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A special form of Rund's h-curvature tensor using $R3$-like Finsler space

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Abstract

The purpose of the present paper is to consider and study a special form of Rund's h-curvature tensor K^i_{ljk} and Berwald's curvature tensor H^i_{ljk} in an $R3$-like C-reducible Finsler space. In this paper, we modify the Rund's h-curvature tensor K^i_{ljk} to special form by using some special Finsler spaces like C-reducible, $R3$-like Finsler spaces.

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1 Introduction

Let $F^n = (M^n, F)$ be an n-dimensional Finsler space with the fundamental function $F(x, y)$. The Finsler Γ_v-connection $R\Gamma$, constructed from the Cartan connection $C\Gamma$, is called the Rund connection. Matsumoto defined the curvature tensor $R\Gamma$ and the concept of special form of Rund's curvature tensor K^i_{ljk} [5]. The author [9] have studied Finsler space with Rund's h-curvature tensor K^i_{ljk} of a special form. Here we extend the study of a special form of Rund's h-curvature tensor using $R3$-like Finsler space and obtain some results. In this paper, the range of indexes varies from 1 to n and the v-covariant and o-covariant derivatives are denoted by $|j$ and $\|j$, respectively.

We use the following notations from [5, 8]:

$$
\begin{aligned}
&(a)\quad g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad g^{ij} = \left(g_{ij}\right)^{-1}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}\\
&(b)\quad C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \quad C^k_{ij} = \frac{1}{2}g^{km}(\dot{\partial}_m g_{ij})\\
&(c)\quad h_{ij} = g_{ij} - l_i l_j, \quad h^i_j = \delta^i_j - l^i l_j\\
&(d)\quad C^i_{lr} y^l = 0, \quad h^i_l y^l = 0\\
&(e)\quad y^i L_k = y^i_k L, \quad g_{ij} y^i y^j = F^2, \quad l_i y^i = F\\
&(f)\quad L_k = y_k L
\end{aligned}
\tag{1.1}
$$

Definition 1.1 (see [1, 2])**.** A Finsler space F^n $(n > 3)$ is called $R3$-like, if the curvature tensor R_{hijk} is written in the form

$$
R_{hijk} = g_{hj} L_{ik} + g_{ik} L_{hj} - g_{hk} L_{ij} - g_{ij} L_{hk} \tag{1.2}
$$

where $L = L_{ij} g^{ij}$ is the tensor.

Definition 1.2 (see [2, 3]). A Finsler space F^n is called C-reducible if it satisfies the equation

$$C_{ijk} = \big(C_i h_{jk} + C_j h_{ki} + C_k h_{ij}\big)/(n+1) \tag{1.3}$$

where $C_i = C_{ijk} g^{jk}$.

Definition 1.3 (see [6]). A Finsler space F^n is called P-reducible if the torsion tensor P_{ijk} is written as

$$P_{ijk} = G_i h_{jk} + G_j h_{ki} + G_k h_{ij} \tag{1.4}$$

where $P_{ijk} = C_{ijk|0}$ and $G_i = C_{i|0}/(n+1)$.

The v-covariant derivative of P-reducible Finsler space is given by [9]

$$P^i_{jk|l} = G^i_{|l} h_{jk} + G_{j|l} h^i_k + G_{k|l} h^i_j \tag{1.5}$$

Definition 1.4 (see [7]). A Finsler space is called Landsberg Finsler space if $C_{ijk|0} = 0$.

We use the following identities from [5, 7, 9]:

$$\begin{aligned}
&\text{(a)}\quad R^i_{jk\|l} - K^i_{ljk} + U_{(jk)}\big\{P^i_{jr} P^r_{kl} + P^i_{kl|j}\big\} = 0\\
&\text{(b)}\quad R^i_{ljk} y^l = R^i_{0jk} = R^i_{jk}\\
&\text{(c)}\quad K^i_{ljk} + K^i_{jkl} + K^i_{klj} = 0\\
&\text{(d)}\quad H^i_{ljk} = R^i_{jk\|l},\\
&\text{(e)}\quad H^i_{ljk} + H^i_{jkl} + H^i_{klj} = 0,\\
&\text{(f)}\quad K^i_{ljk} = -K^i_{lkj}, \quad K_{lijk} = g_{ir} K^r_{ljk}\\
&\text{(g)}\quad h_{ij\|k} = 2C_{ijk} - L^{-2}\big(y_i h_{jk} + y_j h_{ik}\big)\\
&\text{(h)}\quad h^i_{j\|k} = -L^{-2}\big(y_j h^i_k + y^i h_{jk}\big)
\end{aligned} \tag{1.6}$$

where R^i_{jk} is the (v) h-torsion tensor and the suffix '0' means contraction with y^i. The notation $u_{(jk)}$ denotes the interchange on indices j and k and substraction.

We use the following lemma in the next section.

Lemma 1.5 (see [4]). *If the equation $v_{hi} h_{jk} + v_{ij} h_{hk} + v_{jh} h_{ik} = 0$ holds in F^n, then we have (1) $v_{ij} = 0, (n \geq 4)$ and (2) $v_{ij} = v(m_i n_j - m_j n_i)$, with reference to the Moor frame (l^i, m^i, n^i), where v is a scalar.*

2 Special form of Rund's h-curvature tensor K^i_{ljk}

Let F^n be a Finsler space with Rund's h-curvature tensor K^i_{ljk} of the special form [9]

$$K^i_{ljk} = U_{(jk)}\big\{A_{jk} h^i_l + D_{lk} h^i_j + E^i_j h_{kl}\big\} \tag{2.1}$$

where A_{jk}, D_{lj}, E^i_j, F^i_j, G_{lk} are Finsler tensor fields.

Consider h-curvature tensor of the form

$$K^i_{ljk} = R^i_{ljk} - C^i_{lr} R^r_{jk} \tag{2.2}$$

Using equations (1.2) and (1.3) in (2.2), we get

$$\begin{aligned}K^i_{ljk} = \big\{ &\big(\delta^i_j L_{lk} + g_{lk} L^i_j - \delta^i_k L_{lj} - g_{lj} L^i_k\big) \\ &- \big(\delta^r_j L_k + y_k L^r_j - \delta^r_k L_j - y_j L^r_k\big)\big(C^i h_{lr} + C_l h^i_r + C_r h^i_l\big)/(n+1)\big\}\end{aligned}$$

By using some Finsler identities, the above equation can be written as

$$\begin{aligned}K^i_{ljk} = &\big\{ \big(h^i_j + l^i l_j\big) L_{lk} + \big(h_{lk} + l_l l_k\big) L^i_j - \big(h^i_k + l^i l_k\big) L_{lj} - \big(h_{lj} + l_l l_j\big) L^i_k \big\} \\ &- \big\{ C^i h_{lr} \delta^r_j L_k + C_l h^i_r \delta^r_j L_k + C_r h^i_l \delta^r_j L_k + C^i h_{lr} y_k L^r_j C_l h^i_r y_k L^r_j + C_r h^i_l y_k L^r_j \\ &- C^i h_{lr} \delta^r_k L_j - C_l h^i_r \delta^r_k L_j - C_r h^i_l \delta^r_k L_j - C^i h_{lr} y_j L^r_k - C_l h^i_r y_j L^r_k - C_r h^i_l y_j L^r_k \big\}/(n+1)\end{aligned}$$

After simplification and the rearrange the terms, we get

$$\begin{aligned}K^i_{ljk} = &\big\{ h^i_j L_{lk} + h_{lk} L^i_j - h^i_k L_{lj} - h_{lj} L^i_k \big\} + \big\{ l^i l_j L h_{lk} + l_l l_k L h^i_j - l^i l_k L h_{lj} - l_l l_j L h^i_k \big\} \\ &- 2\big\{ C^i h_{lj} L y_k + C_l h^i_j L y_k + C_j h^i_l L y_k - C^i h_{lk} L y_j - C_l h^i_k L y_j - C_k h^i_l L y_j \big\}/(n+1)\end{aligned}$$

$$\begin{aligned}K^i_{ljk} = U_{(jk)} \big[&h^i_l \big(2 C_k L y_j\big)/(n+1) + h^i_j \big(L_{lk} + L l_l l_k - 2 C_l L y_k/(n+1)\big) \\ &+ h_{lk} \big(L^i_j + L l^i l_j + 2 C^i L y_j/(n+1)\big)\big]\end{aligned}$$

In simple form, the above equation can be written as a special form of (2.1) as

$$K^i_{ljk} = U_{(jk)} \big(h^i_l A_{jk} + h^i_j D_{lk} + h_{lk} E^i_j\big) \tag{2.3}$$

where

$$\begin{aligned}A_{jk} &= 2 C_k y_j L/(n+1) \\ D_{lk} &= \big(L_{lk} + L l_l l_k - 2 C_l L y_k/(n+1)\big) \\ E^i_j &= \big(L^i_j + L l^i l_j + 2 C^i L y_j\big)/(n+1)\end{aligned} \tag{2.4}$$

Thus we state the following.

Theorem 2.1. *In an R3-like, C-reducible Finsler space, the h-curvature tensor reduces to special form of Rund's h-curvature tensor (2.3).*

Now we compare the Rund's curvature tensor and h-curvature tensor. Thus, from (2.1) and (2.2), we have

$$R^i_{ljk} - C^i_{lr} R^r_{jk} = \big\{ A_{jk} h^i_l + D_{lk} h^i_j + E^i_j h_{kl} - A_{kj} h^i_l - D_{lj} h^i_k - E^i_k h_{jl} \big\} \tag{2.5}$$

Contracting (2.5) with respect to y^l, y^k and using (1.1d), we get

$$R^i_{j0} = D_{00} h^i_j \tag{2.6}$$

Again contracting (2.6) with respect to i and j, we get

$$R = (n-1) D_{00} \tag{2.7}$$

Now, we will find D_{00}. Consider D_{ij} from the special form (2.3), and contract this with respect to i and j, and by using (1.1e), we have

$$D_{00} = 2 L F^2 \tag{2.8}$$

Substituting (2.8) in (2.7), we have

$$R = 2(n-1) L F^2$$

Thus we state the following.

Theorem 2.2. *If the Rund's h-curvature tensor has the special form (2.1), then the scalar curvature of the space is* $2(n-1)LF^2$.

Let us suppose that F^n is $R3$-like C-reducible Finsler space. Then, by using (1.3) and (2.3), the h-curvature tensor (2.2) can written as

$$\begin{aligned}
R^i_{ljk} &= K^i_{ljk} + C^i_{lr}R^r_{jk} \\
R^i_{ljk} &= U_{(jk)}\big\{A_{jk}h^i_l + D_{lk}h^i_j + E^i_j h_{kl}\big\} \\
&\quad + \big\{\big(\delta^r_j L_k + y_k L^r_j - \delta^r_k Lj - y_j L^r_k\big)\big(C^i h_{lr} + C_l h^i_r + C_r h^i_l\big)/(n+1)\big\} \\
R^i_{ljk} &= \big\{A_{jk} + \big(2C_j L y_k\big)/(n+1)\big\}h^i_l - \big\{A_{kj} + \big(2C_k L y_j\big)/(n+1)\big\}h^i_l \\
&\quad + \big\{D_{lk} + 2C_l L y_k/(n+1)\big\}h^i_j - \big\{D_{lj} + 2C_l L y_j/(n+1)\big\}h^i_k \\
&\quad + \big\{E^i_j - 2C^i L y_j/(n+1)\big\}h_{kl} - \big\{E^i_k - 2C^i L y_k/(n+1)\big\}h_{jl} \\
R^i_{ljk} &= U_{(jk)}\big[\big\{A_{jk} + 2C_j L y_k/(n+1)\big\}h^i_l + \big\{D_{lk} + 2C_l L y_k/(n+1)\big\}h^i_j \\
&\qquad + \big\{E^i_j - 2C^i L y_j/(n+1)\big\}h_{kl}\big] \\
R^i_{ljk} &= U_{(jk)}\big\{Q_{jk}h^i_l + N_{lj}h^i_k + M^i_j h_{kl}\big\}
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
Q_{jk} &= \big\{A_{jk} + 2C_j L y_k/(n+1)\big\} \\
N_{lk} &= \big\{D_{lk} + 2C_l L y_k/(n+1)\big\} \\
M^i_j &= \big\{E^i_j - 2C^i L y_j/(n+1)\big\}
\end{aligned}$$

Thus we have the following.

Theorem 2.3. *In an R3-like C-reducible Finsler space, if the Rund's h-curvature tensor has the special form (2.3), then the Cartan h-curvature tensor* R^i_{ljk} *has the special form (2.9).*

Using the special form of Rund's h-curvature tensor K^i_{ljk} in the Bianchi identity (1.6d), we get

$$\begin{aligned}
&\big(A_{jk} - A_{kj} + D_{kj} - D_{jk}\big)h^i_l + \big(A_{kl} - A_{lk} + D_{lk} - D_{kl}\big)h^i_j \\
&\qquad + \big(A_{lj} - A_{jl} + D_{jl} - D_{lj}\big)h^i_k = 0
\end{aligned} \tag{2.10}$$

Due to Lemma 1.1, equation (2.10) can be written as

$$A_{jk} - A_{kj} = D_{jk} - D_{kj}$$

Thus we have the following.

Theorem 2.4. *If the Rund's h-curvature tensor* K^i_{ljk} *is of the special form (2.3), then both the tensor fields* A_{ij} *and* D_{ij} *are symmetric simultaneously.*

It is also known that a Finsler space is Landsberg space with $P_{ijk} = C_{ijk/0} = 0$. If F^n is Landsberg, then from (1.6a) and (1.6e), we get

$$R^i_{jk\|l} - K^i_{ljk} = 0 \quad \text{or} \quad H^i_{ljk} = K^i_{ljk}$$

Thus we can propose the following.

Corollary 2.5. *If F^n is a Landsberg space and the Rund's h-curvature tensor K^i_{ljk} is of the form (2.3), then Cartan curvature tensor coincides with the Berwald's curvature tensor.*

Now consider h-curvature tensor (1.6a) of the form

$$R^i_{jk\|l} = K^i_{ljk} - U_{(jk)}\{P^i_{jr}P^r_{kl} + P^i_{kl|j}\}$$

From equation (1.6e), the above equation can be written as

$$H^i_{ljk} = K^i_{ljk} - U_{(jk)}\{P^i_{jr}P^r_{kl} + P^i_{kl|j}\} \tag{2.11}$$

Suppose F^n is a P-reducible Finsler space, then by using (1.4), (1.5), (2.3), and (2.11), we have

$$\begin{aligned}
H^i_{ljk} &= U_{(jk)}\{A_{jk}h^i_l + D_{lk}h^i_j + E^i_j h_{kl}\} - U_{(jk)}\{P^i_{jr}P^r_{kl} + P^i_{kl|j}\}\\
H^i_{ljk} &= \{A_{jk}h^i_l + D_{lk}h^i_j + E^i_j h_{kl} - A_{kj}h^i_l - D_{lj}h^i_k - E^i_k h_{jl}\}\\
&\quad - U_{(jk)}\{(G^i h_{jr} + G_j h^i_r + G_r h^i_j)(G^r h_{kl} + G_k h^r_l + G_l h^r_k) + (G_{k|j}h^i_l + G_{l|j}h^i_k + G^i_{|j}h_{kl})\}\\
H^i_{ljk} &= \{A_{jk}h^i_l + D_{lk}h^i_j + E^i_j h_{kl} - A_{kj}h^i_l - D_{lj}h^i_k - E^i_k h_{jl}\} - (G_{k|j}h^i_l + G_{l|j}h^i_k + G^i_{|j}h_{kl})\\
&\quad + (G_{j|k}h^i_l + G_{l|k}h^i_j + G^i_{|k}h_{jl}) - \{G^i G_r h_{jr}h_{kl} + G^i G_k h_{jl} + G_j G^r h^i_r h_{kl} + G_j G_l h^i_k\\
&\quad + G_r G^r h^i_j h_{kl} + G_r G_k h^i_j h^r_l + G_r G_l h^i_j h^r_k - G^i G_r h_{kr}h_{jl} - G^i G_j h_{kl} - G_k G^r h^i_r h_{jl}\\
&\quad - G_k G_l h^i_j - G_r G^r h^i_k h_{jl} - G_r G_j h^i_k h^r_l - G_r G_l h^i_j h^r_k\}\\
H^i_{ljk} &= U_{(jk)}\{T_{jk}h^i_l + M_{lk}h^i_j + N^i_j h_{kl}\}
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
T_{jk} &= A_{jk} - g_{k|j}\\
M_{lk} &= D_{lk} + G_{l|k} + G_k G_l - G_r G^r h_{kl} - G_r G_k h^r_l - G_r G_l h^r_k\\
N^i_j &= E^i_j - G^i_{|j} - G^i G^r h_{jr} - G_j G^r h^i_r + G^i G_j
\end{aligned}$$

Thus we have the following.

Theorem 2.6. *In a P-reducible Finsler space, and the special form of Rund's h-curvature tensor K^i_{ljk} has the special form of Berwald's curvature tensor, then H^i_{ljk} is of the form (2.12).*

Consider the Bianchi identity

$$H^i_{ljk} + H^i_{jkl} + H^i_{klj} = 0 \tag{2.13}$$

Substituting (2.12) in (2.13), we get

$$\begin{aligned}
&\big[(T_{jk} - T_{kj} + M_{kj} - M_{jk})h^i_l + (T_{kl} - T_{lk} + M_{lk} - M_{kl})h^i_j\\
&\quad + (T_{lj} - T_{jl} + M_{jl} - M_{lj})h^i_k\big] = 0
\end{aligned} \tag{2.14}$$

Due to Lemma 1.1, equation (2.14) can be written as

$$T_{jk} - T_{kj} = M_{jk} - M_{kj}$$

Thus we have the following.

Theorem 2.7. *If in a Finsler space F^n the h-curvature tensor H^i_{ljk} is of the form (2.12), then the tensor fields T_{jk} and M_{kj} both are simultaneously symmetric.*

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