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Research Article

Geometry of Noncommutative k-Algebras*

Arvid Siqveland

Buskerud University College, Department of Technology, P.O. Box 251, N-3601 Kongsberg, Norway
Address correspondence to Arvid Siqveland, arvid.siqveland@hibu.no

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Abstract Let X be a scheme over an algebraically closed field k, and let $x \in \operatorname{Spec} R \subseteq X$ be a closed point corresponding to the maximal ideal $\mathfrak{m} \subseteq R$. Then $\hat{\mathcal{O}}_{X,x}$ is isomorphic to the prorepresenting hull, or local formal moduli, of the deformation functor $\operatorname{Def}_{R/\mathfrak{m}} : \underline{\ell} \to \text{Sets}$. This suffices to reconstruct X up to etalé coverings. For a noncommutative k-algebra A the simple modules are not necessarily of dimension one, and there is a geometry between them. We replace the points in the commutative situation with finite families of points in the noncommutative situation, and replace the geometry of points with the geometry of sets of points given by noncommutative deformation theory. We apply the theory to the noncommutative moduli of three-dimensional endomorphisms.

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1 Introduction

There have been several attempts to generalize the ordinary commutative algebraic geometry to the noncommutative situation. The main problem in the direct generalization is the lack of localization of noncommutative k-algebras. This can only be done for Ore sets, and does not give a satisfactory solution to the problem.

In the study of flat deformations of A-modules when A is a commutative, finitely generated k-algebra (k algebraically closed), one realizes that for each maximal ideal $\mathfrak{m}$, putting $V = A/\mathfrak{m}$, the deformation functor Def_V has a (unique up to nonunique isomorphism) prorepresenting hull (local formal moduli) $\hat{H}(V)$ isomorphic to the completed local ring, that is $\hat{H}(V) \cong \hat{A}_{\mathfrak{m}}$, see [5].

In the general situation with A not necessarily commutative, the deformation theory can be directly generalized to families of right (or left) A-modules, see [1] or [3], and we can replace the local complete rings with the local formal moduli of finite subsets of the simple modules. From now on, k denotes an algebraically closed field of characteristic zero. An A-module M is *simple* if it contains no other proper submodules but the zero module (0); it is *indecomposable* if it is not the sum of two proper submodules.

The following results from Eriksen [1] and Laudal [3] are assumed as a basis for this text.

Definition 1. $\mathbf{a}_r$ is the category of r-pointed Artinian k-algebras. An object of this category is an Artinian k-algebra R, together with a pair of structural ring homomorphisms $f : k^r \to R$ and $g : R \to k^r$ with $g \circ f = \mathrm{Id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. The morphisms of $\mathbf{a}_r$ are the ring homomorphims that commute with the structural morphisms.

For any family $V = \{V_1, \dots, V_r\}$ of right A-modules, there is a noncommutative deformation functor $\operatorname{Def}_V : \mathbf{a}_r \to \text{Sets}$. If $\operatorname{Ext}^1_A(V_i, V_j)$ has a finite k-dimension for $1 \le i, j \le r$, Laudal (or equally Eriksen) proves that Def_V has a formal moduli $(\hat{H}, \hat{M}_{\hat{H}})$, unique up to nonunique isomorphism. Given this, the local reconstruction theorem is the following.

Theorem 2 (the generalized Burnside theorem)**.** *Let A be a finite dimensional k-algebra, and let $V = \{V_1, \dots, V_r\}$ be the family of simple right A-modules. Then, the ($\hat{H}$-flat) proversal family $\eta : A \to (\hat{H}_{ij} \otimes_k \operatorname{Hom}_k(V_i, V_j))$ is an isomorphism ($\hat{H}_{ij} = e_i \hat{H} e_j$).*

We will use Laudal and Eriksen's results to define (geometric) formal localizations, and use this to define the noncommutative affine spectrum $\operatorname{Spec} A$. This leads to the definition of a noncommutative variety and its relation to noncommutative moduli. We will end the paper with a classical example, the moduli of 3×3-matrices up to conjugacy.

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2 r-pointed ringed spaces

Lemma 3. *Let A be a finitely generated, commutative k-algebra and $\mathfrak{m}_1$, $\mathfrak{m}_2$ two different maximal ideals with corresponding simple modules $V_i = A/\mathfrak{m}_i$, $i = 1, 2$. Then,* $\operatorname{Ext}^1_A(V_1, V_2) = 0$.

Proof. It is enough to consider

$$A = k\big[x_1, \dots, x_n\big], \quad \mathfrak{m}_1 = \big(x_1, \dots, x_n\big), \quad \mathfrak{m}_2 = \big(x_1 - \alpha_1, \dots, x_n - \alpha_n\big)$$

with $\alpha_1 \neq 0$. First of all, it is well known that

$$\operatorname{Ext}^1_A\left(V_1, V_2\right) = \operatorname{HH}^1\left(A, \operatorname{Hom}_k\left(V_1, V_2\right)\right) = \operatorname{Der}_k\left(A, \operatorname{Hom}_k\left(V_1, V_2\right)\right)/\operatorname{Inner}.$$

The inner derivations are given by $ad_\gamma(x_i) = \gamma x_i - x_i\gamma = \gamma\alpha_i$ in this case, and this determines the (inner) derivations completely. Now, let $\delta : A \to \operatorname{Hom}(V_1, V_2)$ be a derivation. Then, since A is commutative,

$$0 = \delta\big(x_1\big(x_i - \alpha_i\big)\big) = \delta\big(\big(x_i - \alpha_i\big)x_1\big) = -\alpha_i\delta\big(x_1\big) + \delta\big(x_i\big)\alpha_1 \Longrightarrow \delta\big(x_i\big) = \frac{\alpha_i}{\alpha_1}\delta\big(x_1\big) \Longrightarrow \delta = ad_{\frac{\delta(x_1)}{\alpha_1}}$$

which proves that every derivation is inner. $\square$

In the noncommutative case, the above result is obviously no longer true, so that if a scheme should be a classifying space for the simple modules of a noncommutative k-algebra, it should consider *sets* of points and their infinitesimal geometry. This is then necessary for the reconstruction of k-algebras in general. We will see that in some cases this is also sufficient.

3 Matrix algebras

To ease the explicit understanding of noncommutative varieties, we now treat the explicit case here. To introduce notation, we give an example with an obvious generalization.

Example 4. Consider the following matrix variables

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad t_{11}(1) = \begin{pmatrix} t_{11}(1) & 0 \\ 0 & 0 \end{pmatrix}, \quad t_{11}(2) = \begin{pmatrix} t_{11}(2) & 0 \\ 0 & 0 \end{pmatrix},$$

$$t_{12} = \begin{pmatrix} 0 & t_{12} \\ 0 & 0 \end{pmatrix}, \quad t_{21} = \begin{pmatrix} 0 & 0 \\ t_{21} & 0 \end{pmatrix}, \quad t_{22} = \begin{pmatrix} 0 & 0 \\ 0 & t_{22} \end{pmatrix}.$$

The free 2×2 matrix k-algebra *generated* by these elements by ordinary matrix multiplication is then denoted

$$F = \begin{pmatrix} k\big\langle t_{11}(1), t_{11}(2)\big\rangle & t_{12} \\ t_{21} & k\big[t_{22}\big] \end{pmatrix}.$$

Let

$$f_{11} = t_{11}(1)t_{12}t_{21} - t_{11}(2) = \begin{pmatrix} t_{11}(1)t_{12}t_{21} - t_{11}(2) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

We consider the two-sided ideal in F generated by f_{11}, that is $\mathfrak{a} = \langle f_{11}\rangle$, and for the quotient algebra we use the notation

$$Q = F/\mathfrak{a} = \begin{pmatrix} k\big\langle t_{11}(1), t_{11}(2)\big\rangle & t_{12} \\ t_{21} & k\big[t_{22}\big] \end{pmatrix} \Big/ \begin{pmatrix} f_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case $Q = (Q_{ij})$, and $k\langle t_{11}(1), t_{11}(2)\rangle$ maps injective into Q, but $Q_{11} \neq k\langle t_{11}(1), t_{11}(2)\rangle$ as for example $t_{12}t_{21} \in Q_{11}$. However, letting $\langle Q - Q_{ii}\rangle$ be the ideal generated by the matrices in Q with 0 (i, i)-entry, we will write $\widetilde{Q}_{11} = k\langle t_{11}(1), t_{11}(2)\rangle = Q/\langle Q - Q_{11}\rangle$ when necessary.

Let $k^r \to R = (R_{ij})$ be a matrix algebra. We let $\langle R - R_{ii}\rangle$ denote the ideal generated by the matrices in R with 0 (i, i)-entry, and we let $\widetilde{R}_{ii}$ denote the quotient $R/\langle R - R_{ii}\rangle$. We call the algebras $\widetilde{R}_{ii}$ the *diagonal algebras* of the matrix algebra $R = (R_{ij})$. We let $\iota_{ii} : R \to R/\langle R - R_{ii}\rangle = \widetilde{R}_{ii}$ be the canonical morphism, and we let $\tau_{ii} : \widetilde{R}_{ii} \to R$ be the natural inclusion. Then, τ_{ii} obeys the rules for an algebra morphism except for the fact that $\tau_{ii}(1) \neq 1$. Thus, $\tau_{ii}^{-1}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ is an ideal.

Proposition 5. *There is a one to one correspondence between the right (left) maximal ideals in the matrix algebra R and the right (left) maximal ideals in its diagonal algebras.*

Proof. Let $\mathfrak{m} \subset R$ be a maximal ideal. Then, for some i, $1 \leq i \leq n$, $\tau_{ii}^{-1}(\mathfrak{m}) \neq \widetilde{R}_{ii}$ because $1 \in \mathfrak{m}$ otherwise. We see that for $m \in \mathfrak{m}$, $\tau_{ii}(\iota_{ii}(m)) \in \mathfrak{m}$ implying that $\iota_{ii}(m) \in \tau_{ii}^{-1}(\mathfrak{m})$ so that $\mathfrak{m} \subseteq \iota_{ii}^{-1}(\tau_{ii}^{-1}(\mathfrak{m}))$. Because $\mathfrak{m}$ is maximal, $\mathfrak{m} = \iota_{ii}^{-1}(\tau_{ii}^{-1}(\mathfrak{m}))$ and $\tau_{ii}^{-1}(\mathfrak{m})$ is a maximal ideal and together with the canonical surjection ι the correspondence is established. $\square$

4 Geometric localizations

The universal property of the localization L of a commutative k-algebra A in a maximal ideal $\mathfrak{m}$ is a diagram

$$\begin{tikzcd} A \arrow[r, "\rho_L"] \arrow[dr, "\kappa^A"'] & L \arrow[d, "\kappa^L"] \\ & A/\mathfrak{m} \end{tikzcd}$$

such that $\rho_L(a)$ is a unit in L whenever $\kappa_A(a)$ is a unit in $A/\mathfrak{m}$. For any other L' with this property, there exists a unique morphism $\phi : L \to L'$ such that $\rho_{L'} = \rho_L \circ \phi$.

This definition may very well be extended to the noncommutative situation, but it is well known that the localization process works only for Ore sets. In the following, A is a not necessarily commutative k-algebra.

Lemma 6. *V is a simple A module if the structure morphism $\rho : A \twoheadrightarrow \mathrm{End}_k(V)$ is surjective. If k is algebraically closed, the converse holds.*

Proof. Let W be a submodule of V, let $0 \neq w \in W$ be an element, and let $v \in V$ be any element. Let $\phi : V \to V$ be the linear transformation sending w to v and all other elements in a basis for W to 0. Then, $\phi = \rho_a$ for some $a \in A$ because of the surjectivity. Then, $v = \phi(w) = a \cdot w \in W$. This proves that $V = W$ and V is simple. The proof of the converse can be found in the introductory book of Lam [2]. $\square$

Definition 7. Let A be a (not necessarily commutative) k-algebra, and let $V = \{V_1, \ldots, V_n\}$ be simple right A-modules. Then, a k-algebra L is called a localization of A in V if there exists a diagram

$$\begin{tikzcd} A \arrow[r, "\rho_L"] \arrow[dr, "\kappa_i^A"'] & L \arrow[d, "\kappa_i^L"] \\ & \mathrm{Hom}_k\big(V_i, V_i\big) \end{tikzcd}$$

such that $\rho_L(a)$ is a unit in L whenever $\kappa_i^A(a)$ is a unit in $\mathrm{Hom}_k(V_i, V_i)$ for every i, $1 \leq i \leq n$, and if for any other L' with this property, there exists a unique $\phi : L \to L'$ such that $\rho_{L'} = \rho_L \circ \phi$.

Example 8. As an elementary example, let A be commutative and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be maximal ideals. Put $V_i = A/\mathfrak{m}_i$, $1 \leq i \leq n$. Then, $L = \oplus_{i=1}^n A_{\mathfrak{m}_i}$ fulfils the condition of being a localization of A in $V = \{V_i, \ldots, V_n\}$. Notice that the set of simple modules of L are the modules V.

Example 9. Let A be any k-algebra and $V_1, \ldots, V_n$ simple right A-modules. Assume that there exists a k-algebra $L = \oplus L_i \to \oplus \mathrm{Hom}_k(V_i, V_i) = \mathrm{End}_k(V)$ such that each L_i is finitely generated with V_i as the only simple L_i-module. Also assume that $\hat{H}_{L_i}(V_i) \cong \hat{H}_A(V_i)$ and that L_i is miniversal (in the meaning that L_i is an algebraization of $\hat{H}_A(V_i)$). Then $L \cong A_V$, the localization of A in the family V.

Knowing that the local formal moduli exists, we can replace the localizations with this. However, we do not know for certain that algebraizations exist. The (next) best we can do is the following: relaxing to some degree the universal property.

Definition 10. Let A be any k-algebra and $V = \{V_1, \dots, V_n\}$ a family of simple right A-modules. Then, L is called a *prolocalization of A in V* if there exist diagrams

$$\begin{tikzcd} A \arrow[r, "\rho_L"] \arrow[dr, "\kappa_i^A"'] & L \arrow[d, "\kappa_i^L"] \\ & \mathrm{Hom}_k\left(V_i, V_i\right) \end{tikzcd}$$

for each i, $1 \leq i \leq n$, such that $\rho_L(a)$ is a unit in L whenever $\kappa_i^A(a)$ is a unit for each i, and if for each i one has $\hat{H}_L(V_i) \cong \hat{H}_A(V_i)$. One writes $L = \hat{A}_V$ and notices that prolocalizations are not unique.

Lemma 11. *Prolocalizations exist.*

Proof. Note that $L = \oplus_{i=1}^n \hat{H}_A(V_i) \otimes_k \mathrm{End}_k(V_i)$ satisfies the properties of the definition. The homomorphism $\kappa_i^L : L \to \mathrm{End}_k(V_i)$ is surjective and $l \in L$ is a unit whenever $\kappa_i^L(l)$ is a unit in $\mathrm{End}_k(V_i)$ implying that $\{V_1, \dots, V_n\}$ is exactly the set of simple L-modules. Now, let $L_n = \oplus_{i=1}^n \hat{H}_A(V_i)/\operatorname{rad}^n \otimes_k \mathrm{End}_k(V_i)$. Then, by the generalized Burnsides theorem, Theorem 2, we have the matrix algebra $(\hat{H}_{L_n}(i,j) \otimes_k \mathrm{Hom}_k(V_i, V_j)) \cong L_n$, implying in particular that $\hat{H}_{L_n}(i,i) = \hat{H}_{L_n}(V_i) \cong \hat{H}_A/\operatorname{rad}^n(V_i)$. Taking the projective limit, we then end at $\hat{H}_L(V_i) \cong \hat{H}_A(V_i)$ for each i, proving the claim. □

Lemma 12. *Let $V = \{V_1, \dots, V_n\}$ be a set of simple right A-modules. Let $\hat{H}_V$ be the prolocalization of A in V. Then,* $\mathrm{Simp}(\hat{H}_V) = V$.

Proof. Note that $\hat{A}_V = \oplus_{i=1}^n \hat{H}_A(V_i) \otimes_k \mathrm{End}_k(V_i)$ maps surjectively onto $\mathrm{End}_k(V_i)$, so by Lemma 6, V_i is a simple $\hat{A}_V$-module, that is $V \subseteq \mathrm{Simp}(\hat{A}_V)$. It is also obvious that if $a \in \hat{H}_A(V_i) \otimes_k \mathrm{End}_k(V_i)$ maps to a unit in $\mathrm{End}_k(V_i)$, it is itself a unit. Thus, $\hat{H}_A(V_i) \otimes_k \mathrm{End}_k(V_i)$ is a local ring and the general result follows from Proposition 5. □

Now we come to the main point of this section. For moduli situations, we have to be concerned with the geometry between the different simple objects. This also strengthen the universal property of the localizations we consider.

Definition 13 (geometric prolocalizations)**.** Let A be any k-algebra and $V = \{V_1, \dots, V_n\}$ a family of simple right A-modules. Then, L is called a *geometric prolocalization of A in V* if there exists diagrams

$$\begin{tikzcd} A \arrow[r, "\rho_L"] \arrow[dr, "\kappa_i^A"'] & L \arrow[d, "\kappa_i^L"] \\ & \mathrm{End}_k\left(V_i\right) \end{tikzcd}$$

for each i, $1 \leq i \leq n$, such that $\rho_L(a)$ is a unit in L whenever $\kappa_i^A(a)$ is a unit for each i, and if there exists an isomorphism of matrix k-algebras

$$\left(\hat{H}_L(i,j) \otimes_k \mathrm{Hom}_k\left(V_i, V_j\right)\right) \cong \left(\hat{H}_A(i,j) \otimes_k \mathrm{Hom}_k\left(V_i, V_j\right)\right).$$

We write $L = \hat{A}_V^G$, and notice that geometric prolocalizations are not unique.

Lemma 14. *The geometric prolocalization $\hat{A}_V^G$ of A in $V = \{V_1, \dots, V_n\}$ exists, and* $\mathrm{Simp}(\hat{A}_V^G) = V$.

Proof. Put $\hat{A}_V^G = (\hat{H}_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$. Then exactly as above, $\hat{A}_V^G$ fulfils the conditions. Notice that even for a noncommutative k-algebra, $(u+f)(p - pfp + pfpfp - pfpfpfp + \cdots) = 1$ when $f \in \operatorname{rad}(\hat{A}_V^G)$ and p is a right unit of u (we recall that $\operatorname{rad}(\hat{A}_V^G) = \ker \eta$, where $\eta : \hat{A}_V^G \to k^n$ is the natural morphism). □

If a (geometric) prolocation is finitely generated, we will call it an algebraic localization. This then includes the ordinary localizations.

5 Noncommutative schemes

For any set S we consider the subset of the power set consisting of *finite* subsets. We use the notation $P(S) = \{M \subseteq S \mid M \text{ is finite}\}$. We now make the direct generalization of the sheafification to the noncommutative situation:

let A be a not necessarily commutative k-algebra, and put $X = \mathrm{Simp}(A) = \{A\text{-modules } V \mid V \text{ is simple}\}$. The generalization of the topological space of A is the *Jacobson topology*: for $f \in A$, we define $D(f) = \{V \in \mathrm{Simp}\, A \mid \rho(f) : V \to V \text{ is invertible}\}$, where $\rho : A \to \mathrm{End}_k(V)$ is the structure morphism. We have $D(f) \bigcap D(g) = D(fg)$, and so we can let the topology on $\mathrm{Simp}\, A$ be the topology with base of open subsets $D(f)$, $f \in A$.

For $f \in A$, we *define*

$$\hat{A}_f = \left\{ \phi : P\big(D(f)\big) \longrightarrow \coprod_{\underline{c} \in P(D(f))} \hat{A}^G_{\underline{c}} \mid \text{there exists } a \in A,\ n \in \mathbb{N} \text{ such that } \phi(\underline{c}) = a \cdot f^{-n} \right\}.$$

We then define the sheaf of regular, not necessarily commutative, functions on $X = \mathrm{Simp}\, A$ by

$$\widehat{\mathcal{O}}_{\mathrm{Simp}\, A}(U) = \varprojlim_{D(f) \subseteq U} \hat{A}_f.$$

Now if all the $\hat{A}^G_{\underline{c}}$ are algebraizable, that is, there exist algebraic localizations $A^G_{\underline{c}}$ of A for every finite subset $\underline{c}$ with natural and coherent morphisms $A^G_{\underline{c}_1} \to A^G_{\underline{c}_2}$ for each inclusion $\underline{c}_2 \subseteq \underline{c}_1$, we use the same definition and constructions as above (without the hat) and we end up with the following proposition.

Proposition 15. *One has the following:*

(1) $\mathit{\Gamma}(\mathrm{Simp}\, A, \mathcal{O}_{\mathrm{Simp}\, A}) \cong A$;
(2) *if A is commutative, then* $(\mathrm{Simp}\, A, \mathcal{O}_{\mathrm{Simp}\, A}) \cong (\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$.

Proof. (1) We see that $A \cong A_1$ and so this follows by definition.
(2) This follows as

$$A_f = \varprojlim_{D(g) \subseteq D(f)} A_g.$$

□

Definition 16. We call $(\mathrm{Simp}\, A, \hat{\mathcal{O}}_{\mathrm{Simp}\, A})$ an affine scheme, and we say that the set of simple A-modules $|\mathrm{Simp}\, A|$ is a scheme for A. A not necessarily commutative scheme is an r-pointed topological space that can be covered by affine schemes.

6 Relation to moduli problems

Consider any diagram $\underline{c}$ of A-modules, not necessarily finite. On the *set* $|c|$, we define the Jacobson topology generated by the open subsets $D_{\underline{c}}(f)$ for $f \in A$ given by $D_{\underline{c}}(f) = \{V \in |c| : \rho_V(f) \in \mathrm{End}_k(V)^*\}$ where $\rho_V : A \to \mathrm{End}_k(V)$ is the structure morphism and where $End_k(V)^* \subseteq \mathrm{End}_k(V)$ denotes the units in this k-algebra. We let $\hat{O}_V = (\hat{H}(i,j) \otimes_k \mathrm{Hom}_k(V_i, V_j))$ when $V = \{V_1, \ldots, V_n\}$.

Then, we define a sheaf of r-pointed k-algebras on the topological space $|c|$ as follows. At first, let $P(U) = \{\underline{c}_0 \subseteq \underline{c} : |\underline{c}_0| \text{ is finite}, \ \underline{c}_0 \subseteq U\}$. Then, we *define*

$$\hat{O}_f = \left\{ \phi : P\big(D_{\underline{c}}(f)\big) \longrightarrow \coprod_{\underline{c}_0 \in P(D_{\underline{c}}(f))} \hat{O}_{\underline{c}_0} \mid \phi\big(\underline{c}_0\big) = a \cdot f^{-n} \text{ for some } a \in A,\ n \in \mathbb{N} \right\}.$$

Notice that if $D_{\underline{c}}(f)$ is finite, $\hat{O}_f \cong \hat{O}_{D_{\underline{c}}(f)}$. This follows directly from the definition.

Given this, we now define

$$\hat{O}_{\underline{c}}(U) = \varprojlim_{D(f) \subseteq U} \hat{O}_f.$$

Then, $\hat{O}_{\underline{c}}$ is a sheaf by the universal property of projective limits which exists in the category of not necessarily commutative k-algebras.

Proposition 17. *One has* $(D_{\underline{c}}(f), \hat{O}_{D_{\underline{c}}}) \cong (\mathrm{Simp}(\hat{O}_f), \hat{O}_{\mathrm{Simp}(\hat{O}_f)})$.

Proof. When $\underline{c}$ is finite, $\hat{O}_{\underline{c}} \subseteq (\hat{H}_{ij} \otimes \mathrm{Hom}_k(V_i, V_j)) \twoheadrightarrow \mathrm{End}(V)$. As the O-construction is a closure operation and the surjectivity gives simplicity of the representations, dividing out by powers of the radical, using the general Burnside theorem and taking projective limits, the result follows. □

Thus, $(|\underline{c}|, \hat{O}_{|\underline{c}|})$ is a (not necessarily commutative) scheme. Moreover, the natural morphism $\rho : A \to \hat{O}_f$ glues together to a global module $\tilde{\rho}$ on $|\underline{c}|$. By the geometric properties, it is reasonable to call $(|\underline{c}|, \tilde{\rho})$ a moduli for $|\underline{c}|$, the original set of A-modules.

Definition 18. $\underline{c}$ is called an affine scheme for the k-algebra A if $(\underline{c}, O_{\underline{c}}) \cong (\mathrm{Simp}(A), O_{\mathrm{Simp}(A)})$.

7 The noncommutative moduli of rank 3 endomorphisms

In this section, we consider the problem of providing a natural algebraic geometric structure on the set of $n \times n$ Jordan forms. It turns out that there are serious combinatorial difficulties in the general case, and also that the general case would be hard to conclude from, in particular geometrically. The case of 2×2 Jordan forms can be found in [4], but this example is too simple to illustrate the geometry, thus we restrict to the case of 3×3 Jordan forms. The main result of this section is the following.

Theorem 19. *The noncommutative k-algebra*

$$M = \mathrm{M}_3(k)^{\mathrm{GL}_3(k)} = \begin{pmatrix} k\big[s_1, s_2, s_3\big] & \big\langle t_{12}(1), t_{12}(2), t_{12}(3)\big\rangle & \big\langle t_{13}(1), t_{13}(2), t_{13}(3)\big\rangle \\ 0 & k\big[t_1, t_2\big] & \big\langle t_{23}(1), t_{23}(2)\big\rangle \\ 0 & 0 & k[u] \end{pmatrix} /\mathfrak{b},$$

where $\mathfrak{b}$ is the two-sided ideal generated by the relations in the generic case (see below), is the algebraic k-algebra of the affine moduli of the $\mathrm{GL}_3(k)$-orbits of $\mathrm{M}_3(k)$. Thus, it also comes with a universal family, giving the parametrization of the closures of the orbits.

The construction of this structure is based on the noncommutative deformation theory given in [1,3]. Put $M_3(k) = \mathrm{Spec}(A)$, $A = k[x_{ij}]_{1 \le i,j \le 3}$, then $G := \mathrm{GL}_3(k)$ acts on A by conjugacy, that is, $g = (\alpha_{ij}) \in G$ acts linearly on A by $g(x_{ij}) = (\alpha_{ij})(x_{ij})(\alpha_{ij})^{-1}$. Denote this action by $\nabla : G \to \mathrm{Aut}(A)$. Let M be an A-module, and let $\nabla : G \to \mathrm{Aut}(M)$ be an action such that

$$\nabla_g(am) = \nabla_g(a)\nabla_g(m).$$

Then, (M, ∇) is called an $A - G$-module. The category of $A - G$-modules is equivalent to the category of $A[G]$-modules, where $A[G]$ is the skew group ring.

The affine k-algebra of the closure of a G-orbit is, by definition, an $A - G$-module of the form $A/\mathfrak{a}$ where $\mathfrak{a}$ is a G-stable ideal of A, together with the natural G-action:

$$\mathrm{Spec}(A/\mathfrak{a}) \subset \mathrm{Spec}(A).$$

It will turn out that we have three different cases to consider: the closure of the orbits of the Jordan form with all eigenvalues equal (called the generic case in Theorem 19), the closure of the orbits of the Jordan form with only two different eigenvalues and the closure of the orbit of the Jordan form with three different eigenvalues.

(1) All eigenvalues equal

We are considering the Jordan forms

$$M_1^{\lambda} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad M_2^{\lambda} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad M_3^{\lambda} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(2) Two different eigenvalues

The following Jordan forms are possible, letting $\underline{\lambda} = (\lambda_1, \lambda_2)$:

$$M_1^{\underline{\lambda}} = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad M_2^{\underline{\lambda}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

The case

$$M_1^{\underline{\lambda}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad M_2^{\underline{\lambda}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

will be correspondingly.

(3) Three different eigenvalues

The one and only orbit is the orbit of

$$M = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Now we set $M^\lambda = M - \lambda I$,

$$s_1^\lambda = \operatorname{tr}\left(M^\lambda\right), \quad s_2^\lambda = -\left|M_{11}^\lambda\right| - \left|M_{22}^\lambda\right| - \left|M_{33}^\lambda\right|, \quad s_3^\lambda = \left|M^\lambda\right|, \quad s_{ij}^\lambda = \left|M_{ij}^\lambda\right|.$$

For simplicity, we will also use the notation

$$x_{ij}^\lambda = x_{ij}, \quad i \neq j, \quad x_{ii}^\lambda = x_{ii} - \lambda.$$

We then have the following representation of the ideals defining the closures.

Lemma 20. $\mathfrak{a}_1^\lambda = (s_1^\lambda, s_2^\lambda, s_3^\lambda)$, $\mathfrak{a}_2^\lambda = (s_1^\lambda, s_{ij}^\lambda)$, $\mathfrak{a}_3^\lambda = (x_{ij}^\lambda)$.

Proof. $s_1^\lambda = s_1 - 3\lambda$, $s_2^\lambda = s_2 + 2s_1\lambda - 3\lambda^2$, $s_3^\lambda = -\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3$. Thus

$$\begin{aligned} s_1 = 3\lambda \wedge s_2 = -3\lambda^2 \wedge s_3 = \lambda^3 \iff s_1^\lambda &= s_1 - 3\lambda = 0 \wedge s_2^\lambda = s_2 + 2s_1\lambda - 3\lambda^2 = -6\lambda^2 + 2\cdot 3\lambda^2 \\ &= 0 \wedge s_3^\lambda = -\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = -\lambda^3 + 3\lambda^3 - 3\lambda^3 + \lambda^3 = 0. \end{aligned}$$ □

In the case with exactly two different eigenvalues $\lambda_1 \neq \lambda_2$ and $\underline{\lambda} = (\lambda_1, \lambda_2)$, the orbit closures are given by the following.

Lemma 21. $\mathfrak{a}_1^{\underline{\lambda}} = (s_1^{\lambda_1} - (\lambda_2 - \lambda_1), s_2^{\lambda_1}, s_3^{\lambda_1})$, $\mathfrak{a}_2^{\underline{\lambda}} = (s_1^{\lambda_1} - (\lambda_2 - \lambda_1), s_{ij}^{\lambda_1})$.

Proof. From direct computation:

$$\begin{aligned} s_1^{\lambda_1} - \left(\lambda_2 - \lambda_1\right) &= s_1 - 2\lambda_1 - \lambda_2, \\ s_2^{\lambda_1} &= s_2 + 2s_1\lambda_1 - 3\lambda_1^2 = s_2 + 2\left(2\lambda_1 + \lambda_2\right)\lambda_1 - 3\lambda_1^2 = s_2 + 2\lambda_1\lambda_2 + \lambda_1^2, \\ s_3^{\lambda_1} &= -\lambda_1^3 + \left(2\lambda_1 + \lambda_2\right)\lambda_1^2 + \left(-2\lambda_1\lambda_2 - \lambda_1^2\right)\lambda_1 + s_3 = s_3 - \lambda_1^2\lambda_2. \end{aligned}$$ □

And of course, in the case with three different eigenvalues $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, the orbit closure is given by $\mathfrak{a} = (s_3 - \lambda_1\lambda_2\lambda_3, s_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2, s_1 - \lambda_1 - \lambda_2 - \lambda_3)$.

Proposition 22. *The k-dimension of $\operatorname{Ext}^1_{A-G}(V_i, V_j)$ is given as the (i,j) entry in the matrix*

$$\begin{pmatrix} 3 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is true in all three cases, even if the representation of the orbits differs in notation.

Proof. This is more or less straight forward computations, except for two cases.

(1) The reader may check that $(1, \underline{0})$ and $(0, \psi)$,

$$\psi = \begin{pmatrix} x_{33} + x_{22} & x_{21} & -x_{31} \\ x_{12} & x_{11} + x_{33} & x_{32} \\ -x_{13} & x_{23} & x_{22} + x_{11} \end{pmatrix},$$

are both elements in $\operatorname{Ext}^1_{A-G}(V_2, V_2)$ considered in the Yoneda complex.

(2) Writing up the syzygies we find that for $i > j$,

$$\operatorname{ext}^1_{A-G}\left(V_i, V_j\right) \leq \operatorname{ext}^1_A\left(V_i, V_j\right) = 0.$$

See [6] for a detailed computation of all cases. Notice, however, that there does not yet exist a computer program computing this dimension (or invariants in general) under the action of an infinite group. □

8 The local formal moduli

Let $\hat{H}(\{V_i\}_{i=1}^r)$ denote the formal local noncommutative moduli of the modules $\{V_i\}_{i=1}^r$ corresponding to the closures of the orbits. We will compute this k-algebra in the worst case situation, which is seen to be the case where all eigenvalues are equal, and three closures are contained in each other: the generic case.

Let $\phi^\lambda : A \to A$ be the automorphism sending x_{ij} to x_{ij}^λ. This automorphism sends s_i to s_i^λ, $i = 1, 2, 3$, s_{ij} to s_{ij}^λ, $1 \leq i, j \leq 3$. Because ϕ^λ obviously commutes with the group action, that is, because the diagram

$$\begin{tikzcd} A \arrow[r, "\nabla_g"] \arrow[d, "\phi^\lambda"'] & A \arrow[d, "\phi^\lambda"] \\ A \arrow[r, "\nabla_g"'] & A \end{tikzcd}$$

obviously commutes, we get the following, first in the case with three coinciding eigenvalues as follows.

Lemma 23. *For every $\lambda \in k$, let $V_i^\lambda = A/\mathfrak{a}_i^\lambda$. Then*

$$\hat{H}\big(V_1^\lambda, V_2^\lambda, V_3^\lambda\big) \cong \hat{H}\big(V_1^0, V_2^0, V_3^0\big).$$

Proof. The automorphism ϕ^λ transforms every computation with tangent space bases, resolutions and Massey products for $\hat{H}(V_1^0, V_2^0, V_3^0)$ to $\hat{H}(V_1^\lambda, V_2^\lambda, V_3^\lambda)$. □

And in the case with two coinciding eigenvalues as follows.

Lemma 24. *For every $\underline{\lambda} = (\lambda_1, \lambda_2) \in k^2$, $\lambda_1 \neq \lambda_2$, one has that*

$$\hat{H}\Big(V_1^{\underline{\lambda}}, V_2^{\underline{\lambda}}\Big) \cong \hat{H}\Big(V_1^{(0,\lambda_2-\lambda_1)}, V_2^{(0,\lambda_2-\lambda_1)}\Big).$$

Proof. Use the automorphism $\phi^{\lambda_1} : A \to A$ described in the previous section. This automorphism sends $s_1 - (\lambda_2 - \lambda_1)$ to $s_1^{\lambda_1} - (\lambda_2 - \lambda_1)$, s_2 to $s_2^{\lambda_1}$, s_3 to $s_3^{\lambda_1}$ and s_{ij} to $s_{ij}^{\lambda_1}$ for $1 \leq i, j \leq 3$. Thus, the tangent spaces, the resolutions and the computation of Massey Products are isomorphic. □

The computations of the local formal moduli are based on resolutions of the $A-G$-modules and liftings of these. The representation of the Massey products given by obstructions are given previously in [7], the full details in [6].

Because of the lemmas above, we can write up the local formal moduli of every situation V_1, V_2, V_3 corresponding to one eigenvalue, V_1, V_2 corresponding to two different eigenvalues and V corresponding to three different eigenvalues.

Proposition 25. *Let*

$$\hat{T} = \begin{pmatrix} k\big[\big[t_{11}(1), t_{11}(2), t_{11}(3)\big]\big] & \big\langle\big\langle t_{12}(1), t_{12}(2), t_{12}(3)\big\rangle\big\rangle & \big\langle\big\langle t_{13}(1), t_{13}(2), t_{13}(3)\big\rangle\big\rangle \\ 0 & k\big[\big[t_{22}(1), t_{22}(2)\big]\big] & \big\langle\big\langle t_{23}(1), t_{23}(2)\big\rangle\big\rangle \\ 0 & 0 & k\big[\big[t_{33}(1)\big]\big] \end{pmatrix}.$$

Then, the noncommutative local formal moduli of the modules corresponding to the closure of the orbits of the Jordan forms M_1, M_2, M_3 is

$$\hat{T}/\big(f_{ij}(l)\big) = \hat{T}/\mathfrak{b},$$

where $\mathfrak{b}$ is the ideal generated by

$$f_{12}(1) = t_{11}(3)t_{12}(2) - t_{11}(2)t_{12}(3) - t_{12}(2)t_{22}(1) - 3t_{12}(3)t_{22}^2(2) + 2t_{12}(3)t_{22}(1)t_{22}(2),$$

$$f_{12}(2) = t_{11}(3)t_{12}(1) - t_{11}(1)t_{12}(3) - t_{12}(1)t_{22}(1) + t_{12}(3)t_{22}(1)t_{22}^2(2) - 2t_{12}(3)t_{22}^3(2),$$

$$\begin{aligned}
f_{12}(3) &= t_{11}(2)t_{12}(1) - t_{11}(1)t_{12}(2) - 2t_{12}(1)t_{22}(1)t_{22}(2) + 3t_{12}(1)t_{22}^2(2) \\
&\quad + t_{12}(2)t_{22}^2(2)t_{22}(1) - 2t_{12}(2)t_{22}^3(2), \\
f_{13}(1) &= t_{11}(3)t_{13}(2) - t_{11}(2)t_{13}(3) - 3t_{13}(2)t_{33}(1) - t_{12}(2)t_{23}(1) - 3t_{12}(3)t_{23}(2) \\
&\quad + 3t_{13}(3)t_{33}^2(1) - 2t_{12}(1)t_{22}(2)t_{23}(1) - 2t_{12}(2)t_{22}(2)t_{23}(2), \\
f_{13}(2) &= t_{11}(3)t_{13}(1) - t_{11}(1)t_{13}(3) - 3t_{13}(1)t_{33}(1) - t_{12}(1)t_{23}(1) - t_{12}(3)t_{23}(2)t_{33}(1) \\
&\quad - 2t_{12}(3)t_{22}(2)t_{23}(2) + t_{13}(3)t_{33}^3(1), \\
f_{13}(3) &= t_{11}(2)t_{13}(1) - t_{11}(1)t_{13}(2) + 3t_{12}(1)t_{23}(2) - t_{11}(3)t_{13}(1)t_{33}(1) + t_{11}(1)t_{13}(3)t_{33}(1) \\
&\quad + t_{12}(1)t_{23}(1)t_{33}(1) - t_{12}(2)t_{23}(2)t_{33}(1) - 2t_{12}(1)t_{22}(2)t_{23}(1) - 2t_{12}(2)t_{22}(2)t_{23}(2) \\
&\quad + \frac{1}{3}t_{11}(3)t_{13}(2)t_{33}^2(1) - \frac{1}{3}t_{11}(2)t_{13}(3)t_{33}^2(1) - t_{12}(3)t_{23}(2)t_{33}^2(1) - \frac{1}{3}t_{12}(2)t_{23}(1)t_{33}^2(1) \\
&\quad - 6t_{12}(3)t_{22}(2)t_{23}(2)t_{33}(1) - 2t_{12}(3)t_{22}(2)t_{23}(1)t_{33}^2(1), \\
f_{23}(1) &= -t_{22}(1)t_{23}(2) + 3t_{23}(2)t_{33}(1) + t_{23}(1)t_{33}^2(1) - 2t_{22}(2)t_{23}(1)t_{33}(1) + t_{22}^2(2)t_{23}(1).
\end{aligned}$$

Proposition 26. *Let*

$$\hat{T} = \begin{pmatrix} k\big[\big[t_{11}(1), t_{11}(2), t_{11}(3)\big]\big] & \big\langle\big\langle t_{12}(1), t_{12}(2), t_{12}(3)\big\rangle\big\rangle \\ 0 & k\big[\big[t_{22}(1), t_{22}(2)\big]\big] \end{pmatrix}.$$

Then, the noncommutative local formal moduli of the modules corresponding to the closure of the orbits of M_1 and M_2 is

$$\hat{T}/\big(f_{ij}(l)\big),$$

where

$$\begin{aligned}
f_{12}(1) &= t_{11}(3)t_{12}(2) - t_{11}(2)t_{12}(3) - t_{12}(2)t_{22}(1) - 2\lambda t_{12}(3)t_{22}(2) \\
&\quad + 2t_{12}(3)t_{22}(2)t_{22}(1) - 3t_{12}(3)t_{22}^2(2), \\
f_{12}(2) &= t_{11}(3)t_{12}(1) - t_{11}(1)t_{12}(3) - t_{12}(1)t_{22}(1) - \lambda t_{12}(3)t_{22}^2(2) \\
&\quad + t_{12}(3)t_{22}(1)t_{22}^2(2) - 2t_{12}(3)t_{22}^3(2), \\
f_{12}(3) &= t_{11}(2)t_{12}(1) - t_{11}(1)t_{12}(2) + 2\lambda t_{12}(1)t_{22}(2) - 2t_{12}(1)t_{22}(2)t_{22}(1) \\
&\quad + 3t_{12}(1)t_{22}^2(2) - \lambda t_{12}(2)t_{22}^2(2) + t_{12}(2)t_{22}^2(2)t_{22}(1) - 2t_{12}(2)t_{22}^3(2).
\end{aligned}$$

Now, it is also obvious that in the case with three different eigenvalues, the local formal moduli is

$$\hat{T} = k\big[\big[t_{11}(1), t_{11}(2), t_{11}(3)\big]\big].$$

All the relations defining the local formal moduli are polynomials and the choice of defining systems in the computation of this polynomials, the proversal family, is algebraizable (see, e.g., [6]). Thus, we may replace the double brackets with simple brackets and let

$$T = \begin{pmatrix} k\big\langle t_{11}(1), t_{11}(2), t_{11}(3)\big\rangle & \big\langle t_{12}(1), t_{12}(2), t_{12}(3)\big\rangle & \big\langle t_{13}(1), t_{13}(2), t_{13}(3)\big\rangle \\ 0 & k\big\langle t_{22}(1), t_{22}(2)\big\rangle & \big\langle t_{23}(1), t_{23}(2)\big\rangle \\ 0 & 0 & k\big\langle t_{33}(1)\big\rangle \end{pmatrix}.$$

Then, $M = T/\mathfrak{b}$ together with the universal family

$$\rho : A \longrightarrow M \otimes_k \operatorname{End}_k(V)$$

is a moduli for the orbit closures. This follows from Propositions 25 and 26 proving that the restriction to subdiagrams are correct, and from Lemmas 23 and 24 which prove that the family above is universal. Finally, we also need to prove that the points of this k-algebra corresponds to the orbit closures. This will follow from a study of the geometry.

9 The geometry

The endomorphisms with Jordan form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ correspond to the points on the *surface*

$$4s_1^3 s_3 - s_1^2 s_2^2 + 18 s_1 s_2 s_3 - 4s_2^3 + 27 s_3^2 = 0.$$

The forms $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ with coinciding eigenvalues give the *curve*

$$s_2 = -\frac{1}{3} s_1^2 \wedge s_3 = \frac{1}{27} s_1^3.$$

The geometric picture should show three generic points. The case with all three eigenvalues different is well known to be parameterized by the *points* in affine 3-space. A *point* in this affine 3-space, on the *surface*, represents a new 3-dimensional affine space glued onto this point. A *point* on the *curve* on the *surface* represents a new 3-dimensional affine space which is glued onto the point. Outside the curve and the surface, all points are identified.

Necessary conditions for the k-algebra $M = \mathrm{M}_3(k)^{\mathrm{GL}_3(k)}$ to be the affine ring for $M_3(k)/\,\mathrm{GL}_3(k)$ are that the simple modules of this ring are in one-to-one correspondence with the orbits, and that it is closed under forming local formal moduli for finite subsets of the simple modules. In particular, the Ext^1-dimensions must coincide, and the universal family must exist.

Recalling (again) that $\mathrm{Ext}^1_M(V_i, V_j) \cong \mathrm{Der}_k(M, \mathrm{Hom}_k(V_i, V_j))$, we can compute the tangent space dimensions $\mathrm{Ext}^1_M(V_i, V_j)$ by looking at k-derivations δ. The dimension drops if $\delta(f) \neq 0$ for some relation f. Let $V_1(t_{11}(1), t_{11}(2), t_{11}(3))$, $V_2(t_{22}(1), t_{22}(2))$ and $V_3 = t_{33}(1)$ be three points on the diagonal of M. Then, the constant ext^1_M-locus is given as follows:

$(1, 2)$

$$\begin{aligned}
f_{12}(1) &= t_{12}(3)\big(-t_{11}(2) - 3t_{22}^2(2) + 2t_{22}(1)t_{22}(2)\big) + t_{12}(2)\big(t_{11}(3) - t_{22}(1)\big) = 0,\\
f_{12}(2) &= t_{12}(1)\big(t_{11}(3) - t_{22}(1)\big) + t_{12}(3)\big(-t_{11}(1) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2)\big) = 0,\\
f_{12}(3) &= t_{12}(1)\big(t_{11}(2) - 2t_{22}(1)t_{22}(2) + 3t_{22}^2(2)\big) + t_{12}(2)\big(-t_{11}(1) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2)\big) = 0.
\end{aligned}$$

We put

$$t_{11}(1) = s_3, \quad t_{11}(2) = s_2, \quad t_{11}(3) = s_1, \quad t_{22}(1) = \lambda_2, \quad t_{22}(2) = \lambda_1$$

and we get the equations

$$s_1 = \lambda_2, \quad s_2 = 2\lambda_1\lambda_2 - 3\lambda_1^2, \quad s_3 = \lambda_1^2\lambda_2 - 2\lambda_1^3,$$

which is exactly the point

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 - 2\lambda_1 \end{pmatrix}$$

on the *surface*.

$(1, 3)$

$$\begin{aligned}
f_{13}(1) &= t_{13}(2)\big(t_{11}(3) - 3t_{33}(1)\big) + t_{13}(3)\big(-t_{11}(2) + 3t_{33}^2(1)\big) = 0,\\
f_{13}(2) &= t_{13}(1)\big(t_{11}(3) - 3t_{33}(1)\big) + t_{13}(3)\big(-t_{11}(1) + t_{33}^3(1)\big) = 0,\\
f_{13}(3) &= t_{13}(1)\big(t_{11}(2) - t_{11}(3)t_{33}(1)\big) + t_{13}(2)\Big(-t_{11}(1) + \frac{1}{3}t_{11}(3)t_{33}^2(1)\Big)\\
&\quad + t_{13}(3)\Big(t_{11}(1)t_{33}(1) - \frac{1}{3}t_{11}(2)t_{33}^2(1)\Big) = 0.
\end{aligned}$$

We put

$$t_{11}(1) = s_3, \quad t_{11}(2) = s_2, \quad t_{11}(3) = s_1, \quad t_{33}(1) = \lambda_1,$$

and we get the following equations:

$$\begin{aligned} s_1 &= 3\lambda_1 & s_1 &= 3\lambda_1 \\ s_2 &= 3\lambda_1^2 \iff & s_2 &= 3\lambda_1^2 \\ s_3 &= \lambda_1^3 & s_3 &= \lambda_1^3 \\ s_2 &= s_1\lambda_1 \\ s_3 &= \frac{1}{3}s_1\lambda_1^2 \\ s_3\lambda_1 &= \frac{1}{3}s_2\lambda_1^2. \end{aligned}$$

This gives the points on the *curve*

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

$(2,3)$

$$f_{23}(1) = t_{23}(1)\big(t_{33}^2(1) - 2t_{22}(2)t_{33}(1) + t_{22}^2(2)\big) + t_{23}(2)\big(-t_{22}(1) + 3t_{33}(1)\big).$$

On the curve, the above chosen parameters correspond to

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & 3\lambda_1 - 2\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 - 2\lambda_1 \end{pmatrix},$$

that is

$$t_{22}(1) = 3\lambda_1, \quad t_{22}(2) = \lambda_1, \quad t_{33}(1) = \lambda_1.$$

This is true for both equations above:

$$t_{22}(1) = 3t_{33}(1) \iff 3\lambda_1 = 3\lambda_1, \quad 2t_{22}(2)t_{33}(1) = t_{33}^2(1) + t_{22}^2(2) \iff 2\lambda_1^2 = 2\lambda_1^2.$$

Thus, the constant ext^1-locus is preserved on the curve.

The constant ext^1-locus for the local formal moduli for a point on the surface, that is the case with exactly two different eigenvalues, is given by the equations (for simplicity we put $\lambda = 1$)

$$\begin{aligned} f_{12}(1) &= t_{12}(3)\big(-t_{11}(2) - 2t_{22}(2) + 2t_{22}(1)t_{22}(2) - 3t_{22}^2(2)\big) + t_{12}(2)\big(t_{11}(3) - t_{22}(1)\big), \\ f_{12}(2) &= t_{12}(3)\big(-t_{11}(1) - t_{22}^2(2) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2)\big) + t_{12}(1)\big(t_{11}(3) - t_{22}(1)\big), \\ f_{12}(3) &= t_{12}(2)\big(-t_{11}(1) - t_{22}^2(2) + t_{22}(1)t_{22}^2(2) - 2t_{22}^3(2)\big) + t_{12}(1)\big(t_{11}(2) + 2t_{22}(2) - 2t_{22}(1)t_{22}(2) + 3t_{22}^2(2)\big). \end{aligned}$$

We let

$$t_{11}(3) = s_1 + 1, \quad t_{11}(2) = s_2, \quad t_{11}(1) = s_3, \quad t_{22}(2) = \lambda_1, \quad t_{22}(1) = \lambda_2.$$

Then, we get the equations

$$s_1 = \lambda_2 - 1, \quad s_2 = -2\lambda_1 + 2\lambda_1\lambda_2 - 3\lambda_1^2, \quad s_3 = -\lambda_1^2 + \lambda_1^2\lambda_2 - 2\lambda_1^3,$$

which are the surface

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 - 1 - 2\lambda_1 \end{pmatrix}.$$

This gives the picture of the moduli for $\mathrm{GL}_3(k)$ as the affine 3-space, the affine 2-space and the curve and proves the main theorem of the section. Notice that the affine 2-space in the middle is the blowup of the surface along the curve.

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