Ashdin Publishing
Journal of Physical Mathematics
Vol. 2 (2010), Article ID P100803, 22 pages
doi:10.4303/jpm/P100803

On deformed quantum mechanical schemes and $\star$-value equations based on the space-space noncommutative Heisenberg-Weyl group

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Abstract

We investigate the Weyl-Wigner-Gröenewold-Moyal, the Stratonovich, and the Berezin group quantization schemes for the space-space noncommutative Heisenberg-Weyl group. We show that the $\star$-product for the deformed algebra of Weyl functions for the first scheme is different than that for the other two, even though their respective quantum mechanics' are equivalent as far as expectation values are concerned, provided that some additional criteria are imposed on the implementation of this process. We also show that it is the $\star$-product associated with the Stratonovich and the Berezin formalisms that correctly gives the Weyl symbol of a product of operators in terms of the deformed product of their corresponding Weyl symbols. To conclude, we derive the stronger $\star$-valued equations for the 3 quantization schemes considered and discuss the criteria that are also needed for them to exist.

2000 MSC: 81Q99, 81R60, 81S30

1 Introduction

It is well known [15, 16, 27, 34, 35, 36] that for nonrelativistic standard quantum mechanics, the expectation value of an operator on Hilbert space can be formally represented as a statistical-like average of the corresponding Weyl phase-space function with the statistical density given by the Wigner function associated with the density matrix of the quantum state. Moreover, when applying this scheme to a product of two arbitrary operator functions of the quantum position and momentum operators, their corresponding Weyl phase-space function was given by the exponential of the Poisson bidifferential acting on the Weyl equivalent of each of the two operators. This correspondence between the product of quantum operators and the twisted product of their classical phase-space equivalents can be viewed as a deformation of the point product in the algebra $\mathcal{A}$ of C^∞ phase-space functions with the Gröenewold-Moyal multidifferential operator:

$$\star_\hbar := \exp\left[\frac{i\hbar}{2}\Lambda\right] := \exp\left[\frac{i\hbar}{2}\Big(\overleftarrow{\nabla}_{\mathbf{q}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{q}}\Big)\right],$$

inducing this deformation. This concept of a twisted product was given a more general mathematical framework by Bayen et al. in [5], whose proposed deformation quantization paradigm and noncommutative symbol calculus led to an autonomous reformulation of quantum theory directly in terms of phase-space functions, composed via the twisted or $\star$-product, instead of operators and Hilbert space states.

While applications of the original Weyl-Wigner-Gröenewold-Moyal (WWGM) formalism were restricted to the description of systems in flat phase space, the systems under consideration in the more general deformation quantization scheme possess an intrinsic group of symmetries with the phase-space being an homogenous manifold on which the group of transformations acts transitively [2, 4, 12, 17, 18, 22, 23, 24, 25, 26]. This implies then the possibility of extending the phase space approach to the "quantization" of curved spaces. However, for the various known versions of deformation theory, there are a large variety of $\star$-products which in turn imply, in general, different quantum mechanical theories for the same problem.

In order to deal with such nonuniqueness and arrive at a $\star$-product that would ensure the physical equivalence of deformation quantization with the ordinary quantum mechanics, the need for supplementary conditions has been suggested, so that the linear bijective mapping between operators on Hilbert space and classical functions on phase space can be implemented by a kernel operator which satisfies a number of physically sensible postulates thus hopefully providing a scheme to single out the most adequate symbol calculus from the many that have and could be proposed.

Moreover, such nonuniqueness becomes manifest even for quantum deformation schemes with known equivalent $\star$-products in flat space-time standard quantum mechanics, when space-space and/or space-time noncommutativity is incorporated into the formalism. This noncommutative quantum mechanics and the behavior of classical fields, defined as functions of noncommutative spatial variables, have been the object of a great deal of attention in the last years. Physicists became attracted to the more mathematical aspects of deformation quantization with the hope that such theories would provide the tools needed to remove the singularities in physical field theories without the need of renormalization. Although these expectations have not materialized up to now, noncommutative field theory and its quantum mechanical minisuperspace have led to many new and interesting results. In particular, in the context of string theory, there has been a lot of interest in studying solitonic solutions of noncommutative field theory [3, 13, 14, 19, 30]. Also motivated by that work, but in a somewhat different direction, coherent structures in the form of noncommutative solitons and vortices were studied by the authors in a recent collaboration [21]. It was shown there that the noncommutativity of the spatial variables, when averaged with vortex or plateau-type coherent states, induced an effective lattice structure of Landau cells whose distribution and size depended on the coherent states considered. This shows that the effect of the noncommutativity on coherent structures, with an amplitude comparable to the scale parameter θ of noncommutativity of the $\star$-product, is to induce a behavior of classical structures in a physical lattice whose dynamics can be described in terms of a Peierls-Nabarro potential. It would not be unreasonable to expect that such dynamical creation of lattice structures as an effect of the noncommutativity on coherent states, which mathematically would be reflected in the replacement of differential field equations by equations of differences, could be related to another important quantization scheme known as loop quantum gravity. This final objective forms part of an ongoing program initiated in [21], and it is within that much wider context that the present work is intended.

Thus, in order to arrive at an identification of the $\star$-product appropriate for the above mentioned program, we will here specifically start by extending the WWGM procedure in order to analyze a space-space noncommutative Heisenberg-Weyl algebra (again, noncommutativity being understood here as a nonvanishing commutator between the operators of spatial coordinates or momenta) in order to obtain the generalization of the well-known expressions of the Heisenberg-Weyl algebra of usual quantum mechanics. Afterwards, we will apply to this same Lie algebra two quantization formalisms which are purportedly more general and that were developed to provide a quantization scheme even for curved spaces. The first one started with the work of Stratonovich [31] and was further developed elsewhere [8, 11, 33]. The second corresponds to the Berezin geometric quantization program of covariant and contravariant symbols for Kähler manifolds [6]. Finally, we derive the additional specific requirements that need to be imposed on these different schemes, in order to obtain $\star$-valued equations which constitute a stronger quantization requirement, as they relate eigenvalues of the physical states appearing in the density matrix to the Weyl equivalents of the operator observables.

2 The WWGM phase-space quantum mechanics based on the space-space noncommutative Heisenberg-Weyl Lie algebra

By a space-space (and/or momentum-momentum) noncommutative Heisenberg-Weyl algebra, we understand [29] the algebra of position and momentum operators satisfying the commutation relations:

$$\big[\hat{R}_i, \hat{R}_j\big] = i\theta_{ij}\hat{I}, \quad \big[\hat{P}_i, \hat{P}_j\big] = i\hbar\bar{\theta}_{ij}\hat{I}, \quad \big[\hat{R}_i, \hat{P}_j\big] = i\hbar\delta_{ij}\hat{I}, \tag{2.1}$$

where $\hat{R}_i$, $\hat{P}_i$, $i = 1, \dots, d$ are the components of the position and momentum quantum operators, respectively, with component eigenvalues on $\mathbb{R}^d$, the identity $\hat{I}$ is the central element of the algebra, and θ_{ij} and $\bar{\theta}_{ij}$ are evidently antisymmetric matrices, which in the most general case can be functions of the generators of the above algebra. For our present purposes and algebraic simplicity, in what follows, we will set $\bar{\theta}_{ij} = 0$ and $d = 2$ and consider only the zeroth order constant term of the Taylor expansion of $\theta_{12} \equiv \theta$.

From an intrinsically noncommutative operator point of view, the development of a formulation for the quantum mechanics based on the above Heisenberg-Weyl algebra of operators requires first a specification of a representation for the generators of the algebra, second a specification of the Hamiltonian which governs the time evolution of the system, and last a specification of the Hilbert space on which these operators and the other observables of the theory act. As for the choice of the Hilbert space, a reasonable assumption is that it can be taken to be the same as that for the corresponding system in the usual quantum mechanics, but for a realization of the space-space noncommutative Heisenberg-Weyl algebra, because of the noncommutativity (2.1), we cannot use configuration space as a basis. We can use, however, for a basis either of the eigenkets $|p_1, p_2\rangle$, $|q_1, p_2\rangle$, $|q_2, p_1\rangle$, of the commuting pairs of observables $(\hat{P}_1, \hat{P}_2)$, $(\hat{R}_1, \hat{P}_2)$, or $(\hat{R}_2, \hat{P}_1)$, respectively, or any combination of the (R, P) such that they form a complete set of commuting observables.

Specifically, we choose as the realization of our Heisenberg-Weyl algebra the one based on $|q_1, p_2\rangle$. The construction follows standard procedures (cf., e.g., [20]) and it is detailed in [29]. We then have that $\hat{R}_2$ in this basis is realized by

$$\hat{R}_2 = -i\theta\partial_{q_1} + i\hbar\partial_{p_2}, \tag{2.2}$$

$$\hat{P}_1 = -i\hbar\partial_{q_1}. \tag{2.3}$$

The representations for the remainder of the generators $\hat{R}_1$ and $\hat{P}_2$ of the algebra are obviously just multiplicative. Note that the change of basis $|q_1, p_2\rangle \to |q_2, p_1\rangle$ follows directly from the transition function $\langle q_1, p_2 | q_2, p_1 \rangle$, which is derived [1] by noting that

$$\begin{aligned}\big\langle q_1, p_2|\hat{R}_2|q_2, p_1\big\rangle &= q_2\big\langle q_1, p_2|q_2, p_1\big\rangle = i\big(\hbar\partial_{p_2} - \theta\partial_{q_1}\big)\big\langle q_1, p_2|q_2, p_1\big\rangle,\\ \big\langle q_1, p_2|\hat{P}_1|q_2, p_1\big\rangle &= p_1\big\langle q_1, p_2|q_2, p_1\big\rangle = -i\hbar\partial_{q_1}\big\langle q_1, p_2|q_2, p_1\big\rangle.\end{aligned}$$

Combining these two expressions yields

$$\big(\hbar q_2 - \theta p_1\big)\big\langle q_1, p_2|q_2, p_1\big\rangle = i\hbar\partial_{p_2}\big\langle q_1, p_2|q_2, p_1\big\rangle,$$

which can be readily solved to give, after normalization, the following:

$$\big\langle q_1, p_2|q_2, p_1\big\rangle = \frac{1}{2\pi\hbar}\exp\left[-\frac{i}{\hbar}\left(q_2 p_2 - \frac{\theta}{\hbar}p_1 p_2 - q_1 p_1\right)\right]. \tag{2.4}$$

Since the displacement operators $\{(2\pi\hbar)^{-1}\exp[\frac{i}{\hbar}(\mathbf{y}\cdot\hat{\mathbf{R}} + \mathbf{x}\cdot\hat{\mathbf{P}})]\}$, where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, form a complete orthonormal set in the space-space noncommutative Heisenberg algebra any Schrödinger operator (which may depend explicitly on time), $A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t)$ can be written as follows:

$$A\big(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t\big) = \iint d\mathbf{x}\, d\mathbf{y}\alpha(\mathbf{x}, \mathbf{y}, t)\exp\left[\frac{i}{\hbar}\left(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}}\right)\right], \tag{2.5}$$

where the c-function $\alpha(\mathbf{x}, \mathbf{y}, t)$ is determined by

$$\alpha(\mathbf{x}, \mathbf{y}, t) = (2\pi\hbar)^{-2}\,\mathrm{Tr}\left\{A\big(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t\big)\exp\left[-\frac{i}{\hbar}\big(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}}\big)\right]\right\}. \tag{2.6}$$

The Weyl function corresponding to the quantum operator $A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t)$ is then given by

$$\begin{aligned}W_A\big(\mathbf{p}, \mathbf{q}, t\big) &= \iint d\mathbf{x}\, d\mathbf{y}\alpha(\mathbf{x}, \mathbf{y}, t)\exp\left[\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p} + \mathbf{y}\cdot\mathbf{q})\right]\\ &= \iint dx_1 dy_2 e^{\frac{i}{\hbar}(x_1 p_1 + y_2 q_2)}\\ &\quad\times\left\langle q_1 - \frac{x_1}{2} - \frac{\theta y_2}{2\hbar}, p_2 + \frac{y_2}{2}|\hat{A}|q_1 + \frac{x_1}{2} + \frac{\theta y_2}{2\hbar}, p_2 - \frac{y_2}{2}\right\rangle.\end{aligned} \tag{2.7}$$

To derive the expectation value of a product of two Schrödinger operators, one writes the expectation value of the product in terms of the von Neumann density matrix $\boldsymbol{\rho}$ as follows:

$$\big\langle \hat{A}_1\hat{A}_2\big\rangle = \mathrm{Tr}\,\big[\boldsymbol{\rho}\hat{A}_1\hat{A}_2\big], \tag{2.8}$$

and evaluates the trace in the above chosen basis. Thus by using completeness of the basis $|q_1, p_2\rangle$ and substituting (2.5) for the operators $\hat{A}_1$ and $\hat{A}_2$, equation (2.8) then becomes

$$\begin{aligned}\big\langle \hat{A}_1\hat{A}_2\big\rangle = \int & d\mathbf{x}\, d\mathbf{y}\, d\mathbf{u}\, d\mathbf{v}\, dq_1\, dp_2\, dq_1'\, dp_2'\, dq_1''\, dp_2''\big\langle q_1, p_2|\boldsymbol{\rho}|q_1', p_2'\big\rangle\alpha_1(\mathbf{x}, \mathbf{y}, t)\alpha_2(\mathbf{u}, \mathbf{v}, t)\\ &\times\big\langle q_1', p_2'|e^{\frac{i}{\hbar}(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}})}|q_1'', p_2''\big\rangle\big\langle q_1'', p_2''|e^{\frac{i}{\hbar}(\mathbf{u}\cdot\hat{\mathbf{P}} + \mathbf{v}\cdot\hat{\mathbf{R}})}|q_1, p_2\big\rangle.\end{aligned}$$

Moreover, resorting to the Baker-Campbell-Hausdorff theorem, making use of (2.4), and performing the integrals over q_1', p_2', q_1'' and p_2'', we obtain

$$\begin{aligned}\big\langle \hat{A}_1\hat{A}_2\big\rangle = \int d\mathbf{x}\,d\mathbf{y}\,d\mathbf{u}\,d\mathbf{v}\,dq_1\,dp_2\Big\langle q_1,p_2|\boldsymbol{\rho}|q_1-x_1-u_1 \\ -\frac{v_2\theta}{\hbar}-\frac{y_2\theta}{\hbar},p_2+y_2+v_2\Big\rangle\alpha_1(\mathbf{x},\mathbf{y},t)\alpha_2(\mathbf{u},\mathbf{v},t) \\ \times\exp\Bigg[\frac{i}{\hbar}\Big(y_1q_1-y_1u_1+v_1q_1+x_2p_2+x_2v_2+u_2p_2 \\ -\frac{y_1x_1}{2}+\frac{y_2x_2}{2}-\frac{v_1u_1}{2}+\frac{u_2v_2}{2}\Big)\Bigg] \\ \times\exp\Bigg[\frac{i}{\hbar}\Big(-\frac{\theta}{\hbar}y_1v_2-\frac{\theta}{2\hbar}y_1y_2-\frac{\theta}{2\hbar}v_1v_2\Big)\Bigg].\end{aligned}\tag{2.9}$$

Making now the change of variables $q_1=\xi$, $p_2=\eta$ and substituting $\alpha_1(\mathbf{x},\mathbf{y},t)$ and $\alpha_2(\mathbf{u},\mathbf{v},t)$ in terms of their corresponding Weyl functions, equation (2.9) becomes

$$\begin{aligned}\big\langle \hat{A}_1\hat{A}_2\big\rangle = \Big(\frac{1}{2\pi\hbar}\Big)^8\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,d\mathbf{x}\,d\mathbf{y}\,d\mathbf{u}\,d\mathbf{v}\,d\xi\,d\eta \\ \times\Big\langle \xi,\eta|\boldsymbol{\rho}|\xi-x_1-u_1-\frac{v_2\theta}{\hbar}-\frac{y_2\theta}{\hbar},\eta+y_2+v_2\Big\rangle \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)\exp\Bigg[\frac{i}{\hbar}y_1\Big(\xi-u_1-\frac{\theta}{\hbar}v_2-\frac{x_1}{2}-\frac{\theta}{2\hbar}y_2-q_1\Big)\Bigg] \\ \times\exp\Bigg[\frac{i}{\hbar}v_1\Big(\xi-\frac{u_1}{2}-\frac{\theta}{2\hbar}v_2-q_1'\Big)\Bigg]e^{\frac{i}{\hbar}v_2(x_2+\frac{u_2}{2}-q_2')}e^{\frac{i}{\hbar}y_2(\frac{x_2}{2}-q_2)} \\ \times e^{-\frac{i}{\hbar}x_1p_1}e^{-\frac{i}{\hbar}u_1p_1'}e^{-\frac{i}{\hbar}x_2(p_2-\eta)}e^{-\frac{i}{\hbar}u_2(p_2'-\eta)}.\end{aligned}$$

Next, we integrate over y_1, x_2, v_1, u_2, u_1, v_2, ξ, and η to get

$$\begin{aligned}\langle \hat{A}_1\hat{A}_2\rangle = \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ \times\Big\langle 2q_1'-q_1-\frac{x_1}{2}-\frac{\theta y_2}{2\hbar},2p_2'-p_2+\frac{y_2}{2}|\boldsymbol{\rho}|q_1-\frac{x_1}{2}-\frac{\theta y_2}{2\hbar},p_2+\frac{y_2}{2}\Big\rangle \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}y_2q_2}e^{-\frac{i}{\hbar}x_1p_1} \\ \times e^{-\frac{i}{\hbar}q_2'(2p_2-2p_2'-y_2)}e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1-\frac{2\theta}{\hbar}p_2+\frac{2\theta}{\hbar}p_2'-x_1)}.\end{aligned}$$

Observe now that this expression can also be written as follows:

$$\begin{aligned}\big\langle \hat{A}_1\hat{A}_2\big\rangle = \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ \times\Bigg[e^{\frac{\theta y_2}{\hbar}\partial_{x_1}}\Big\langle 2q_1'-q_1-\frac{x_1}{2},2p_2'-p_2+\frac{y_2}{2}|\boldsymbol{\rho}|q_1-\frac{x_1}{2},p_2+\frac{y_2}{2}\Big\rangle\Bigg] \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}y_2q_2}e^{-\frac{i}{\hbar}x_1p_1} \\ \times e^{-\frac{i}{\hbar}q_2'(2p_2-2p_2'-y_2)}e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1-\frac{2\theta}{\hbar}p_2+\frac{2\theta}{\hbar}p_2'-x_1)},\end{aligned}$$

and after integrating by parts, we obtain

$$\begin{aligned}\big\langle \hat{A}_1\hat{A}_2\big\rangle = \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ \times\left\langle 2q_1'-q_1-\frac{x_1}{2}, 2p_2'-p_2+\frac{y_2}{2}|\boldsymbol{\rho}|q_1-\frac{x_1}{2}, p_2+\frac{y_2}{2}\right\rangle \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}y_2q_2}e^{-\frac{i}{\hbar}q_2'(2p_2-2p_2'-y_2)}e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1)} \\ \times e^{\frac{i}{\hbar}x_1(p_1'-p_1)}e^{-\frac{i}{\hbar^2}\theta y_2(p_1'-p_1)}e^{\frac{2i}{\hbar^2}\theta p_1'(p_2-p_2')}.\end{aligned} \tag{2.10}$$

To reconstruct the star product that should arise from this formulation, we use the following identities:

$$\begin{aligned}&e^{-\frac{\theta}{\hbar}p_1'\partial_{q_2'}}e^{\frac{i}{\hbar}q_2'y_2} = e^{\frac{i}{\hbar}q_2'y_2}e^{-\frac{i\theta}{\hbar^2}y_2p_1'}, \quad e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}e^{-\frac{i}{\hbar}q_2y_2} = e^{-\frac{i}{\hbar}q_2y_2}e^{\frac{i\theta}{\hbar^2}y_2p_1},\\ &e^{-\frac{\theta}{\hbar}p_1'\partial_{q_2'}}e^{-\frac{2i}{\hbar}q_2'(p_2-p_2')} = e^{-\frac{2i}{\hbar}(p_2-p_2')(q_2'-\frac{\theta}{\hbar}p_1')},\end{aligned}$$

so that (2.10) becomes

$$\begin{aligned}\langle \hat{A}_1\hat{A}_2\rangle = \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ \times\left\langle 2q_1'-q_1-\frac{x_1}{2}, 2p_2'-p_2+\frac{y_2}{2}|\boldsymbol{\rho}|q_1-\frac{x_1}{2}, p_2+\frac{y_2}{2}\right\rangle \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1)}e^{\frac{i}{\hbar}x_1(p_1'-p_1)} \\ \times e^{-\frac{\theta}{\hbar}p_1'\partial_{q_2'}}\big(e^{\frac{i}{\hbar}q_2'y_2}e^{-\frac{2i}{\hbar}q_2'(p_2-p_2')}\big)\big(e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}e^{-\frac{i}{\hbar}q_2y_2}\big).\end{aligned}$$

After integrating by parts, the above equation reads

$$\begin{aligned}\big\langle \hat{A}_1\hat{A}_2\big\rangle = \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}d\mathbf{q}d\mathbf{p}'d\mathbf{q}'dx_1dy_2 \\ \times\left\langle 2q_1'-q_1-\frac{x_1}{2}, 2p_2'-p_2+\frac{y_2}{2}|\boldsymbol{\rho}|q_1-\frac{x_1}{2}, p_2+\frac{y_2}{2}\right\rangle \\ \times W_{A_1}\Big(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,T\Big)W_{A_2}\Big(\mathbf{p}',q_1',q_2'+\frac{\theta}{\hbar}p_1',T\Big)e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1)} \\ \times e^{\frac{i}{\hbar}x_1(p_1'-p_1)}e^{\frac{i}{\hbar}y_2(q_2'-q_2)}e^{-\frac{2i}{\hbar}q_2'(p_2-p_2')}.\end{aligned}$$

Now make the following change of variables:

$$\begin{aligned}&x_1 = 2q_1-2z_1, \quad y_2 = 2z_2-2p_2, \quad q_1' = q_1+\mu_1,\\ &q_2' = q_2+\mu_2, \quad p_1' = p_1+\nu_1, \quad p_2' = p_2+\nu_2\end{aligned}$$

to obtain

$$\begin{aligned}\big\langle \hat{A}_1\hat{A}_2\big\rangle = \frac{16}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mu_1\,d\mu_2\,d\nu_1\,d\nu_2\,dz_1\,dz_2 \\ \times\big\langle z_1+2\mu_1, z_2+2\nu_2|\boldsymbol{\rho}|z_1,z_2\big\rangle e^{-\frac{2i}{\hbar}\mu_1p_1}e^{\frac{2i}{\hbar}\nu_2q_2} \\ \times e^{-\frac{2i}{\hbar}\nu_1(\mu_1-q_1+z_1)}e^{-\frac{2i}{\hbar}\mu_2(p_2-\nu_2-z_2)}W_{A_1}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right) \\ \times e^{\nu_1\partial_{p_1}}e^{\nu_2\partial_{p_2}}e^{\mu_1\partial_{q_1}}e^{\mu_2\partial_{q_2}}W_{A_2}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right).\end{aligned} \tag{2.11}$$

But

$$
\begin{aligned}
e^{\frac{2i}{\hbar}q_1\nu_1}e^{\nu_1\overrightarrow{\partial}_{p_1}}W_{A_2} &= e^{\frac{2i}{\hbar}q_1\nu_1}e^{-\frac{i\hbar}{2}\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{p_1}}W_{A_2},\\
e^{\frac{2i}{\hbar}q_2\nu_2}e^{\nu_2\overrightarrow{\partial}_{p_2}}W_{A_2} &= e^{\frac{2i}{\hbar}q_2\nu_2}e^{-\frac{i\hbar}{2}\overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{p_2}}W_{A_2},\\
e^{-\frac{2i}{\hbar}p_1\mu_1}e^{\mu_1\overrightarrow{\partial}_{q_1}}W_{A_2} &= e^{-\frac{2i}{\hbar}p_1\mu_1}e^{\frac{i\hbar}{2}\overleftarrow{\partial}_{p_1}\overrightarrow{\partial}_{q_1}}W_{A_2},\\
e^{-\frac{2i}{\hbar}p_2\mu_2}e^{\mu_2\overrightarrow{\partial}_{q_2}}W_{A_2} &= e^{-\frac{2i}{\hbar}p_2\mu_2}e^{\frac{i\hbar}{2}\overleftarrow{\partial}_{p_2}\overrightarrow{\partial}_{q_2}}W_{A_2},
\end{aligned}
$$

which, when substituted into (2.11) and integrated by parts, results in

$$
\begin{aligned}
\langle \hat{A}_1\hat{A}_2\rangle = \frac{16}{(2\pi\hbar)^4}\int & d\mathbf{p}\,d\mathbf{q}\,d\mu_1\,d\mu_2\,d\nu_1\,d\nu_2\,dz_1\,dz_2\\
&\times \langle z_1+2\mu_1, z_2+2\nu_2|\boldsymbol{\rho}|z_1,z_2\rangle e^{-\frac{2i}{\hbar}\mu_1 p_1}e^{\frac{2i}{\hbar}\nu_2 q_2}\\
&\times e^{-\frac{2i}{\hbar}\nu_1(\mu_1-q_1+z_1)}e^{-\frac{2i}{\hbar}\mu_2(p_2-\nu_2-z_2)}\\
&\times \left[W_{A_1}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\star_\hbar W_{A_2}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\right].
\end{aligned}
\tag{2.12}
$$

Last, integrating over ν_1, μ_2, μ_1, and ν_2 and performing the final change of variables $z_1 = q_1+\frac{s_1}{2}$, $z_2 = p_2+\frac{s_2}{2}$, equation (2.12) takes the following form:

$$
\begin{aligned}
\langle \hat{A}_1\hat{A}_2\rangle = \frac{1}{(2\pi\hbar)^2}\int & d\mathbf{p}\,d\mathbf{q}\,ds_1\,ds_2\left\langle q_1-\frac{s_1}{2},p_2-\frac{s_2}{2}|\boldsymbol{\rho}|q_1+\frac{s_1}{2},p_2+\frac{s_2}{2}\right\rangle e^{\frac{i}{\hbar}s_1p_1}\\
&\times e^{-\frac{i}{\hbar}s_2q_2}\left[W_{A_1}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\star_\hbar W_{A_2}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\right].
\end{aligned}
\tag{2.13}
$$

Recalling the definition of the Wigner function:

$$
\rho_w(\mathbf{p},\mathbf{q}) := \frac{1}{(2\pi\hbar)^2}\int ds_1\,ds_2\left\langle q_1-\frac{s_1}{2},p_2-\frac{s_2}{2}|\boldsymbol{\rho}|q_1+\frac{s_1}{2},p_2+\frac{s_2}{2}\right\rangle e^{\frac{i}{\hbar}s_1p_1}e^{-\frac{i}{\hbar}s_2q_2}, \tag{2.14}
$$

equation (2.13) may be expressed in the following compact form:

$$
\langle \hat{A}_1\hat{A}_2\rangle = \int d\mathbf{p}\,d\mathbf{q}\,\rho_w(\mathbf{p},\mathbf{q})\left[W_{A_1}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\star_\hbar W_{A_2}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\right], \tag{2.15}
$$

where

$$
\star_\hbar := \exp\left[\sum_{i=1,2}\frac{i\hbar}{2}\left(\overleftarrow{\partial}_{q_i}\overrightarrow{\partial}_{p_i}-\overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{q_i}\right)\right]. \tag{2.16}
$$

Consequently, in the phase-space formulation of quantum mechanics based on the algebra (2.1), the algebra of Weyl functions is deformed by a $\star$-product defined by

$$
\begin{aligned}
W_{A_1}\star W_{A_2} := m\circ\Big[& e^{\sum_{i=1,2}\frac{i\hbar}{2}(\partial_{q_i}\otimes\partial_{p_i'}-\partial_{q_i'}\otimes\partial_{p_i})}\circ e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}\\
&\otimes e^{\frac{\theta}{\hbar}p_1'\partial_{q_2'}}W_{A_1}(\mathbf{p},\mathbf{q})\otimes W_{A_2}(\mathbf{p}',\mathbf{q}')\Big]_{\mathbf{q},\mathbf{p}=\mathbf{q}',\mathbf{p}'}.
\end{aligned}
\tag{2.17}
$$

In addition, by a similar calculation to the one above, we can show that the Weyl symbol:

$$
W_\rho(\mathbf{p},\mathbf{q}) = (2\pi\hbar)^{-2}\int d\mathbf{x}\,d\mathbf{y}\ \mathrm{Tr}\left[\boldsymbol{\rho}e^{-\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{P}+\mathbf{y}\cdot\mathbf{R})}\right]e^{\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p}+\mathbf{y}\cdot\mathbf{q})} \tag{2.18}
$$

associated with the density matrix $\boldsymbol{\rho}$ is related to the Wigner function by

$$W_\rho(\mathbf{p},\mathbf{q}) = e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_w(\mathbf{p},\mathbf{q}). \tag{2.19}$$

Hence for the space-space noncommutative Heisenberg-Weyl algebra, the Weyl symbol of the density matrix and the Wigner function as defined in (2.14) are not the same, contrary from what is the case for the usual quantum mechanics Heisenberg algebra:

$$W_\rho(\mathbf{p},\mathbf{q}) \xrightarrow{\theta\to 0} \rho_w(\mathbf{p},\mathbf{q}).$$

Note now that if we substitute (2.19) into (2.15) and integrate by parts, we get

$$\begin{aligned}
\langle \hat{A}_1\hat{A}_2\rangle &= \int d\mathbf{p}\,d\mathbf{q} W_\rho(\mathbf{p},\mathbf{q}) e^{-\frac{\theta}{\hbar}p_1\overrightarrow{\partial}_{q_2}}\Big[W_{A_1}\Big(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\Big)\star_\hbar W_{A_2}\Big(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\Big)\Big] \\
&= \int d\mathbf{p}\,d\mathbf{q} W_\rho(\mathbf{p},\mathbf{q}) e^{-\frac{\theta}{\hbar}p_1\overrightarrow{\partial}_{q_2}} \\
&\quad \times \Bigg[W_{A_1}\Bigg(p_1-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}, p_2-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_2}, q_1, q_2+\frac{i\hbar}{2}\overrightarrow{\partial}_{p_2}+\frac{\theta}{\hbar}\Bigg(p_1-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}\Bigg),t\Bigg) \\
&\qquad \times W_{A_2}\Bigg(\mathbf{p},q_1-\frac{i\hbar}{2}\overleftarrow{\partial}_{p_1}, q_2+\frac{\theta}{\hbar}p_1,t\Bigg)\Bigg] \\
&= \int d\mathbf{p}\,d\mathbf{q} W_\rho(\mathbf{p},\mathbf{q})\big[W_{A_1}(\mathbf{p},\mathbf{q},t)\star_\theta\circ\star_\hbar W_{A_2}(\mathbf{p},\mathbf{q},t)\big],
\end{aligned} \tag{2.20}$$

where

$$\star_\theta\circ\star_\hbar := e^{\frac{i\theta}{2}(\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{q_2}-\overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{q_1})}\circ\exp\Bigg[\sum_{i=1,2}\frac{i\hbar}{2}\Big(\overleftarrow{\partial}_{q_i}\overrightarrow{\partial}_{p_i}-\overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{q_i}\Big)\Bigg]. \tag{2.21}$$

Clearly, the expectation values obtained from (2.13) and (2.20) are the same. However, since for the space-space noncommutative Heisenberg-Weyl algebra the Wigner function associated with the density matrix $\hat{\boldsymbol{\rho}}$ and its corresponding Weyl symbol are not the same, the twistings in (2.18) and (2.20) of the product of Weyl symbols of two arbitrary operators do not agree in general. Their explicit forms are obviously basis dependent as well as dependent on whether averaging is done relative to the Wigner function or the Weyl symbol of the density matrix.

Furthermore, given the two different $\star$-products (2.17) and (2.21) of a pair of Weyl-symbols, it is pertinent to inquire which of them corresponds to the Weyl-symbol of a product of two operators. To answer this question univocally, we need to make use of (2.4), (2.6), and (2.7). After a rather lengthy but fairly direct calculation, one can show that

$$W_{A_1A_2} = W_{A_1}\star_\theta\circ\star_\hbar W_{A_2}. \tag{2.22}$$

So, for the quantum mechanics based on the space-space noncommutative Heisenberg-Weyl Lie group, we need to make iterative use of (2.22) for the calculation of Weyl-symbols corresponding to quantum operators. In particular, note that the Weyl-symbol corresponding to an operator $\hat{A}_1 = \hat{A}_1(\hat{\mathbf{P}})$ which is a function only of the momenta operators is given by the c-function $W_{A_1}(\mathbf{p})$ having the same functional form as the quantum operator, as it is the case in the usual WWGM quantum mechanics. However, for q-functions of the position operators, this is not always true for the space-space noncommutative Heisenberg-Weyl group, as can be easily seen, when consider, for example, the Weyl-symbol associated with the operator $\hat{R}_1\hat{R}_2$, for which (2.22) yields $W_{R_1R_2} = (q_1 + i\frac{\theta}{2}\partial_{q_2})q_2 = q_1q_2 + i\frac{\theta}{2}$.

From a statistical point of view, both the Wigner function (2.14) and the Weyl symbol (2.18) for the density matrix admit a quasiprobabilistic interpretation, although the projected density probabilities are not all the same. Indeed, projecting (2.14) onto the plane $q_1 - p_2$ (i.e., integrating over q_2, p_1) immediately yields

$$\int dp_1\, dq_2 \rho_w(\mathbf{p},\mathbf{q}) = \langle q_1, p_2|\hat{\boldsymbol{\rho}}|q_1, p_2\rangle,$$

while projecting onto the $q_2 - p_1$ plane by making use of (2.4) results in

$$\int dp_2\, dq_1 \rho_w(\mathbf{p},\mathbf{q}) = \langle q_2 + (\theta/\hbar)p_1, p_1|\hat{\boldsymbol{\rho}}|q_2 + (\theta/\hbar)p_1, p_1\rangle.$$

However, if we perform the same calculations for the corresponding Weyl symbol, we find

$$\int dp_1\, dq_2 W_\rho(\mathbf{p},\mathbf{q}) = \langle q_1, p_2|\hat{\boldsymbol{\rho}}|q_1, p_2\rangle, \qquad \int dp_2\, dq_1 W_\rho(\mathbf{p},\mathbf{q}) = \langle q_2, p_1|\hat{\boldsymbol{\rho}}|q_2, p_1\rangle.$$

Let us now see how the above results compare with the ones resulting from applying the Stratonovich-Weyl correspondence and the Berezin geometric quantization to the space-space noncommutative Heisenberg-Weyl Lie group.

3 The Stratonovich-Weyl correspondence for the space-space noncommutative Heisenberg-Weyl Lie group

In order to make our discussion self-contained and fix notation, we begin by summarizing the essential elements of the Stratonovich-Weyl correspondence. For a considerably more ample presentation of this formalism, we refer the reader to the work in [8, 11, 31, 33].

Let X be an even dimensional homogenous space given by the quotient G/H, where G is a simply connected Lie group (of finite dimension n) describing the dynamical symmetry of a given quantum system, and $H \subset G$ its isotropy subgroup. If X is given a Kählerian structure, then it can be interpreted as the phase space of a classical dynamical system. The mapping $\Omega \to |\Omega\rangle\langle\Omega|$, where $\Omega = \Omega(g)$ is a point in X and $g \in G$, is the geometric quantization for this system [6].

The Stratonovich generalization of the standard Gröenewold-Moyal quantization to quantum systems possessing and intrinsic group G of symmetries is based on the following postulates:

(i) linearity: there is a one-to-one map $\hat{A} \to W_A(\Omega)$;
(ii) reality: $W_{A^\dagger}(\Omega) = [W_A(\Omega)]^*$;
(iii) standardization: $\int_X d\mu(\Omega) W_A(\Omega) = \mathrm{Tr}\,\hat{A}$, where $d\mu(\Omega)$ is the invariant space measure;
(iv) traciality: $\int_X d\mu(\Omega) W_{A_1}(\Omega) W_{A_2}(\Omega) = \mathrm{Tr}(\hat{A}_1\hat{A}_2)$.
(v) covariance: $W_{g\cdot A}(\Omega) = W_A(g^{-1}\cdot\Omega)$, where $g \cdot A$ denotes the adjoint action of a unitary irreducible representation π of G on $\hat{A}$.

A function $W_A(\Omega)$ satisfying these five properties is known as the Stratonovich-Weyl (SW) symbol associated with a quantum operator $\hat{A}$ acting on Hilbert space. The linearity map is implemented by means of the generalized Weyl rule:

$$W_A(\Omega) = \mathrm{Tr}\left[\hat{A}\Delta(\Omega)\right], \tag{3.1}$$

where $\Delta(\Omega)$ is the Stratonovich-Weyl Kernel which is an operator-valued function on X. By virtue of the tracial property, we have that

$$\mathrm{Tr}\left[\hat{A}\Delta(\Omega)\right] = \int_x d\mu(\Omega')W_A(\Omega')W_{\Delta(\Omega)}(\Omega') = \int_X d\mu(\Omega')\,\mathrm{Tr}\left[\hat{A}\Delta(\Omega')\right]W_{\Delta(\Omega)}(\Omega'), \quad (3.2)$$

where $W_{\Delta(\Omega)}(\Omega')$ is the Weyl-equivalent of the Stratonovich Kernel. From (3.2), we infer that

$$\Delta(\Omega) = \int_X d\mu(\Omega')\Delta(\Omega')W_{\Delta(\Omega)}(\Omega'), \quad (3.3)$$

so that the function

$$K(\Omega,\Omega') := W_{\Delta(\Omega)}(\Omega') = \mathrm{Tr}\left[\Delta(\Omega)\Delta(\Omega')\right] \quad (3.4)$$

behaves as a Dirac delta function on the manifold X. Consequently, making use of this property, the Weyl rule (3.1) may be inverted to give the following:

$$\hat{A} = \int_X d\mu(\Omega)W_A(\Omega)\Delta(\Omega). \quad (3.5)$$

Furthermore, from (3.1), (3.3), and (3.4), the SW-postulates (ii)–(v) translate to the following conditions on the SW-kernel operator:

(iib) $\Delta(\Omega) = [\Delta(\Omega)]^\dagger$, $\forall\Omega\in X$;
(iiib) $\int_X d\mu(\Omega)\Delta(\Omega) = I$;
(ivb) $\int_X d\mu(\Omega')\,\mathrm{Tr}[\Delta(\Omega)\Delta(\Omega')]\Delta(\Omega') = \Delta(\Omega)$;
(vb) $\Delta(g\cdot\Omega) = \pi(g)\Delta(\Omega)\pi(g)^{-1}$.

In terms of the formalism of coherent states [9, 10, 28], we have that, whenever the Peter-Weyl theorem applies [11, 33], the SW kernel $\Delta(\Omega)$, satisfying the above conditions, can be given explicitly as [8]

$$\Delta(\Omega) = \sum_\nu Y^*_\nu(\Omega)D_\nu = \sum_\nu Y_\nu(\Omega)D^\dagger_\nu. \quad (3.6)$$

Here,

$$D_\nu := \int_X d\mu(\Omega)Y_\nu(\Omega)|\Omega\rangle\langle\Omega| \quad (3.7)$$

denotes a set of operators acting on the Hilbert space $\mathcal{H}$. The harmonic functions $Y_\nu(\Omega)$, which form a complete orthonormal basis in $L^2(X,\mu)$, are eigenfunctions of the Laplace-Beltrami operator $(\delta d + d\delta)$ associated with the space X, while the index ν is, in general, a composite label. We would like to stress here, as it should have already become evident from our previous considerations, that since we are always going from the quantum mechanics of operators and Hilbert space to classical phase space averages, our Weyl correspondences are surjective and therefore unique maps (to a given quantum operator there corresponds a unique Weyl function, which corresponds to the case $s = 0$ for the families of operators and functions considered in [8]).

Note now that when substituting (3.6) and (3.7) in (3.1), we get

$$W_A(\Omega) = \sum_\nu Y^*_\nu(\Omega)\mathcal{A}_\nu = \sum_\nu Y_\nu(\Omega)\tilde{\mathcal{A}}_\nu,$$

where

$$\mathcal{A}_\nu = \mathrm{Tr}(\hat{A}D_\nu), \quad \tilde{\mathcal{A}}_\nu = \mathrm{Tr}\left(\hat{A}D_\nu^\dagger\right).$$

The generalized twisted product of two SW-symbols follows directly from (3.5) and the above and is given by

$$\begin{aligned} W_A(\Omega) \star_S W_B(\Omega) &:= W_{AB}(\Omega) := \mathrm{Tr}\left[\hat{A}\hat{B}\,\Delta(\Omega)\right] \\ &= \int_X d\mu(\Omega') \int_X d\mu(\Omega'') W_A(\Omega') W_B(\Omega'') L(\Omega,\Omega',\Omega''), \end{aligned} \tag{3.8}$$

where the tri-kernel $L(\Omega,\Omega',\Omega'')$ is defined by

$$L(\Omega,\Omega',\Omega'') := \mathrm{Tr}\left[\Delta(\Omega)\Delta(\Omega')\Delta(\Omega'')\right]. \tag{3.9}$$

We are now ready to apply these results of the general formalism to the space-space noncommutative Heisenberg-Weyl alggroup H_5, defined by the nilpotent Lie algebra (2.1), for the particular case ($d = 2$, $\bar{\theta}_{ij} = 0$) considered in the previous section. In terms of bosonic creation and destruction operators and holomorphic coordinates, appropriate for calculating the SW kernel, and symbols in terms of coherent states, the Lie algebra of the generators of H_5 is given by

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij},\ i = 1,2, \quad \left[\hat{a}_i, \hat{a}_j\right] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0,\ i = 1,2, \tag{3.10}$$

where

$$\begin{aligned} \hat{a}_1 &= \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_1 + \frac{\theta}{2\hbar}\hat{P}_2 + i\hat{P}_1\Big), \quad \hat{a}_1^\dagger = \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_1 + \frac{\theta}{2\hbar}\hat{P}_2 - i\hat{P}_1\Big), \\ \hat{a}_2 &= \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_2 - \frac{\theta}{2\hbar}\hat{P}_1 + i\hat{P}_2\Big), \quad \hat{a}_2^\dagger = \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_2 - \frac{\theta}{2\hbar}\hat{P}_1 - i\hat{P}_2\Big). \end{aligned} \tag{3.11}$$

The group elements are therefore of the following form:

$$g(s,\alpha,\beta) = e^{(isI + \alpha\hat{a}_1^\dagger - \bar{\alpha}\hat{a}_1 + \beta\hat{a}_2^\dagger - \bar{\beta}\hat{a}_2)},$$

where $\alpha, \beta \in \mathbb{C}$, and $\bar{\alpha}$, $\bar{\beta}$ denotes complex conjugation. Clearly, here $X = H_5/U(1) = \mathbb{C}^2$, and the invariant measure is

$$d\mu(\Omega) = \pi^{-2} d^2\alpha\, d^2\beta.$$

The Glauber coherent states are

$$|\Omega\rangle := |\alpha,\beta\rangle = D(\alpha,\beta)|0\rangle$$

with $D(\alpha,\beta)$ denoting the displacement operator:

$$D(\alpha,\beta) := e^{(\alpha\hat{a}_1^\dagger - \bar{\alpha}\hat{a}_1 + \beta\hat{a}_2^\dagger - \bar{\beta}\hat{a}_2)}. \tag{3.12}$$

Since the harmonic functions in this case are the exponentials:

$$Y_\nu(\Omega) := Y_{(\xi,\eta)}(\alpha,\beta) = \exp\left(\xi\bar{\alpha} - \bar{\xi}\alpha + \eta\bar{\beta} - \bar{\eta}\beta\right), \tag{3.13}$$

so that

$$\Delta(\alpha,\beta) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\xi \int_{\mathbb{C}} d^2\eta D(\xi,\eta)\exp\left(\bar{\xi}\alpha - \xi\bar{\alpha} + \bar{\eta}\beta - \eta\bar{\beta}\right); \tag{3.14}$$

the expectation value of a quantum operator $\hat{A}$ is given by

$$\langle \hat{A}\rangle = \mathrm{Tr}\left[\hat{\boldsymbol{\rho}}\hat{A}\right] = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha \int_{\mathbb{C}} d^2\beta W_\rho(\alpha,\beta)W_A(\alpha,\beta), \tag{3.15}$$

where

$$W_\rho(\alpha,\beta) = \mathrm{Tr}\left[\Delta(\alpha,\beta)\hat{\boldsymbol{\rho}}\right] \tag{3.16}$$

is the SW-symbol corresponding to the density matrix operator $\hat{\boldsymbol{\rho}}$.

We can now make use of (3.8) and (3.9) together with (3.12) and (3.14) to get an explicit expression for the twisted product of two SW-symbols based on the quotient space $\mathbb{C}^2 = H_5/U(1)$. Thus, noting that since the $\hat{a}_1$, $\hat{a}_1^\dagger$ commute with the $\hat{a}_2$, $\hat{a}_2^\dagger$, we can write the displacement operator as $D(\alpha,\beta) = D(\alpha)D(\beta)$, and the tri-kernel as $L(\alpha,\alpha',\alpha'';\beta,\beta',\beta'') = L(\alpha,\alpha',\alpha'')L(\beta,\beta',\beta'')$. Moreover, using also repeatedly the coherent states properties:

$$D(\xi)|\beta\rangle = e^{i\,\mathrm{Im}(\xi\bar{\beta})}|\xi+\beta\rangle, \tag{3.17}$$

$$\langle\alpha|\alpha'\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\alpha'|^2-2\bar{\alpha}\alpha')}, \tag{3.18}$$

we find

$$L(\alpha,\alpha',\alpha'') = 4\exp\left[4i\big(\alpha_2'\alpha_1 - \alpha_1'\alpha_2 + \alpha_1'\alpha_2'' - \alpha_2'\alpha_1'' + \alpha_1''\alpha_2 - \alpha_2''\alpha_1\big)\right],$$

and an analogous expression for $L(\beta,\beta',\beta'')$.

Consequently,

$$\begin{aligned} &W_A(\alpha,\beta)\star_S W_B(\alpha,\beta) \\ &\quad= \frac{16}{\pi^4}\int_{\mathbb{C}} d^2\alpha'' \int_{\mathbb{C}} d^2\alpha' e^{4i\alpha_1'(\alpha_2''-\alpha_2)} e^{4i\alpha_2'(\alpha_1-\alpha_1'')} e^{4i(\alpha_1''\alpha_2-\alpha_2''\alpha_1)} \\ &\quad\times \int_{\mathbb{C}} d^2\beta'' \int_{\mathbb{C}} d^2\beta' e^{4i\beta_1'(\beta_2''-\beta_2)} e^{4i\beta_2'(\beta_1-\beta_1'')} e^{4i(\beta_1''\beta_2-\beta_2''\beta_1)} W_A(\alpha',\beta')W_B(\alpha'',\beta''). \end{aligned} \tag{3.19}$$

Making next the change of variables $\alpha_1'' = \alpha_1+\eta_1$, $\alpha_2'' = \alpha_2+\eta_2$, $\beta_1'' = \beta_1+\xi_1$, $\beta_2'' = \beta_2+\xi_2$, we can write

$$\begin{aligned} &W_A(\alpha,\beta)\star_S W_B(\alpha,\beta) \\ &\quad= \frac{16}{\pi^4}\int_{\mathbb{C}}\cdots\int_{\mathbb{C}} d\eta_1 d\eta_2 d\xi_1 d\xi_2 d\alpha_1' d\alpha_2' d\beta_1' d\beta_2' e^{4i(\alpha_1'-\alpha_1)\eta_2} \\ &\quad\times e^{-4i(\alpha_2'-\alpha_2)\eta_1} e^{4i(\beta_1'-\beta_1)\xi_2} e^{-4i(\beta_2'-\beta_2)\xi_1} W_A(\alpha_1,\alpha_2,\beta_1,\beta_2) \\ &\quad\times e^{(\eta_1\vec{\partial}_{\alpha_1}+\eta_2\vec{\partial}_{\alpha_2}+\xi_1\vec{\partial}_{\beta_1}+\xi_2\vec{\partial}_{\beta_2})} W_B(\alpha_1,\alpha_2,\beta_1,\beta_2). \end{aligned} \tag{3.20}$$

We can change the last exponential in the above equation into a bidifferential by noting that

$$e^{4i(\alpha_1'-\alpha_1)\eta_2} e^{\eta_2\vec{\partial}_{\alpha_2}} W_B\big(\alpha_1,\alpha_2,\beta_1,\beta_2\big) = e^{4i(\alpha_1'-\alpha_1)\eta_2} e^{-\frac{i}{4}\overleftarrow{\partial}_{\alpha_1'}\overrightarrow{\partial}_{\alpha_2}} W_B\big(\alpha_1,\alpha_2,\beta_1,\beta_2\big),$$

and similarly for the other terms. Hence, substituting the results in (3.19), integrating by parts, and integrating over the remaining variables in the integrand, we finally arrive at

$$\begin{aligned} &W_A(\alpha,\beta) \star_S W_B(\alpha,\beta) \\ &\quad := W_A(\alpha,\beta) e^{\frac{i}{4}(\overleftarrow{\partial}_{\alpha_1}\overrightarrow{\partial}_{\alpha_2} - \overleftarrow{\partial}_{\alpha_2}\overrightarrow{\partial}_{\alpha_1} + \overleftarrow{\partial}_{\beta_1}\overrightarrow{\partial}_{\beta_2} - \overleftarrow{\partial}_{\beta_2}\overrightarrow{\partial}_{\beta_1})} W_B(\alpha,\beta). \end{aligned} \tag{3.21}$$

Now, substituting this result into (3.15), we obtain the expectation value of a product of quantum operators derived according to the Stratonovich-Weyl correspondence in the context of the space-space noncommutative Heisenberg-Weyl group. Moreover, since the alternate calculation in the previous section was done based on the Lie algebra of the same group and since the Stratonovich phase-space formulation was purported to be a generalization of the later to physical systems with Lie group symmetries which, evidently include the one common to the two approaches, a coincidence of results would then appear natural. In order to verify this conjecture we first need to convert the holomorphic variables in (3.15), (3.16), and (3.21) into phase-space variables. That is, we need to make the substitutions:

$$\begin{aligned} &\alpha_1 \longrightarrow \frac{1}{\sqrt{2\hbar}}\Big(q_1 + \frac{\theta}{2\hbar}p_2\Big), \quad \alpha_2 \longrightarrow \frac{1}{\sqrt{2\hbar}}p_1, \\ &\beta_1 \longrightarrow \frac{1}{\sqrt{2\hbar}}\Big(q_2 - \frac{\theta}{2\hbar}p_1\Big), \quad \beta_2 \longrightarrow \frac{1}{\sqrt{2\hbar}}p_2. \end{aligned} \tag{3.22}$$

Hence,

$$\begin{aligned} &\partial_{\alpha_1} = \sqrt{2\hbar}\partial_{q_1}, \quad \partial_{\alpha_2} = \sqrt{2\hbar}\Big(\frac{\theta}{2\hbar}\partial_{q_2} + \partial_{p_1}\Big), \\ &\partial_{\beta_1} = \sqrt{2\hbar}\partial_{q_2}, \quad \partial_{\beta_2} = \sqrt{2\hbar}\Big(-\frac{\theta}{2\hbar}\partial_{q_1} + \partial_{p_2}\Big), \end{aligned} \tag{3.23}$$

from where the Stratonovich twist bidifferential expressed in terms of phase-space variables takes the following form:

$$\star_S = \star_\theta \circ \star_\hbar. \tag{3.24}$$

Furthermore, making use of (3.12), (3.13), (3.16), and (3.14), we have

$$\begin{aligned} W_\rho(\alpha,\beta) &= \mathrm{Tr}\left[\Delta(\alpha,\beta)\hat{\boldsymbol{\rho}}\right] \\ &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\xi \int_{\mathbb{C}} d^2\eta \, \mathrm{Tr}\left[e^{(\xi\hat{a}_1^\dagger - \bar{\xi}\hat{a}_1 + \eta\hat{a}_2^\dagger - \bar{\eta}\hat{a}_2)}\hat{\boldsymbol{\rho}}\right] \exp\left(\bar{\xi}\alpha - \xi\bar{\alpha} + \bar{\eta}\beta - \eta\bar{\beta}\right). \end{aligned}$$

Evaluating now the trace in the above expression relative to the mixed phase-space basis $\{|q_1, p_2\rangle\}$ and after a fairly lengthy but straightforward calculation, we arrive at

$$\begin{aligned} W_\rho(\alpha,\beta) = 4 \iint & dq_1' dp_2' e^{2i\alpha_2(2\alpha_1 - \sqrt{\frac{2}{\hbar}}q_1' - \frac{\theta}{\hbar\sqrt{2\hbar}}p_2')} e^{-2i\beta_1(2\beta_2 - \sqrt{\frac{2}{\hbar}}p_2')} \\ & \times \Big\langle q_1', p_2' |\hat{\boldsymbol{\rho}}| 2\sqrt{2\hbar}\alpha_1 - q_1' - \frac{2\theta}{\sqrt{2\hbar}}\beta_2, -p_2' + 2\sqrt{2\hbar}\beta_2 \Big\rangle. \end{aligned}$$

Finally, making the change of variables:

$$q_1' = \sqrt{2\hbar}\alpha_1 - \frac{\lambda_1}{2} - \frac{\theta}{\sqrt{2\hbar}}\beta_2, \quad p_2' = \beta_2 - \frac{\lambda_2}{2}$$

yields

$$W_\rho(\alpha,\beta) = \iint d\lambda_1 d\lambda_2 e^{\frac{2i\alpha_2}{\sqrt{2\hbar}}(\lambda_1+\frac{\theta}{2\hbar}\lambda_2)} e^{-\frac{2i\beta_1\lambda_2}{\sqrt{2\hbar}}}$$
$$\times \left\langle \sqrt{2\hbar}\alpha_1 - \frac{\lambda_1}{2} - \frac{\theta}{\sqrt{2\hbar}}\beta_2, \beta_2 - \frac{\lambda_2}{2} |\hat{\boldsymbol{\rho}}| \sqrt{2\hbar}\alpha_1 + \frac{\lambda_1}{2} - \frac{\theta}{\sqrt{2\hbar}}\beta_2, \beta_2 + \frac{\lambda_2}{2} \right\rangle.$$

In terms of phase-space variables, this result reads

$$W_\rho\big(\alpha(p_1,q_2),\beta(q_1,p_2)\big)$$
$$= e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}} \iint d\lambda_1 d\lambda_2 e^{\frac{i}{\hbar}(p_1\lambda_1 - q_2\lambda_2)} \left\langle q_1 - \frac{\lambda_1}{2}, p_2 - \frac{\lambda_2}{2} |\hat{\boldsymbol{\rho}}| q_1 + \frac{\lambda_1}{2}, p_2 + \frac{\lambda_2}{2} \right\rangle. \tag{3.25}$$

If we now compare (3.21), (3.24), and (3.25) with (2.20), (2.21), (2.14), and (2.19) of the previous section, we see that for the space-space noncommutative Weyl-Heisenberg Lie group the quantum mechanics resulting from both formalisms are equivalent provided that in the calculation of the expectation values, we derive the phase-space averages by combining the appropriate $\star$-product for the evaluation of Weyl-symbols with the appropriate Wigner function or Weyl-symbol associated with the density matrix for the problem, according to the above referred formulas.

4 The Berezin quantization procedure by means of involution operators and its application to the space-space noncommutative Heisenberg-Weyl algebra

This quantization scheme arises from the basic property that for homogenous symmetric spaces, there is an involutive automorphism of G acting on them. Such is the case for $X = H_5/U(1)$, where the involution automorphisms are reflections around each point. Recalling equations (2.5), (2.6) in Section 2, we see that the Weyl function is the Fourier transform of the α function in (2.5), while the Fourier transform of the unitary displacement operators $\{(2\pi\hbar)^{-1}\exp[\frac{i}{\hbar}(\mathbf{y}\cdot\hat{\mathbf{R}} + \mathbf{x}\cdot\hat{\mathbf{P}})]\}$ is indeed reflections. It is thus natural to write [6, 22, 23, 24, 25, 26]

$$\hat{A} = \int_X d\mu(x) w_A(x) \hat{U}(x) \tag{4.1}$$

as a generalization of (2.5). Here, $\hat{U}(x)$ is the unitary operator corresponding to the group element that performs reflections around the point $x \in X$.

As noted by the authors in [22, 23, 24, 25, 26], the use of the reflection operator provides a way to circumvent the situation when a Fourier transform on X cannot be consistently defined. The function $w_A(x)$ appearing in (4.1) corresponds to the Weyl contravariant symbol which is, in general, different from the Weyl covariant symbol defined as:

$$\tilde{w}_A(x) := \mathrm{Tr}\left[\hat{A}\hat{U}(x)\right].$$

Berezin also showed that there exists a bijective map relating $w_A, \tilde{w}_A$ to the usual contravariant and covariant symbols P_A, Q_A, respectively, whose expressions are given by

$$\hat{A} = \int_X d\mu(x) P_A(x) |x\rangle\langle x|, \quad Q_A(x) = \langle x|\hat{A}|x\rangle,$$

where $\{|x\rangle\}$ corresponds to an overcomplete basis of normalized states tagged by points in X.

Thus in order to implement this quantization formalism, we must first determine what will be in our case the reflection operator $\hat{U}(x)$. To this end, we will make use of the Hilbert space spanned by the coherent states of the last section, which in fact constitute an overcomplete basis. Each coherent state $|\alpha,\beta\rangle = |\alpha\rangle \otimes |\beta\rangle$ is tagged by a point $(\alpha,\beta) \in \mathbb{C}^2 = X$.

We may now construct the reflection operator $\hat{U}(\alpha,\beta)$ by acting transitively on the reflection operator around the origin $\hat{U}(0,0)$ with the unitary operator associated to $g \in G$. From the properties of the algebra (3.10), it is clear that $\hat{U}(\alpha,\beta) = \hat{U}(\alpha) \otimes \hat{U}(\beta)$, where each $\hat{U}(\alpha)$ acts on a copy of $\mathbb{C}$. Then for simplicity, we will reduce the calculation to one copy of $\mathbb{C}$ and obtain the final result just by taking the direct product of the two copies. Thus, following Berezin, consider a complex line bundle L over $\mathbb{C}$ with fiber metric $e^{-K(v,\bar{v})}$, where $K(v,\bar{v}) = v\bar{v}$ is the Kähler potential. The Hilbert space $\mathcal{H}$ consists of holomorphic sections of L with inner product:

$$\langle f|g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2v\ \bar{f}(v)g(v)e^{-v\bar{v}},$$

where the holomorphic section $f(v)$ denotes the evaluation:

$$f(v) = \langle v|f\rangle.$$

The coherent state $|\alpha\rangle$, expressed in the Fock-Bargmann representation $\mathfrak{F}$, is given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

Hence,

$$\langle v|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n\bar{v}^n}{n!} = e^{-\frac{1}{2}|\alpha|^2+\alpha\bar{v}}. \tag{4.2}$$

Making use of the identity resolution:

$$\mathbb{I} = \frac{1}{\pi}\int_{\mathbb{C}} d^2v\ e^{-|v|^2}|v\rangle\langle v|, \tag{4.3}$$

we can write the left hand of (4.2) as follows:

$$\alpha(v) = \frac{1}{\pi}\int_{\mathbb{C}} d^2v'\langle v|v'\rangle e^{-|v'|^2}\alpha(v').$$

It is easy to show that this last expression becomes an identity if we set $\langle v|v'\rangle := B(v',\bar{v}) = e^{v'\bar{v}}$ and make use of (4.2) on both sides of the equation. Moreover, it also follows that $B(v',\bar{v})$ satisfies the following properties:

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2v'\ e^{-|v'|^2}B(v',\bar{v})f(v') = f(v), \quad \frac{1}{\pi}\int_{\mathbb{C}} d^2v'\ e^{-|v'|^2}B(v,\bar{v}')B(v',\bar{u}) = B(v,\bar{u}). \tag{4.4}$$

Thus $B(v',\bar{v})$ is the Bergman reproducing kernel [7], and in the $\mathfrak{F}$ representation space, the quantity $\pi\delta(v,v') := B(v',\bar{v})e^{-|v'|^2}$ acts as a Dirac delta function under integration.

Let us now define the operator $\hat{U}(0)$ by

$$\hat{U}(0) := \frac{1}{\pi}\int_{\mathbb{C}} d^2v\ e^{-|v|^2}|-v\rangle\langle v|.$$

To show that this is the reflection operator around the origin, we take the action of $\hat{U}(0)$ over any arbitrary state $|v'\rangle$ and use the above definition of the delta function action:

$$\hat{U}(0)|v'\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2v e^{-|v|^2}|-v\rangle\langle v|v'\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2v\, e^{-|v|^2}B(v',\bar{v})|-v\rangle = |-v'\rangle.$$

With the above results, we are now in a position to calculate the more general operator $\hat{U}(\zeta)$. This is done by noticing that by taking the unitary transformation $\hat{D}(\zeta)\hat{U}(0)\hat{D}^\dagger(\zeta)$, where $\hat{D}(\zeta)$ is the unitary displacement operator representation of the H_3 group acting on coherent states according to (3.17). Since $\hat{U}(0)$ is an involution, $\hat{D}(\zeta)$ induces displacements and $(\hat{D}(\zeta)\hat{U}(0)\hat{D}^\dagger(\zeta))^2 = \mathbb{I}$, the operator $\hat{U}(\zeta)$ must correspond to a reflection around $\zeta \in \mathbb{C}$. To show this, we first use (3.12) to obtain the explicit form of the operator $\hat{U}(\zeta) := \hat{D}(\zeta)\hat{U}(0)\hat{D}^\dagger(\zeta)$:

$$\hat{U}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2v e^{-|v|^2}\hat{D}(\zeta)|-v\rangle\langle v|\hat{D}^\dagger(\zeta).$$

Making now use of (4.2) in order to express the arbitrary ket $|v\rangle$ in terms of the normalized coherent state basis:

$$|v\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, e^{(-\frac{1}{2}|\alpha|^2+\bar{\alpha}v)}|\alpha\rangle,$$

and applying (3.17) on the coherent state $|\alpha\rangle$ yields

$$\hat{D}(\zeta)|v\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, e^{(-\frac{1}{2}|\alpha|^2+\bar{\alpha}v+i\,\mathrm{Im}(\zeta\bar{\alpha}))}|\alpha+\zeta\rangle. \tag{4.5}$$

Furthermore, making use of (4.5) and the properties of the Bergman kernel in (4.4), we obtain after some fairly straightforward calculations the expression:

$$\hat{U}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, e^{(\zeta\bar{\alpha}-\bar{\zeta}\alpha)}|\alpha+\zeta\rangle\langle\zeta-\alpha|.$$

Finally, making the change of variables $\zeta - \alpha = \rho$ yields

$$\hat{U}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\rho\, e^{\bar{\zeta}\rho-\bar{\rho}\zeta}|2\zeta-\rho\rangle\langle\rho|. \tag{4.6}$$

We next use this expression to repeat a similar calculation to the one we did above in order to obtain $\hat{U}(0)$. Thus, taking the action of the operator $\hat{U}(\zeta)$ on an arbitrary state $|v\rangle$ and expanding the coherent state $|2\zeta-\rho\rangle$ in (4.6) in terms of $|v\rangle$, by making use of (4.2) and (4.3), we get

$$\hat{U}(\zeta)|v\rangle = \frac{1}{\pi^2}e^{-2|\zeta|^2}\int_{\mathbb{C}} d^2v'\, e^{-|v'|^2}e^{2\bar{v}'\zeta}|v'\rangle\int_{\mathbb{C}} d^2\rho\, e^{-|\rho|^2}e^{v\bar{\rho}}e^{(2\bar{\zeta}-\bar{v}')\rho},$$

which when resorting repeatedly to equation (4.4) gives

$$\hat{U}(\zeta)|v\rangle = e^{2(\bar{\zeta}v-|\zeta|^2)}|2\zeta-v\rangle. \tag{4.7}$$

The function inside the ket in the above equation can be rewritten as $2(\zeta-v)+v$ to make evident the fact that this is the reflection of the point v around ζ. To complete the proof, we check that $\hat{U}(\zeta)$ is indeed an involution. This follows directly by once more acting with $\hat{U}(\zeta)$ on (4.7). Accordingly, we obtain

$$\hat{U}(\zeta)^2|v\rangle = \hat{U}(\zeta)\big[e^{2(\bar{\zeta}v-|\zeta|^2)}|2\zeta - v\rangle\big] = e^{2(\bar{\zeta}v-|\zeta|^2)}e^{2\bar{\zeta}(2\zeta-v)}e^{-2|\zeta|^2}|2\zeta-(2\zeta-v)\rangle = |v\rangle.$$

As we mentioned at the beginning of this section, the Weyl contravariant and covariant symbols are not the same in general. We will show, however, that for the symmetric homogenous space treated here this is not the case. Indeed, making the change $\hat{U}(\zeta) \to 2\hat{U}(\zeta) \equiv \hat{V}(\zeta)$ in (4.1), the latter reduces to (3.5) and consequently $w_A = W_A = \tilde{w}_A$ in which case both symbols are equal. This follows from equation (4.6) and observing that by using our previous results, we can write the identity as follows:

$$e^{\bar{\zeta}\rho-\bar{\rho}\zeta}|2\zeta-\rho\rangle = \frac{1}{2\pi}\int_{\mathbb{C}} d^2\lambda\, e^{\bar{\lambda}\zeta-\bar{\zeta}\lambda}e^{\frac{1}{2}(\bar{\rho}\lambda-\bar{\lambda}\rho)}|\lambda+\rho\rangle.$$

Moreover, the coherent state $e^{\frac{1}{2}(\bar{\rho}\lambda-\bar{\lambda}\rho)}|\lambda+\rho\rangle$ is nothing else but $\hat{D}(\lambda)|\rho\rangle$, so we can replace this into (4.6), and the operator $\hat{V}(\zeta) = 2\hat{U}(\zeta)$ takes now the following form:

$$\hat{V}(\zeta) = \frac{1}{\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}} d^2\lambda d^2\rho\, e^{\bar{\lambda}\zeta-\bar{\zeta}\lambda}\hat{D}(\lambda)|\rho\rangle\langle\rho|. \tag{4.8}$$

Finally, observe that in this last expression the quantity $\frac{1}{\pi}\int_{\mathbb{C}} d^2\rho\,|\rho\rangle\langle\rho|$ is just the identity operator in terms of normalized coherent states. It is then obvious that (4.8) reduces simply to

$$\hat{V}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\lambda\, e^{\bar{\lambda}\zeta-\bar{\zeta}\lambda}\hat{D}(\lambda), \tag{4.9}$$

which allows us to conclude that $\hat{V}(\alpha)\otimes\hat{V}(\beta) \equiv \Delta(\alpha,\beta)$ as seen from (3.14). This argument demonstrates that for the Heisenberg-Weyl algebra (2.1), the SW formalism as well as that of Berezin provide the same quantization scheme.

It is interesting to observe that because both the SW and the Berezin formalisms are based on complex valued holomorphic states and non-Hermitian operators, defined in turn by means of creation and destruction operators, the noncommutativity of the observables in the algebra (2.1) is hidden in the definition of those creation and destruction operators. So, as long as we remain in the complex domain, their quantum mechanics for the ordinary and the Heisenberg-Weyl algebras (2.1) appear as indistinguishable (see, e.g., (3.19)). It should also be clear from our presentation so far that there are a variety of Bopp maps that can be chosen to construct creation and destruction operators from phase-space operator observables. In our construction (see (3.11)), we have chosen a map that keeps the algebra of $\hat{a}$ and $\hat{a}^\dagger$ unchanged, as this choice allows us to use all the machinery of standard WWGM up to the point where we re-express the final results in terms of real dynamical phase-space variables.

Moreover, it is known that for the WWGM quantum mechanics, there is a ⋆-value equation which is a result stronger than the one providing the phase-space expectation values for operators and products of operators on Hilbert space. Indeed, it is fairly straightforward to show that (see, e.g., [32]) the star-value equation:

$$W_H(\mathbf{p},\mathbf{q}) \star_\hbar \rho_w = E\rho_w$$

is a necessary and sufficient condition for the weaker expectation value relation:

$$\iint d\mathbf{p}d\mathbf{q}W_H(\mathbf{p},\mathbf{q})\,\rho_w = \iint d\mathbf{p}d\mathbf{q}W_H(\mathbf{p},\mathbf{q}) \star_\hbar \rho_w$$

to follow. Here, $W_H(\mathbf{p},\mathbf{q})$ is the Weyl-symbol associated with the Hamiltonian operator $\hat{H}$ satisfying the eigenvalue equation $\hat{H}|\Psi\rangle = E|\psi\rangle$, $|\Psi\rangle$ is a pure energy state, and ρ_w is the Wigner function corresponding to the pure state density matrix $\hat{\boldsymbol{\rho}} = |\psi\rangle\langle\psi|$. We will investigate next if similar $\star$-valued equations exist for the quantum mechanical formulations on the Weyl-Heisenberg group consider above, and whether their equivalence stands for such stronger equations.

5 Star-value equations for phase-space quantum mechanics based on the space-space noncommutative Heisenberg-Weyl group

Given a Hamiltonian $\hat{H}(\hat{\mathbf{P}},\hat{\mathbf{R}})$ for a quantum mechanical system where $\hat{\mathbf{P}},\hat{\mathbf{R}}$ satisfy the algebra (2.1) (with $i,j = 1,2$ and $\bar{\theta} = 0$) and the pure state density matrix $\hat{\boldsymbol{\rho}} = |\psi\rangle\langle\psi|$, we can consider star-value equations associated with the $\star$-products (2.17) or (2.21). Let us begin by considering first the $\star$-product in (2.17) between the Weyl-symbol corresponding to $\hat{H}$ and the Weyl-symbol corresponding to the density matrix $\hat{\boldsymbol{\rho}}$. We get (after resorting to (2.19) in order to obtain the last equality):

$$\begin{aligned} W_H \star W_\rho &= m \circ \Big[e^{\sum_{i=1,2} \frac{i\hbar}{2}(\partial_{q_i}\otimes\partial_{p'_i} - \partial_{q'_i}\otimes\partial_{p_i})} \circ e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} \\ &\qquad \otimes e^{\frac{\theta}{\hbar}p'_1\partial_{q'_2}} W_H(\mathbf{p},\mathbf{q}) \otimes W_\rho(\mathbf{p}',\mathbf{q}') \Big]_{\mathbf{q},\mathbf{p}=\mathbf{q}',\mathbf{p}'} \\ &= \big(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} W_H\big) \star_\hbar \big(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} W_\rho\big) = \big(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} W_H\big) \star_\hbar \rho_w. \end{aligned}$$

Note that in general,

$$e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} W_H(\mathbf{p},\mathbf{q}) = W_H\Big(\mathbf{p}, q_1, q_2 + \frac{\theta}{\hbar}p_1\Big),$$

which says the following: calculate first the Weyl-symbol corresponding to the Hamiltonian operator by applying (2.17) repeatedly, followed by the displacement of the q_2 argument by the exponential on the left hand side of the above expression. Hence,

$$W_H \star W_\rho = W_H\Big(\mathbf{p}, q_1, q_2 + \frac{\theta}{\hbar}p_1\Big) \star_\hbar \rho_w.$$

Substituting now the expression (2.14) for the Wigner function and (2.16) for the $\star_\hbar$-product, we have

$$\begin{aligned} W_H \star W_\rho &= (2\pi\hbar)^{-2} \iint ds_1 ds_2 \psi\Big(q_1 - \frac{s_1}{2}, p_2 - \frac{s_2}{2}\Big)\psi^*\Big(q_1 + \frac{s_1}{2}, p_2 + \frac{s_2}{2}\Big) \\ &\quad \times \Big[\hat{W}_H\Big(q_1, q_2 + \frac{i\hbar}{2}\overrightarrow{\partial}_{p_2} + \frac{\theta}{\hbar}\Big(p_1 - \frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}\Big); p_1 - \frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}, p_2\Big) \\ &\qquad \times e^{\frac{i}{\hbar}s_1(p_1 + \frac{i\hbar}{2}\overleftarrow{\partial}_{q_1})} e^{-\frac{i}{\hbar}s_2(q_2 - \frac{i\hbar}{2}\overleftarrow{\partial}_{p_2})}\Big] \\ &= (2\pi\hbar)^{-2} \iint ds_1 ds_2 \psi\Big(q_1 - \frac{s_1}{2}, p_2 - \frac{s_2}{2}\Big)\psi^*\Big(q_1 + \frac{s_1}{2}, p_2 + \frac{s_2}{2}\Big) \\ &\quad \times \Big[\hat{W}_H\Big(q_1 - \frac{s_1}{2}, q_2 + \frac{i\hbar}{2}\overrightarrow{\partial}_{p_2} + \frac{\theta}{\hbar}p_1 - \frac{i\theta}{2}\overrightarrow{\partial}_{q_1}; p_1 - \frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}, p_2 - \frac{s_2}{2}\Big) \\ &\qquad \times e^{\frac{i}{\hbar}s_1 p_1} e^{-\frac{i}{\hbar}s_2 q_2}\Big]. \end{aligned}$$

If we now note that we can make the following replacement of the q_2 and p_1 arguments in W_H inside the square brackets:

$$q_2 \longrightarrow i\hbar\partial_{s_2}, \quad p_1 \longrightarrow -i\hbar\partial_{s_1},$$

and integrate by parts, we arrive at

$$W_H \star W_\rho = (2\pi\hbar)^{-2} \iint ds_1 ds_2 e^{\frac{i}{\hbar}s_1 p_1} e^{-\frac{i}{\hbar}s_2 q_2} \times \left[\hat{W}_H\left(q_1 - \frac{s_1}{2}, -i\hbar\partial_{s_2} + \frac{i\hbar}{2}\overrightarrow{\partial}_{p_2} + i\theta\partial_{s_1} - \frac{i\theta}{2}\overrightarrow{\partial}_{q_1}; i\hbar\partial_{s_1} - \frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}; p_2 - \frac{s_2}{2}\right) \times \psi\left(q_1 - \frac{s_1}{2}, p_2 - \frac{s_2}{2}\right)\psi^*\left(q_1 + \frac{s_1}{2}, p_2 + \frac{s_2}{2}\right)\right].$$

Observe next that making the identifications:

$$\hat{Q}_1 := q_1 - \frac{s_1}{2}, \quad \hat{\Pi}_1 := i\hbar\partial_{s_1} - \frac{i\hbar}{2}\partial_{q_1}, \qquad \hat{\Pi}_2 := p_2 - \frac{s_2}{2}, \quad \hat{Q}_2 := -i\hbar\partial_{s_2} + \frac{i\hbar}{2}\partial_{p_2} + \frac{\theta}{\hbar}\hat{\Pi}_1, \tag{5.1}$$

we obtain a realization for the Heisenberg-Weyl algebra:

$$\left[\hat{Q}_1, \hat{Q}_2\right] = i\theta, \quad \left[\hat{Q}_i, \hat{\Pi}_j\right] = i\hbar\delta_{ij}, \quad \left[\hat{\Pi}_1, \hat{\Pi}_2\right] = 0.$$

Observe also that the operator $\hat{W}_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)$ annihilates any function of $q_1 + \frac{s_1}{2}$ and $p_2 + \frac{s_2}{2}$. Hence,

$$W_H \star W_\rho = (2\pi\hbar)^{-2} \iint ds_1 ds_2 e^{\frac{i}{\hbar}s_1 p_1} e^{-\frac{i}{\hbar}s_2 q_2} \psi^*\left(q_1 + \frac{s_1}{2}, p_2 + \frac{s_2}{2}\right) \times \left[\hat{W}_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)\psi\left(q_1 - \frac{s_1}{2}, p_2 - \frac{s_2}{2}\right)\right]. \tag{5.2}$$

Furthermore, consider the eigenvalue equation:

$$\hat{H}\left(\hat{R}_1, \hat{R}_2; \hat{P}_1, \hat{P}_2\right)|\psi\rangle = E|\psi\rangle. \tag{5.3}$$

Since the operators $\hat{\mathbf{P}}$, $\hat{\mathbf{R}}$ satisfy the algebra (2.1) (with $i, j = 1, 2$ and $\bar{\theta} = 0$), the projection of (5.3) with the bra $\langle R_1, P_2|$ yields (making use of (2.2))

$$\hat{H}\left(R_1, -i\theta\partial_{R_1} + i\hbar\partial_{P_2}; -i\hbar\partial_{R_1}, P_2\right)\langle R_1, P_2|\psi\rangle = E\langle R_1, P_2|\psi\rangle. \tag{5.4}$$

Setting now

$$R_1 \equiv \hat{Q}_1 = q_1 - \frac{s_1}{2}, \quad P_2 \equiv \hat{\Pi}_2 = p_2 - \frac{s_2}{2},$$

and comparing the expression for $\hat{R}_2 = -i\theta\partial_{R_1} + i\hbar\partial_{P_2}$ in (5.4) with $\hat{Q}_2$ in (5.1), we get

$$\partial_{R_1} = \frac{1}{2}\partial_{q_1} - \partial_{s_1}, \quad \partial_{P_2} = \frac{1}{2}\partial_{p_2} - \partial_{s_2}.$$

However, also comparing the $\hat{R}_2$ in (5.4) with (2.2) yields

$$\partial_{q_1} = \partial_{R_1}, \quad \partial_{p_2} = \partial_{P_2},$$

from where it also clearly follows

$$\partial_{s_1} = -\frac{1}{2}\partial_{R_1}, \quad \partial_{s_2} = -\frac{1}{2}\partial_{P_2}.$$

Substituting the above into (5.4) and comparing with (5.1), we arrive at

$$\hat{H}\big(\hat{Q}_1, \hat{Q}_2; \hat{\Pi}_1, \hat{\Pi}_2\big)\langle Q_1, \Pi_2|\psi\rangle = E\langle Q_1, \Pi_2|\psi\rangle,$$

so, if we could make the identification $\hat{H}(\hat{Q}_1, \hat{Q}_2; \hat{\Pi}_1, \hat{\Pi}_2) = W_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)$, we would then have that (5.2) would immediately imply that

$$\begin{aligned} &W_H \star W_\rho \\ &\quad = (2\pi\hbar)^{-2} E \iint ds_1 ds_2 e^{\frac{i}{\hbar}s_1 p_1} e^{-\frac{i}{\hbar}s_2 q_2} \psi^*\Big(q_1 + \frac{s_1}{2}, p_2 + \frac{s_2}{2}\Big)\psi\Big(q_1 - \frac{s_1}{2}, p_2 - \frac{s_2}{2}\Big) = E\rho_w, \end{aligned}$$

or

$$W_H\Big(\mathbf{p}; q_1, q_2 + \frac{\theta}{\hbar}p_1\Big) \star_\hbar \rho_w(\mathbf{p}, \mathbf{q}) = E\rho_w. \tag{5.5}$$

Note, however, that the feasibility of this identification requires that $\hat{H}(\hat{Q}_1, \hat{Q}_2; \hat{\Pi}_1, \hat{\Pi}_2)$ and $W_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)$ should be of the same functional form for their operator arguments, but, according to our discussion following equation (2.22), this will only be possible for Hamiltonians having the Weyl symmetrized ordering of operators.

The corresponding expression of the $\star$-value equation for the product $W_H \star_\theta \circ \star_\hbar W_\rho$ follows immediately by recalling (see the argument given in the paragraph following equation (4.9)) that in holomorphic coordinates, the $\star$-value equation does not see the noncommutativity:

$$W_H(\alpha, \beta) \star_S W_\rho(\alpha, \beta) = EW_\rho \equiv W_H(\alpha, \beta) \star_\hbar W_\rho(\alpha, \beta) = EW_\rho.$$

Thus, when going back to phase-space variables by making use of (3.22) and (3.25) yields

$$\begin{aligned} &W_H\left(\frac{1}{\sqrt{2\hbar}}\Big(q_1 + \frac{\theta}{\sqrt{2\hbar}}p_2\Big), \frac{1}{\sqrt{2\hbar}}\Big(q_2 - \frac{\theta}{\sqrt{2\hbar}}p_1\Big), \frac{1}{\sqrt{2\hbar}}p_1, \frac{1}{\sqrt{2\hbar}}p_2\right) \\ &\qquad \times \star_\theta \circ \star_\hbar e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}} \rho_w\big(q_1, q_2, p_1, p_2\big) \\ &\quad = Ee^{-\frac{\theta}{\hbar}p_1\partial_{q_2}} \rho_w\big(q_1, q_2, p_1, p_2\big). \end{aligned} \tag{5.6}$$

Evidently, the two $\star$-valued equations (5.5) and (5.6) are different, even that the weaker expectation values resulting from them are the same. This difference may turn out to be important for certain problems in deformation quantization such as the ones mentioned in the introduction.

Acknowledgement

This work was supported in part by CONACyT Project UA7899-F.

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Received July 31, 2010
Revised September 27, 2010