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Research Article

Phase Spaces and Deformation Theory*

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Abstract We have previously introduced the notion of non-commutative phase space (algebra) associated to any associative algebra, defined over a field. The purpose of the present paper is to prove that this construction is useful in non-commutative deformation theory for the construction of the versal family of finite families of modules. In particular, we obtain a much better understanding of the obstruction calculus, that is, of the Massey products.

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1 Introduction

In [8], we sketched a physical "toy model," where the space-time of classical physics became a section of a universal fiber space $\tilde{E}$, defined on the moduli space $\underline{H} = \mathrm{Simp}(H)$ of the physical systems we chose to consider (in this case, the systems composed of an observer and an observed, both sitting in a Euclidean 3-space). This moduli space was called the *time-space*. Time, in this mathematical model, was defined to be a metric ρ on the time-space, measuring all possible infinitesimal changes of *the state* of the objects in the family we are studying. This gave us a model of relativity theory, in which the set of all (relative) velocities turned out to be a projective space. Dynamics was introduced into this picture, via the general construction, for any associative algebra A, of a *phase space* $\mathrm{Ph}(A)$. This is a universal pair of a homomorphism of algebras, $\iota : A \to \mathrm{Ph}(A)$, and a derivation, $d : A \to \mathrm{Ph}(A)$, such that for any homomorphism of A into a k-algebra R, the derivations of A in R are induced by unique homomorphisms $\mathrm{Ph}(A) \to R$, composed with d. Iterating this $\mathrm{Ph}(-)$-construction, we obtained a limit morphism $\iota(n) : \mathrm{Ph}^n(A) \to \mathrm{Ph}^\infty(A)$ with image $\mathrm{Ph}^{(n)}(A)$, and a universal derivation $\delta \in \mathrm{Der}_k(\mathrm{Ph}^\infty(A), \mathrm{Ph}^\infty(A))$, the *Dirac*-derivation. A general *dynamical structure of order* n is now a two-sided δ-ideal σ in $\mathrm{Ph}^\infty(A)$ inducing a surjective homomorphism $\mathrm{Ph}^{(n-1)}(A) \to \mathrm{Ph}^\infty(A)/\sigma =: A(\sigma)$.

In [8] and later in [10], we have shown that, associated to any such *time space* H with a fixed dynamical structure $H(\sigma)$, there is a kind of "Quantum field theory". In particular, we have stressed the point that, if H is the affine ring of a moduli space of the objects we want to study, the ring $\mathrm{Ph}^\infty(H)$ is the complete ring of observables, containing the parameters not only of the iso-classes of the objects in question, but also of all dynamical parameters. The choice made by fixing the dynamical structure σ, and reducing to the k-algebra $H(\sigma)$, would classically correspond to the introduction of a parsimony principle (e.g. to the choice of some Lagrangian).

The purpose of this paper is to study this phase-space construction in greater detail. There is a natural descending filtration of two-sided ideals, $\{\mathcal{F}_n\}_{0\leq n}$ of $\mathrm{Ph}^\infty(A)$. The corresponding quotients $\mathrm{Ph}^n(A)/\mathcal{F}_n$ are finite dimensional vector spaces, and considered as affine varieties; these are our non-commutative Jet-spaces.

We will first see, in Section 2, that we may extend the usual prolongation-projection procedure of Elie Cartan to this non-commutative setting, and obtain a framework for the study of general systems of (non-commutative) PDEs; see also [9].

In Section 3, we present a short introduction to non-commutative deformations of modules, and the generalized Massey products, as exposed in [4,5].

Then, in Section 4, the main part of the paper follows: the construction for finitely generated associative algebras A of the versal family of the non-commutative deformation functor of any finite family of finitely dimensional A-modules, based on the phase-space of a *resolution* of the k-algebra A.

Notice that our $\mathrm{Ph}^\infty(A)$ is a non-commutative analogue of the notion of higher differentials treated in many texts (see [1] and the more recent paper [2]).

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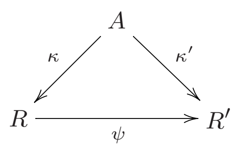
2 Phase spaces and the Dirac derivation

Given a k-algebra A, denote by $A/k - \underline{\text{alg}}$ the category where the objects are homomorphisms of k-algebras $\kappa : A \to R$, and the morphisms $\psi : \kappa \to \kappa'$ are commutative diagrams:



and consider the functor

$$\text{Der}_k(A, -) : A/k - \underline{\text{alg}} \longrightarrow \underline{\text{Sets}}$$

defined by $\text{Der}_k(A, \kappa) := \text{Der}_k(A, R)$. It is representable by a k-algebra-morphism, $\iota : A \to \text{Ph}(A)$ with a *universal family* given by a universal derivation $d : A \to \text{Ph}(A)$. It is easy to construct $\text{Ph}(A)$. In fact, let $\pi : F \to A$ be a surjective homomorphism of algebras, with $F = k\langle t_1, t_2, \ldots, t_r\rangle$, freely generated by the t_is, and put $I = \ker \pi$. Let,

$$\text{Ph}(A) = k\langle t_1, t_2, \ldots, t_r, dt_1, dt_2, \ldots, dt_r\rangle/(I, dI),$$

where dt_i is a formal variable. Clearly there is a homomorphism $i_0' : F \to \text{Ph}(A)$ and a derivation $d' : F \to \text{Ph}(A)$, defined by putting $d'(t_i) = \text{cl}(dt_i)$, the equivalence class of dt_i. Since i_0' and d' both kill the ideal I, they define a homomorphism $i_0 : A \to \text{Ph}(A)$ and a derivation $d : A \to \text{Ph}(A)$. To see that i_0 and d have the universal property, let $\kappa : A \to R$ be an object of the category $A/k - \underline{\text{alg}}$. Any derivation $\xi : A \to R$ defines a derivation $\xi' : F \to R$, mapping t_i to $\xi'(t_i)$. Let $\rho_{\xi'} : k\langle t_1, t_2, \ldots, t_r, dt_1, dt_2, \ldots, dt_r\rangle \to R$ be the homomorphism defined by

$$\rho_{\xi'}(t_i) = \kappa\big(\pi(t_i)\big), \quad \rho_{\xi'}(dt_i) = \xi\big(\pi(t_i)\big),$$

where $\rho_{\xi'}$ sends I and dI to zero, and so defines a homomorphism $\rho_\xi : \text{Ph}(A) \to R$, such that the composition with $d : A \to \text{Ph}(A)$ is ξ. The unicity is a consequence of the fact that the images of i_0 and d generate $\text{Ph}(A)$ as k-algebra.

Clearly $\text{Ph}(-)$ is a covariant functor on $k - \underline{\text{alg}}$, and we have the identities,

$$\begin{aligned} d_* &: \text{Der}_k(A, A) = \text{Mor}_A\big(\text{Ph}(A), A\big), \\ d^* &: \text{Der}_k\big(A, \text{Ph}(A)\big) = \text{End}_A\big(\text{Ph}(A)\big), \end{aligned}$$

with the last one associating d to the identity endomorphism of $\text{Ph}(A)$. In particular, we see that i_0 has a cosection, $\sigma_0 : \text{Ph}(A) \to A$, corresponding to the trivial (zero) derivation of A.

Let now V be a right A-module, with the structure morphism $\rho(V) =: \rho : A \to \text{End}_k(V)$. We obtain a universal derivation:

$$u(V) =: u : A \longrightarrow \text{Hom}_k\big(V, V \otimes_A \text{Ph}(A)\big),$$

defined by $u(a)(v) = v \otimes d(a)$. Using the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \text{Hom}_A\big(V, V \otimes_A \text{Ph}(A)\big) \longrightarrow \text{Hom}_k\big(V, V \otimes_A \text{Ph}(A)\big) \\ &\xrightarrow{\iota} \text{Der}_k\Big(A, \text{Hom}_k\big(V, V \otimes_A \text{Ph}(A)\big)\Big) \xrightarrow{\kappa} \text{Ext}^1_A\big(V, V \otimes_A \text{Ph}(A)\big) \longrightarrow 0, \end{aligned}$$

we obtain the non-commutative Kodaira-Spencer class

$$c(V) := \kappa\big(u(V)\big) \in \text{Ext}^1_A\big(V, V \otimes_A \text{Ph}(A)\big),$$

inducing the Kodaira-Spencer morphism

$$g : \Theta_A := \text{Der}_k(A, A) \longrightarrow \text{Ext}^1_A(V, V)$$

via the identity d_*. If $c(V) = 0$, then the exact sequence above proves that there exist a $\nabla \in \text{Hom}_k(V, V \otimes_A \text{Ph}(A))$ such that $u = \iota(\nabla)$. This is just another way of proving that $c(V)$ is the obstruction for the existence of a connection,

$$\nabla : \text{Der}_k(A, A) \longrightarrow \text{Hom}_k(V, V).$$

It is well known, I think, that in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting

$$\mathrm{ch}^i(V) := 1/i!\, c^i(V) \in \mathrm{Ext}^i_A\big(V, V \otimes_A \mathrm{Ph}(A)\big)$$

and that if $c(V) = 0$, the curvature $R(\nabla)$ of the connection ∇ induces a curvature class in a generalized Lie-algebra cohomology:

$$R_\nabla \in H^2\big(k, A; \Theta_A, \mathrm{End}_A(V)\big).$$

Any $\mathrm{Ph}(A)$-module W, given by its structure map,

$$\rho(W)^1 =: \rho^1 : \mathrm{Ph}(A) \longrightarrow \mathrm{End}_k(W),$$

corresponds bijectively to an induced A-module structure $\rho : A \to \mathrm{End}_k(W)$, together with a derivation $\delta_\rho \in \mathrm{Der}_k(A, \mathrm{End}_k(W))$, defining an element $[\delta_\rho] \in \mathrm{Ext}^1_A(W, W)$. Fixing this last element, we find that the set of $\mathrm{Ph}(A)$-module structures on the A-module W is in one-to-one correspondence with $\mathrm{End}_k(W)/\mathrm{End}_A(W)$. Conversely, starting with an A-module V and an element $\delta \in \mathrm{Der}_k(A, \mathrm{End}_k(V))$, we obtain a $\mathrm{Ph}(A)$-module V_δ. It is then easy to see that the kernel of the natural map

$$\mathrm{Ext}^1_{\mathrm{Ph}(A)}\big(V_\delta, V_\delta\big) \longrightarrow \mathrm{Ext}^1_A(V, V)$$

induced by the linear map

$$\mathrm{Der}_k\big(\mathrm{Ph}(A), \mathrm{End}_k\big(V_\delta\big)\big) \longrightarrow \mathrm{Der}_k\big(A, \mathrm{End}_k(V)\big)$$

is the quotient

$$\mathrm{Der}_A\big(\mathrm{Ph}(A), \mathrm{End}_k\big(V_\delta\big)\big)/\,\mathrm{End}_k(V)$$

and the image is a subspace $[\delta_\rho]^\perp \subseteq \mathrm{Ext}^1_A(V, V)$, which is rather easy to compute; see examples below.

Remark 1. Defining *time* as a metric on the moduli space, $\mathrm{Simp}(A)$, of simple A-modules, in line with the philosophy of [8], noticing that $\mathrm{Ext}^1_A(V, V)$ is the tangent space of $\mathrm{Simp}(A)$ at the point corresponding to V, we see that the non-commutative space $\mathrm{Ph}(A)$ also parametrizes the set of *generalized momenta*, that is, the set of pairs of a point $V \in \mathrm{Simp}(A)$, and a tangent vector at that point.

Example 2. (i) Let $A = k[t]$, then obviously, $\mathrm{Ph}(A) = k\langle t, dt\rangle$ and d is given by $d(t) = dt$, such that for $f \in k[t]$, we find $d(f) = J_t(f)$ with the notations of [7], that is, the non-commutative derivation of f with respect to t. One should also compare this with the non-commutative Taylor formula of loc.cit. If $V \simeq k^2$ is an A-module, defined by the matrix $X \in M_2(k)$, and $\delta \in \mathrm{Der}_k(A, \mathrm{End}_k(V))$ is defined in terms of the matrix $Y \in M_2(k)$, then the $\mathrm{Ph}(A)$-module V_δ is the $k\langle t, dt\rangle$-module defined by the action of the two matrices $X, Y \in M_2(k)$, and we find

$$e^1_V := \dim_k \mathrm{Ext}^1_A(V, V) = \dim_k \mathrm{End}_A(V) = \dim_k \big\{Z \in M_2(k) \mid [X, Z] = 0\big\},$$
$$e^1_{V_\delta} := \dim_k \mathrm{Ext}^1_{\mathrm{Ph}(A)}\big(V_\delta, V_\delta\big) = 8 - 4 + \dim\big\{Z \in M_2(k) \mid [X, Z] = [Y, Z] = 0\big\}.$$

We have the following inequalities:

$$2 \le e^1_V \le 4 \le e^1_{V_\delta} \le 8.$$

(ii) Let $A = k[t_1, t_2]$, then we find

$$\mathrm{Ph}(A) = k\big\langle t_1, t_2, dt_1, dt_2\big\rangle/\big(\big[t_1, t_2\big], \big[dt_1, t_2\big] + \big[t_1, dt_2\big]\big).$$

In particular, we have a surjective homomorphism

$$\mathrm{Ph}(A) \longrightarrow k\big\langle t_1, t_2, dt_1, dt_2\big\rangle/\big(\big[t_1, t_2\big], \big[dt_1, dt_2\big], \big[t_i, dt_i\big] - 1\big),$$

with the right-hand side algebra being the Weyl algebra. This homomorphism exists in all dimensions. We also have a surjective homomorphism,

$$\mathrm{Ph}(A) \longrightarrow k\big[t_1, t_2, \xi_1, \xi_2\big],$$

that is, onto the affine algebra of the classical phase-space.

The phase-space construction may, of course, be iterated. Given the k-algebra A, we may form the sequence $\{\mathrm{Ph}^n(A)\}_{0\leq n}$, defined inductively by

$$\mathrm{Ph}^0(A) = A, \quad \mathrm{Ph}^1(A) = \mathrm{Ph}(A), \dots, \mathrm{Ph}^{n+1}(A) := \mathrm{Ph}\left(\mathrm{Ph}^n(A)\right).$$

Let $i_0^n : \mathrm{Ph}^n(A) \to \mathrm{Ph}^{n+1}(A)$ be the canonical imbedding, and let $d_n : \mathrm{Ph}^n(A) \to \mathrm{Ph}^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $\mathrm{Ph}^n(A) \to \mathrm{Ph}^{n+2}(A)$, there exists by universality a homomorphism $i_1^{n+1} : \mathrm{Ph}^{n+1}(A) \to \mathrm{Ph}^{n+2}(A)$, such that

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

Notice that we compose functions and functors from left to right. Clearly, we may continue this process constructing new homomorphisms

$$\left\{i_j^n : \mathrm{Ph}^n(A) \longrightarrow \mathrm{Ph}^{n+1}(A)\right\}_{0\leq j\leq n}$$

with the property

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

Notice also that we have the "bi-gone" $i_0^0 i_0^1 = i_0^0 i_1^1$ and the "hexagone"

$$i_0^1 i_0^2 = i_0^1 i_1^2, \quad i_1^1 i_0^2 = i_0^1 i_2^2, \quad i_1^1 i_1^2 = i_1^1 i_2^2$$

and, in general,

$$i_p^n i_q^{n+1} = i_{q-1}^n i_p^{n+1} \quad (p < q), \quad i_p^n i_p^{n+1} = i_p^n i_{p+1}^{n+1}, \quad i_p^n i_q^{n+1} = i_q^n i_{p+1}^{n+1} \quad (q < p)$$

which is all easily proved by composing with i_0^{n-1} and d_{n-1}. Thus, the $\mathrm{Ph}^*(A)$ is a semi-cosimplicial algebra with a cosection onto A. Therefore, for any object

$$\kappa : A \longrightarrow R \in A/k - \underline{\mathrm{alg}}$$

the semi-cosimplicial algebra above induces a semi-simplicial k-vector space, $\mathrm{Der}_k(\mathrm{Ph}^*(A), R)$, and one should be interested in its homology.

The system of k-algebras and homomorphisms of k-algebras $\{\mathrm{Ph}^n(A), i_j^n\}_{n,0\leq j\leq n}$ has an inductive (direct) limit, $\mathrm{Ph}^\infty(A)$, together with homomorphisms $i_n : \mathrm{Ph}^n(A) \to \mathrm{Ph}^\infty(A)$ satisfying

$$i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \dots, n.$$

Moreover, the family of derivations $\{d_n\}_{0\leq n}$ define a unique derivation $\delta : \mathrm{Ph}^\infty(A) \to \mathrm{Ph}^\infty(A)$, such that $i_n \circ \delta = d_n \circ i_{n+1}$. Put

$$\mathrm{Ph}^{(n)}(A) := im\, i_n \subseteq \mathrm{Ph}^\infty(A).$$

The k-algebra $\mathrm{Ph}^\infty(A)$ has a descending filtration of two-sided ideals, with $\{\mathcal{F}_n\}_{0\leq n}$ given inductively by

$$\mathcal{F}_1 = \mathrm{Ph}^\infty(A) \cdot im(\delta) \cdot \mathrm{Ph}^\infty(A),$$
$$\delta\mathcal{F}_n \subseteq \mathcal{F}_{n+1}, \quad \mathcal{F}_{n_1}\mathcal{F}_{n_2} \cdot \mathcal{F}_{n_r} \subseteq \mathcal{F}_n, \quad n_1 + \cdots + n_r = n,$$

such that the derivation δ induces derivations $\delta_n : \mathcal{F}_n \to \mathcal{F}_{n+1}$. Using the canonical homomorphism $i_n : \mathrm{Ph}^n(A) \to \mathrm{Ph}^\infty(A)$, we pull the filtration $\{\mathcal{F}_p\}_{0\leq p}$ back to $\mathrm{Ph}^n(A)$, not bothering to change the notation.

Definition 3. Let $\mathcal{D}(A) := \varprojlim_{n\geq 1} \mathrm{Ph}^\infty(A)/\mathcal{F}_n$ be the completion of $\mathrm{Ph}^\infty(A)$ in the topology given by the filtration $\{\mathcal{F}_n\}_{0\leq n}$. The k-algebra $\mathrm{Ph}^\infty(A)$ will be referred to as the k-algebra of higher differentials, and $\mathcal{D}(A)$ will be called the k-algebra of formalized higher differentials. Put

$$\mathcal{D}_n := \mathcal{D}_n(A) := \mathrm{Ph}^\infty(A)/\mathcal{F}_{n+1}.$$

Clearly, δ defines a derivation on $\mathcal{D}(A)$, and an isomorphism of k-algebras

$$\epsilon := \exp(\delta) : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$$

and, in particular, an algebra homomorphism

$$\tilde{\eta} := \exp(\delta) : A \longrightarrow \mathcal{D}(A),$$

inducing the algebra homomorphisms

$$\tilde{\eta}_n : A \longrightarrow \mathcal{D}_n(A)$$

which, by killing, in the right-hand side algebra, the image of the maximal ideal, $\mathfrak{m}(\underline{t})$, of A corresponding to a point $\underline{t} \in \mathrm{Simp}_1(A)$, induces a homomorphism of k-algebras

$$\tilde{\eta}_n(\underline{t}) : A \longrightarrow \mathcal{D}_n(A)(\underline{t}) := \mathcal{D}_n / \big(\mathcal{D}_n \mathfrak{m}(\underline{t}) \mathcal{D}_n\big)$$

and an injective homomorphism

$$\tilde{\eta}(\underline{t}) : A \longrightarrow \varprojlim_{n \geq 1} \mathcal{D}_n(A)(\underline{t});$$

see [8]. More generally, let A be a finitely generated k-algebra and let $\rho : A \to \mathrm{End}_k(V)$ be an n-dimensional representation (e.g. a point of $\mathrm{Simp}_n(A)$) corresponding to a two-sided ideal $\mathfrak{m} = \ker \rho$ of A. Then $\tilde{\eta}$ induces a homomorphism

$$\tilde{\eta}(\mathfrak{m}) : A \longrightarrow \mathcal{D}/(\mathcal{D}\mathfrak{m}\mathcal{D})$$

and we will be interested in the image; see Section 4.

The k-algebras $\mathrm{Ph}^n(A)$ are our generalized jet spaces. In fact, any homomorphism of A-algebras

$$P_n : \mathrm{Ph}^n(A) \longrightarrow A$$

composed with

$$\delta^n : A \longrightarrow \mathrm{Ph}^n(A)$$

is a usual differential operator of order $\leq n$ on A. Notice also the commutative diagram

$$\begin{array}{ccccccc}
A \xrightarrow{d^{n-1}} \mathrm{Ph}^{n-1}(A) & \xrightarrow{d} & \mathrm{Ph}^n(A) & \xrightarrow{d} & \mathrm{Ph}^{n+1}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{F}_{p-1} & \xrightarrow{d} & \mathcal{F}_p & \xrightarrow{d} & \mathcal{F}_{p+1} & \xrightarrow{d} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{F}_{p-1}/\mathcal{F}_p & \xrightarrow{d} & \mathcal{F}_p/\mathcal{F}_{p+1} & \xrightarrow{d} & \mathcal{F}_{p+1}/\mathcal{F}_{p+2} & \xrightarrow{d} & \cdots .
\end{array}$$

Here the upper vertical morphisms are injective, with the lower line being the sequence of *symbols*.

It is easy to see that the differential operators form an associative k-algebra, $\mathrm{Diff}(A)$. In fact, assume two differential operators

$$P_m : \mathrm{Ph}^m(A) \longrightarrow A, \quad P_n : \mathrm{Ph}^n(A) \longrightarrow A,$$

and consider the functorially defined diagram

$$\begin{array}{ccccc}
A & \xrightarrow{d^m} & \mathrm{Ph}^m(A) & \xrightarrow{d^n} & \mathrm{Ph}^{m+n}(A) \\
 & & \downarrow{\scriptstyle P_m} & & \downarrow{\scriptstyle \mathrm{Ph}^n(P_m)} \\
 & & A & \xrightarrow{d^n} & \mathrm{Ph}^n(A) \\
 & & & & \downarrow{\scriptstyle P_n} \\
 & & & & A,
\end{array}$$

then the product is defined by the composition

$$P_m P_n = \mathrm{Ph}^{(n)}\big(P_m\big) \circ P_n.$$

Let now V be, as above, a right A-module, with structure morphism $\rho(V) : A \to \mathrm{End}_k(V)$. Consider the linear map

$$\iota_n := id \otimes \big(i_1 \circ \cdots \circ i_n\big) : V \otimes_A \mathrm{Ph}\, A \longrightarrow V \otimes_A \mathrm{Ph}^{n+1}, \quad n \geq 0.$$

Assume that the non-commutative Kodaira-Spencer class, defined above,

$$c(V) := \kappa\big(u(V)\big) \in \mathrm{Ext}^1_A\big(V, V \otimes_A \mathrm{Ph}(A)\big),$$

vanishes. Then, as we know, there exist a connection, that is, a linear map

$$\nabla_0 \in \mathrm{Hom}_k\big(V, V \otimes_A \mathrm{Ph}(A)\big)$$

such that $u(V) = \iota(\nabla_0)$. It is also easy to see that this connection induces higher-*order connections*, that is, k-linear maps,

$$\nabla(n) \in \mathrm{Hom}_k\big(V \otimes_A \mathrm{Ph}^n(A), V \otimes_A \mathrm{Ph}^{n+1}(A)\big), \quad n \geq 0,$$

defined by

$$\nabla(n)(v \otimes f) = \iota_n\big(\nabla_0(v)\big) i_0(f) + v \otimes d_n(f).$$

In fact, we just have to prove that $\nabla(n)$ is well defined, that is, we have to prove that

$$\nabla(n)(va \otimes f) = \nabla(n)(v \otimes af), \quad \forall a \in A,\ f \in \mathrm{Ph}^n(A).$$

Noticing that

$$\iota_n\big(v \otimes d_0(a)\big) = v \otimes d_n(a),$$

where we have put $a := i_0 \circ \cdots \circ i_0(a)$, we find

$$\begin{aligned}
\nabla(n)(va \otimes f) &= \iota_n\big(\nabla_0(va)\big) i_0(f) + va \otimes d_n(f) \\
&= \iota_n\big(\nabla_0(v) i_0(a) + v \otimes da\big) i_0(f) + v \otimes a d_n f \\
&= \iota_n\big(\nabla_0(v)\big) i_0(af) + v \otimes d_n a i_0(f) + v \otimes a d_n f \\
&= \nabla(n)(v \otimes af).
\end{aligned}$$

These higher-order connections will induce a diagram

$$\begin{array}{ccccccc}
V \otimes_A \mathrm{Ph}^{n-1}(A) & \xrightarrow{\nabla(n-1)} & V \otimes_A \mathrm{Ph}^n(A) & \xrightarrow{\nabla(n)} & V \otimes_A \mathrm{Ph}^{n+1}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
V \otimes_A \mathcal{F}_{p-1} & \xrightarrow{\nabla(n-1)} & V \otimes_A \mathcal{F}_p & \xrightarrow{\nabla(n)} & V \otimes_A \mathcal{F}_{p+1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
V \otimes_A \mathcal{F}_{p-1}/\mathcal{F}_p & \xrightarrow{\nabla(n-1)} & V \otimes_A \mathcal{F}_p/\mathcal{F}_{p+1} & \xrightarrow{\nabla(n)} & V \otimes_A \mathcal{F}_{p+1}/\mathcal{F}_{p+2} & \longrightarrow & \cdots,
\end{array}$$

where the lower line is the sequence of *symbols*. Notice that

$$\nabla^n \in \mathrm{Hom}_k\big(V, V \otimes \mathrm{Ph}^n(A)\big),$$

as given above, by definition has the property that for all $a \in A$ and all $v \in V$ we have

$$\nabla^n(va) = \nabla^n(v)a + \nabla^{n-1}(v)da + \cdots + v \otimes d^n(a).$$

Assume, in particular, that V and the A-module W are free of ranks p and q, respectively. Let $\{P_{i,j}\}_{i=1,\dots,p,\ j=1,\dots,q}$ be a family of A-homomorphisms $\mathrm{Ph}^n(A) \to A$, defining a *generalized differential operator*

$$\mathfrak{D} := \begin{pmatrix} P_{1,1} & \cdots & P_{1,p} \\ \cdots & & \\ P_{q,1}, & \cdots & P_{q,p} \end{pmatrix} \circ \nabla^n : V \longrightarrow W.$$

The solution space of $\mathfrak{D}$ is by definition $\mathbf{S}(\mathfrak{D}) := \ker \mathfrak{D}$. There are natural generalizations of this set-up, which we will, hopefully, return to in a later paper, extending the classical prolongation-projection method of Elie Cartan to this non-commutative setting. See Example 4 for the commutative analogue.

In [8], we introduced the notion of a *dynamical structure* for a k-algebra A, as a two-sided δ-stable ideal $\sigma \subset \mathrm{Ph}^\infty(A)$, or equivalently as the corresponding quotient $A(\sigma)$ of the δ-algebra $\mathrm{Ph}^\infty(A)$. Any such $A(\sigma)$ will be given in terms of a sequence of ideals, $\sigma_n \subset \mathrm{Ph}^n(A)$ $(n \geq 0)$, with the property that $d(\sigma_n) \subset \sigma_{n+1}$. The *solution space* of such a system, should be considered as the non-commutative scheme parametrized by $A(\sigma)$, that is, as the geometric system of all simple representations of $A(\sigma)$; see [6].

This is, in a sense, dual to the classical theory of PDEs, as we will show by considering the following example, leaving the general situation to the hypothetical paper referred to above.

Example 4 (see [9])**.** (i) Let $A = k[t_1, t_2, \dots, t_n]$, and consider the situation corresponding to a *free particle* (see [8]) that is, where we have obtained $A(\sigma)$ by killing $d^2 t_i$, for every $i = 1, 2, \dots, n$, then the commutativization $A(\sigma)_k^{\mathrm{com}}$ of $A(\sigma)_k := \mathrm{Ph}(A)/\mathcal{F}_{k+1}$ is a free A-module generated by the basis

$$\{dt_{i_1} dt_{i_2} \cdots dt_{i_r}\}_{i_1 \leq i_2 \leq \cdots \leq i_r,\ r \leq k}.$$

Put $|\underline{i}| = r$ if $\underline{i} = \{i_1, i_2, \dots, i_r\}$. The dual basis $\{p_{\underline{i}}\}_{i_1 \leq i_2 \leq \cdots \leq i_r,\ r \leq k}$ may be identified with a basis $D_{\underline{i}}$ of the A-module of all (classical higher-order) differential operators of order less or equal to k. In fact, consider the composition

$$\tilde{\eta} : A \longrightarrow \mathcal{D}(A) \longrightarrow A(\sigma)_k,$$

then, for $f \in A$ we have

$$p_{\underline{i}}\big(\tilde{\eta}(f)\big) = \frac{1}{\mu_1! \mu_2! \cdots \mu_s!} D_{j_1}^{\mu_1} D_{j_2}^{\mu_2} \cdots D_{j_r}^{\mu_r}(f),$$

where we assume

$$j_1 = i_1 = i_2 = \cdots = i_{\mu_1} < j_2 = i_{\mu_1+1} = i_{\mu_1+2} = \cdots = i_{\mu_1+\mu_2} < \cdots < i_{\mu_1+\cdots+\mu_s} = j_r$$

and where $D_{i_p}^{\mu_p}$ is μ_pth-order derivation with respect to t_{i_p}. If $\{i_1, i_2, \dots, i_r\} = \emptyset$, we let $D_{\underline{i}}$ to be the identity operator on A.

Now, consider the commutativization of $A(\sigma)$, as a k-linear space, and for every $k \geq 1$,

$$\mathcal{E}_k := A(\sigma)_k^{\mathrm{com}}$$

as a family of affine spaces fibered over $\mathrm{Simp}_1(A)$,

$$\pi_k : \mathcal{E}_k \longrightarrow \mathrm{Spec}(A).$$

This family is defined by the homomorphism of k-algebras

$$A \longrightarrow \mathcal{O}\big(\mathcal{E}_k\big) := A\big[p_{\underline{i}}\big], \quad [\underline{i}] \leq k.$$

Let $P_q(\underline{t}, p_{\underline{i}}) \in \mathcal{O}(\mathcal{E}_{k_q})$, $q = 1, \dots, d$, then the system of equations

$$P_q = 0, \quad q = 1, \dots, d$$

is a system of partial differential equations (an SPDE, for short). Suppose there is a *solution*, that is, an $f \in A$, such that

$$P_q\big(D_{\underline{i}}(f)\big) = 0, \quad q = 1, \dots, d,$$

then, for every j, we must have

$$D_j\big(P_q\big(D_{\underline{i}}(f)\big)\big) = 0$$

which amounts to extending the SPDE by, including together with $P_q \in \mathcal{O}(\mathcal{E}_{k_q})$, the polynomials

$$D_j P_q := \frac{\partial P_q}{\partial t_j} + \sum_{\underline{i}} \frac{\partial P_q}{\partial p_{\underline{i}}}\, p_{\underline{i}+j} \in \mathcal{O}\big(\mathcal{E}_{k_q+1}\big),$$

where it should be clear how to interpret the indices. Let us denote by $\mathcal{P}$ the extended family of polynomials,

$$\big\{D_{j_l}\cdots D_{j_1}P_q \mid P_q \in \mathcal{O}\big(\mathcal{E}_{k_q}\big)\big\}_{j_l,q\geq 0}$$

and let $\mathfrak{p}_m \subset O(\mathcal{E}_m)$ be the ideal, generated by the polynomials in $\mathcal{P}$, contained in $O(\mathcal{E}_m)$. Denote by $S_m := S_m\mathcal{P} \subset \mathcal{E}_m$ the corresponding subvariety. Clearly, the canonical map $\mathcal{E}_{m+l} \to \mathcal{E}_m$ induced by the trivial derivation of $\mathrm{Ph}^m(A)$ has a canonical restriction $pl_l : S_{m+l} \to S_m$. Denote also by $\pi_k : S_k \to \mathrm{Spec}(A)$ the restriction of the morphism $\pi_k : \mathcal{E}_m \to \mathrm{Spec}(A)$, defined above, to S_k. Classically, the system is called regular if all π_k are fiber bundles, so smooth, for all $k \geq 1$. Now, for any closed point of $\mathrm{Spec}(A)$, that is, for any point $\underline{t} \in \mathrm{Simp}_1(A)$, consider the sequence of fibers over $\underline{t}$, and the corresponding sequence of maps $pl_1(\underline{t}) : S_{m+1}(\underline{t}) \to S_m(\underline{t})$. An element $\tilde{f} \in \varprojlim_m S_m(\underline{t})$ corresponds exactly to an element $\hat{f} \in \hat{A}_{\underline{t}}$, for which,

$$P_q\big(D_{\underline{i}}(\hat{f})\big) = 0, \quad q = 1,\ldots,d,$$

that is, to a formal solution of the SPDE. Thus, the projective limit of schemes $\mathcal{SP}(\underline{t}) := \varprojlim_m S_m(\underline{t})$ is the space of formal solutions of the SPDE at $\underline{t} \in \mathrm{Simp}_1(A)$.

A fundamental problem in the classical theory of PDE is then the following.

Find necessary and sufficient conditions on the SPDE $\{P_q\}_{q=1,\ldots,d}$ for $\mathcal{SP}(\underline{t})$ to be non-empty, and find, based on $\{P_l\}_l$, its structure. In particular, compute its dimension $\sigma(\underline{t})$.

We will not, here, venture into this vast theory, but just add one remark. The solution space is in fact a family, with parameter-space $\mathrm{Simp}_1(A)$. Given any point $\underline{t} \in \mathrm{Simp}_1(A)$, the (formal) scheme, $\mathcal{SP}(\underline{t})$, of formal solutions may have deformations. We might want to compute the formal moduli $\underline{H}$, and relate the given family to the corresponding mini-versal family.

The tangent space of $\underline{H}$ is given as

$$A^1\big(k, \mathcal{O}\big(\mathcal{S}(\mathcal{P})(\underline{t}), \mathcal{O}\big(\mathcal{S}(\mathcal{P})(\underline{t})\big)\big)\big) = \mathrm{Hom}_{\mathcal{O}(\mathcal{E})(\underline{t})}\big(\mathfrak{p}(\underline{t}), \mathcal{O}\big(\mathcal{S}(\mathcal{P})(\underline{t})\big)\big)/\,\mathrm{Der};$$

see [3]. A tangent at the point $\underline{t}$ of $\mathrm{Simp}_1(A)$ is the value at $\underline{t}$ of a linear combination of the fundamental vector fields, the derivations $\{D_j\}$ of A. The map between the tangent space of the given family and the tangent space of H is then easily seen to be the following:

$$\eta : T_{\mathrm{Simp}_1(A),\underline{t}} \longrightarrow \mathrm{Hom}_{\mathcal{O}(\mathcal{E})(\underline{t})}\big(\mathfrak{p}(\underline{t}), \mathcal{O}\big(\mathcal{S}(\mathcal{P})(\underline{t})\big)\big)/\,\mathrm{Der},$$

where $\eta(D_j)$ is the class of the map, associating a $P \in \mathfrak{p}$ to the class at $\underline{t}$ of $D_j(P)$. The image of the tangent at $\underline{t}$ of $\mathrm{Simp}_1(A)$, corresponding to D_j, in the tangent space of H, is zero if this map is a derivation. Now, this is exactly what we have arranged, together with any $P \in \mathfrak{p}$, and also including

$$D_j P := \frac{\partial P}{\partial t_j} + \sum_{\underline{i}} \frac{\partial P}{\partial p_{\underline{i}}}\, p_{\underline{i}+j}$$

in the ideal $\mathfrak{p}$. Thus, the map η is trivial, and the given pro-family is formally constant, as one probably should have suspected! Moreover, it is easy to see that if $pl_1 : S_{k+1}(\underline{t}) \to S_k(\underline{t})$ has a local section, then $\pi_k : S_k \to \mathrm{Spec}(A)$ is formally constant at $\underline{t} \in \mathrm{Spec}(A)$. The basic problem is to find computable conditions under which the constancy of π_k implies the surjectivity of pl_1, and thereby the non-triviality of $\mathcal{S}(\mathcal{P})(\underline{t})$.

We will, hopefully, come back to these questions in a later paper.

(ii) Let $A = k[t]/(t^2)$, then

$$\begin{aligned}
\mathrm{Ph}(A) &= k\langle t, dt\rangle/\big(t^2, tdt + dtt\big),\\
\mathrm{Ph}^{(2)}(A) &= k\big\langle t, dt, d^2t\big\rangle/\big(t^2, tdt + dtt, td^2t + 2dt^2 + d^2tt\big),\\
\mathrm{Ph}^{\infty}(A) &= k\big\langle t, dt, \ldots, d^nt, \ldots\big\rangle/\big(t^2, tdt + dtt, \ldots, td^nt + ndtd^{n-1}t + \cdots + d^ntt, \ldots\big),
\end{aligned}$$

and it is easy to see that $\eta(t) = \sum_n 1/n!d^n(t)$ is non-zero in $\mathcal{D}/(\mathcal{D}\,(t)\mathcal{D})$, and, of course, $\eta(t)^2 = 0$. In particular, there is a homomorphism onto

$$\mathcal{D}/\big(\mathcal{D}(t)\mathcal{D}\big) \longrightarrow k[dt]/\big(dt^2\big) \simeq A.$$

(iii) Let now $A = k[x,y]/(x^3 - y^2)$, compute $\mathcal{D}$, and see that $dy^2 = 0$ in $\mathcal{D}/(\mathcal{D}\,(x,y)\mathcal{D})$, so that there are no natural surjective homomorphisms $\mathcal{D}/(\mathcal{D}\,(x,y)\mathcal{D}) \to A$. The map $\tilde{\eta} := \exp(\delta) : A \to \mathcal{D}$ is, however, injective. The difference between examples (i) and (ii) is, of course, due to the fact that in the first case A is graded, and in the second it is not; see Section 4.

3 Non-commutative deformations of families of modules

In [5,6,7], we introduced non-commutative deformations of families of modules of non-commutative k-algebras, and the notion of *swarm* of right modules (or more generally of objects in a k-linear abelian category). Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras

$$k^r \xrightarrow{\iota} R \xrightarrow{\pi} k^r$$

such that the composition of ι and π is the identity. Any such *r-pointed k-algebra* R is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $\operatorname{Rad}(R) := \ker \pi$, such that $R/\operatorname{Rad}(R) \simeq k^r$. The dual k-vector space of $\operatorname{Rad}(R)/\operatorname{Rad}(R)^2$ is called the tangent space of R.

For $r = 1$, there is an obvious inclusion of categories $\underline{l} \subseteq \underline{a}_1$, where $\underline{l}$, as usual, denotes the category of commutative local Artinian k-algebras with residue field k.

Fix a (not necessarily commutative) associative k-algebra A and consider a right A-module M. The ordinary deformation functor $\operatorname{Def}_M : \underline{l} \to \underline{\text{Sets}}$ is then defined. Assuming $\operatorname{Ext}^i_A(M,M)$ has a finite k-dimension for $i = 1, 2$, it is well known (see [12] or [5]) that Def_M has a pro-representing hull H, *the formal moduli of* M. Moreover, the tangent space of H is isomorphic to $\operatorname{Ext}^1_A(M,M)$, and H can be computed in terms of $\operatorname{Ext}^i_A(M,M)$, $i = 1, 2$, and their *matric* Massey products; see [5].

In the general case, consider a finite family $\mathcal{V} = \{V_i\}_{i=1}^r$ of right A-modules. Assume that $\dim_k \operatorname{Ext}^1_A(V_i, V_j) < \infty$. Any such family of A-modules will be called a *swarm*. We will define a deformation functor $\operatorname{Def}_{\mathcal{V}} : \underline{a}_r \to \underline{\text{Sets}}$ generalizing the functor Def_M above. Given an object $\pi : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the k-vector space and the left R-module $(R_{i,j} \otimes_k V_j)$. It is easy to see that

$$\operatorname{End}_R\big(\big(R_{i,j} \otimes_k V_j\big)\big) \simeq \big(R_{i,j} \otimes_k \operatorname{Hom}_k\big(V_i, V_j\big)\big).$$

Clearly, π defines a k-linear and left R-linear map

$$\pi(R) : \big(R_{i,j} \otimes_k V_j\big) \longrightarrow \oplus_{i=1}^r V_i,$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\pi}(R) : \big(R_{i,j} \otimes_k \operatorname{Hom}_k\big(V_i, V_j\big)\big) \longrightarrow \oplus_{i=1}^r \operatorname{End}_k\big(V_i\big).$$

The right A-module structure on the V_is is defined by a homomorphism of k-algebras:

$$\eta_0 : A \longrightarrow \oplus_{i=1}^r \operatorname{End}_k\big(V_i\big) \subset \big(\operatorname{Hom}_k\big(V_i, V_j\big)\big) =: \operatorname{End}_k(V).$$

Notice that this homomorphism also provides each $\operatorname{Hom}_k(V_i, V_j)$ with an A-bimodule structure. Let $\operatorname{Def}_{\mathcal{V}}(R) \in \underline{\text{Sets}}$ be the set of isoclasses of homomorphisms of k-algebras,

$$\eta' : A \longrightarrow \big(R_{i,j} \otimes_k \operatorname{Hom}_k\big(V_i, V_j\big)\big)$$

such that $\tilde{\pi}(R) \circ \eta' = \eta_0$, where the equivalence relation is defined by inner automorphisms in the k-algebra $(R_{i,j} \otimes_k \operatorname{Hom}_k(V_i, V_j))$ inducing the identity on $\oplus_{i=1}^r \operatorname{End}_k(V_i)$. One easily proves that $\operatorname{Def}_{\mathcal{V}}$ has the same properties as the ordinary deformation functor and we may prove the following theorem (see [5]).

Theorem 5. *The functor* $\mathrm{Def}_{\mathcal{V}}$ *has a pro-representable hull, that is, an object* H *of the category of pro-objects* $\hat{\underline{a}}_r$ *of* $\underline{a}_r$*, together with a versal family*

$$\tilde{V} = \big(H_{i,j} \otimes V_j\big) \in \varprojlim_{n\geq 1} \mathrm{Def}_{\mathcal{V}}\big(H/\mathfrak{m}^n\big),$$

where $\mathfrak{m} = \mathrm{Rad}(H)$*, such that the corresponding morphism of functors on* $\underline{a}_r$

$$\kappa : \mathrm{Mor}(H, -) \longrightarrow \mathrm{Def}_{\mathcal{V}},$$

defined for $\phi \in \mathrm{Mor}(H, R)$ *by* $\kappa(\phi) = R \otimes_{\phi} \tilde{V}$*, is smooth and an isomorphism on the tangent level. Moreover,* H *is uniquely determined by a set of matric Massey products defined on subspaces*

$$D(n) \subseteq \mathrm{Ext}^1\left(V_i, V_{j_1}\right) \otimes \cdots \otimes \mathrm{Ext}^1\left(V_{j_{n-1}}, V_k\right)$$

with values in $\mathrm{Ext}^2(V_i, V_k)$*.*

The right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := \mathrm{End}_H(\tilde{V}) = \big(H_{i,j} \otimes \mathrm{Hom}_k\left(V_i, V_j\right)\big)$$

and the k-algebra $O(\mathcal{V})$ acts on the family of A-modules $\mathcal{V} = \{V_i\}$, extending the action of A. If $\dim_k V_i < \infty$, for all $i = 1, \ldots, r$, the operation of associating $(O(\mathcal{V}), \mathcal{V})$ to $(A, \mathcal{V})$ turns out to be a closure operation.

Moreover, we prove the crucial result.

Theorem 6 (a generalized Burnside theorem)**.** *Let A be a finite dimensional k-algebra, with k being an algebraically closed field. Consider the family* $\mathcal{V} = \{V_i\}_{i=1}^r$ *of all simple A-modules, then*

$$\eta : A \longrightarrow O(\mathcal{V}) = \big(H_{i,j} \otimes \mathrm{Hom}_k\left(V_i, V_j\right)\big)$$

is an isomorphism.

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum H_0 of the formal moduli H; see [3]. The tangent space of H_0 is determined by a family of subspaces

$$\mathrm{Ext}^1_0\left(V_i, V_j\right) \subseteq \mathrm{Ext}^1_A\left(V_i, V_j\right), \quad i \neq j,$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathcal{V}$, and by a subspace

$$T_0(\varDelta) \subseteq \prod_i \mathrm{Ext}^1_A\left(V_i, V_i\right)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathcal{V} = \{V_i\}_{i=1}^r$; see [4]. If $\mathcal{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some Artinian k-algebra A, we show that the above notion of *almost split sequence* coincides with that of Auslander; see [11].

Using this we consider, in [5,7], the general problem of classification of iterated extensions of a family of modules $\mathcal{V} = \{V_i,\}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family $\mathcal{V}$, and extension type given by a directed representation graph $\varGamma$. The main result is the following; see [7].

Proposition 7. *Let A be any k-algebra and* $\mathcal{V} = \{V_i\}_{i=1}^r$ *any swarm of A-modules, such that*

$$\dim_k \mathrm{Ext}^1_A\left(V_i, V_j\right) < \infty, \quad \forall i, j = 1, \ldots, r.$$

- (i) *Consider an iterated extension* E *of* $\mathcal{V}$*, with representation graph* $\varGamma$*. Then there exists a morphism of* k*-algebras* $\phi : H\mathcal{V} \to k[\varGamma]$ *such that* $E \simeq k[\varGamma]\otimes_{\phi}\tilde{V}$ *as right A-algebras.*
- (ii) *The set of equivalence classes of iterated extensions of* $\mathcal{V}$ *with representation graph* $\varGamma$ *is a quotient of the set of closed points of the affine algebraic variety* $\underline{A}[\varGamma] = \mathrm{Mor}(H\mathcal{V}, k[\varGamma])$*.*
- (iii) *There is a versal family* $\tilde{V}[\varGamma]$ *of A-modules defined on* $\underline{A}[\varGamma]$*, containing as fibers all the isomorphism classes of iterated extensions of* $\mathcal{V}$ *with representation graph* $\varGamma$*.*

To any, not necessarily finite, swarm $\underline{c} \subset \underline{\mathrm{mod}}(A)$ of right-A-modules, we have associated two associative k-algebras (see [6,7]):

$$O\big(|\underline{c}|, \pi\big) = \varprojlim_{\mathcal{V} \subset |\underline{c}|} O(\mathcal{V})$$

and a sub-quotient $\mathcal{O}_\pi(\underline{c})$, together with natural k-algebra homomorphisms

$$\eta\big(|\underline{c}|\big) : A \longrightarrow O\big(|\underline{c}|, \pi\big)$$

and $\eta(\underline{c}) : A \to \mathcal{O}_\pi(\underline{c})$ with the property that the A-module structure on $\underline{c}$ is extended to an $\mathcal{O}$-module structure in an optimal way. We then defined an *affine non-commutative scheme* of right A-modules to be a swarm $\underline{c}$ of right A-modules, such that $\eta(\underline{c})$ is an isomorphism. In particular, we considered, for finitely generated k-algebras, the swarm $\mathrm{Simp}^*_{<\infty}(A)$ consisting of the finite dimensional simple A-modules, and the *generic* point A, together with all morphisms between them. The fact that this is a swarm, that is for all objects $V_i, V_j \in \mathrm{Simp}_{<\infty}$ we have $\dim_k \mathrm{Ext}^1_A(V_i, V_j) < \infty$, is easily proved. We have in [7] proved the following result (see [7, Proposition 4.1] for the definition of the notion of *geometric* k-algebra)

Proposition 8. *Let A be a geometric k-algebra, then the natural homomorphism*

$$\eta\big(\mathrm{Simp}^*(A)\big) : A \longrightarrow \mathcal{O}_\pi\big(\mathrm{Simp}^*_{<\infty}(A)\big)$$

is an isomorphism, that is, $\mathrm{Simp}^*_{<\infty}(A)$ *is a scheme for A.*

In particular, $\mathrm{Simp}^*_{<\infty}(k\langle x_1, x_2, \ldots, x_d\rangle)$ is a scheme for $k\langle x_1, x_2, \ldots, x_d\rangle$. To analyze the local structure of $\mathrm{Simp}_n(A)$, we need the following lemma (see [7, Lemma 3.23]).

Lemma 9. *Let $\mathcal{V} = \{V_i\}_{i=1,\ldots,r}$ be a finite subset of* $\mathrm{Simp}_{<\infty}(A)$*, then the morphism of k-algebras,*

$$A \longrightarrow O(\mathcal{V}) = \big(H_{i,j} \otimes_k \mathrm{Hom}_k\big(V_i, V_j\big)\big)$$

is topologically surjective.

Proof. Since the simple modules V_i $(i = 1, \ldots, r)$ are distinct, there is an obvious surjection

$$\eta_0 : A \longrightarrow \prod_{i=1,\ldots,r} \mathrm{End}_k\big(V_i\big).$$

Put $\mathfrak{r} = \ker \eta_0$, and consider for $m \geq 2$ the finite dimensional k-algebra, $B := A/\mathfrak{r}^m$. Clearly, $\mathrm{Simp}(B) = \mathcal{V}$ so that by the generalized Burnside theorem (see [5, Theorem 3.4]) we find

$$B \simeq O^B(\mathcal{V}) := \big(H^B_{i,j} \otimes_k \mathrm{Hom}_k\big(V_i, V_j\big)\big).$$

Consider the commutative diagram

$$\begin{array}{ccccc}
A & \longrightarrow & \big(H^A_{i,j} \otimes_k \mathrm{Hom}_k\big(V_i, V_j\big)\big) =: O^A(\mathcal{V}) & & \\
\downarrow & & \downarrow & \searrow & \\
B & \longrightarrow & \big(H^B_{i,j} \otimes_k \mathrm{Hom}_k\big(V_i, V_j\big)\big) & \xrightarrow{\alpha} & O^A(\mathcal{V})/\mathfrak{m}^m,
\end{array}$$

where all morphisms are natural. In particular α exists since $B = A/\mathfrak{r}^m$ maps into $O^A\mathcal{V}/\operatorname{Rad}^m$, and therefore induces the morphism α commuting with the rest of the morphisms. Consequently, α has to be surjective, and we have proved the contention. □

Example 10. As an example of what may occur in rank infinity, we will consider the invariant problem $\mathbf{A}^1_{\mathbf{C}}/\mathbf{C}^*$. Here we are talking about the algebra $A = \mathbf{C}[x](\mathbf{C}^*)$ crossed product of $\mathbf{C}[x]$ with the group $\mathbf{C}^*$. If $\lambda \in \mathbf{C}^*$, the product in A is given by $x \times \lambda = \lambda \times \lambda^{-1}x$. There are two "points" (i.e. orbits) modeled by the obvious origin $V_0 := A \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}(0))$, and by $V_1 := A \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}[x, x^{-1}])$. We may also choose the two points $V_0 := \mathbf{C}(0), V_1 := \mathbf{C}[x]$, in line with the definitions of [6]. Obviously, $\mathbf{C}[x]$ corresponds to the closure of the orbit $\mathbf{C}[x, x^{-1}]$. This choice is the best if we want to make visible the adjacencies in the quotient, and we will therefore treat both cases.

We need to compute

$$\operatorname{Ext}_A^p\left(V_i, V_j\right), \quad p = 1, 2,\ i, j = 1, 2.$$

Now,

$$\operatorname{Ext}_A^1\left(V_i, V_j\right) = \operatorname{Der}_{\mathbf{C}}\left(A, \operatorname{Hom}_{\mathbf{C}}\left(V_i, V_j\right)\right)/\operatorname{Triv}, \quad i, j = 1, 2$$

and since x acts as zero on V_1, and $\mathbf{C}^*$ acts as identity on V_1 and as a homogenous multiplication on V_0, we find

$$\operatorname{Der}_k\left(A, \operatorname{Hom}_k\left(V_0, V_0\right)\right)/\operatorname{Triv} = \operatorname{Der}_k\left(A, \operatorname{Hom}_k\left(V_0, V_0\right)\right) = \operatorname{Der}_{\mathbf{C}}\left(A, \mathbf{C}(0)\right).$$

Any $\delta \in \operatorname{Der}_k(A, \mathbf{C}(0))$ is determined by its values $\delta(x), \delta(\lambda) \in \mathbf{C}(0) \mid \lambda \in \mathbf{C}^*$. Moreover, since in A we have $(\lambda) \times (\lambda^{-1}x) = x \times (\lambda)$, we find

$$\delta(\lambda\mu) = \delta(\lambda) + \delta(\mu), \quad \delta\big((\lambda) \times \left(\lambda^{-1}x\right)\big) = \delta\big(x \times (\lambda)\big).$$

The left-hand side of the last equation is $\delta((\lambda^{-1}x)) = \lambda^{-1}\delta(x)$, and the right-hand side is $\delta(x)$, and since this must hold for all $\lambda \in \mathbf{C}^*$, we must have $\delta(x) = 0$. Moreover, since $\delta(\lambda\mu) = \delta(\lambda) + \delta(\mu)$, it is clear that the continuity of δ implies that δ must be equal to $\alpha \ln(|\cdot|)$, for some $\alpha \in \mathbf{C}$. (To simplify the writing, we will put $\log := \ln(|\cdot|)$.) Therefore,

$$\operatorname{Ext}_A^1\left(V_0, V_0\right) = \operatorname{Der}_k\left(A, \operatorname{Hom}_{\mathbf{C}}\left(V_0, V_0\right)\right) = \mathbf{C}.$$

The cup-product of this class, $\log \cup \log$, sits in $HH^2(A, \mathbf{C}(0)) = \operatorname{Ext}_a^2(V_0, V_0)$, and is given by the 2-cocycle

$$(\lambda, \mu) \longrightarrow \log(\lambda) \times \log(\mu).$$

This is seen to be a boundary, that is, there exists a map $\psi : \mathbf{C}^* \to \mathbf{C}(0)$, such that for all $\lambda, \mu \in \mathbf{C}^*$ we have

$$\log(\lambda) \times \log(\mu) = \psi(\lambda) - \psi(\lambda\mu) + \psi(\mu).$$

Just put $\psi_{1,1} := \psi_2 = -1/2\log^2$. Therefore, the cup product is zero, and if we, in general, put

$$\psi_n := \psi_{1,1,\ldots,1} = (-)^{n+1}1/(n!)\log^n, \quad n \geq 1,$$

where n is the number of 1s in the first index, then computing the Massey products of the element $\log \in \operatorname{Ext}_A^1(V_0, V_0)$, we find the nth Massey product

$$\left[\log, \log, \ldots, \log\right] = \left\{(\lambda, \mu) \longrightarrow \sum_{p=1,\ldots,n-1} \psi_p\psi_{n-p}\right\}$$

and this is easily seen to be the boundary of the 1-cochain

$$\psi_{n+1} = (-)^{n+2}1/\big((n+1)!\big)\log^{n+1}.$$

Therefore, all Massey products are zero. Of course, we have not yet proved that they could be different from zero, that is, we have not computed the *obstruction* group $\operatorname{Ext}_A^2(V_0, V_0)$ and found it non-trivial! Now this is unnecessary.

Now, assume first $V_0 = \mathbf{C}[x, x^{-1}]$, then every

$$\delta \in \operatorname{Ext}_A^1\left(V_0, V_0\right) = \operatorname{Der}_{\mathbf{C}}\left(A, \operatorname{Hom}_{\mathbf{C}}\left(V_0, V_0\right)\right)/\operatorname{Triv}$$

is determined by the values of $\delta(x)$ and $\delta(\lambda)$, $\lambda \in \mathbf{C}^*$. Since $\operatorname{Ext}^1_{\mathbf{C}[x]}(V_0, V_1) = 0$, we may find a trivial derivation such that subtracting from δ we may assume $\delta(x) = 0$. But then the formula

$$\delta(x \times \lambda) = \delta\big(\lambda \times \left(\lambda^{-1}x\right)\big)$$

implies

$$x\delta(\lambda) = \delta(\lambda)\big(\lambda^{-1}x\big)$$

from which it follows that

$$\delta(\lambda)\big(x^p\big) = \big(\lambda^{-1}x\big)^p\delta(\lambda)(1).$$

Now, since $\lambda\mu = \mu\lambda$ in $\mathbf{C}^*$, we find

$$\left(\lambda^{-1}\mu x\right)^p \delta(\lambda)(1)(\mu x) = \left(\lambda\mu^{-1}x\right)^p \delta(\lambda)(1)(\lambda x)$$

which should hold for any pair of $\mu, \lambda \in \mathbf{C}^*$, and any p. This obviously implies $\delta = 0$.

This argument shows not only that

$$\operatorname{Ext}^1_A\left(V_1, V_1\right) = \operatorname{Der}_{\mathbf{C}}\left(A, \operatorname{Hom}_{\mathbf{C}}\left(V_1, V_1\right)\right)/\operatorname{Triv} = 0$$

when $V_1 = \mathbf{C}[x, x^{-1}]$, but also when $V_1 = \mathbf{C}[x]$. Finally, we find that the formula above,

$$x\delta(\lambda) = \delta(\lambda)\left(\lambda^{-1}x\right),$$

shows that for

$$\delta \in \operatorname{Ext}^1_A\left(V_1, V_0\right) = \operatorname{Der}_{\mathbf{C}}\left(A, \operatorname{Hom}_{\mathbf{C}}\left(V_1, V_0\right)\right)/\operatorname{Triv}$$

we have $\delta(\lambda)(xx^p) = 0$ for all p. Therefore,

$$\operatorname{Ext}^1_A\left(V_1, V_0\right) = \operatorname{Der}_{\mathbf{C}}\left(A, \operatorname{Hom}_{\mathbf{C}}\left(V_1, V_0\right)\right)/\operatorname{Triv} = 0$$

when $V_1 = \mathbf{C}[x, x^{-1}]$. However, when $V_1 = \mathbf{C}[x]$, we find that δ with $\delta(\lambda)(1) \neq 0$ and with $\delta(\lambda)(x^p) = 0$, for $p \geq 1$, survives. These will, as above, give rise to a logarithm of the real part of $\mathbf{C}^*$. Therefore, in this case $\operatorname{Ext}^1_A(V_1, V_0) = \mathbf{C}$. The miniversal families look like

$$H = \begin{pmatrix} \mathbf{C}\left[[t]\right] & 0 \\ 0 & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x, x^{-1}]$, and like

$$H = \begin{pmatrix} \mathbf{C}\left[[t]\right] & 0 \\ \langle \mathbf{C} \rangle & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x]$.

4 The infinite phase space construction and Massey products

Let, as above, $\mathcal{V} = \{V_i\}_{i=1,\dots,r}$ be a family of A-modules. To compute the relevant cohomology for the deformation theory, that is, the $\operatorname{Ext}^*_A(V_i, V_j)$, we may use the Leray spectral sequence of [3], together with the formulas

$$\operatorname{Ext}^n_A\left(V_i, V_j\right) = HH^{n+1}\left(k, A; \operatorname{Hom}_k\left(V_i, V_j\right)\right),$$
$$HH^{n+1}(k, A; W) = \varprojlim_{\underline{\text{Free}}/A}{}^{(n)} \operatorname{Der}_k(-, W), \quad n > 0,$$

where W is any A-bimodule. Choose a surjective morphism $\mu : F \to A$ of a free k-algebra F onto A, and put $I = \ker\mu$, then we find that

$$HH^3(k, A; W) = \varprojlim_{\underline{\text{Free}}/A}{}^{(2)} \operatorname{Der}_k(-, W) = \operatorname{Hom}_F\left(I/I^2, \ker\mu\right)/\operatorname{Der},$$
$$\operatorname{Ext}^2_A\left(V_i, V_j\right) = \operatorname{Hom}_F\left(I/I^2, \operatorname{Hom}_k\left(V_i, V_j\right)\right)/\operatorname{Der},$$

where Der is the restriction of the derivations, $\operatorname{Der}_k(F, -)$, to I/I^2. Moreover, consider a commutative diagram of homomorphisms of algebras, in which $\tilde{\rho}$ is not yet included

$$\begin{array}{ccccc} A & \longleftarrow & F & \longleftarrow & I \\ {\scriptstyle\rho}\downarrow & \searrow^{\tilde{\rho}} & \downarrow{\scriptstyle\rho'} & & \\ S & \underset{\pi}{\longleftarrow} & R, & & \end{array}$$

and where $J := \ker\pi$ has square 0. The composition map $O : I/I^2 \to \ker\pi$ induces an element $o \in HH^3(k, A; J)$, independent upon the choice of ρ'. If this (obstruction) element vanishes, then O is the restriction to I of a derivation $\xi : F \to \ker\pi$. Subtracting this from ρ', we may assume that $\rho'(I) = 0$, so there exists a lifting $\tilde{\rho}$ of ρ. If there exists a lifting $\tilde{\rho}$, then we may obviously assume that $O = 0$.

Now, let $\{\psi_{i,j}(l) \in \operatorname{Der}_k(A; V_i, V_j)\}_{l=1,\dots,d_{i,j}}$ represent a basis of $\operatorname{Ext}^1_A(V_i, V_j)$, and let $E_{i,j} := \{t_{i,j}(l)\}_{l=1,\dots,d_{i,j}}$ denote the dual basis. Consider, the free matrix k-algebra (quiver) $(T^1_{i,j})$, generated in slot (i, j) by the (formal) elements of $E_{i,j}$. There is a unique homomorphism

$$\pi : T^1 := \big(T_{i,j}\big) \longrightarrow \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \cdot & \cdot & \cdots & 0 \\ 0 & \cdot & \cdots & k \end{pmatrix}.$$

Denote by the same letter the completion of T^1 with respect to the powers of the radical $\operatorname{Rad}(T^1) := \ker\pi$. Then $T^1 \in \underline{\hat{a}}_r$. Consider the k-algebra and the π-induced homomorphism

$$\pi_1 : \big(T^1_{i,j} \otimes_k \operatorname{Hom}_k\big(V_i, V_j\big)\big) \longrightarrow \big(\operatorname{Hom}_k\big(V_i, V_i\big)\big).$$

Clearly, π_1 splits, and it is easy to see that

$$\xi : A \longrightarrow \big(T^1_{i,j} \otimes \operatorname{Hom}_k\big(V_i, V_j\big)\big)$$

defined by

$$\xi = \sum_{i,j,l} t_{i,j}(l)\psi_{i,j}(l)$$

is a derivation, $\xi : A \to ((T^1/\operatorname{Rad}^2(T^1))_{i,j} \otimes_k \operatorname{Hom}_k(V_i, V_j))$, therefore inducing a unique homomorphism, $\tilde{\rho}_1$, makes the following diagram commute

$$\begin{array}{ccccc} A & \xrightarrow{\;id+\delta\;} & \mathcal{D}_1(A) & \longleftarrow & \mathcal{D}_2(A) \\ \downarrow{\scriptstyle\rho} & \searrow{\scriptstyle\tilde{\rho}_1} & \downarrow{\scriptstyle\rho_1} & & \\ \big(\operatorname{Hom}_k\big(V_i, V_i\big)\big) & \longleftarrow & \big(\big(T^1/\operatorname{Rad}^2\big(T^1\big)\big)_{i,j} \otimes_k \operatorname{Hom}_k\big(V_i, V_j\big)\big). & & \end{array}$$

Now, we would have liked to extend this diagram, completing it with commuting homomorphisms,

$$\begin{array}{ccc} A & \xrightarrow{\;\exp(\delta)\;} & \mathcal{D}_n(A) \\ \downarrow{\scriptstyle\tilde{\rho}_1} & & \downarrow{\scriptstyle\rho_n} \\ \big(T^1(1)_{i,j} \otimes_k \operatorname{Hom}_k\big(V_i, V_j\big)\big) & \longleftarrow & \big(H(n)_{i,j} \otimes_k \operatorname{Hom}_k\big(V_i, V_j\big)\big), \end{array}$$

where

$$\big(T^1(n) := T^1/\operatorname{Rad}^{n+1}\big(T^1\big)\big), \quad \big(H(n) := H/\operatorname{Rad}^{n+1}(H)\big).$$

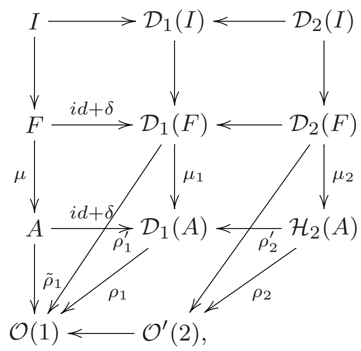
However, as will be clear in the next construction, the obvious continuation of this procedure does not work. In fact, the formalized higher differentials $\mathcal{D}(A)$ is not really the natural phase-space to work with for all purposes. In an obvious sense it is too homogenous. We are therefore led to the construction of a kind of *projective resolution* of A. Consider as above a surjective homomorphism, $\mu : F \to A$, with $F = k\langle x_1, x_2, \dots, x_s\rangle$ a free k-algebra, and $I = \ker\mu$. Obviously $\operatorname{Ph}^{(p)}(F)$, for $p \geq 1$, are also free, and $\operatorname{Ph}^{(p+1)}(F)$ is a free $\operatorname{Ph}^{(p)}(F)$-algebra. Let $\exp(\delta) : F \to \mathcal{D}(F)$ be defined as in Section 2 by

$$\exp(\delta) = id + d + 1/2d^2 + \cdots$$

and denote by $\eta_p : F \to \mathcal{D}_p(F)$ the induced homomorphism. Define

$$\mathcal{H}_p := \mathcal{D}_p(F)/\big(i_0(I), \eta_p(I)\big).$$

Clearly, $\mathcal{H}_p = \mathcal{D}_p(A)$, $\mathcal{H} = \operatorname{proj\,lim} \mathcal{H}_p$, for $p = 0, 1$. For $p \geq 2$, there are only natural surjective homomorphisms, $\kappa_p : \mathcal{H}_p \to \mathcal{D}_p(A)$. By functoriality, the diagram above induces another commutative diagram, which may be completed to the commutative diagram (ρ_2' and ρ_2 not yet included)



where we, in expectation of later constructions, put

$$\begin{aligned} H(1) &= T^1/\operatorname{Rad}\left(T^1\right)^2, \quad H'(2) = T^1/\operatorname{Rad}\left(T^1\right)^3, \\ \mathcal{O}(n) &:= \left(H(n)_{i,j} \otimes_k \operatorname{Hom}_k\left(V_i, V_j\right)\right), \quad n \geq 1, \\ \mathcal{O}'(n) &:= \left(H'(n)_{i,j} \otimes_k \operatorname{Hom}_k\left(V_i, V_j\right)\right), \quad n \geq 2. \end{aligned}$$

Now the map

$$(id + \delta) \circ \mu_1 \circ \rho_1 : I \longrightarrow \mathcal{O}(1)$$

is zero, and the resulting map $\tilde{\rho}_1 : A \to \mathcal{O}(1)$ is, as deformation of the family $\mathcal{V}$, the universal family at the tangent level. Since $\mathrm{Ph}^{(n+1)}(F)$ is a free algebra over $\mathrm{Ph}^{(n)}(F)$, there is lifting ρ_2'. We want an induced ρ_2. Consider the composition

$$O' := \exp(\delta) \circ \rho_2 : F \longrightarrow \mathcal{O}'(2)$$

lifting $\mu \circ \tilde{\rho}_1$. The restriction to I vanishes on I^2 and induces a map

$$O(2) : I/I^2 \longrightarrow \left(\left(\operatorname{Rad}\left(T^1\right)^2/\operatorname{Rad}\left(T^1\right)^3\right)_{i,j} \otimes_k \operatorname{Hom}_k\left(V_i, V_j\right)\right).$$

It is easily seen to be F-linear, both from left and right, and so it induces the obstruction

$$o_2 \in \left(\left(\operatorname{Rad}\left(T^1\right)^2/\operatorname{Rad}\left(T^1\right)^3\right)_{i,j} \otimes_k \operatorname{Ext}^2_A\left(V_i, V_j\right)\right)$$

independent upon the choice of extension ρ_2'. Now

$$\left(\left(\operatorname{Rad}\left(T^1\right)^2/\operatorname{Rad}\left(T^1\right)^3\right)_{i,j} \otimes_k \operatorname{Ext}^2_A\left(V_i, V_j\right)\right)$$

may be identified with

$$\operatorname{Hom}_k\left(\left(\operatorname{Ext}^2_A\left(V_i, V_j\right)^*\right), \left(\operatorname{Rad}\left(T^1\right)^2/\operatorname{Rad}(H)^3\right)_{i,j}\right)$$

which is a subspace of

$$\operatorname{Mor}_{\underline{a}_r}\left(k^r \oplus \left(\operatorname{Ext}^2_A\left(V_i, V_j\right)^*\right), T^1/\operatorname{Rad}\left(T^1\right)^3\right).$$

Denote by T^2 the free matrix algebra (quiver), in $\hat{\underline{a}_r}$, generated by $\operatorname{Ext}^2_A(V_i, V_j)^*$, just like the construction of T^1 above, such that

$$T^2/\operatorname{Rad}\left(T^2\right)^2 = k^r \oplus \left(\operatorname{Ext}^2_A\left(V_i, V_j\right)^*\right).$$

We may now state and prove the main result of this paper.

Theorem 11. (i) *For any finite family of (finite dimensional) A-modules, $\mathcal{V} := \{V_i\}_{i=1,\dots,r}$, there is a homomorphism $\tilde{\rho}$, making the following diagram commutative*

$$\begin{array}{ccc} A & \xrightarrow{\exp(\delta)} & \mathcal{H} \\ {\scriptstyle \rho_V}\downarrow & & \downarrow{\scriptstyle \bar{\rho}} \\ \mathrm{End}_k(V) & \longleftarrow & \big(H_{i,j} \otimes \mathrm{Hom}_k\,(V_i, V_j)\big), \end{array}$$

such that the versal family $\tilde{\rho} = \exp(\delta) \circ \bar{\rho}$.

(ii) *Moreover, $H = (H_{i,j})$ may be constructed recursively, as a quotient of $T^1 = (T^1_{i,j})$, by annihilating a series of obstructions, o_n, defining a morphism in $\underline{a}_r$, $o : T^2 \to T^1$, such that $H \simeq T^1 \otimes_{T^2} k^r$.*

Proof. We have above constructed an obstruction for lifting ρ_1 to a ρ_2. It is a unique element;

$$o_2 \in \mathrm{Mor}_{\underline{a}_r}\Big(k^r \oplus \Big(\mathrm{Ext}^2_A\,(V_i, V_j)^*\Big) T^1/\,\mathrm{Rad}\,\big(T^1\big)^3\Big).$$

Obviously, the image

$$o_2\Big(\Big(\mathrm{Ext}^2_A\,(V_i, V_j)^*\Big) \subset T^1/\,\mathrm{Rad}\,\big(T^1\big)^3\Big)$$

generates an ideal of T^1, contained in $\mathrm{Rad}(T^1)^2$. Call it σ_2, and put

$$H(2) = T^1/\big(\mathrm{Rad}\,\big(T^1\big)^3 + \sigma_2\big).$$

Then, there is a commutative diagram

$$\begin{array}{ccccccc} I & \longrightarrow & \mathcal{D}_1(I) & \longleftrightarrow & \mathcal{D}_2(I) & & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ F & \xrightarrow{\eta_1} & \mathcal{D}_1(F) & \longleftarrow & \mathcal{D}_2(F) & & \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \\ A & \xrightarrow{\eta_1} & \mathcal{D}_1(A) & \longleftarrow & \mathcal{H}_2(A) & & \\ {\scriptstyle \tilde{\rho}_1}\downarrow & \swarrow{\scriptstyle \rho_1} & & \swarrow{\scriptstyle \bar{\rho}_2} & & & \\ \mathcal{O}_1 & \longleftarrow & \mathcal{O}_2 = \big(H(2)_{i,j} \otimes \mathrm{Hom}_k\,(V_i, V_j)\big). & & & & \end{array}$$

In fact, since we have divided out with the obstruction, we know that the morphism

$$O(2) : I/I^2 \longrightarrow \Big(\mathrm{Rad}\,\big(T^1\big)^2/\,\mathrm{Rad}\,\big(T^1\big)^3 + \sigma_2\Big)_{i,j} \otimes_k \mathrm{Hom}_k\,(V_i, V_j)$$

is the restriction of a derivation

$$\psi_2' : F \longrightarrow \Big(\mathrm{Rad}\,\big(T^1\big)^2/\,\mathrm{Rad}\,\big(T^1\big)^3 + \sigma_2\Big)_{i,j} \otimes_k \mathrm{Hom}_k\,(V_i, V_j).$$

Now change the morphism ρ_2', to ρ_2'' mapping $d^2 x_i$ to $\rho_2'(d^2 x_i) - 2\psi_2(x_i)$. It is easily seen that for this new morphism, $\eta_2 \circ \rho_2'$ is zero, restricted to I, proving the existence of $\bar{\rho}_2$. Recall that $\mathcal{D}_1(A) = \mathcal{H}_1$.

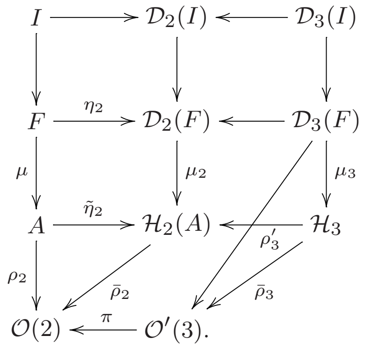
Now $\bar{\rho}_2$ defines $\tilde{\rho}_2 := \eta_2 \circ \rho_2$. Let σ_3' be the two-sided ideal in T^1 generated by

$$\mathrm{Rad}\,\big(T^1\big)^4 + \mathrm{Rad}\,\big(T^1\big)\sigma_2 + \sigma_2\,\mathrm{Rad}\,\big(T^1\big)$$

and let us put

$$H'(3) := T^1/\sigma_3', \quad \mathcal{O}'(3) := \big(H'(3) \otimes_k \mathrm{Hom}_k\,(V_i, V_j)\big).$$

The diagram above induces a commutative diagram, ρ_3', constructed as above, but where $\bar{\rho}_3$ is the problem,



Consider now the map $\eta_3 \circ \rho_3' : I \to \mathcal{O}'(3)$ ending up in

$$\left(\left(\operatorname{Rad}\left(T^1\right)^3 + \sigma_2\right)/\sigma_3\right)_{i,j} \otimes_k \operatorname{Hom}_k\left(V_i, V_j\right)$$

which clearly is killed by $\operatorname{Rad}(\mathcal{O}'(3))$, and therefore really is a matrix of vector spaces, as an $\mathcal{O}'(3)$-module. As above, this map is easily seen to be a left and right linear map as F-modules, F acting on $\mathcal{O}(3)$ via $\tilde{\eta}_3 : F \to \mathcal{D}_3(F)$. Moreover, the induced element

$$\begin{aligned} o_3 &\in \left(\left(\operatorname{Rad}\left(T^1\right)^3 + \sigma_2\right)/\sigma_3'\right)_{i,j} \otimes_k \operatorname{Ext}_A^2\left(V_i, V_j\right) \\ &= \left(\operatorname{Hom}_k\left(\operatorname{Ext}_A^2\left(V_i, V_j\right)^*, \left(\operatorname{Rad}\left(T^1\right)^3 + \sigma_2\right)/\sigma_3'\right)_{i,j}\right) \end{aligned}$$

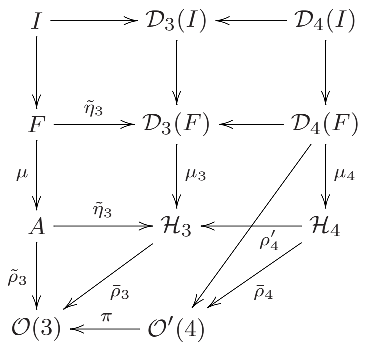
is independent on the choice of ρ_3'. Now, we define $H(3) := H'(3)/\sigma_3$, where σ_3 is defined by the image of o_3, and define $\kappa : \mathcal{O}'(3) \to \mathcal{O}(3)$ as above. Since, by functoriality, the morphism

$$\eta_3 \circ \rho_3'\kappa : I \longrightarrow \mathcal{O}(3)$$

must induce the zero element in the corresponding

$$\left(\operatorname{Ext}_A^1\left(V_i, V_j\right) \otimes \left(\left(\operatorname{Rad}\left(T^1\right)^3 + \sigma_2\right)/\sigma_3'\right)_{i,j}\right)/im o_3,$$

it must be the restriction of a derivation $\xi : F \to \mathcal{O}(3)$. Now change ρ_3' by sending d^3x_i to $\rho_3'(d^3x_i) - 3!\psi_3(x_i)$, leaving the other values of the parameters unchanged. Then, a little calculation shows that the new ρ_3' maps each $\eta_3(f)$, $f \in I$, to zero, inducing a morphism $\bar{\rho}_3 : \mathcal{H}_3 \to \mathcal{O}(3)$. We now have a new situation, given by a commutative diagram, not yet including ρ_4,



and it is clear how to proceed. This proves (i), and the rest is a consequence of the general theorem [3, Theorem 4.2.4]. □

We cannot replace $\mathcal{H}$ by $\mathcal{D}$. This follows from the trivial Example 4(iii) above. However, if we are in a graded situation, things are nicer.

Corollary 12. *Assume that A is a finitely generated, graded, in degree 1, k-algebra, and assume that $\mathcal{V}$ is a family of graded A-modules. Then there is a corresponding graded formal moduli $(H_{i,j})^{\mathrm{gr}}$, and there is a commutative diagram,*

$$\begin{array}{ccc} A & \xrightarrow{\exp(\delta)} & \mathcal{D} \\ {\scriptstyle \rho_V}\downarrow & & \downarrow{\scriptstyle \bar{\rho}^{\mathrm{gr}}} \\ \mathrm{End}_k(V) & \longleftarrow & \left(H_{i,j}^{\mathrm{gr}} \otimes \mathrm{Hom}_k\left(V_i, V_j\right)\right), \end{array}$$

such that the graded versal family $\tilde{\rho}^{\mathrm{gr}} = \exp(\delta) \circ \bar{\rho}^{\mathrm{gr}}$.

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