{C}^H(S) \rightarrow \mathbb{R}$ is uniquely determined on the closure of the set of all linear combinations of conjugacy classes of $\pi_1(S)$. We do not know if this closure is equal to all of $\mathcal{C}^H(S)$ (this seems unlikely), but it can be shown to contain all the Hölder geodesic currents that we will encounter in this article, including the image of the embedding $\mathcal{Z}(\hat{\lambda}; \mathbb{R}) \rightarrow \mathcal{C}^H(S)$ that will appear in §7.2.

The following statement will be particularly important in our characterization of which relative tangent cycles can occur as shearing cycles of Hitchin characters.

PROPOSITION 7.4. *Let $\alpha \in \mathcal{C}^m(S)$ be a non-zero measure geodesic current. Then,*

$$\ell_a^g(\alpha) > 0$$

for every Hitchin character $\varrho \in \text{Hit}_n(S)$ and every $a=1, 2, \dots, n-1$.

Proof. This is a simple consequence of the Anosov property of Theorem 3.1.

For this, we need to remind the reader of the construction of the length functions $\ell_a^g: \mathcal{C}^H(S) \rightarrow \mathbb{R}$ in [15], taking Remark 7.2 into account. As in §3.1, consider the geodesic flow $(g_t)_{t \in \mathbb{R}}$ on the unit tangent bundle T^1S (for an arbitrary metric of negative curvature) and its flat lift to a flow $(G_t)_{t \in \mathbb{R}}$ on the vector bundle $T^1S \times_{\varrho'} \mathbb{R}^n$, twisted by a homomorphism $\varrho': \pi_1(S) \rightarrow \text{SL}_n(\mathbb{R})$ lifting ϱ . In addition, choose a riemannian metric $\|\cdot\|$ on the vector bundle $T^1S \times_{\varrho'} \mathbb{R}^n \rightarrow T^1S$.

The vector bundle $T^1S \times_{\varrho'} \mathbb{R}^n \rightarrow T^1S$ splits as a direct sum of line bundles $L_a \rightarrow T^1S$ as in §3.1. For $a=1, 2, \dots, n$, this data provides a function $f_a: T^1S \rightarrow \mathbb{R}$ defined by the property that, for $x \in T^1S$,

$$f_a(x) = - \left(\frac{d}{dt} \log \|G_t(v_a(x))\|_{g_t(x)} \right) \Big|_{t=0},$$

where $v_a(x)$ is an arbitrary non-zero vector in the fiber $L_a(x)$ of the line bundle $L_a \rightarrow T^1S$. For a measure geodesic current $\alpha \in \mathcal{C}^m(S)$, the length $\ell_a^g(\alpha)$ is then defined as the integral

$$\ell_a^g(\alpha) = \int_{T^1S} (f_a - f_{a+1}) \alpha \times dt$$

of the function $f_a - f_{a+1}$ with respect to the measure $\alpha \times dt$ on T^1S that, locally, is the product of the transverse measure α for the geodesic flow $(g_t)_{t \in \mathbb{R}}$ with the measure dt along the orbits of this geodesic flow.

The measure $\alpha \times dt$ is invariant under the geodesic flow. Therefore, for every $t_0 > 0$,

$$\begin{aligned} \int_{T^1S} f_a \alpha \times dt &= \int_{T^1S} f_a \circ g_u \alpha \times dt \\ &= \frac{1}{t_0} \int_{T^1S} \int_0^{t_0} f_a \circ g_u \, du \, \alpha \times dt \\ &= \frac{1}{t_0} \int_{T^1S} \int_0^{t_0} -\frac{d}{du} \log \|G_u(v_a(x))\|_{g_u(x)} \, du \, \alpha \times dt(x) \\ &= \frac{1}{t_0} \int_{T^1S} \log \frac{\|v_a(x)\|_x}{\|G_{t_0}(v_a(x))\|_{g_{t_0}(x)}} \, \alpha \times dt(x), \end{aligned}$$

so that

$$\ell_a^e(\alpha) = \frac{1}{t_0} \int_{T^1S} \log \frac{\|v_a(x)\|_x}{\|G_{t_0}(v_a(x))\|_{g_{t_0}(x)}} \frac{\|G_{t_0}(v_{a+1}(x))\|_{g_{t_0}(x)}}{\|v_{a+1}(x)\|_x} \, \alpha \times dt(x).$$

Theorem 3.1 provides constants $A, B > 0$ such that

$$\log \frac{\|v_a(x)\|_x}{\|G_{t_0}(v_a(x))\|_{g_{t_0}(x)}} \frac{\|G_{t_0}(v_{a+1}(x))\|_{g_{t_0}(x)}}{\|v_{a+1}(x)\|_x} \geq \log A + Bt_0$$

for every $t_0 > 0$. In particular, this integrand is strictly positive for t_0 large enough, and it follows that the integral $\ell_a^e(\alpha)$ is strictly positive. □

7.2. Shearing cycles and length functions

We now consider a special type of Hölder geodesic current.

We saw in §4.1 that a positive tangent cycle $\mu \in \mathcal{Z}(\hat{\lambda}; \mathbb{R})$ determines a transverse measure for $\hat{\lambda}$. A general tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}; \mathbb{R})$ determines a transverse Hölder distribution, which lifts to a Hölder geodesic current $\alpha \in \mathcal{C}^H(S)$ [7], [6]. This provides an embedding $\mathcal{Z}(\hat{\lambda}; \mathbb{R}) \subset \mathcal{C}^H(S)$, and the length functions $\ell_a^e: \mathcal{C}^H(S) \rightarrow \mathbb{R}$ of the previous section restrict to linear functions $\ell_a^e: \mathcal{Z}(\hat{\lambda}; \mathbb{R}) \rightarrow \mathbb{R}$.

The following statement connects these length functions to the homological interpretation of relative tangent cycles in §4.5, based on a train-track neighborhood \hat{U} of the orientation cover $\hat{\lambda}$. Proposition 4.5 provides isomorphisms $\mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) \cong H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$ and $\mathcal{Z}(\hat{\lambda}; \mathbb{R}) \cong H_1(\hat{U}; \mathbb{R})$.

THEOREM 7.5. *Let $\varrho \in \text{Hit}_n(S)$ be a Hitchin character with shearing cycle*

$$\sigma^e \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \subset \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1}).$$

Then, for every $a = 1, 2, \dots, n - 1$, the a -th component $\sigma_a^e \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) \cong H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$ of σ^e is related to the length function $\ell_a^e: \mathcal{Z}(\hat{\lambda}; \mathbb{R}) \rightarrow \mathbb{R}$ by the property that

$$\ell_a^e(\alpha) = [\alpha] \cdot [\sigma_a^e]$$

for every tangent cycle $\alpha \in \mathcal{Z}(\hat{\lambda}; \mathbb{R}) \cong H_1(\widehat{U}; \mathbb{R})$, where \cdot denotes the algebraic intersection number of relative homology classes in the train-track neighborhood \widehat{U} of the orientation cover $\hat{\lambda}$.

In the case where $n=2$, this statement is implicit in [42], and explicitly expressed in [5, Theorem E] in terms of the Thurston intersection form on the space $\mathcal{Z}(\lambda; \mathbb{R})$ of tangent cycles.

We will split the proof of Theorem 7.5 into several lemmas.

We first give a homological interpretation of the length function $\ell_a^o: \mathcal{Z}(\hat{\lambda}; \mathbb{R}) \rightarrow \mathbb{R}$, by connecting it to a certain closed 1-form Ω_a defined on \widehat{U} . Because of the lack of transverse regularity of the geodesic lamination $\hat{\lambda}$ and of the line bundles $L_a \rightarrow T^S$, we however need a version of closed 1-form that is weaker than the usual differentiable definition.

7.2.1. Topological closed 1-forms

Remember that a differential form is closed if and only if it is locally exact. This motivates the following definition. A *topological closed 1-form* Ω on a topological space X is defined by a family $\{(V_i, F_i)\}_{i \in I}$ such that the following conditions hold:

- (1) each V_i is open in X , and $X = \bigcup_{i \in I} V_i$;
- (2) each $F_i: V_i \rightarrow \mathbb{R}$ is a continuous function;
- (3) for every $i, j \in I$, the function $F_i - F_j$ is locally constant on $V_i \cap V_j$.

By definition, two such families $\{(V_i, F_i)\}_{i \in I}$ and $\{(V_{i'}, F_{i'})\}_{i' \in I'}$ define the same topological closed 1-form when, for every $i \in I$ and $i' \in I'$, the function $F_i - F_{i'}$ is locally constant on $V_i \cap V_{i'}$. In other words, a topological closed 1-form Ω is defined as an equivalence class of families $\{(V_i, F_i)\}_{i \in I}$ satisfying the above three conditions. Equivalently, a topological closed 1-form is a maximal family $\{(V_i, F_i)\}_{i \in I}$ satisfying the above three conditions.

The motivation of the definition comes of course from the case where X is a differentiable manifold and all functions F_i are differentiable. Such a family of charts $\{(V_i, F_i)\}_{i \in I}$ uniquely determines a closed differential 1-form Ω by the property that $\Omega|_{V_i} = dF_i$ on each chart $V_i \subset X$.

The definition of a topological closed 1-form Ω is specially designed so that we can integrate Ω over every oriented continuous arc k . Indeed, if Ω is defined by a family of charts $\{(V_i, F_i)\}_{i \in I}$, the arc k can be split into a finite family of arcs k_1, k_2, \dots, k_p such that each k_j is contained in some chart V_{i_j} . Then, the integral of Ω over the arc k is defined as

$$\int_k \Omega = \sum_{j=1}^p (F_{i_j}(x_j^+) - F_{i_j}(x_j^-)),$$

where $x_j^+, x_j^- \in V_{i_j}$ are the positive and negative endpoints of the arc k_j . It is immediate that this definition is independent of the decomposition of k into subarcs k_j (and of the chart V_{i_j} containing each individual k_j), and that the integral $\int_k \Omega$ is invariant under homotopy of k fixing its endpoints.

The construction is of course designed to coincide with the usual definition when Ω is a closed differential 1-form and the arc k is differentiable.

The invariance of the integral $\int_k \Omega$ under homotopy of k fixing endpoints yields the following result.

LEMMA 7.6. *A topological closed 1-form Ω on the space X defines a cohomology class $[\Omega] \in H^1(X; \mathbb{R})$, represented by the singular cochain assigning the integral $\int_k \Omega$ to each continuous arc (=singular 1-simplex) k .*

When X is a metric space, we say that a topological closed 1-form Ω is *Hölder continuous* if it can be defined by a family of charts $\{(V_i, F_i)\}_{i \in I}$, where each function $F_i: V_i \rightarrow \mathbb{R}$ is Hölder continuous. The terminology is convenient but somewhat abusive, because such a form is only, locally, the “differential” of a Hölder continuous function.

7.2.2. A homological interpretation of the lengths of tangent cycles

The length functions ℓ_a^g are constructed in [15] by using the differential 1-forms $\omega_a = f_a dt$, defined on the orbits of the geodesic flow in the unit tangent bundle T^1S , that we encountered in the proof of Proposition 7.4.

The (geodesic) leaves of $\hat{\lambda}$ have canonical lifts to orbits of the geodesic flow in T^1S , so that the 1-forms ω_a induce 1-forms along the leaves of $\hat{\lambda}$, which we will also denote by ω_a . We want to extend the forms ω_a to the train-track neighborhood \hat{U} of $\hat{\lambda}$.

We first extend ω_a to the subset $\hat{\lambda}$.

LEMMA 7.7. *There exists a Hölder continuous topological closed 1-form Ω_a on the orientation cover $\hat{\lambda}$ whose restriction to each leaf of $\hat{\lambda}$ is equal to the form $\omega_a = f_a dt$ above.*

Proof. We need to remember the precise definition of the form ω_a . We fix a riemannian metric $\|\cdot\|$ on the flat bundle $T^1S \tilde{\times}_{\varrho'} \mathbb{R}^n$, where $\varrho': \pi_1(S) \rightarrow \text{SL}_n(\mathbb{R})$ lifts the Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$. Each $x \in \hat{\lambda}$ determines a point (x, v) in the unit tangent bundle T^1S by considering the unit tangent vector $v \in T_x^1S$ of the (oriented) leaf of $\hat{\lambda}$ passing through x . This enables us to identify $\hat{\lambda}$ to a subset of T^1S , which is a union of orbits of the geodesic flow.

Locally choose a non-zero section $x \mapsto v_a(x)$ of the restriction of the line bundle L_a over this subset $\hat{\lambda}$, in such a way that the section is flat over each of the leaves of $\hat{\lambda}$. This

restriction is possible because the line bundle is flat over these leaves, by invariance of L_a under the flat lift of the geodesic flow (see Theorem 3.1). The choice of the local section $x \mapsto v_a(x)$ locally defines a function F_a on $\hat{\lambda}$ by the property that

$$F_a(x) = -\log \|v_a(x)\|_x,$$

which is smooth along each leaf of $\hat{\lambda}$. Then ω_a is defined as $\omega_a = dF_a$ on each leaf of $\hat{\lambda}$. The construction is designed so that ω_a does not depend on the choice of the local section $v_a(x)$. However, its “antidifferential” F_a does, and the crux of the argument is to choose the local sections $x \mapsto v_a(x)$ so that locally F_a only changes by a constant as one moves from one chart to another, as required in the definition of topological closed 1-forms. We will use the slithering maps of §5.1 for this purpose.

Every $x \in \hat{\lambda}$ admits a small neighborhood $V_x \subset \hat{U}$ such that, if we lift V_x to an open subset \tilde{V}_x in the universal covering \tilde{S} , the leaves of $\tilde{\lambda} \subset \tilde{S}$ that meet \tilde{V}_x are parallel in the following sense: given any three such leaves g, g' and g'' , one of them separates the other two in \tilde{S} . The leaves of $\tilde{\lambda} \cap \tilde{V}_x$ are oriented by the canonical orientation of the leaves of $\hat{\lambda} \cap V_x$. As in §5.1, the lift of the line bundle L_a associates a line $\tilde{L}_a(g) \subset \mathbb{R}^n$ with each oriented leaf of $\tilde{\lambda}$.

Pick an arbitrary non-zero vector $\tilde{v}_a^x(x) \in \tilde{L}_a(g_x)$ for the leaf g_x of $\tilde{\lambda}$ that contains the lift $\tilde{x} \in \tilde{V}_x$ of $x \in V_x$. Then, for every $y \in V_x \cap \hat{\lambda}$, define $\tilde{v}_a^x(y) = \Sigma_{g_y g_x}(\tilde{v}_a^x(x)) \in \tilde{L}_a(g_y)$, where $\Sigma_{g_y g_x}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the slithering map of §5.1. This map \tilde{v}_a projects to a section v_a of the line bundle L_a over $V_x \cap \hat{\lambda}$, which is flat along the leaves of $V_x \cap \hat{\lambda}$ because $\tilde{v}_a^x(y)$ depends only on the leaf g_y , and which is Hölder continuous because of the Hölder continuity property of the slithering map $\Sigma_{g_y g_x}$ (condition (2) of Proposition 5.1). We can then consider the Hölder continuous function $F_a^x: V_x \cap \hat{\lambda} \rightarrow \mathbb{R}$ defined by

$$F_a^x(y) = -\log \|v_a^x(y)\|_y.$$

We claim that the family $\{(V_x \cap \hat{\lambda}, F_a^x)\}_{x \in \hat{\lambda}}$ defines a Hölder continuous topological 1-form Ω_a . Namely, we will prove that $F_a^x - F_a^{x'}$ is locally constant near each $y \in V_x \cap V_{x'} \cap \hat{\lambda}$.

Consider such a point $y \in V_x \cap V_{x'} \cap \hat{\lambda}$. We arranged that the leaves of $\tilde{\lambda}$ that meet \tilde{V}_x are parallel to each other. By Lemma 5.6, it follows that for every $z \in V_x \cap V_{x'} \cap \hat{\lambda}$

$$\tilde{v}_a^x(z) = \Sigma_{g_z g_x}(\tilde{v}_a^x(x)) = \Sigma_{g_z g_y} \circ \Sigma_{g_y g_x}(\tilde{v}_a^x(x)) = \Sigma_{g_z g_y}(\tilde{v}_a^x(y)).$$

In the construction of the local sections v_a^x and $v_a^{x'}$, we can choose the lift $\tilde{V}_{x'} \subset \tilde{S}$ so that it contains the lift $\tilde{y} \in \tilde{V}_x$ of $y \in V_x \cap V_{x'}$. In particular, the vectors $\tilde{v}_a^x(y)$ and $\tilde{v}_a^{x'}(y)$ are both in the line $\tilde{L}_a(g_y)$ associated with the leaf of $\tilde{\lambda}$ passing through \tilde{y} , and there exists

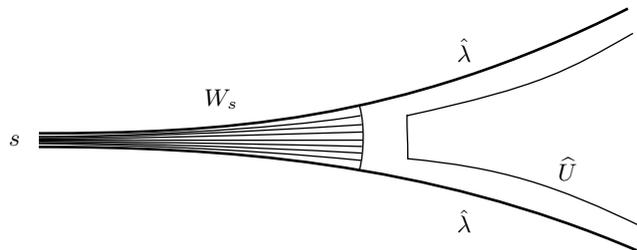


Figure 9. A spike s of $\widehat{U} \setminus \widehat{\lambda}$ and its neighborhood W_s .

a non-zero number $c \in \mathbb{R}$ such that $\tilde{v}_a^{x'}(y) = c\tilde{v}_a^x(y)$. Each $z \in V_x \cap V_{x'} \cap \widehat{\lambda}$ that is sufficiently close to y lifts to the same point \tilde{z} in $\tilde{V}_{x'}$ and in \tilde{V}_x , and as before,

$$\tilde{v}_a^{x'}(z) = \Sigma_{g_z g_y}(\tilde{v}_a^{x'}(y)) = \Sigma_{g_z g_y}(c\tilde{v}_a^x(y)) = c\tilde{v}_a^x(z).$$

It follows that $v_a^{x'}(z) = cv_a^x(z)$ for $z \in \widehat{\lambda}$ sufficiently close to y .

This proves that

$$F_a^x(z) - F_a^{x'}(z) = -\log \|v_a^x(z)\|_z + \log \|v_a^{x'}(z)\|_z = -\log \|v_a^x(z)\|_z + \log \|cv_a^x(z)\|_z = \log |c|$$

for every $z \in \widehat{\lambda}$ near y . As a consequence, we have that $F_a^x - F_a^{x'}$ is locally constant near each $y \in V_x \cap V_{x'} \cap \widehat{\lambda}$.

This concludes the proof that the family $\{(V_x \cap \widehat{\lambda}, F_a^x)\}_{x \in \widehat{\lambda}}$ defines a Hölder continuous topological closed 1-form Ω_a on $\widehat{\lambda}$. This form Ω_a was specially designed so that its restriction to the leaves of $\widehat{\lambda}$ coincides with the original 1-form ω_a . \square

We now extend Ω_a to the whole train-track neighborhood \widehat{U} .

LEMMA 7.8. *There exists a Hölder continuous topological closed 1-form Ω_a on the train-track neighborhood \widehat{U} of $\widehat{\lambda}$ whose restriction to each leaf of $\widehat{\lambda}$ is equal to the form ω_a .*

Proof. Each spike s of $\widehat{U} \setminus \widehat{\lambda}$ admits a neighborhood W_s foliated by half-geodesics converging to s ; see Figure 9. Choose these neighborhoods so that they are pairwise disjoint, and so that each of them is delimited by a differentiable (say) arc.

The union W of $\widehat{\lambda}$ and of these neighborhoods W_s is then foliated by complete geodesics (namely leaves of $\widehat{\lambda}$) and complete half-geodesics (in the neighborhoods W_s of the spikes). The slithering maps of §5.1 straightforwardly extend to the preimage of W in the universal cover \tilde{S} , by elementary slitherings on each component of the preimage of the W_s . This enables us to use the argument of the proof of Lemma 7.7 to extend the topological closed 1-form Ω_a to W .

Each component A of $\widehat{U} \setminus W$ is an annulus, bounded on one side by a component of $\partial\widehat{U}$ and on the other side by a dodecagon, made up of six arcs contained in leaves of $\widehat{\lambda}$ and of six arcs delimiting the W_s associated with the six spikes of the component of $\widehat{U} \setminus \widehat{\lambda}$ that contains A . As a consequence, there is a Hölder continuous retraction $r: \widehat{U} \rightarrow W$. We can then pull back by r the topological closed 1-form Ω that we have constructed on W to extend it to \widehat{U} .

Since the retraction r fixes W , this does not change the restriction of Ω to the leaves of $\widehat{\lambda}$, which consequently coincides with the form $\omega_a = f_a dt$. □

By Lemma 7.6, the topological closed 1-form Ω_a provided by Lemma 7.8 defines a cohomology class $[\Omega_a] \in H^1(\widehat{U}; \mathbb{R})$. Also, a tangent cycle $\alpha \in \mathcal{Z}(\widehat{\lambda}; \mathbb{R})$ determines a homology class $[\alpha] \in H_1(\widehat{U}; \mathbb{R})$ by Proposition 4.2. The following statement connects these to the length $\ell_a^g(\alpha) \in \mathbb{R}$.

LEMMA 7.9. *For every tangent cycle $\alpha \in \mathcal{Z}(\widehat{\lambda}; \mathbb{R})$, the length $\ell_a^g(\alpha)$ is equal to the evaluation*

$$\ell_a^g(\alpha) = \langle [\Omega_a] - [\Omega_{a+1}], [\alpha] \rangle$$

of the cohomology class $[\Omega_a] - [\Omega_{a+1}] \in H^1(\widehat{U}; \mathbb{R})$ over the homology class $[\alpha] \in H_1(\widehat{U}; \mathbb{R})$.

Proof. The tangent cycle $\alpha \in \mathcal{Z}(\widehat{\lambda}; \mathbb{R})$ defines a transverse Hölder distribution for the geodesic lamination $\widehat{\lambda}$; see [7] and [6]. As in [38], we can then interpret the data of the geodesic lamination $\widehat{\lambda}$ endowed with this transverse Hölder distribution as a closed de Rham current in \widehat{U} . The homology class of $H_1(\widehat{U}; \mathbb{R})$ defined by this de Rham current is exactly the class $[\alpha]$ introduced in Proposition 4.2.

By definition, the length $\ell_a^g(\alpha)$ is obtained by locally integrating the differential form $\omega_a - \omega_{a+1}$ over the leaves of $\widehat{\lambda}$, and then integrating the corresponding function of the leaves of $\widehat{\lambda}$ with respect to the transverse Hölder distribution defined by α . See [15] for precise details, using a suitable partition of unity for T^1S . Since ω_a is the restriction of the topological closed 1-form Ω to the leaves of $\widehat{\lambda}$, this construction is identical to the expression of [38] for the evaluation of $[\Omega_a - \Omega_{a+1}] \in H^1(\widehat{U}; \mathbb{R})$ over the homology class $[\alpha] \in H_1(\widehat{U}; \mathbb{R})$ represented by the de Rham current $\alpha \in \mathcal{C}^H(\widehat{\lambda})$. □

7.2.3. A homological interpretation of the shearing cycle σ_a^g

We now relate the shearing cycle σ_a^g to the topological closed 1-form Ω_a provided by Lemma 7.8.

For this, consider an arc k in \widehat{U} that is tightly transverse to $\widehat{\lambda}$. As usual, orient k to the right of the leaves of $\widehat{\lambda}$, and lift k to an oriented arc \tilde{k} in the universal cover \widetilde{S} .

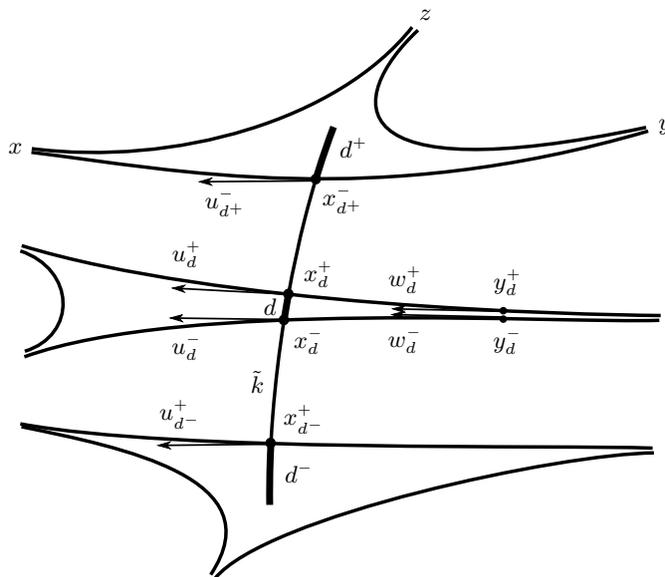


Figure 10. Integrating $\tilde{\Omega}_a$ over a tightly transverse arc \tilde{k} .

Consistently with the canonical orientation of the leaves of $\hat{\lambda}$, we orient the leaves of $\tilde{\lambda}$ that meet \tilde{k} to the left of \tilde{k} .

We first consider a component d of $\tilde{k} \setminus \tilde{\lambda}$ that does not contain any of the two endpoints of k . In particular, the positive and negative endpoints x_d^+ and x_d^- of d belong to $\tilde{\lambda}$.

The tangent of the oriented leaf of $\tilde{\lambda}$ passing through x_d^\pm determines an element $u_d^\pm \in T^1\tilde{S}$ of the unit tangent bundle of \tilde{S} . If g_d^\pm denotes the leaf of $\tilde{\lambda}$ passing through x_d^\pm and if we use the same letter to denote the projection $d \subset k \subset \hat{U}$ of the arc $d \subset \tilde{k} \subset \tilde{S}$, we now connect the integral $\int_d \Omega_a$ to the elementary slithering map $\Sigma_{g_d^+ g_d^-} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The riemannian metric on the vector bundle $T^1S \times_{\rho'} \mathbb{R}^n$ used in the definition of the forms $\Omega_a = f_a dt$ along $\hat{\lambda}$ defines, for each $u \in T^1\tilde{S}$, a norm $\|\cdot\|_u$ on \mathbb{R}^n .

LEMMA 7.10. *Let k be an arc in \hat{U} that is tightly transverse to $\hat{\lambda}$, and let d be a component of $k \setminus \hat{\lambda}$ that contains none of the two endpoints of k . Then,*

$$\int_d \Omega_a = \log \frac{\|v_a(u_d^-)\|_{u_d^-}}{\|\Sigma_{g_d^+ g_d^-}(v_a(u_d^-))\|_{u_d^+}}$$

for any non-zero vector $v_a(u_d^-)$ in the line $\tilde{L}_a(u_d^-)$.

The property is rather immediate when Ω_a is the form provided by our proof of Lemmas 7.7 and 7.8, and when d is sufficiently small. However, we need a proof that is

valid for all components d of $k \setminus \hat{\lambda}$ that are disjoint from the endpoints. Our argument will be valid for all forms Ω_a satisfying the conclusions of Lemma 7.8, not just the one provided by our proof.

Proof. The two leaves g_d^+ and g_d^- are asymptotic. We can therefore find points $y_d^+ \in g_d^+$ and $y_d^- \in g_d^-$ which are arbitrarily close to each other. Let w_d^\pm be the element of the unit tangent bundle $T^1\tilde{S}$ determined by the tangent of the oriented geodesic g_d^\pm at the point y_d^\pm . See Figure 10. We can then deform d to an arc consisting of the arc from x_d^- to y_d^- in the leaf g_d^- , followed by a short arc from y_d^- to y_d^+ , and completed by the arc from y_d^+ to x_d^+ in g_d^+ . Then, by definition of the restriction ω_a of Ω_a to the leaves of $\hat{\lambda}$,

$$\int_d \Omega_a = \int_{x_d^-}^{y_d^-} \Omega_a + \int_{y_d^-}^{y_d^+} \Omega_a + \int_{y_d^+}^{x_d^+} \Omega_a = \log \frac{\|v_a(u_d^-)\|_{u_d^-}}{\|v_a(u_d^-)\|_{w_d^-}} + \int_{y_d^-}^{y_d^+} \Omega_a + \log \frac{\|v_a(u_d^+)\|_{w_d^+}}{\|v_a(u_d^+)\|_{u_d^+}}$$

for arbitrary non-zero vectors $v_a(u_d^+) \in \tilde{L}_a(u_d^+)$ and $v_a(u_d^-) \in \tilde{L}_a(u_d^-)$. In particular, we can choose $v_a(u_d^+) = \Sigma_{g_d^+ g_d^-}(v_a(u_d^-))$, in which case

$$\int_d \Omega_a = \log \frac{\|v_a(u_d^-)\|_{u_d^-}}{\|\Sigma_{g_d^+ g_d^-}(v_a(u_d^-))\|_{u_d^+}} - \log \frac{\|v_a(u_d^-)\|_{w_d^-}}{\|\Sigma_{g_d^+ g_d^-}(v_a(u_d^-))\|_{w_d^+}} + \int_{y_d^-}^{y_d^+} \Omega_a.$$

Now, we let the points y_d^+ and y_d^- tend to the common endpoint of g_d^+ and g_d^- in such a way that the distance from y_d^+ to y_d^- tends to zero. Looking at the projections to S , the integral $\int_{y_d^-}^{y_d^+} \Omega_a$ tends to zero by continuity of the functions F_a that locally define Ω_a , while the quotient

$$\frac{\|v_a(u_d^-)\|_{u_d^-}}{\|\Sigma_{g_d^+ g_d^-}(v_a(u_d^-))\|_{u_d^+}}$$

tends to 1 (compare Lemma 5.2 and use the ρ -equivariance of the riemannian metric $\|\cdot\|$). It follows that

$$\int_d \Omega_a = \log \frac{\|v_a(u_d^-)\|_{u_d^-}}{\|\Sigma_{g_d^+ g_d^-}(v_a(u_d^-))\|_{u_d^+}}$$

for any non-zero vector $v_a(u_d^-) \in \tilde{L}_a(u_d^-)$. \square

We will now choose preferred vectors $v_a(u_d^-) \in \tilde{L}_a(u_d^-)$.

Let d^+ and d^- be the components of $\tilde{k} \setminus \tilde{\lambda}$ that contain the positive and negative endpoints of \tilde{k} , respectively. In particular, their endpoints $x_{d^+}^-$ and $x_{d^-}^+$ are the points of $\tilde{k} \cap \tilde{\lambda}$ that are closest to the positive and negative endpoints in \tilde{k} , respectively. As usual, let $u_{d^\pm}^\mp \in T^1\tilde{S}$ be defined by the vector tangent to the (oriented) leaf $g_{d^\pm}^\mp$ of $\tilde{\lambda}$ passing through $x_{d^\pm}^\mp$. See Figure 10.

The flag map $\mathcal{F}_\rho: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ associates several lines of \mathbb{R}^n with the vector $u_{d^+}^- \in T^1 \tilde{S}$. This includes the n lines $\tilde{L}_a(u_{d^+}^-) = \mathcal{F}_\rho(x)^{(a)} \cap \mathcal{F}_\rho(y)^{(n-a+1)}$ of §3.1, defined by the flags $\mathcal{F}_\rho(x)$ and $\mathcal{F}_\rho(y)$ respectively associated with the positive endpoint x and the negative endpoint y of the leaf $g_{d^+}^-$. We can also consider the line $\mathcal{F}_\rho(z)^{(1)}$ of the flag $\mathcal{F}_\rho(z)$ associated with the third vertex z of the triangle component of $\tilde{S} \setminus \tilde{\lambda}$ that contains d^+ . Pick a non-trivial vector $v(u_{d^+}^-)$ in this line $\mathcal{F}_\rho(z)^{(1)}$, and let $v_a(u_{d^+}^-) \in \tilde{L}_a(u_{d^+}^-)$ be the projection of $v(u_{d^+}^-)$ parallel to the $\tilde{L}_b(u_{d^+}^-)$ with $b \neq a$.

In particular, the quantity

$$\frac{\|v_a(u_{d^+}^-)\|_{u_{d^+}^-}}{\|v_{a+1}(u_{d^+}^-)\|_{u_{d^+}^-}}$$

is independent of the choice of the vector $v(u_{d^+}^-) \in \mathcal{F}_\rho(z)^{(1)}$. Note that this ratio is finite and positive by genericity of the flag triple $(\mathcal{F}_\rho(x), \mathcal{F}_\rho(y), \mathcal{F}_\rho(z))$.

We can introduce similar definitions at the point $x_{d^-}^+$ of $\tilde{k} \cap \tilde{\lambda}$ that is closest to the negative endpoint of \tilde{k} . Considering the triangle component of $\tilde{S} \setminus \tilde{\lambda}$ that contains the negative endpoint of \tilde{k} , this leads to a well-defined positive ratio

$$\frac{\|v_a(u_{d^-}^+)\|_{u_{d^-}^+}}{\|v_{a+1}(u_{d^-}^+)\|_{u_{d^-}^+}}.$$

LEMMA 7.11. *Let k be an arc in \hat{U} that is tightly transverse to $\hat{\lambda}$. Then, for the above definitions,*

$$\sigma_a^\rho(k) = \int_{k \setminus (d^+ \cup d^-)} (\Omega_a - \Omega_{a+1}) + \log \frac{\|v_a(u_{d^+}^-)\|_{u_{d^+}^-}}{\|v_{a+1}(u_{d^+}^-)\|_{u_{d^+}^-}} - \log \frac{\|v_a(u_{d^-}^+)\|_{u_{d^-}^+}}{\|v_{a+1}(u_{d^-}^+)\|_{u_{d^-}^+}}.$$

Note that the notation is ambiguous in the special case where $u_{d^+}^- = u_{d^-}^+$, which occurs when the arc k crosses $\hat{\lambda}$ in only one point. We will leave to the reader the easy task of lifting the ambiguity in this case.

Proof. By a well-known result of Birman–Series [1], the intersection $\tilde{k} \cap \tilde{\lambda}$ has Hausdorff dimension zero. Since the topological closed 1-form Ω_a is Hölder continuous, it follows that

$$\int_{k \setminus (d^+ \cup d^-)} \Omega_a = \sum_d \int_d \Omega_a,$$

where the sum is over all components d of $\tilde{k} \setminus \tilde{\lambda}$ that are different from d^+ and d^- . (The critical property is that the image of a set of Hausdorff dimension zero under a Hölder continuous function has Hausdorff dimension zero, and in particular has Lebesgue measure zero in \mathbb{R} .)

We now apply Lemma 7.10 while choosing $v_a(u_d^-) = \Sigma_{g_d^- g_d^+}(v_a(u_d^+)) \in L_a(u_d^-)$, where $v_a(u_d^+) \in \tilde{L}_a(u_d^+)$ is determined as above by the vertices of the triangle component of $\tilde{S} \setminus \tilde{\lambda}$ that contains d^+ . Then,

$$\int_{k \setminus (d^+ \cup d^-)} \Omega_a = \sum_d \log \frac{\|\Sigma_{g_d^- g_d^+}(v_a(u_d^+))\|_{u_d^-}}{\|\Sigma_{g_d^+ g_d^-}(v_a(u_d^-))\|_{u_d^+}}$$

by observing that

$$\Sigma_{g_d^+ g_d^-}(v_a(u_d^-)) = \Sigma_{g_d^+ g_d^-} \circ \Sigma_{g_d^- g_d^+}(v_a(u_d^+)) = \Sigma_{g_d^+ g_d^+}(v_a(u_d^+)).$$

If g_x denotes the oriented leaf of $\tilde{\lambda}$ passing through $x \in \tilde{k} \cap \tilde{\lambda}$ and if $u_x \in T^1 \tilde{S}$ is the unit vector tangent to g_x at x , the map $x \mapsto \|\Sigma_{g_x g_d^+}(v_a(u_d^+))\|_{u_x}$ is Hölder continuous, because g_x depends Lipschitz continuously on x by [13, §5.2.6], and because the slithering map $\Sigma_{g g_d^+}$ is a Hölder continuous function of the leaf g by Proposition 5.1. Using again the fact that $\tilde{k} \cap \tilde{\lambda}$ has Hausdorff dimension zero, it follows that

$$\int_{k \setminus (d^+ \cup d^-)} \Omega_a = \sum_d \log \frac{\|\Sigma_{g_d^- g_d^+}(v_a(u_d^+))\|_{u_d^-}}{\|\Sigma_{g_d^+ g_d^-}(v_a(u_d^-))\|_{u_d^+}} = \log \frac{\|v_a(u_d^+)\|_{u_d^+}}{\|\Sigma_{g_d^+ g_d^+}(v_a(u_d^+))\|_{u_d^+}}.$$

By construction, the slithering map $\Sigma_{g_d^- g_d^+}$ sends $\tilde{L}_a(u_d^+)$ to $\tilde{L}_a(u_d^-)$. In particular, there exists a non-zero number μ_a such that $\Sigma_{g_d^- g_d^+}(v_a(u_d^+)) = \mu_a v_a(u_d^-)$. Then,

$$\begin{aligned} \int_{k \setminus (d^+ \cup d^-)} (\Omega_a - \Omega_{a+1}) &= \log \frac{\|v_a(u_d^+)\|_{u_d^+}}{\|v_{a+1}(u_d^+)\|_{u_d^+}} - \log \frac{\|v_a(u_d^-)\|_{u_d^-}}{\|v_{a+1}(u_d^-)\|_{u_d^-}} - \log \left| \frac{\mu_a}{\mu_{a+1}} \right| \\ &= \log \frac{\|v_a(u_d^+)\|_{u_d^+}}{\|v_{a+1}(u_d^+)\|_{u_d^+}} - \log \frac{\|v_a(u_d^-)\|_{u_d^-}}{\|v_{a+1}(u_d^-)\|_{u_d^-}} + \sigma_a^e(k), \end{aligned}$$

by definition of the shearing parameter $\sigma_a^e(k)$ in §5.2 (and using Lemma 1.6). □

LEMMA 7.12. *For every homology class $[\alpha] \in H_1(\widehat{U}; \mathbb{R})$,*

$$\langle [\Omega_a] - [\Omega_{a+1}], [\alpha] \rangle = [\alpha] \cdot [\sigma_a^e].$$

Proof. We already observed, in the proof of Proposition 4.5, that $H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$ admits a basis where each element is represented by a generic tie of \widehat{U} . We can therefore write the image $[\alpha] \in H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$ as a linear combination $[\alpha] = \sum_i \mu_i [k_i]$ of classes represented by generic ties k_i , with coefficients $\mu_i \in \mathbb{R}$.

Recall that the ties of \widehat{U} are oriented to the right for the canonical orientation of the leaves of $\hat{\lambda}$. In particular, the components of the horizontal boundary $\partial_h \widehat{U}$ are of two

types: those components where the orientation of the ties point outside of \widehat{U} , and those where it points inside. Also, because of this orientation convention,

$$[\alpha] \cdot [\sigma_a^g] = \sum_i \mu_i [k_i] \cdot [\sigma_a^g] = \sum_i \mu_i \sigma_a^g(k_i)$$

by definition of the homology class $[\sigma_a^g] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$ associated with the relative tangent cycle $\sigma_a^g \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$ by Proposition 4.5.

We now modify each arc k_i by a homotopy respecting $\widehat{\lambda}$ and $\partial_h \widehat{U}$ to obtain an arc k'_i such that the following holds: for every component C of the horizontal boundary $\partial_h \widehat{U}$, there is an arc $k_C \subset \widehat{U}$ such that, for every arc k'_i with an endpoint in C , the component of $k'_i \setminus \widehat{\lambda}$ containing this endpoint is equal to k_C . The only case where this regrouping of arcs near the horizontal boundary requires some care is when the original tie k_i meets $\widehat{\lambda}$ in one point; in this special situation, one needs to first choose the relevant arcs k_C so that $k'_i = k_i$, and then modify the other k_j accordingly.

Now, by Lemma 7.11,

$$\begin{aligned} \sum_i \mu_i \sigma_a^g(k'_i) &= \sum_i \mu_i \int_{k'_i} (\Omega_a - \Omega_{a+1}) - \sum_i \mu_i \int_{d_i^+} (\Omega_a - \Omega_{a+1}) - \sum_i \mu_i \int_{d_i^-} (\Omega_a - \Omega_{a+1}) \\ &\quad + \sum_i \mu_i \log \frac{\|v_a(u_{d_i^+}^-)\|_{u_{d_i^+}^-}}{\|v_{a+1}(u_{d_i^+}^-)\|_{u_{d_i^+}^-}} - \sum_i \mu_i \log \frac{\|v_a(u_{d_i^-}^+)\|_{u_{d_i^-}^+}}{\|v_{a+1}(u_{d_i^-}^+)\|_{u_{d_i^-}^+}}, \end{aligned}$$

where d_i^+ and d_i^- are the components of $k'_i \setminus \widehat{\lambda}$ containing the positive and negative endpoints of k'_i , respectively. In particular, each d_i^\pm is equal to one of the arcs k_C associated with the components C of the horizontal boundary $\partial_h \widehat{U}$.

The key observation is now that $[\alpha] = \sum_i \mu_i [k_i] \in H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$ comes from an element of $H_1(\widehat{U}; \mathbb{R})$, and in particular has boundary zero. This implies that, for each component C of $\partial_h \widehat{U}$ where the ties point outwards, the sum of the μ_i such that k_i has an endpoint in C is equal to zero; equivalently, the μ_i such that $d_i^+ = k_C$ add up to zero. Similarly, for each component C of $\partial_h \widehat{U}$ where the ties point inwards, the sum of the coefficients μ_i such that $d_i^- = k_C$ is equal to zero.

This implies that most terms cancel out in the above sum, and that

$$\sum_i \mu_i \sigma_a^g(k'_i) = \sum_i \mu_i \int_{k'_i} (\Omega_a - \Omega_{a+1}) = \langle [\Omega_a] - [\Omega_{a+1}], [\alpha] \rangle.$$

For the second equality note that, because the μ_i for which the positive (resp. negative) endpoint of k'_i is in a given component C of $\partial_h \widehat{U}$ add up to zero, the chain $\sum_i \mu_i k'_i$ is closed and represents the class $[\alpha] \in H_1(\widehat{U}; \mathbb{R})$.

This proves that

$$[\alpha] \cdot [\sigma_a] = \sum_i \mu_i \sigma_a(k_i) = \sum_i \mu_i \sigma_a(k'_i) = \langle [\Omega_a] - [\Omega_{a+1}], [\alpha] \rangle. \quad \square$$

The combination of Lemmas 7.9 and 7.12 completes the proof of Theorem 7.5.

COROLLARY 7.13. *Let μ be a non-trivial transverse measure for the orientation cover $\hat{\lambda}$, and let $[\mu] \in H_1(\hat{U}; \mathbb{R})$ be its associated homology class as in §4.3. Then,*

$$[\mu] \cdot [\sigma_a^\varrho] > 0$$

for each component $\sigma_a^\varrho \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) \cong H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$ of the shearing cycle

$$\sigma^\varrho \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$$

of a Hitchin character $\varrho \in \text{Hit}_n(S)$.

Proof. This is an immediate consequence of Theorem 7.5 and Proposition 7.4. \square

8. Parametrizing Hitchin components

In §3 and §5.2, we associated certain invariants with a Hitchin character $\varrho \in \text{Hit}_n(S)$.

The first type of invariants are the triangle invariants $\tau_{abc}^\varrho(s)$, defined as s ranges over the slits of λ and as $a, b, c \geq 1$ range over all integers such that $a+b+c=n$. Noting that there are $\frac{1}{2}(n-1)(n-2)$ such triples (a, b, c) and $12(g-1)$ slits of $S \setminus \lambda$, we can combine all these invariants into a single map

$$\text{Hit}_n(S) \longrightarrow \mathbb{R}^{6(g-1)(n-1)(n-2)}.$$

The second invariant is the shearing cycle $\sigma^\varrho \in \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$, which provides a map

$$\text{Hit}_n(S) \longrightarrow \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n) \cong \mathbb{R}^{18(g-1)(n-1)},$$

where the isomorphism is given by Proposition 4.7.

Combining these two maps, we define

$$\Phi: \text{Hit}_n(S) \longrightarrow \mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n) \cong \mathbb{R}^{6(g-1)(n+1)(n-1)},$$

which sends each Hitchin character $\varrho \in \text{Hit}_n(S)$ to its triangle invariants and its shearing cycle. We proved in Corollary 6.8 that this map Φ is injective. We will show that Φ induces a homeomorphism between $\text{Hit}_n(S)$ and an open convex polyhedral cone \mathcal{P} contained in a linear subspace of $\mathbb{R}^{6(g-1)(n+1)(n-1)}$.

LEMMA 8.1. *The above map*

$$\Phi: \text{Hit}_n(S) \longrightarrow \mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$$

is continuous.

Proof. The key property is that the flag curve $\mathcal{F}_\varrho: \partial_\infty \widetilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ depends continuously on the Hitchin homomorphism $\varrho: \pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$, and is uniformly Hölder continuous as ϱ ranges over a compact subset of the space of homomorphisms $\pi_1(S) \rightarrow \text{PSL}_n(\mathbb{R})$. These two properties follow from the application to the setup of §3.1 of the classical structural stability theorems for Anosov flows, and Hölder continuity properties for their stable and unstable foliations; see for instance [24, §18 and §19].

The continuity property immediately shows that the triangle invariants $\tau_{abc}^\varrho(s)$ depend continuously on ϱ .

The case of the shearing cycle $\sigma^\varrho \in \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$ requires an additional argument, because its construction relies on the slithering maps $\Sigma_{gg'}: \mathbb{R}^n \rightarrow \mathbb{R}^n$. The uniform Hölder continuity property makes the estimates used in the construction of slithering maps in §5.1 uniform, and guarantees uniform convergence in this construction. It follows that, for any two leaves g and g' of $\tilde{\lambda}$, the slithering map $\Sigma_{gg'}$ depends continuously on ϱ . After this, the continuous dependence of the flag map \mathcal{F}_ϱ on ϱ is enough to prove that σ^ϱ depends continuously on ϱ . □

8.1. Constraints between invariants

There are clear constraints on the image of Φ . The first one is the following consequence of Lemma 1.1, which we have already encountered in Lemma 3.4.

Triangle rotation condition. If the spikes of the component T of $S \setminus \lambda$ are indexed as s, s' and s'' in counterclockwise order around T , then

$$\tau_{abc}^\varrho(s) = \tau_{bca}^\varrho(s') = \tau_{cab}^\varrho(s'').$$

The second constraint comes from the quasi-additivity property of the shearing cycle σ^ϱ . Recall that the lack of additivity of the a th component $\sigma_a^\varrho \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ of $\sigma^\varrho \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \subset \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ is measured by its boundary $\partial\sigma_a^\varrho$, which associates a number $\partial\sigma_a^\varrho(\hat{s}) \in \mathbb{R}$ with each spike \hat{s} of the orientation cover $\hat{\lambda}$ of the geodesic lamination λ . The spikes \hat{s} can be *positive* or *negative*, according to whether the canonical orientation of the leaves of $\hat{\lambda}$ orients the two leaves that are adjacent to \hat{s} towards \hat{s} or away from \hat{s} (namely, according to whether \hat{s} sits at the positive or negative end of these leaves).

The following constraint comes from the computation of $\partial\sigma_a^\ell$ provided by Proposition 5.15 and Lemma 5.14.

Shearing cycle boundary condition. For every positive slit s^+ of $\hat{\lambda}$ projecting to a slit s of λ ,

$$\partial\sigma_a^\ell(s^+) = \sum_{b+c=n-a} \tau_{abc}^\ell(s).$$

Note that this property for positive slits, combined with the equivariance property of $\sigma^\ell \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \subset \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$ with respect to the covering involution of the cover $\hat{\lambda} \rightarrow \lambda$, determines $\partial\sigma_a^\ell$ on negative slits. More precisely,

$$\partial\sigma_a^\ell(s^-) = - \sum_{b+c=a} \tau_{(n-a)bc}^\ell(s)$$

for every negative slit s^- of $\hat{\lambda}$ projecting to a slit s of λ .

The last condition is provided by Corollary 7.13.

Positive intersection condition.

$$[\mu] \cdot [\sigma_a^\ell] > 0$$

for every non-trivial transverse measure μ for $\hat{\lambda}$, where

$$[\mu] \in H_1(\widehat{U}; \mathbb{R}) \quad \text{and} \quad [\sigma_a^\ell] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$$

are the homology classes respectively defined by $\mu \in \mathcal{Z}(\hat{\lambda}; \mathbb{R})$ and by the a th component $\sigma_a^\ell \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ of the shearing cycle $\sigma^\ell \in \mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1}) \subset \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}^{n-1})$, and where \cdot denotes the algebraic intersection in \widehat{U} .

Let \mathcal{P} be the set of pairs (τ, σ) such that

(1) τ is a function which associates a number $\tau_{abc}(s) \in \mathbb{R}$ with each triple of integers $a, b, c \geq 1$ with $a+b+c=n$, and with each slit s of λ ;

(2) $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ is a tangent cycle for λ valued in the coefficient bundle $\widehat{\mathbb{R}}^{n-1}$ and relative to the slits of λ ; in particular, σ is defined by $n-1$ relative tangent cycles $\sigma_a \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$;

(3) τ and σ satisfy the above triangle rotation condition, shearing cycle boundary condition and positive intersection condition.

We will call a function $\tau \in \mathbb{R}^{6(g-1)(n-1)(n-2)}$ as in (1) a *triangle data function*. It is *rotation-invariant* when it satisfies the triangle rotation condition.

PROPOSITION 8.2. *The space \mathcal{P} is an open convex polyhedral cone in a linear subspace of $\mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ of dimension $2(g-1)(n^2-1)$.*

Proof. The transverse measures for the geodesic lamination $\hat{\lambda}$ form a positive cone over a finite-dimensional simplex [25], [35]. It therefore suffices to check the positive intersection condition on the vertices of this simplex (corresponding to ergodic measures). This reduces the positive intersection condition to finitely many linear inequalities, in fact at most $6(g-1)$ such inequalities by the main result of [35]. As a consequence, \mathcal{P} is an open convex polyhedral cone in the linear subspace of $\mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{C}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$ defined by the triangle rotation condition and the shearing cycle boundary condition. We need to compute its dimension, which will require a few lemmas.

The triangle rotation condition divides the dimension of the space of triangle data functions by 3, in the sense that the space of rotation-invariant triangle data functions $\tau \in \mathbb{R}^{6(g-1)(n-1)(n-2)}$ is isomorphic to $\mathbb{R}^{2(g-1)(n-1)(n-2)}$. Indeed, if we pick a spike s_j for each triangle component T_j of $S \setminus \lambda$, such a rotation-invariant τ is completely determined by the $2(g-1)(n-1)(n-2)$ numbers $\tau_{abc}(s_j)$. We will use this observation to denote by $\mathbb{R}^{2(g-1)(n-1)(n-2)}$ the space of all rotation-invariant triangle data functions τ .

Consider the linear subspace $\mathcal{L} \subset \mathbb{R}^{2(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ consisting of all pairs (τ, σ) , where τ is a rotation-invariant triangle data function and σ is a twisted tangent cycle for λ relative to its slits, and where τ and σ satisfy the shearing cycle boundary condition.

To analyze \mathcal{L} , we introduce a new vector space $\mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$, consisting of all functions $\xi: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}^{n-1}$. For $a=1, 2, \dots, n-1$, we denote the a th component of such a $\xi \in \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ by $\xi_a: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}$. The definition of the space \mathcal{L} can then be expressed in terms of two maps

$$\partial: \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \longrightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1}) \quad \text{and} \quad \Xi: \mathbb{R}^{2(g-1)(n-1)(n-2)} \longrightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1}).$$

The first map $\partial: \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ is the usual boundary map of §4.4, and associates with a relative cycle $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ the restriction

$$\partial\sigma: \{\text{positive slits of } \hat{\lambda}\} = \{\text{slits of } \lambda\} \longrightarrow \mathbb{R}^{n-1}$$

of its boundary $\partial\sigma$ to positive slits of the orientation cover $\hat{\lambda}$. (Recall that this restriction completely determines $\partial\sigma$ by the definition of twisted relative tangent cycles, since $\partial\sigma_a(s^-) = -\partial\sigma_{n-a}(s^+)$ when the negative slit s^- of $\hat{\lambda}$ projects to the same slit of λ as the positive slit s^+ .)

The second map $\Xi: \mathbb{R}^{2(g-1)(n-1)(n-2)} \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ associates with each rotation-invariant triangle data function $\tau \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$ the function $\xi^\tau: \{\text{slits of } \lambda\} \rightarrow \mathbb{R}^{n-1}$ defined by the property that

$$\xi_a^\tau(s) = \sum_{b+c=n-a} \tau_{abc}(s) \in \mathbb{R}$$

for every slit s of λ and every $a=1, 2, \dots, n-1$.

Then the subspace \mathcal{L} consists of all pairs $(\tau, \sigma) \in \mathbb{R}^{2(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ such that $\partial\sigma = \Xi(\tau)$ in $\mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$.

LEMMA 8.3. *The image of the boundary map $\partial: \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ consists of all $\xi \in \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ such that*

$$\sum_{s \text{ slit of } \lambda} \xi_a(s) = \sum_{s \text{ slit of } \lambda} \xi_{n-a}(s)$$

for every $a=1, 2, \dots, n-1$. This image has codimension $\lfloor \frac{1}{2}(n-1) \rfloor$ in $\mathcal{Z}(\text{slits}; \mathbb{R}^{n-1}) \cong \mathbb{R}^{12(g-1)(n-1)}$.

Proof. This is an immediate consequence of the homological interpretation of twisted relative tangent cycles in §4.6, and more precisely of the isomorphism $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1})$ constructed there.

This construction is well behaved with respect to the boundary maps ∂ in the following sense. There is a unique isomorphism $\mathcal{Z}(\text{slits}; \mathbb{R}^{n-1}) \cong H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1})$ defined as follows: this isomorphism associates with $\xi \in \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ the element of $H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1}) \subset H_0(\partial_v \widehat{U}; \mathbb{R}^{n-1})$ that assigns to each component of $\partial_v \widehat{U}$ facing a positive slit s^+ the weight $\xi(s) \in \mathbb{R}^{n-1}$ associated by ξ with the projection s of s^+ (and, by definition of the twisting of the coefficient bundle $\widetilde{\mathbb{R}}^n$, assigns to the component of $\partial_v \widehat{U}$ facing a negative slit s^- projecting to s the vector $-\bar{\xi}(s) \in \mathbb{R}^{n-1}$ whose a th coordinate is $-\bar{\xi}_a(s) = -\xi_{n-a}(s)$). Then, for these isomorphisms $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1})$ and $\mathcal{Z}(\text{slits}; \mathbb{R}^{n-1}) \cong H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1})$, the boundary homomorphism $\partial: \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ corresponds to the homological boundary $\partial: H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}) \rightarrow H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1})$.

Lemma 8.3 is then an immediate consequence of the long exact sequence

$$\dots \rightarrow H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}) \rightarrow H_0(\partial_v U; \widetilde{\mathbb{R}}^{n-1}) \rightarrow H_0(U; \widetilde{\mathbb{R}}^{n-1}) \rightarrow H_0(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}),$$

using the properties that, because \widehat{U} is connected and $\partial_v \widehat{U}$ is non-empty,

$$\dim H_0(U; \widetilde{\mathbb{R}}^{n-1}) = \lfloor \frac{1}{2}(n-1) \rfloor$$

and $H_0(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}) = 0$. □

LEMMA 8.4. *For $n > 3$, the image of $\Xi: \mathbb{R}^{2(g-1)(n-1)(n-2)} \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ consists of all $\xi \in \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$ such that*

$$\xi_{n-1}(s_1) = 0 \quad \text{and} \quad \xi_1(s_1) = \sum_{a=2}^{n-2} \left(\frac{a-1}{n-3} - 1 \right) \xi_a(s_1) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \xi_a(s_2) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \xi_a(s_3),$$

whenever s_1, s_2 and s_3 are the three spikes of the same component T of $S \setminus \lambda$. In particular, the image of Ξ has dimension $12(g-1)(n-3)$.

When $n=3$, the image of $\Xi: \mathbb{R}^{4(g-1)} \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^2)$ consists of all $\xi \in \mathcal{Z}(\text{slits}; \mathbb{R}^2)$ such that

$$\xi_2(s_1) = 0 \quad \text{and} \quad \xi_1(s_1) = \xi_1(s_2) = \xi_1(s_3),$$

whenever s_1, s_2 and s_3 are the three spikes of the same component T of $S \setminus \lambda$. In particular, the image of Ξ then has dimension $4(g-1)$.

When $n=2$, the image of $\Xi: \mathbb{R}^0 \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R})$ is of course trivial.

Proof. If $\xi^\tau = \Xi(\tau)$ for a rotation-invariant function $\tau \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$, then by definition $\xi_{n-1}^\tau(s) = \sum_{b+c=1} \tau_{(n-1)bc}(s) = 0$ for every slit s , since all indices b and c are supposed to be at least 1.

Less trivially, if $n > 3$ and if s_1, s_2 and s_3 are the three spikes of a same component T of $S \setminus \lambda$, in this order counterclockwise around T ,

$$\begin{aligned} & \sum_{a=2}^{n-2} \frac{a-1}{n-3} \xi_a^\tau(s_1) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \xi_a^\tau(s_2) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \xi_a^\tau(s_3) \\ &= \sum_{a=1}^{n-2} \frac{a-1}{n-3} \sum_{b+c=n-a} \tau_{abc}(s_1) + \sum_{a=1}^{n-2} \frac{a-1}{n-3} \sum_{b+c=n-a} \tau_{abc}(s_2) + \sum_{a=1}^{n-2} \frac{a-1}{n-3} \sum_{b+c=n-a} \tau_{abc}(s_3) \\ &= \sum_{a=1}^{n-2} \frac{a-1}{n-3} \sum_{b+c=n-a} \tau_{abc}(s_1) + \sum_{b=1}^{n-2} \frac{b-1}{n-3} \sum_{a+c=n-b} \tau_{abc}(s_1) + \sum_{c=1}^{n-2} \frac{c-1}{n-3} \sum_{a+b=n-c} \tau_{abc}(s_1) \\ &= \sum_{a,b,c} \left(\frac{a-1}{n-3} + \frac{b-1}{n-3} + \frac{c-1}{n-3} \right) \tau_{abc}(s_1) \\ &= \sum_{a,b,c} \tau_{abc}(s_1) \\ &= \sum_{a=1}^{n-1} \xi_a^\tau(s_1), \end{aligned}$$

where the second equality uses the rotation invariance of τ . It follows that

$$\xi_1^\tau(s_1) = \sum_{a=2}^{n-2} \left(\frac{a-1}{n-3} - 1 \right) \xi_a^\tau(s_1) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \xi_a^\tau(s_2) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \xi_a^\tau(s_3).$$

As a consequence, any function $\xi = \Xi(\tau)$ in the image of Ξ satisfies the relations of Lemma 8.4.

Conversely, as a ranges from 2 to $n-2$ and s ranges over all slits of λ , the functions $\tau \mapsto \xi_a^\tau(s)$ are linearly independent in the space $\mathbb{R}^{2(g-1)(n-1)(n-2)}$ of rotation-invariant

triangle data functions τ . Indeed, this follows from a simple computation focusing on the coefficients of the terms $\tau_{1bc}(s)$ and $\tau_{2bc}(s)$ in any linear relation between these functions.

The dimension computation then follows from the fact that λ has $12(g-1)$ slits. This completes the proof of Lemma 8.4 in the case considered, when $n > 3$.

The proof is much simpler when $n=3$, as the triangle data function τ assigns only one number $\tau_{111}(s)$ to each slit s . This makes the argument in this case completely straightforward. □

LEMMA 8.5. *The intersection $\text{im}(\partial) \cap \text{im}(\Xi)$ of the images $\text{im}(\partial) = \partial(\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}))$ and $\text{im}(\Xi) = \Xi(\mathbb{R}^{2(g-1)(n-1)(n-2)})$ has dimension*

$$\begin{cases} 12(g-1)(n-3) - \lfloor \frac{1}{2}(n-1) \rfloor, & \text{if } n > 3, \\ 4g-5, & \text{if } n = 3. \end{cases}$$

As in Lemma 8.4, the case $n=2$ is trivial since $\text{im}(\Xi) = 0$ in this case.

Proof. This is an immediate consequence of the characterization of these images in Lemmas 8.3 and 8.4. Indeed, one very easily checks that the restrictions of the $\lfloor \frac{1}{2}(n-1) \rfloor$ relations of Lemma 8.3 to the image $\text{im}(\Xi)$ are linearly independent. □

We now return to the subspace $\mathcal{L} \subset \mathbb{R}^{2(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$, consisting of all pairs (τ, σ) such that $\partial\sigma = \Xi(\tau)$ in $\mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$. The maps Ξ and ∂ induce the same linear map $\mathcal{L} \rightarrow \mathcal{Z}(\text{slits}; \mathbb{R}^{n-1})$, associating $\Xi(\tau) = \partial\sigma$ with each $(\tau, \sigma) \in \mathcal{L}$. The image of this map is $\text{im}(\partial) \cap \text{im}(\Xi)$, and its kernel is the direct sum of $\ker \Xi$ and $\ker \partial$. Note that $\ker \partial$ is just the space $\mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1})$ of closed tangent cycles. Therefore, by combining Lemma 8.5, Lemma 8.4 and Proposition 4.7,

$$\begin{aligned} \dim \mathcal{L} &= \dim \text{im}(\partial) \cap \text{im}(\Xi) + \dim \ker \Xi + \dim \ker \partial \\ &= 12(g-1)(n-3) - \lfloor \frac{1}{2}(n-1) \rfloor + 2(g-1)(n-1)(n-2) \\ &\quad - 12(g-1)(n-3) + 6(g-1)(n-1) + \lfloor \frac{1}{2}(n-1) \rfloor \\ &= 2(g-1)(n^2-1), \end{aligned}$$

when $n > 3$.

When $n=3$, the same argument gives that

$$\dim \mathcal{L} = (4g-5) + 0 + (12g-11) = 16(g-1),$$

which is equal to $2(g-1)(n^2-1)$ in this case as well.

Since \mathcal{P} is an open convex polyhedral cone in the space \mathcal{L} , this concludes the proof of Proposition 8.2. □

COROLLARY 8.6. *The map $\Phi: \text{Hit}_n(S) \rightarrow \mathcal{P}$ is a local homeomorphism.*

Proof. The map Φ is continuous by Lemma 8.1, and injective by Corollary 6.8. By the invariance of the domain theorem, it is therefore a local homeomorphism, since $\text{Hit}_n(S)$ and \mathcal{P} have the same dimension by Proposition 8.2. \square

8.2. An estimate from the positive intersection condition

This section is devoted to a key estimate (Lemma 8.8) which will play a crucial role in our proof, completed in the next section §8.3, that the above map $\Phi: \text{Hit}_n(S) \rightarrow \mathcal{P}$ is a global homeomorphism.

For the preimage $\tilde{\lambda}$ of the maximal geodesic lamination λ in the universal cover \tilde{S} of S , we want to quantify the complexity of the relative position with respect to each other of the components T of the complement $\tilde{S} \setminus \tilde{\lambda}$. For this, we choose a train-track neighborhood U of λ , with preimage \tilde{U} in \tilde{S} .

We also select an oriented arc \tilde{k} tightly transverse to $\tilde{\lambda}$ in \tilde{S} ; recall that this means that \tilde{k} is transverse to the leaves of $\tilde{\lambda}$ and that, for each component T of $\tilde{S} \setminus \tilde{\lambda}$, the intersection $T \cap \tilde{k}$ is either empty, or an arc containing an endpoint of \tilde{k} , or an arc joining two distinct components of ∂T . As in §4.7, using Proposition 4.1, we can arrange by a homotopy respecting $\tilde{\lambda}$ that \tilde{k} is contained in \tilde{U} .

Let T be a component of $\tilde{S} \setminus \tilde{\lambda}$ that meets \tilde{k} , and does not contain any of the endpoints of \tilde{k} . Then $\tilde{k} \cap T$ consists of a single arc, since \tilde{k} is tightly transverse to $\tilde{\lambda}$, and $\tilde{k} \cap T$ can be joined to the complement $T \setminus \tilde{U}$ by a path contained in T . We define the *divergence radius* $r(T) \geq 1$ of T with respect to \tilde{U} and \tilde{k} as the minimum number of branches of \tilde{U} that are met by a path joining $\tilde{k} \cap T$ to the complement $T \setminus \tilde{U}$ in T .

LEMMA 8.7. *For every integer r_0 , the number of triangles T with divergence radius $r(T) = r_0$ is uniformly bounded, independently of r_0 .*

Proof. Instead of counting the components T of $\tilde{S} \setminus \tilde{\lambda}$ meeting \tilde{k} , it is easier to count the components of $\tilde{k} \setminus \tilde{\lambda}$. Cutting \tilde{k} into smaller arcs if necessary, we may assume without loss of generality that \tilde{k} is sufficiently short that it projects to an arc k embedded in S . Then there is a natural correspondence between the components of $\tilde{k} \setminus \tilde{\lambda}$ and those of $k \setminus \lambda$. For each component d of $k \setminus \lambda$, let T_d be the component of $\tilde{S} \setminus \tilde{\lambda}$ that contains the component of $\tilde{k} \setminus \tilde{\lambda}$ corresponding to d , and define $r(d) = r(T_d)$. We need to show that the number of components d of $k \setminus \lambda$ with $r(d) = r_0$ is uniformly bounded.

As e ranges over all branches of the train-track neighborhood U , the components of $e \setminus \lambda$ form a family of rectangles R_i whose union is equal to $U \setminus \lambda$. In particular, this

decomposes $U \setminus \lambda$ in two pieces:

- (1) the union of the finitely many rectangles R_i that meet the boundary ∂U ;
- (2) $12(g-1)$ infinite chains of rectangles $R_{i_1} \cup R_{i_2} \cup \dots \cup R_{i_k} \cup \dots$, where each R_{i_k} shares with $R_{i_{k+1}}$ a side contained in a tie of U , that form the spikes of $U \setminus \lambda$.

Compare Proposition 4.1 and Figure 3.

If d is a component of $k \setminus \lambda$ whose divergence radius $r(d)$ is equal to 1, then it meets one of the finitely many rectangles R_i of (1) above. The number of components of $k \setminus \lambda$ meeting a given rectangle R_i is uniformly bounded, by a constant depending on the diameter of the rectangles R_i and on the minimum distance between \tilde{k} and its iterates under the action of $\pi_1(S)$. Therefore, there are only finitely many components of $k \setminus \lambda$ with divergence radius 1.

If d is a component of $k \setminus \lambda$ with $r(d) > 1$, it is contained in one of the spikes $R_{i_1} \cup R_{i_2} \cup \dots \cup R_{i_k} \cup \dots$ as in (2) above. In fact, d meets the $(r(d)-1)$ -th rectangle $R_{i_{r(d)-1}}$ of this spike, by definition of the divergence radius $r(d)$. Since the number of components of $k \setminus \lambda$ meeting each R_i is uniformly bounded, and since there are only $12(g-1)$ spikes, it follows that for $r_0 > 1$ the number of components d of $k \setminus \lambda$ with $r(d) = r_0$ is uniformly bounded. □

To explain the divergence radius terminology, consider the two sides of T that meet \tilde{k} . These two leaves of $\tilde{\lambda}$ follow the same train route in \tilde{U} over a length of approximately $r(T)$ branches (up to a bounded error term) before diverging at some switch of \tilde{U} .

The side of the oriented arc \tilde{k} where this divergence occurs will greatly matter. There are two possibilities for the two sides of T meeting \tilde{k} : Either they are asymptotic on the left-hand side of \tilde{k} , or they are asymptotic on the right-hand side. We say that T *points to the left* of \tilde{k} in the first case, and *points to the right* in the second case. We already encountered this terminology in §4.7 and §6.2.

Finally, remember that $\hat{\lambda}$ denotes the orientation cover of λ , and that the covering map $\hat{\lambda} \rightarrow \lambda$ uniquely extends to a cover $\hat{U} \rightarrow U$ for some train-track neighborhood \hat{U} of $\hat{\lambda}$.

Let T_0 be the component of $\tilde{S} \setminus \tilde{\lambda}$ containing the negative endpoint of \tilde{k} . Using the point of view of §4.7, a relative tangent cycle $\sigma \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ associates a number $\sigma(T_0, T) \in \mathbb{R}$ with each component T of $\tilde{S} \setminus \tilde{\lambda}$.

LEMMA 8.8. *Suppose that the relative tangent cycle $\sigma \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R}) \cong H_1(\hat{U}, \partial_v \hat{U}; \mathbb{R})$ satisfies the positive intersection property of §8.1: $[\mu] \cdot [\sigma] > 0$ for every non-trivial transverse measure μ for $\hat{\lambda}$, defining a homology class $[\mu] \in H_1(\hat{U}; \mathbb{R})$. Then, there exists a constant $C > 0$ such that, for all but finitely many components T of $\tilde{S} \setminus \tilde{\lambda}$ meeting \tilde{k} ,*

- $\sigma(T_0, T) \geq Cr(T)$ if T points to the right of \tilde{k} ;
- $\sigma(T_0, T) \leq -Cr(T)$ if T points to the left of \tilde{k} .

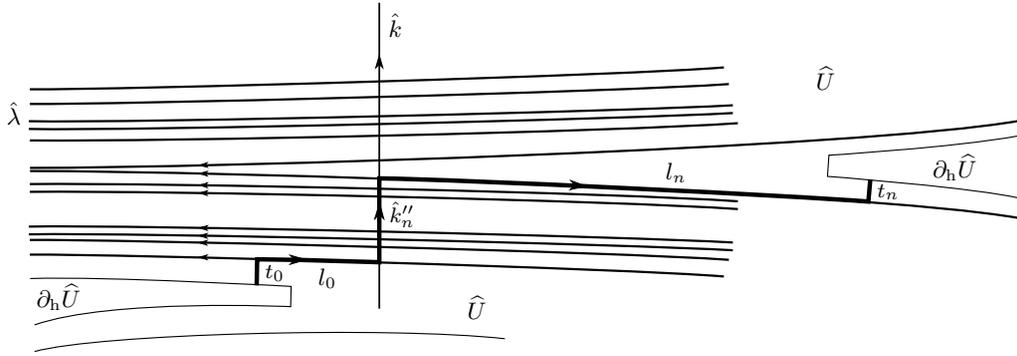


Figure 11. The arc $\hat{k}'_n = \hat{k}''_n \cup l_0 \cup l_n \cup t_0 \cup t_n$.

Proof. Pick a tie k_e in each branch e of the train-track neighborhood \hat{U} . Then, for each transverse measure μ for $\hat{\lambda}$, define

$$\|\mu\| = \sum_e \mu(k_e),$$

where the sum is over all branches e of \hat{U} . This defines a norm $\|\cdot\|$ on the space $\mathcal{M}(\hat{\lambda}) \subset \mathcal{Z}(\hat{\lambda}; \mathbb{R})$ of transverse measures for $\hat{\lambda}$. The space of transverse measures of norm 1 is compact for the weak* topology and, since σ satisfies the positive intersection condition, there consequently exists a number $\varepsilon > 0$ such that $[\mu] \cdot [\sigma] \geq \varepsilon$ for every transverse measure μ with $\|\mu\| = 1$. Let us show that the conclusion of the lemma holds for every $C < \varepsilon$.

For this, we use a proof by contradiction. Suppose that the property does not hold. Then, there exists a sequence of distinct components T_n of $\tilde{S} \setminus \tilde{\lambda}$ meeting \tilde{k} such that $\sigma(T_0, T_n) < Cr(T_n)$ if T_n points to the right of \tilde{k} , and $\sigma(T_0, T_n) > -Cr(T_n)$ if it points to the left. Passing to a subsequence if necessary, we can arrange that either all T_n point to the right, or they all point to the left.

Let us focus attention on the case where all T_n point to the left, in which case $\sigma(T_0, T_n) > -Cr(T_n)$ for every n . The other case will be similar.

Let \tilde{k}_n be the subarc of \tilde{k} going from the negative endpoint of \tilde{k} to an arbitrary point of $\tilde{k} \cap T_n$. Let k_n be the projection of $\tilde{k}_n \subset \tilde{U}$ to U . Among the two lifts of k_n to the cover \hat{U} of U , let \hat{k}_n be the one where the canonical orientation of the leaves of $\hat{\lambda}$ points to the left for the orientation of \hat{k}_n coming from the orientation of \tilde{k} . (We are here using the fact that \tilde{k} is tightly transverse to $\tilde{\lambda}$.) In particular, \hat{k}_n is tightly transverse to $\hat{\lambda}$ in \hat{U} , and $\sigma(T_0, T) = \sigma(\hat{k}_n)$ by the construction of §4.7.

Let $[\hat{k}_n] \in H_1(\hat{U}, \partial_h \hat{U}; \mathbb{R})$ be the relative homology class associated with \hat{k}_n as in Step 2 of the proof of Proposition 4.5. Namely, $[\hat{k}_n]$ is represented by an arc $\hat{k}'_n \subset \hat{U}$ with $\partial \hat{k}'_n \subset \partial_h \hat{U}$ that is made up of the following five pieces: the arc \hat{k}''_n obtained from \hat{k}_n by

removing the two components of $\hat{k}_n \setminus \hat{\lambda}$ that contain its endpoints; two arcs l_0 and l_n in the leaves of $\hat{\lambda}$ that contain the endpoints of \hat{k}_n'' ; two arcs t_0 and t_n contained in ties of \widehat{U} , with one endpoint in the horizontal boundary $\partial_h \widehat{U}$, with the other endpoint in $\hat{\lambda}$, and whose interior is disjoint from $\hat{\lambda}$. We choose the indexing so that l_n joins the positive endpoint of \hat{k}_n'' to the negative endpoint of t_n , and l_0 joins the positive endpoint of t_0 to the negative endpoint of \hat{k}_n'' . In addition, we can arrange that t_0 and l_0 are independent of n . See Figure 11.

By Step 2 of the proof of Proposition 4.5, the homology classes $[\sigma] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$ and $[\hat{k}_n] = [\hat{k}_n'] \in H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$ are such that

$$[\hat{k}_n'] \cdot [\sigma] = \sigma(\hat{k}_n) = \sigma(T_0, T_n).$$

By definition of the divergence radius $r(T_n)$, the arc l_n crosses approximately $r(T_n)$ branches of \widehat{U} (counted with multiplicity). Because the triangles T_n are all distinct, $r(T_n)$ tends to infinity as n tends to ∞ by Lemma 8.7. Passing to a subsequence if necessary, the standard weak* compactness argument provides a non-trivial transverse measure μ for $\hat{\lambda}$ such that

$$\int_k \mu = \lim_{n \rightarrow \infty} \frac{1}{r(T_n)} \#(k \cap l_n)$$

for every arc k transverse to $\hat{\lambda}$, where $\#(k \cap l_n)$ denotes the number of points of $k \cap l_n$. In addition, $\|\mu\| = 1$ by definition of the norm $\|\cdot\|$.

Note that $\hat{k}_n' \setminus l_n$ has uniformly bounded length. In addition, the orientation of l_n coming from the orientation of \hat{k}_n' is opposite to the canonical orientation of the leaf of $\hat{\lambda}$ that contains it. Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{r(T_n)} [\hat{k}_n'] = -[\mu]$$

in $H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$. Then, intersecting with the class $[\sigma] \in H_1(\widehat{U}, \partial_v \widehat{U}; \mathbb{R})$ represented by $\sigma \in \mathcal{Z}(\hat{\lambda}, \text{slits}; \mathbb{R})$ gives

$$[\mu] \cdot [\sigma] = - \lim_{n \rightarrow \infty} \frac{1}{r(T_n)} [\hat{k}_n'] \cdot [\sigma] = - \lim_{n \rightarrow \infty} \frac{1}{r(T_n)} \sigma(T_0, T_n) \leq C,$$

since $\sigma(T_0, T_n) > -Cr(T_n)$ by hypothesis.

Therefore, we have constructed a transverse measure μ for $\hat{\lambda}$ such that $[\mu] \cdot [\sigma] \leq C$ and $\|\mu\| = 1$. But this contradicts our hypothesis that $C < \varepsilon \leq [\mu] \cdot [\sigma]$ for any such μ , and provides the contradiction sought when all T_n point to the left of \tilde{k} .

The argument is similar when all T_n point to the right. The only difference is that the transverse measure μ then constructed represents the homology class

$$[\mu] = \lim_{n \rightarrow \infty} \frac{1}{r(T_n)} [\hat{k}_n']$$

in $H_1(\widehat{U}, \partial_h \widehat{U}; \mathbb{R})$, because the orientation of l_n now coincides with the canonical orientation of the leaf of $\widehat{\lambda}$ containing it. Since the inequality $\sigma(T_0, T_n) < Cr(T_n)$ is also reversed, this again provides a transverse measure μ for $\widehat{\lambda}$ such that $[\mu] \cdot [\sigma] < C < \varepsilon$ and $\|\mu\|=1$, concluding the proof in this case as well. \square

COMPLEMENT 8.9. *The conclusion of Lemma 8.8 holds when σ is replaced by any σ' in a small neighborhood of σ in the finite-dimensional vector space $\mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$.*

Proof. By compactness of the space of transverse measures μ with $\|\mu\|=1$, we can choose $\varepsilon > 0$ so that $[\mu] \cdot [\sigma'] \geq \varepsilon$ for every $\sigma' \in \mathcal{Z}(\widehat{\lambda}, \text{slits}; \mathbb{R})$ sufficiently close to σ and every transverse measure μ with $\|\mu\|=1$. Then the proof shows that the conclusion of Lemma 8.8 holds for any such σ' and $C < \varepsilon$. \square

8.3. Realization of invariants, and parametrization of $\text{Hit}_n(S)$

At the beginning of §8, we introduced the map

$$\Phi: \text{Hit}_n(S) \longrightarrow \mathcal{P} \subset \mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$$

that associates its triangle invariants and shearing cycle with a Hitchin character. We showed in §8.1 that the image of Φ is contained in the convex polyhedral cone \mathcal{P} defined by the triangle rotation condition, the shearing cycle boundary condition and the positive intersection condition. We also showed in Corollary 8.6 that $\Phi: \text{Hit}_n(S) \rightarrow \mathcal{P}$ is a local homeomorphism.

PROPOSITION 8.10. *The map $\Phi: \text{Hit}_n(S) \rightarrow \mathcal{P}$ is proper.*

Proof. We need to prove the following property: Let $\{\varrho_i\}_{i \in \mathbb{N}}$ be a sequence in $\text{Hit}_n(S)$ such that $\{\Phi(\varrho_i)\}_{i \in \mathbb{N}} = \{(\tau^{\varrho_i}, \sigma^{\varrho_i})\}_{i \in \mathbb{N}}$ converges to a point $(\tau^\infty, \sigma^\infty) \in \mathcal{P}$; then the sequence $\{\varrho_i\}_{i \in \mathbb{N}}$ admits a converging subsequence.

For this, we will revisit our proof that a Hitchin character is determined by its triangle invariants and its shearing cycle, as in §6.2. In that proof, we showed that the fundamental group $\pi_1(S)$ is generated by elements γ of the type described in Lemma 6.4, and then proved that

$$\varrho_i(\gamma) = \left(\prod_{T \in \mathcal{T}_{g_0(\gamma h_0)}}^{\leftarrow} (\Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, T)} \circ \widehat{\Sigma}_T^i \circ \Theta_{E_0 F_0}^{-\sigma^{\varrho_i}(T_0, T)}) \right)^{-1} \circ \Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, \gamma T_0)} \circ \Phi_0^i \in \text{PGL}_n(\mathbb{R}),$$

with the notation of Lemma 6.7, except that $\widehat{\Sigma}_T^i$ and Φ_0^i were respectively called $\widehat{\Sigma}'_T$ and Φ_0 there. Recall that the terms $\Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, T)}$ depend only on the shearing cycles σ^{ϱ_i} , whereas $\widehat{\Sigma}_T^i$ and Φ_0^i depend only on the triangle invariants of ϱ_i . Also, the expression of

$\widehat{\Sigma}_T^i$ given in the proof of Lemma 6.6 involves two cases, according to whether T points to the left or to the right as seen from T_0 , as defined there (or again in §8.2).

LEMMA 8.11. *There exists a constant C , independent of T , such that*

$$\|\Theta_{E_0 F_0}^{\sigma^{\ell_i}(T_0, T)} \circ \widehat{\Sigma}_T^i \circ \Theta_{E_0 F_0}^{-\sigma^{\ell_i}(T_0, T)} - \text{Id}_{\mathbb{R}^n}\| \leq C \max_a e^{-(n-1)\sigma_a^{\ell_i}(T_0, T)},$$

if T points to the right between T_0 and γT_0 (as seen from T_0), and

$$\|\Theta_{E_0 F_0}^{\sigma^{\ell_i}(T_0, T)} \circ \widehat{\Sigma}_T^i \circ \Theta_{E_0 F_0}^{-\sigma^{\ell_i}(T_0, T)} - \text{Id}_{\mathbb{R}^n}\| \leq C \max_a e^{(n-1)\sigma_a^{\ell_i}(T_0, T)},$$

if T points to the left.

Proof. Fix for \mathbb{R}^n a basis in which the a th term belongs to the line $E_0^{(a)} \cap F_0^{(n-a+1)}$. Then, by definition, the matrix of $\Theta_{E_0 F_0}^{\sigma^{\ell_i}(T_0, T)}$ in this basis is diagonal, with diagonal entries $e^{u_1}, e^{u_2}, \dots, e^{u_n}$, where u_1, u_2, \dots, u_n are uniquely determined by the properties that $u_a - u_{a+1} = \sigma_a^{\ell_i}(T_0, T)$ and $\sum_{a=1}^n u_a = 0$.

Consider for instance the case where T points to the left. Then the map $\widehat{\Sigma}_T^i$ respects the flag E_0 , and acts by the identity on each of the lines $E_0^{(a)} / E_0^{(a-1)}$. Therefore, in the above basis for \mathbb{R}^n , the matrix A of $\widehat{\Sigma}_T^i$ is upper triangular with all diagonal entries equal to 1.

By construction, the map $\widehat{\Sigma}_T^i$ is completely determined by, and depends continuously on, the triangle invariants $\tau_{abc}^{\ell_i}(s)$ associated with the slit s of λ that is the projection of the spike of T delimited by the two components of ∂T that separate T_0 from γT_0 . Since these triangle invariants converge to $\tau_{abc}^\infty(s)$, we conclude that each ab -entry A_{ab} of the matrix A is uniformly bounded by a constant C . We already observed that $A_{ab} = 0$ if $a > b$ and $A_{aa} = 1$.

Multiplying matrices, we conclude that for $a < b$ the ab -entry of the matrix of

$$\Theta_{E_0 F_0}^{\sigma^{\ell_i}(T_0, T)} \circ \widehat{\Sigma}_T^i \circ \Theta_{E_0 F_0}^{-\sigma^{\ell_i}(T_0, T)} - \text{Id}_{\mathbb{R}^n}$$

is equal to $A_{ab}e^{u_a - u_b}$ and bounded by

$$|A_{ab}|e^{u_a - u_b} \leq C e^{u_a - u_b} = C e^{-\sum_{c=a}^{b-1} (u_{c+1} - u_c)} = C e^{\sum_{c=a}^{b-1} \sigma_c^{\ell_i}(T_0, T)} \leq C \max_c e^{(n-1)\sigma_c^{\ell_i}(T_0, T)}.$$

The other entries of this matrix are 0 since $A_{ab} = 0$ if $a > b$, and since $A_{aa} = 1$.

This proves the estimate required when the triangle T points to the left.

The proof is almost identical when T points to the right, except that the matrix A is now lower triangular. □

We now use the property that the limit $(\tau^\infty, \sigma^\infty) \in \mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$ actually belongs to the polyhedron \mathcal{P} , and more precisely the fact that the relative tangent cycle $\sigma^\infty \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$ satisfies the positive intersection condition.

LEMMA 8.12. For $\gamma \in \pi_1(S)$ as above, the $\varrho_i(\gamma) \in \text{PSL}_n(\mathbb{R})$ are bounded independently of i .

Proof. Because σ^∞ satisfies the positive intersection condition, the combination of Lemma 8.8, Complement 8.9 and Lemma 8.11 provides constants $C, D > 0$ such that, in the expression

$$\varrho_i(\gamma) = \left(\prod_{T \in \mathcal{T}_{g_0(\gamma h_0)}}^{\leftarrow} (\Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, T)} \circ \widehat{\Sigma}_T^i \circ \Theta_{E_0 F_0}^{-\sigma^{\varrho_i}(T_0, T)}) \right)^{-1} \circ \Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, \gamma T_0)} \circ \Phi_0^i,$$

the contribution of each triangle T is such that

$$\| \Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, T)} \circ \widehat{\Sigma}_T^i \circ \Theta_{E_0 F_0}^{-\sigma^{\varrho_i}(T_0, T)} - \text{Id}_{\mathbb{R}^n} \| \leq C e^{-Dr(T)},$$

for the divergence radius $r(T)$ defined in §8.2. In addition, for every integer $r_0 \geq 1$, Lemma 8.7 shows that the number of triangles T such that $r(T) = r_0$ is bounded independently of r_0 . It follows that the product

$$\prod_{T \in \mathcal{T}_{g_0(\gamma h_0)}}^{\leftarrow} (\Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, T)} \circ \widehat{\Sigma}_T^i \circ \Theta_{E_0 F_0}^{-\sigma^{\varrho_i}(T_0, T)})$$

converges and is uniformly bounded.

By construction, the remaining terms $\Theta_{E_0 F_0}^{\sigma^{\varrho_i}(T_0, \gamma T_0)}$ and Φ_0^i are completely determined by, and depends continuously on, the triangle and shearing invariants of ϱ_i . Since these invariants converge, it follows that these two terms are also uniformly bounded. \square

Lemma 8.12 shows that the sequence $(\varrho_i(\gamma))_{i \in \mathbb{N}}$ admits a converging subsequence in $\text{PSL}_n(\mathbb{R})$. Doing this for all γ in the finite set of generators for $\pi_1(S)$ provided by Lemma 6.4, we conclude that the sequence $(\varrho_i)_{i \in \mathbb{N}}$ admits a converging subsequence in $\text{Hit}_n(S)$.

Therefore, every sequence of $\text{Hit}_n(S)$ whose image under Φ converges in the polyhedron \mathcal{P} admits a converging subsequence in $\text{Hit}_n(S)$. This proves that the map $\Phi: \text{Hit}_n(S) \rightarrow \mathcal{P}$ is proper, and concludes the proof of Proposition 8.10. \square

THEOREM 8.13. The map $\Phi: \text{Hit}_n(S) \rightarrow \mathcal{P}$ is a homeomorphism from the Hitchin component $\text{Hit}_n(S)$ to the polyhedron $\mathcal{P} \subset \mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^n)$.

Proof. The map Φ is a local homeomorphism by Corollary 8.6, and proper by Proposition 8.10. Since Φ is injective by Corollary 6.8 and since the convex polytope \mathcal{P} is connected, this proves that Φ is a homeomorphism. \square

Remark 8.14. The formulas of §6.2, in particular Lemma 6.7, provide an explicit construction for the inverse map $\Phi^{-1}: \mathcal{P} \rightarrow \text{Hit}_n(S)$. The boundedness estimates that we just used in the proof of Lemma 8.12 show that the infinite products involved in these formulas do converge. This immediately proves that this inverse map Φ^{-1} is real analytic.

It can be shown that the forward map Φ is also analytic, using the fact [10] that the flag curve $\mathcal{F}_\varrho: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ depends real analytically on the homomorphism ϱ . However, this is beyond the scope of this article.

8.4. Constraints among triangle invariants, and on shearing cycles

The shearing cycle boundary condition does more than connecting the boundary of the shearing cycle σ^ϱ of a Hitchin character $\varrho \in \text{Hit}_n(S)$ to its triangle invariants $\tau_{abc}^\varrho(s)$. It also puts constraints between the triangle invariants themselves, and restricts the twisted relative tangent cycles that can occur as shearing cycles of Hitchin characters. As a complement to Theorem 8.13, this section is devoted to emphasizing these somewhat unexpected phenomena, which we already encountered in Lemmas 8.3 and 8.4.

COROLLARY 8.15. *A rotation-invariant triangle data function $\tau \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$ is the triangle invariant τ^ϱ of a Hitchin character $\varrho \in \text{Hit}_n(S)$ if and only if*

$$\sum_{s \text{ slit of } \lambda} \sum_{b+c=n-a} \tau_{abc}(s) = \sum_{s \text{ slit of } \lambda} \sum_{b+c=a} \tau_{(n-a)bc}(s)$$

for every $a=1, 2, \dots, n-1$.

As a consequence, the triangle invariants of Hitchin characters form a linear subspace of codimension $\lfloor \frac{1}{2}(n-1) \rfloor$ in the space $\mathbb{R}^{2(g-1)(n-1)(n-2)}$ of all rotation-invariant triangle data functions.

Proof. Theorem 8.13 shows that τ is the triangle invariant of a Hitchin character if and only if there exists a relative cycle $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ such that the pair (τ, σ) satisfies the shearing boundary condition, and such that σ satisfies the positive intersection condition.

The proof of Proposition 8.2, and in particular Lemmas 8.3 and 8.5, takes care of the first constraint. More precisely, with the notation of that proof, there exists $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ such that (τ, σ) satisfies the shearing boundary cycle condition if and only if $\Xi(\tau)$ belongs to the image $\text{im}(\partial)$. Lemma 8.3 shows that this is equivalent to the condition stated in Corollary 8.15, while Lemma 8.5 shows that $\Xi^{-1}(\text{im}(\partial))$ has codimension $\lfloor \frac{1}{2}(n-1) \rfloor$ in $\mathbb{R}^{2(g-1)(n-1)(n-2)}$.

The only thing left to prove is that the positive intersection condition has no impact on this property. Namely, if there exists $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ such that (τ, σ) satisfies the

shearing cycle boundary condition, the relative tangent cycle σ can be chosen so that, in addition, it satisfies the positive intersection condition.

For this, we will use the existence of a closed twisted tangent cycle $\sigma_0 \in \mathcal{Z}(\lambda; \widehat{\mathbb{R}}^{n-1})$ that satisfies the positive intersection condition. An easy way to construct such a tangent cycle is to consider the shearing cycle $\sigma_0 = \sigma^{\varrho_0} \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ of a Hitchin character $\varrho_0 \in \text{Hit}_2(S) \subset \text{Hit}_n(S)$ coming from a discrete homomorphism $\varrho: \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{R}) \subset \text{PSL}_n(\mathbb{R})$. All triangle invariants $\tau_{abc}^{\varrho_0}(s)$ of such a Hitchin character are equal to zero; the easiest way to see this is to apply Lemma 1.1 and to observe that, for every triangle component of $\widetilde{S} \setminus \widetilde{\lambda}$ with vertices \tilde{s}, \tilde{s}' and \tilde{s}'' , there is an element of $\text{PGL}_n(\mathbb{R})$ coming from an element of $\text{PGL}_2(\mathbb{R})$ that fixes the flag $\mathcal{F}_{\varrho_0}(\tilde{s}) \in \text{Flag}(\mathbb{R}^n)$ and exchanges $\mathcal{F}_{\varrho_0}(\tilde{s})$ and $\mathcal{F}_{\varrho_0}(\tilde{s})$. It therefore follows from the shearing cycle boundary condition that $\partial\sigma_0 = 0$, namely that σ_0 is closed. And σ_0 satisfies the positive intersection condition by Corollary 7.13.

If the rotation-invariant triangle data function $\tau \in \mathbb{R}^{2(g-1)(n-1)(n-2)}$ satisfies the conditions of Corollary 8.15, we just showed that there exists $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ such that (τ, σ) satisfies the shearing cycle boundary condition. For $c > 0$ sufficiently large, $\sigma + c\sigma_0$ satisfies the positive intersection condition since this property holds for σ_0 and since the space of transverse measures for $\hat{\lambda}$ is finite-dimensional [25], [35]. In addition, the pair $(\tau, \sigma + c\sigma_0)$ satisfies the shearing cycle boundary condition since $\partial(\sigma + c\sigma_0) = \partial\sigma$, and the triangle rotation condition by choice of τ . As a consequence, Theorem 8.13 provides a Hitchin character $\varrho \in \text{Hit}_n(S)$ whose triangle invariant τ^ϱ is τ , and whose shearing cycle σ^ϱ is equal to $\sigma + c\sigma_0$. □

There is a similar characterization of the shearing cycles of Hitchin characters.

COROLLARY 8.16. *Let $n > 3$. For a twisted relative tangent cycle $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ and for $a = 1, 2, \dots, n-1$, let $\partial\sigma_a$ be the a -th component of its boundary $\partial\sigma: \{\text{slits of } \hat{\lambda}\} \rightarrow \mathbb{R}^{n-1}$. Then, σ is the shearing cycle σ^ϱ of a Hitchin character $\varrho \in \text{Hit}_n(S)$ if and only if σ satisfies the positive intersection condition and*

$$\begin{aligned} \partial\sigma_{n-1}(s_1^+) &= 0, \\ \partial\sigma_1(s_1^+) &= \sum_{a=2}^{n-2} \left(\frac{a-1}{n-3} - 1 \right) \partial\sigma_a(s_1^+) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \partial\sigma_a(s_2^+) + \sum_{a=2}^{n-2} \frac{a-1}{n-3} \partial\sigma_a(s_3^+), \end{aligned}$$

whenever s_1^+, s_2^+ and s_3^+ are positive slits of the orientation cover $\hat{\lambda}$ that project to the three spikes of the same component T of $S \setminus \lambda$.

As a consequence, the shearing cycles of Hitchin characters form an open convex polyhedral cone in a linear subspace of codimension $24(g-1)$ of

$$\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong \mathbb{R}^{18(g-1)(n-1)}.$$

Proof. This is a consequence of Lemma 8.4 and Proposition 4.7. □

COROLLARY 8.17. *When $n=3$, a twisted relative tangent cycle $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^2)$ is the shearing cycle σ^ϱ of a Hitchin character $\varrho \in \text{Hit}_3(S)$ if and only if σ satisfies the positive intersection condition and*

$$\partial\sigma_2(s_1^+) = 0 \quad \text{and} \quad \partial\sigma_1(s_1^+) = \partial\sigma_1(s_2^+) = \partial\sigma_1(s_3^+) = 0,$$

whenever s_1^+ , s_2^+ and s_3^+ are positive slits of the orientation cover $\widehat{\lambda}$ that project to the three spikes of the same component T of $S \setminus \lambda$. As a consequence, the shearing cycles of Hitchin characters form an open convex polyhedral cone in a subspace of codimension $20(g-1)$ of $\mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^2) \cong \mathbb{R}^{36(g-1)}$.

When $n=2$, a twisted relative tangent cycle $\sigma \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}) = \mathcal{Z}(\lambda, \text{slits}; \mathbb{R})$ is the shearing cycle σ^ϱ of a Hitchin character $\varrho \in \text{Hit}_2(S)$ if and only if σ is closed (namely $\sigma \in \mathcal{Z}(\lambda; \mathbb{R})$) and satisfies the positive intersection condition. The shearing cycles of Hitchin characters (=fuchsian representations) consequently form an open convex polyhedral cone in a subspace of codimension $12(g-1)$ of $\mathcal{Z}(\lambda, \text{slits}; \mathbb{R}) \cong \mathbb{R}^{18(g-1)}$.

Proof. This also follows from Lemma 8.4 and Proposition 4.7. □

We conclude this article by giving, in the next two sections, two brief applications of the machinery developed in this article. In particular, these applications require the full generality of geodesic laminations (as opposed to the much simpler case of geodesic laminations with finitely many leaves considered in [9]).

9. The action of pseudo-Anosov homeomorphisms on the Hitchin component

Let $\varphi: S \rightarrow S$ be a pseudo-Anosov homeomorphism of the surface S . We can use our parametrization of $\text{Hit}_n(S)$ to show that the action of φ on the Hitchin component $\text{Hit}_n(S)$ is concentrated in a relatively small factor of $\text{Hit}_n(S)$, where it acts linearly. This section is only intended as an illustration of the possible applications of the main results of the article; we are consequently limiting its scope to avoid making an already long article much longer.

The pseudo-Anosov property of φ is usually expressed in terms of transverse measured foliations on the surface S [40], [17]. It will be more convenient to use the point of view of [14], so that the homeomorphism $\varphi: S \rightarrow S$ is (isotopic to) a pseudo-Anosov homeomorphism if there exist a geodesic lamination λ^s , a transverse measure μ^s for λ^s and a number $R > 1$ such that, after an isotopy of φ , the following conditions hold:

- (1) each component of the complement of the topological support λ^s of μ^s is a topological disk;

- (2) $\varphi(\lambda^s) = \lambda^s$;
- (3) the pull back $\varphi^*(\mu^s)$ of the transverse measure μ^s is equal to $R\mu^s$.

The homomorphism $\varphi: S \rightarrow S$ acts on the character variety $\mathcal{X}_{\text{PSL}_n(\mathbb{R})}(S)$ as $\varrho \mapsto \varrho \circ \varphi_*$, where $\varphi_*: \pi_1(S) \rightarrow \pi_1(S)$ is any homomorphism induced by φ (by choosing a path joining the base point to its image under φ). When $\varrho \in \mathcal{X}_{\text{PSL}_n(\mathbb{R})}(S)$ comes from a Teichmüller character of $\text{Hit}_2(S)$, it is immediate that so does $\varrho \circ \varphi_*$. By connectedness, it follows that the action $\varrho \mapsto \varrho \circ \varphi_*$ respects the Hitchin component $\text{Hit}_n(S)$.

Replacing φ by one of its powers does not significantly change its dynamics.

LEMMA 9.1. *There exists an integer $k > 0$ and a maximal geodesic lamination λ^+ containing λ^s such that $\varphi^k(\lambda^+) = \lambda^+$ after isotopy of φ^k . In addition, φ^k can be chosen so that it respects each slit of λ^+ .*

Proof. The homeomorphism φ permutes the finitely many slits of λ^s . Therefore, there exists k such that φ^k respects each slit.

Let λ^+ be any maximal geodesic lamination containing λ^s . Because each component of $S \setminus \lambda^s$ is a topological disk, or more precisely an ideal polygon, λ^+ is obtained from λ^s by adding finitely many diagonal leaves joining spikes of these polygons. Since φ^k respects each slit of λ^s , namely each spike of $S \setminus \lambda^s$, it can easily be isotoped to respect these diagonal leaves (as well as λ^s). By construction, φ^k respects each slit of λ^+ . \square

We can now use the maximal geodesic lamination λ^+ to construct a parametrization of the Hitchin component $\text{Hit}_n(S)$ by the polytope

$$\mathcal{P} \subset \mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$$

as in Theorem 8.13.

Because φ^k respects the geodesic lamination λ^+ , it acts on $\mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ as follows. Lift φ to a homeomorphism $\tilde{\varphi}: \tilde{S} \rightarrow \tilde{S}$ of the universal cover \tilde{S} ; in particular, $\tilde{\varphi}^k$ respects the pre-image $\tilde{\lambda}^+$ of λ^+ . Then, using the point of view of §4.7, define a linear map $\varphi_*^k: \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \rightarrow \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ by the property that $\varphi_*^k(\alpha)(T, T') = \alpha(\tilde{\varphi}^k(T), \tilde{\varphi}^k(T'))$ for any two components T and T' of $\tilde{S} \setminus \tilde{\lambda}^+$.

PROPOSITION 9.2. *For the homeomorphism*

$$\Phi: \text{Hit}_n(S) \longrightarrow \mathcal{P} \subset \mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$$

provided by Theorem 8.13, the action of φ^k on $\text{Hit}_n(S)$ corresponds to the restriction to \mathcal{P} of the product of the identity $\text{Id}_{\mathbb{R}^{6(g-1)(n-1)(n-2)}}$ and of the linear action of φ^k on $\mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$.

Proof. For $\varrho \in \text{Hit}_n(S)$, we need to compare the triangle invariants $\tau_{abc}^{\varrho \circ \varphi_*^k}(s)$ and the shearing cycle $\sigma^{\varrho \circ \varphi_*^k} \in \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ of $\varrho \circ \varphi_*^k$ to those of ϱ .

Lift φ to a homeomorphism $\tilde{\varphi}: \tilde{S} \rightarrow \tilde{S}$ of the universal cover \tilde{S} , which is equivariant with respect to $\varphi_*: \pi_1(S) \rightarrow \pi_1(S)$ in the sense that $\tilde{\varphi}(\gamma x) = \varphi_*(\gamma)\tilde{\varphi}(x)$ for every $x \in \tilde{S}$ and $\gamma \in \pi_1(S)$. It is well known what $\tilde{\varphi}$ extends to a homeomorphism of $\tilde{S} \cap \partial_\infty \tilde{S}$. The flag maps \mathcal{F}_ϱ and $\mathcal{F}_{\varrho \circ \varphi_*^k}: \partial_\infty \tilde{S} \rightarrow \text{Flag}(\mathbb{R}^n)$ are then related by the property that $\mathcal{F}_{\varrho \circ \varphi_*^k} = \mathcal{F}_\varrho \circ \tilde{\varphi}^k$. Going back to the definitions of these invariants and remembering that φ^k respects each slit of λ^+ , it immediately follows that ϱ and $\varrho \circ \varphi_*^k$ have the same triangle invariants $\tau_{abc}^{\varrho \circ \varphi_*^k}(s) = \tau_{abc}^\varrho(s)$, and that $\sigma^{\varrho \circ \varphi_*^k} = \varphi_*^k(\sigma^\varrho)$. □

This is better described in terms of the map $\pi: \text{Hit}_n(S) \rightarrow \mathbb{R}^{6(g-1)(n-1)(n-2)}$ corresponding to the projection of $\text{Hit}_n(S) \cong \mathcal{P}$ to the first factor of $\mathbb{R}^{6(g-1)(n-1)(n-2)} \times \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$. Namely, π associates its triangle invariants $\tau_{abc}^\varrho(s)$ with a Hitchin character $\varrho \in \text{Hit}_n(S)$. The image $\mathcal{L} = \pi(\text{Hit}_n(S))$ is the vector space of dimension

$$2(g-1)(n-1)(n-2) - \lfloor \frac{1}{2}(n-1) \rfloor$$

determined by Corollary 8.15. This defines a fibration $\pi: \text{Hit}_n(S) \rightarrow \mathcal{L}$, where the fiber $\pi^{-1}(\tau)$ above each $\tau \in \mathcal{L}$ is a convex polyhedral cone of dimension

$$3(g-1)(n-1) + \lfloor \frac{1}{2}(n-1) \rfloor$$

in $\mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong \mathbb{R}^{18(g-1)(n-1)}$.

Then, Proposition 9.2 states that the action of φ^k on $\text{Hit}_n(S)$ respects each fiber $\pi^{-1}(\tau)$, and acts on each of these polyhedral cones $\pi^{-1}(\tau) \subset \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ by restriction of $\varphi_*^k: \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \rightarrow \mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1})$.

If U is a train-track neighborhood of λ^+ , the endomorphism φ_*^k of $\mathcal{Z}(\lambda^+, \text{slits}; \widehat{\mathbb{R}}^{n-1}) \cong H_1(U, \partial_v U; \widehat{\mathbb{R}}^{n-1})$ can be explicitly described in terms of a classical object associated with the pseudo-Anosov homeomorphism φ , namely the incidence matrix of φ with respect to the train track U (see for instance [17, Exposés 9–10]). However, this would take us beyond the intended scope of this article.

10. Length functions of measured laminations

One of the motivations for this article is to extend to the Hitchin component the differential calculus of lengths of simple closed curves that was developed for hyperbolic geometry in [39], [42], [6], [5].

For a Hitchin character $\varrho \in \text{Hit}_n(S)$, the length functions $\ell_1^\varrho, \ell_2^\varrho, \dots, \ell_{n-1}^\varrho$, of [15] and §7.1 can be restricted to Thurston’s space $\widehat{\mathcal{ML}}(S)$ of measured geodesic laminations.

There is just a little subtlety, which is that the geodesic currents discussed in §7.1 form a completion of the set of homotopy classes of *oriented* closed curves, whereas $\mathcal{ML}(S)$ completes the set of homotopy classes of *unoriented* simple closed curves.

An unoriented simple closed curve γ in S defines two oriented curves γ^* and γ^{**} , one for each orientation, and therefore two measure geodesic currents that we also denote by $\gamma^*, \gamma^{**} \in \mathcal{C}^m(S)$. Then there is a unique continuous embedding $\iota: \mathcal{ML}(S) \rightarrow \mathcal{C}^m(S)$ that is homogeneous, in the sense that $\iota(t\mu) = t\iota(\mu)$ for every $\mu \in \mathcal{ML}(S)$ and every $t > 0$, and such that $\iota(\gamma) = \frac{1}{2}(\gamma^* + \gamma^{**})$ for every simple closed curve $\gamma \in \mathcal{ML}(S)$; see for instance [3]. Combining this embedding with $\ell_a^e: \mathcal{C}^m(S) \rightarrow \mathbb{R}$ defines, for each $a = 1, 2, \dots, n-1$, a length function $\ell_a^e: \mathcal{ML}(S) \rightarrow \mathbb{R}$. The definition, and in particular the introduction of the factor $\frac{1}{2}$, is designed so that when $n=2$ the function ℓ_1^e coincides with Thurston’s length function $\ell^e: \mathcal{ML}(S) \rightarrow \mathbb{R}$ for the hyperbolic metric on S associated with $\varrho \in \text{Hit}_2(S)$, which plays a fundamental role in hyperbolic geometry; see for instance [40], [17], [39], [3], [33] for a few applications of this length function ℓ^e .

Because $\ell_a^e(\gamma^{**}) = \ell_{n-a}^e(\gamma^*)$, the length functions ℓ_a^e and ℓ_{n-a}^e coincide on $\mathcal{ML}(S)$ so that, in practice, we have only $\lfloor \frac{1}{2}n \rfloor$ length functions $\ell_a^e: \mathcal{ML}(S) \rightarrow \mathbb{R}$.

The space $\mathcal{ML}(S)$ of measured geodesic laminations is homeomorphic to $\mathbb{R}^{6(g-1)}$, but admits no differentiable structure that is respected by the action of the mapping class group. As a consequence, we cannot use the standard concepts of differential calculus on this space.

However, $\mathcal{ML}(S)$ is naturally endowed with a piecewise integral linear structure; this means that it admits an atlas locally modelling it over $\mathbb{R}^{6(g-1)}$ where the coordinate changes are piecewise linear and where the linear pieces of these coordinate changes have integer coefficients [39], [36]. In particular, because a piecewise linear map does not have a tangent map, a consequence of the piecewise linear structure is that $\mathcal{ML}(S)$ admits a well-defined tangent space $T_\mu \mathcal{ML}(S)$ at each point $\mu \in \mathcal{ML}(S)$.

Each tangent space $T_\mu \mathcal{ML}(S)$ is homeomorphic to $\mathbb{R}^{6(g-1)}$ and is homogeneous, in the sense that there is a well-defined multiplication of tangent vectors by non-negative numbers, but it is not always a vector space. Indeed, there exist points $\mu \in \mathcal{ML}(S)$ where the tangent space $T_\mu \mathcal{ML}(S)$ admits no vector space structure which is respected by all coordinate charts; a typical example of such points are the positive real multiples of simple closed curves, which are dense in $\mathcal{ML}(S)$. Conversely, at a measured geodesic lamination μ whose support is a maximal geodesic lamination, the piecewise integral linear structure does define a natural vector space structure on the tangent space $T_\mu \mathcal{ML}(S)$; these μ form a subset of full measure in $\mathcal{ML}(S)$. See [42], for instance.

THEOREM 10.1. *For a Hitchin character $\varrho \in \text{Hit}_n(S)$ and for every $a=1, 2, \dots, \lfloor \frac{1}{2}n \rfloor$, the length function $\ell_a^{\varrho}: \mathcal{ML}(S) \rightarrow \mathbb{R}$ admits a tangent map $T_{\mu}\ell_a^{\varrho}: T_{\mu}\mathcal{ML}(S) \rightarrow \mathbb{R}$ at each $\mu \in \mathcal{ML}(S)$, in the following sense. For $\mu \in \mathcal{ML}(S)$ and $v \in T_{\mu}\mathcal{ML}(S)$, let $t \mapsto \alpha_t$ be a curve in $\mathcal{ML}(S)$ such that $\alpha_0 = \mu$ and the right-hand-side tangent derivative*

$$\left. \frac{d}{dt^+} \alpha_t \right|_{t=0}$$

exists and is equal to v . Then

$$\left. \frac{d}{dt^+} \ell_a^{\varrho}(\alpha_t) \right|_{t=0} = T_{\mu}\ell_a^{\varrho}(v) \in \mathbb{R}.$$

Proof. See [15, §3.2]. □

The proof of Theorem 10.1 relies on two key ingredients: the analytic interpretation [7], [6] of tangent vectors $v \in T_{\mu}\mathcal{ML}(S)$ as a certain type of Hölder geodesic currents as in §7.1; and the continuity of the length function $\ell_a^{\varrho}: \mathcal{C}^H(S) \rightarrow \mathbb{R}$ for the Hölder topology, proved in [15]. In particular, $T_{\mu}\ell_a^{\varrho}(v)$ is equal to the a th length $\ell_a^{\varrho}(v)$ of the Hölder geodesic current $v \in \mathcal{C}^H(S)$ associated with $v \in T_{\mu}\mathcal{ML}(S)$.

The results of the current paper, and in particular Theorem 7.5, provide a description of the tangent map $T_{\mu}\ell_a^{\varrho}$ on the faces of $T_{\mu}\mathcal{ML}(S)$.

This is based on a more combinatorial interpretation, also developed in [6] and [7], of tangent vectors $v \in T_{\mu}\mathcal{ML}(S)$ as tangent cycles for geodesic laminations λ containing the support λ_{μ} of μ ; these tangent cycles must satisfy a certain positivity condition (unrelated to the positive intersection condition of §8.1). This decomposes the tangent space $T_{\mu}\mathcal{ML}(S)$ into a family of cones F_{λ} , indexed by geodesic laminations λ containing the support λ_{μ} of μ , where F_{λ} consists of those tangent vectors $v \in T_{\mu}\mathcal{ML}(S)$ that can be described as tangent cycles for λ . In particular, each F_{λ} is naturally identified to a convex polyhedral cone in the vector space $\mathcal{Z}(\lambda; \mathbb{R})$ of all tangent cycles for λ , and the homogeneous and additive structure induced on the cone F_{λ} by the vector space structure of $\mathcal{Z}(\lambda; \mathbb{R})$ is compatible with the piecewise linear structure of $\mathcal{ML}(S)$. The F_{λ} are the faces of $T_{\mu}\mathcal{ML}(S)$ for the piecewise linear structure of $\mathcal{ML}(S)$. See [42] for a slightly different approach.

In the generic case where the support λ_{μ} of $\mu \in \mathcal{ML}(S)$ is maximal, there is only one face in $T_{\mu}\mathcal{ML}(S)$, namely $F_{\lambda_{\mu}}$. This face $F_{\lambda_{\mu}}$ is equal to the whole vector space $\mathcal{Z}(\lambda_{\mu}; \mathbb{R})$ of tangent cycles for λ_{μ} .

Because of the positivity condition involved in the interpretation of tangent vectors $v \in T_{\mu}\mathcal{ML}(S)$ as tangent cycles for geodesic laminations, it is quite possible that different geodesic laminations λ and λ' define the same face $F_{\lambda} = F_{\lambda'}$. The correspondence $\lambda \mapsto F_{\lambda}$

can be made bijective by restricting attention to chain recurrent geodesic laminations [42], [7]. Instead of chain recurrent geodesic laminations, we will focus on the case where the geodesic lamination λ is maximal, as it is better adapted to our purposes. Every geodesic lamination λ' is contained in a maximal geodesic lamination λ , so that every face of $T_\mu\mathcal{ML}(S)$ is contained in a face F_λ associated with a maximal geodesic lamination λ . Note that, although λ is maximal, the dimension of the associated face F_λ may be significantly smaller than the dimension $6(g-1)$ of $T_\mu\mathcal{ML}(S)$.

THEOREM 10.2. *The tangent map $T_\mu\ell_a^g: T_\mu\mathcal{ML}(S) \rightarrow \mathbb{R}$ is linear on each face of $T_\mu\mathcal{ML}(S)$.*

More precisely, if the face $F_\lambda \subset T_\mu\mathcal{ML}(S)$ is associated with a maximal geodesic lamination λ , if we interpret the tangent vector $v \in F_\lambda$ as a tangent cycle for λ , and if $\sigma^g \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$ is the shearing cycle of ϱ , then

$$T_\mu\ell_a^g(v) = [\sigma_a^g] \cdot [v]$$

where, as in §4.5 and §7.2, the dot \cdot denotes the algebraic intersection number in a train-track neighborhood \widehat{U} of the orientation cover $\widehat{\lambda}$ of λ , $[\sigma_a^g] \in H_1(\widehat{U}, \partial_v\widehat{U}; \mathbb{R})$ is the a -th component of the twisted relative homology class

$$[\sigma^g] \in H_1(U, \partial_v U; \widetilde{\mathbb{R}}^{n-1}) \subset H_1(\widehat{U}, \partial_v\widehat{U}; \mathbb{R}^{n-1})$$

defined by $\sigma^g \in \mathcal{Z}(\lambda, \text{slits}; \widehat{\mathbb{R}}^{n-1})$, and $[v] \in H_1(\widehat{U}; \mathbb{R})$ is the homology class represented by $v \in \mathcal{Z}(\lambda; \mathbb{R}) \subset \mathcal{Z}(\widehat{\lambda}; \mathbb{R})$.

Proof. We already observed that $T_\mu\ell_a^g(v) = \ell_a^g(v)$ where the right-hand side interprets v as a tangent cycle for λ and involves the function $\ell_a^g: \mathcal{Z}(\widehat{\lambda}) \rightarrow \mathbb{R}$ introduced in §7.1. The formula then occurs as a special case of Theorem 7.5. \square

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