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Infinite Lie algebras and dual pairs in 4D CFT models

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To Ludwig Faddeev with appreciation and friendship

Abstract

It is known that there are no local scalar Lie fields in more than two dimensions. Bilocal fields, however, which naturally arise in conformal operator product expansions, do generate infinite Lie algebras. It is demonstrated that these Lie algebras of local observables admit (highly reducible) unitary positive energy representations in a Fock space. The multiplicity of their irreducible components is governed by a compact gauge group. The mutually commuting observable algebra and gauge group form a dual pair in the sense of Howe. In a theory of local scalar fields of conformal dimension two in four space-time dimensions the associated dual pairs are constructed and classified. The talk reviews joint work of B. Bakalov, N. M. Nikolov, K.-H. Rehren, and the author.

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1 Introduction

In his (undelivered) AMS Einstein Lecture [12] Freeman Dyson divides mathematicians into birds and frogs. “Birds”, he says, “delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs . . . delight in the details of particular objects, and they solve problems one at a time.” Dyson identifies himself with a frog; perhaps, he has chosen the unattractive image to hide his bias? In the introduction to his book [7, 8] Ivan Cherednik is less diplomatic: the concrete facts, special functions, numbers, . . . are for him real, while general theories, “conceptual mathematics” belong to the imaginary axis. After some age it looks easier to use one’s knowledge to find and display connections among different developments – i.e., to stay imaginary. I find it hard to imitate Ludwig Faddeev who, at 75, still stays firmly on the real axis.

In the present talk I review some of the results of [2, 3, 41, 42, 43, 44, 45] on 4D *conformal field theory* (CFT) models, which turn out to relate two independent developments: Roger Howe’s theory of dual pairs, [21, 22, 23, 24], and a representation-theoretic version of the Doplicher-Haag-Roberts’ (DHR) theory of superselection sectors and compact gauge groups [11, 20]. I will first briefly recall Howe’s and DHR’s theories, then I will explain how some 2D CFT technics can be extended to four space-time dimensions (in spite of persistent doubts that this is at all

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possible), and finally we will see how our study of 4D CFT models provides a bridge between the two independent developments.

If we pretend to play birds, it is legitimate to ask what have to do conformal quantum field theory (QFT) models with real physics? After all, if there are discrete masses in the world (in particular, if we are there), what is conformal invariance good for? Not having a completely convincing answer to this question I will end this introduction on a speculative note.

Progress in physics demands idealizations. The law of inertia which marked the beginning of classical mechanics requires neglecting friction. Maybe it makes sense to begin with an idealized world without masses? It is, in fact, dimensional transmutation, a quantum effect linked to renormalization, which provides a hope of solving the problem with the mass gap in Yang-Mills theory [14]. It is also not an accident that some of the most attractive current attempts to understand the generation of mass in the standard model start with a conformally invariant classical Lagrangian [6, 15, 40].

1.1 Reductive dual pairs

As the Wikipedia is telling us, the notion of a (reductive) dual pair was introduced by Roger Howe in an influential preprint of the 1970s that was eventually published in [23, 24]. It was previewed in two earlier papers of Howe [21, 22], highlighting the role of the Heisenberg group and the applications of dual pairs to physics. Howe is clearly a bird: for him a dual pair, the counterpart for groups and for Lie algebras of the mutual commutants of von Neumann algebras [20], is a (highly structured) concept that plays a unifying role in such widely different topics as Weil's metaplectic group approach [52, 53] to θ functions and automorphic forms (an important chapter in number theory) and the quantum mechanical Heisenberg group along with the description of massless particles in terms of the ladder representations of $U(2,2)$ [39], among others (in physics).

Howe begins in [22] with a $2n$-dimensional real symplectic manifold $\mathcal{W} = \mathcal{V} + \mathcal{V}'$ where $\mathcal{V}$ is spanned by n symbols a_i, $i = 1, \dots, n$, called *annihilation operators* and $\mathcal{V}'$ is spanned by their conjugate, the *creation operators* a_i^* satisfying the canonical commutation relations (CCR)

$$[a_i, a_j] = 0 = [a_i^*, a_j^*], \quad [a_i, a_j^*] = \delta_{ij}.$$

The commutator of two elements of the real vector space $\mathcal{W}$ being a real number defines a (nondegenerate, skew-symmetric) bilinear form on it which vanishes on $\mathcal{V}$ and on $\mathcal{V}'$ separately and for which $\mathcal{V}'$ appears as the dual space to $\mathcal{V}$ (the space of linear functionals on $\mathcal{V}$). The real symplectic Lie algebra $\mathrm{sp}(2n, \mathbb{R})$ spanned by anti-Hermitian quadratic combinations of a_i and a_j^* acts by commutators on $\mathcal{W}$ preserving its reality and the above bilinear form. This action extends to the *Fock space* $\mathcal{F}$ (unitary, irreducible) representation of the CCR. It is, however, only exponentiated to the double cover of $Sp(2n, \mathbb{R})$, the *metaplectic group* $Mp(2n)$ (that is not a matrix group – i.e., has no faithful finite-dimensional representation; we can view its Fock space, called by Howe [22] *oscillator representation* as the defining one). Two subgroups G and G' of $Mp(2n)$ are said to form a *(reductive) dual pair* if they act reductively on $\mathcal{F}$ (that is automatic for a unitary representation) and each of them is the full centralizer of the other in $Mp(2n)$. We note that the oscillator representation of $Mp(2n)$ has a *minimality* property [26, 27] that keeps attracting the attention of both physicists and mathematicians — see, e.g., [18, 33, 34].

1.2 Local observables determine a compact gauge group

Observables (unlike charge carrying fields) are left invariant by (global) gauge transformations. This is, in fact, a key property of a gauge symmetry or a *superselection rule* as defined by Wick, Wightman, and Wigner back in 1952 [54]. It required the nontrivial vision of Rudolf Haag to predict in the 1960s that a local net of observable algebras should determine the compact gauge group that governs the structure of its superselection sectors (for a review and references to the original work, see [20]). It took over 20 years and the courage and dedication of Haag's (then) young collaborators, Doplicher and Roberts [11], to carry out this program to completion. They proved that all superselection sectors of a local QFT $\mathcal{A}$ with a mass gap are contained in the vacuum representation of a canonically associated (graded local) field extension $\mathcal{E}$, and they are in a one-to-one correspondence with the unitary irreducible representations (IRs) of a compact gauge group of internal symmetries of $\mathcal{E}$, so that $\mathcal{A}$ consists of the fixed points of $\mathcal{E}$ under G.

2 How do 2D CFT methods work in higher dimensions?

A number of reasons are given why 2-dimensional conformal field theory is, in a way, exceptional so that extending its methods to higher dimensions appears to be hopeless.

(1) The $2D$ conformal group is infinite dimensional: it is the direct product of the diffeomorphism groups of the left and right (compactified) light rays. (In the Euclidean picture it is the group of analytic and antianalytic conformal mappings.) By contrast, for $D > 2$, according to the Liouville theorem, the quantum mechanical conformal group in D space-time dimensions is finite (in fact, $(D+1)(D+2)/2$)-dimensional: it is (a covering of) the spin group $\mathrm{Spin}(D, 2)$.

(2) The representation theory of affine Kac-Moody algebras [28] and of the Virasoro algebra [31] is playing a crucial role in constructing soluble $2D$ models of (rational) CFT. There are, on the other hand, no local Lie fields in higher dimensions: after an inconclusive attempt by Robinson [48] (criticized in [36]) this was proven for scalar fields by Baumann [4].

(3) The light cone in two dimensions is the direct product of two light rays. This geometric fact is the basis of splitting $2D$ variables into right- and left-movers' *chiral variables*. No such splitting seems to be available in higher dimensions.

(4) There are chiral algebras in $2D$ CFT whose *local currents* satisfy the axioms of *vertex algebras*[2] and have rational correlation functions. It was believed for a long time that they have no physically interesting higher-dimensional CFT analog.

(5) Furthermore, the chiral currents in a $2D$ CFT on a torus have elliptic correlation functions [55], the 1-point function of the stress energy tensor appearing as a modular form (these can be also interpreted as finite temperature correlation functions and a thermal energy mean value on the Riemann sphere). Again, there seemed to be no good reason to expect higher-dimensional analogs of these attractive properties.

We will argue that each of the listed features of $2D$ CFT does have, when properly understood, a higher-dimensional counterpart.

(1) The presence of a conformal anomaly (a nonzero Virasoro central charge c) tells us that the infinite conformal symmetry in $1+1$ dimension is, in fact, broken. What is actually used in $2D$ CFT are the (conformal) *operator product expansions* (OPEs) which can be derived for any D

[2] As a mathematical subject vertex algebras were anticipated by Frenkel and Kac [17] and introduced by Borcherds [5]; for reviews and further references, see, e.g., [16, 29].

and allow to extend the notion of a primary field (for instance, with respect to the stress-energy tensor).

(2) For $D = 4$, infinite-dimensional Lie algebras are generated by *bifields* $V_{ij}(x_1, x_2)$ which naturally arise in the OPE of a (finite) set of (say, Hermitian, scalar) local fields ϕ_i of dimension d (> 1):

$$\begin{aligned} &\left(x_{12}^2\right)^d \phi_i(x_1)\phi_j(x_2) = N_{ij} + x_{12}^2 V_{ij}(x_1, x_2) + O\Big(\left(x_{12}^2\right)^2\Big), \\ &x_{12} = x_1 - x_2, \quad x^2 = \mathbf{x}^2 - {x^0}^2, \quad N_{ij} = N_{ji} \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where V_{ij} are defined as (infinite) sums of OPE contributions of (twist two) conserved local tensor currents (and the real symmetric matrix (N_{ij}) is positive definite). We say more on this in what follows (reviewing results of [2, 3, 42, 43, 44, 45]).

(3) We will exhibit a factorization of higher-dimensional intervals by using the following parametrization of the conformally compactified space-time [46, 47, 51, 50]:

$$\bar{M} = \left\{ z_\alpha = e^{it} u_\alpha,\ \alpha = 1, \dots, D;\ t, u_\alpha \in \mathbb{R};\ u^2 = \sum_{\alpha=1}^{D} u_\alpha^2 = 1 \right\} = \frac{\mathbb{S}^{D-1} \times \mathbb{S}^1}{\{1, -1\}}. \tag{2.2}$$

The real interval between two points $z_1 = e^{it_1} u_1$, $z_2 = e^{it_2} u_2$ is given by

$$z_{12}^2 \left(z_1^2 z_2^2\right)^{-1/2} = 2\big(\cos t_{12} - \cos\alpha\big) = -4 \sin t_+ \sin t_-, \quad z_{12} = z_1 - z_2, \tag{2.3}$$

$$t_\pm = 1/2\big(t_{12} \pm \alpha\big), \quad u_1 \cdot u_2 = \cos\alpha, \quad t_{12} = t_1 - t_2. \tag{2.4}$$

Thus t_+ and t_- are the compact picture counterparts of “left” and “right” chiral variables (see [47]). The factorization of $2D$ cross ratios into chiral parts again has a higher-dimensional analog [10]:

$$s := \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} = u_+ u_-, \quad t := \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} = \big(1 - u_+\big)\big(1 - u_-\big), \quad x_{ij} = x_i - x_j, \tag{2.5}$$

which yields a separation of variables in the d’Alembert equation (cf. Remark 3.1) One should, in fact, be able to derive the factorization (2.5) from (2.3).

(4) It turns out that the requirement of *global conformal invariance* (GCI) in Minkowski space together with the standard Wightman axioms of local commutativity and energy positivity entails the rationality of correlation functions in any even number of space-time dimensions [41]. Indeed, GCI and local commutativity of Bose fields (for space-like separations of the arguments) imply the *Huygens principle* and, in fact, the strong (algebraic) locality condition

$$\left(x_{12}^2\right)^n \big[\phi_i(x_1), \phi_j(x_2)\big] = 0 \quad \text{for } n \text{ sufficiently large},$$

a condition only consistent with the theory of free fields for an even number of space-time dimensions. It is this Huygens locality condition which allows the introduction of higher-dimensional vertex algebras [1, 46, 47].

(5) Local GCI fields have elliptic thermal correlation functions with respect to the (differences of) conformal time variables in any even number of space-time dimensions; the corresponding energy mean values in a Gibbs (KMS) state (see, e.g., [20]) are expressed as linear combinations of modular forms [47].

The rest of the paper is organized as follows. In Section 3 we reproduce the general form of the 4-point function of the bifield V and the leading term in its conformal partial wave expansion. The case of a theory of scalar fields of dimension $d = 2$ is singled out, in which the bifields (and the unit operator) close a commutator algebra. In Section 4 we classify the arising infinite-dimensional Lie algebras $\mathcal{L}$ in terms of the three real division rings $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. In Section 5 we formulate the main result of [2, 3] on the Fock space representations of the Lie algebra $\mathcal{L}(\mathbb{F})$ coupled to the (dual, in the sense of Howe [22]) compact gauge group $U(N, \mathbb{F})$ where N is the central charge of $\mathcal{L}$.

3 Four-point functions and conformal partial wave expansions

The conformal bifields $V(x_1, x_2)$ of dimension $(1, 1)$ which arise in the OPE (2.1) (as sums of integrals of conserved tensor currents) satisfy the d'Alembert equation in each argument [43]; we will call them *harmonic bifields*. Their correlation functions depend on the dimension d of the local scalar fields ϕ. For $d = 1$ one is actually dealing with the theory of a free massless field. We will, therefore, assume $d > 1$. A basis $\{f_{\nu i},\ \nu = 0, 1, \ldots, d-2,\ i = 1, 2\}$ of invariant amplitudes $F(s, t)$ such that

$$\big\langle 0 \mid V_1\big(x_1, x_2\big) V_2\big(x_3, x_4\big) \mid 0 \big\rangle = \frac{1}{\rho_{13}\rho_{24}} F(s, t), \quad \rho_{ij} = x_{ij}^2 + i0x_{ij}^0,\ x^2 = \boldsymbol{x}^2 - \big(x^0\big)^2 \tag{3.1}$$

is given by

$$\begin{aligned} \big(u_+ - u_-\big) f_{\nu 1}(s, t) &= \frac{u_+^{\nu+1}}{\big(1 - u_+\big)^{\nu+1}} - \frac{u_-^{\nu+1}}{\big(1 - u_-\big)^{\nu+1}}, \\ \big(u_+ - u_-\big) f_{\nu 2}(s, t) &= (-1)^\nu \big(u_+^{\nu+1} - u_-^{\nu+1}\big), \quad \nu = 0, 1, \ldots, d-2, \end{aligned} \tag{3.2}$$

where $u_\pm$ are the "chiral variables" (2.5);

$$\begin{aligned} &f_{01} = \frac{1}{t}, \quad f_{02} = 1, \quad f_{11} = \frac{1 - s - t}{t^2}, \quad f_{12} = t - s - 1, \\ &f_{21} = \frac{(1-t)^2 - s(2-t) + s^2}{t^3}, \quad f_{\nu 2}(s, t) = \frac{1}{t} f_{\nu 1}\left(\frac{s}{t}, \frac{1}{t}\right), \end{aligned} \tag{3.3}$$

$f_{\nu,i}$ $(i = 1, 2)$ corresponding to *single pole terms* [45] in the 4-point correlation functions $w_{\nu i}(x_1, \ldots, x_4) = f_{\nu i}(s, t)/\rho_{13}\,\rho_{24}$:

$$\begin{aligned} w_{01} &= \frac{1}{\rho_{14}\,\rho_{23}}, \quad w_{02} = \frac{1}{\rho_{13}\rho_{24}}, \\ w_{11} &= \frac{\rho_{13}\,\rho_{24} - \rho_{14}\,\rho_{23} - \rho_{12}\,\rho_{34}}{\rho_{14}^2\,\rho_{23}^2}, \quad w_{12} = \frac{\rho_{14}\,\rho_{23} - \rho_{13}\,\rho_{24} - \rho_{12}\,\rho_{34}}{\rho_{13}^2\,\rho_{24}^2}, \\ w_{21} &= \frac{\big(\rho_{13}\,\rho_{24} - \rho_{14}\,\rho_{23}\big)^2 - \rho_{12}\,\rho_{34}\,\big(2\,\rho_{13}\,\rho_{24} - \rho_{14}\,\rho_{23}\big) + \rho_{12}^2\,\rho_{34}^2}{\rho_{14}^3\,\rho_{23}^3}, \\ w_{22} &= \frac{\big(\rho_{14}\,\rho_{23} - \rho_{13}\,\rho_{24}\big)^2 - \rho_{12}\,\rho_{34}\,\big(2\,\rho_{14}\,\rho_{23} - \rho_{13}\,\rho_{24}\big) + \rho_{12}^2\,\rho_{34}^2}{\rho_{13}^3\,\rho_{24}^3}. \end{aligned} \tag{3.4}$$

We have $w_{\nu 2} = P_{34} w_{\nu 1} (= P_{12} w_{\nu 1})$ where P_{ij} stands for the substitution of the arguments x_i and x_j. Clearly, for $x_1 = x_2$ (or $s = 0$, $t = 1$) only the amplitudes f_{0i} contribute to the 4-point function (3.1). It has been demonstrated in [44] that the lowest angular momentum (ℓ) contribution to $f_{\nu i}$ corresponds to $\ell = \nu$. The corresponding OPE of the bifield V starts with a local scalar field ϕ of dimension $d = 2$ for $\nu = 0$; with a conserved current j_μ (of $d = 3$) for $\nu = 1$; with the stress energy tensor $T_{\lambda\mu}$ for $\nu = 2$. Indeed, the amplitude $f_{\nu 1}$ admits an expansion in twist two[3] *conformal partial waves* $\beta_\ell(s,t)$ [9] starting with (for a derivation, see [44, Appendix B])

$$\beta_\nu(s,t) = \frac{G_{\nu+1}(u_+) - G_{\nu+1}(u_-)}{u_+ - u_-}, \quad G_\mu(u) = u^\mu F(\mu,\mu;2\mu;u). \tag{3.5}$$

Remark 3.1. Equations (3.2)–(3.5) provide examples of solutions of the d'Alembert equation in any of the arguments x_i, $i = 1,2,3,4$. In fact, the general conformal covariant (of dimension 1 in each argument) such solution has the form of the right-hand side of (3.1) with

$$F(s,t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

Remark 3.2. We note that albeit each individual conformal partial wave is a transcendental function (like (3.5)) the sum of all such twist two contributions is the rational function $f_{\nu 1}(s,t)$.

It can be deduced from the analysis of 4-point functions that the commutator algebra of a set of harmonic bifields generated by OPE of scalar fields of dimension d can only close on the V's and the unit operator for $d = 2$. In this case the bifields V are proven, in addition, to be *Huygens bilocal* [45].

Remark 3.3. In general, irreducible positive energy representations of the (connected) conformal group are labeled by triples $(d; j_1, j_2)$ including the dimension d and the Lorentz weight (j_1, j_2) $(2j_i \in \mathbb{N})$ [37]. It turns out that for $d = 3$ there is a spin-tensor bifield of weight $((3/2; 1/2, 0), (3/2; 0, 1/2))$ whose commutator algebra does close; for $d = 4$ there is a conformal tensor bifield of weight $((2;1,0),(2;0,1))$. These bifields may be termed *left-handed*: they are analogs of chiral $2D$ currents; a set of bifields invariant under space reflections would also involve their right-handed counterparts (of weights $((3/2; 0, 1/2), (3/2; 1/2, 0))$ and $((2;0,1),(2;1,0))$, respectively).

4 Infinite-dimensional Lie algebras and real division rings

Our starting point is the following result of [45].

Proposition 4.1. *The harmonic bilocal fields V arising in the OPEs of a (finite) set of local Hermitian scalar fields of dimension $d = 2$ can be labeled by the elements M of an unital algebra $\mathcal{M} \subset \mathrm{Mat}(L, \mathbb{R})$ of real matrices closed under transposition, $M \to {}^tM$, in such a way that the following commutation relations (CR) hold:*

$$\begin{aligned}\left[V_{M_1}(x_1,x_2), V_{M_2}(x_3,x_4)\right] &= \Delta_{13} V_{{}^tM_1 M_2}(x_2,x_4) + \Delta_{24} V_{M_1 {}^tM_2}(x_1,x_3)\\ &\quad + \Delta_{23} V_{M_1 M_2}(x_1,x_4) + \Delta_{14} V_{M_2 M_1}(x_3,x_2)\\ &\quad + \mathrm{tr}\left(M_1 M_2\right)\Delta_{12,34} + \mathrm{tr}\left({}^tM_1 M_2\right)\Delta_{12,43};\end{aligned} \tag{4.1}$$

[3] The twist of a symmetric traceless tensor is defined as the difference between its dimension and its rank. All conserved symmetric tensors in $4D$ have twist two.

here Δ_{ij} *is the free field commutator,* $\Delta_{ij} := \Delta^+_{ij} - \Delta^+_{ji}$*, and* $\Delta_{12,ij} = \Delta^+_{1i}\,\Delta^+_{2j} - \Delta^+_{i1}\,\Delta^+_{j2}$ *where* $\Delta^+_{ij} = \Delta^+(x_i - x_j)$ *is the* 2*-point Wightman function of a free massless scalar field.*

We call the set of bilocal fields closed under the CR (4.1) a *Lie system*. The types of Lie systems are determined by the corresponding *t-algebras* – i.e., real associative matrix algebras $\mathcal{M}$ closed under transposition. We first observe that each such $\mathcal{M}$ can be equipped with a *Frobenius inner product*

$$\langle M_1, M_2\rangle = \mathrm{tr}\left({}^tM_1 M_2\right) = \sum_{ij} \left(M_1\right)_{ij}\left(M_2\right)_{ij},$$

which is symmetric, positive definite, and has the property $\langle M_1M_2, M_3\rangle = \langle M_1, M_3{}^tM_2\rangle$. This implies that for every right ideal $\mathcal{I} \subset \mathcal{M}$ its orthogonal complement is again a right ideal while its transposed ${}^t\mathcal{I}$ is a left ideal. Therefore, $\mathcal{M}$ is a *semisimple* algebra so that every module over $\mathcal{M}$ is a direct sum of irreducible modules.

Let now $\mathcal{M}$ be irreducible. It then follows from Schur's lemma (whose real version [35] is richer but less popular than the complex one) that its commutant $\mathcal{M}'$ in $\mathrm{Mat}(L,\mathbb{R})$ coincides with one of the three *real division rings* (or not necessarily commutative *fields*): the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$, and the noncommutative division ring $\mathbb{H}$ of quaternions. In each case the Lie algebra of bilocal fields is a central extension of an infinite-dimensional Lie algebra that admits a discrete series of highest weight representations[4].

It was proven, first in the theory of a single scalar field ϕ (of dimension two) [42], and eventually for an arbitrary set of such fields [45], that the bilocal fields V_M can be written as linear combinations of normal products of free massless scalar fields $\varphi_i(x)$:

$$V_M\left(x_1, x_2\right) = \sum_{i,j=1}^{L} M^{ij} : \varphi_i\left(x_1\right)\varphi_j\left(x_2\right) : . \tag{4.2}$$

For each of the above types of Lie systems V_M has a canonical form, namely,

$$\begin{aligned}
\mathbb{R} &: V\left(x_1, x_2\right) = \sum_{i=1}^{N} : \varphi_i\left(x_1\right)\varphi_i\left(x_2\right) :, \\
\mathbb{C} &: W\left(x_1, x_2\right) = \sum_{j=1}^{N} : \varphi^*_j\left(x_1\right)\varphi_j\left(x_2\right) :, \\
\mathbb{H} &: Y\left(x_1, x_2\right) = \sum_{m=1}^{N} : \varphi^+_m\left(x_1\right)\varphi_m\left(x_2\right),
\end{aligned} \tag{4.3}$$

where φ_i are real, φ_j are complex, and φ_m are quaternionic valued fields (corresponding to (3.2) with $L = N, 2N$, and $4N$, respectively). We will denote the associated infinite-dimensional Lie algebra by $\mathcal{L}(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

[4]Finite-dimensional simple Lie groups G with this property have been extensively studied by mathematicians (for a review and references, see [13]); for an extension to the infinite-dimensional case, see [49]. If Z is the center of G and K is a closed maximal subgroup of G such that K/Z is compact, then G is characterized by the property that (G, K) is a *Hermitian symmetric pair*. Such groups give rise to *simple space-time symmetries* in the sense of [38] (see also earlier work — in particular by Günaydin — cited there).

Remark 4.2. We note that the quaternions (represented by 4×4 real matrices) appear both in the definition of Y – i.e., of the matrix algebra $\mathcal{M}$, and of its commutant $\mathcal{M}'$, the two mutually commuting sets of imaginary quaternionic units ℓ_i and r_j corresponding to the splitting of the Lie algebra so(4) of real skew-symmetric 4×4 matrices into a direct sum of “a left and a right” so(3) Lie subalgebras:

$$\begin{aligned}
&\ell_1 = \sigma_3 \otimes \epsilon, \quad \ell_2 = \epsilon \otimes \mathbf{1}, \quad \ell_3 = \ell_1\ell_2 = \sigma_1 \otimes \epsilon, \\
&\left(\ell_j\right)_{\alpha\beta} = \delta_{\alpha 0}\delta_{j\beta} - \delta_{\alpha j}\delta_{0\beta} - \varepsilon_{0j\alpha\beta}, \quad \alpha,\beta = 0,1,2,3, \ j = 1,2,3, \\
&r_1 = \epsilon \otimes \sigma_3, \quad r_2 = \mathbf{1} \otimes \epsilon, \quad r_3 = r_1 r_2 = \epsilon \otimes \sigma_1,
\end{aligned} \tag{4.4}$$

where σ_k are the Pauli matrices, $\epsilon = i\sigma_2, \varepsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor normalized by $\varepsilon_{0123} = 1$. We have

$$\begin{aligned}
Y(x_1,x_2) &= V_0(x_1,x_2)\mathbf{1} + V_1(x_1,x_2)\ell_1 + V_2(x_1,x_2)\ell_2 + V_3(x_1,x_2)\ell_3 \\
&= Y(x_2,x_1)^+ \quad \left(\ell_i^+ = -\ell_i, \ [\ell_i, r_j] = 0\right), \\
V_\kappa(x_1,x_2) &= \sum_{m=1}^{N} :\varphi_m^\alpha(x_1)\left(\ell_\kappa\right)_{\alpha\beta}\varphi_m^\beta(x_2):, \quad \ell_0 = \mathbf{1}.
\end{aligned}$$

In order to determine the Lie algebra corresponding to the CR (4.1) in each of the three cases (4.4) we choose a discrete basis and specify the topology of the resulting infinite matrix algebra in such a way that the generators of the conformal Lie algebra (most importantly, the conformal Hamiltonian H) belong to it. The basis, say (X_{mn}) where m,n are multiindices, corresponds to the expansion [50] of a free massless scalar field φ in creation and annihilation operators of fixed energy states

$$\varphi(z) = \sum_{\ell=0}^{\infty}\sum_{\mu=1}^{(\ell+1)^2}\left(\left(z^2\right)^{-\ell-1}\varphi_{\ell+1,\mu} + \varphi_{-\ell-1,\mu}\right)h_{\ell\mu}(z),$$

where $(h_{\ell\mu}(z), \ \mu = 1,\ldots,(\ell+1)^2)$ form a basis of homogeneous harmonic polynomials of degree ℓ in the complex 4-vector z (of the parametrization (2.2) of $\bar{M}$). The generators of the conformal Lie algebra su$(2,2)$ are expressed as infinite sums in X_{mn} with a finite number of diagonals (cf. [2, Appendix B]). The requirement su$(2,2) \subset \mathcal{L}$ thus restricts the topology of $\mathcal{L}$ implying that the last (c-number) term in (4.1) gives rise to a nontrivial central extension of $\mathcal{L}$.

The analysis of [2, 3] yields the following.

Proposition 4.3. *The Lie algebras $\mathcal{L}(\mathbb{F})$, $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$ are* 1*-parameter central extensions of appropriate completions of the following inductive limits of matrix algebras:*

$$\mathbb{R} : \mathrm{sp}(\infty,\mathbb{R}) = \lim_{n\to\infty} \mathrm{sp}(2n,\mathbb{R}),$$

$$\mathbb{C} : u(\infty,\infty) = \lim_{n\to\infty} u(n,n),$$

$$\mathbb{H} : \mathrm{so}^*(4\infty) = \lim_{n\to\infty} \mathrm{so}^*(4n).$$

In the free field realization (4.3) *the suitably normalized central charge coincides with the positive integer N.*

5 Fock space representation of the dual pair $\mathcal{L}(\mathbb{F}) \times U(N, \mathbb{F})$

To summarize the discussion of the last section, there are three infinite-dimensional irreducible Lie algebras, $\mathcal{L}(\mathbb{F})$ that are generated in a theory of GCI scalar fields of dimension $d = 2$ and correspond to the three real division rings $\mathbb{F}$ (Proposition 4.3). For an integer central charge N they admit a free field realization of type (4.2) and a Fock space representation with (compact) gauge group $U(N, \mathbb{F})$:

$$U(N, \mathbb{R}) = O(N), \quad U(N, \mathbb{C}) = U(N), \quad U(N, \mathbb{H}) = Sp(2N) \ \big(= USp(2N) \big). \tag{5.1}$$

It is remarkable that this result holds in general.

Theorem 5.1. *(i) In any unitary irreducible positive energy representation (UIPER) of $\mathcal{L}(\mathbb{F})$ the central charge N is a positive integer.*

(ii) All UIPERs of $\mathcal{L}(\mathbb{F})$ are realized (with multiplicities) in the Fock space $\mathcal{F}$ of $N dim_{\mathbb{R}}\mathbb{F}$ free Hermitian massless scalar fields.

(iii) The ground states of equivalent UIPERs in $\mathcal{F}$ form irreducible representations of the gauge group $U(N, \mathbb{F})$ (5.1). This establishes a one-to-one correspondence between UIPERs of $\mathcal{L}(\mathbb{F})$ occurring in the Fock space and the irreducible representations of $U(N, \mathbb{F})$.

The *proof* of this theorem for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is given in [2] (the proof of (i) is already contained in [42]); the proof for $\mathbb{F} = \mathbb{H}$ is given in [3].

Remark 5.2. Theorem 5.1 is also valid — and its proof becomes technically simpler — for a 2-dimensional chiral theory (in which the local fields are functions of a single complex variable). For $\mathbb{F} = \mathbb{C}$ the representation theory of the resulting infinite-dimensional Lie algebra $u(\infty, \infty)$ is then essentially equivalent to that of the vertex algebra $W_{1+\infty}$ studied in [30] (see the introduction to [2] for a more precise comparison).

Theorem 5.1 provides a link between two parallel developments, one in the study of highest weight modules of reductive Lie groups (and of related dual pairs — see Section 1.1) [13, 25, 32, 49] (and [22, 23, 24]), the other in the work of Haag-Doplicher-Roberts [11, 20] on the theory of (global) gauge groups and superselection sectors, see Section 1.2. (They both originate — in the talks of Irving Segal and Rudolf Haag, respectively — at the same Lille 1957 conference on mathematical problems in quantum field theory.) Albeit the settings are not equivalent the results match. The observable algebra (in our case, the commutator algebra generated by the set of bilocal fields V_M) determines the (compact) gauge group and the structure of the superselection sectors of the theory. (For a more careful comparison between the two approaches, see [2, Sections 1 and 4].)

The infinite-dimensional Lie algebra $\mathcal{L}(\mathbb{F})$ and the compact gauge group $U(N, \mathbb{F})$ appear as a rather special (limit-) case of a *dual pair* in the sense of Howe [22, 23, 24]. It would be interesting to explore whether other (inequivalent) pairs would appear in the study of commutator algebras of (spin)tensor bifields (discussed in Remark 3.3) and of their supersymmetric extension (e.g., a limit as $m, n \to \infty$ of the series of Lie superalgebras $\mathrm{osp}(4m^* \mid 2n)$ studied in [19]).

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