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Double cross biproduct and bi-cycle bicrossproduct Lie bialgebras

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Abstract

We construct double cross biproduct and bi-cycle bicrossproduct Lie bialgebras from braided Lie bialgebras. The main results generalize Majid's matched pair of Lie algebras and Drinfeld's quantum double and Masuoka's cross product Lie bialgebras.

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1 Introduction

As an infinitesimal or semiclassical structures underlying the theory of quantum groups, the notion of Lie bialgebras was introduced by Drinfeld in his remarkable report [3], where he also introduced the double Lie bialgebra $D(\mathfrak{g})$ as an important construction. Some years later, the theory of matched pairs of Lie algebras $(\mathfrak{g}, \mathfrak{m})$ was introduced by Majid in [4]. Its bicrossed product (or double cross sum) $\mathfrak{m} \bowtie \mathfrak{g}$ is more general than Drinfeld's classical double $D(\mathfrak{g})$ because $\mathfrak{g}$ and $\mathfrak{m}$ need not have the same dimension and the actions need not be strictly coadjoint ones.

Since then it was found that many other structures in Hopf algebras can be constructed in the infinitesimal setting, see [5] and the references cited therein. Also in [6], Majid introduced the concept of braided Lie bialgebras and proved the bosonisation theorem (see Theorem 4.4) associating braided Lie bialgebras to ordinary Lie bialgebras. Examples of braided Lie bialgebras were also given there. On the other hand, there is a close relation between extension theory and cross product Lie bialgebras, see Masuoka [7].

Natural and interesting questions are the following: Are there any more general results on this to uniform all the existing results together? What conditions will be needed? We give answers to these questions in the present paper. Majid's and Masuoka's results will be generalized as corollaries of our main results.

This paper is organized as follows. In Section 2, we recall some definitions and fix some notations. In Section 3, we review the notion of matched pairs of Lie coalgebras as dual version of Majid's matched pairs of Lie algebras. In Section 4, we construct double cross biproduct Lie bialgebras through two braided Lie bialgebras. In Section 5, we construct bicycle bicrossproduct Lie bialgebras, which is a generalization of Masuoka's cross product Lie bialgebras. The main results are stated in Theorems 4.6 and 5.7.

Throughout this paper, all vector spaces will be over a fixed field of character zero. The identity map of a vector space V is denoted by $\mathrm{id}_V : V \to V$.

2 Preliminaries

Definition 2.1 (see [8])**.** A *Lie coalgebra* is a vector space L equipped with a linear map $\delta : L \to L \otimes L$, called a cobracket, satisfying the co-anticommutativity and the co-Jacobi identity:

$$\text{(CL1):} \quad \delta = -\tau\delta,$$
$$\text{(CL2):} \quad \big(1+\xi+\xi^2\big)(\delta\otimes \mathrm{id})\delta = 0,$$

where $\tau : L\otimes L \to L\otimes L$, $\xi : L\otimes L\otimes L \to L\otimes L\otimes L$ are twistings defined by

$$\tau(u\otimes v) = v\otimes u, \quad \xi(u\otimes v\otimes w) = v\otimes w\otimes u.$$

We would like to use the sigma notation $\delta(a) = \sum a_1\otimes a_2 = a_1\otimes a_2$ for all $a\in L$ to denote the cobracket, then the above conditions can be written as

$$\text{(CL1):} \quad \sum a_1\otimes a_2 = -\sum a_2\otimes a_1,$$
$$\text{(CL2):} \quad \sum a_{11}\otimes a_{12}\otimes a_2 + \sum a_{12}\otimes a_2\otimes a_{11} + \sum a_2\otimes a_{11}\otimes a_{12} = 0.$$

In fact, (CL2) is equivalent to one of the following two equations by (CL1):

$$\sum a_1\otimes a_{21}\otimes a_{22} + \sum a_{21}\otimes a_{22}\otimes a_1 + \sum a_{22}\otimes a_1\otimes a_{21} = 0,$$
$$\sum a_1\otimes a_{21}\otimes a_{22} = \sum a_{11}\otimes a_{12}\otimes a_2 - \sum a_{11}\otimes a_2\otimes a_{12}.$$

Definition 2.2. A Lie bialgebra H is a vector space equipped simultaneously with a Lie algebra structure $(H,[\,,\,])$ and a Lie coalgebra (H,δ) structure such that the following compatibility condition is satisfied:

$$\text{(LB):} \quad \delta\big([a,b]\big) = \sum \big[a,b_1\big]\otimes b_2 + \sum b_1\otimes \big[a,b_2\big] + \sum a_1\otimes \big[a_2,b\big] + \sum \big[a_1,b\big]\otimes a_2,$$

and we denote it by $(H,[\,,\,],\delta)$.

Note that (LB) is equal to

$$\delta\big([a,b]\big) = \sum[a,b_1]\otimes b_2 + \sum b_1\otimes[a,b_2] - \sum[b,a_1]\otimes a_2 - \sum a_1\otimes[b,a_2].$$

We can also write them as ad-actions on tensors by

$$\delta\big([a,b]\big) = a\triangleright\delta(b) - b\triangleright\delta(a) = a\triangleright\delta(b) + \delta(a)\triangleleft b.$$

Let A, H be both Lie algebras and Lie coalgebras; for $a,b\in A$, $h,g\in H$, denote maps

$$\alpha : H\otimes A\to A, \quad \beta : H\otimes A\to H, \quad \phi : A\to H\otimes A, \quad \psi : H\to A\otimes H$$

by

$$\alpha(h\otimes a) = h\triangleright a, \quad \beta(h\otimes a) = h\triangleleft a, \quad \phi(a) = \sum a_{(-1)}\otimes a_{(0)}, \quad \psi(h) = \sum h_{(0)}\otimes h_{(1)}.$$

We now fix some notions.

For a Lie algebra H and a linear map $\alpha : H\otimes A\to A$ such that

$$\alpha\big([\,,\,]\otimes \mathrm{id}_A\big) = \alpha\big(\mathrm{id}_H\otimes\alpha\big) - \alpha\big(\mathrm{id}_H\otimes\alpha\big)\big(\tau\otimes \mathrm{id}_A\big),$$

$$[h,g]\triangleright a=h\triangleright(g\triangleright a)-g\triangleright(h\triangleright a),\quad \text{for } h,g\in H,\ a\in A,$$

then (A,α) is called a left H-module. For a Lie coalgebra H and a linear map $\phi: A\to H\otimes A$ such that

$$(\delta_H\otimes \mathrm{id}_A)\phi=(\mathrm{id}_H\otimes\phi)\phi-(\tau\otimes \mathrm{id}_A)(\mathrm{id}_H\otimes\phi)\phi,$$
$$\sum a_{(-1)1}\otimes a_{(-1)2}\otimes n_{(0)}=\sum a_{(-2)}\otimes a_{(-1)}\otimes a_{(0)}-\sum a_{(-1)}\otimes a_{(-2)}\otimes a_{(0)},$$

then (A,ϕ) is called a left H-comodule. If H and A are Lie algebras, A is a left H-module and $\alpha(\mathrm{id}_H\otimes[\,,\,])=[\,,\,](\alpha\otimes\mathrm{id}_A)+[\,,\,](\mathrm{id}_H\otimes\alpha)(\tau\otimes\mathrm{id}_A)$, $h\triangleright[a,b]=[h\triangleright a,b]+[a,h\triangleright b]$, then $(A,[\,,\,],\alpha)$ is called a left H-module Lie algebra. If H is a Lie coalgebra and A is a Lie algebra, A is a left H-comodule and $\phi[\,,\,]=(\mathrm{id}_H\otimes[\,,\,])(\phi\otimes\mathrm{id}_A)+(\mathrm{id}_H\otimes[\,,\,])(\tau\otimes\mathrm{id}_A)(\mathrm{id}_A\otimes\phi)$, $\phi([a,b])=\sum a_{(-1)}\otimes[a_{(0)},b]+\sum b_{(-1)}\otimes[a,b_{(0)}]$, then A is called a left H-comodule Lie algebra. Right Lie (co)module and Lie (co)module Lie (co)algebra can be defined similarly, see also [5].

3 Matched pairs of Lie algebras and Lie coalgebras

The general theory of matched pairs of Lie algebras was introduced by Majid in his Ph.D. thesis and published in [4]. He summarized his theory in his book [5]. In this section, we review the notion of matched pairs of Lie coalgebras, which is the dual version of Majid's matched pairs of Lie algebras. We will use it to construct double matched pairs in the next section.

Definition 3.1 (see [5, Definition 8.3.1])**.** Assume that A and H are Lie algebras. If (A,α) is a left H-module, (H,β) is a right A-module, and the following (BB1) and (BB2) hold, then (A,H,α,β) (or (A,H)) is called a *matched pair of Lie algebras*:

$$\text{(BB1)}:\quad h\triangleright[a,b]=[h\triangleright a,b]+[a,h\triangleright b]+(h\triangleleft a)\triangleright b-(h\triangleleft b)\triangleright a,$$
$$\text{(BB2)}:\quad [h,g]\triangleleft a=[h,g\triangleleft a]+[h\triangleleft a,g]+h\triangleleft(g\triangleright a)-g\triangleleft(h\triangleright a)$$

Lemma 3.2 (see [4])**.** *Let (A,H) be a matched pair of Lie algebras, then we get a new Lie algebra on the vector space $A\oplus H$ with bracket given by*

$$\big[(a,h),(b,g)\big]=\big([a,b]+h\triangleright b-g\triangleright a,[h,g]+h\triangleleft b-g\triangleleft a\big).$$

We denoted it by $A\bowtie H$.

Proposition 3.3 (see [5, Proposition 8.3.4])**.** *Assume that A and H are Lie bialgebras, (A,H) is a matched pair of Lie algebras, A is a left H-module Lie coalgebra, and H is a right A-module Lie coalgebra. If*

$$\big(\mathrm{id}_H\otimes\alpha\big)\big(\delta_H\otimes\mathrm{id}_A\big)+\big(\beta\otimes\mathrm{id}_A\big)\big(\mathrm{id}_H\otimes\delta_A\big)=0,$$

i.e.,

$$\text{(AA1)}:\quad \sum h_1\otimes h_2\triangleright a+\sum h\triangleleft a_1\otimes a_2=0,$$

then $A\bowtie H$ becomes a Lie bialgebra.

Example 3.4. A skew-pairing between Lie bialgebras is a linear map $\omega : H \otimes A \to k$ such that (see [5])

$$\omega([h,g],a) = \omega(h,a_1)\omega(g,a_2), \quad \omega(h,[a,b]) = \omega(h_1,b)\omega(h_2,a).$$

For such a skew-pair, we can define a right A-module of H as $h \triangleleft a = \omega(h_2,a)h_1$, since the right-hand sides of

$$h \triangleleft [a,b] = \omega\big(h_2,[a,b]\big)h_1 = \omega(h_{21},b)\omega(h_{22},a)h_1 = \omega(h_2,a)\omega(h_{12},b)h_{11}$$

and

$$(h \triangleleft a) \triangleleft b - (h \triangleleft b) \triangleleft a = \omega(h_2,b)\omega(h_{12},a)h_{11}$$

are equal because H is a Lie coalgebra. Similarly, there is a left H-module of A by $h \triangleright a = \omega(h,a_2)a_1$. It is easy to see that they form a matched pair of Lie algebras and the condition in Proposition 3.3 holds. In this case $A \bowtie H$ becomes a Lie bialgebra which we denote by $A \bowtie_\omega H$. This is a generalization of Drinfeld's double Lie bialgebra when $A = H^{*op}$.

Definition 3.5. Two Lie coalgebras (A,H) form a *matched pair of Lie coalgebras* if (A,ϕ) is a left H-comodule and (H,ψ) is a right A-comodule, obeying the conditions

$$\text{(BB3):} \quad (\mathrm{id} \otimes \delta)\phi = (\phi \otimes \mathrm{id})\delta + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\delta + (\psi \otimes \mathrm{id})\phi + (\mathrm{id} \otimes \tau)(\psi \otimes \mathrm{id})\phi,$$
$$\text{(BB4):} \quad (\delta \otimes \mathrm{id})\psi = (\mathrm{id} \otimes \psi)\delta + (\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \psi)\delta + (\mathrm{id} \otimes \phi)\psi + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\psi.$$

In sigma notation, the above conditions are

$$\begin{aligned}\text{(BB3):} \quad & \sum a_{(-1)} \otimes a_{(0)1} \otimes a_{(0)2} \\ &= \sum a_{1(-1)} \otimes a_{1(0)} \otimes a_2 + \sum a_{2(-1)} \otimes a_1 \otimes a_{2(0)} \\ &\quad + \sum a_{(-1)(0)} \otimes a_{(-1)(1)} \otimes a_{(0)} - \sum a_{(-1)(0)} \otimes a_{(0)} \otimes a_{(-1)(1)}, \\ \text{(BB4):} \quad & \sum h_{(0)1} \otimes h_{(0)2} \otimes h_{(1)} \\ &= \sum h_1 \otimes h_{2(0)} \otimes h_{2(1)} + \sum h_{1(0)} \otimes h_2 \otimes h_{1(1)} \\ &\quad + \sum h_{(0)} \otimes h_{(1)(-1)} \otimes h_{(1)(0)} - \sum h_{(1)(-1)} \otimes h_{(0)} \otimes h_{(1)(0)}.\end{aligned}$$

Lemma 3.6. *Let (A,H) be a matched pair of Lie coalgebras, we define $D = A \blacktriangleright\!\!\blacktriangleleft H$ as the vector space $A \oplus H$ with Lie cobracket $\delta_D(a) = (\delta_A + \phi - \tau\phi)(a)$, $\delta_D(h) = (\delta_H + \psi - \tau\psi)(h)$, that is,*

$$\delta_D(a) = \sum a_1 \otimes a_2 + \sum a_{(-1)} \otimes a_{(0)} - \sum a_{(0)} \otimes a_{(-1)},$$
$$\delta_D(h) = \sum h_1 \otimes h_2 + \sum h_{(0)} \otimes h_{(1)} - \sum h_{(1)} \otimes h_{(0)},$$

this makes $A \blacktriangleright\!\!\blacktriangleleft H$ into a Lie coalgebra.

Proof. It is easy to show that δ_D satisfies the coanticommutitive condition. We now intend to show $(1+\xi+\xi^2)(\delta_D \otimes \mathrm{id})\delta_D(a) = 0$. By definition of δ_D we have

$$(\delta_D \otimes \mathrm{id})\delta_D(a) = \delta_D\big(a_1\big) \otimes a_2 + \delta_D\big(a_{(-1)}\big) \otimes a_{(0)} - \delta_D\big(a_{(0)}\big) \otimes a_{(-1)}$$

$$
\begin{aligned}
&= a_{11} \otimes a_{12} \otimes a_2(1) + a_{1(-1)} \otimes a_{1(0)} \otimes a_2(2) \\
&\quad - a_{1(0)} \otimes a_{1(-1)} \otimes a_2(3) + a_{(-1)1} \otimes a_{(-1)2} \otimes a_{(0)}(4) \\
&\quad + a_{(-1)(0)} \otimes a_{(-1)(1)} \otimes a_{(0)}(5) - a_{(-1)(1)} \otimes a_{(-1)(0)} \otimes a_{(0)}(6) \\
&\quad - a_{(0)1} \otimes a_{(0)2} \otimes a_{(-1)}(7) - a_{(0)(-1)} \otimes a_{(0)(0)} \otimes a_{(-1)}(8) \\
&\quad + a_{(0)(0)} \otimes a_{(0)(-1)} \otimes a_{(-1)}(9),
\end{aligned}
$$

$$
\begin{aligned}
\xi(\delta_D \otimes \mathrm{id})\delta_D(a) &= a_{12} \otimes a_2 \otimes a_{11}(10) + a_{1(0)} \otimes a_2 \otimes a_{1(-1)}(11) \\
&\quad - a_{1(-1)} \otimes a_2 \otimes a_{1(0)}(12) + a_{(-1)2} \otimes a_{(0)} \otimes a_{(-1)1}(13) \\
&\quad + a_{(-1)(1)} \otimes a_{(0)} \otimes a_{(-1)(0)}(14) - a_{(-1)(0)} \otimes a_{(0)} \otimes a_{(-1)(1)}(15) \\
&\quad - a_{(0)2} \otimes a_{(-1)} \otimes a_{(0)1}(16) - a_{(0)(0)} \otimes a_{(-1)} \otimes a_{(0)(-1)}(17) \\
&\quad + a_{(0)(-1)} \otimes a_{(-1)} \otimes a_{(0)(0)}(18),
\end{aligned}
$$

$$
\begin{aligned}
\xi^2(\delta_D \otimes \mathrm{id})\delta_D(a) &= a_2 \otimes a_{11} \otimes a_{12}(19) + a_2 \otimes a_{1(-1)} \otimes a_{1(0)}(20) \\
&\quad - a_2 \otimes a_{1(0)} \otimes a_{1(-1)}(21) + a_{(0)} \otimes a_{(-1)1} \otimes a_{(-1)2}(22) \\
&\quad + a_{(0)} \otimes a_{(-1)(0)} \otimes a_{(-1)(1)}(23) - a_{(0)} \otimes a_{(-1)(1)} \otimes a_{(-1)(0)}(24) \\
&\quad - a_{(-1)} \otimes a_{(0)1} \otimes a_{(0)2}(25) - a_{(-1)} \otimes a_{(0)(-1)} \otimes a_{(0)(0)}(26) \\
&\quad + a_{(-1)} \otimes a_{(0)(0)} \otimes a_{(0)(-1)}(27).
\end{aligned}
$$

It follows from the braided Jacobi identity of A that $(1) + (10) + (19) = 0$. Since A is a left H-comodule, $(13) + (9) - (26) = 0$, $(4) - (17) + (27) = 0$, $(22) - (8) + (18) = 0$. By the condition (BB3) of matched pair of braided Lie coalgebras, $-(16)+(2)-(21)+(5)-(24) = 0$, $-(25)+(11)-(3)+(14)-(6) = 0$, $-(7)+(20)-(12)+(23)-(15) = 0$. Therefore, (CL2) holds on A. Similarly, (CL2) holds on H. Hence (CL2) holds on the direct sum space $A \oplus H$. □

Proposition 3.7. *Let (A, H) be a matched pair of Lie coalgebras with both of them Lie bialgebras, and A is left H-comodule Lie algebra, H is right A-comodule Lie algebra, such that*

$$\big([\,,\,]_H \otimes \mathrm{id}_A\big)\big(\mathrm{id}_H \otimes \phi\big) + \big(\mathrm{id}_H \otimes [\,,\,]_A\big)\big(\varphi \otimes \mathrm{id}_A\big) = 0,$$

i.e.,

$$(\mathrm{AA2}): \quad \sum \big[h, a_{-1}\big] \otimes a_0 + \sum h_0 \otimes \big[h_1, a\big] = 0,$$

then the direct sum Lie algebra structure makes $A \bowtie H$ into a Lie bialgebra. We call it the double cross coproduct Lie bialgebra.

Proof. The Lie coalgebra structure is as in Lemma 3.6. The Lie algebra structure is as in $A \oplus H$. We only check the condition (LB). There are four cases: $A \otimes A$, $H \otimes H$, $A \otimes H$, and $H \otimes A$. We only check the first two cases. For $a, b \in A$,

$$
\begin{aligned}
\delta_D\big([a, b]\big) &= \delta_A\big([a, b]\big) + \phi\big([a, b]\big) - \tau\phi\big([a, b]\big), \\
a \triangleright \delta_D(b) + \delta_D(a) \triangleleft b &= a \triangleright (\delta + \phi - \tau\phi)(b) + (\delta + \phi - \tau\phi)(a) \triangleleft b \\
&= \big[a, b_1\big] \otimes b_2(1) + b_1 \otimes \big[a, b_2\big](2) + b_{(-1)} \otimes \big[a, b_{(0)}\big](3) \\
&\quad - \big[a, b_{(0)}\big] \otimes b_{(-1)}(4) + a_1 \otimes \big[a_2, b\big](5) + \big[a_1, b\big] \otimes a_2(6) \\
&\quad + a_{(-1)} \otimes \big[a_{(0)}, b\big](7) - \big[a_{(0)}, b\big] \otimes a_{(-1)}(8).
\end{aligned}
$$

Now because A is a Lie bialgebra, we get $\delta([a,b]) = (1)+(2)+(5)+(6)$. Since A is a left H-comodule Lie algebra, $\phi([a,b]) = (7)+(3)$, $\tau\phi([a,b]) = (8)+(4)$. Hence the compatibility condition (LB) is valid on A. For $x, y \in H$,

$$\begin{aligned}
\delta_D\big([h,g]\big) &= \delta\big([h,g]\big) + \psi\big([h,g]\big) - \tau\psi\big([h,g]\big),\\
h \triangleright \delta_D(g) + \delta_D(h) \triangleleft g &= h \triangleright (\delta + \psi - \tau\psi)(g) + (\delta + \psi - \tau\psi)(h) \triangleleft g\\
&= \big[h, g_1\big] \otimes g_2 (1) + g_1 \otimes \big[h, g_2\big](2) + \big[h, g_{(0)}\big] \otimes g_{(1)}(3)\\
&\quad - \big[h, g_{(0)}\big] \otimes g_{(1)}(4) + h_1 \otimes \big[h_2, g\big](5) + \big[h_1, g\big] \otimes h_2(6)\\
&\quad + \big[h_{(0)}, g\big] \otimes h_{(1)}(7) - h_{(1)} \otimes \big[h_{(0)}, g\big](8).
\end{aligned}$$

Now because H is a Lie bialgebra, we get $\delta([a,b]) = (1)+(2)+(5)+(6)$. Since H is a right A-comodule Lie algebra, $\phi([a,b]) = (7)+(3)$, $\tau\phi([a,b]) = (8)+(4)$. Hence the compatibility condition (LB) is valid on H. □

Example 3.8. A skew-copairing between Lie bialgebras is a linear map $R : k \to H \otimes A$ which is denoted by

$$R = R^1 \otimes R^2 = r = r^1 \otimes r^2 \in H \otimes A$$

such that

$$R^1 \otimes \delta\big(R^2\big) = \big[R^1, r^1\big] \otimes r^2 \otimes R^2, \quad \delta\big(R^1\big) \otimes R^2 = R^1 \otimes r^1 \otimes \big[R^2, r^2\big].$$

For such a skew-copair, we can define a right A-comodule of H by $\psi(h) = [h, R^1] \otimes R^2$, a left H-module of A by $\phi(a) = R^1 \otimes [a, R^2]$. It is easy to see that they form a matched pair of Lie coalgebras and the condition in Proposition 3.7 holds. In this case $A \bowtie H$ becomes a Lie bialgebra which we denote by $A \bowtie^R H$. This is a generalization of Drinfeld's codouble Lie bialgebra when $A = H^{*cop}$.

4 Yetter-Drinfeld modules and double cross biproduct

The concept of Yetter-Drinfeld modules over Lie bialgebras was introduced by Majid in [6] (where he call it Lie crossed modules), which he used to construct biproduct Lie bialgebras. This is a Lie version of biproduct Hopf algebras introduced by Radford in [9].

Definition 4.1. Let H be simultaneously a Lie algebra and a Lie coalgebra. If V is a left-H module and left H-comodule, satisfying

$$\text{(YD1)}: \quad \phi(h \triangleright v) = \big[h, v_{(-1)}\big] \otimes v_{(0)} + v_{(-1)} \otimes h \triangleright v_{(0)} + h_1 \otimes h_2 \triangleright v,$$

then V is called a left Yetter-Drinfeld module over H.

We denote the category of Yetter-Drinfeld modules over H by ${}^H_H\mathcal{M}$. It can be shown that ${}^H_H\mathcal{M}$ forms a monoidal category if H is Lie bialgebra [6].

Definition 4.2. Let A be simultaneously a Lie algebra and Lie coalgebra. If V is a right A-module and right A-comodule, satisfying

$$\text{(YD2)}: \quad \psi(v \triangleleft a) = v_{(0)} \otimes \big[v_{(1)}, a\big] + v_{(0)} \triangleleft a \otimes v_{(1)} + v \triangleleft a_1 \otimes a_2,$$

then V is called a right Yetter-Drinfeld module over A.

We denote the category of Yetter-Drinfeld modules over A by $\mathcal{M}_A^A$.

The next condition was also introduced by Majid in [6], which is a modification of the condition (LB).

Definition 4.3. If A is a Lie algebra and a Lie coalgebra and H is a right Yetter-Drinfeld module over A, we call H a *braided Lie bialgebra* in $\mathcal{M}_A^A$, if the following condition is satisfied:

$$\delta\big([h,g]\big)=\big[h,g_1\big]\otimes g_2+g_1\otimes\big[h,g_2\big]+h_1\otimes\big[h_2,g\big]+\big[h_1,g\big]\otimes h_2-s(h\otimes g),\quad \text{(LBS) for } H,$$

where

$$s(h\otimes g)=g_{(-1)}\triangleright h\otimes g_{(0)}+h_{(0)}\otimes h_{(-1)}\triangleright g-h_{(-1)}\triangleright g\otimes h_{(0)}-g_{(0)}\otimes g_{(-1)}\triangleright h.$$

Theorem 4.4 (see [6, Theorem 3.7])**.** *If A is a Lie bialgebra and H is a braided Lie bialgebra in $\mathcal{M}_A^A$, then the biproduct $A \rtimes\!\!\!\lhd H$ form an ordinary Lie bialgebra.*

In the following, we construct the double cross biproduct of braided Lie bialgebras. Firstly, we give conditions for A to be a braided Lie bialgebra in ${}_H^H\mathcal{M}$.

$$\delta\big([a,b]\big)=\big[a,b_1\big]\otimes b_2+b_1\otimes\big[a,b_2\big]+a_1\otimes\big[a_2,b\big]+\big[a_1,b\big]\otimes a_2-s(a\otimes b),\quad \text{(LBS) for } A,$$

where

$$s(a\otimes b)=b_{(-1)}\triangleright a\otimes b_{(0)}+a_{(0)}\otimes a_{(-1)}\triangleright b-a_{(-1)}\triangleright b\otimes a_{(0)}-b_{(0)}\otimes b_{(-1)}\triangleright a.$$

Definition 4.5. Let A, H be both Lie algebras and Lie coalgebras, obeying the following conditions:

$$\begin{aligned}
&\text{(BB5):}\quad \delta_A(h\triangleright a)=h\triangleright a_1\otimes a_2+a_1\otimes h\triangleright a_2+h_{(0)}\triangleright a\otimes h_{(1)}-h_{(1)}\otimes h_{(0)}\triangleright a,\\
&\text{(BB6):}\quad \delta_H(h\triangleleft a)=h_1\otimes h_2\triangleleft a+h_1\triangleleft a\otimes h_2+a_{(-1)}\otimes h\triangleleft a_{(0)}-h\triangleleft a_{(0)}\otimes a_{(-1)},\\
&\text{(BB7):}\quad \phi\big([a,b]\big)=a_{(-1)}\otimes\big[a_{(0)},b\big]+b_{(-1)}\otimes\big[a,b_{(0)}\big]+a_{(-1)}\triangleleft b\otimes a_{(0)}-b_{(-1)}\triangleleft a\otimes b_{(0)},\\
&\text{(BB8):}\quad \psi\big([h,g]\big)=\big[h,g_{(0)}\big]\otimes g_{(1)}+\big[h_{(0)},g\big]\otimes h_{(1)}+g_{(0)}\otimes h\triangleright g_{(1)}-h_{(0)}\otimes g\triangleright h_{(1)},\\
&\text{(YDB):}\quad \phi(h\triangleright a)+\psi(h\triangleleft a)=\big[h,a_{(-1)}\big]\otimes a_{(0)}+a_{(-1)}\otimes h\triangleright a_{(-1)}+h_1\otimes h_2\triangleright a\\
&\qquad\qquad\qquad +h_{(0)}\otimes\big[h_{(1)},a\big]+h_{(0)}\triangleleft a\otimes h_{(1)}+h\triangleleft a_1\otimes a_2,
\end{aligned}$$

then we call (A, H) a *double matched pair*.

Note that (BB5) and (BB8) have appeared in [5, Proposition 8.3.5] when Majid constructs $A \blacktriangleright\!\!\lhd H$. Now we give the main result of this section.

Theorem 4.6. *Let (A, H) be matched pair of Lie algebras and Lie coalgebras and let (A, H) be double matched pair, A is a braided Lie bialgebra in ${}_H^H\mathcal{M}$, H is a braided Lie bialgebra in $\mathcal{M}_A^A$, define the double cross biproduct of A and H, denoted by $A \bowtie H$, $A \bowtie H = A \bowtie H$ as Lie algebra, $A \bowtie H = A \blacktriangleright\!\!\blacktriangleleft H$ as Lie coalgebra, then $A \bowtie H$ becomes a Lie bialgebra.*

Proof. First we check the axiom (LB) on $H \otimes A$. For $h \in H, a \in A$, by (BB5), (BB6), (YDB) we get the third equality below:

$$\begin{aligned}
\delta_D\big([h,a]\big)&=\delta_A(h\triangleright a)+\phi(h\triangleright a)-\tau\phi(h\triangleright a)+\delta_H(h\triangleleft a)+\psi(h\triangleleft a)-\tau\psi(h\triangleleft a)\\
&=\delta_A(h\triangleright a)+\delta_H(h\triangleleft a)+\phi(h\triangleright a)+\psi(h\triangleleft a)-\tau\big(\phi(h\triangleright a)+\psi(h\triangleleft a)\big)
\end{aligned}$$

$$
\begin{aligned}
&= h \triangleright a_1 \otimes a_2(1) + a_1 \otimes h \triangleright a_2(2) + h_{(0)} \triangleright a \otimes h_{(1)}(3) - h_{(1)} \otimes h_{(0)} \triangleright a(4)\\
&\quad + h_1 \otimes h_2 \triangleleft a(5) + h_1 \triangleleft a \otimes h_2(6) + a_{(-1)} \otimes h \triangleleft a_{(0)}(7) - h \triangleleft a_{(0)} \otimes a_{(-1)}(8)\\
&\quad + \big[h, a_{(-1)}\big] \otimes a_{(0)}(9) + a_{(-1)} \otimes h \triangleright a_{(-1)}(10) + h_1 \otimes h_2 \triangleright a(11)\\
&\quad + h_{(0)} \otimes \big[h_{(1)}, a\big](12) + h_{(0)} \triangleleft a \otimes h_{(1)}(13) + h \triangleleft a_1 \otimes a_2(14)\\
&\quad - a_{(0)} \otimes \big[h, a_{(-1)}\big](15) - h \triangleright a_{(-1)} \otimes a_{(-1)}(16) - h_2 \triangleright a \otimes h_1(17)\\
&\quad - \big[h_{(1)}, a\big] \otimes h_{(0)}(18) - h_{(1)} \otimes h_{(0)} \triangleleft a(19) - a_2 \otimes h \triangleleft a_1(20)\\
&= h \triangleright (\delta + \phi - \tau\phi)(a) + (\delta + \psi - \tau\psi)(h) \triangleleft a = h \triangleright \delta_D(a) + \delta_D(h) \triangleleft a,
\end{aligned}
$$

where

$$
\begin{aligned}
h \triangleright \delta(a) &= (1) + (17) + (2) + a_1 \otimes h \triangleleft a_2 = (1) + (17) + (2) - (20),\\
h \triangleright \phi(a) &= (5) + (6) + (13),\ h \triangleright \tau\phi(a) = (9) + (14) + (8),\\
\delta(h) \triangleleft a &= (7) + (11) - (10) + (12),\ \psi(h) \triangleleft a = (15) + (3) + (16),\\
\tau\psi(h) \triangleleft a &= (4) + (19) + (18),
\end{aligned}
$$

thus the fourth equality.

Next we investigate the case of (LB) on $A \otimes A$:

$$
\begin{aligned}
\delta_D\big([a,b]\big) &= \delta_A\big([a,b]\big) + \phi\big([a,b]\big) - \tau\phi\big([a,b]\big)\\
&= \big[a, b_1\big] \otimes b_2(1) + b_1 \otimes \big[a, b_2\big](2) + a_1 \otimes \big[a_2, b\big](3) + \big[a_1, b\big] \otimes a_2(4)\\
&\quad - b_{(-1)} \triangleright a \otimes b_{(0)}(5) - a_{(0)} \otimes a_{(-1)} \triangleright b(6) + a_{(-1)} \triangleright b \otimes a_{(0)}(7)\\
&\quad + b_{(0)} \otimes b_{(-1)} \triangleright a(8) + a_{(-1)} \otimes \big[a_{(0)}, b\big](9) + b_{(-1)} \otimes \big[a, b_{(0)}\big](10)\\
&\quad + a_{(-1)} \triangleleft b \otimes a_{(0)}(11) - b_{(-1)} \triangleleft a \otimes b_{(0)}(12) - \big[a_{(0)}, b\big] \otimes a_{(-1)}(13)\\
&\quad - \big[a, b_{(0)}\big] b \otimes {}_{(-1)}(14) - a_{(0)} \otimes a_{(-1)} \triangleleft b(15) + b_{(0)} \otimes b_{(-1)} \triangleleft a(16)\\
&= a \triangleright (\delta + \phi - \tau\phi)(b) + (\delta + \phi - \tau\phi)(a) \triangleleft b = a \triangleright \delta_D(b) + \delta_D(a) \triangleleft b,
\end{aligned}
$$

where by (LBS) we get $\delta_A([a,b]) = (1) + \cdots + (8)$; by (BB7) we get $\phi([a,b]) = (9) + (10) + (11) - (12)$. In the next equality,

$$
\begin{aligned}
&a \triangleright \delta(b) = (1) + (2), \quad a \triangleright \phi(b) = -(5) - (12) + (10), \quad a \triangleright \tau\phi(b) = (14) - (8) - (16),\\
&\delta(a) \triangleleft b = (3) + (4), \quad \phi(a) \triangleleft b = (9) + (7) + (11), \quad \tau\phi(a) \triangleleft b = (6) + (15) + (13).
\end{aligned}
$$

□

We remark at this moment how Theorem 4.6 generalized Majid's Theorem 4.4. On the one hand, we get an ordinary Lie bialgebra through two braided Lie bialgebras A and H as in Theorem 4.6, here A need not be a Lie bialgebra as in Theorem 4.4. On the other hand, when the maps β and ψ are zero maps, Theorem 4.6 reduces to Theorem 4.4.

5 Bicycle bicrossproduct Lie bialgebras

This section is absolutely new, we construct bi-cycle bicrossproduct Lie bialgebras, which is a generalization of double cross biproduct. Let A and H be both simultaneously Lie algebras and Lie coalgebras, A is a left H-module and a left H-comodule, H is a right A-module and a right A-comodule, denote maps

$$\sigma : H\otimes H \to A, \quad \theta : A\otimes A \to H, \quad P : A \to H\otimes H, \quad Q : H \to A\otimes A$$

by

$$\sigma(h,g)\in A, \quad \theta(a,b)\in H, \quad P(a)=\sum a_{[1]}\otimes a_{[2]}\in H\otimes H, \quad Q(h)=\sum h_{\langle 1\rangle}\otimes h_{\langle 2\rangle}\in A\otimes A,$$

we always omit the sum notation.

Let H be a Lie algebra and for a left H-module A, an antisymmetric bilinear map $\sigma : H\otimes H \to A$ is a cocycle on H if and only if

$$\text{(CC1)}: \quad h\triangleright\sigma(g,l)+g\triangleright\sigma(l,h)+l\triangleright\sigma(h,g)=\sigma\big([h,g],l\big)+\sigma\big([g,l],h\big)+\sigma\big([l,h],g\big).$$

Let A be a Lie algebra and for a right A-module H, an antisymmetric bilinear map $\theta : A\otimes A \to H$ is a cocycle on A if and only if

$$\text{(CC2)}: \quad \theta(a,b)\triangleleft c+\theta(b,c)\triangleleft a+\theta(c,a)\triangleleft b=\theta\big(a,[b,c]\big)+\theta\big(b,[c,a]\big)+\theta\big(c,[a,b]\big).$$

Let H be a Lie coalgebra and for a left H-comodule A, an antisymmetric bilinear map $P : A \to H\otimes H$ is a cycle on A if and only if

$$\begin{aligned}\text{(CC3)}: \quad & a_{(-1)}\otimes a_{(0)[1]}\otimes a_{(0)[2]}+a_{(0)[1]}\otimes a_{(0)[2]}\otimes a_{(-1)}+a_{(0)[2]}\otimes a_{(-1)}\otimes a_{(0)[1]}\\ &=a_{[1]1}\otimes a_{[1]2}\otimes a_{[2]}+a_{[1]2}\otimes a_{[2]}\otimes a_{[1]1}+a_{[2]}\otimes a_{[1]1}\otimes a_{[1]2}.\end{aligned}$$

Let A be a Lie coalgebra and for a left A-comodule H, an antisymmetric bilinear map $Q : H \to A\otimes A$ is a cycle on H if and only if

$$\begin{aligned}\text{(CC4)}: \quad & h_{(0)\langle 1\rangle}\otimes h_{(0)\langle 2\rangle}\otimes h_{(1)}+h_{(1)}\otimes h_{(0)\langle 1\rangle}\otimes h_{(0)\langle 2\rangle}+h_{(0)\langle 2\rangle}\otimes h_{(1)}\otimes h_{(0)\langle 1\rangle}\\ &=h_{\langle 1\rangle}\otimes h_{\langle 2\rangle 1}\otimes h_{\langle 2\rangle 2}+h_{\langle 2\rangle 2}\otimes h_{\langle 1\rangle}\otimes h_{\langle 2\rangle 1}+h_{\langle 2\rangle 1}\otimes h_{\langle 2\rangle 2}\otimes h_{\langle 1\rangle}.\end{aligned}$$

In the following definitions, we introduced the concept of cocycle Lie algebras and cycle Lie coalgebras, which are in fact not really ordinary Lie algebras and Lie coalgebras, but weaker structures than them.

Definition 5.1. (i) Let σ be an anticommutativity map on a vector space H satisfying (CC1)(we call it a cocycle over H), equipped with a anticommutativity map $[\,,\,] : H\otimes H \to H$, satisfying the following cocycle Jacobi identity:

$$\text{(CC5)}: \quad \big[[h,g],l\big]+\big[[g,l],h\big]+\big[[l,h],g\big]=h\triangleleft\sigma(g,l)+g\triangleleft\sigma(l,h)+l\triangleleft\sigma(h,g),$$

then we call H a left cocycle Lie algebra.

(ii) Let θ be an anticommutativity map on a vector space A satisfying (CC2)(we call it a cocycle over A), equipped with a anticommutativity map $[\,,\,] : A\otimes A \to A$, satisfying the following cocycle Jacobi identity:

$$\text{(CC6)}: \quad \big[[a,b],c\big]+\big[[b,c],a\big]+\big[[c,a],b\big]=\theta(a,b)\triangleright c+\theta(b,c)\triangleright a+\theta(c,a)\triangleright b,$$

then we call A a right cocycle Lie algebra.

(iii) Let P be an anticocommutativity map on a vector space H satisfying (CC3) (we call it a cycle over H), equipped with a anticommutativity map $\delta : H \to H \otimes H$, satisfying the following cycle co-Jacobi identity:

$$\begin{aligned}\text{(CC7):}\quad & h_{11} \otimes h_{12} \otimes h_2 + h_{12} \otimes h_2 \otimes h_{11} + h_2 \otimes h_{11} \otimes h_{12}\\ &= h_{(0)} \otimes h_{(1)[1]} \otimes h_{(1)[2]} + h_{(1)[1]} \otimes h_{(1)[2]} \otimes h_{(0)} + h_{(1)[2]} \otimes h_{(0)} \otimes h_{(1)[1]},\end{aligned}$$

then we call H a left cycle Lie coalgebra.

(iv) Let Q be an anticocommutativity map on a vector space A satisfying (CC4) (we call it a cocycle over A), equipped with a anticommutativity map $\delta : A \to A \otimes A$, satisfying the following cycle co-Jacobi identity:

$$\begin{aligned}\text{(CC8):}\quad & a_{11} \otimes a_{12} \otimes a_2 + a_{12} \otimes a_2 \otimes a_{11} + a_2 \otimes a_{11} \otimes a_{12}\\ &= a_{(-1)\langle 1\rangle} \otimes a_{(-1)\langle 2\rangle} \otimes a_{(0)} + a_{(-1)\langle 2\rangle} \otimes a_{(0)} \otimes a_{(-1)\langle 1\rangle} + a_{(0)} \otimes a_{(-1)\langle 1\rangle} \otimes a_{(-1)\langle 2\rangle},\end{aligned}$$

then we call A a right cycle Lie coalgebra.

Although the structure of cocycle Lie algebras and cycle Lie coalgebras may be interesting, we do not intend to devote on it. What we need is only the conditions from (CC5) to (CC8) when both sides of them become zero for H and A are ordinary Lie algebras and Lie coalgebras.

Definition 5.2. A *cocycle cross product system* is a pair of Lie algebras A and H, where A is a left H-module, H is a right A-module, $\sigma : H \otimes H \to A$ is a cocycle on H, $\theta : A \otimes A \to H$ is a cocycle on A and the following conditions are satisfied:

$$\begin{aligned}
&\text{(TM1):} && [h, g] \triangleright a + \big[\sigma(h, g), a\big] = h \triangleright (g \triangleright a) - g \triangleright (h \triangleright a) + \sigma(h, g \triangleleft a) + \sigma(h \triangleleft a, g),\\
&\text{(TM2):} && h \triangleleft [a, b] + \big[h, \theta(a, b)\big] = (h \triangleleft a) \triangleleft b - (h \triangleleft b) \triangleleft a + \theta(h \triangleright a, b) + \theta(a, h \triangleright b),\\
&\text{(TBB1):} && h \triangleright [a, b] + \sigma\big(h, \theta(a, b)\big) = [h \triangleright a, b] + [a, h \triangleright b] + (h \triangleleft a) \triangleright b - (h \triangleleft b) \triangleright a,\\
&\text{(TBB2):} && [h, g] \triangleleft a + \theta\big(\sigma(h, g), a\big) = [h, g \triangleleft a] + [h \triangleleft a, g] + h \triangleleft (g \triangleright a) - g \triangleleft (h \triangleright a).
\end{aligned}$$

Lemma 5.3. *Let (A, H) be a cocycle cross product system of Lie algebras, then $A_{\alpha,\theta}\#_{\beta,\sigma}H$ becomes a Lie algebra with brackets given by*

$$\big[(a, h), (b, g)\big]_D = \big([a, b] + h \triangleright b - g \triangleright a + \sigma(h, g), [h, g] + h \triangleleft b - g \triangleleft a + \theta(a, b)\big).$$

Proof. We should see

$$\big[h, [a, b]_D\big]_D + \big[a, [b, h]_D\big]_D + \big[b, [h, a]_D\big]_D = 0.$$

In fact,

$$\begin{aligned}
\big[h, [a, b]_D\big]_D &= h \triangleright [a, b] + h \triangleleft [a, b] + \big[h, \theta(a, b)\big],\\
\big[a, [b, h]_D\big]_D &= -[a, h \triangleright b] - \theta(a, h \triangleright b) + (h \triangleleft b) \triangleright a + (h \triangleleft b) \triangleleft a,\\
\big[b, [h, a]_D\big]_D &= [b, h \triangleright a] + \theta(b, h \triangleright a) - (h \triangleleft a) \triangleright b - (h \triangleleft a) \triangleleft b.
\end{aligned}$$

By (TM2) we get the result. The other cases can be easily checked too. □

Definition 5.4. A *cycle cross coproduct system* is a pair of Lie coalgebras A and H, where A is left H-comodule, H is right A-comodule, $P: A\to H\otimes H$ is a cycle on A, $\theta: A\otimes A\to H$ is a cocycle on A, $Q: H\to A\otimes A$ is a cycle over H and the following conditions are satisfied:

$$\begin{aligned}
\text{(TM3):}\quad & a_{(-1)1}\otimes a_{(-1)2}\otimes a_{(0)}+a_1\otimes a_{2[1]}\otimes a_{1[2]}\\
&=a_{(-2)}\otimes a_{(-1)}\otimes a_{(0)}-a_{(-1)}\otimes a_{(-2)}\otimes a_{(0)}\\
&\quad+a_{[1](-1)}\otimes a_{[1](0)}\otimes a_{[2]}+a_{[2](-1)}\otimes a_{[1]}\otimes a_{[2](0)},\\
\text{(TM4):}\quad & h_{(0)}\otimes h_{(1)1}\otimes h_{(1)2}+h_1\otimes h_{2\langle 1\rangle}\otimes h_{1\langle 2\rangle}\\
&=h_{(0)}\otimes h_{(1)}\otimes h_{(2)}-h_{(0)}\otimes h_{(2)}\otimes h_{(1)}\\
&\quad+h_{\langle 1\rangle(-1)}\otimes h_{\langle 1\rangle(0)}\otimes h_{\langle 2\rangle}+h_{\langle 2\rangle(-1)}\otimes h_{\langle 1\rangle}\otimes h_{\langle 2\rangle(0)},\\
\text{(TBB3):}\quad & a_{(-1)}\otimes a_{(0)1}\otimes a_{(0)2}+a_{[1]}\otimes a_{[2]\langle 1\rangle}\otimes a_{[2]\langle 2\rangle}\\
&=a_{1(-1)}\otimes a_{1(0)}\otimes a_2+a_{2(-1)}\otimes a_1\otimes a_{2(0)}\\
&\quad+a_{(-1)(0)}\otimes a_{(-1)(1)}\otimes a_{(0)}-a_{(-1)(0)}\otimes a_{(0)}\otimes a_{(-1)(1)},\\
\text{(TBB4):}\quad & h_{(0)1}\otimes h_{(0)2}\otimes h_{(1)}+h_{\langle 1\rangle[1]}\otimes h_{\langle 1\rangle[2]}\otimes h_{\langle 2\rangle}\\
&=h_1\otimes h_{2(0)}\otimes h_{2(1)}+h_{1(0)}\otimes h_2\otimes h_{1(1)}\\
&\quad+h_{(0)}\otimes h_{(1)(-1)}\otimes h_{(1)(0)}-h_{(1)(-1)}\otimes h_{(0)}\otimes h_{(1)(0)}.
\end{aligned}$$

Lemma 5.5. *Let (A,H) be a cycle cross coproduct system of Lie algebras. Define*

$$D=A^{\phi,P}\#^{\psi,Q}H$$

as the vector space $A\oplus H$ with the Lie cobracket

$$\delta_D(a)=(\delta_A+\phi-\tau\phi+P)(a),\quad \delta_D(h)=(\delta_H+\psi-\tau\psi+Q)(h),$$

that is,

$$\begin{aligned}
\delta_D(a)&=a_1\otimes a_2+a_{(-1)}\otimes a_{(0)}-a_{(0)}\otimes a_{(-1)}+P(a),\\
\delta_D(h)&=h_1\otimes h_2+h_{(0)}\otimes h_{(1)}-h_{(1)}\otimes h_{(0)}+Q(h),
\end{aligned}$$

this make $A^{\phi,P}\#^{\psi,Q}H$ into a Lie coalgebra.

Proof. We only check the braided co-Jacobi identity on A. By definition of δ_D we have

$$\begin{aligned}
(\delta_D\otimes\mathrm{id})\delta_D(a)&=\delta_D(a_1)\otimes a_2+\delta_D(a_{(-1)})\otimes a_{(0)}-\delta_D(a_{(0)})\otimes a_{(-1)}\\
&=a_{11}\otimes a_{12}\otimes a_2(a1)+a_{1(-1)}\otimes a_{1(0)}\otimes a_2(a2)\\
&\quad-a_{1(0)}\otimes a_{1(-1)}\otimes a_2(a3)+a_{(-1)1}\otimes a_{(-1)2}\otimes a_{(0)}(a4)\\
&\quad+a_{(-1)(0)}\otimes a_{(-1)(1)}\otimes a_{(0)}(a5)-a_{(-1)(1)}\otimes a_{(-1)(0)}\otimes a_{(0)}(a6)\\
&\quad-a_{(0)1}\otimes a_{(0)2}\otimes a_{(-1)}(a7)-a_{(0)(-1)}\otimes a_{(0)(0)}\otimes a_{(-1)}(a8)\\
&\quad+a_{(0)(0)}\otimes a_{(0)(-1)}\otimes a_{(-1)}(a9)+a_{1[1]}\otimes a_{1[2]}\otimes a_2(a10)\\
&\quad+a_{(-1)\langle 1\rangle}\otimes a_{(-1)\langle 2\rangle}\otimes a_{(0)}(a11)-a_{(0)[1]}\otimes a_{(0)[2]}\otimes a_{(-1)}(a12)\\
&\quad+a_{[1]1}\otimes a_{[1]2}\otimes a_{[2]}(a13)+a_{[1](0)}\otimes a_{[1](1)}\otimes a_{[2]}(a14)\\
&\quad-a_{[1](1)}\otimes a_{[1](0)}\otimes a_{[2]}(a15)+a_{[1]\langle 1\rangle}\otimes a_{[1]\langle 2\rangle}\otimes a_{[2]}(a16),
\end{aligned}$$

$$\begin{aligned}
\xi^2(\delta_D\otimes \mathrm{id})\delta_D(a) = {} & a_2\otimes a_{11}\otimes a_{12}(b1)+a_2\otimes a_{1(-1)}\otimes a_{1(0)}(b2)\\
& -a_2\otimes a_{1(0)}\otimes a_{1(-1)}(b3)+a_{(0)}\otimes a_{(-1)1}\otimes a_{(-1)2}(b4)\\
& +a_{(0)}\otimes a_{(-1)(0)}\otimes a_{(-1)(1)}(b5)-a_{(0)}\otimes a_{(-1)(1)}\otimes a_{(-1)(0)}(b6)\\
& -a_{(-1)}\otimes a_{(0)1}\otimes a_{(0)2}(b7)-a_{(-1)}\otimes a_{(0)(-1)}\otimes a_{(0)(0)}(b8)\\
& +a_{(-1)}\otimes a_{(0)(0)}\otimes a_{(0)(-1)}(b9)+a_2\otimes a_{1[1]}\otimes a_{1[2]}(b10)\\
& +a_{(0)}\otimes a_{(-1)\langle 1\rangle}\otimes a_{(-1)\langle 2\rangle}(b11)-a_{(-1)}\otimes a_{(0)[1]}\otimes a_{(0)[2]}(b12)\\
& +a_{[2]}\otimes a_{[1]1}\otimes a_{[1]2}(b13)+a_{[2]}\otimes a_{[1](0)}\otimes a_{[1](1)}(b14)\\
& -a_{[2]}\otimes a_{[1](1)}\otimes a_{[1](0)}(b15)+a_{[2]}\otimes a_{[1]\langle 1\rangle}\otimes a_{[1]\langle 2\rangle}(b16),\\
\xi(\delta_D\otimes \mathrm{id})\delta_D(a) = {} & a_{12}\otimes a_2\otimes a_{11}(c1)+a_{1(0)}\otimes a_2\otimes a_{1(-1)}(c2)\\
& -a_{1(-1)}\otimes a_2\otimes a_{1(0)}(c3)+a_{(-1)2}\otimes a_{(0)}\otimes a_{(-1)1}(c4)\\
& +a_{(-1)(1)}\otimes a_{(0)}\otimes a_{(-1)(0)}(c5)-a_{(-1)(0)}\otimes a_{(0)}\otimes a_{(-1)(1)}(c6)\\
& -a_{(0)2}\otimes a_{(-1)}\otimes a_{(0)1}(c7)-a_{(0)(0)}\otimes a_{(-1)}\otimes a_{(0)(-1)}(c8)\\
& +a_{(0)(-1)}\otimes a_{(-1)}\otimes a_{(0)(0)}(c9)+a_{1[2]}\otimes a_2\otimes a_{1[1]}(c10)\\
& +a_{(-1)\langle 2\rangle}\otimes a_{(0)}\otimes a_{(-1)\langle 1\rangle}(c11)-a_{(0)[2]}\otimes a_{(-1)}\otimes a_{(0)[1]}(c12)\\
& +a_{[1]2}\otimes a_{[2]}\otimes a_{[1]1}(c13)+a_{[1](1)}\otimes a_{[2]}\otimes a_{[1](0)}(c14)\\
& -a_{[1](0)}\otimes a_{[2]}\otimes a_{[1](1)}(c15)+a_{[1]\langle 2\rangle}\otimes a_{[2]}\otimes a_{[1]\langle 1\rangle}(c16).
\end{aligned}$$

It follows from the condition (CC8) on A that $(a11)+(b11)+(c11)=(a1)+(b1)+(c1)=0$; by (CC3) we have $(a13)-(b13)+(c13)-(b12)-(a12)-(c12)=0$; by (TM3) we get

$$\begin{aligned}
&(a10)-(b14)-(c15)+(a4)-(b8)+(c9)=0,\\
&(b10)-(c14)-(a15)+(b4)-(c8)+(a9)=0,\\
&(c10)-(a14)-(b15)+(c4)-(a8)+(b9)=0;
\end{aligned}$$

by (TBB3) we have

$$\begin{aligned}
&-(b7)-(b16)+(a2)-(c3)+(a5)-(c6)=0,\\
&-(c7)-(c16)+(b2)-(a3)+(b5)-(a6)=0,\\
&-(a7)-(a16)+(c2)-(b3)+(c5)-(b6)=0.
\end{aligned}$$

Therefore, (CL2) holds on A. Similarly, (CL2) holds on H. □

The following conditions are needed by the next theorem. Note that (TBB5), ..., (TBB8) are extended from (BB5), ..., (BB8); (TLB3) and (TLB4) are extended from (LBS); (TYD) from (YDB). Here (TLB1) and (TLB2) are new ones.

$$\begin{aligned}
\text{(TBB5):}\quad & \delta_A(h\triangleright a)+Q(h\triangleleft a)\\
&= h\triangleright a_1\otimes a_2+a_1\otimes h\triangleright a_2+h_{(0)}\triangleright a\otimes h_{(1)}-h_{(1)}\otimes h_{(0)}\triangleright a\\
&\quad +h_{\langle 1\rangle}\otimes \big[h_{\langle 2\rangle},a\big]+\big[h_{\langle 1\rangle},a\big]\otimes h_{\langle 2\rangle}+\sigma\big(h,a_{(-1)}\big)\otimes a_{(0)}-a_{(0)}\otimes\sigma\big(h,a_{(-1)}\big),\\
\text{(TBB6):}\quad & \delta_H(h\triangleleft a)+P(h\triangleright a)\\
&= h_1\otimes h_2\triangleleft a+h_1\triangleleft a\otimes h_2+a_{(-1)}\otimes h\triangleleft a_{(0)}-h\triangleleft a_{(0)}\otimes a_{(-1)}\\
&\quad +\big[h,a_{[1]}\big]\otimes a_{[2]}+a_{[1]}\otimes\big[h,a_{[2]}\big]+h_{(0)}\otimes\theta\big(h_{(1)},a\big)-\theta\big(h_{(1)},a\big)\otimes h_{(0)},
\end{aligned}$$

$$\begin{aligned}
\text{(TBB7):}\quad & \phi\big([a,b]\big)+\psi\theta(a,b)\\
&= a_{(-1)}\otimes\big[a_{(0)},b\big]+b_{(-1)}\otimes\big[a,b_{(0)}\big]+a_{(-1)}\triangleleft b\otimes a_{(0)}-b_{(-1)}\triangleleft a\otimes b_{(0)}\\
&\quad+\theta\big(a,b_1\big)\otimes b_2+\theta\big(a_1,b\big)\otimes a_2+a_{[1]}\otimes a_{[2]}\triangleright b-b_{[1]}\otimes b_{[2]}\triangleright a,\\
\text{(TBB8):}\quad & \psi\big([h,g]\big)+\phi\sigma[h,g]\\
&= \big[h,g_{(0)}\big]\otimes g_{(1)}+\big[h_{(0)},g\big]\otimes h_{(1)}+g_{(0)}\otimes h\triangleright g_{(1)}-h_{(0)}\otimes g\triangleright h_{(1)}\\
&\quad+h_1\otimes\sigma\big(h_2,g\big)+g_1\otimes\sigma\big(h,g_2\big)+h\triangleleft g_{\langle 1\rangle}\otimes g_{\langle 2\rangle}-g\triangleleft h_{\langle 1\rangle}\otimes h_{\langle 2\rangle},\\
\text{(TLB1):}\quad & \delta\theta(a,b)+P[a,b]\\
&= a_{(-1)}\otimes\theta\big(a_{(0)},b\big)+b_{(-1)}\otimes\theta\big(a,b_{(0)}\big)-\theta\big(a,b_{(0)}\big)\otimes b_{(-1)}-\theta\big(a_{(0)},b\big)\otimes a_{(-1)}\\
&\quad+a_{[1]}\otimes a_{[2]}\triangleleft b+a_{[1]}\triangleleft b\otimes a_{[2]}-b_{[1]}\triangleleft a\otimes b_{[2]}-b_{[1]}\otimes b_{[2]}\triangleleft a,\\
\text{(TLB2):}\quad & \delta\sigma(h,g)+Q[h,g]\\
&= \sigma\big(h_{(0)},g\big)\otimes h_{(1)}+\sigma\big(h,g_{(0)}\big)\otimes g_{(1)}-h_{(1)}\otimes\sigma\big(h_{(0)},g\big)-g_{(1)}\otimes\sigma\big(h,g_{(0)}\big)\\
&\quad+h\triangleright g_{\langle 1\rangle}\otimes g_{\langle 2\rangle}+g_{\langle 1\rangle}\otimes h\triangleright g_{\langle 2\rangle}-h_{\langle 1\rangle}\otimes g\triangleright h_{\langle 2\rangle}-g\triangleright h_{\langle 1\rangle}\otimes h_{\langle 2\rangle},\\
\text{(TLB3):}\quad & \delta\big([a,b]\big)+Q\theta(a,b)\\
&= \big[a,b_1\big]\otimes b_2+b_1\otimes\big[a,b_2\big]+a_1\otimes\big[a_2,b\big]+\big[a_1,b\big]\otimes a_2\\
&\quad-b_{(-1)}\triangleright a\otimes b_{(0)}-a_{(0)}\otimes a_{(-1)}\triangleright b+a_{(-1)}\triangleright b\otimes a_{(0)}+b_{(0)}\otimes b_{(-1)}\triangleright a,\\
\text{(TLB4):}\quad & \delta\big([h,g]\big)+Q\theta(h,g)\\
&= \big[h,g_1\big]\otimes g_2+g_1\otimes\big[h,g_2\big]+h_1\otimes\big[h_2,g\big]+\big[h_1,g\big]\otimes h_2\\
&\quad-h_{(0)}\otimes g\triangleright h_{(1)}-h\triangleleft g_{(1)}\otimes g_{(0)}+g_{(0)}\otimes h\triangleleft g_{(1)}+g\triangleleft h_{(1)}\otimes h_{(0)},\\
\text{(TYB):}\quad & \phi(h\triangleright a)+\psi(h\triangleleft a)\\
&= \big[h,a_{(-1)}\big]\otimes a_{(0)}+a_{(-1)}\otimes h\triangleright a_{(-1)}+h_1\otimes h_2\triangleright a+h_{(0)}\otimes\big[h_{(1)},a\big]\\
&\quad+h_{(0)}\triangleleft a\otimes h_{(1)}+h\triangleleft a_1\otimes a_2+a_{[1]}\otimes\sigma\big(h,a_{[2]}\big)+h_{\langle 1\rangle}\otimes\theta\big(h_{\langle 2\rangle},a\big).
\end{aligned}$$

Definition 5.6. (i) A *left bi-cycle braided Lie bialgebras* H is simultaneously a left cocycle Lie algebra and a left cycle Lie coalgebra satisfying the above condition (TLB4).

(ii) A *right bi-cycle braided Lie bialgebras* A is simultaneously a right cocycle Lie a algebra and right cycle Lie coalgebra satisfying the above condition (TLB3).

The next theorem says that we can get an ordinary Lie bialgebra from two bi-cycle braided Lie bialgebras.

Theorem 5.7. *Let (A,H) be a cocycle cross product system and a cycle cross coproduct system, then the cocycle cross product Lie algebra and cycle cross coproduct Lie coalgebra fit together to form an ordinary Lie bialgebra if the conditions (TBB5)–(TBB8), (TLB1)–(TLB4), and (TYB) are satisfied. We call it the bi-cycle bicrossproduct Lie bialgebra and denote it by $A^{\phi,P}_{\alpha,\theta}\#^{\psi,Q}_{\beta,\sigma}H$.*

Proof. We investigate the case of (LB) on $A\otimes A$:

$$\begin{aligned}
\delta_D\big([a,b]_D\big) &= \delta_D\big([a,b]+\theta(a,b)\big)\\
&= \delta\big([a,b]\big)+\phi\big([a,b]\big)-\tau\phi\big([a,b]\big)+P\big([a,b]\big)\\
&\quad+\delta\theta(a,b)+\psi\theta(a,b)-\tau\psi\theta(a,b)+Q\theta(a,b).
\end{aligned}$$

Denote the right-hand side terms by $(a),(b),\ldots,(h)$:

$$\begin{aligned}
a\triangleright\delta_D(b)+\delta_D(a)\triangleleft b &= a\triangleright(\delta+\phi-\tau\phi+P)(b)+(\delta+\phi-\tau\phi+P)(a)\triangleleft b\\
&= \big[a,b_1\big]\otimes b_2(1)+b_1\otimes\big[a,b_2\big](2)-b_{(-1)}\triangleright a\otimes b_{(0)}(3)\\
&\quad -b_{(-1)}\triangleleft a\otimes b_{(0)}(4)+b_{(-1)}\otimes\big[a,b_{(0)}\big](5)-\big[a,b_{(0)}\big]\otimes b_{(-1)}(6)\\
&\quad +b_{(0)}\otimes b_{(-1)}\triangleright a(7)+b_{(0)}\otimes b_{(-1)}\triangleleft a(8)+a_1\otimes\big[a_2,b\big](9)\\
&\quad +\big[a_1,b\big]\otimes a_2(10)+a_{(-1)}\otimes\big[a_{(0)},b\big](11)+a_{(-1)}\triangleright b\otimes a_{(0)}(12)\\
&\quad +a_{(-1)}\triangleleft b\otimes a_{(0)}(13)-a_{(0)}\otimes a_{(-1)}\triangleright b(14)-a_{(0)}\otimes a_{(-1)}\triangleleft b(15)\\
&\quad -\big[a_{(0)},b\big]\otimes a_{(-1)}(16)+\theta\big(a,b_1\big)\otimes b_2(17)+b_1\otimes\theta\big(a,b_2\big)(18)\\
&\quad +b_{(-1)}\otimes\theta\big(a,b_{(0)}\big)(19)-\theta\big(a,b_{(0)}\big)\otimes b_{(-1)}(20)-b_{[1]}\triangleright a\otimes b_{[2]}(21)\\
&\quad -b_{[1]}\triangleleft a\otimes b_{[2]}(22)-b_{[1]}\otimes b_{[2]}\triangleright a(23)-b_{[1]}\otimes b_{[2]}\triangleleft a(24)\\
&\quad +a_1\otimes\theta\big(a_2,b\big)(25)+\theta\big(a_1,b\big)\otimes a_2(26)+b_{(-1)}\otimes\theta\big(a_{(0)},b\big)(27)\\
&\quad -\theta\big(a_{(0)},b\big)\otimes b_{(-1)}(28)+a_{[1]}\otimes a_{[2]}\triangleright b(29)+a_{[1]}\otimes a_{[2]}\triangleleft b(30)\\
&\quad +a_{[1]}\triangleright b\otimes a_{[2]}(31)+a_{[1]}\triangleleft b\otimes a_{[2]}(32).
\end{aligned}$$

Then by (TLB3) we get

$$(a)+(h)=(1)+(2)+(9)+(10)-(3)-(14)+(12)+(7);$$

by (TBB7) we get

$$\begin{aligned}
(b)+(f)&=(11)+(5)+(13)-(4)+(17)+(26)+(29)-(23),\\
(c)+(g)&=(16)+(6)+(8)-(13)-(18)-(25)-(31)+(21);
\end{aligned}$$

by (TLB1) we get

$$(d)+(e)=(27)+(19)-(20)-(28)+(30)+(32)-(22)-(24).$$

We investigate the case of (LB) on $H\otimes H$:

$$\begin{aligned}
\delta_D\big([h,g]_D\big) &= \delta_D\big([h,g]+\sigma(h,g)\big)\\
&= \delta\big([h,g]\big)+\psi\big([h,g]\big)-\tau\psi\big([h,g]\big)+Q\big([h,g]\big)\\
&\quad +\delta\sigma(h,g)+\phi\sigma(h,g)-\tau\phi\sigma(h,g)+P\sigma(h,g).
\end{aligned}$$

Denote the right-hand side terms by $(a),(b),\ldots,(h)$:

$$\begin{aligned}
h\triangleright\delta_D(g)+\delta_D(h)\triangleleft g &= h\triangleright\big(\delta+\psi-\tau\psi+Q\big)(g)+\big(\delta+\psi-\tau\psi+Q\big)(h)\triangleleft g\\
&= \big[h,g_1\big]\otimes g_2(1)+g_1\otimes\big[h,g_2\big](2)+\big[h,g_{(0)}\big]\otimes g_{(1)}(3)\\
&\quad +g_{(0)}\otimes h\triangleright g_{(1)}(4)+g_{(0)}\otimes h\triangleleft g_{(1)}(5)-h\triangleright g_{(1)}\otimes g_{(0)}(6)\\
&\quad -h\triangleleft g_{(1)}\otimes g_{(0)}(7)-g_{(1)}\otimes\big[h,g_{(0)}\big](8)+h_1\otimes\big[h_2,g\big](9)\\
&\quad +\big[h_1,g\big]\otimes h_2(10)-h_{(0)}\otimes g\triangleright h_{(1)}(11)-h_{(0)}\otimes g\triangleleft h_{(1)}(12)\\
&\quad +\big[h_{(0)},g\big]\otimes h_{(1)}(13)-h_{(1)}\otimes\big[h_{(0)},g\big](14)+g\triangleright h_{(1)}\otimes h_{(0)}(15)\\
&\quad +g\triangleleft h_{(1)}\otimes h_{(0)}(16)+\sigma\big(h,g_1\big)\otimes g_2(17)+g_1\otimes\sigma\big(h,g_2\big)(18)
\end{aligned}$$

$$
\begin{aligned}
&+\sigma\big(h, g_{(0)}\big)\otimes g_{(1)}(19)-g_{(1)}\otimes\sigma\big(h, g_{(0)}\big)(20)+h\triangleright g_{\langle 1\rangle}\otimes g_{\langle 2\rangle}(21)\\
&+h\triangleleft g_{\langle 1\rangle}\otimes g_{\langle 2\rangle}(22)+g_{\langle 1\rangle}\otimes h\triangleright g_{\langle 2\rangle}(23)+g_{\langle 1\rangle}\otimes h\triangleleft g_{\langle 2\rangle}(24)\\
&+h_1\otimes\sigma\big(h_2, g\big)(25)+\sigma\big(h_1, g\big)\otimes h_2(26)+\sigma\big(h_{(0)}, g\big)\otimes h_{(1)}(27)\\
&-h_{(1)}\otimes\sigma\big(h_{(0)}, g\big)(28)-h_{\langle 1\rangle}\otimes g\triangleright h_{\langle 2\rangle}(29)-h_{\langle 1\rangle}\otimes g\triangleleft h_{\langle 2\rangle}(30)\\
&-g\triangleright h_{\langle 1\rangle}\otimes h_{\langle 2\rangle}(31)-g\triangleleft h_{\langle 1\rangle}\otimes h_{\langle 2\rangle}(32).
\end{aligned}
$$

Then by (TLB4) we get

$$(a)+(h)=(1)+(2)+(9)+(10)-(12)-(7)+(5)+(16);$$

by (TBB8) we get

$$
\begin{aligned}
(b)+(f)&=(3)+(13)+(4)-(11)+(25)+(18)+(22)-(32),\\
(c)+(g)&=(8)+(14)+(6)-(15)-(26)-(17)-(24)+(30);
\end{aligned}
$$

by (TLB2) we get

$$(d)+(e)=(27)+(19)-(20)-(28)+(21)+(23)-(29)-(31).$$

We now check the axiom (LB) on $H\otimes A$. For $h\in H, a\in A$, we get the equality below:

$$
\begin{aligned}
\delta_D\big([h,a]_D\big)&=\delta_D(h\triangleright a)+\delta_D(h\triangleleft a)\\
&=\delta_A(h\triangleright a)+\phi(h\triangleright a)-\tau\phi(h\triangleright a)+P(h\triangleright a)\\
&\quad+\delta_H(h\triangleleft a)+\psi(h\triangleleft a)-\tau\psi(h\triangleleft a)+Q(h\triangleleft a).
\end{aligned}
$$

Denote the right-hand side terms by $(a),(b),\ldots,(h)$:

$$
\begin{aligned}
h\triangleright\delta_D(a)+\delta_D(h)\triangleleft a&=h\triangleright(\delta+\phi-\tau\phi+P)(a)+(\delta+\psi-\tau\psi+Q)(h)\triangleleft a\\
&=h\triangleright a_1\otimes a_2(1)+h\triangleleft a_1\otimes a_2(2)+a_1\otimes h\triangleright a_2(3)\\
&\quad+a_1\otimes h\triangleleft a_2(4)+\big[h, a_{(-1)}\big]\otimes a_{(0)}(5)+a_{(-1)}\otimes h\triangleright a_{(0)}(6)\\
&\quad+a_{(-1)}\otimes h\triangleleft a_{(0)}(7)-h\triangleright a_{(0)}\otimes a_{(-1)}(8)-h\triangleleft a_{(0)}\otimes a_{(-1)}(9)\\
&\quad-a_{(0)}\otimes\big[h, a_{(-1)}\big](10)+h_1\otimes h_2\triangleright a(11)+h_1\otimes h_2\triangleleft a(12)\\
&\quad+h_1\triangleright a\otimes h_2(13)+h_1\triangleleft a\otimes h_2(14)+h_{(0)}\otimes\big[h_{(1)}, a\big](15)\\
&\quad+h_{(0)}\triangleright a\otimes h_{(1)}(16)+h_{(0)}\triangleleft a\otimes h_{(1)}(17)-h_{(1)}\otimes h_{(0)}\triangleright a(18)\\
&\quad-h_{(1)}\otimes h_{(0)}\triangleleft a(19)-\big[h_{(1)}, a\big]\otimes h_{(0)}(20)+\sigma\big(h, a_{(-1)}\big)\otimes a_{(0)}(21)\\
&\quad-a_{(0)}\otimes\sigma\big(h, a_{(-1)}\big)(22)+\big[h, a_{[1]}\big]\otimes a_{[2]}(23)+\sigma\big(h, a_{[1]}\big)\otimes a_{[2]}(24)\\
&\quad+a_{[1]}\otimes\big[h, a_{[2]}\big](25)+a_{[1]}\otimes\sigma\big(h, a_{[2]}\big)(26)+h_{(0)}\otimes\theta\big(h_{(1)}, a\big)(27)\\
&\quad-\theta\big(h_{(1)}, a\big)\otimes h_{(0)}(28)+h_{\langle 1\rangle}\otimes\theta\big(h_{\langle 2\rangle}, a\big)(29)+h_{\langle 1\rangle}\otimes\big[h_{\langle 2\rangle}, a\big](30)\\
&\quad+\big[h_{\langle 1\rangle}, a\big]\otimes h_{\langle 2\rangle}(31)+\theta\big(h_{\langle 1\rangle}, a\big)\otimes h_{\langle 2\rangle}(32).
\end{aligned}
$$

Then by (TBB5) we get

$$(a)+(h)=(1)+(3)+(16)-(18)+(29)+(31)+(21)-(22);$$

by (TBB6) we get

$$(d) + (e) = (12) + (14) + (7) - (8) + (23) + (25) + (27) - (28);$$

by (TYB) we get

$$(b) + (f) = (5) + (6) + (11) + (15) + (17) + (2) + (26) + (32),$$
$$(c) + (g) = (10) + (9) - (13) + (20) + (19) - (4) - (26) - (30). \qquad \square$$

When the maps σ, θ, P, Q are zeros in Theorem 5.7, we get Theorem 4.6. When the maps ϕ, ψ, P, Q are zeros in Theorem 5.7, we get [7, Proposition 1.8]. In fact, any maps of α, β, ϕ, ψ, σ, θ, P, Q can be zero and then we get other types of propositions. What we need to do is to let the corresponding maps in the conditions in Theorem 5.7 be zero, we left this to the reader.

Added remarks

After finishing the work, the author finds that the notion of "braided Lie bialgebra" has been introduced earlier by Sommerhäuser in his unpublished paper [10] to give another construction of symmetrizable Kac-Moody algebras, where he calls it "Yetter-Drinfeld Lie algebra." On the other hand, our construction is somewhat the Lie version of the construction of bialgebra in [1, 2], so this paper can have another name "Cross Product Lie Bialgebras."

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