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Research Article

Symmetry Invariance, Anticommutativity and Nilpotency in BRST Approach to QED: Superfield Formalism

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Abstract We provide the geometrical interpretation for the Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry invariance of the Lagrangian density of a four $(3+1)$-dimensional (4D) interacting $\mathrm{U}(1)$ gauge theory within the framework of superfield approach to BRST formalism. This interacting theory, where there is an explicit coupling between the $\mathrm{U}(1)$ gauge field and matter (Dirac) fields, is considered on a $(4,2)$-dimensional supermanifold parametrized by the four spacetime variables $x^\mu (\mu = 0,1,2,3)$ and a pair of Grassmannian variables θ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0$, $\theta\bar{\theta} + \bar{\theta}\theta = 0$). We express the Lagrangian density and (anti-)BRST charges in the language of the superfields and show that (i) the (anti-)BRST invariance of the 4D Lagrangian density is equivalent to the translation of the super Lagrangian density along the Grassmannian direction(s) (θ and/or $\bar{\theta}$) of the $(4,2)$-dimensional supermanifold such that the outcome of the above translation(s) is zero, and (ii) the anticommutativity and nilpotency of the (anti-)BRST charges are the automatic consequences of our superfield formulation.

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1 Introduction

The usual superfield approach [1,2,3,4,6,19,20,21] to BRST formalism has been very successfully applied to the case of 4D (non-)Abelian 1-form ($A^{(1)} = dx^\mu A_\mu$) gauge theories. In this approach, one constructs a super curvature 2-form $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} + i\tilde{A}^{(1)} \wedge \tilde{A}^{(1)}$ by exploiting the Maurer-Cartan equation in the language of the super 1-form gauge connection $\tilde{A}^{(1)}$ and the super exterior derivative $\tilde{d} = dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\theta\partial_\theta + d\bar{\theta}\partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) that are defined on a $(4,2)$-dimensional supermanifold, parametrized by the superspace variables $Z^M = (x^\mu, \theta, \bar{\theta})$ (with the super derivatives $\partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}})$).

The above super curvature is subsequently equated to the ordinary 2-form curvature $F^{(2)} = dA^{(1)} + iA^{(1)} \wedge A^{(1)}$ (with $d = dx^\mu \partial_\mu$ and $A^{(1)} = dx^\mu A_\mu$) defined on the 4D ordinary flat Minkowskian spacetime manifold that is parametrized by the ordinary spacetime variable $x^\mu (\mu = 0,1,2,3)$. This restriction, popularly known as the horizontality condition, leads to the derivation of the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of the 4D (non-)Abelian 1-form gauge theories.

The key reasons behind the emergence of the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields, due to the above horizontality condition[1] (HC), are

(i) the nilpotency of the (super) exterior derivatives $(\tilde{d})d$ which play very important roles in the above HC, and
(ii) the super 1-form connection $\tilde{A}^{(1)} = dZ^M \tilde{A}_M$ involves the vector superfield $\tilde{A}_M$ that consists of the multiplet superfields $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$ which are nothing but the generalizations of the gauge and (anti-)ghost fields $(A_\mu, C, \bar{C})$. The latter are the basic fields of the 4D (non-)Abelian 1-form gauge theories.

The above equality (i.e. $\tilde{F}^{(2)} = F^{(2)}$), due to the HC[2], implies that the ordinary curvature 2-form $F^{(2)}$ remains unaffected by the presence of the Grassmannian coordinates θ and $\bar{\theta}$ (with $\theta^2 = \bar{\theta}^2 = 0$, $\theta\bar{\theta} + \bar{\theta}\theta = 0$) of the superspace variable Z^M. This restriction, however, does not shed any light on the derivation of the nilpotent (anti-)BRST symmetry transformations that are associated with the matter (e.g. Dirac, complex scalar, etc.) fields of an interacting 4D (non-)Abelian 1-form gauge theory.

[1] This condition has been christened as the soul-flatness condition in [18].
[2] It is clear that, for the Abelian 1-form theory, we have $\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)}$ and $F^{(2)} = dA^{(1)}$.

In a recent set of papers [10, 11, 12, 13, 14], the above HC has been extended so as to derive the nilpotent (anti-)BRST symmetry transformations for the matter (or analogous) fields within the framework of the superfield approach to BRST formalism without spoiling the cute geometrical interpretations for the nilpotent (anti-)BRST symmetry transformations (and their corresponding generators) that emerge from the application of the HC *alone*.

In fact, in a set of a couple of papers [13, 14], we have been able to generalize the HC by a gauge invariant restriction (GIR) on the matter superfields (defined on the above $(4, 2)$-dimensional supermanifold) which enables us to derive the nilpotent (anti-)BRST symmetry transformations *together* for the gauge, (anti-)ghost and matter fields of a 4D interacting (non-)Abelian gauge theory in one stroke. In this *single* GIR on the matter superfields of the above supermanifold, the (super) covariant derivatives and their intimate connection with the (super) curvature 2-forms, play a very decisive role.

In the earlier works on the superfield formulation [21, 20, 19, 2, 1, 3, 4, 6, 18, 10, 11, 12, 13, 14], the nilpotent (anti-)BRST symmetry invariance of the physical 4D Lagrangian density of the (non-)Abelian 1-form gauge theories has *not yet* been captured. In our previous endeavours [7, 8, 9], we have attempted to capture the nilpotent symmetry invariance of the 2D (non-)Abelian 1-form gauge theories (without any interaction with matter fields) within the framework of the superfield approach to BRST formalism. However, these theories are found to be topological in nature and they are endowed with the nilpotent (anti-)BRST as well as nilpotent *(anti-)co-BRST* symmetry transformations.

In our very recent paper [15], we have been able to provide the geometrical interpretation for the (anti-)BRST invariance of the 4D free (non-)Abelian 1-form gauge theories (where there is no interaction with the matter fields) within the framework of the superfield formalism. In this work [15], we have also provided the reasons behind the uniqueness of the above symmetry transformations (and their invariances) and furnished the logical arguments for the non-existence of the on-shell nilpotent (anti-)BRST symmetry transformations *together* for the 4D non-Abelian 1-form gauge theory.

The central theme of our present paper is to generalize the key results of our earlier work in [15] to the more general case of an interacting $\mathrm{U}(1)$ gauge theory where there is an explicit coupling between the 1-form $\mathrm{U}(1)$ gauge field and the Noether conserved current constructed with the matter (Dirac) fields. We find that the GIR on the matter (Dirac) superfields (defined on the above $(4, 2)$-dimensional supermanifold) enables us to derive the exact nilpotent (anti-)BRST symmetry transformations for the matter (Dirac) fields which can *never* be obtained by exploiting the HC *alone*. The above GIR also provides a meeting-ground for the HC of the usual superfield formalism [1, 2, 3, 4, 6, 18, 19, 20, 21] and a gauge *invariant* condition on the matter superfields.

For our central objective of encapsulating the (anti-)BRST invariance of the 4D Lagrangian density of the interacting $\mathrm{U}(1)$ gauge theory within the framework of the superfield approach to BRST formalism, the following key points are of utmost importance, namely;

(i) the application of the HC enables us to demonstrate that the kinetic energy term of the $\mathrm{U}(1)$ gauge field remains independent of the Grassmannian variables when it is expressed in terms of the gauge superfields (that are obtained after the application of the HC), and

(ii) the application of the above GIR on the matter superfields enables us to show that all the terms containing the matter (Dirac) superfields are independent of the Grassmannian variables when they are expressed in terms of the superfields that are obtained after the application of the HC as well as the GIR on the matter superfields (see, e.g. (4.8) below).

The above key restrictions (i.e. HC and GIR) enable us to express the total Lagrangian density of the 4D interacting $\mathrm{U}(1)$ gauge theory with Dirac fields in the language of the superfields, in such a manner that, ultimately, a partial derivative w.r.t. θ and/or $\bar{\theta}$ on it becomes zero. In other words, the total super Lagrangian density (defined on the $(4, 2)$-dimensional supermanifold) becomes independent of the Grassmannian variables.

The above observation, in turn, implies that the corresponding 4D Lagrangian density of the parent theory (defined on the 4D ordinary spacetime manifold) becomes automatically (anti-)BRST invariant. Stated in the language of geometry on the above supermanifold, the translation of the above super Lagrangian density (defined in terms of the superfields obtained after the application of the HC and GIR) along either of the Grassmannian directions (i.e. θ and/or $\bar{\theta}$) of the above supermanifold becomes zero. This result is consistent with our earlier observation in [15].

The main motivating factors that have contributed to our curiosity to carry out the present investigation are as follows. First, it is very important for us to generalize our earlier work [15] to the case where there is an explicit coupling between the $\mathrm{U}(1)$ gauge field and matter (Dirac) fields. Second, it is a challenge to check the validity of the geometrical interpretations, provided for the (anti-)BRST invariance in our earlier work [15], to the present *interacting* case. Third, we explicitly express the (anti-)BRST charges in terms of the superfields and prove their nilpotency and anticommutativity properties. Finally, our present attempt is a modest step towards our main goal of applying the superfield formalism to the case of higher p-form ($p \geq 2$) gauge theories which have become important in the context of string theories.

Our present paper is organized as follows.

In Section 2, we give a brief synopsis of the nilpotent (anti-)BRST symmetry invariance of the Lagrangian density of a 4D interacting $\mathrm{U}(1)$ gauge theory where there is an explicit coupling between the $\mathrm{U}(1)$ gauge field and the Noether conserved current, constructed with the help of the Dirac fields.

We exploit, in Section 3, the horizontality condition (HC) to express the kinetic energy, gauge-fixing, and ghost terms in the language of the superfields (derived after the application of HC).

Section 4 deals with a GIR on the matter superfields to obtain the (anti-)BRST symmetry transformations for the matter fields and to express the kinetic term, interaction term and mass term of the Dirac fields in the language of the superfields, obtained after the application of the HC and GIR.

In Section 5, we express the (anti-)BRST charges in the language of the superfields obtained after the application of the HC and GIR. This exercise enables us to prove the nilpotency and anticommutativity of the above charges in a simple manner from which the geometrical meanings ensue.

Finally, in Section 6, we make some concluding remarks and point out a few future directions for further investigations.

Appendix A is devoted to the derivation of the nilpotent (anti-)BRST symmetry transformations *together* for the gauge, matter and (anti-)ghost fields of the theory from a *single* GIR on the matter superfields.

2 Nilpotent (anti-)BRST symmetry invariance in QED: Lagrangian formalism

We begin with the following nilpotent (anti-)BRST symmetry invariant Lagrangian density of the 4D interacting Abelian 1-form $\mathrm{U}(1)$ gauge theory in the Feynman gauge[3] (see, e.g. [18])

$$\begin{aligned}\mathcal{L}_b &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi + B(\partial\cdot A) + \frac{1}{2}B^2 - i\partial_\mu\bar{C}\partial^\mu C \\ &\equiv \mathcal{L}_b^{(g)} + \mathcal{L}_b^{(d)}.\end{aligned} \tag{2.1}$$

In the above, the kinetic energy term for the 1-form gauge field is constructed with the help of the curvature tensor $F_{\mu\nu}$ which is derived from the 2-form $F^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu)F_{\mu\nu}$. The latter emerges (i.e. $F^{(2)} = dA^{(1)}$) when the exterior derivative $d = dx^\mu\partial_\mu$ (with $d^2 = 0$) acts on the 1-form connection $A^{(1)} = dx^\mu A_\mu$ that defines the gauge potential A_μ of our present theory. The Nakanishi-Lautrup auxiliary scalar field B is invoked to linearize the gauge-fixing term $[-(1/2)(\partial\cdot A)^2]$. The latter requires, for the nilpotent (anti-)BRST symmetry invariance in the theory, the fermionic (i.e. $C^2 = \bar{C}^2 = 0, C\bar{C} + \bar{C}C = 0$) (anti-)ghost fields $(\bar{C})C$ which play a central role in the proof of the unitarity of the theory. The covariant derivative $D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi$ generates the interaction term (i.e. $-e\bar{\psi}\gamma^\mu A_\mu\psi$) between the gauge field A_μ and Dirac fields ψ and $\bar{\psi}$ (with mass m and charge e). Here γ^μ's are the standard hermitian 4×4 Dirac matrices in 4D.

The above Lagrangian (2.1) has been split into two parts $\mathcal{L}_b^{(g)}$ and $\mathcal{L}_b^{(d)}$ for our later convenience. The former corresponds to the kinetic term, gauge-fixing term and Faddeev-Popov ghost term for the 1-form gauge and fermionic (anti-)ghost fields of the theory and the latter corresponds to all the terms in the Lagrangian density that necessarily possess the Dirac fields.

The following off-shell nilpotent ($s_{(a)b}^2 = 0$) and anticommuting ($s_b s_{ab} + s_{ab} s_b = 0$) infinitesimal (anti-)BRST symmetry transformations $s_{(a)b}$[4]

$$s_b A_\mu = \partial_\mu C, \quad s_b C = 0, \quad s_b\bar{C} = iB, \quad s_b B = 0, \quad s_b\psi = -ieC\psi, \quad s_b\bar{\psi} = -ie\bar{\psi}C, \quad s_b F_{\mu\nu} = 0, \tag{2.2}$$

$$s_{ab} A_\mu = \partial_\mu\bar{C}, \quad s_{ab}\bar{C} = 0, \quad s_{ab} C = -iB, \quad s_{ab} B = 0, \quad s_{ab}\psi = -ie\bar{C}\psi, \quad s_{ab}\bar{\psi} = -ie\bar{\psi}\bar{C}, \quad s_{ab} F_{\mu\nu} = 0, \tag{2.3}$$

leave the above Lagrangian density (2.1) quasi-invariant because it transforms to a total spacetime derivative under the above transformations (cf. (5.1) below). The key features of the above transformations are:

[3] We follow here the convention and notations such that the flat 4D Minkowski spacetime manifold is characterized by a metric $\eta_{\mu\nu}$ with the signatures $(+1, -1, -1, -1)$ so that $\square = \eta_{\mu\nu}\partial^\mu\partial^\nu = (\partial_0)^2 - (\partial_i)^2$ and $P\cdot Q = \eta_{\mu\nu}P^\mu Q^\nu \equiv P_0 Q_0 - P_i Q_i$ is the dot product between two non-null 4-vectors P_μ and Q_μ. The Greek indices $\mu, \nu, \kappa, \ldots = 0, 1, 2, 3$ correspond to the spacetime directions and the Latin indices $i, j, k, \ldots = 1, 2, 3$ stand only for the space directions on the above 4D flat Minkowskian spacetime manifold.

[4] We follow here the notation and convention adopted in [10,11,12,13,14]. In their totality, the (anti-)BRST symmetry transformations $\delta_{(A)B}$ are the product (i.e. $\delta_{(A)B} = \eta s_{(a)b}$) of an anticommuting ($\eta C == C\eta$, $\eta\psi = -\psi\eta$ etc.) spacetime independent parameter η and $s_{(a)b}$ such that the operator form of the nilpotency of $\delta_{(A)B}$ is traded with that of $s_{(a)b}$.

(i) the curvature tensor, owing its origin to the cohomological operator $d = dx^\mu \partial_\mu$ (with $d^2 = 0$), remains invariant under both the above nilpotent symmetry transformations (i.e. $s_{(a)b} F_{\mu\nu} = 0$),
(ii) the total terms involving the Dirac fields (i.e. $\bar\psi(i\gamma^\mu D_\mu - m)\psi$) also remain invariant under $s_{(a)b}$ (i.e. $s_{(a)b}[\bar\psi(i\gamma^\mu D_\mu - m)\psi] = 0$),
(iii) the nilpotency (i.e. $d^2 = 0$) of the exterior derivative $d = dx^\mu \partial_\mu$ is replicated in the language of the nilpotency of the above symmetry transformations $s_{(a)b}$ (i.e. $s^2_{(a)b} = 0$), and
(iv) there is a deep connection between the exterior derivative and the above nilpotent transformations which will be exploited in the superfield approach to BRST formalism (see Sections 3 and 4 and the appendix below).

It can be checked that the gauge-fixing and Faddeev-Popov ghost terms of the Lagrangian density (2.1) can be written, in the exact form(s), as

$$-s_b\left[i\bar C\left\{(\partial\cdot A) + \frac{1}{2}B\right\}\right],\; s_{ab}\left[iC\left\{(\partial\cdot A) + \frac{1}{2}B\right\}\right],\; s_b s_{ab}\left(\frac{i}{2}A_\mu A^\mu + \frac{1}{2}C\bar C\right), \tag{2.4}$$

modulo some total spacetime derivative terms which do not affect the dynamics of the theory. The above expressions provide a simple proof for the nilpotent symmetry invariance of the Lagrangian density (2.1) because of

(i) the nilpotency of the transformations $s_{(a)b}$ (i.e. $s^2_{(a)b} = 0$),
(ii) the invariance of the curvature term (i.e. $s_{(a)b} F_{\mu\nu} = 0$), and
(iii) the invariance of the terms involving Dirac fields (i.e. $s_{(a)b}[\bar\psi(i\gamma^\mu D_\mu - m)\psi] = 0$) under the nilpotent (anti-)BRST symmetry transformations $s_{(a)b}$.

3 Symmetry transformations for the gauge and ghost fields: horizontality condition

We exploit here the usual superfield approach [1,2,3,4,6,18,19,20,21] to obtain the nilpotent (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of the Lagrangian density (2.1). To this end in mind, first of all, we generalize the 4D local fields $(A_\mu(x), C(x), \bar C(x))$ to the superfields $(\mathcal{B}_\mu(x,\theta,\bar\theta), \mathcal{F}(x,\theta,\bar\theta), \bar{\mathcal{F}}(x,\theta,\bar\theta))$ that are defined on a $(4,2)$-dimensional supermanifold, parametrized by the superspace variables $Z^M = (x^\mu, \theta, \bar\theta)$. In terms of these superfields, we can define a super 1-form connection as

$$\tilde A^{(1)} = dZ^M \tilde A_M = dx^\mu \mathcal{B}_\mu + d\theta \bar{\mathcal{F}} + d\bar\theta \mathcal{F}, \tag{3.1}$$

where $\tilde A_M$ is the vector superfield with the component multiplet fields as $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$ and $dZ^M = (dx^\mu, d\theta, d\bar\theta)$ is the superspace differential.

The above component superfields can be expanded in terms of the basic fields $(A_\mu, C, \bar C)$, auxiliary field B and secondary fields as (see, e.g. [1,2,3,4])

$$\begin{aligned}
\mathcal{B}_\mu(x,\theta,\bar\theta) &= A_\mu(x) + \theta \bar R_\mu(x) + \bar\theta R_\mu(x) + i\theta\bar\theta S_\mu(x),\\
\mathcal{F}(x,\theta,\bar\theta) &= C(x) + i\theta \bar B(x) + i\bar\theta \mathcal{B}(x) + i\theta\bar\theta s(x),\\
\bar{\mathcal{F}}(x,\theta,\bar\theta) &= \bar C(x) + i\theta \bar{\mathcal{B}}(x) + i\bar\theta B(x) + i\theta\bar\theta \bar s(x),
\end{aligned} \tag{3.2}$$

where the secondary fields are $\bar B(x), \mathcal{B}(x), \bar{\mathcal{B}}(x), s(x), \bar s(x)$. It will be noted that, in the limit $(\theta,\bar\theta) \to 0$, we retrieve our basic fields $(A_\mu, C, \bar C)$. In the above expansion, the bosonic and fermionic component fields do match with each-other. The exact expressions for the secondary fields can be derived in terms of the basic fields of the theory if we exploit the celebrated HC.

Let us recall our observation after the (anti-)BRST symmetry transformations (2.2) and (2.3). These transformations $s_{(a)b}$ owe their origin to the exterior derivative $d = dx^\mu\partial_\mu$ which plays a very important role in the application of the HC. To this end in mind, let us generalize the 4D ordinary d to its counterpart on the $(4,2)$-dimensional supermanifold, as

$$d \to \tilde d = dZ^M \partial_M = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar\theta \partial_{\bar\theta}, \quad \partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar\theta}). \tag{3.3}$$

The HC is the requirement that the super 2-form $\tilde F^{(2)} = \tilde d \tilde A^{(1)}$, defined on the $(4,2)$-dimensional supermanifold, should be equal (i.e. $\tilde F^{(2)} = F^{(2)}$) to the ordinary 4D 2-form $F^{(2)} = dA^{(1)}$. Physically, this amounts to the restriction that the gauge (i.e. (anti-)BRST) invariant quantities, which are the components of the curvature tensor $F_{\mu\nu}$, should remain *unaffected* by the presence of the Grassmannian variables θ and $\bar\theta$.

The explicit computations for $\tilde{F}^{(2)}$ (from (3.1) and (3.3)) and its subsequent equality with the ordinary 4D 2-form $F^{(2)}$, leads to the following relationships between the secondary fields and basic fields of the theory [10,11, 12,13,14]

$$R_\mu = \partial_\mu C, \quad \bar{R}_\mu = \partial_\mu \bar{C}, \quad S_\mu = \partial_\mu B \equiv -\partial_\mu \bar{B}, \quad B + \bar{B} = 0, \quad \mathcal{B} = 0, \quad \bar{\mathcal{B}} = 0, \quad s = 0, \quad \bar{s} = 0. \tag{3.4}$$

Insertion of these values into the expansion (3.2) of the superfields leads to the following final expansion

$$\begin{aligned}
\mathcal{B}_\mu^{(h)}(x,\theta,\bar{\theta}) &= A_\mu(x) + \theta\partial_\mu\bar{C}(x) + \bar{\theta}\partial_\mu C(x) + i\theta\bar{\theta}\partial_\mu B(x) \\
&\equiv A_\mu(x) + \theta\big(s_{ab}A_\mu(x)\big) + \bar{\theta}\big(s_b A_\mu(x)\big) + \theta\bar{\theta}\big(s_b s_{ab} A_\mu(x)\big), \\
\mathcal{F}^{(h)}(x,\theta,\bar{\theta}) &= C(x) - i\theta B(x) \\
&\equiv C(x) + \theta\big(s_{ab}C(x)\big) + \bar{\theta}(s_b C(x)) + \theta\bar{\theta}\big(s_b s_{ab} C(x)\big), \\
\bar{\mathcal{F}}^{(h)}(x,\theta,\bar{\theta}) &= \bar{C}(x) + i\bar{\theta}B(x) \\
&\equiv \bar{C}(x) + \theta\big(s_{ab}\bar{C}(x)\big) + \bar{\theta}\big(s_b\bar{C}(x)\big) + \theta\bar{\theta}\big(s_b s_{ab}\bar{C}(x)\big),
\end{aligned} \tag{3.5}$$

where the following points are important, namely;

(i) the superscript (h) on the superfields corresponds to the superfields obtained after the application of the HC,

(ii) the transformations $s_b C = 0$ and $s_{ab}\bar{C} = 0$ have been taken into account in the above uniform expansions,

(iii) the super curvature tensor $\tilde{F}_{\mu\nu}^{(h)} = \partial_\mu \mathcal{B}_\nu^{(h)} - \partial_\nu \mathcal{B}_\mu^{(h)}$ is found to be equal to the ordinary curvature tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ because all the Grassmannian dependent terms cancel out to become zero, and

(iv) there is a very intimate relationship between the (anti-)BRST symmetry transformations acting on a 4D field and the translational generators (along the Grassmannian directions of the supermanifold) acting on the corresponding $(4,2)$-dimensional superfield obtained after the application HC. Mathematically, this statement can be expressed as

$$\operatorname*{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\tilde{\Omega}^{(h)}(x,\theta,\bar{\theta}) = s_b\Omega(x), \quad \operatorname*{Lim}_{\bar{\theta}\to 0}\frac{\partial}{\partial\theta}\tilde{\Omega}^{(h)}(x,\theta,\bar{\theta}) = s_{ab}\Omega(x), \tag{3.6}$$

where $\Omega(x)$ is the ordinary 4D generic local field and $\tilde{\Omega}^{(h)}(x,\theta,\bar{\theta})$ is the corresponding generic superfield derived after the application of the HC.

The above superfields would be used to express the kinetic term for the $\mathrm{U}(1)$ gauge field, gauge-fixing term for the photon field and Faddeev-Popov ghost terms for the (anti-)ghost fields of the theory. These can be expressed in the following *three* different and distinct forms (see, e.g. [15])

$$\begin{aligned}
\tilde{\mathcal{L}}_{(g)}^{(1)} &= -\frac{1}{4}\tilde{F}_{\mu\nu}^{(h)}\tilde{F}^{\mu\nu(h)} + \operatorname*{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\Big[-i\bar{\mathcal{F}}^{(h)}\Big(\partial^\mu\mathcal{B}_\mu^{(h)} + \frac{1}{2}B\Big)\Big], \\
\tilde{\mathcal{L}}_{(g)}^{(2)} &= -\frac{1}{4}\tilde{F}_{\mu\nu}^{(h)}\tilde{F}^{\mu\nu(h)} + \operatorname*{Lim}_{\bar{\theta}\to 0}\frac{\partial}{\partial\theta}\Big[+i\mathcal{F}^{(h)}\Big(\partial^\mu\mathcal{B}_\mu^{(h)} + \frac{1}{2}B\Big)\Big], \\
\tilde{\mathcal{L}}_{(g)}^{(3)} &= -\frac{1}{4}\tilde{F}_{\mu\nu}^{(h)}\tilde{F}^{\mu\nu(h)} + \frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\Big[+\frac{i}{2}\mathcal{B}^{\mu(h)}\mathcal{B}_\mu^{(h)} + \frac{1}{2}\mathcal{F}^{(h)}\bar{\mathcal{F}}^{(h)}\Big],
\end{aligned} \tag{3.7}$$

where the subscript (g) on the above super Lagrangians stands for the terms in the 4D Lagrangian density that correspond to the gauge and (anti-)ghost fields. In other words, we have encapsulated the kinetic term, gauge-fixing term and Faddeev-Popov ghost term of the 4D Lagrangian density (2.1) in the language of the superfields derived after the application of the HC.

It is very interesting to note that the (anti-)BRST invariance of the kinetic term, gauge-fixing term and Faddeev-Popov ghost term can be captured in the language of the translations of the above super Lagrangian densities $\tilde{\mathcal{L}}_{(g)}^{(1,2,3)}$ along the Grassmannian directions. Mathematically, the nilpotent BRST invariance can be expressed as follows

$$s_b\mathcal{L}_b^{(g)} = 0 \Longleftrightarrow \operatorname*{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\tilde{\mathcal{L}}_{(g)}^{(1)} = 0, \quad s_b^2 = 0 \Longleftrightarrow \Big(\operatorname*{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\Big)^2 = 0, \tag{3.8}$$

$$s_b\mathcal{L}_b^{(g)} = 0 \Longleftrightarrow \frac{\partial}{\partial\bar{\theta}}\tilde{\mathcal{L}}_{(g)}^{(3)} = 0, \qquad s_b^2 = 0 \Longleftrightarrow \Big(\frac{\partial}{\partial\bar{\theta}}\Big)^2 = 0. \tag{3.9}$$

Similarly, the anti-BRST invariance of the kinetic, gauge-fixing and ghost terms can be expressed, in the language of the superfields, as follows

$$s_{ab}\mathcal{L}_b^{(g)} = 0 \Longleftrightarrow \underset{\bar\theta\to 0}{\text{Lim}}\,\frac{\partial}{\partial\theta}\tilde{\mathcal{L}}^{(2)}_{(g)} = 0, \quad s_{ab}^2 = 0 \Longleftrightarrow \Big(\underset{\bar\theta\to 0}{\text{Lim}}\,\frac{\partial}{\partial\theta}\Big)^2 = 0, \tag{3.10}$$

$$s_{ab}\mathcal{L}_b^{(g)} = 0 \Longleftrightarrow \frac{\partial}{\partial\theta}\tilde{\mathcal{L}}^{(3)}_{(g)} = 0, \qquad s_{ab}^2 = 0 \Longleftrightarrow \Big(\frac{\partial}{\partial\theta}\Big)^2 = 0. \tag{3.11}$$

Thus, we note that the Grassmannian independence of the super Lagrangian densities, defined in terms of the superfields on the $(4, 2)$-dimensional supermanifold, automatically implies the (anti-)BRST invariance of the $4D$ Lagrangian density defined in terms of $4D$ local fields taking their values on a $4D$ flat Minkowskian spacetime manifold.

4 (Anti-)BRST symmetry transformations for Dirac fields: gauge invariant condition

Unlike the superfields $\mathcal{B}_\mu(x,\theta,\bar\theta), \mathcal{F}(x,\theta,\bar\theta), \bar{\mathcal{F}}(x,\theta,\bar\theta)$ that form a vector supermultiplet (cf. the previous section), the fermionic matter superfields $\Psi(x,\theta,\bar\theta)$ and $\bar\Psi(x,\theta,\bar\theta)$ (which are the generalizations of the $4D$ matter Dirac fields $\psi(x)$ and $\bar\psi(x)$ of the Lagrangian density (2.1) onto the $(4, 2)$-dimensional supermanifold) do not belong to any supermultiplet. Thus, it appears that there is no connection between the superfields $(\mathcal{B}_\mu, \mathcal{F}, \bar{\mathcal{F}})$ and the matter superfields $(\Psi(x,\theta,\bar\theta), \bar\Psi(x,\theta,\bar\theta))$. However, there is one *gauge invariant* relationship in which the matter superfields do 'talk' with the super gauge connection $\tilde{A}^{(1)}$ of (3.1). We exploit this relationship and impose the following GIR on the matter superfields

$$\bar\Psi(x,\theta,\bar\theta)\Big[\tilde d + ie\tilde{A}^{(1)}_{(h)}\Big]\Psi(x,\theta,\bar\theta) = \bar\psi(x)\Big(d + ieA^{(1)}\Big)\psi(x). \tag{4.1}$$

The r.h.s. of the above equation is a $U(1)$ gauge (i.e. (anti-)BRST) invariant quantity because it is connected with the covariant derivative.

The relationship (4.1) is interesting on the following grounds. First, it is a gauge (i.e. (anti-)BRST) invariant quantity. Thus, it is physical in some sense. Second, it will be noted that, on the l.h.s. of (4.1), we have $\tilde{A}^{(1)}_{(h)}$ which is derived after the application of HC[5]. Thus, HC of the previous section and GIR of our present section (cf. (4.1)) are intimately connected. In fact, the explicit form of $\tilde{A}^{(1)}_{(h)} = dx^\mu \mathcal{B}^{(h)}_\mu + d\theta\bar{\mathcal{F}}^{(h)} + d\bar\theta\mathcal{F}^{(h)}$ is such that the whole expansion of (3.5) is going to play a very decisive role in the determination of the exact nilpotent (anti-)BRST symmetry transformations for the matter fields in the language of the (anti-)ghost fields and matter fields themselves.

To find out the impact of the above restriction, we have to expand the matter superfields along the Grassmannian θ and $\bar\theta$ directions of the $(4, 2)$-dimensional supermanifold as follows

$$\Psi(x,\theta,\bar\theta) = \psi(x) + i\theta\bar b_1(x) + i\bar\theta b_2(x) + i\theta\bar\theta f(x), \quad \bar\Psi(x,\theta,\bar\theta) = \bar\psi(x) + i\theta\bar b_2(x) + i\bar\theta b_1(x) + i\theta\bar\theta\bar f(x), \tag{4.2}$$

where $\psi(x)$ and $\bar\psi(x)$ are the 4D basic Dirac fields of the Lagrangian density (2.1) and $b_1, \bar b_1, b_2, \bar b_2, f, \bar f$ are the secondary fields which will be expressed in terms of the basic fields of the Lagrangian density (2.1) due to the above GIR (4.1). In the limit $(\theta,\bar\theta)\to 0$, we retrieve our basic $4D$ local fields ψ and $\bar\psi$ and bosonic $(b_1, b_2, \bar b_1, \bar b_2)$ and fermionic $(\psi, \bar\psi, f, \bar f)$ degrees of freedom do match in the above expansion. This is consistent with the basic requirements of a true supersymmetric field theory.

The explicit computation of the l.h.s. of GIR (4.1) and its subsequent comparison with the r.h.s., yields the following relationship between the secondary fields and the basic fields of the theory (see [13] for details)

$$b_1 = -e\bar\psi C, \quad b_2 = -eC\psi, \quad \bar b_1 = -e\bar C\psi, \quad \bar b_2 = -e\bar\psi\bar C, \quad f = -ie[B + e\bar C C]\psi, \quad \bar f = +ie\bar\psi[B + eC\bar C]. \tag{4.3}$$

The insertions of the above values into the expansion of the matter superfields (4.2), lead to the following explicit expansions

$$\begin{aligned}
\Psi^{(G)}(x,\theta,\bar\theta) &= \psi(x) + \theta\big(-ie\bar C\psi(x)\big) + \bar\theta\big(-ieC\psi(x)\big) + \theta\bar\theta\big[e(B + e\bar C C)\psi(x)\big]\\
&\equiv \psi(x) + \theta\big(s_{ab}\psi(x)\big) + \bar\theta\big(s_b\psi(x)\big) + \theta\bar\theta\big(s_b s_{ab}\psi(x)\big),\\
\bar\Psi^{(G)}(x,\theta,\bar\theta) &= \bar\psi(x) + \theta\big(-ie\bar\psi(x)\bar C\big) + \bar\theta\big(-ie\bar\psi(x)C\big) + \theta\bar\theta\big[-e\bar\psi(x)(B + eC\bar C)\big]\\
&\equiv \bar\psi(x) + \theta\big(s_{ab}\bar\psi(x)\big) + \bar\theta\big(s_b\bar\psi(x)\big) + \theta\bar\theta\big(s_b s_{ab}\bar\psi(x)\big).
\end{aligned} \tag{4.4}$$

[5] The HC is basically a gauge *covariant* restriction for the discussion of the non-Abelian gauge theory. It, however, reduces to a GIR for the Abelian $U(1)$ gauge theory.

The superscript (G) on the above superfields denotes the fact that they have been obtained after the application of the GIR (cf. equation (4.1)).

It is very interesting to note, at this stage, that the GIR (cf. (4.1)) on the matter superfields leads to

(i) the exact and unique derivation of the nilpotent (anti-)BRST symmetry transformations for the matter fields $\psi(x)$ and $\bar\psi(x)$, and
(ii) the geometrical interpretation for the (anti-)BRST symmetry transformations as the translational generators along the Grassmannian directions θ and $\bar\theta$ of the above supermanifold. As a consequence, we obtain the analogue of the equation (3.6), as

$$\lim_{\theta\to 0}\frac{\partial}{\partial\bar\theta}\tilde\Omega^{(G)}(x,\theta,\bar\theta)=s_b\Omega(x),\quad \lim_{\bar\theta\to 0}\frac{\partial}{\partial\theta}\tilde\Omega^{(G)}(x,\theta,\bar\theta)=s_{ab}\Omega(x), \tag{4.5}$$

where $\tilde\Omega^{(G)}(x,\theta,\bar\theta)$ stands for the expansions (4.4) for the matter superfields, obtained after the application of the GIR (4.1). The generic field $\Omega(x)$ stands for the 4D matter Dirac fields $\psi(x)$ and $\bar\psi(x)$.

There is an interesting consequence due to our expansion in (4.4). It is straightforward to check that the following equation

$$\bar\Psi^{(G)}(x,\theta,\bar\theta)\Psi^{(G)}(x,\theta,\bar\theta)=\bar\psi(x)\psi(x), \tag{4.6}$$

is automatically satisfied. This observation implies, ultimately, that the equation (4.1) can be re-expressed as

$$\bar\Psi^{(G)}(x,\theta,\bar\theta)\Big[\tilde d+ie\tilde A^{(1)}_{(h)}-M\Big]\Psi^{(G)}(x,\theta,\bar\theta)=\bar\psi(x)\big(d+ieA^{(1)}-m\big)\psi(x). \tag{4.7}$$

Physically, the above restriction implies that the total Dirac part of the Lagrangian density (2.1) (i.e. $\mathcal{L}^{(d)}$) remains unaffected due to the presence of the Grassmannian variables on the $(4,2)$-dimensional supermanifold, on which, our 4D interacting $\mathrm{U}(1)$ gauge theory (with Dirac fields) has been generalized. The above observation implies that the super Lagrangian density for the Dirac fields, using GIR (4.1) and super expansion (4.4), can be written as

$$\tilde{\mathcal{L}}^{(d)}=\bar\Psi^{(G)}(x,\theta,\bar\theta)\big(i\gamma^M D^{(h)}_M-m\big)\Psi^{(G)}(x,\theta,\bar\theta)\equiv\mathcal{L}^{(d)}, \tag{4.8}$$

where γ^M's are some non-trivial generalization of the 4×4 Dirac matrices γ^μ to the $(4,2)$-dimensional supermanifold and $\gamma^M D^{(h)}_M$ is defined as[6]

$$\gamma^M D^{(h)}_M=\gamma^\mu\Big(\partial_\mu+ie\mathcal{B}^{(h)}_\mu\Big)+C_\theta\Big(\partial_\theta+ie\bar{\mathcal{F}}^{(h)}\Big)+C_{\bar\theta}\Big(\partial_{\bar\theta}+ie\mathcal{F}^{(h)}\Big). \tag{4.9}$$

Here the superfields $\mathcal{B}^{(h)}_\mu$, $\mathcal{F}^{(h)}$ and $\bar{\mathcal{F}}^{(h)}$ are the expanded form of equation (3.5) that have been obtained after the application of HC.

It is straightforward to capture now the (anti-)BRST invariance (i.e. $s_{(a)b}[\bar\psi(i\gamma^\mu D_\mu-m)\psi]=0$) of the Dirac part of the Lagrangian density (2.1) in the language of the superfields. This can be expressed as follows

$$s_b\mathcal{L}^{(d)}=0\Longleftrightarrow\lim_{\theta\to 0}\frac{\partial}{\partial\bar\theta}\tilde{\mathcal{L}}^{(d)}=0,\quad s_{ab}\mathcal{L}^{(d)}=0\Longleftrightarrow\lim_{\bar\theta\to 0}\frac{\partial}{\partial\theta}\tilde{\mathcal{L}}^{(d)}=0. \tag{4.10}$$

5 Nilpotent and anticommuting (anti-)BRST charges: superfield formulation

It is straightforward to check that the total Lagrangian density (2.1) transforms, under the (anti-)BRST symmetry transformations (cf. (2.2) and (2.3)), as

$$s_{ab}\mathcal{L}_b=\partial_\mu\big[B\partial^\mu\bar C\big],\quad s_b\mathcal{L}_b=\partial_\mu\big[B\partial^\mu C\big]. \tag{5.1}$$

The Noether (anti-)BRST conserved currents $J^\mu_{(a)b}$ (i.e. $\partial_\mu J^\mu_i=0$, $i=b,ab$), that ensue due to the above symmetry transformations, are

$$J^\mu_{ab}=B\partial^\mu\bar C-F^{\mu\nu}\partial_\nu\bar C-e\bar\psi\gamma^\mu\bar C\psi,\quad J^\mu_b=B\partial^\mu C-F^{\mu\nu}\partial_\nu C-e\bar\psi\gamma^\mu C\psi, \tag{5.2}$$

[6] The $(4,2)$-dimensional representation $\gamma^M\equiv(\gamma^\mu,C_\theta,C_{\bar\theta})$ serves our purpose in equation (4.9). Here C_θ and $C_{\bar\theta}$ are fermionic in nature and they reduce to zero in the limit $(\theta,\bar\theta)\to 0$. The exact form of C_θ and $C_{\bar\theta}$ is *not* essential for our purposes because, irrespective of their form, we obtain $(\partial_\theta+ie\bar{\mathcal{F}}^{(h)})\Psi^{(G)}=0$ and $(\partial_{\bar\theta}+ie\mathcal{F}^{(h)})\Psi^{(G)}=0$ when (4.9) is inserted into (4.8) for the verification of the equality (see, also, [13] for more details).

which lead to the derivation of the following conserved charges

$$Q_{ab} = \int d^3x \big[B\dot{\bar{C}} - \dot{B}\bar{C}\big], \quad Q_b = \int d^3x \big[B\dot{C} - \dot{B}C\big]. \tag{5.3}$$

In the derivation of the above simple expressions for $Q_{(a)b}$ as well as in the proof of the conservation of currents, the following equations of motion

$$\begin{aligned} &\partial_\mu F^{\mu\nu} = \partial^\nu B + e\bar{\psi}\gamma^\nu\psi, \quad \Box C = \Box\bar{C} = \Box B = 0, \\ &i\gamma^\mu(\partial_\mu\psi) = m\psi + e\gamma^\mu A_\mu\psi, \quad i(\partial_\mu\bar{\psi})\gamma^\mu = -m\bar{\psi} - e\bar{\psi}\gamma^\mu A_\mu, \end{aligned} \tag{5.4}$$

that emerge from the Lagrangian density (2.1), have been exploited.

To understand the nilpotency and anticommutativity of the (anti-)BRST charges, we have to express them in terms of the superfields that have been obtained after the application of the HC and GIR (cf. (3.5) and (4.4)). For instance, it can be checked that the (anti-)BRST charges can be written as

$$Q_b = i\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\Big(\int d^3x \Big[\mathcal{B}_0^{(h)} F^{(h)}\Big]\Big), \quad Q_{ab} = i\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta}\Big(\int d^3x \Big[\mathcal{B}_0^{(h)} \bar{F}^{(h)}\Big]\Big). \tag{5.5}$$

The above expressions immediately imply (cf. (3.8)–(3.11))

$$\frac{\partial}{\partial\bar{\theta}}Q_b = 0, \quad \frac{\partial}{\partial\theta}Q_b = 0, \quad \frac{\partial}{\partial\bar{\theta}}Q_{ab} = 0, \quad \frac{\partial}{\partial\bar{\theta}}Q_{ab} = 0, \tag{5.6}$$

because of the fact that $\partial_\theta\partial_{\bar{\theta}} + \partial_{\bar{\theta}}\partial_\theta = 0$ and $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$. In the language of the (anti-)BRST symmetry transformations (cf. (3.6) and (4.5)), we have

$$s_b Q_b = 0, \quad s_{ab} Q_b = 0, \quad s_b Q_{ab} = 0, \quad s_{ab} Q_{ab} = 0, \tag{5.7}$$

which imply the nilpotency (i.e. $Q_{(a)b}^2 = 0$) and anticommutativity (i.e. $Q_b Q_{ab} + Q_{ab} Q_b = 0$) properties of the conserved (anti-)BRST charges because $s_b Q_b = -i\{Q_b, Q_b\} = 0$, $s_{ab} Q_b = -i\{Q_b, Q_{ab}\} = 0$, $s_{ab} Q_{ab} = -i\{Q_{ab}, Q_{ab}\} = 0$, $s_b Q_{ab} = -i\{Q_{ab}, Q_b\} = 0$, etc. It is interesting to point out that the (anti-)BRST charges (5.5) can be equivalently expressed as

$$Q_{ab} = i\int d^3x \int d^2\theta \Big[\mathcal{B}_0^{(h)} \bar{F}^{(h)}\Big], \quad Q_b = i\int d^3x \int d^2\theta \Big[\mathcal{B}_0^{(h)} F^{(h)}\Big], \tag{5.8}$$

where we have adopted $d^2\theta = d\bar{\theta} d\theta$. Ultimately, the above expressions imply the following interesting relationships

$$i s_b s_{ab}\big[A_0\bar{C}\big] = B\dot{\bar{C}} - \dot{B}\bar{C}, \quad i s_b s_{ab}\big[A_0 C\big] = B\dot{C} - \dot{B}C, \tag{5.9}$$

which establish the (anti-)BRST invariance of the (anti-)BRST charges.

The BRST charge Q_b, expressed in (5.5), can be also written in the following two distinctly different ways, namely;

$$\begin{aligned} Q_b &= i\operatorname*{Lim}_{\theta\to 0}\frac{\partial}{\partial\bar{\theta}}\int d^3x \Big[\dot{\bar{F}}^{(h)} F^{(h)} - i\mathcal{B}_0^{(h)} B(x)\Big] \equiv i\int d^3x \int d\bar{\theta}\Big[\dot{\bar{F}}^{(h)} F^{(h)} - i\mathcal{B}_0^{(h)} B(x)\Big], \\ Q_b &= -i\operatorname*{Lim}_{\bar{\theta}\to 0}\frac{\partial}{\partial\theta}\int d^3x \Big[\dot{F}^{(h)} F^{(h)}\Big] \equiv -i\int d^3x \int d\theta\Big[\dot{F}^{(h)} F^{(h)}\Big]. \end{aligned} \tag{5.10}$$

The above expressions demonstrate the followings

(i) the nilpotency property because $\partial_\theta^2 = \partial_{\bar{\theta}}^2 = 0$,

(ii) the (anti-)BRST invariance of the BRST charge Q_b because of the validity of (5.6) as well as the sanctity of the following expressions

$$Q_b = i\int d^3x s_b\big[\dot{\bar{C}}C - iA_0 B\big] \Longrightarrow s_b Q_b = 0, \quad Q_b = -i\int d^3x s_{ab}\big[\dot{C}C\big] \Longrightarrow s_{ab} Q_b = 0, \tag{5.11}$$

(iii) the anticommutativity property because $s_{ab} Q_b = -i\{Q_b, Q_{ab}\} = 0$. Thus, we note that the nilpotency and anticommutativity properties of the BRST charge becomes quite simple in the language of the superfield formulation when QED is considered on the $(4, 2)$-dimensional supermanifold.

In exactly similar fashion, we can express the anti-BRST charge as

$$Q_{ab} = -i \operatorname*{Lim}_{\bar\theta \to 0} \frac{\partial}{\partial \theta} \int d^3x \Big[\dot F^{(h)} \bar F^{(h)} + i \mathcal{B}_0^{(h)} B(x) \Big] \equiv -i \int d^3x \int d\theta \Big[\dot F^{(h)} \bar F^{(h)} + i \mathcal{B}_0^{(h)} B(x) \Big],$$
$$Q_{ab} = i \operatorname*{Lim}_{\theta \to 0} \frac{\partial}{\partial \bar\theta} \int d^3x \Big[\dot{\bar F}^{(h)} \bar F^{(h)} \Big] \equiv i \int d^3x \int d\bar\theta \Big[\dot{\bar F}^{(h)} \bar F^{(h)} \Big]. \tag{5.12}$$

The above expressions automatically imply the validity of relations quoted in (5.6). In other words, the (anti-)BRST charges (and their corresponding symmetry transformations) are always found to be absolutely anticommuting and nilpotent of order two within the framework of our present superfield approach to BRST formalism (as far as QED with Dirac fields is concerned).

6 Conclusions

In our present endeavor, we have concentrated on the (anti-)BRST invariance of the Lagrangian density of a 4D interacting $\mathrm{U}(1)$ gauge theory with Dirac fields. As in our earlier work [15] on the 4D *free* (non-)Abelian 1-form gauge theories (having no interaction with matter fields), we find that the Grassmannian independence of the super Lagrangian densities (that are expressed in terms of the superfields obtained after the application of the HC and GIR) is a sure guarantee that the corresponding 4D Lagrangian density would respect the nilpotent (anti-)BRST symmetry invariance.

In the language of the geometry on the $(4, 2)$-dimensional supermanifold, if the translation of the super Lagrangian densities along

(i) the $\bar\theta$-direction of the above supermanifold (without any translation along the θ-direction) is zero, the corresponding 4D Lagrangian density would possess the nilpotent BRST invariance,
(ii) the θ-direction of the above supermanifold (without any shift along the $\bar\theta$-direction) is zero, there will be nilpotent anti-BRST invariance for the 4D Lagrangian density of the theory, and
(iii) the θ- and $\bar\theta$-directions (one followed by the other; either ways) is zero, there would be existence of the (anti-)BRST symmetry invariance *together* for the 4D Lagrangian density of the theory.

We have been able to show in our present work (as well as in our earlier works [7, 8, 9, 10, 11, 12, 13, 14, 15]) that the nilpotent internal (anti-)BRST symmetry transformations $s_{(a)b}$ for the 4D theories are very intimately connected with the translational generators $(\partial_\theta, \partial_{\bar\theta})$ along the Grassmannian directions of the $(4, 2)$-dimensional supermanifold. Thus, one of the key features of our superfield approach to BRST formalism is the sure guarantee that the nilpotency (i.e. $s^2_{(a)b} = 0$) and the anticommutativity (ie. $s_b s_{ab} + s_{ab} s_b = 0$) properties would always be satisfied by the (anti-)BRST symmetry transformations $s_{(a)b}$.

We have already noted that, the above specific features are the integral ingredients of our superfield approach to BRST formalism because the translational generators $(\partial_\theta, \partial_{\bar\theta})$ always obey the nilpotency property $(\partial^2_\theta = 0, \partial^2_{\bar\theta} = 0)$ as well as the anticommutativity property (i.e. $\partial_\theta \partial_{\bar\theta} + \partial_{\bar\theta} \partial_\theta = 0$). We have been able to shed more light on it through the (anti-)BRST charges when we have expressed them in terms of the superfields and translational generators (see, Section 5). The anticommutativity and nilpotency ensue automatically.

The important geometrical consequences of these GIRs on the matter superfields (cf. (4.1) and (A.1) below) are as follows

(i) the Grassmannian independence of the Dirac part of the super Lagrangian density ($\tilde{\mathcal{L}}^{(d)}$) due to the application of (4.1), and
(ii) the Grassmannian independence of the kinetic energy term for the $\mathrm{U}(1)$ gauge superfield and the super Lagrangian density $\tilde{\mathcal{L}}^{(d)}$ due to the application of the GIR (A.1) (see the appendix below).

In fact, the GIR (A.1) (see the appendix below) on the matter superfields provides a generalization of the HC because it leads to results that are obtained due to the application of HC and GIR (4.1) separately.

One of the highlights of our present investigation is the simplicity and beauty that have been brought in for the (anti-)BRST invariance of the Lagrangian density of the 4D interacting $\mathrm{U}(1)$ gauge theory within the framework of the superfield approach to BRST formalism. It is nice to point out that our present work has *already* been generalized to

(i) the case of interacting Abelian $\mathrm{U}(1)$ gauge theory where there is coupling between the gauge field and complex scalar fields [17], and
(ii) the case of the interacting 4D non-Abelian gauge theory (with Dirac fields) which happens to be more general than our present theory [16].

We have also devoted time on the nilpotent (anti-)BRST as well as the nilpotent (anti-)co-BRST invariance of the 4D free Abelian 2-form gauge theory in the Lagrangian formalism [5]. It would be interesting endeavor to capture the nilpotent (anti-)BRST (as well as (anti-)co-BRST) invariance of the 4D (non-)Abelian 2-form (and still higher form gauge theories) within the framework of the superfield approach to BRST formalism. These are some of the issues that are presently under investigation.

Appendix A (Anti-)BRST transformations for all the fields from a single GIR on matter superfield

In this appendix, we concisely recapitulate some of the key points connected with the derivation of the results of Sections 3 and 4 from a single GIR on the superfields, defined on the $(4, 2)$-dimensional supermanifold [15]. This GIR also owes its origin to the (super) covariant derivatives but, in a form, that is quite different from (4.1). The explicit form of this GIR is

$$\bar{\Psi}(x,\theta,\bar{\theta})\tilde{D}\tilde{D}\Psi(x,\theta,\bar{\theta}) = \bar{\psi}(x)DD\psi(x), \tag{A.1}$$

where $\tilde{D}$ and D are the covariant derivatives defined on the $(4, 2)$-dimensional supermanifold and 4D flat Minkowski spacetime manifold, respectively[7]. These are defined as follows

$$\tilde{D} = \tilde{d} + ie\tilde{A}^{(1)}, \quad D = d + ieA^{(1)}, \quad d = dx^\mu \partial_\mu, \quad A^{(1)} = dx^\mu A_\mu, \tag{A.2}$$

where all the quantities, in the above, have been taken from the earlier Sections 3 and 4. For instance, equations (3.1), (3.3) and (4.2) have been used.

The above restriction is a GIR on the superfields (defined on the above supermanifold) because of the fact that the r.h.s. of (A.1) is

$$\bar{\psi}DD\psi = \frac{ie}{2}(dx^\mu \wedge dx^\nu)\bar{\psi}F_{\mu\nu}\psi \equiv ie\bar{\psi}F^{(2)}\psi. \tag{A.3}$$

It is straightforward to check that the above quantity is $\mathrm{U}(1)$ gauge invariant. A noteworthy point, at this stage, is that the r.h.s. is a 2-form with the differentials (i.e. $dx^\mu \wedge dx^\nu$) in the spacetime variables *only*. However, the l.h.s. of the equation (5.1) contains all the differential 2-forms in terms of the superspace variables. It is obvious that all the coefficients of the differentials in the Grassmannian variables of the l.h.s. will be set equal to zero.

It has been clearly demonstrated in our earlier work [15] that the outcomes of the above equality in (A.1) (i.e. GIR) are the relationships that we have already obtained separately and independently in (3.4) and (4.3). Thus, the expansions of the superfields, ultimately, reduce to the forms which can be expressed in terms of the appropriate (anti-)BRST symmetry transformations $s_{(a)b}$ as quoted in the key equations (3.5) and (4.4).

It is obvious, from our above discussions, that the total (anti-)BRST invariant Lagrangian density (2.1), defined in terms of the local fields (taking their values on the 4D flat spacetime manifold), can be recast in terms of the superfields (defined on the $(4, 2)$-dimensional supermanifold) by adding super Lagrangian densities given in equations (3.7) and (4.8) as

$$\tilde{\mathcal{L}}_T = \tilde{\mathcal{L}}^{(1,2,3)}_{(g)} + \tilde{\mathcal{L}}^{(d)}. \tag{A.4}$$

Now the nilpotent (anti-)BRST invariance of the Lagrangian density (2.1) can be expressed as (4.10) with the replacement: $\tilde{\mathcal{L}}^{(d)} \to \tilde{\mathcal{L}}_T$. Finally, we conclude that the Grassmannian independence of the super Lagrangian density encodes the (anti-)BRST invariance of the 4D Lagrangian density.

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