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Research Article

On Graded Global Dimension of Color Hopf Algebras

Yan-Hua Wang

Department of Applied Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, China
Address correspondence to Yan-Hua Wang, yhw@mail.shufe.edu.cn

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Abstract In this paper, we prove the fundamental theorem of color Hopf module similar to the fundamental theorem of Hopf module. As an application, we prove that the graded global dimension of a color Hopf algebra coincides with the projective dimension of the trivial module $\mathbb{K}$.

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1 Introduction

Let G be a group. The notion of color Hopf algebras first appeared in the book of Montgomery [6, 10.5.11]. The most important examples are Li-Zhang's twisted Hopf algebras in [4], universal enveloping algebras of Lie superalgebras and universal enveloping algebras of color Lie algebras in [1] (or [2,6,9,11]). Roughly speaking, a color Hopf algebra means a G-graded algebra and G-graded coalgebra satisfying some compatibility conditions. Its unique difference from a Hopf algebra is that the comultiplication $\triangle : A \to A \otimes A$ is an algebra homomorphism, not for the componentwise multiplication on $A \otimes A$, but for the twisted multiplication on $A \otimes A$ by Lusztig's rule.

Lorenz-Lorenz proved that the global dimension of a Hopf algebra is exactly the projective dimension of the trivial module $\mathbb{K}$; see [5, Section 2.4]. One may ask a similar question for color Hopf algebras. Following Schauenburg [10] and Doi [3], we prove the fundamental theorem of color Hopf module. As an application, we show that the graded global dimension of a color Hopf algebra coincides with the graded projective dimension of the trivial module $\mathbb{K}$, which also is equal to the projective dimension of $\mathbb{K}$.

The paper is organized as follows: in Section 2, we provide some background material for color Hopf algebras. In Section 3, we prove the fundamental theorem of color Hopf module; and we prove the main theorem: let A be a color Hopf algebra, then the graded global dimensional of A is equal to the (graded) projective dimensional of right A-module $\mathbb{K}$, where $\mathbb{K}$ is viewed as the trivial graded right A-module via the counit of A; see Theorem 9.

Throughout, $\mathbb{K}$ will be a field. All algebras and coalgebras are over $\mathbb{K}$. All unspecified spaces (algebras, coalgebras, etc.) are graded by the group G, all unadorned Hom and $\otimes$ are taken over $\mathbb{K}$. $\mathbb{K}^{\times}$ denotes $\mathbb{K} \setminus \{0\}$.

2 Preliminaries

Let G be a group with identity element e. We will write G as a multiplication group. An associative algebra A with unit 1_A is said to be *G-graded* if there is a family $\{A_g \mid g \in G\}$ of subspaces of A such that $A = \oplus_{g\in G} A_g$ with $1_A \in A_e$ and $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. Any element $a \in A_g$ is called *a homogenous element* of degree g, and we write $|a| = g$. In this paper, all unspecified elements are homogenous.

A *graded right A-module M* is a right A-module with a decomposition $M = \oplus_{g\in G} M_g$ such that $M_g A_h \subseteq M_{gh}$. We denote the module as $M \otimes A \to M$, $m \otimes a \to ma$ for any $m \in M$, $a \in A$. Let M and N be graded right A-modules. Define

$$\mathrm{Hom}_{A-\mathrm{gr}}(M, N) = \big\{ f \in \mathrm{Hom}_A(M, N) \mid f\big(M_g\big) \subseteq N_g,\ \forall g \in G \big\}.$$

We obtain the category of graded right A-modules, denoted by A-gr; for details see [8]. A module M is said to be a *gr-free module* if M is isomorphic to a direct sum of graded modules of the form $A(g)$; see [7, page 5]. In the following, we will refer to projective objects of A-gr as *gr-projective modules*.

Recall from [12] that a *graded coalgebra C* is a graded $\mathbb{K}$-space $C = \oplus_{g\in G} C_g$ with counit $\epsilon : C \to \mathbb{K}$ and comultiplication $\triangle : C \to C \otimes C$ satisfying the following conditions: $\triangle(C_g) \subseteq \sum_{h\in G} C_{gh^{-1}} \otimes C_h$ and $\epsilon(C_g) = 0$ for $g \neq e$, $g \in G$.

A *graded right A-comodule* M is a right A-comodule with a decomposition $M = \oplus_{g\in G} M_g$ such that $\rho : M \to M \otimes A$, where $\rho(m_x) = \sum_{g\in G} m_{xg^{-1}} \otimes a_g$ for any $m_x \in M$.

A *bicharacter* $\chi : G \times G \to \mathbb{K}^{\times}$ means

$$\chi(g, hl) = \chi(g, h)\chi(g, l), \quad \chi(gh, l) = \chi(g, l)\chi(h, l),$$

where $g, h, l \in G$ and $\mathbb{K}^{\times}$ is the multiplication group of the unit in $\mathbb{K}$.

Definition 1. A color Hopf algebra A is a 6-tuple $(A, m, u, \triangle, \epsilon, S)$ such that

(G1) $A = \oplus_{g\in G} A_g$ is a graded algebra with multiplication $m : A \otimes A \to A$ and the unit map $u : \mathbb{K} \to A$. In the meantime, $(A, \triangle, \epsilon)$ is a graded coalgebra with respect to the same grading;

(G2) the counit $\epsilon : A \to \mathbb{K}$ and comultiplication $\triangle : A \to A \otimes A$ are algebra maps in the sense that

$$\epsilon(ab) = \epsilon(a)\epsilon(b),$$
$$\triangle(ab) = \sum \chi(|a_2|, |b_1|) a_1 b_1 \otimes a_2 b_2, \quad a, b \in A; \tag{2.1}$$

(G3) the antipode $S : A \to A$ is a graded map such that

$$\sum a_1 S(a_2) = \epsilon(a) = \sum S(a_1) a_2$$

for all homogenous elements $a \in A$, where $\triangle(a) = \sum a_1 \otimes a_2$.

Remark 2. The antipode preserves the degree, that is, $|S(a)| = |a|$ for all homogenous $a \in A$.

The antipode of color Hopf algebras has similar results with Hopf algebras; see [1] (compare with [12, page 74], and [4, Theorem 2.10]).

Lemma 3. *Let A be a color Hopf algebra, then the antipode S satisfies*

$$S(ab) = \chi(|a|, |b|) S(b) S(a), \quad a, b \in A,$$
$$\triangle(S(a)) = \sum \chi(|a_1|, |a_2|) S(a_2) \otimes S(a_1), \quad a \in A. \tag{2.2}$$

3 Graded global dimension of color Hopf algebras

Let M be a graded right A-comodule. The *coinvariants* of M form the set

$$M^{coA} = \{m \in M \mid \rho(m) = m \otimes 1\}.$$

Note that M^{coA} is a graded subspace of M.

Definition 4. Let A be a color Hopf algebra. A graded right color Hopf module is a graded $\mathbb{K}$-space M such that

(1) M is a graded right A-module;
(2) M is a graded right A-comodule with comodule map $\rho : M \to M \otimes A$ defined by $\rho(m) = \sum m_0 \otimes m_1$;
(3) ρ is a right A-module map, that is

$$\rho(ma) = \sum \chi(|m_1|, |a_1|) m_0 a_1 \otimes m_1 a_2. \tag{3.1}$$

Example 5. Let M be a graded $\mathbb{K}$-space. Then we define on $M \otimes A$ a graded right A-module structure by $(m \otimes a)b = m \otimes ab$ for any $m \in M$, $a, b \in A$, and a graded right A-comodule structure given by the map $\rho : M \otimes A \to M \otimes A \otimes A$, $\rho(m \otimes a) = \sum m \otimes a_1 \otimes a_2$ for any $m \in M$, $a \in A$. Thus $M \otimes A$ becomes a graded right color Hopf module with these two structures. Indeed

$$\begin{aligned}
\rho((m \otimes a)b) &= \rho(m \otimes ab) \\
&= \sum m \otimes (ab)_1 \otimes (ab)_2 \\
&= \sum \chi(|a_2|, |b_1|) m \otimes a_1 b_1 \otimes a_2 b_2 \quad \text{by (2.1)} \\
&= \sum \chi(|a_2|, |b_1|) (m \otimes a_1) b_1 \otimes a_2 b_2 \\
&= \sum \chi(|(m \otimes a)_1|, |b_1|) (m \otimes a)_0 b_1 \otimes (m \otimes a)_1 b_2.
\end{aligned}$$

Lemma 6. *Let A be a color Hopf algebra. If $a, b \in A$ are homogenous, then*

$$\epsilon(a)\chi\big(|a|, |b|\big) = \epsilon(a). \tag{3.2}$$

Proof. If $|a| \neq e$, then $\epsilon(a) = 0$ and hence the equation holds. If $|a| = e$, then $\chi(|a|, |b|) = 1$, thus $\epsilon(a)\chi(|a|, |b|) = \epsilon(a)$. □

The following theorem can be viewed as the fundamental theorem of color Hopf module (compare with [12, page 84]).

Theorem 7. *Let A be a color Hopf algebra and M be a graded right color Hopf module. Then $M \cong M^{coA} \otimes A$ is a graded right color Hopf module, where $M^{coA} \otimes A$ is a trivial right color Hopf module. In particular, M is a graded free right color Hopf module.*

Proof. Consider the map $\alpha : M \to M$ defined by $\alpha(m) = \sum m_0 S(m_1)$ for any $m \in M$. If $m \in M$, then

$$\begin{aligned}
\rho\big(\alpha(m)\big) &= \rho\Big(\sum m_0 S(m_1)\Big) \\
&= \sum \chi\big(|m_1|, \big|\big(S(m_2)\big)_1\big|\big) m_0 \big(S(m_2)\big)_1 \otimes m_1 \big(S(m_2)\big)_2 \quad \text{by (3.1)} \\
&= \sum \chi\big(|m_1|, |m_3|\big)\chi\big(|m_2|, |m_3|\big) m_0 S(m_3) \otimes m_1 S(m_2) \\
&\qquad \text{by (2.2) and } S \text{ preserve the degree} \\
&= \sum \chi\big(|m_1||m_2|, |m_3|\big) m_0 S(m_3) \otimes m_1 S(m_2) \\
&= \sum \chi\big(|m_1|, |m_2|\big) m_0 S(m_2) \otimes \epsilon(m_1) \\
&= \sum m_0 S(m_2) \otimes \epsilon(m_1) \quad \text{by (3.2)} \\
&= \sum m_0 S(m_1) \otimes 1 \\
&= \alpha(m) \otimes 1.
\end{aligned}$$

Thus $\alpha(m) \in M^{coA}$.

It makes then sense to define the map $F : M \to M^{coA} \otimes A$ by $F(m) = \sum \alpha(m_0) \otimes m_1$, for all $m \in M$. Define map $G : M^{coA} \otimes A \to M$ by $G(m \otimes a) = ma$, for all $m \in M^{coA}$, $a \in A$. We will show that F is the inverse of G. Indeed, if $m \in M^{coA}$ and $a \in A$, then $\rho(ma) = \sum \chi(|1_A|, |a_1|) ma_1 \otimes 1_A a_2 = \sum ma_1 \otimes a_2$. Thus,

$$\begin{aligned}
(F \circ G)(m \otimes a) &= F(ma) \\
&= \sum \alpha\big((ma)_0\big) \otimes (ma)_1 \\
&= \sum \alpha\big(ma_1\big) \otimes a_2 \\
&= \sum \big(ma_1\big)_0 S\big(\big(ma_1\big)_1\big) \otimes a_2 \\
&= \sum \big(ma_1\big) S\big(a_2\big) \otimes a_3 \\
&= \sum m\epsilon\big(a_1\big) \otimes a_2 \\
&= m \otimes a, \\
(G \circ F)(m) &= \sum G\big(\alpha(m_0) \otimes m_1\big) \\
&= \sum G\big(m_0 S(m_1) \otimes m_2\big) \\
&= \sum m_0 S(m_1) m_2 \\
&= \sum m_0 \epsilon(m_1) \\
&= m.
\end{aligned}$$

Hence, $G \circ F = \mathrm{id}_M$ and $F \circ G = \mathrm{id}_{M^{coA} \otimes A}$.

It remains to show that G is a morphism of a graded color Hopf module, that is, it is a morphism of a graded right A-module and a morphism of a graded right A-comodule.

The first assertion is clear since

$$G\big((m\otimes a)b\big)=G\big(m\otimes ab\big)=m(ab)=(ma)b=G(m\otimes a)b.$$

In order to show that G is a morphism of a graded right A-comodule, we have to prove that

$$(\rho\circ G)(m\otimes a)=(G\otimes \mathrm{id})\rho(m\otimes a).$$

This is immediate since for $m\otimes a\in M^{coA}\otimes A$ we have

$$\begin{aligned}(\rho\circ G)(m\otimes a)&=\rho(ma)\\&=\sum ma_1\otimes a_2\\&=\sum(G\otimes \mathrm{id})\big(m\otimes a_1\otimes a_2\big)\\&=\sum(G\otimes \mathrm{id})\rho(m\otimes a).\end{aligned}$$

This ends the proof. □

Proposition 8. *Let A be a color Hopf algebra and M be a graded right A-module. Then $M\otimes A$ is a graded right color Hopf module using comodule map $\rho=id_M\otimes\triangle$.*

Proof. Define the graded right A-module structure of $M\otimes A$ as

$$(m\otimes a)b=\sum\chi\big(|a|,|b_1|\big)mb_1\otimes ab_2,\quad \forall m\in M,\ a,b\in A.$$

Indeed, $M\otimes A$ is a graded right A-module and for any $a,b,c\in A$, $m\in M$, we have

$$\begin{aligned}\big((m\otimes a)b\big)c&=\sum\chi\big(|a|,|b_1|\big)\big(mb_1\otimes ab_2\big)c\\&=\sum\chi\big(|a|,|b_1|\big)\chi\big(|a||b_2|,|c_1|\big)mb_1c_1\otimes ab_2c_2\\&=\sum\chi\big(|a|,|b_1|\big)\chi\big(|a|,|c_1|\big)\chi\big(|b_2|,|c_1|\big)mb_1c_1\otimes ab_2c_2,\\(m\otimes a)(bc)&=\sum\chi\big(|a|,|(bc)_1|\big)m(bc)_1\otimes a(bc)_2\\&=\sum\chi\big(|a|,|b_1||c_1|\big)\chi\big(|b_2|,|c_1|\big)mb_1c_1\otimes ab_2c_2\quad\text{by (2.1)}.\end{aligned}$$

Since

$$\sum\chi\big(|a|,|b_1|\big)\chi\big(|a|,|c_1|\big)\chi\big(|b_2|,|c_1|\big)=\sum\chi\big(|a|,|b_1||c_1|\big)\chi\big(|b_2|,|c_1|\big),$$

we have $((m\otimes a)b)c=(m\otimes a)(bc)$. Thus $M\otimes A$ is a graded right A-module.

Define the graded right A-comodule of $M\otimes A$ as

$$\rho(m\otimes a)=\sum(m\otimes a)_0\otimes(m\otimes a)_1=\sum\big(m\otimes a_1\big)\otimes a_2.$$

Then $M\otimes A$ is a graded right A-comodule since

$$\begin{aligned}(id\otimes\triangle)\rho(m\otimes a)&=(id\otimes\triangle)\Big(\sum\big(m\otimes a_1\big)\otimes a_2\Big)\\&=\sum\big(m\otimes a_1\big)\otimes a_2\otimes a_3,\\(\rho\otimes id)\rho(m\otimes a)&=(\rho\otimes id)\Big(\sum\big(m\otimes a_1\big)\otimes a_2\Big)\\&=\sum\big(m\otimes a_1\big)\otimes a_2\otimes a_3,\\(id\otimes\epsilon)\rho(m\otimes a)&=\sum m\otimes a_1\otimes\epsilon\big(a_2\big)=m\otimes a\otimes 1.\end{aligned}$$

Thus $M\otimes A$ is a right A-comodule.

Moreover, $M \otimes A$ is a graded right color Hopf module. Since

$$\begin{aligned}
\rho\big((m \otimes a)b\big) &= \sum \chi\big(|a|, |b_1|\big)\rho\big(mb_1 \otimes ab_2\big) \\
&= \sum \chi\big(|a_1||a_2|, |b_1|\big)\chi\big(|a_2|, |b_2|\big) mb_1 \otimes a_1b_2 \otimes a_2b_3 \quad \text{by (2.1)} \\
&= \sum \chi\big(|a_1|, |b_1|\big)\chi\big(|a_2|, |b_1|\big)\chi\big(|a_2|, |b_2|\big) mb_1 \otimes a_1b_2 \otimes a_2b_3 \\
&= \sum \chi\big(|a_2|, |b_1|\big)\big(m \otimes a_1\big)b_1 \otimes a_2b_2 \\
&= \sum \chi\big(|(m \otimes a)_1|, |b_1|\big)(m \otimes a)_0 b_1 \otimes (m \otimes a)_1 b_2.
\end{aligned}$$

This completes the proof. □

We will refer to projective objects of graded A-module as gr-projective modules. Taking the notations of [7], we denote the graded global dimensional of A as gr. gl. dim A.

Theorem 9. *Let A be a color Hopf algebra. Then one has*

$$\text{gr. gl.dim } A = \text{gr. p.dim}_A \mathbb{K} = \text{p.dim}_A \mathbb{K},$$

where gr. gl.dim A and gr. p. dim A denote the graded global dimension and graded projective dimension of A, respectively; p.dim A denotes the projective dimension of A.

Proof. Consider the projective resolution of $\mathbb{K}$ in the category of graded right A-modules:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{K} \longrightarrow 0.$$

Assume that M is a graded right A-module. Then for any graded right A-module P, we have a graded right A-module structure on $M \otimes P$ with the action given by

$$(m \otimes p)a = \sum \chi\big(|p|, |a_1|\big) ma_1 \otimes pa_2, \ m \in M, \ p \in P, \ a \in A.$$

In this way, we obtain an exact sequence of graded right A-modules

$$\cdots \longrightarrow M \otimes P_1 \longrightarrow M \otimes P_0 \longrightarrow M \otimes \mathbb{K} \cong M \longrightarrow 0.$$

We claim that this is a projective resolution of M and this will complete the proof.

Now we recall the degree-shift functor on A-gr. Let $g \in G$ and $M = \oplus_{g\in G} M_g$ be a graded right A-module. We can define a new graded right A-module $M(g)$ which has the same module structure with M, and has the gradation given by $M(g)_h = M_{gh}$ for all $h \in G$ (see [7,8]). Indeed, if P is a projective graded right A-module, then P is a direct summand in a free graded right A-module, thus $P \oplus X \simeq \oplus_{g\in G} A(g)^{(I_g)}$ as a graded right A-module for some graded right A-module X and some set I. Then

$$(M \otimes P) \oplus (M \otimes X) \simeq \oplus_{g\in G}\big(M \otimes A(g)\big)^{(I_g)},$$

where it is enough to show that each $M \otimes A(g)$ is projective. Note $M \otimes A(g) = (M \otimes A)(g)$, so we only prove that $M \otimes A$ is projective. But this is true since $M \otimes A$ has a graded right color Hopf module structure if we take the graded right A-module structure and graded right A-comodule structure as Proposition 8.

The last equality gr. $\text{p.dim}_A \mathbb{K} = \text{p.dim}_A \mathbb{K}$ is derived from [7, I.2.7]. □

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