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Nonassociative quantum theory on octooctonion algebra

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Abstract

Using octooctonions (i.e., octonions with octonion coefficients $\mathbb{O}\times\mathbb{O}$), this paper expresses select findings from nonassociative quantum theory in harmonized notation: Nonrelativistic and relativistic spin operators, Pauli and Dirac matrices, Dirac equation with electromagnetic and gravitational field, and dimensional reduction from quaternionic spin. A generalization of the dimensional reduction program is proposed to argue that octooctonion algebra is wide enough to model a speculated quantum theory that contains all symmetries of the Standard Model, together with four-dimensional Euclidean quantum gravity. The most narrow candidate for such a formulation consists of four generalized Dirac matrices and a four-dimensional operator space with associated fields and charges. Algebraic properties of this relation will be discussed, together with a landscape choice between all possible octooctonionic relations of similar kind.

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1 Introduction

Recent development towards a quantum theory on nonassociative algebra addressed many important items, typically on octonionic algebras, including octonionic electrodynamics, geometric relation between the light cone and Heisenberg uncertainty [1], four-dimensional Euclidean operator quantum gravity [2], unobservability in nonassociative parts of quantum operators [3], nonassociative decomposition of supersymmetric momentum and spin operators [4, 5], and dynamic operator formulation with nonassociative Hamiltonian [6]. While these individual works propose to answer relevant questions within their scope, the difference in notation and conventions makes it difficult at times to piece these together to potentially form a consistent and ultimately complete quantum theory on nonassociative background.

Working towards this goal, this paper uses a common notation across select findings, and merges these into a prototype formulation on octooctonions. The similarity to an earlier proposed dimensional reduction construction [7] is used to argue for a generalized construction on octooctonion algebra. It is shown how this offers enough flexibility to include electromagnetic and gravitational fields, while the remaining freedoms in the formulation will expose desired SU(2) and SU(3) symmetries.

Open questions include general solvability of a differential equation on nonassociative algebra, investigation into the behavior of fields that keep the formulation invariant under all of its symmetries, understanding properties of landscape choices other than the one in this paper, and more generally, understanding nonassociative quantum theory used herein, as compared to traditional quantum field theory. As with any proposed unified description of the fundamental forces, the origin of the hierarchy problem remains uncertain. It will be argued that 4D Euclidean quantum gravity used in this paper satisfies, at least in principle, an earlier solution [8] of the hierarchy problem that requires a nonlocal Machian response to all matter within the horizon of a local experiment, explaining the weakness of gravity as compared to the electroweak scale.

It is concluded that further examination of the description in this paper will give a good understanding of applicability for octooctonion modeling of forces of nature, and possibly give further insight into the meaning of quantum theory on nonassociative background in general.

2 Octooctonions

Octooctonions, $\mathbb{O} \otimes \mathbb{O}$, are octonions with octonion coefficients. An octonion basis is

$$b_{\mathbb{O}} := \big\{e_0, e_1, e_2, \dots, e_7\big\},$$

with $e_0 \equiv 1$ the identity element under multiplication, and $\{e_1, e_2, \dots, e_7\}$ the anticommutative, nonreal octonion basis elements. Multiplication between the octonion basis elements e_μ is defined as

$$\begin{aligned} &e_\mu e_\nu = \epsilon_{\mu\nu\rho} e_\rho - \delta_{\mu\nu}, \\ &\epsilon_{\mu\nu\rho} = +1 \quad \text{for } \mu\nu\rho \in \{123, 145, 176, 246, 257, 347, 365\}. \end{aligned} \tag{2.1}$$

Here, the $\{e_\mu, e_\nu, e_\rho\}$ form associative 3-cycles.

An octooctonion A can then be written as

$$A := \sum_{\mu=0}^{7} \vec{a}_\mu e_\mu.$$

The $\vec{a}_\mu$ are also octonions each. This can be expressed to real number coefficients $a_{\mu\dot{\nu}}$ as

$$A := \sum_{\mu=0}^{7} \sum_{\dot{\nu}=0}^{7} a_{\mu\dot{\nu}} e_\mu e_{\dot{\nu}}.$$

The index $\dot{\nu}$ is written with a dot to indicate that its associated octonion basis $e_{\dot{\nu}}$ is separate from the octonion basis with undotted indices. Instead, the octonion basis element with dotted index is chosen to represent the octonion basis of the coefficient:

$$\vec{a}_\mu := \sum_{\dot{\nu}=0}^{7} a_{\mu\dot{\nu}} e_{\dot{\nu}}.$$

Octonion basis elements with dotted indices commute, associate, and distribute with basis elements with undotted indices, i.e.,

$$e_{\dot{\nu}} e_\mu = e_\mu e_{\dot{\nu}}, \quad e_{\dot{\nu}}\big(e_\mu e_\rho\big) = \big(e_{\dot{\nu}} e_\mu\big) e_\rho,$$

and so on. The octooctonion basis elements can be abbreviated as

$$b_{\mathbb{O}\otimes\mathbb{O}} := \big\{e_0, e_1, e_2, \dots, e_7\big\} \otimes \big\{e_{\dot{0}}, e_{\dot{1}}, e_{\dot{2}}, \dots, e_{\dot{7}}\big\} = \big\{e_\mu e_{\dot{\nu}}\big\} \quad \text{where } \mu, \nu \in \{0, 1, \dots, 7\}.$$

Here, the real axis corresponds to $e_0 e_{\dot{0}}$.

Each octooctonion is expressed as a pair of basis elements, one dotted and one undotted. If only one octooctonion basis element is written, the other is inferred, i.e.,

$$e_\mu \equiv e_\mu e_{\dot{0}}, \quad e_{\dot{\mu}} \equiv e_{\dot{\mu}} e_0.$$

3 Nonassociative decomposition of the spin operator

3.1 Chirality of (split-)octonions

Multiplication in octonion and split-octonion algebra is generally nonassociative, but governed by seven associative triplets $\{q_l, q_m, q_n\}$. Together with anticommutation rules, the choice of triplets fixes the multiplication table of the (split-)octonion. The associative triplets for split-octonions in [4] are

$$q_l q_m = q_n, \quad \text{where } lmn \in \{123, 156, 174, 264, 275, 345, 376\}.$$

Here, only the first split-octonion triplet $\{123\}$ is purely imaginary ($q_1^2 = q_2^2 = q_3^2 = -1$) and cyclic ($q_1q_2 = q_3$, $q_2q_3 = q_1$, $q_3q_1 = q_2$), as compared to the octonions where every associative triplet is cyclic.

The associative 3-cycles of the octonion basis used in this paper (equation (2.1) above) corresponds to associative triplets of a chosen split-octonion subalgebra in the octooctonions, e.g.,

$$\begin{aligned} b_{\text{split-}\mathbb{O}} &:= \{e_0 e_{\dot 0}, e_0 e_{\dot 1}, e_0 e_{\dot 2}, e_0 e_{\dot 3}, e_1 e_{\dot 4}, e_1 e_{\dot 5}, e_1 e_{\dot 6}, e_1 e_{\dot 7}\} \equiv \{1, e_{\dot 1}, e_{\dot 2}, e_{\dot 3}, e_1 e_{\dot 4}, e_1 e_{\dot 5}, e_1 e_{\dot 6}, e_1 e_{\dot 7}\}, \\ a_l a_m &= a_n \quad \text{for all } a_l, a_m, a_n \in b_{\text{split-}\mathbb{O}} \text{ when } lmn \in \{123, 145, 176, 246, 257, 347, 365\}. \end{aligned}$$

When mapping split-octonion basis elements from [4] to the notation here, $a_l \leftrightarrow q_m$, there must necessarily be a change of some of the associative triplets into the opposite sign. The reason for this may be understood as the two multiplication tables having opposite chirality (a similar concept between octonion multiplication tables is called left/right octonions in [9]).

The following mapping is now chosen:

$$\{1, q_1, q_2, q_3, \ q_4, q_5, q_6, q_7\} \longmapsto \{1, e_{\dot 1}, e_{\dot 2}, e_{\dot 3}, e_1 e_{\dot 5}, e_1 e_{\dot 6}, e_1 e_{\dot 7}, e_1 e_{\dot 4}\}, \tag{3.1}$$

$$\{123, \overline{156}, 174, \overline{264}, 275, \overline{345}, 376\} \longmapsto \{123, 145, \overline{176}, 246, \overline{257}, 347, \overline{365}\}, \tag{3.2}$$

$$\imath \longmapsto e_1 e_{\dot 0} \equiv e_1. \tag{3.3}$$

In this notation, equation (3.1) expresses the mapping of basis elements, and equation (3.2) overlines the associative triplets that are changed into the opposite chirality (i.e., from left-handed to right-handed, or vice versa). Equation (3.3) maps the complex basis $\imath$ onto $e_1 e_{\dot 0}$, which is itself not part of the split-octonion basis, but satisfies the algebraic relations of the split-octonions when paired with a dotted octonion basis element.

3.2 Nonrelativistic spin operator

In the notation of this paper, the nonrelativistic spin operators from [4] become

$$\begin{aligned} \hat{s}_i &= \frac{\imath}{2} q_i = -\frac{\imath}{4} \epsilon_{ijk} q_{(j+3)} q_{(k+3)}, \quad i, j, k \in \{1, 2, 3\}, \\ \hat{s}_1 &= \frac{\imath}{2} q_1 = -\frac{\imath}{4}\big(q_5 q_6 - q_6 q_5\big) \longmapsto -\frac{e_1}{4}\big(e_1 e_{\dot 6} e_1 e_{\dot 7} - e_1 e_{\dot 7} e_1 e_{\dot 6}\big) = \frac{e_1}{4}\big(e_{\dot 6} e_{\dot 7} - e_{\dot 7} e_{\dot 6}\big), \\ \hat{s}_2 &= \frac{\imath}{2} q_2 = -\frac{\imath}{4}\big(q_6 q_4 - q_4 q_6\big) \longmapsto -\frac{e_1}{4}\big(e_1 e_{\dot 7} e_1 e_{\dot 5} - e_1 e_{\dot 5} e_1 e_{\dot 7}\big) = \frac{e_1}{4}\big(e_{\dot 7} e_{\dot 5} - e_{\dot 5} e_{\dot 7}\big), \\ \hat{s}_3 &= \frac{\imath}{2} q_3 = \frac{\imath}{4}\big(q_5 q_4 - q_4 q_5\big) \longmapsto -\frac{e_1}{4}\big(e_1 e_{\dot 5} e_1 e_{\dot 6} - e_1 e_{\dot 6} e_1 e_{\dot 5}\big) = \frac{e_1}{4}\big(e_{\dot 5} e_{\dot 6} - e_{\dot 6} e_{\dot 5}\big). \end{aligned}$$

Due to the different chirality of the multiplication tables, the chirality of the $\hat{s}_i$ is effectively changed as well. This has no significance for the formulation, which is symmetric under this change.

In abbreviated form,

$$\hat{s}_i = \frac{e_1}{4}\epsilon_{ijk} e_{(\dot{j+4})} e_{(\dot{k+4})}, \quad i, j, k \in \{1, 2, 3\}, \tag{3.4}$$

the spin operator is decomposed into octonion basis elements with $[e_{(\dot{j+4})}]^2 = [e_{(\dot{k+4})}]^2 = -1$. This could also be written using split-octonion basis elements, i.e.,

$$\hat{s}_i = -\frac{e_1}{4}\epsilon_{ijk}\big[e_1 e_{(\dot{j+4})}\big]\big[e_1 e_{(\dot{k+4})}\big],$$

since the $e_1^2 = -1$ cancel each other out.

A factor $1 \pm e_1$ yields another valid decomposition of (3.4) with

$$\big(1 \pm e_1\big)^2 = \pm 2e_1, \tag{3.5}$$

$$\hat{s}_i = \pm\frac{1}{4}\epsilon_{ijk}\left[\frac{1}{2}\big(1 \pm e_1\big)e_{(\dot{j+4})}\right]\left[\frac{1}{2}\big(1 \pm e_1\big)e_{(\dot{k+4})}\right], \quad i, j, k \in \{1, 2, 3\}. \tag{3.6}$$

3.3 Relativistic spin operator

The relativistic spin operator from [5] can be obtained from choosing the decomposition of non-relativistic spin operator (3.6) with minus sign "−", then allowing the indices to run $\{0, 1, 2, 3\}$ (instead of $\{1, 2, 3\}$), and choosing the opposite sign "+" for the R^0 element, i.e.,

$$i_\mu \longmapsto e_{\dot{\mu}} \quad \text{for } \mu \in \{1, \ldots, 7\}, \tag{3.7}$$

$$i_0 \longmapsto e_1, \tag{3.8}$$

$$R^0 = \frac{i_4}{2}\big(1 + i_0\big) \longmapsto \frac{e_{\dot{4}}}{2}\big(1 + e_1\big), \tag{3.9}$$

$$R^j = \frac{i_{(j+4)}}{2}\big(1 - i_0\big) \longmapsto \frac{e_{(\dot{j+4})}}{2}\big(1 - e_1\big) \quad \text{for } j \in \{1, 2, 3\}. \tag{3.10}$$

Whereas equation (3.10) forms a nonassociative decomposition of the nonrelativistic spin operator, the new zero-component (3.9) is now understood as the relativistic generalization of the spin operator $M^{\mu\nu}$, namely

$$M^{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & e_{\dot{1}} & e_{\dot{2}} & e_{\dot{3}} \\ -e_{\dot{1}} & 0 & e_1 e_{\dot{3}} & -e_1 e_{\dot{2}} \\ -e_{\dot{2}} & -e_1 e_{\dot{3}} & 0 & e_1 e_{\dot{1}} \\ -e_{\dot{3}} & e_1 e_{\dot{2}} & -e_1 e_{\dot{1}} & 0 \end{pmatrix}.$$

This satisfies the relations from [4] (for details, see the reference in [5]):

$$\begin{aligned} \big[R^\mu, R^\nu\big] &= 2M^{\mu\nu}, \\ \big[M^{\mu\nu}, M^{\rho\sigma}\big] &= i_0\big(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\big) \\ &\longmapsto e_1\big(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\big), \\ \big(R^\mu, R^\nu, R^\rho\big) &= \big(R^\mu R^\nu\big)R^\rho - R^\mu\big(R^\nu R^\rho\big) = 2\varepsilon^{\mu\nu\rho\sigma}R_\sigma. \end{aligned}$$

4 Dirac equation on nonassociative algebra

4.1 Pauli matrices

The Pauli matrices are

$$\sigma^0 \equiv 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Commutator relations between the σ^j $(j,k,l \in \{1,2,3\})$ are

$$\sigma^j \sigma^k = \imath \epsilon^{jkl} \sigma^l + \delta^{jk} \sigma^0, \quad \left(\sigma^j\right)^2 \equiv \sigma^j \sigma^j = \sigma^0.$$

The σ^0 and σ_0 are now mapped to the real axis $(e_0 e_{\dot 0})$, and the σ^j to a pair of basis elements $e_m e_{\dot n}$ or $e_n e_{\dot m}$, where $m \in \{1,\ldots,7\}$ is a fixed number, and $n \in \{i,j,k\}$, where $e_i e_j = e_k$ (and $\{i,j,k\}$ ordered, and $i \neq j \neq k \neq i$). Specifically,

$$\imath \longmapsto e_m, \quad \sigma^0 \longmapsto e_0 e_{\dot 0} \equiv 1, \quad \sigma^1 \longmapsto e_m e_{\dot j}, \quad \sigma^2 \longmapsto e_m e_{\dot k}, \quad \sigma^3 \longmapsto e_m e_{\dot l}.$$

For example

$$\sigma^1 \sigma^2 \longmapsto e_m e_{\dot j} e_m e_{\dot k} = -e_0 e_{\dot l} = \left(e_m e_m\right) e_{\dot l} = e_m \left(e_m e_{\dot l}\right) \longmapsto \imath \sigma^3,$$

and so on.

4.2 Dirac matrices

The Dirac matrices γ^μ and unit matrix I have the relation

$$\left\{\gamma^\mu, \gamma^\nu\right\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \eta^{\mu\nu} I. \tag{4.1}$$

They can, e.g., be written with Pauli matrices as

$$\gamma^0 := \text{choice of } \left\{ \underbrace{\begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}}_{\text{Dirac basis}}, \underbrace{\begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}}_{\text{Weyl basis}} \right\}, \quad \gamma^j := \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Many other representations are possible.

Using a fixed number $m \in \{1,\ldots,7\}$ and four distinct indices $j,k,l,n \in \{1,\ldots,7\}$, the defining relation for Dirac matrices (4.1) can be satisfied with

$$\gamma^0 \longmapsto e_j e_{\dot m}, \quad \gamma^1 \longmapsto e_k, \quad \gamma^2 \longmapsto e_l, \quad \gamma^3 \longmapsto e_n. \tag{4.2}$$

Whereas all choices for the Pauli matrices above (Section 4.1) were algebraically equivalent, this is not the case anymore for the possible choices for the Dirac matrices. There are two algebraically different choices for a quadruplet $\{j,k,l,n\}$:

(1) the quadruplet $\{j,k,l,n\}$ consists of mutually nonassociative basis elements; e.g., $\{e_4, e_5, e_6, e_7\}$;
(2) the set $\{j,k,l,n\}$ contains one associative three-cycle (e.g., $\{e_1, e_2, e_3\}$).

4.3 A note on Dirac matrices and nonassociativity

The defining relation for Dirac matrices (equation (4.1) above) can be expressed in octonionic algebras, since an octonion basis contains 7 mutually anticommutative basis elements; however, it is not possible to choose 4 independent basis elements that are also *associative*, because only associative triplets exist. Therefore, any octonionic representation of the Dirac matrices must include nonassociative parts that are not present in current textbook formulations in physics.

One speculation is that dropping the nonassociative parts of formulations today restricts physical calculations to one or two forces (e.g., electroweak interaction), but introduce problematic divergences when trying to merge more forces (e.g., strong force problems when trying perturbation theory with a coupling constant of order 1; or unification of electromagnetism, weak or strong force with gravity). It is speculated that by keeping the nonassociative parts, one can unify more forces with fewer problems. A nonperturbative description of the strong force could be possible, and unobservability of individual quarks results from nonassociative unobservable operators (see, e.g., [10]).

4.4 Dirac equation with electromagnetic field

The Dirac matrices can be chosen:

$$\gamma^0 \longmapsto -e_1 e_{\dot 1}, \quad \gamma^1 \longmapsto e_{\dot 7}, \quad \gamma^2 \longmapsto -e_{\dot 6}, \quad \gamma^3 \longmapsto e_{\dot 5}. \tag{4.3}$$

Using the electromagnetic field $A^\mu = (A^0, \vec{A})$, charge q, and writing the four-dimensional complex wave function Ψ into real parts ψ^{r}_μ and imaginary parts ψ^{i}_μ, the Dirac equation with electromagnetic field (from [2, equation 6]) can be expressed in octooctonions:

$$(\nabla - m)\Psi = 0, \tag{4.4}$$

$$\begin{aligned} \nabla \longmapsto\, & e_{\dot 1}\partial_0 + e_1 e_{\dot 1} q A_0 - e_1 e_{\dot 7}\partial_1 + e_{\dot 7} q A_1 + e_1 e_{\dot 6}\partial_2 - e_{\dot 6} q A_2 - e_1 e_{\dot 5}\partial_3 + e_{\dot 5} q A_3 \\ & = \big[e_{\dot 1}\big(\partial_0 + e_1 q A_0\big)\big] + e_1\big[-e_{\dot 7}\big(\partial_1 + e_1 q A_2\big) + e_{\dot 6}\big(\partial_2 + e_1 q A_2\big) - e_{\dot 5}\big(\partial_3 + e_1 q A_3\big)\big], \end{aligned} \tag{4.5}$$

$$m \longmapsto m \equiv e_0 e_{\dot 0} m, \tag{4.6}$$

$$\begin{aligned} \Psi \longmapsto\, & \psi^{\mathrm{r}}_0 + e_{\dot 1}\psi^{\mathrm{i}}_0 + e_{\dot 2}\psi^{\mathrm{r}}_1 + e_{\dot 3}\psi^{\mathrm{i}}_1 + e_1 e_{\dot 4}\psi^{\mathrm{r}}_2 - e_1 e_{\dot 5}\psi^{\mathrm{i}}_2 - e_1 e_{\dot 6}\psi^{\mathrm{r}}_3 - e_1 e_{\dot 7}\psi^{\mathrm{i}}_3 \\ & = \big(\psi^{\mathrm{r}}_0 + e_{\dot 1}\psi^{\mathrm{i}}_0 + e_{\dot 2}\psi^{\mathrm{r}}_1 + e_{\dot 3}\psi^{\mathrm{i}}_1\big) + e_1\big(e_{\dot 4}\psi^{\mathrm{r}}_2 - e_{\dot 5}\psi^{\mathrm{i}}_2 - e_{\dot 6}\psi^{\mathrm{r}}_3 - e_{\dot 7}\psi^{\mathrm{i}}_3\big). \end{aligned} \tag{4.7}$$

4.5 Dirac equation with electromagnetic and gravitational field

Per [2] the generalization of equations (4.5) and (4.7) above, using a real number α, is

$$\nabla \longmapsto \big[e_{\dot 1}\big(\partial_0 + e_1 q A_0\big)\big] + \exp\big(\alpha e_1\big)\big[-e_{\dot 7}\big(\partial_1 + e_1 q A_1\big) + e_{\dot 6}\big(\partial_2 + e_1 q A_2\big) - e_{\dot 5}\big(\partial_3 + e_1 q A_3\big)\big],$$

$$\Psi \longmapsto \big(\psi^{\mathrm{r}}_0 + e_{\dot 1}\psi^{\mathrm{i}}_0 + e_{\dot 2}\psi^{\mathrm{r}}_1 + e_{\dot 3}\psi^{\mathrm{i}}_1\big) + \exp\big(\alpha e_1\big)\big(e_{\dot 4}\psi^{\mathrm{r}}_2 - e_{\dot 5}\psi^{\mathrm{i}}_2 - e_{\dot 6}\psi^{\mathrm{r}}_3 - e_{\dot 7}\psi^{\mathrm{i}}_3\big).$$

The Dirac matrices corresponding to a hypothetical, purely gravitational interaction $\alpha = 0$ then simply are

$$\gamma^0_{\mathrm{gr}} \longmapsto -e_{\dot 1}, \quad \gamma^1 \longmapsto e_{\dot 7}, \quad \gamma^2 \longmapsto -e_{\dot 6}, \quad \gamma^3 \longmapsto e_{\dot 5}. \tag{4.8}$$

5 Dimensional reduction program (DRP)

The field-free Dirac equation used in the dimensional reduction program [7] (for short DRP) demonstrates three apparent fermion generations, each consisting of one massless spin 1/2 particle with only one helicity, and one massive spin 1/2 particle. It has the form

$$\widetilde{P}\psi = 0, \tag{5.1}$$

$$\gamma_0\gamma_\mu p^\mu \psi = 0, \quad \text{with } \mu \in \{0, 1, \ldots, 9\}, \tag{5.2}$$

$$\gamma_0\gamma_\mu = \begin{pmatrix} -\tilde{\sigma}_\mu & 0 \\ 0 & \sigma_\mu \end{pmatrix}, \tag{5.3}$$

using generalized Pauli matrices

$$\begin{aligned} &\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2\ldots8} := \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix}, \\ &\sigma_9 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\sigma}_{2\ldots8} := \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \end{aligned} \tag{5.4}$$

The $q \in \{e_1, \ldots, e_7\}$ are nonreal octonion basis elements.

Compared to the traditional Dirac equation, the imaginary basis element $\imath$ is generalized to the 7 octonion basis elements. When selecting a preferred basis element, e.g., e_4, the DRP takes advantage of the remaining associative 3-cycle $\{e_1, e_2, e_3\}$ for eigenvalue $\lambda_\pm$ calculation of (5.1):

$$\lambda_\pm = \frac{1}{2}, \quad \psi_+ = \begin{pmatrix} A \\ kD \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} kB \\ C \end{pmatrix}. \tag{5.5}$$

Here, $k \in \{e_1, e_2, e_3\}$, and A, B, C, D any real number. This is in contrast to the traditional Dirac equation, where no choice of k is possible, and therefore B and D both must be zero. In the DRP [7], the three choices of k are interpreted as three generations of fermions with electrons

$$e_\uparrow = \begin{pmatrix} 1 \\ k \end{pmatrix}, \quad e_\downarrow = \begin{pmatrix} -k \\ 1 \end{pmatrix},$$

neutrinos (massless) moving in z-direction, with a single helicity state

$$\nu_z = \begin{pmatrix} 0 \\ k \end{pmatrix}, \quad \nu_{-z} = \begin{pmatrix} k \\ 0 \end{pmatrix},$$

and a new, single complex massless spin-1/2 particle, with the opposite helicity

$$\emptyset_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

5.1 DRP in octooctonions

The classical Dirac matrices have been modeled above in octooctonions (equation (4.2)) as

$$\gamma^0 \longmapsto e_j e_{\dot{m}}, \quad \gamma^1 \longmapsto e_k, \quad \gamma^2 \longmapsto e_l, \quad \gamma^3 \longmapsto e_n.$$

In order to generalize the imaginary basis element of γ^2, as done in the DRP (equation (5.4)), the Dirac equation must be modified to become of the form

$$\gamma^0 \longmapsto e_j, \quad \gamma^1 \longmapsto e_k, \quad \gamma^2 \longmapsto e_l e_{\dot{m}}, \quad \gamma^3 \longmapsto e_n.$$

This could only be achieved by Wick-rotating the time axis $x_0 \mapsto e_{\dot{m}}[x_0]$ and also the y axis $x_2 \mapsto e_{\dot{m}}[x_2]$. Then, the octonionic generalization of γ_2 would be the choice of $e_{\dot{m}}$.

In this formulation, the algebraic generators $e_{\dot{m}}$ in the equation of motion (the Dirac equation) would be the same as the generators of the Wick rotation of the time and one spacial axis, also $e_{\dot{m}}$. This would couple the algebraic generators of a nonassociative background to physical parameters, which is problematic when attempting to insert a physical field: Modeling dynamic interaction would likely become ambiguous.

One could expect that if the DRP is key to describing fermion generations on octonionic background, then octooctonions as proposed here will not be successful for the description of nature. Conversely, if octooctonions are key to nonassociative quantum theory, then the DRP needs to be modified.

5.2 A proposed generalized DRP

The argument for a proposed generalized dimensional reduction program, to model fermion generations on octooctonions, is as follows:

- It may be understood as a special case, to pick the classical γ^2 and expand its associated imaginary basis element to octonions, as done in the DRP of [7]. A generalization allows any of the classical γ matrices to be expanded in a way that allows for a quaternionic degree of freedom after selecting a preferred nonassociative basis element.
- Applying such a generalization to γ^0 pairs the Dirac matrices from four-dimensional Euclidean quantum gravity (4D EQG; equation (4.8)) with the classical Dirac matrices for electromagnetism (equation (4.3)).
- From this formulation, there is no preference about which relation is fundamental: The 4D EQG equation of motion or the classical Dirac equation on Minkowskian spacetime.
- Since the 4D EQG equation of motion allows a generalized DRP to be applied to any of its γ^μ, it will now be assumed to be fundamental (as opposed to the classical point of view, which has Minkowskian spacetime as preferred, fundamental geometry of the field-free vacuum).
- Such a formulation may, at least in principle, be wide enough to include the symmetries of the Standard Model, and is therefore concluded to be of interest for the description of nature.

A preliminary look at

$$\gamma^0 \longmapsto e_j e_{\dot{\mu}}, \quad \gamma^1 \longmapsto e_k, \quad \gamma^2 \longmapsto e_l e_{\dot{\nu}}, \quad \gamma^3 \longmapsto e_n,$$

with $\mu, \nu \in \{0, \ldots, 7\}$, yields the 4D EQG equation of motion for $\{\mu = \nu = 0\}$, the classical Dirac equation for $\{\mu \neq 0,\ \nu = 0\}$, and a generalized DRP on four-dimensional Euclidean background for $\{\mu = 0,\ \nu \neq 0\}$.

In the case $\{\mu \neq 0,\ \nu \neq 0,\ \mu \neq \nu\}$, there will always be two associative 3-cycles $\{e_r, e_s, e_t\}$ such that any two basis elements in $\{e_\mu, e_\nu, e_r, e_s, e_t\}$ anticommute, and therefore reproduce the required eigenvalue/eigenfunction combinations from equation (5.5) above. It is therefore still possible to describe three fermion generations as in [7], from a slightly generalized DRP on octooctonions.

5.3 Supplying electromagnetic and gravitational fields

On a background

$$\gamma^0 \longmapsto e_j, \quad \gamma^1 \longmapsto e_k, \quad \gamma^2 \longmapsto e_l, \quad \gamma^3 \longmapsto e_n,$$

with $\{j, k, l\}$ an associative 3-cycle, the generalized DRP on octooctonions allows for factors $\{E^0, E^1, E^2, E^3\}$, each of which may be a linear combination of terms to basis elements $\{1, e_{\dot\mu}, e_{\dot\nu}, e_{\dot\rho}, e_{\dot\sigma}\}$ (in the most general case), where the $\{e_{\dot\mu}, e_{\dot\nu}, e_{\dot\rho}, e_{\dot\sigma}\}$ form a nonassociative octonionic quadruplet. The remaining 3 basis elements with dotted index, which are not contained in this quadruplet, must necessarily be an associative 3-cycle, and therefore allow the eigenvalue calculation as in (5.5).

The generalized Dirac equation is then

$$(\nabla - m)\Psi = 0, \quad \nabla = \sum_{i=0}^{3} \gamma^i E^i.$$

Mass m is positive and real. The wave function Ψ is a general octooctonion.

In this notation, the classical Dirac equation of a charge q with electromagnetic field A_μ emerges from

$$E^0_{\mathrm{EM}} := e_{\dot 4}\big(\partial_0 + e_{\dot 4} q A_0\big), \quad E^1 := \partial_1 + e_{\dot 4} q A_1, \quad E^2 := \partial_2 + e_{\dot 4} q A_2, \quad E^3 := \partial_3 + e_{\dot 4} q A_3.$$

Here, $e_{\dot 4}$ is an arbitrary choice of octonion basis element, out of a nonassociative quadruplet $\{e_{\dot 4}, e_{\dot 5}, e_{\dot 6}, e_{\dot 7}\}$. It applies the generalized DRP to the γ^0 component of the equation of motion.

4D EQG would then be described as [2]

$$E^0_{\mathrm{GR}} := \partial_0 + e_{\dot 4} q A_0,$$

This allows for a simple, one-parameter unification of E_{EM} and E_{GR} through a real mixing angle α as

$$E^0_{\mathrm{GR,\,EM}} := \exp\big(\alpha e_{\dot 4}\big)\big(\partial_0 + e_{\dot 4} q A_0\big).$$

5.4 Algebraic choices

So far it has been shown that previously proposed equations of motion of a spin 1/2 particle in electromagnetic and gravitational fields [2] can be rewritten using octooctonion algebra, which also allows for a generalized DRP after [7] that proposes to model fermion generations. This section will examine the algebraic choices and remaining freedoms to find whether such an approach would be wide enough to model the symmetries of the Standard Model (namely, SU(2) flavor and SU(3) color symmetry).

The field-free equation of motion,

$$(\nabla - m)\Psi = 0, \quad \nabla = \sum_{i=0}^{3} \gamma^i E^i,$$

modeled gamma matrices to octonion basis elements

$$\gamma^0 \longmapsto e_j, \quad \gamma^1 \longmapsto e_k, \quad \gamma^2 \longmapsto e_l, \quad \gamma^3 \longmapsto e_n,$$

where $\{e_j, e_k, e_l\}$ is an associative 3-cycle. This a first algebraic choice, because the gamma matrices could just as well be mapped to a nonassociative 4-tuple (as discussed in Section 4.2 above). Without field, the E^i are chosen as linear derivatives, ∂_i:

$$E^i_{\text{free}} := \partial_i \quad \text{with } i \in \{0, 1, 2, 3\}.$$

So far, no dotted octonion basis element (other than the unit element $e_{\dot{0}}$) has been used.

A field was then introduced after choosing an arbitrary nonreal octonion basis element from the dotted indices, $e_{\dot{4}}$, and the equation of motion is required to remain invariant when rotating the wave function Ψ along $e_{\dot{4}}$, i.e., under U(1) symmetry with respect to a preferred octonion element:

$$\Psi \longmapsto \exp\left(e_{\dot{4}} q\phi\right)\Psi'$$

where the real number q is a charge, and $\partial\phi/\partial_i = A_i$ a four-potential.

The invariant equation of motion then becomes:

$$E^i_{\text{U}(1)} := \partial_i + e_{\dot{4}} q A_i.$$

In comparison, the classical Dirac equation with electromagnetic field is exactly

$$E^0_{\text{EM}} := e_{\dot{4}}\left(\partial_0 + e_{\dot{4}} q A_0\right) = e_{\dot{4}} E^0_{\text{U}(1)}, \quad E^j_{\text{EM}} := E^j_{\text{U}(1)} \quad \text{with } j \in \{1, 2, 3\}.$$

The same preferred octonion basis element $e_{\dot{4}}$ that was used for modeling U(1) invariance is also applied to the E^0 component only. This was interpreted as generalized DRP, applied on the γ^0 "time" component.

The algebraic choices so far can be summarized as follows:

(1) "Landscape choice": The gamma matrices are modeled by four octonion basis elements $\{e_j, e_k, e_l, e_n\}$ that contain an associative 3-cycle $\{e_j, e_k, e_l\}$ and a single nonassociative basis element e_n. This choice is algebraically distinct from choosing a nonassociative 4-tuple. It is also physically different from choosing more (or fewer?) than four gamma matrices.

(2) "U(1) choice": The basis element $e_{\dot{4}}$ is chosen as the preferred basis element to require U(1) invariance of the equation of motion. This results in the 4D EQG equation of motion with field.

(3) "GR-EM choice": Application of the generalized dimensional reduction program (DRP) on the time component (γ^0), using the same preferred basis octonion $e_{\dot{4}}$ as was chosen for U(1) invariance, results in the classical Dirac equation with electromagnetic field.

5.5 Remaining freedoms ("symmetries")

Algebraic choices correspond to remaining degrees of freedom that take advantage of the symmetries of the formulation. Any symmetry within the octonions is available for exploit to require invariance of the equation of motion. In octooctonions, the dotted and undotted indices are mostly algebraically separate, they only mix at the real axis, $e_\mu^2 = e_{\dot{\mu}}^2 = -e_0 e_{\dot{0}} \equiv -1$. Therefore, the symmetries between different choices of dotted and undotted nonreal octonion basis elements each are described by G_2, the smallest exceptional Lie group.

The following symmetries govern octonion algebra in general (see, e.g., [11]):

- transformation between the seven nonreal basis elements of an octonion, G_2,
- transformation between six octonion basis elements that leaves one preferred basis element unchanged, SU(3),
- rotation within an associative 3-cycle $\{e_j, e_k, e_l\}$, which leaves the remaining nonassociative 4-tuple invariant, SU(2),
- one-parameter rotation in the space of one octonion basis $\{e_j\}$, U(1).

The corresponding freedoms, or symmetries of the formulation, then are the following:

(1) "Landscape freedom": The gamma matrices are modeled through undotted octonion indices. Because any octonion basis element could be changed, the symmetries between the different choices are described by G_2. This includes simple one-element rotation U(1), rotation within an associative 3-cycle SU(2), and rotation between six octonion basis elements that leave a preferred seventh basis element unchanged SU(3).

(2) "Preferred basis freedom": The preferred basis element $e_{\dot{4}}$ was chosen for requiring U(1) invariance for electromagnetism and gravitation, and also for the generalized DRP on γ^0 to mix these two forces. Then the choices for the remaining six dotted octonion basis elements is then described by SU(3) symmetry in general. Requiring the equation of motion to be invariant under such SU(3) symmetry would have a nontrivial impact on the A_i four-potential, which is modeled on dotted indices as well.

(3) "DRP choice (spacial)": The generalized DRP can be applied on any of the E^1, E^2, and E^3 components, using a combination of dotted indices from a chosen nonassociative 4-tuple $\{e_{\dot{4}}, e_{\dot{5}}, e_{\dot{6}}, e_{\dot{7}}\}$. This would leave a freedom within the preferred associative 3-cycle $\{e_{\dot{1}}, e_{\dot{2}}, e_{\dot{3}}\}$ with its SU(2) symmetry, for modeling fermion generations. This also would have a direct, nontrivial effect on the existing A_i potential.

5.6 Preliminary summary

Table 5.1 shows a schematic overview of the remaining freedoms of the formulation, after selecting $\{e_1, e_2, e_3\}$ as the associative 3-cycle that models γ^i, $\{e_{\dot{1}}, e_{\dot{2}}, e_{\dot{3}}\}$ as the 3-cycle to model fermion generations after [7], and e_7 and $e_{\dot{7}}$ as the preferred octonion basis for mixing electromagnetism (EM) and gravitation (GR). There is

- in the γ^i, an SU(2) invariance when rotating the chosen associative 3-cycle,
- in the γ^i, an SU(3) invariance when keeping the preferred octonion basis element fixed and varying all others,
- in the E^i, an SU(2) invariance in the associative 3-cycle that is proposed to model fermion generations, and

Octonion index	1	2	3	4	5	6	7
γ^i	γ^1	γ^2	γ^3				γ^0
E^i				E^1	E^2	E^3	E^0
Symmetry	SU(2)						U(1)
	SU(3)						U(1)
	G_2						
Force	weak?						EM, GR
	strong?						EM, GR

Table 5.1. Schematic overview of remaining freedoms ("symmetries") and speculative force assignment, after a chosen octonion basis selection for the γ^i and E^i.

- in the E^i, an SU(3) invariance when keeping the preferred octonion basis from electromagnetism and gravitation fixed and rotating all others.

It has long been proposed that quark fields could be regarded as octonionic fields [12, 13], which probably is the first application of nonassociative algebra for description of the strong force. Also, use of octonions for modeling the SU(2) gauge field was considered in the comprehensive introduction [14]. These approaches motivate speculative force assignments ("weak?", "strong?") in Table 5.1, and a claim that the octooctonion formulation presented here may be sufficiently wide for modeling the remaining forces of the Standard Model.

6 Notes

6.1 Expectation values from nonassociative background

In classical quantum mechanics, the expression for obtaining expectation values of an operator ∇ on a system modeled by wave functions Ψ is $\langle\Psi|\nabla|\Psi\rangle$ On octonions, however, this expression becomes ambiguous due to their nonassociativity, as nonassociative parts of the operator result in unobservables [3, 10]. Satisfying only a Dirac equation of the form $(\nabla - m)\Psi = 0$, therefore, cannot answer the question regarding general observation outcomes. For comparison, the Dirac equation with electromagnetic field was modeled in [15] on split-octonions, using expressions with three octonion basis elements, and then associating field and wave function differently as compared to space and time derivatives. While this formulation cannot be brought into the form $(\nabla - m)\Psi = 0$, it could well be speculated that the difference in formulation relates to the question for finding general solutions.

There are distinct cases where the octonionic expression can be reduced to associative matrix formulations, namely, for electromagnetism and gravity. When calculating mixing effects between forces on nonassociative background, or between fermion generations modeled by the DRP, the question remains how the proposed differential equations can actually be solved. This question is independent from symmetries that leave the formulation itself invariant (which can be answered on the algebra level).

6.2 A note on the landscape choice

After restricting the algebra to octooctonions, the fundamental algebraic choice was made to use four γ^i as generators of the background, and four E^i to generate motion and dynamics. Also, the γ^i were chosen to contain an associative 3-cycle, whereas the E^i consist of a nonassociative 4-tuple. The formulation was tied to earlier descriptions of electromagnetism and gravity, as well as, a proposal for modeling fermion generations. The remaining algebraic freedoms then exposed the desired symmetries to speculate that the general formulation may be wide enough for modeling all forces of the Standard Model.

This landscape choice is motivated by its potential ability to describe nature, i.e., because it “may work”. However, exclusion of other possible landscapes has not been examined yet: What dynamic behavior would emerge when the γ^i would instead be chosen from a nonassociative 4-tuple? What if the E^i contain an associative 3-cycle? What if not four, but a different number of γ^i or E^i are used?

These open questions need to be addressed to specify the motivation for looking at octooctonion algebra more clearly. If one wants to understand how exactly the physical principles of today’s descriptions of nature emerge from nonassociative quantum theory, it must be clarified better what the driving principles are to precede such emergence.

6.3 Machian approach to the hierarchy problem

An interpretation of Mach’s principle was offered in [8] to reduce Planck’s scale to the electroweak scale by assuming a nonlocal Machian response of the universe in local experiments. It required for the gravitational flux of a body to be distributed over all matter inside the horizon. Gravitation in this paper is modeled on four-dimensional Euclidean background, i.e., to a metric:

$$ds^2 = dt^2 + |d\vec{x}|^2.$$

Following the argument from [16] that positive definiteness of the norm relates to physical observables, the model here would allow for a speculated purely gravitational process to happen outside the light cone. It could span the entire universe $d\vec{x}$ with no observer time dt passing, yet be allowable since Euclidean norm is always positive definite. It could therefore model the required strong nonlocal feedback of all matter in the universe to effectively weaken the gravitational force locally in the referenced Machian approach.

7 Summary and outlook

In this paper, octooctonion algebra was used to express select recent findings towards a quantum theory on nonassociative algebra in harmonized notation. This included a nonassociative decomposition of the spin operator, the Dirac equation with fields for electromagnetism and four-dimensional Euclidean quantum gravity, and a generalized dimensional reduction program for modeling fermion generations. Algebraic choices and remaining freedoms (“symmetries”) were examined and found to expose the desired properties to justify consideration for modeling the forces of the Standard Model.

Further examination will include ways to supply weak or strong fields, how to unambiguously predict expectation values from quantum theory on nonassociative background, clarifying the physical principles that may drive a particular landscape selection on octooctonion background and generally understanding the origin of the hierarchy between Planck scale and electroweak scale.

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