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On nonergodic property of Bose gas with weak pair interaction

D. V. PROKHORENKO

Institute of Spectroscopy RAS, 142190 Troitsk, Moskow Region, Russia
E-mail: prokhordv@yandex.ru

Dedicated to the memory of my father V. D. Prokhorenko

Abstract

In this paper we prove that Bose gas with weak pair interaction is nonergodic system. In order to prove this fact we consider the divergences in some nonequilibrium diagram technique. These divergences are analogous to the divergences in the kinetic equations discovered by Cohen and Dorfman. We develop the general theory of renormalization of such divergences and illustrate it with some simple examples. The fact that the system is nonergodic leads to the following consequence: to prove that the system tends to the thermal equilibrium we should take into account its behavior on its boundary. In this paper we illustrate this thesis with the Bogoliubov derivation of the kinetic equations.

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1 Introduction

In this paper we study divergences in some nonequilibrium diagram technique which is analogous to the Keldysh diagram technique. It is more or less evident that these divergences are the same as the divergences in the kinetic equations discovered by Cohen and Dorfman [1, 2, 3]. We develop the general theory of renormalization of such divergences analogously to the Bogoliubov-Parasiuk R-operation method [4, 5, 6]. Our main result can be formulated as follows: for a wide class of Bose systems in the sense of formal power series on coupling constant there exists non-Gibbs functional $\langle\cdot\rangle$, commuting with the number of particle operator such that the correlators $\langle\Psi^{\pm}(t,x_1)\cdots\Psi^{\pm}(t,x_n)\rangle$ are translation invariant, do not depend on t, and satisfy the weak cluster property. Here $\Psi^{\pm}$ are the fields operators and the weak cluster property means the following:

$$\lim_{|a|\to\infty}\int_{R^{3n}}\big\langle\Psi^{\pm}\big(t,x_1+\delta_1e_1a\big)\cdots\Psi^{\pm}\big(t,x_n+\delta_ne_1a\big)\big\rangle f\big(x_1,\dots,x_n\big)d^3x_1\cdots d^3x_n$$
$$=\int_{R^{3n}}\big\langle\Psi^{\pm}\big(t,x_{i_1}\big)\cdots\Psi^{\pm}\big(t,x_{i_k}\big)\big\rangle\big\langle\Psi^{\pm}\big(t,x_{i_k}\big)\cdots\Psi^{\pm}\big(t,x_{i_n}\big)\big\rangle f\big(x_1,\dots,x_n\big)d^3x_1\cdots d^3x_n,$$

there $\delta_i\in\{1,0\}$, $i=1,2\dots,n$ and

$$\begin{aligned}&i_1<i_2<\cdots<i_k,\quad i_{k+1}<i_{k+2}<\cdots<i_n,\\&\big\{i_1,i_2,\dots,i_k\big\}=\big\{i=1,2\cdots n\mid\delta_i=0\big\}\neq\emptyset,\\&\big\{i_{k+1},i_{k+2},\dots,i_n\big\}=\big\{i=1,2\cdots n\mid\delta_i=1\big\}\neq\emptyset.\end{aligned}$$

$f(x_1,\dots,x_n)$ is a test function, and e_1 is a unit vector parallel to the x-axis. This statement is a simple consequence of the theorem from Section 6.

Let us prove that the existence of such functionals implies nonergodic property of the system. We consider the problem only on classical level. The accurate consideration for the quantum case can be found in Section 10. Suppose that our system is ergodic, i.e., there are no first integrals of the system except energy. Then, the distribution function depends only on energy. We can represent the distribution function $f(E)$ as follows:

$$f(E) = \sum c_\alpha \delta\big(E - E_\alpha\big),$$

where the sum can be continuous (integral). Let 1 be some enough large but finite subsystem of our system. Let 2 be a subsystem obtained from 1 by translation on the vector $\vec{l}$ of sufficiently large length parallel to the x-axis. Let 12 be a union of the subsystems 1 and 2. Let ρ_1, ρ_2, and ρ_{12} be distribution functions for the subsystems 1, 2, and 12, respectively. Let Γ_1, Γ_2, and Γ_{12} be points of the phase spaces for the subsystem 1, 2, and 12, respectively. By the same method as the method used for the derivation of the Gibbs distribution we find

$$\rho_{12} = \sum c_\alpha d_\alpha \frac{e^{-\frac{E_{\Gamma_1}}{T_\alpha}}}{Z_\alpha} \frac{e^{-\frac{E_{\Gamma_2}}{T_\alpha}}}{Z_\alpha}, \quad d_\alpha > 0,\ \forall \alpha,$$

in the obvious notation. But the weak cluster property implies that $\rho_{12} = \rho_1 \rho_2$. Therefore all the coefficients c_α are equal to zero except one. We find that $f(E) = c\delta(E - E_0)$ for some constants c and E_0. So each finite subsystem of our system can be described by Gibbs formula and we obtain a contradiction.

Nonergodic property means that there is no thermalization in infinite Bose-gas system.

This fact implies to prove that the system tends to thermal equilibrium we should take into account the behavior of the system on its boundary. Indeed if a system has no boundary the system is infinite.

To illustrate this fact we will study Bogoliubov derivation of kinetic equations [7]. When one derives BBGKI-chain, one neglects some boundary terms. If one takes into account this boundary terms and uses the Bogoliubov method of derivation of the kinetic equations, one finds that these boundary terms compensate the scattering integral.

I think that the dependence of behavior of the system on boundary can be observed for small systems such as nanosystems or biological systems.

Note that our main result is closely related with the so-called Prigogin hypothesis which states that the infinite-dimensional Liouville dynamics cannot be derived from the Hamilton dynamics. The Prigogin hypothesis is proven in [8].

The paper is organized as follows. In Section 2 we introduce the notion of the algebra of canonical commutative relations and develop a useful representation for some class of the states on this algebra. In Section 3 we describe the von Neumann dynamics for the states. In Section 4 we describe a useful representation for the von Neumann dynamics—the dynamics of correlations. In Section 5 we describe the decomposition of the kinetic evolution operator by the so-called trees of correlations. In Section 6 we describe the general form of the counterterms which subtract the divergences in the nonequilibrium perturbation theory. In Section 7 we describe the so-called Friedrichs diagrams. In Section 8 we describe the Bogoliubov-Parasiuk prescriptions and formulate our main theorem. In Section 9 we prove our main theorem. In Section 10 we derive the nonergodic property of Bose gas with weak pair interaction from our main result. In Section 11 we consider one example related to our general theory. In Section 12 we reconsider the Bogoliubov derivation of the Boltzmann equation. This example illustrates the main thesis of this paper: to prove that the system tends to the thermal equilibrium one has to take into account its behavior on its boundary. Section 13 is a conclusion.

2 The algebra of canonical commutative relations

Let $S(\mathbb{R}^3)$ be a Schwatrz space of test functions (infinitely-differentiable functions decaying at infinity faster than any inverse polynomial with all its derivatives). The algebra of canonical commutative relations $\mathcal{C}$ is a unital algebra generated by symbols $a^+(f)$ and $a(f)$, $f \in S(\mathbb{R}^3)$ satisfying the following canonical commutative relations:

(a) $a^+(f)$ is a linear functional of f,

(b) $a(f)$ is an antilinear functional of f,

$$\big[a(f), a(g)\big] = \big[a^+(f), a^+(f)\big] = 0, \quad \big[a(f), a^+(g)\big] = \langle f, g\rangle,$$

where $\langle f, g\rangle$ is a standard scalar product in $L^2(\mathbb{R}^3)$, $\langle f, g\rangle := \int d^3x f^*(x)g(x)$.

Let ρ_0 be a Gauss state on $\mathcal{C}$ defined by the following correlator:

$$\rho_0\big(a^+(k)a^+(k')\big) = \rho_0\big(a(k)a(k')\big) = 0, \quad \rho_0\big(a^+(k)a(k')\big) = n(k)\delta(k-k'),$$

where $n(k)$ is a real-valued function from the Schwartz space. In the case then

$$n(k) = \frac{e^{-\beta(\omega(k)-\mu)}}{1 - e^{-\beta(\omega(k)-\mu)}},$$

where $\mu \in \mathbb{R}$, $\mu < 0$, ρ_0 is called the Plank state. Here $\omega(k) = \frac{k^2}{2}$.

Let $\mathcal{C}'$ be a space of linear functionals on $\mathcal{C}$, and $\mathcal{C}'_{+,1}$ a set of all states on $\mathcal{C}$. Let us make the GNS construction corresponding to the algebra $\mathcal{C}$ and the Gauss state ρ_0. We obtain the set $(\mathcal{H}, D, \hat{}, \rangle)$ consisting of the Hilbert space $\mathcal{H}$, the dense linear subspace D in $\mathcal{H}$, the representation $\hat{}$ of $\mathcal{C}$ by means of the linear operators from D to D, and the cyclic vector $\rangle \in D$, i.e., the vector such that $\hat{C}\rangle = D$. This set satisfies the following condition: $\forall a \in \mathcal{C}$, $\langle \hat{a}\rangle = \rho_0(a)$. Below we will omit the symbol $\hat{}$, i.e., we will write a instead of $\hat{a}$.

Let us introduce the field operators:

$$\Psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{ikx} a(k)dk, \quad \Psi^+(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ikx} a^+(k)dk.$$

We say that the state ρ on $\mathcal{C}$ satisfies the weak cluster property if

$$\begin{aligned}
&\lim_{a\to\infty} \int \big\langle \Psi^\pm\big(t, x_1 + \delta_1 e_1 a\big) \cdots \Psi^\pm\big(t, x_n + \delta_n e_1 a\big)\big\rangle f\big(x_1, \ldots, x_n\big) d^3x_1 \cdots d^3x_n \\
&\quad = \int \big\langle \Psi^\pm\big(t, x_{i_1}\big) \cdots \Psi^\pm\big(t, x_{i_k}\big)\big\rangle \big\langle \Psi^\pm\big(t, x_{i_k}\big) \cdots \Psi^\pm\big(t, x_{i_n}\big)\big\rangle f\big(x_1, \ldots, x_n\big) d^3x_1 \cdots d^3x_n,
\end{aligned}$$

where $\delta_i \in \{1, 0\}$, $i = 1, 2 \cdots n$ and

$$\begin{aligned}
&i_1 < i_2 < \cdots < i_k, \quad i_{k+1} < i_{k+2} < \cdots < i_n, \\
&\big\{i_1, i_2, \ldots, i_k\big\} = \big\{i = 1, 2 \cdots n \mid \delta_i = 0\big\} \neq \emptyset, \\
&\big\{i_{k+1}, i_{k+2}, \ldots, i_n\big\} = \big\{i = 1, 2 \cdots n \mid \delta_i = 1\big\} \neq \emptyset.
\end{aligned}$$

$f(x_1, \ldots, x_n)$ is a test function. e_1 is an unit vector parallel to the x-axis.

Definition 2.1. The vector of the form

$$\int v(p_1,\dots,p_n)a^{\pm}(p_1)\cdots a^{\pm}(p_n)\rangle d^3p_1\cdots d^3p_n,\quad v(p_1,\dots,p_n)\in S(\mathbb{R}^{3n}),\tag{2.1}$$

is called a finite vector. The finite linear combination of the vectors of the form (2.1) is also called a finite vector.

Let $f(x_1,\dots,x_k\mid y_1,\dots,y_l\mid v_1,\dots,v_m\mid w_1,\dots,w_n)$ be a function of the form

$$\begin{aligned}&f(x_1,\dots,x_k\mid y_1,\dots,y_l\mid v_1,\dots,v_m\mid w_1,\dots,w_n)\\&\quad=g(x_1,\dots,x_k\mid y_1,\dots,y_l\mid v_1,\dots,v_m\mid w_1,\dots,w_n)\delta\left(\sum_{i=1}^k x_i-\sum_{j=1}^m w_j-\sum_{f=1}^l y_f+\sum_{g=1}^n v_g\right),\end{aligned}$$

where g is a function from Schwartz space.

Consider the following functional on $\mathcal{C}$:

$$\begin{aligned}\rho_f(A):=&\int\prod_{i=1}^k dx_i\prod_{j=1}^l dx_j\prod_{f=1}^m dv_f\prod_{g=1}^n dw_g\\&\times f(x_1,\dots,x_n\mid y_1,\dots,y_l\mid v_1,\dots,v_m\mid w_1,\dots,w_n)\\&\times\rho_0\big(:a(x_1)\cdots a(x_n)\overbrace{a^+(y_1)\cdots a^+(y_l):A:a(v_1)\cdots a(v_m)}a^+(w_1)\cdots a^+(w_n):\big).\end{aligned}\tag{2.2}$$

Here the symbol $:\overbrace{(..):A:(..)}:$ means that when one transforms the previous expression to the normal form according to the Gauss property of ρ_0, one must neglect all correlators $\rho_0(a^{\pm}(x_1)a^{\pm}(x_n))$ such that $a^{\pm}(x_1)$ and $a^{\pm}(x_n)$ both do not come from A.

Let $\widetilde{\mathcal{C}'}$ be a subspace in $\mathcal{C}'$ spanned on the functionals just defined.

Now let us introduce a useful method for the representation of the states just defined.

Let $\mathcal{C}_2=\mathcal{C}_+\otimes\mathcal{C}_-$, where $\mathcal{C}_+$ and $\mathcal{C}_-$ are the algebras of canonical commutative relations. The algebras $\mathcal{C}_\pm$ are generated by the generators $a_\pm(k)$, $a^+_\pm(k)$, respectively, satisfying the following relations:

$$\begin{aligned}&[a^+_+(k),a^+_+(k')]=[a_+(k),a_+(k')]=0,\quad [a^+_-(k),a^+_-(k')]=[a_-(k),a_-(k')]=0,\\&[a_+(k),a^+_+(k')]=\delta(k-k'),\quad [a_-(k),a^+_-(k')]=\delta(k-k'),\quad [a^\pm_+(k),a^\pm_-(k)]=0.\end{aligned}$$

Here we put by definition $a^-_\pm:=a_\pm$. Let us consider the following Gauss functional ρ'_0 on $\mathcal{C}_2$ defined by its two-point correlator:

$$\begin{aligned}&\rho'_0(a^\pm_-(k)a^\pm_-(k'))=\rho_0(a^\pm(k)a^\pm(k')),\quad \rho'_0(a^\pm_+(k)a^\pm_+(k'))=\rho_0(a^\mp(k')a^\mp(k)),\\&\rho'_0(a^+_+(k)a^-_-(k'))=\rho'_0(a^-_+(k)a^+_-(k'))=0,\quad \rho'_0(a^-_+(k)a^-_-(k'))=n(k)\delta(k-k'),\\&\rho'_0(a^+_+(k)a^+_-(k'))=(1+n(k))\delta(k-k').\end{aligned}$$

One can prove that the functional ρ'_0 is a state.

Let us make the GNS construction corresponding to the state ρ'_0 and the algebra $\mathcal{C}_2$. We obtain the set $(\mathcal{H}',\tilde{D},\hat{},\rangle)$ consisting of the Hilbert space $\mathcal{H}'$, the dense linear subspace $\tilde{D}$ in $\mathcal{H}'$, the representation $\hat{}$ of $\mathcal{C}_2$ by means of the linear operators from $\tilde{D}$ to $\tilde{D}$, and the cyclic vector $\rangle\in\tilde{D}$, i.e., the vector such that $\hat{C}\rangle=\tilde{D}$. This set satisfies the following condition: $\forall a\in\mathcal{C}_2$, $\langle\hat{a}\rangle=\rho'_0(a)$. Below we will omit the symbol $\hat{}$, i.e., we will write a instead of $\hat{a}$.

Now we can rewrite the functional, defined in (2.2) ρ_f as follows: $\rho_f(A) = \langle A' S_f \rangle$, where A' is an element of $\mathcal{C}_2$ such that it contains only the operators a_-, a_-^+ and can be represented through a_-, a_-^+ in the same way as A can be represented through a, a^+. S_f is an element of $\mathcal{C}_2$ of the form

$$S_f = \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g f(x_1, \dots, x_n \mid y_1, \dots, y_l \mid v_1, \dots, v_m \mid w_1, \dots, w_n)$$
$$\times : a_+^+(x_1) \cdots a_+^+(x_n) a_+(y_1) \cdots a_+(y_l) a_-(v_1) \cdots a_-(v_m) a_-^+(w_1) \cdots a_-^+(w_n) : . \quad (2.3)$$

Here the symbol $: \cdots :$ is a normal ordering with respect to the state ρ_0'.

Denote by $\tilde{D}'$ the space dual to $\tilde{D}$. We just construct the injection from C' into $\tilde{D}'$. Denote its image by $\tilde{\mathcal{H}}'$.

By definition the space $\mathcal{C}''$ is a space of all functionals on $\mathcal{C}$ which can be represented as finite linear combinations of the following functionals: $\rho(A) = \langle A' : S_{f_1} \cdots S_{f_n} : \rangle$. Here A' is an element of $\mathcal{C}_2$ such that it contains only the operators a_-, a_-^+ and can be represented through a_-, a_-^+ in the same way as A can be represented through a, a^+ and S_{f_i} are the elements of the form (2.3). Denote by $\tilde{\mathcal{H}}''$ the subspace in $\tilde{D}'$ spanned on the vectors $: S_{f_1} \cdots S_{f_n} : \rangle$ (in obvious sense).

There exists an involution $\star$ on $\tilde{\mathcal{H}}'$ defined by the following formula:

$$\Big\{ \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g f(x_1, \dots, x_n \mid y_1, \dots, y_l \mid v_1, \dots, v_m \mid w_1, \dots, w_n)$$
$$\times : a_+^+(x_1) \cdots a_+^+(x_n) a_+(y_1) \cdots a_+(y_l) a_-(v_1) \cdots a_-(v_m) a_-^+(w_1) \cdots a_-^+(w_n) : \rangle \Big\}^\star$$
$$= \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g f^*(x_1, \dots, x_n \mid y_1, \dots, y_l \mid v_1, \dots, v_m \mid w_1, \dots, w_n)$$
$$\times : a_-^+(x_1) \cdots a_-^+(x_n) a_-(y_1) \cdots a_-(y_l) a_+(v_1) \cdots a_+(v_m) a_+^+(w_1) \cdots a_+^+(w_n) : \rangle.$$

We define the involution $\star$ on $\mathrm{Hom}(\tilde{H}', \tilde{H}')$ by the following equation: $(a|f\rangle)^\star = a^\star(|f\rangle)^*$, where $a \in \mathrm{Hom}(\tilde{H}', \tilde{H}')$ and $|f\rangle \in \tilde{H}'$.

We also define the involution $\star$ on $\mathcal{C}^2$ by the following equation:

$$\Big\{ \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g f(x_1, \dots, x_n \mid y_1, \dots, y_l \mid v_1, \dots, v_m \mid w_1, \dots, w_n)$$
$$\times : a_+^+(x_1) \cdots a_+^+(x_n) a_+(y_1) \cdots a_+(y_l) a_-(v_1) \cdots a_-(v_m) a_-^+(w_1) \cdots a_-^+(w_n) : \Big\}^\star$$
$$= \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g f^*(x_1, \dots, x_n \mid y_1, \dots, y_l \mid v_1, \dots, v_m \mid w_1, \dots, w_n)$$
$$\times : a_-^+(x_1) \cdots a_-^+(x_n) a_-(y_1) \cdots a_-(y_l) a_+(v_1) \cdots a_+(v_m) a_+^+(w_1) \cdots a_+^+(w_n) :,$$

where $f(x_1, \dots, x_k \mid y_1, \dots, y_l \mid v_1, \dots, v_m \mid w_1, \dots, w_n)$ is a test function of its arguments. Note that the involution on $\mathrm{Hom}(\tilde{H}', \tilde{H}')$ extends the involution on $\mathcal{C}^2$. We say that the element $a \in \mathcal{C}_2$ is real if $a^\star = a$. The involution on $\tilde{H}''$ can be defined in a similar way.

3 The von Neumann dynamics

Suppose that our system is described by the following Hamiltonian: $H = H_0 + \lambda V$, where

$$H_0 = \int d^3k\big(\omega(k) - \mu\big)a^+(k)a(k),$$
$$V = \int d^3p_1 d^3p_2 d^3q_1 d^3q_2 v(p_1, p_2 \mid q_1, q_2)\delta(p_1+p_2-q_1-q_2)a^+(p_1)a^+(p_2)a(q_1)a(q_2).$$

Here the kernel $v(p_1, p_2 \mid q_1, q_2)$ belongs to the Schwartz space of test functions. To point out the fact that H is represented through the operators a^+, a^- we will write $H(a^+, a^-)$.

The von Neumann dynamics takes place in the space $\tilde{\mathcal{H}}''$ and is defined by the following differential equation: $\frac{d}{dt}|f\rangle = \mathcal{L}|f\rangle$, where the von Neumann operator has the form

$$\mathcal{L} = -iH\big(a_-^+, a_-^-\big) + iH^\dagger\big(a_+^+, a_+^-\big),$$

where we put by definition

$$\left(\int \prod_{i=1}^n dp_i \prod_{j=1}^m dq_j v(p_1, \dots, p_n \mid q_1, \dots, q_m) : a^+(p_1) \cdots a^+(p_n)a(q_1) \cdots a(q_n) :\right)^\dagger$$
$$= \int \prod_{i=1}^n dp_i \prod_{j=1}^m dq_j v(p_1, \dots, p_n \mid q_1, \dots, q_m)^* : a^+(p_1) \cdots a^+(p_n)a(q_1) \cdots a(q_n) : .$$

Let us divide the von Neumann operator into the free operator $\mathcal{L}$ and the interaction $\mathcal{L}_{\text{int}}$, $\mathcal{L} = \mathcal{L}_0 + \lambda\mathcal{L}_{\text{int}}$, where

$$\mathcal{L}_0 = -iH_0\big(a_-^+, a_-^-\big) + iH_0^\dagger\big(a_+^+, a_+^-\big), \quad \mathcal{L}_{\text{int}} = -iH_{\text{int}}\big(a_-^+, a_-^-\big) + iH_{\text{int}}^\dagger\big(a_+^+, a_+^-\big).$$

Note that the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are real (with respect the involution $\star$).

Let us introduce kinetic evolution operator (in the interaction representation)

$$U(t'', t') = e^{-\mathcal{L}_0 t''} e^{\mathcal{L}(t''-t')} e^{\mathcal{L}_0 t'}.$$

After differentiating with respect to t we find the differential equation for $U(t, t')$.

$$\frac{d}{dt}U(t, t') = \mathcal{L}_{\text{int}}(t)U(t, t'),$$

where $\mathcal{L}_{\text{int}}(t) = e^{-\mathcal{L}_0 t}\mathcal{L}_{\text{int}} e^{\mathcal{L}_0 t}$. So the state $\rangle_\rho$ under consideration in the space $\tilde{\mathcal{H}}''$ in the interaction representation has the form $\rangle_\rho = T\exp\big(\int_{-\infty}^0 \mathcal{L}_{\text{int}}(t)dt\big)\rangle$, where T is the time-ordering operator.

Note that we have a linear map from $\tilde{H}''$ into $\widetilde{\mathcal{C}}'$. It is easy to see that the von Neumann dynamics are in agreement with the Heizenberg dynamics in $\mathcal{C}'$.

4 Dynamics of correlations

Let us construct some new representation of the von Neumann dynamics useful for the renormalization program. This representation is called the dynamics of correlations. The ideas of the dynamics of correlations belong to Prigogin [9]. The dynamics of correlations take place in the space $\mathcal{H}_c := \bigoplus_0^\infty \operatorname{sym} \otimes^n \tilde{\mathcal{H}}'$. Now let us describe how the operators $\mathcal{L}_0^c$ and $\mathcal{L}_{\text{int}}^c$ act in the space $\mathcal{H}_c$.

Let us define the actions of operators $\mathcal{L}_0^c$ and $\mathcal{L}_{\text{int}}^c$ which correspond to the operators $\mathcal{L}_0$ and $\mathcal{L}_{\text{int}}$.

By definition all the spaces $\otimes^n\tilde{\mathcal{H}}'$ are invariant under the actions of operators $\mathcal{L}_0^c$. Note that the space $\tilde{\mathcal{H}}'$ is invariant under the action of operator $\mathcal{L}_0$. Let us denote the restriction of $\mathcal{L}_0$ to the space $\tilde{\mathcal{H}}'$ by the symbol $\mathcal{L}_0'$. By definition the restriction of $\mathcal{L}_0^c$ to the each subspace $\text{sym}\otimes^n\tilde{\mathcal{H}}'$ of $\mathcal{H}_c$ has the form

$$\mathcal{L}_0' \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \mathbf{1} \otimes \mathcal{L}_0' \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathcal{L}_0'.$$

Now let us define $\mathcal{L}_{\text{int}}^c$. Let $|f\rangle \in \mathcal{H}_c$ belong to the subspace $\otimes^n\tilde{\mathcal{H}}'$ and have the form

$$|f\rangle = \sum_{i=0}^{m} f_1^i\rangle \otimes \cdots \otimes f_n^i\rangle,$$

where f_j^i has the form

$$f_i^j = \int \prod_{i=1}^{k} dx_i \prod_{j=1}^{l} dx_j \prod_{f=1}^{m} dv_f \prod_{g=1}^{n} dw_g f\big(x_1, \ldots, x_n \mid y_1, \ldots, y_l \mid v_1, \ldots, v_m \mid w_1, \ldots, w_n\big)$$
$$\times : a_+^+\big(x_1\big) \cdots a_+^+\big(x_n\big) a_+\big(y_1\big) \cdots a_+\big(y_l\big) a_-\big(v_1\big) \cdots a_-\big(v_m\big) a_-^+\big(w_1\big) \cdots a_-^+\big(w_n\big) : . \quad (4.1)$$

By definition $\mathcal{L}_{\text{int}}^{c,l}|f\rangle = 0$ if $l > n$. Let us consider the following vector in $\tilde{\mathcal{H}}''$: $\sum_{i=1}^m : \prod_{j=1}^n f_j^i :\rangle$. Let us transform the expression $\mathcal{L}_{\text{int}} \sum_{i=1}^m : \prod_{j=1}^n f_j^i :$ to the normal form. Let us denote by h_l the sum of all the terms in the previous expression such that exactly l operators f_j^i couple with $\mathcal{L}_{\text{int}}$. We find that $h_l\rangle$ has the form

$$h_l\rangle = \sum_{i=1}^{k} : g_1^i \cdots g_{n-l+1}^i :\rangle$$

for some k. Here g_k^i has the form of the right-hand side of (4.1). Now let us consider the following vector:

$$|f\rangle_l^c = \text{sym} \sum_{i=1}^{k} : g_1^i :\rangle \otimes \cdots \otimes : g_{n-l+1}^i :\rangle,$$

where we define symmetrization operator as follows:

$$\text{sym}\big(f_1 \otimes \cdots \otimes f_n\big) = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma_1} \otimes \cdots \otimes f_{\sigma(n)}.$$

(S_n—the group of permutation of n elements.) Put by definition $\mathcal{L}_{\text{int}}^{c,l}|f\rangle = |f\rangle_l^c$. One can prove that this definition is correct. Analogously, in the expression $\mathcal{L}_{\text{int}} \sum_{i=1}^m : \prod_{j=1}^n f_j^i :\rangle$ let us keep only the terms such that $\mathcal{L}_{\text{int}}$ does not couple with any of f_j^i. Let us write the sum of such terms as follows: $\sum_{i=1}^f : \prod_{j=1}^{n+1} h_j^i :\rangle$. Here $h_j^i\rangle$ has the form of the right-hand side of (4.1). Let $|h\rangle$ be a vector in $\text{sym}\otimes^{n+1}\tilde{\mathcal{H}}'$ defined as follows: $|h\rangle = \text{sym} \sum_{i=1}^f \bigotimes_{j=1}^{n+1} : h_j^i :\rangle$. Put by definition $\mathcal{L}_{\text{int}}^{c,0}|f\rangle = |h\rangle$.

We have the evident linear map $F : \mathcal{H}_c \to \tilde{\mathcal{H}}''$ which assigns to each vector $\text{sym} : f_1 :\rangle \otimes \cdots \otimes : f_n :\rangle$ the vector $: f_1 \cdots f_n :\rangle$. Denote by U^c the evolution operator in interaction representation in the dynamics of correlation. The following statement describes the relation between the von Neumann dynamics and the dynamics of correlations.

Statement. The following relation holds:

$$F \circ U^c(t', t'') = U(t', t'') \circ F.$$

5 The tree of correlations

The useful representation of dynamics in $\mathcal{H}_c$ is a decomposition by the so-called trees of correlations.

Definition 5.1. A graph is a triple $T = (V, R, f)$, where V, R are finite sets called the set of vertices and set of lines, respectively, and f is a map:

$$h : R \to V^{(2)} \cup V \times \{+\} \cup V \times \{-\},$$

where $V^{(2)}$ is a set of all disordered pairs (v_1, v_2), $v_1, v_2 \in V$ such that $v_1 \neq v_2$.

If $(v_1, v_2) = f(r)$ for some $r \in R$, we say that the vertices v_1 and v_2 are connected by a line r. If $f(r) = (v_1, v_2)$, $v_1, v_2 \in V$, we say that the line r is internal.

Remark 5.2. We use this unusual definition of graphs only in purpose of this section to simplify our notations.

Definition 5.3. The graph Γ is called connected graph if for two any vertices v, v' there exists a sequence of vertices $v = v_0, v_1, \dots, v_n = v'$ such that $\forall i = 0, \dots, n-1$ the vertices v_i and v_{i+1} are connected by some line.

By definition we say that the line r is an internal line if $f(r) = (v_1, v_2)$ for some vertices v_1 and v_2.

For each graph Γ we define its connected components by the obvious way.

Definition 5.4. We say that the graph Γ is a tree or an acyclic graph if the number of its connected components increases after removing an arbitrary line.

Definition 5.5. The elements of the set $f^{-1}(V \times \{-\})$ we call the shoots. Put by definition $R_{\mathrm{sh}} = f^{-1}(V \times \{-\})$. The elements of the set $f^{-1}(V \times \{+\})$ we call the roots. Put by definition $R_{\mathrm{root}} = f^{-1}(V \times \{+\})$.

Definition 5.6. Directed tree is a triple $(T, \Phi_v, \Phi_{\mathrm{sh}})$, where T is a tree and Φ_v and Φ_{sh} are the following maps:

$$\Phi_v : V \to \{1, 2, \dots \sharp V\}, \quad \Phi_{\mathrm{sh}} : R_{\mathrm{sh}} \to \big\{1, 2, \dots, \sharp R_{\mathrm{sh}}\big\}.$$

Definition 5.7. We will consider the two directed trees $(T,\Phi_v,\Phi_{\mathrm{sh}})$ and $(T',\Phi'_v,\Phi'_{\mathrm{sh}})$ as identical if we can identify the sets of lines R and R' of T and T', respectively, and identify the sets of vertices V and V' of T and T', respectively, such that after these identification the trees T and T' become the same, the functions Φ_v and Φ'_v become the same, and the functions Φ_{sh} and Φ'_{sh} become the same.

Denote by $r(T)$ the number of roots of T and by $s(T)$ the number of shoots of T. Below, we will denote each directed tree $(T, \Phi_v, \Phi_{\mathrm{sh}})$ by the same symbol T as a tree omitting the reference to Φ_v, Φ_{sh} and write simply tree instead of the directed tree.

We say that the connected directed tree T is right if there exists exactly one line from $f^{-1}(V \times \{+\})$.

We say that the tree T is right if each of its connected components is right.

The vertex v of the tree T is called a root vertex if $(v, +) \in f^{-1}(R)$.

To point out the fact that some object A corresponds to a tree T we will often write A_T. For example, we will write $T = (V_T, R_T, f_T)$ instead of $T = (V, R, f)$.

Definition 5.8. For each connected right tree T there exists an essential partial ordering on the set of its vertices. Let us describe it by induction on the number of its vertices. Suppose that we have defined this relation for all right trees such that the number of their vertices is less than or equal to $n-1$. Let T be a right tree such that the number of its vertices is equal to n. Let $v_{\max}$ be a root vertex of T. Put by definition that the vertex $v_{\max}$ is a maximal vertex. Let $v_1, \ldots, v_k$ be all of its children, i.e., the vertices connected with $v_{\max}$ by lines. By definition each vertex $v_i < v_{\max}, i = 1, \ldots, k$. We can consider the vertices $v_1, \ldots, v_k$ as a root vertices of some directed trees $T_i, i = 1, \ldots, k$. By definition the set of vertices of T_i consists of all vertices v which can be connected with v_i by some path $v = v'_1, \ldots, v'_l = v_i$ such that $v_{\max} \neq v'_j$ for all $j = 1, \ldots, l$. The incident relations on T_i are induced by incident relations on T. Put by definition that $\forall (i,j)$, $i, j = 1, \ldots, k$, $i \neq j$ and for any two vertices $v'_1 \in T_i$ and $v'_2 \in T_j$, $v'_1 \not< v'_2$. If $v'_1, v'_2 \in T_i$ for some T_i, we put $v'_1 \leqslant v'_2$ in T if and only if $v'_1 \leqslant v'_2$ in the sense of ordering on T_i. We put also $v < v_{\max}$ for every vertex $v \neq v_{\max}$. These relations are enough to define the partial ordering on T.

If the tree T has several connected components, we define a partial ordering at each its connected components as previously and put $v_1 \not> v_2$ if v_1 and v_2 do not belong to the same connected component of T.

Below without loss of generality we suppose that for each tree of correlation T and its line r the pair $(v_1, v_2) = f(r)$ satisfies the inequality $v_1 > v_2$.

Definition 5.9. The tree of correlations C is a triple $C = (T, \varphi, \vec{\tau})$, where T is a directed tree, $\vec{\tau}$ is a map from $R \setminus R_{\mathrm{sh}}$ to $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x \geq 0\}$:

$$\vec{\tau} : R \setminus R_{\mathrm{sh}} \longrightarrow \mathbb{R}^+, \quad r \longmapsto \tau(r), \quad \big(\tau(r)\big)_{r \in \mathbb{R}} = \vec{\tau}(r),$$

and φ is a map which assigns to each vertex v of T an element $\varphi(v) \in \operatorname{Hom}\big(\bigotimes_{r \to v} \tilde{\mathcal{H}}', \tilde{\mathcal{H}}'\big)$ of a space of linear maps from $\bigotimes_{r \to v} \tilde{\mathcal{H}}'$ to $\tilde{\mathcal{H}}'$.

In $\bigotimes_{(r \to v)} \tilde{\mathcal{H}}'$ the tensor product is taken over all lines r such that $r \to v$. Let v be a vertex of the tree T. If $f(r) = (v', v)$ for some vertex v' or $f(r) = (v, +)$, we say that the line comes from the vertex v and write $r \leftarrow v$. If $f(r) = (v, v')$ for some vertex v' or $f(r) = (v, -)$, we say that the line comes into the vertex v and write $r \to v$.

Definition 5.10. Let $(T, \varphi, \vec{\tau})$ be a tree of correlations such that for each vertex v $\varphi(v) = \mathcal{L}^{c, l_v}_{\mathrm{int}}$, where l_v is a number of lines coming into v. We call this tree the von Neumann tree and denote it by $T_{\vec{\tau}}$. We also say that φ is a von Neumann vertex function.

Definition 5.11. To each tree of correlations $(T, \varphi, \vec{\tau})$ we assign an element $U^t_{T,\varphi}(\vec{\tau}) \in \operatorname{Hom}\big(\bigoplus_{R_{\mathrm{sh}}} \tilde{\mathcal{H}}', \bigoplus_{R_{\mathrm{root}}} \tilde{\mathcal{H}}'\big)$ by the following way.

If T is disconnected, then

$$U^t_{T,\varphi}(\vec{\tau}) f_1 \otimes \cdots \otimes f_n = \bigotimes_{CT} \Big\{ U^t_{CT, C\varphi}(C\vec{\tau}) \bigotimes_{i \in R_{\mathrm{sh}}(CT)} f_i \Big\}.$$

Here the number of connected components of T is equal to n, and connected components of T are denoted by CT. $C\varphi$ and $C\vec{\tau}$ are the restrictions of φ and $\vec{\tau}$ to the sets of vertices and lines of CT, respectively. $R_{\mathrm{sh}}(CT)$ is a set of shoots of CT. Now let T be a connected tree. To define $U^t_{T,\varphi}(\vec{\tau}) \bigotimes_{r \in R_{\mathrm{sh}}} f_r$ by induction it is enough to consider the following two cases.

Case (1). The tree T has no shoots.

(a) Suppose that the tree T has more than one vertex. Let $v_{\min}$ be some minimal vertex of T and v_0 a vertex such that a unique line r_0 comes from $v_{\min}$ into v_0. Let T' be a tree obtained from T by removing the vertex v_{min} of T. Let $\vec{\tau}'$ be a restriction of $\vec{\tau}$ to $R \setminus \{r_0\}$. Let φ' be a function, defined on $V \setminus \{v_{min}\}$ as follows: $\varphi'(v) = \varphi(v)$ if $v \neq v_0$ and

$$\varphi'(v_0) \bigotimes_{r \to v_0;\ r \neq r_0} f_r = \varphi(v_0) \bigotimes_{r \to v_0} h_r,$$

where $h_r = f_r$ if $r \neq r_0$, and $h_{r_0} = e^{\mathcal{L}_0 \tau(r_0)} \varphi(v_{\min})$.

Put by definition $U^t_{T,\varphi}(\vec{\tau})\rangle = U^t_{T',\varphi'}(\vec{\tau}')\rangle$,

(b) The tree T has only one vertex $v_{\min}$. Then

$$U^t_{T,\varphi}(\vec{\tau}) = e^{-(t-\tau)\mathcal{L}_0} \varphi(v_{\min}).$$

Case (2). The tree T has a shoot r_0 coming into the vertex v_0. In this case instead of the tree $(T, \varphi, \vec{\tau})$ we consider the tree $(T', \varphi', \vec{\tau}')$, where the tree T' has the same vertices as T, the set of lines of T is obtained by removing the line r_0 from the set of lines of T', the function $\vec{\tau}'$ is a restriction of the function $\vec{\tau}$ to the set of lines of T', and the function φ' is defined as follows:

$$\varphi'(v) = \varphi(v), \text{ if } v \neq v_0, \quad \varphi'(v_0) \bigotimes_{r \to v_0;\ r \neq r_0} h_r = \varphi(v_0) \bigotimes_{r \to v_0} g_r,$$

where $g_r = h_r$, if $r \neq r_0$, $g_r = e^{\mathcal{L}_0(t-t_r)} f_{r_0}$. Here we put $t_r = \sum \tau_{r'}$, where the sum is taken over all lines r' which forms decreasing way coming from $+$ to v_0. Put by definition

$$U^t_{T,\varphi}(\vec{\tau})|f\rangle := U^t_{T',\varphi'}(\vec{\tau'})|f'\rangle,$$

where $|f'\rangle = \bigotimes_{r \in (R_{\text{sh}})_{T'}} f_r$.

Let $(T, \varphi, \vec{\tau})$ be some tree of correlations. We can identify the tensor product $\bigotimes_{r \in R_{\text{sh}}} \tilde{\mathcal{H}}'_r$ with $\bigotimes_{i=1}^{\text{sh}(T)} \tilde{\mathcal{H}}'$ and the tensor product $\bigotimes_{r \in R_{\text{root}}} \tilde{\mathcal{H}}'_r$ with $\bigotimes_{i=1}^{r(T)} \tilde{\mathcal{H}}'$. Using these identifications let us consider an operator $V^t_{T,\varphi}(\vec{\tau}) : \mathcal{H}^c \to \mathcal{H}^c$ defined by the following formula:

$$V^t_{T,\varphi} = \text{sym} \circ U^t_{T,\varphi} \circ P_{\text{sh}(T)},$$

where $P_{\text{sh}(T)}$ is a projection of $\mathcal{H}_c$ to sym $\bigotimes_{i=1}^{\text{sh}(T)} \tilde{\mathcal{H}}'$.

Remark 5.12. If $(T, \varphi, \vec{\tau})$ is a von Neumann tree of correlations, then we will shortly denote the operators $U^t_{(T,\varphi)}$ and $V^t_{(T,\varphi)}$ by U^t_T and V^t_T, respectively.

The following theorem holds.

Theorem 5.13. *The following representation for the evolution operators holds (in the sense of formal power series on coupling constant λ):*

$$U^c(t', t'') = \sum_T \frac{\lambda^{n_T}}{n_T!} \int_{\forall r \in R_{sh}\ t - t_r > t''} V^t_T(\vec{\tau}) d\vec{\tau}.$$

Here n_T is a number of vertices of the directed tree T.

6 The general theory of renormalization of $U(t,-\infty)\rangle$

In the present section we by using the decomposition of correlations dynamics by trees describe the general structure of counterterms of $U(t,-\infty)\rangle$, which subtract the divergences from $U(t,-\infty)\rangle$. We will prove in Section 10 below that there exist divergences in the theory. Note that the structure of R-operation for the processes at large times for some class of systems has been considered in [10].

Let T be a tree. Let us give a definition of its right subtree.

Definition 6.1. Let $v_1,\dots,v_n$ be vertices of T such that $\forall i,j=1,\dots,n$, $i\neq j$, $v_i \not< v_j$. Let us define subtree $T_{v_1,\dots,v_n}$. By definition the set of vertices $V_{T_{v_1,\dots,v_n}}$ of $T_{v_1,\dots,v_n}$ consists of all vertices v such that $v<v_i$ for some $i=1,\dots,n$.

The set $R_{T_{v_1,\dots,v_n}}$ of all lines of the tree $T_{v_1,\dots,v_n}$ consists of all lines r of R_T such that $h(r)=(v'',v')$ and $v',v''\leqslant v_i$ for some $i=1,\dots,n$. The incident relations on $T_{v_1,\dots,v_n}$ are induced by the incident relations of T except the following point: if the line r comes from the vertex v into v_i, $i=1,\dots,n$, we put $f_{T_{\{v_1,\dots,v_n\}}}(r)=(v,+)$. In this case the line r is a root and the vertex v is a root vertex of the tree $V_{T_{\{v_1,\dots,v_n\}}}$. The tree $T_{v_1,\dots,v_n}$ is called a right subtree of T.

The Bogoliubov-Parasiuk renormalization prescription. Let us define the following operator:

$$W_{r_0}(t)=\bigotimes_{r\in R_{\mathrm{root}}(T)} Z_{r,r_0}(t),$$

where by definition $Z_{r,r_0}(t)=1$, if $r\neq r_0$, $Z_{r,r_0}(t)=e^{-\mathcal{L}_0 t}$. We say that the amplitudes $\{A_{T,\varphi}\}$ are time-translation invariant amplitudes if for each tree T and for each its root line r_0 $W_{r_0}(t)A_{T,\varphi}=A_{T,\varphi}$.

For each set of amplitudes $A_{T,\varphi}$ put by definition $A_{T,\varphi}\rangle=F\circ A_{T,\varphi}$, where T is an arbitrary tree without shoots.

Now let us formulate our main result.

Theorem 6.2. *There exists a procedure called renormalization which to each tree T without shoots assign the amplitudes $\Lambda_{T,\varphi}$ satisfying to the following properties (a)–(e):*

(a) If the tree T is not connected and $\{CT\}$ is a set of its connected components, while $\{C\varphi\}$ is a set of its restriction of φ to CT, $\Lambda_{T,\varphi}=\bigotimes\Lambda_{CT,C\varphi}$ in obvious notations.

(b) The amplitudes $\Lambda_{T,\varphi}$ are real, i.e., $(\Lambda_{T,\varphi})^\star=\Lambda_{T,\varphi^*}$

(c) The amplitude $\Lambda_{T,\varphi}$ satisfies the property of time-translation invariance.

It has been proven that

$$U(t,-\infty)\rangle=\sum_T\frac{\lambda^{n_T}}{n_T}\int d\vec{\tau}\,U_T^t(\vec{\tau})\rangle. \tag{6.1}$$

In the last formula the summation is taken over all trees T without shoots.

Let T be a tree without shoots and T' a right subtree of T in the described before sense. Let us define the amplitude $\Lambda_{T',\varphi}\star U^t_{T,\varphi}(\vec{\tau})$. Let by definition $T\backslash T'$ be a tree obtained by removing from the set V_T all the vertices of T' and from the set R_T all the internal lines of T'. In (6.1) $\vec{\tau}$ is a map from $R_{T\backslash T'}$ into $\mathbb{R}^+$.

We can consider the amplitude $U^t_{T\setminus T'}$ *as a map*

$$\bigotimes_{(R_{T\setminus T'})_{sh}} \tilde{H}' \longrightarrow \bigotimes_{(R_{T\setminus T'})_{root}} \tilde{H}'.$$

By using this identification we simply put

$$\Lambda_{T',\varphi} \star U^t_{T,\varphi}(\vec{\tau}) = U^t_{T\setminus T',\varphi}(\vec{\tau})\Lambda_{T',\varphi}.$$

Now let us define the renormalized amplitudes, by means of the counterterms Λ_T*, by the following formula:*

$$\big(R_\Lambda U\big)(t,-\infty) = \sum_T \frac{\lambda^{n_T}}{n_T!} \sum_{T'\subseteq T} \int \Lambda_{T'} \star U^t_T(\vec{\tau})\, d\vec{\tau}.$$

(d) The renormalized amplitudes $(R_\Lambda U)(t,-\infty)\rangle$ *are finite.*

(e) Let T *be an arbitrary connected tree without shoots. Consider the following element of* $\mathcal{H}'$*:* $a := \sum_T \sum_{T'\subseteq T} \int \Lambda_{T'} \star U^t_T(\vec{\tau})\rangle d\vec{\tau}$*. We can represent the element* a *as follows:*

$$a = \sum_{k,l,f,g=0}^{\infty} \int w_m\big(x_1,\dots,x_k \mid y_1,\dots,y_l \mid v_1,\dots,v_f \mid w_1,\dots,w_g\big)$$
$$:\prod_{i=1}^{k_m} a^+_+(x_i)\, dx_i \prod_{i=1}^{l_m} a_+(y_i)\, dy_i \prod_{i=1}^{f_m} a_-(v_i)\, dv_i \prod_{i=1}^{g_m} a^+_-(w_i) dw_i :\rangle.$$

Let $\tilde{w}_{k,l,f,g}(z_1,\dots,z_n)$ $(n = k_m + l_m + f_m + g_m)$ *be a Fourier transform of* $w_{k,l,f,g}(x_1,\dots,x_k \mid y_1,\dots,y_l \mid v_1,\dots,v_f \mid w_1,\dots,w_g)$*. Then*

$$\int dz_1,\dots,dz_n \tilde{w}_{k,l,f,g}\big(z_1 + s(1)e_1 a,\dots,z_n + s(n)e_1 a\big) f\big(z_1,\dots,z_n\big)$$

tends to zero as a as $a \to +\infty$*. Here* $s(i)$ *are the numbers from* $\{0,1\}$ *and there exist numbers* $i,j,\ i,j = 1,\dots,n$ *such that* $s(i) = 0$*,* $s(j) = 1$ *for some* $i,j = 1,\dots,n$*.* $f(z_1,\dots,z_n)$ *is a test function.* e_1 *is a unit vector parallel to the* x*-axis.*

Remark 6.3. The property (d) implies the weak cluster property of the functional $(RU)(0,-\infty)$.

This theorem is a simple consequence of the Theorem-Construction from Section 8.

The renormalized amplitudes satisfy the following properties.

Property 1. For each $t \in \mathbb{R}$

$$\big(R_\Lambda U\big)(t,-\infty)\rangle = e^{-\mathcal{L}_0 t}\big(R_\Lambda U\big)(0,-\infty)\rangle.$$

This property simply follows from the definition of $(R_\Lambda U)(t,-\infty)\rangle$ and means that the functional $(R_\Lambda U)(t,-\infty)\rangle$ is a stationary state.

Property 2. $(R_\Lambda U\big)(t,-\infty)\rangle = U(t,0)\big(R_\Lambda U)(0,-\infty)\rangle$.

This property follows from the following representation of $(R_\Lambda U)(t,-\infty)\rangle$:

$$\big(R_\Lambda U\big)(t,-\infty)\rangle = U(t,-\infty)\mathcal{I}\rangle,$$

where $\mathcal{I}\rangle = \sum_T \frac{1}{n_T!}\Lambda_T\rangle$, and the sum in the last formula is taken over all von Neumann trees without shoots. Property 2 means that the functional $(R_\Lambda U)(t,-\infty)\rangle$ satisfies the von Neumann dynamics.

Remark 6.4. The existence of the stationary translation invariant functional satisfying the weak cluster property follows from the previous theorem and Properties 1 and 2.

7 The Friedrichs diagrams

Now let us start to give a constructive description of the counterterms Λ_T such that the amplitude $R(U)(t,-\infty)\rangle$ is finite, and the counterterms Λ_T satisfy the properties (a)–(e) from the previous section.

At first we represent $U^t_{T,\varphi}(\vec{\tau})$, where T is some tree without shoots, as a sum taken over all the so-called Friedrichs graphs Φ concerned with T.

Definition 7.1. A Friedrichs graph Φ_T concerned with the directed tree T without shoots is a set $(\tilde{V}, R, Or, f^+, f^-, g)$, where $\tilde{V}$ is a union of the set of vertices of T and the set $\{\oplus\}$. Recall that there is a partial order on V_T. We define a partial order on the set $\tilde{V}$ if we put $\forall v \in V_T \ \oplus > v$. f^+ and f^- are the maps $f^+, f^- : R \to V$ such that $f^+(r) > f^-(r)$. Or is a map $R \to \{+,-\}$ called an orientation. g is a function which to each pair (v,r), $v \in V_T$, $r \in R$ such that $f^+(r) = v$ or $f^-(r) = v$ assigns $+$ or $-$. The graph $(\tilde{V}, R, Or, f^+, f^-, g)$ must satisfy the following property: if we consider $\oplus$ as a vertex, the obtained graph is connected.

If $f^+(r) = v$, we write $r \to v$, and if $f^-(r) = v$, we write $r \leftarrow v$.

If we want to point out that the object B concerned with the graph Φ, we will write B_Φ. For example, we will write V_Φ and R_Φ for the sets of vertices and lines of Φ, respectively.

At the picture we will represent the elements of V by points and the element $\oplus$ by $\oplus$. We will represent the elements of R by lines. The line r connects the vertices $f^+(r)$ and $f^-(r)$ at the picture. We will represent orientation $Or(r)$ by arrow on r. If $Or(r) = +$, the arrow is directed from $f^-(r)$ to $f^+(r)$. If $Or(r) = -$, the arrow is directed from $f^+(r)$ to $f^-(r)$. To represent the map $g : (r,v) \to \{+,-\}$ we will draw the symbol $g((r,v))$ ($+$ or $-$) near each shoot (r,v). At the picture a shoot (r,v) is a small segment of the line r near v.

Definition 7.2. The Friedrichs diagram Γ is a set (T, Φ, φ, h), where T is a tree, Φ is a Friedrichs graph, φ is a map which assigns to each vertex v of T a function of momenta $\{p_r \mid r \in R_\Phi\}$ of the form

$$\varphi_v(\cdots p_{r \leftrightarrows v} \cdots) = \psi_v(\cdots p_{r \leftrightarrows v} \cdots) \prod_{S_i} \delta\left(\sum_{j=1}^{j_i} \pm p_i^j\right),$$

where ψ_v is a test function of momenta coming into (from) the vertex v. $\{S_i\}_{i=1}^{n_v}$ is a decomposition of the set of shoots of v into n_v of disjunctive nonempty sets S_i, $p_i^1, \ldots, p_i^{j_i}$ are momenta corresponding to the shoots from $\{S_i\}$, h is a function which assigns to each pair $v \in V$, $r \in R$ such that $f^+(r) \geq v \geq f_-(r)$ a real positive number $h(v,r)$.

It will be clear that it is enough to consider only the diagrams Γ such that for each of its vertex v and set $S_i \in \{S_i\}_{i=1}^{n_v}$ there exists a line r such that $(r, f^-(r)) \in S_i$.

To each Friedrichs diagram $\Gamma = (T, \Phi, \varphi)$ we assign an element of $\tilde{\mathcal{H}}''_c$ of the form

$$U^t_{(T,\Phi,\varphi)}(\vec{\tau}) = \int \cdots dp_{\text{ext}} \cdots U^t_\Gamma(\cdots p_{r_{\text{ext}}} \cdots) : \cdots a^{\pm}_{\pm}(p_{r_{\text{ext}}}) \cdots :\rangle.$$

Here $p_{r_{\text{ext}}}$ are momenta of external lines, i.e., such lines r that $f^+(r) = \oplus$. We choose the lower index of $a^{\pm}_{\pm}(p_{r_{\text{ext}}})$ by the following rule. Let v be a vertex such that $f^-(r_{\text{ext}}) = v$. If $g((r,v)) = +$, we choose $+$ as a lower index, and if $g((r,v)) = -$, we choose $-$ as a lower index. We choose the upper index of $a^{\pm}_{\pm}(p_{r_{\text{ext}}})$ by the following rule. If the lower index of $a^{\pm}_{\pm}(p_{r_{\text{ext}}})$ is $\{-\}$, then the upper index is equal $+$ if the corresponding line comes from the vertex v and this index is equal $-$ if the corresponding line comes into the vertex v. If the lower index of $a^{\pm}_{\pm}(p_{r_{\rm ext}})$ is $\{+\}$, then the upper index is equal $-$ if the corresponding line comes from the vertex v and this index is equal $+$ if the corresponding line comes into the vertex v.

Now let us describe the amplitude $U^t_\Gamma(\cdots p_{\rm ext}\cdots)$. By definition we have

$$\begin{aligned} U^0_\Gamma(\vec\tau)\big(\cdots p_{\rm ext}\cdots\big) &= \int_{r\in R_{in}} \prod_v \varphi_v\big(\cdots p_{r\leftrightarrows v}\cdots\big) \\ &\times \prod_{r\in R_\Gamma} e^{iOr(r)p_r^2(\sum_{r_T\in(R_T)_r}\tau_{r_T}+\sum_{v\in V_r}h(v,r))}dp_r \\ &\times \prod_{r\in R} G\big(Or(r), g\big((r,f^+(r))\big), g\big((r,f^-(r))\big)\big)(p). \end{aligned}$$

Let us describe the elements of this formula. R_Γ is a set of all lines of diagram Γ. Symbol $r\leftrightarrows v$ denotes that the line r comes into (from) the vertex v. In the expression $\psi_v(\cdots p_{r\leftrightarrows v}\cdots)\delta\big(\sum_{r\leftrightarrows v}\pm p_r\big)$ we take the upper sign $+$ if the line r comes into the vertex v and we take lower sign $-$ in the opposite case. The symbol R_T denotes the set of lines of the tree T from the triple (T,Φ,φ) and symbol r_T means the line from R_T. The symbol V_r denotes the set of all vertices v such that $f^+(r)\geq v\geq f^-(r)$. The symbol $(R_T)_r$ denotes the set of all lines r_T of R_T such that the increasing path coming from $f^-(r)$ into $f^+(r)$ contains r_T. $G(Or(r),g(f^+(r)),g(f^-(r)))(p)$ is a factor defined as follows:

$$\begin{aligned} &G\big(Or(r), g\big(f^+(r)\big), g\big(f^-(r)\big)\big)(p)\delta(p-p') \\ &\quad= \rho_0'\left(a^{\mathrm{sgn}(-Or(r)g((r,f^+(r))))}_{g((r,f^+(r)))}(p), a^{\mathrm{sgn}(Or(r)g((r,f^-(r))))}_{g((r,f^-(r)))}(p')\right). \end{aligned}$$

Below we will simply write $G_r(p)$ instead of $G(Or(r),g(f^+(r)),g(f^-(r)))(p)$.

It is evident that we can represent $U^0_T(\vec\tau)$ as a sum taken over some Friedrichs diagrams Γ corresponding to the tree T of the quantities $U^0_\Gamma(\vec s)$.

Now let us define the quotient diagrams.

Definition 7.3. Let $\Gamma=(T,\Phi,\varphi,h)$ be a Friedrichs diagram, $A\subset R_T$ a subset of the set R_T of lines of T, and $\vec\tau$ a map from R_T into $\mathbb{R}^+$.

We define the quotient diagram $\Gamma_{A\vec\tau}:=(T_A,\Phi_A,\varphi_{A\vec\tau},h_A)$ in the following way. To obtain the tree T_A we must tighten all lines from A into points. To obtain Φ_A we must remove all loops obtained by tightening all lines from A into the point.

Now let us define $\varphi_{A\vec\tau}$. Joint all the vertices of T to A. We obtain a tree denoted by ${}^AT$. Let $\{C^AT\}$ be a set of all connected components of ${}^AT$. Let v_0 be a vertex of Φ_A corresponding to the connected component C^AT of ${}^AT$. Put by definition

$$\varphi_\Gamma\big(\cdots p_{r\leftrightarrows v}\cdots\big)_{A\vec\tau} = \int\prod_{v\in V}\varphi_v\big(\cdots\pm p_{r\leftrightarrows v}\cdots\big)\prod_{r\in R_{in}} e^{iOr(r)p_r^2(\sum_{r\in(R_T)_r}\tau_{r_T}+\sum_{v\in V_r}h(v,r))}.$$

Let us point out the notations in the previous formula. R_{in} is a set of all lines of Φ_A such that $f^+(r)$ and $f^-(r)$ are the vertices of C^AT. $(R_T)_r$ denotes the set of all lines r_T of R_T such that the increasing path coming from $f^-(r)$ into $f^+(r)$ contains r_T. The symbol V_r denotes the set of all vertices v such that $f^+(r)\geq v\geq f^-(r)$. $h_A(v_0,r)=\sum_{v\in V_{C^AT}}h(v,r)+\sum_{r_T\in A;\ r_T\in(R_T)_r}\tau_{r_T}$.

Definition 7.4. Let Γ be a Friedrichs diagram. Let $\mathcal{F}_\Gamma$ be a space of all functions of external momenta of the diagram Γ of the form $\psi(\cdots p_{\text{ext}}\cdots)$, where $\psi(\cdots p_{\text{ext}}\cdots)$ is a test function of external momenta.

We denote the convolution of the amplitude $A_\Gamma(\vec{\tau})(\cdots p_{\text{ext}}\cdots)$ with the function $f\in\mathcal{F}_\Gamma$ by $A_\Gamma(\vec{\tau})[f]$.

8 The Bogoliubov-Parasiuk renormalization prescriptions

Let for each Friedrichs diagram $\Gamma=(T,\Phi,\varphi)$ $A_\Gamma(\vec{\tau})(\cdots p_{\text{ext}}\cdots)$ be some amplitude. Fix some diagram Γ and let T' be some right subtree of the tree T corresponding to Γ. Let $\Gamma_{T'}$ be a restriction of the diagram Γ on T' in obvious sense. Define the amplitude $A_{\Gamma_{T'}}\star U_\Gamma(\cdots p_{\text{ext}}\cdots)$ by the following formula:

$$A_{\Gamma_{T'}}\star U_\Gamma\big(\cdots p_{\text{ext}}\cdots\big)=\int\prod_{r\in R'}\Big\{e^{iOr(r)p_r^2(\sum_{r_T\in(R'_T)_r}\tau_{r_T}+\sum_{v\in V'_r}h(v,r))}\Big\}\\ \times\prod_{v\in V'}\varphi_v\big(\cdots p_{r\rightleftarrows v}\cdots\big)A_{\Gamma_{T'}}(\cdots p\cdots).$$

In this formula V' is a set of all vertices v such that v is not a vertex of $V_{T'}$, R' is a set of all lines r of Φ_Γ such that $f^+(r)$ is not a vertex of T'. $(R'_T)_r$ is a set of all lines r_T of T such that r_T is not a line of T' and there exists an increasing path on T coming from $f^-(r)$ into $f^+(r)$ such that this path contains r_T. V'_r is a set of all vertices v of T such that v is not a vertex of T' and $f^+(r)\geq v\geq f^-(r)$.

Let $A_\Gamma(\vec{\tau})(p)$ be some amplitude. Put by definition

$$\hat{A}_\Gamma\big(s_1,\ldots,s_n\big)(p):=A_\Gamma\left(\frac{1}{s_1},\ldots,\frac{1}{s_n}\right)(p)\prod_{i=1}^{n}\frac{1}{s_i^2},$$

where n is a number of lines of T_Γ. Below we will consider the amplitudes $\hat{A}_\Gamma(\vec{s})[f]$ as distributions on $(\mathbb{R}^+)^n$, i.e., as an element of the space of tempered distributions $S'((\mathbb{R}^+)^n)$. Let $\psi(\vec{s})$ be a test function from $S((\mathbb{R}^+)^n)$. We denote the convolution of the amplitude $\hat{A}_\Gamma(\vec{s})[f]$ and the function $\psi(\vec{s})$ by

$$\big\langle\hat{A}_\Gamma\big(\vec{s}\big)[f],\psi\big(\vec{s}\big)\big\rangle:=\int_{(\mathbb{R}^+)^n}d\vec{s}\,\hat{A}_\Gamma\big(\vec{s}\big)[f]\psi\big(\vec{s}\big).$$

The Bogoliubov-Parasiuk prescriptions. It will be clear below that we can take into account only the diagrams Γ such that for each line r_T of the corresponding tree of correlations T $\sharp R_{r_T}\geq 3$. Here R_{r_T} is a set of all lines r of Γ such that the increasing path on T which connects $f^-(r)$ and $f^+(r)$ contains r_T. Below we will consider only such diagrams. Other diagrams can be simply subtracted by some counterterms Λ_T.

According to the Bogoliubov-Parasiuk prescriptions we must to each diagram Γ (corresponding to the connected tree) assign the counterterm amplitude $\hat{C}_\Gamma(\vec{s})[f]$ $f\in\mathcal{F}_\Gamma$ satisfying the following properties.

(a) (Locality). $\hat{C}_\Gamma(\vec{s})[f]$ is a finite linear combination of δ functions centered at zero and their derivatives.

(b) Let Γ be a Friedrichs diagram and T a corresponding tree of correlations. Let $A\subseteq R_T$ and T' is some right subtree of T such that

(1) all lines r_T of T such that r_T is not a line of $R_{T'}$ belong to A,
(2) all the root lines of T' do not belong to A.

Then $\hat{C}_{\Gamma_{A\vec{\tau}}}(\vec{s})[f] = (\hat{C}_{\Gamma'_{A'\vec{\tau}}} \star \hat{U}_\Gamma)(\vec{s})[f]$, where $A' := A \cap (R_{T'})$ and Γ' is a restriction of Γ on T'.

(c) $\hat{C}_\Gamma(\vec{s})[f] = -\mathbb{T}\big(\sum_{\emptyset\subset A\subset R_{T_\Gamma}} \hat{C}_{\Gamma_{A\vec{\tau}}}(\vec{s})[f] + \hat{U}_\Gamma(\vec{s})[f]\big)$, where $\vec{\tau} = (\tau_1,\ldots,\tau_n) = (\frac{1}{s_1},\ldots,\frac{1}{s_n})$, the symbol $\subset$ means here the strong inclusion and $\mathbb{T}$ is some subtract operator.

(d) The amplitudes $\hat{C}_\Gamma(\vec{s})[f]$ satisfy the property of time-translation invariance, i.e.,

$$e^{i\sum_{r\in(R_{\mathrm{root}})_\Gamma} Or(r)p_r^2 t}\hat{C}_\Gamma\big(\tau_1,\ldots,\tau_n\big)[f] = \hat{C}_\Gamma\big(\tau_1+t,\ldots,\tau_n\big)[f].$$

(e) Let Γ be a Friedrichs diagram. Let

$$\hat{R}'_\Gamma\big(\vec{s}\big)[f] := \hat{U}_\Gamma\big(\vec{s}\big)[f] + \sum_{\emptyset\subset A\subset R_{T_\Gamma}} \hat{C}_{\Gamma_{A\vec{\tau}}}\big(\vec{s}\big)[f],$$

$$\hat{R}_\Gamma\big(\vec{s}\big)[f] := \hat{U}_\Gamma\big(\vec{s}\big)[f] + \sum_{\emptyset\subset A\subset R_{T_\Gamma}} \hat{C}_{\Gamma_{A\vec{\tau}}}\big(\vec{s}\big)[f] + \hat{C}_\Gamma\big(\vec{s}\big)[f].$$

The amplitudes $\hat{R}_\Gamma(\vec{s})$ are well-defined distributions on $(\mathbb{R}^+)^n$.

(f) The amplitudes $\hat{R}_\Gamma(\vec{s})$ satisfy the weak cluster property. This property means the following. Let $f(\cdots p_{\mathrm{ext}}\cdots)$ be a test function. Then

$$\int dp\hat{R}_\Gamma\big(\vec{s}\big)\big(\cdots p_{\mathrm{ext}}\cdots\big)f\big(\cdots p_{\mathrm{ext}}\cdots\big)e^{ia\sum_{r\in A}p_r^1} \longrightarrow 0,$$

as $a\to\infty$. Here p_r^1 is a projection of p_r to the x-axis.

Put by definition for each diagram Γ

$$\Lambda_\Gamma\big(\vec{\tau}\big) = \sum_{A\subseteq R_{T_\Gamma}}{}' C_{\Gamma_{A\vec{\tau}}},$$

where $'$ in the sum means that all the root lines of T_Γ do not belong to A.

Put $\Lambda_T = \sum_{\Gamma\sim T}\int_{(\mathbb{R}^+)^n} d\vec{\tau}\Lambda_\Gamma(\vec{\tau})(\cdots p_{\mathrm{ext}}\cdots)\cdots a^{\pm}_{\pm}(p_{\mathrm{ext}})\cdots\rangle$, where the symbol $\Gamma\sim T$ means that the sum is taken over all diagrams corresponding to T with suitable combinatoric factors. Suppose that the properties (a)–(f) are satisfied. Then Λ_T are the counterterms needed in Section 8. Not that the state corresponding to $(RU)(t,-\infty)\rangle$ will obviously commute with the number of particle operator.

Theorem-Construction. *It is possible to find such a subtract operator $\mathbb{T}$ such that there exist counterterms $\hat{C}_\Gamma$ satisfying the properties (a)–(f).*

Note that it is not necessarily for us to use nonreal counterterms. Indeed the evolution operator is real, so after renormalization we can simply take $\mathrm{Re}(RU)(t,-\infty)\rangle$.

9 Proof of the theorem-construction

In this section we prove the Theorem-Construction from the previous section. Note that to prove our theorem we will use some ideas of the papers [11, 12, 13].

Before we prove our theorem let us prove the following.

Lemma 9.1. *Let $L_1 = S(\mathbb{R}^k)$, $L_2 = S((\mathbb{R}^+)^n)$, $k, n = 1, 2, \ldots$. Let $A(p)$ be some nonzero quadratic form on $\mathbb{R}^k$. Let T_t^1, $t \geq 0$ be a one-parameter semigroup acting in L_1 defined as follows: $T_t^1 : f(\cdots p \cdots) \mapsto e^{iA(p)t} f(\cdots p \cdots)$.*

Let T_t^2 $t \geq 0$ be some infinitely differentiable semigroup of continuous operators in L_2.

Let M be a subspace of finite codimension in L_2. Suppose that M is invariant under the action of T_t^2, i.e., $\forall t > 0$ $T_t^2 M \subset M$.

Suppose that there exist the linear independent vectors $f_1, \ldots f_l$ in L_2 such that

$$\mathrm{Lin}\big\{\{f_1, \ldots, f_l\}, M\big\} = L_2, \quad M \cap \mathrm{Lin}\big\{f_1, \ldots, f_l\big\} = 0.$$

For each $i = 1, \ldots, l$ and $t \geq 0$ we have

$$T_t^2 f_i = f_i + a_{i-1} f_{i-1} + \cdots + a_1 f_1 + f,$$

for some coefficients $a_{i-1}, \ldots, a_1$ and the element $f \in M$.

Let g be a functional on $L_1 \otimes M$ such that g is continuous with respect to the topology on $S(\mathbb{R}^k) \times S((\mathbb{R}^+)^n)$. Suppose that $\forall f \in L_1 \otimes M$ and $\forall t > 0$ $\langle g, T_t f\rangle = \langle g, f\rangle$ where $T_t = T_t^1 \otimes T_t^2$.

Then, there exists a continuous extension $\tilde{g}$ of g on $S(\mathbb{R}^k) \times S((\mathbb{R}^+)^n)$ such that $\forall f \in L_1 \otimes L_2$ and $t > 0$ $\langle \tilde{g}, T_t f\rangle = \langle \tilde{g}, f\rangle$.

By definition we say that the functional h on $L_1 \otimes L_2$ is invariant if $\forall t > 0$ and $\forall f \in L_1 \otimes L_2$ $\langle h, T_t f\rangle = \langle h, f\rangle$.

Proof of Lemma 9.1. At first we extend our functional g to the invariant functional $\tilde{g}$ on $L_1 \otimes L_2$ and then we prove that $\tilde{g}$ is continuous.

Let N be a subspace of L_1 of all functions of the form $A(p)f(p)$, where $f(p)$ is a test function. Let $M_1 = \mathrm{Lin}\{M \cup \{f_1\}\}$. Let $h := \big(\frac{d}{dt} T_t^2\big)|_{t=0} f_1$. Let k be a continuous functional on N defined as follows:

$$\big\langle k, \varphi(p) A(p)\big\rangle = -\big\langle g, \varphi(p) \otimes h\big\rangle. \tag{9.1}$$

Let $\tilde{k}$ be an arbitrary continuous extension of k on whole space L_1. The existence of such continuation follows from Malgrange's preparation theorem [14]. Now we define the continuous functional $\tilde{g}_1$ on $L_1 \otimes M_1$ as follows:

$$\tilde{g}_1|_{L_1 \otimes M} = g|_{L_1 \otimes M}, \quad \big\langle \tilde{g}_1, f \otimes f_1\big\rangle = \big\langle \tilde{k}, f\big\rangle, \quad \forall f \in L_1.$$

According to (9.1) we find that $\tilde{g}_1$ is an invariant extension of g on $L_1 \otimes M_1$. Step by step we can extend by the same procedure the functional g to the functionals $\tilde{g}_2, \ldots, \tilde{g}_l$ on $L_1 \otimes M_1, \ldots, L_l \otimes M_l$, respectively, where $M_2 = \mathrm{Lin}\{M \cup \{f_1, f_2\}\}, \ldots, M_l = \mathrm{Lin}\{M \cup \{f_1, f_2, \ldots, f_l\}\}$, respectively. Just constructed functional is separately continuous so it is continuous. The lemma is proved. □

Sketch of the proof of the theorem. We will prove the theorem by induction on the number of lines of the tree of correlations T_Γ corresponding to the diagram Γ. It is evident that it is enough to consider only the diagrams with connected tree of correlations.

The base of induction is evident. Suppose that the theorem is proved for all diagrams of order $< n$. (Order is a number of lines of the tree of correlations.)

Let us give some definitions. Let $\xi(t)$ be a smooth function on $[0,+\infty)$ such that $0 \leq \xi(t) \leq 1$, $\xi(t)=1$ in some small neighborhood of zero and $\xi(t)=0$ if $t > \frac{1}{3n}$. Let us define a decomposition of unit $\{\eta_A(\vec{s}) \mid A \subset \{1,\dots,n\}\}$ by the formula

$$\eta_A(\vec{s}) = \prod_{i \notin A} \xi(s_i) \prod_{i \in A} \big(1-\xi(s_i)\big).$$

Let $\psi(x)$ be some test function on real line such that $\psi(t) \geq 0$, $\int \psi(t)dt = 1$, and $\psi(t)=0$ if $|t| > \frac{1}{10}$. Put by definition $\delta_\lambda(x-\lambda) = \frac{x}{\lambda^2}\psi\big(\frac{x-\lambda}{\lambda}\big)$. We have $\int_0^{+\infty} d\lambda \delta_\lambda(x-\lambda) = 1$. Let $S_N((\mathbb{R}^+)^n)$, $N=1,2,\dots$, be a subspace of $S((\mathbb{R}^+)^n)$ of all functions f such that f has a zero of order $\geq N$ at zero. Let $\Psi(\vec{s})$ be a function of $S((\mathbb{R}^+)^n)$.

We have

$$\begin{aligned} &\big\langle \hat{R}_\Gamma(\vec{s})[f], \Psi(\vec{s})\big\rangle \\ &\quad = \sum_{A \subset \{1,\dots,n\}} \int_0^{+\infty} d\lambda \lambda^{n-1} \int_{(\mathbb{R}^+)^n} d\vec{s}\, \hat{R}_{\Gamma_{A\lambda\vec{s}}}\big(\vec{s}|_{\{1,\dots n\}\setminus A}\big)\delta_1\big(1-|\vec{s}|\big)\Psi\big(\lambda\vec{s}\big)\eta_A\big(\vec{s}\big). \end{aligned} \tag{9.2}$$

The inner integral in (9.2) converges according to the inductive assumption. Therefore if $\Psi(\vec{s}) \in S_N((\mathbb{R}^+)^n)$ and N is large enough, the integral at the right-hand side of (9.2) converges. So $\langle \hat{R}_\Gamma(\vec{s})[f], \Psi(\vec{s})\rangle$ defines a separately continuous functional on $S(\mathbb{R}^{3f}) \otimes S_N((\mathbb{R}_+)^n)$. $f = l-1$, where l is a number of external lines of Γ. To define a subtract operator $\mathbb{T}$ we must extend the functional $\langle \hat{R}_\Gamma(\vec{s})[f], \Psi(\vec{s})\rangle$ to the space $S(\mathbb{R}^{3f}) \otimes S((\mathbb{R})^n)$ such that extended functional will satisfy time-translation invariant property. To obtain this extension we use the lemma. In our case $L_1 = S(\mathbb{R}^{3f})$, $L_2 = S((\mathbb{R}^+)^n)$, $A(p) = -\sum_{r \in R_{\text{ext}}} Or(r)p_r^2$. T_t^2 is an operator acting in the $S((\mathbb{R}_+)^n)$ as follows:

$$\begin{aligned} &T_t^2 f\big(s_1,\dots,s_n\big) = f\Big(\frac{s_1}{1-s_1 t}, s_2,\dots,s_n\Big) && \text{if } s_i < \frac{1}{t}, \\ &T_t^2 f\big(s_1,\dots,s_n\big) = 0 && \text{if } s_i \geq \frac{1}{t}. \end{aligned}$$

The basis $\{f_1,\dots f_l\}$ from the lemma is $\{s_1^{m_1}\cdots s_n^{m_n}\eta_\emptyset(\vec{s})\}$, $m_1,\dots,m_n = 1,2,3\dots$, $m_1+m_2+\cdots+m_n \leq N$ lexicographically ordered. We can now apply our lemma directly.

Now let us prove the weak cluster property. Let $p \in \mathbf{R}^3$. Denote by p^1, p^2, p^3 the projections of p to the x, y, z-axis, respectively. To prove the weak cluster property it is enough to prove the following statement: for each connected diagram Γ the function $\langle F_\Gamma(\vec{s})(\cdots p_{\text{ext}}\cdots), \Psi(\vec{s})\rangle$ defined by

$$\delta\Big(\sum \pm p_{\text{ext}}\Big)\big\langle F_\Gamma\big(\vec{s}\big)\big(\cdots p_{\text{ext}}\cdots\big), \Psi\big(\vec{s}\big)\big\rangle = \big\langle \hat{R}_\Gamma\big(\vec{s}\big)\big(\cdots p_{\text{ext}}\cdots\big), \Psi\big(\vec{s}\big)\big\rangle$$

is a distribution of variables $\cdots p_{\text{ext}}^2 \cdots p_{\text{ext}}^3 \cdots$ (constrained by momentum conservation law) which depends on $\cdots p_{\text{ext}}^1 \cdots$ (constrained by momentum conservation law) by the continuously differentiable way. We will prove this statement by induction on the number of lines of the corresponding tree of correlations. The base of induction is evident. Suppose that the statement is proved for all the trees of correlations such that the number of their lines is less than n. Let Γ be a diagram such that the number of the lines of the corresponding tree of correlations is equal to n. It is evident that if $\Psi(\vec{s})$ has a zero of enough high order at zero, then $\langle \hat{F}_\Gamma(\vec{s})(\cdots p_{\text{ext}}\cdots), \Psi(\vec{s})\rangle$ belongs to the required class (its enough to use our construction with decomposition of unit). Therefore we need to solve by induction the system of equations of the form

$$\left(i\sum \pm (p_{\text{ext}})^2\right)\langle F_\Gamma(\vec{s})(\cdots p_{\text{ext}}\cdots), \Psi(\vec{s})\rangle = \langle F_\Gamma(\vec{s})(\cdots p_{\text{ext}}\cdots), \frac{d}{dt}T_t^2\Psi(\vec{s})\rangle.$$

According to Malgrange's preparation theorem [14] we can choose the solution $\langle F_\Gamma(\vec{s})(\cdots p_{\text{ext}}\cdots), \Psi(\vec{s})\rangle$ such that it belongs to the required class if $\langle F_\Gamma(\vec{s})(\cdots p_{\text{ext}}\cdots)$, $\frac{d}{dt}T_t^2\Psi(\vec{s})\rangle$ belongs to the required class. Therefore the statement is proved. So our theorem is proved.

10 Derivation of nonergodic property from main result

Let us prove (more accurately as in introduction) that our system (Bose gas with weak pair interaction in thermodynamical limit) is nonergodic system.

Let us recall definition of ergodicity [16].

Definition 10.1. Consider a quantum system described by Hamiltonian H. This system is said to be ergodic if the spectrum of H is simple.

This definition is equivalent to the following.

Definition 10.2. A quantum system described by Hamiltonian H is said to be ergodic if each bounded operator commuting with H is a function of H.

The generalization of this definition to the case when there exist some additional commuting first integrals is obvious [16].

It is to difficult to define a Hilbert space and Hamiltonian (as a self-adjoint operator in Hilbert space of states) of the system, in thermodynamical limit. So we give some new definition of ergodicity for this case which can be considered as some variant of last definition. Let us introduce some useful notations. Let V be an algebra of all Wick monomials with kernels from the Schwartz space, i.e., V is a linear space of all expressions of the form

$$\int w(p_1,\dots,p_n \mid q_1,\dots,q_m)\prod_{i=1}^{n} a^+(p_i)dp_i\prod_{j=1}^{m} a(q_i)dq_i, \quad w\in S(\mathbb{R}^{3(n+m)}),$$

where the multiplication is defined by canonical commutative relations. Let V' be an algebraic dual of V. We say that the functional $\rho\in V$ is a stationary functional if $\forall v\in V$ $\rho([H,v])=0$. Here H is a Hamiltonian of our Bose gas. We say that the functional $\rho\in V$ is a translation-invariant functional if $\forall v\in V$ $\rho([\vec{P},v])=0$, where $\vec{P}$ is an operator of momentum of our system. We say that the functional $\rho\in V$ commutes with the number of particle operator if $\forall v\in V$ $\rho([N,v])=0$, where N ia a number of particle operator. Note that $\forall v\in V$ $[H,v]$, $[N,v]$, $[\vec{P},v]\in V$. Denote by V_s the linear space of all Wick monomials of the form $[H,v_0]+[N,v_1]+[\vec{P},\vec{v}]$, $v_0,v_1,\vec{v}\in V$. Denote by V_s' the space of all translation-invariant stationary states commuting with the number of particle operator. We have $\forall f\in V'(f\in V_s' \Leftrightarrow \forall v\in V_s\ f(v)=0)$. Now let us introduce a notion of Gibbsian states.

Let β, $\mu\in\mathbb{R}$, $\beta>0$, $\vec{v}\in\mathbb{R}^3$. We define Gibbsian state on V formally by the following formula:

$$\langle a\rangle_{\beta,\mu,\vec{v}} = \frac{1}{Z_{\beta,\mu,\vec{v}}}\operatorname{tr}\left(ae^{-\beta(H-\mu N+\vec{v}\vec{P})}\right),$$

where $a \in V$, N is a particle number operator and Z is the so-called statistical sum:

$$Z_{\beta,\mu,\vec{v}} = \mathrm{tr}\left(e^{-\beta(H-\mu N+\vec{v}\vec{P})}\right).$$

This states corresponds to canonical distributions. Note that one of the basis statement of statistical mechanics states that there no difference which distribution we use: canonical or microcanonical distribution. Bellow we will omit $\vec{v}$, $\vec{P}$ at all formulas to simplify our notations.

Let V'_G be a subspace spanned by all Gibbsian states, i.e., V'_G is a set of all functionals $\langle\cdot\rangle$ on V of the form $\langle\cdot\rangle = \sum_\alpha c_\alpha \langle\cdot\rangle_{\beta_\alpha,\mu_\alpha}$, where the sum is understood in some generalized sense, for example, it may be continuous (integral). It is evident that $V'_G \subseteq V'_s$.

Now we can give the definition of ergodicity for Bose gas in thermodynamical limit.

Definition 10.3. We say that our system is ergodic if each translation invariant stationary state can be represented as a superposition of Gibbsian states, i.e., $V'_s = V'_G$.

After these previous discussions let us start to prove our statement. Recall that we find non-Gibbsian real stationary translation-invariant functional $\langle\cdot\rangle$ constructed as a formal power series on coupling constant λ satisfying the weak cluster property. This functional can be represented as follows: $\langle\cdot\rangle = \langle\cdot\rangle_0 + \langle\cdot\rangle_1 + \cdots$, where $\langle\cdot\rangle_0$ is a functional of zero order of coupling constant λ, $\langle\cdot\rangle_0$ is a functional of first order of coupling constant λ and e.c.t.

Suppose that our system is ergodic.

We do not suppose that the series $\langle\cdot\rangle_0 + \langle\cdot\rangle_1 + \cdots$ converges, but we will work with it formally as with convergent series and find explicit formulas for $\langle\cdot\rangle$ under the assumption of ergodicity. Let us illustrate formal manipulation that we will use by several examples.

Example 10.4. Let us calculate the sum $\sum_{i=0}^{\infty} x^i$. We do not suppose that $|x| < 1$. Denote by S the sum of this series. We have

$$S = \sum_{i=0}^{\infty} x^i = 1 + \sum_{i=1}^{\infty} x^i = 1 + x\sum_{i=0}^{\infty} x^i = 1 + xS.$$

Therefore

$$S = 1 + xS, \quad (1-x)S = 1, \quad s = \frac{1}{1-x}.$$

Example 10.5. Let us calculate the sum $\sum_{i=1}^{\infty} \frac{1}{i^2}$. Consider the function $\frac{\sin x}{x} = a_0 x + a_1 x + a_2 x^2 + \cdots$ as a polynomial of infinite degree. Let us use the Viete theorem for this "polynomial". The roots of $\frac{\sin x}{x}$ are $x_i = i\pi$, $i \in \mathbb{Z} \setminus \{0\}$. According to the Viete theorem we have

$$\prod_{i\in\mathbb{Z}\setminus\{0\}} x_i = C, \qquad \sum_{j\in\mathbb{Z}\setminus\{0\}} \prod_{i\in\mathbb{Z}\setminus\{0,j\}} x_i = 0, \qquad \sum_{i<j,\ i,j\in\mathbb{Z}\setminus\{0\}} \prod_{k\in\mathbb{Z}\setminus\{0,i,j\}} x_k = -\frac{C}{6}.$$

We find from these equations that $\sum_{i\in\mathbb{Z}\setminus\{0\}} \frac{1}{x_i} = 0$ and

$$\sum_{i<j,\ i,j\in\mathbb{Z}\setminus\{0\}} \frac{1}{x_i x_j} = 1/2\left(\sum_{i\in\mathbb{Z}\setminus\{0\}} \frac{1}{x_i}\right)^2 - 1/2 \sum_{i\in\mathbb{Z}\setminus\{0\}} \frac{1}{x_i^2} = -1/2 \sum_{i\in\mathbb{Z}\setminus\{0\}} \frac{1}{x_i^2} = -\frac{1}{6}.$$

Therefore $\sum_{i=1,2,\ldots} \frac{1}{x_i^2} = \frac{1}{6}$. But $x_n = \pi n$, so we finally have $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$. Such formal manipulation was widely used by Euler and others. Suppose that the set of such formal rules is enough large from one hand and does not contain a contradiction from the other hand. These rules we call the Euler rules. If we can find the “sum” of some series by using the Euler rules, then this series is called convergent in Euler sense. The “sum” of this series is called a sum in Euler sense.

Let us prove that our functional $\langle\cdot\rangle$ can be represented as follows (under the assumption of ergodicity): $\langle\cdot\rangle = \sum_\alpha c_\alpha \langle\cdot\rangle_{\beta_\alpha,\mu_\alpha}$, where c_α are the “sums” of probably divergent series. The convergence (in the Euler sense) of this series will be proven below (under the assumption of ergodicity). The sum can be continuous (integral).

Let $\{e_\alpha, \alpha \in \mathfrak{A}\}$ be a Hamele basis of V_s, $V_s = \mathrm{Lin}\{e_\alpha, \alpha \in \mathfrak{A}\}$. Let $\{e_\beta, \beta \in \mathfrak{B}\}$, $\mathfrak{A} \cap \mathfrak{B} = \emptyset$ be a completion of $\{e_\alpha, \alpha \in \mathfrak{A}\}$ to the Hamele basis of V, i.e., $\{e_\alpha, \alpha \in \mathfrak{A}\} \cup \{e_\beta, \beta \in \mathfrak{B}\}$ be a Hamele basis of V. $\forall \gamma \in \mathfrak{A} \cup \mathfrak{B}$ let f_γ be an element of V' such that $f_\gamma(e_\gamma) = 1$ and $f_\gamma(e_{\gamma'}) = 0$ if $\gamma \neq \gamma'$, $\gamma' \in \mathfrak{A} \cup \mathfrak{B}$.

An arbitrary functional ρ from V' now can be represented as a sum $\rho = \sum_{\alpha\in\mathfrak{A}} l_\alpha f_\alpha + \sum_{\beta\in\mathfrak{B}} l_\beta f_\beta$, where l_α, l_β are arbitrary numbers. Note that for arbitrary l_α, l_β the right-hand side of last equation is well defined because $\forall v \in V$ $f_\gamma(v) \neq 0$ only for finite number of elements $\gamma \in \mathfrak{A} \cup \mathfrak{B}$. It is obvious now that an arbitrary element $f \in V'_s \subseteq V'_G$ (ergodicity) can be represented as follows: $f = \sum_{\beta\in\mathfrak{B}} l_\beta f_\beta$, where l_β are arbitrary numbers.

For all $i = 0, 1, 2, \ldots$ we have the following representations: $\langle\cdot\rangle_i = \sum_{\alpha\in\mathfrak{A}} s^i_\alpha f_\alpha + \sum_{\beta\in\mathfrak{B}} s^i_\beta f_\beta$. But $\forall \alpha \in \mathfrak{A}$ we have $\sum_{i=0}^\infty s^n_\alpha = \sum_{i=0}^\infty \langle e_\alpha\rangle_i = \langle e_\alpha\rangle = 0$, because $e_\alpha \in V_s$ and $\langle\cdot\rangle$ is a translation invariant stationary functional. Therefore $\forall a \in V$

$$\begin{aligned}
\langle a\rangle &= \sum_{i=0}^\infty \langle\cdot\rangle_i = \sum_{i=0}^\infty \left(\sum_{\alpha\in\mathfrak{A}} s^i_\alpha f_\alpha(a) + \sum_{\beta\in\mathfrak{B}} s^i_\beta f_\beta(a) \right) \\
&= \sum_{i=0}^\infty \sum_{\alpha\in\mathfrak{A}} s^i_\alpha f_\alpha(a) + \sum_{i=0}^\infty \sum_{\beta\in\mathfrak{B}} s^i_\beta f_\beta(a) \\
&= \sum_{\alpha\in\mathfrak{A}} \left(\sum_{i=0}^\infty s^i_\alpha \right) f_\alpha(a) + \sum_{i=0}^\infty \sum_{\beta\in\mathfrak{B}} s^i_\beta f_\beta(a) \\
&= \sum_{i=0}^\infty \left(\sum_{\beta\in\mathfrak{B}} s^i_\beta f_\beta \right)(a)
\end{aligned}$$

in Euler sense. Finally $\langle a\rangle = \sum_{i=0}^\infty \langle a\rangle'_i$, where we put $\langle\cdot\rangle'_i = \sum_{\beta\in\mathfrak{B}} s^i_\beta f_\beta \in V'_G$ and our statement is proved.

Let 1 be some enough large but finite subsystem of our system. Let 2 be a subsystem obtained from 1 by translation on the vector $\vec{l}$ of sufficiently large length parallel to the x-axis. Let 12 be a union of the subsystems 1 and 2. Let U_1, U_2, and U_{12} be density matrices for the subsystems 1, 2, and 12, respectively (which correspond to $\langle\cdot\rangle$).

Let $\{\varphi^1_{n,N},\ n, N = 1, 2, \ldots\}$ be a basis of eigenvectors of Hamilton operator for subsystem 1. Let$\{\varphi^2_{m,M},\ m, M = 1, 2, \ldots\}$ be a basis of eigenvectors of Hamilton operator for subsystem 2. N, M are the number of particles in systems 1, 2, respectively. Then the basis of eigenvectors of Hamiltonian for subsystem 12 is $\{\psi_{n,N,m,M} = \varphi^1_{n,N} \otimes \varphi^2_{m,M},\ n, N, m, M = 1, 2, \ldots\}$. According to the assumption that the systems 1,2 are enough large we find the following expression:

$$U_{1,2} = \sum c_\alpha \frac{e^{-\frac{H_{1,2} - \mu_\alpha N_{1,2}}{T_\alpha}}}{Z_\alpha}$$

for density matrix of subsystems 1,2 in obvious notations. We have also the following expression for the density matrix of subsystem 12 (if $l=+\infty$):

$$U_{12}=\sum c_\alpha \frac{e^{-\frac{H_1-\mu_\alpha N_1}{T_\alpha}}}{Z_\alpha}\otimes\frac{e^{-\frac{H_2-\mu_\alpha N_2}{T_\alpha}}}{Z_\alpha}.$$

But if $l=+\infty$, the weak cluster property implies that $U_{12}=U_1\otimes U_2$. This leads to the following relation:

$$\sum_\alpha c_\alpha \frac{e^{-\frac{E_{n,N}-\mu_\alpha N}{T_\alpha}}}{Z_\alpha}\frac{e^{-\frac{E_{m,M}-\mu_\alpha M}{T_\alpha}}}{Z_\alpha}=\sum_\alpha c_\alpha \frac{e^{-\frac{E_{n,N}-\mu_\alpha N}{T_\alpha}}}{Z_\alpha}\sum_\beta c_\beta \frac{e^{-\frac{E_{m,M}-\mu_\beta M}{T_\beta}}}{Z_\beta}.$$

Here $\{E_{n,N},\ n,N=1,2,\ldots\}$ is a set of eigenvalues of $H_{1,2}$. But the set of sequences $\{e^{-\frac{E_{n,N}-\mu_\alpha N}{T_\alpha}}\}$ is linear independent if for all two indices α,β, such that $\alpha\neq\beta$, $(T_\alpha,\mu_\alpha)\neq(T_\beta,\mu_\beta)$. So for each α we have

$$c_\alpha\frac{e^{-\frac{E_{n,N}-\mu_\alpha N}{T_\alpha}}}{Z_\alpha}=c_\alpha\sum_\beta c_\beta\frac{e^{-\frac{E_{n,N}-\mu_\beta N}{T_\beta}}}{Z_\beta}.$$

According to the linear independence of $\{e^{-\frac{E_{n,N}-\mu_\alpha N}{T_\alpha}}\}$ we find that $\forall\alpha\ c_\alpha=c_\alpha^2$. So the series representing c_α are convergent in the Euler sense and $\forall\alpha$, $c_\alpha=0,1$.

But we have $\sum_\alpha c_\alpha=\operatorname{tr}U=1$. So for some β, $c_\beta=1$ and $c_\alpha=0$ if $\alpha\neq\beta$. We see that $\langle\cdot\rangle$ is a canonical Gibbsian distribution $\langle\cdot\rangle=\langle\cdot\rangle_{\beta,\mu}$ for some inverse temperature β and chemical potential μ.

Let us calculate now $\langle a(k)a^+(k')\rangle$. We have constructed $\langle\cdot\rangle$ by some Gauss state ρ_0 described by some test function $n(k)$. It follows from our construction of $\langle\cdot\rangle$ that

$$\langle a(k)a^+(k')\rangle=n(k)\delta(k-k').$$

But if momentum k is sufficiently large, we can neglect by potential energy and find

$$\langle a(k)a^+(k')\rangle=\mathrm{const}\,e^{-\beta\frac{k^2}{2}}\delta(k-k').$$

If we chose $n(k)$ such that $n(k)$ tends to zero as $k\to\infty$ slowly than each Gauss function, we obtain a contradiction. This contradiction proves nonergodic property of our system.

Now let us discuss the so-called the Boltzmann ergodic hypothesis (1871). Let $\langle\cdot\rangle$ be a translation-invariant stationary functional on V such that $\forall t\in\mathbb{R}$ the functionals $\langle e^{itH}(\cdot)e^{-itH}\rangle$ are well defined. The Boltzmann hypothesis states that for each such functional there exists an element $\langle\cdot\rangle'\in V'_G$ such that $\forall a\in V$,

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\langle e^{itH}ae^{-itH}\rangle dt=\langle a\rangle'.$$

We see that according to $V'_s\neq V'_G$ the Boltzmann hypothesis does not hold.

11 Examples, chain diagrams

In this section we consider by direct calculation some class of divergent diagrams in Keldysh diagram technique. At first let us introduce the basis notion of the Keldysh diagram technique.

Let us introduce the Green functions for the system

$$\rho\big(T\big(\Psi_H^{\pm}(t_1,x_1),\dots,\Psi_H^{\pm}(t_n,x_n)\big)\big).$$

Symbol H near $\Psi^{\pm}$ means here that $\Psi_H^{\pm}$ are Heizenberg operators.

We require in nonequilibrium diagram technique the following representation for the Green functions:

$$\rho\big(T\big(\Psi_H^{\pm}(t_1,x_1),\dots,\Psi_H^{\pm}(t_n,x_n)\big)\big)=\rho_0\big(S^{-1}T\big(\Psi_0^{\pm}(t_1,x_1),\dots,\Psi_0^{\pm}(t_n,x_n)S\big)\big).$$

The symbol 0 near $\Psi^{\pm}$ means here that $\Psi_0^{\pm}$ are operators in the Dirac representation (representation of interaction). The S-matrix has the form $S=T\exp(-i\int_{-\infty}^{+\infty}V(t)dt)$ and $S^{-1}=\tilde{T}\exp(i\int_{-\infty}^{+\infty}V(t)dt)$. $\tilde{T}$ is a symbol of the antichronological ordering here. ρ_0 is some Gauss state defined by density function $n(k)$ as usual.

Let us recall the basic elements of nonequilibrium diagram technique. The vertices coming from T-exponent are marked by symbol $-$. The vertices coming from $\tilde{T}$-exponent are marked by symbol $+$. There exist four types of propagators:

$$\begin{aligned}
G_0^{+-}(t_1-t_2,x_1-x_2)&=\rho_0\big(\Psi(t_1,x_1)\Psi^{+}(t_2,x_2)\big),\\
G_0^{-+}(t_1-t_2,x_1-x_2)&=\rho_0\big(\Psi^{+}(t_2,x_2)\Psi(t_1,x_1)\big),\\
G_0^{--}(t_1-t_2,x_1-x_2)&=\rho_0\big(T\big(\Psi(t_1,x_1)\Psi^{+}(t_2,x_2)\big)\big),\\
G_0^{++}(t_1-t_2,x_1-x_2)&=\rho_0\big(\tilde{T}\big(\Psi(t_1,x_1)\Psi^{+}(t_2,x_2)\big)\big).
\end{aligned}$$

Let us write the table of propagators:

$$\begin{aligned}
G_0^{+-}(t,x)&=\int\frac{d^4k}{(2\pi)^4}(2\pi)\delta\big(\omega-\omega(k)\big)\big(1+n(k)\big)e^{-i(\omega t-kx)},\\
G_0^{-+}(t,x)&=\int\frac{d^4k}{(2\pi)^4}(2\pi)\delta\big(\omega-\omega(k)\big)n(k)e^{-i(\omega t-kx)},\\
G_0^{--}(t,x)&=i\int\frac{d^4k}{(2\pi)^4}\{\frac{1+n(k)}{\omega-\omega(k)+i0}-\frac{n(k)}{\omega-\omega(k)-i0}\}e^{-i(\omega t-kx)},\\
G_0^{++}(t,x)&=i\int\frac{d^4k}{(2\pi)^4}\{\frac{n(k)}{\omega-\omega(k)+i0}-\frac{1+n(k)}{\omega-\omega(k)-i0}\}e^{-i(\omega t-kx)}.
\end{aligned}$$

11.1 Divergences

A typical example of divergent diagram is pictured in Figure 1.

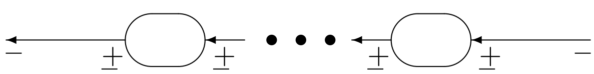

Figure 1.

The ovals represent the sum of one-particle irreducible diagrams. These diagrams are called chain diagrams. Let us suppose that all divergences of self-energy parts (ovals) are subtracted. The divergences arise from the fact that singular supports of propagators coincide. At first we consider diagrams with one self-energy insertion (one-chain diagram). These diagrams are pictured in Figure 2.

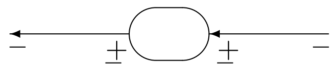

Figure 2.

These diagrams are analogous to one-loop diagrams in quantum field theory.

The aim of this section is to prove that the Green functions can be made finite by the following renormalization of the asymptotical state:

$$\rho_0(\cdot) \to \frac{1}{Z}\rho_0\Big(e^{-\int_{-\infty}^{+\infty} h(t)dt}(\cdot)\Big),$$

where $h = \int h(k)a^+(k)a(k)d^3k$, $h(k)$ is a real-valued function and $Z = \rho_0\big(e^{-\int_{-\infty}^{+\infty} h(t)dt}\big)$.

11.2 Proof of the existence of divergences in the theory

Suppose that there are no divergences in the Keldysh diagram technique if $n(k) \neq \frac{1}{e^{\alpha\frac{k^2}{2}+\beta}+1}$ for any positive α, β. Therefore the Green function

$$\rho\big(S^{-1}\big(S\Psi_0^+(t_1,x_1)\Psi_0(t_2,x_2)\big)\big)$$

is translation invariant. So the density matrix

$$\rho_t(x_1,x_2) := \rho\big(S^{-1}\big(S\Psi_0^+(t,x_1)\Psi_0(t,x_2)\big)\big)$$

is an integral of motion. Let $\rho_t(k) = \int d^3x\rho_t(0,x)e^{ikx}$. In zero order of perturbation theory $\rho(k) = n(k)$. But if there are no divergences in Keldysh diagram technique, it is possible (see [15]) to derive the following kinetic equation for $\rho(k)$:

$$\frac{\partial\rho_t(k)}{\partial t} = \int w(p,p_1 \mid p_2,p_3)\big\{\big(1+\rho(p)\big)\big(1+\rho(p_1)\big)\rho(p_2)\rho(p_3) - \rho(p)\rho(p_1)\big(1+\rho(p_2)\big)\big(1+\rho(p_3)\big)\big\}.$$

The right-hand side of this equation is equal to zero only if $\rho(k) = \frac{1}{e^{\alpha\frac{k^2}{2}+\beta}+1}$ for some α,β ($\alpha > 0, \beta > 0$). But $n(k) = \rho(k)$ in zero order of perturbation theory, so $n(k)$ has a Bose-Einstein form. This contradiction proves our statement.

11.3 Regularization

Let us now introduce regularization. Note that

$$\frac{1}{x+i\varepsilon} = \frac{x}{x^2+\varepsilon} - \pi\frac{i}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}.$$

Therefore we use the following regularization:

$$\delta\big(\omega-\omega(k)\big) \longrightarrow \frac{1}{\pi}\frac{\varepsilon}{\big(\omega-\omega(k)\big)^2+\varepsilon^2} =: \delta_\varepsilon\big(\omega-\omega(k)\big),$$

$$\mathcal{P}\bigg(\frac{1}{\omega-\omega(k)}\bigg) \longrightarrow \frac{\omega-\omega(k)}{\big(\omega-\omega(k)\big)^2+\varepsilon^2} =: \mathcal{P}_\varepsilon\bigg(\frac{1}{\omega-\omega(k)}\bigg).$$

11.4 Some simple relation on the Green functions

Lemma 11.1. *The following equalities hold:*

$$G^{--}\big(t_1-t_2,x_1-x_2\big)^\star = G^{++}\big(t_2-t_1,x_2-x_1\big), \tag{11.1}$$

$$G^{+-}\big(t_1-t_2,x_1-x_2\big)^\star = G^{+-}\big(t_2-t_1,x_2-x_1\big). \tag{11.2}$$

Proof. We have

$$\begin{aligned} G^{--}\big(t_1-t_2,x_1-x_2\big)^\star &= \rho_0\big(T\big(\Psi_H\big(t_1,x_1\big)\Psi_H^+\big(t_2,x_2\big)\big)\big)^\star \\ &= \rho_0\big(\tilde{T}\big(\Psi_H^+\big(t_1,x_1\big)\Psi_H\big(t_2,x_2\big)\big)\big) \\ &= G^{++}\big(t_2-t_1,x_2-x_1\big). \end{aligned}$$

So the equality (11.1) is proved. We have

$$\begin{aligned} G^{+-}\big(t_1-t_2,x_1-x_2\big)^\star &= \rho_0\big(\Psi_H\big(t_1,x_1\big)\Psi_H^+\big(t_2,x_2\big)\big)^\star \\ &= \rho_0\big(\Psi_H\big(t_2,x_2\big)\Psi_H^+\big(t_1,x_1\big)\big) \\ &= G^{+-}\big(t_2-t_1,x_2-x_1\big). \end{aligned}$$

So the equality (11.2) is proved.

The lemma is proved. □

It is easy to prove the following.

Lemma 11.2. *The following equalities hold:*

$$G^{+-}(t,x) = \theta(t)G^{--}(t,x)+\theta(-t)G^{++}(t,x),$$

$$G^{-+}(t,x) = \theta(t)G^{++}(t,x)+\theta(-t)G^{--}(t,x).$$

Let us introduce the following matrix:

$$G = \begin{Vmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{Vmatrix}.$$

Let us introduce a similar matrix for the self-energy operator:

$$\Sigma = \begin{Vmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{Vmatrix}.$$

Dyson equations in Fourier representation have the form $G = G_0 + G_0\Sigma G$. We have from these equations that $\Sigma = G_0^{-1} - C^{-1}$, or in the matrix form

$$\Sigma = \frac{1}{\det G_0}\begin{Vmatrix} G_0^{--} & -G_0^{+-} \\ -G_0^{-+} & G_0^{++} \end{Vmatrix} - \frac{1}{\det G}\begin{Vmatrix} G^{--} & -G^{+-} \\ -G^{-+} & G^{++} \end{Vmatrix}. \tag{11.3}$$

It follows from Lemma 9.1 that

$$G^{++}(\omega,p)=G^{--}(\omega,p)^{\star}\quad G^{+-}(\omega,p)\quad =G^{+-}(\omega,p)^{\star},\quad G^{-+}(\omega,p)=G^{-+}(\omega,p)^{\star}.$$

Therefore $\det G_0, \det G$ are real and we have the following lemma.

Lemma 11.3.

$$\Sigma^{--}\big(t_1-t_2,x_1-x_2\big)^{\star}=\Sigma^{++}\big(t_2-t_1,x_2-x_1\big),$$
$$\Sigma^{+-}\big(t_1-t_2,x_1-x_2\big)^{\star}=\Sigma^{+-}\big(t_2-t_1,x_2-x_1\big).$$

The following lemma holds.

Lemma 11.4.

$$\Sigma^{++}(\omega,p)+\Sigma^{--}(\omega,p)=-\Sigma^{-+}(\omega,p)-\Sigma^{+-}(\omega,p).$$

Proof. The statement of the lemma follows from the Dyson equation (11.3) and the following two obvious equalities:

$$G^{++}(\omega,p)+G^{--}(\omega,p)=G^{-+}(\omega,p)+G^{+-}(\omega,p),$$
$$G_0^{++}(\omega,p)+G_0^{--}(\omega,p)=G_0^{-+}(\omega,p)+G_0^{+-}(\omega,p).$$

□

11.5 Calculation of the propagators in one-chain approximation

Lemma 11.5. *The following limit equalities hold (in the sense of distributions):*

$$\lim_{\varepsilon\to 0}\left(\delta_\varepsilon^2(x)-\frac{1}{2\pi\varepsilon}\delta_\varepsilon(x)\right)=0,\quad \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\delta_\varepsilon(x)-\frac{1}{\varepsilon}\delta(x)\right)=\mathrm{reg},$$
$$\lim_{\varepsilon\to 0}\left\{\frac{1}{\pi}\frac{1}{x^2+\varepsilon^2}-\frac{1}{\varepsilon}\delta(x)\right\}=\mathrm{reg},\quad \lim_{\varepsilon\to 0}\delta_\varepsilon(x)\mathcal{P}_\varepsilon\left(\frac{1}{x}\right)=\mathrm{reg},\quad x\in\mathbb{R}.$$

Here reg *means some correct distribution.*

Proof. Let $f(x)$ be some test function with compact support. We have

$$\int\delta_\varepsilon^2(x)f(x)=\frac{1}{\pi^2}\int\frac{1}{\big(x^2+\varepsilon^2\big)^2}f(x)\,dx$$
$$=\frac{1}{\pi^2}\int\frac{\varepsilon^2}{\big(x^2+\varepsilon^2\big)^2}\big\{f(0)+xf'(0)+x^2\psi(x)\big\}\,dx$$

for some smooth bounded function $\psi(x)$. We have

$$\int\delta_\varepsilon^2(x)f(x)=\frac{1}{\varepsilon\pi^2}\int\frac{1}{\big(x^2+1\big)^2}\big\{f(0)+\varepsilon xf'(0)+\varepsilon^2x^2\psi(\varepsilon x)\big\}$$
$$=\frac{1}{\pi^2}\left\{\frac{1}{\varepsilon}\int\frac{1}{(x^2+1)^2}\,dx\right\}f(0)+O(\varepsilon).$$

But $\int\frac{1}{\big(x^2+1\big)^2}\,dx=\frac{\pi}{2}$. So $\int\delta_\varepsilon^2(x)f(x)=\frac{1}{2\pi\varepsilon}f(0)+O(\varepsilon)$. So the first equality is proved. One can prove the other three equalities in the same way. □

Therefore we see from the Lemmas 9.1 and 11.2 that we can consider only the function $G^{--}(t,x)$. But the function $G^{--}(t,x)$ can be represented as a sum of chain diagrams. At first let us consider the diagrams with one self-energy insertion (one-chain diagram). We have $G_\varepsilon^{--} = \sum_{i,j=\pm} H_\varepsilon^{ij}$, where the diagrams for H_ε^{ij} are presented in Figure 2. We have the following representation for the divergent parts of these diagrams:

$$\begin{aligned}&\big(H_\varepsilon^{--}\big)_{\mathrm{div}}(\omega,p) + \big(H_\varepsilon^{++}\big)_{\mathrm{div}}(\omega,p)\\ &\quad = 2\pi\Sigma^{--}(\omega,p)n(p)\big(1+n(p)\big)\frac{1}{\varepsilon}\delta\big(\omega-\omega(p)\big)\\ &\quad\quad + 2\pi\Sigma^{++}(\omega,p)n(p)\big(1+n(p)\big)\frac{1}{\varepsilon}\delta\big(\omega-\omega(p)\big).\end{aligned}$$

We see that the divergent part of these two diagrams is real (because $\Sigma^{--} = (\Sigma^{++})^*$).

Let us consider the singular part of other two diagrams presented in Figure 3.

Figure 3.

We have

$$\begin{aligned}&\big(H_\varepsilon^{-+}\big)_{\mathrm{div}}(\omega,p) + \big(H_\varepsilon^{+-}\big)_{\mathrm{div}}(\omega,p)\\ &\quad = \pi(2\pi)\big(1+2n(p)\big)\big(1+n(p)\big)\delta_\varepsilon^2\big(\omega-\omega(p)\big)\Sigma^{-+}(\omega,p)\\ &\quad\quad + \pi(2\pi)\big(1+2n(p)\big)n(p)\delta_\varepsilon^2\big(\omega-\omega(p)\big)\Sigma^{+-}(\omega,p)\\ &\quad = \pi\big(1+2n(p)\big)\big(1+n(p)\big)\frac{1}{\varepsilon}\delta\big(\omega-\omega(p)\big)\Sigma^{-+}(\omega,p)\\ &\quad\quad + \pi\big(1+2n(p)\big)n(p)\frac{1}{\varepsilon}\delta\big(\omega-\omega(p)\big)\Sigma^{+-}(\omega,p) + O(\varepsilon).\end{aligned}$$

We see that $(H_\varepsilon^{--})_{\mathrm{div}}(\omega,p) + (H_\varepsilon^{++})_{\mathrm{div}}(\omega,p)$, $(H_\varepsilon^{-+})_{\mathrm{div}}(\omega,p) + (H_\varepsilon^{+-})_{\mathrm{div}}(\omega,p)$ are real.

We will use the dotted line for lines which connect creation-annihilation operators with operators arising from the vertex: $\int h(k)a^+(k)a(k)d^3k$ (see Figure 4).

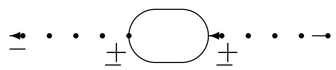

Figure 4.

So the divergences in $G_\varepsilon^{--} = \sum_{i,j=\pm} H_\varepsilon^{ij}$ can be subtracted by the following counterterm:

$$\begin{aligned}h(p) &= \Sigma^{++}(\omega,p) + \Sigma^{--}(\omega,p)\\ &\quad + \frac{\big(1+2n(p)\big)}{2n(p)\big(1+n(p)\big)}\big\{\big(1+n(p)\big)\Sigma^{-+}(\omega,p) + n(p)\Sigma^{+-}(\omega,p)\big\}.\end{aligned}$$

By using Lemma 11.4 we have

$$h(p) = \frac{1+2n(p)}{2\big(1+n(p)\big)n(p)}\big\{\big(1+n(p)\big)\Sigma^{-+}(\omega,p) - n(p)\Sigma^{+-}(\omega,p)\big\}.$$

The left-hand side of this equation can be rewritten as follows (in approximation used in [15]):

$$h(p) = \frac{1+2n(p)}{2n(p)\big(1+n(p)\big)} St(p),$$

where $St(p)$ is a scattering integral. So $h(p) \neq 0$ for nonequilibrium matter.

Analogously one can consider two-chain diagrams presented in Figure 5 by direct calculation and prove that the divergences can be subtracted by the counterterms of the asymptotical state.

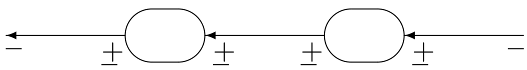

Figure 5.

Note that there arises the phenomenon of overlapping divergences in this example.

12 Notes on Bogoliubov derivation of Boltzmann equations

In this section we study the problem of boundary conditions in Bogoliubov derivation of kinetic equations [7]. Let us consider N particles in $\mathbb{R}^3$. Let q_i be a coordinate of particle number i, and p_i a momentum of particle number i, $i = 1, \dots, N$. Suppose that particles interact by means of the pair potential $V(q_i - q_j)$. We suppose that V belongs to the Schwartz space. Let $x_i = (p_i, q_i)$ be a point in the phase space Γ. Let $f(x_1, \dots, x_n)$ be a distribution function of N particles. If we want to point out that $f(x_1, \dots, x_N)$ depends on t, we will write $f(x_1, \dots, x_N \mid t)$. Let

$$f_1(x_1) = \int dx_2 \cdots dx_N f(x_1, \dots, x_n), \quad f_2(x_1, x_2) = \int dx_3, \dots, dx_N f(x_1, \dots, x_N)$$

be marginal distribution functions. Put by definition

$$\rho_1(x_1) = N f_1(x_1), \quad \rho_2(x_1, x_2) = N^2 f_2(x_1, x_2).$$

If A is a function on the phase space Γ, $\Gamma = \mathbb{R}^{6N}$ and $A = \sum_{i=1}^N \mathcal{A}(x_i)$, then

$$\langle A \rangle = \int f(x_1, \dots, x_n) A(x_1, \dots, x_N) = N \int dx_1 \mathcal{A}(x_1) f(x_1) = \int dx \mathcal{A}(x) \rho_1(x).$$

Now if A is a function on the phase space $A = \sum_{i \neq j} \mathcal{A}(x_i, x_j)$ in the limit of large N, we find $\langle A \rangle = \int dx_1 dx_2 \rho_2(x_1, x_2) \mathcal{A}(x_1, x_2)$. Let us introduce also three-particle distribution function:

$$f_3(x_1, x_2, x_3) = \int dx_4 \cdots dx_N f(x_1, \dots, x_N).$$

Let us derive the equation for $f(x)$. At first let us write equation of motion for $f(x_1, \dots, x_N)$. We have

$$\frac{\partial}{\partial t} f(x_1, \dots, x_N \mid t) + \sum_{i=1}^{n} \frac{p_i}{m} \nabla_i f(x_1, \dots, x_n \mid t) - \sum_{i \neq j} \frac{\partial V(q_i - q_j)}{\partial q_i} \frac{\partial f(x_1, \dots, x_n \mid t)}{\partial p_i} = 0.$$

This equation is only an infinitesimal form of the Liouville theorem. Let us multiply this equation by N and integrate over $dx_2, \ldots, dx_N$. Suppose that $f(x_1, \ldots, x_N)$ is a function of rapid decay of momenta. This assumption allows to integrate over p_i by parts. We find

$$\begin{aligned}&\frac{\partial}{\partial t}\rho_1(x_1 \mid t) + \frac{p}{m}\nabla\rho_1(x_1, t) + \int dx_2 \frac{p_2}{m}\frac{\partial}{\partial q_2}\rho_2(x_1, x_2 \mid t)\\ &\qquad = \int dx_2 \frac{\partial V(q_1 - q_2)}{\partial q_1}\frac{\partial \rho_2(x_1, x_2 \mid t)}{\partial p_1}.\end{aligned} \tag{12.1}$$

Note that we have kept here boundary term. Let us now talk about derivation of kinetic equation. According to the standard prescription we put $\rho_3(x_1, x_2, x_3) = 0$ in equation for $\rho_2(x_1, x_2)$. We find the following equation for ρ_2:

$$\frac{d}{dt}\rho_2\big(x_1(t), x_2(t) \mid t\big) = 0, \tag{12.2}$$

where $(x_1(t), x_2(t))$ is a solution of corresponding two-body problem.

Condition of correlation breaking. We consider only translation-invariant matter in purpose of simplicity. Usual correlation-breaking condition has the form

$$\rho_2(x_1, x_2 \mid 0) = h\big(p_1'(x_1, x_2)\big)h\big(p_2'(x_1, x_2)\big).$$

Here h is a function on momenta-space of one particle. We consider only translation-invariant gas, so h depends only of momentum.

${p'}_1(x_1, x_2)$ and ${p'}_2(x_1, x_2)$ are momenta of particles 1 and 2 at $t = -\infty$ if at $t = 0$ their coordinates and momenta were x_1 and x_2, respectively.

Proposition 12.1.

$$\frac{\partial}{\partial t}\rho_2(x_1, x_2) = 0. \tag{12.3}$$

Indeed, according to (12.2)*,* $\rho_2(x_1, x_2 \mid t) = \rho_2(x_1^0, x_2^0 \mid 0)$*, where* x_1^0 *and* x_2^0 *are phase coordinates of particles 1 and 2, respectively, at a moment* $t = 0$*. Therefore*

$$\rho_2(x_1, x_2 \mid t) = h\big(p_1'(x_1^0, x_2^0)\big)h\big(p_2'(x_1^0, x_2^0)\big).$$

But the points x_1^0 *and* x_2^0 *come to the points* x_1 *and* x_2 *after the time* t*. So*

$$\begin{aligned}&\big(p_1'(x_1^0, x_2^0), p_2'(x_1^0, x_2^0)\big) = \big(p_1'(x_1, x_2), p_2'(x_1, x_2)\big),\\ &\rho_2(x_1, x_2 \mid t) = h\big(p_1'(x_1^0, x_2^0)\big)h\big(p_2'(x_1^0, x_2^0)\big) = h\big(p_1'(x_1, x_2)\big)h\big(p_2'(x_1, x_2)\big) = \rho_2(x_1, x_2 \mid 0).\end{aligned}$$

In result $\rho_2(x_1, x_2 \mid t) = \rho_2(x_1, x_2 \mid 0)$*. The proposition is proved.*

It follows from equations (12.2) and (12.3) that

$$\begin{aligned}&\left(\frac{p_1}{m}\nabla_1 + \frac{p_2}{m}\nabla_2\right)f_2(x_1, x_2 \mid t)\\ &\qquad = \left(\frac{\partial V(q_1 - q_2)}{\partial q_1}\frac{\partial}{\partial p_1} + \frac{\partial V(q_1 - q_2)}{\partial q_2}\frac{\partial}{\partial p_2}\right)f_2(x_1, x_2 \mid t).\end{aligned} \tag{12.4}$$

The function $h(p)$ can be found from the following equation:

$$\rho_1(x) = \lim_{N\to\infty} \frac{1}{N} \int \rho_2\big(x_1, x_2\big)\, dx_2.$$

But in zero order of gas parameter the particles are free and $\rho_1(x) = h(x)$. Formula (12.4) is usually used for the transformation of the right-hand side of equation (12.1) to the scattering integral. On the other hand the equation $\frac{\partial}{\partial t}\rho_2(x_1, x_2) = 0$ shows that there is no irreversible evolution in the system. From an other point of view we will show that the last term in the left-hand side of (12.1) is equal to the scattering integral.

For simplicity we will show the case $v_1 = 0$, $v = \frac{p}{m}$. The general case can be reduced to this case by means of Galilei transformation. So let us consider the integral

$$I = \lim_{R\to\infty} \int d^3p_2 \int_{V_R} dq_2 \frac{p_2}{m} \frac{\partial}{\partial q_2} \rho_2\big(0, 0, p_2, q_2\big),$$

where V_R is a ball of radius R with the center at zero. Let us integrate over dq_2 by using Gauss theorem. We find

$$I = \lim_{R\to\infty} \int d^3p_2 \int_{S_R} dS \frac{p_2}{m} \cos\psi \rho_2\big(0, 0, p_2, q_2\big).$$

Here S_R is a boundary of V_R and ψ is an angle between two rays: the first of them is parallel to p_2, the second starts from zero and passes throw q_2. We have

$$I = \lim_{R\to\infty} \int d^3p_2 \int_{S_R} dS \frac{p_2}{m} \cos\psi h\big(p_1'(0, 0)\big) h\big(p_2'\big(p_2, q_2\big)\big).$$

Let us suppose that the particles scatter only, then they are not too far from each other. Then $h(p_1'(0, 0))h(p_2'(p_2, q_2)) = \rho_1(p_2)\rho_1(0)$ for all $q_2 \in S_R \setminus \mathcal{O}$, where $\mathcal{O}$ is a small neighborhood of the point $q_0 := \frac{p_2}{|p_2|} R \in S_R$. The diameter of $\mathcal{O}$ is approximately equal to the diameter of $\operatorname{supp} V$. Therefore the integral I is not equal to zero and is equal to

$$\begin{aligned} I = &\int d^3p_2 \frac{p_2}{m} \int 2\pi b\, db \\ &\times \big\{\rho_1\big(p_1'\big(p_2, q_2(b)\big), (0, 0)\big)\rho_1\big(p_2'\big(p_2, q_2(b)\big), (0, 0)\big) - \rho_1\big(p_2\big)\rho_1(0)\big\}, \end{aligned} \tag{12.5}$$

where $b := q_2 - q_0$. But the right-hand side of (12.5) is a usual scattering integral.

Therefore if we keep boundary terms in BBGKI-chain, we obtain the kinetic equations without scattering integral.

13 Conclusion

In the present paper we have developed the general theory of the renormalization of nonequilibrium diagram technique. To study this problem we have used some ideas of the theory of R-operation developed by N. N. Bogoliubov and O. S. Parasiuk.

We illustrate our ideas by simple examples of one- and two-chain diagrams in Keldysh diagram technique.

We want to illustrate in this paper the following general thesis: to prove that the system tends to the thermal equilibrium, one should take into account its behavior on its boundary. In the last section we have shown that some boundary terms in BBGKI-chain which are usually neglected in Bogoliubov derivation of kinetic equation compensate scattering integral in kinetic equation.

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