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On Hom-type algebras

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Abstract

Hom-algebras are generalizations of algebras obtained using a twisting by a linear map. But there is a priori a freedom on where to twist. We enumerate here all the possible choices in the Lie and associative types and study the relations between the obtained algebras. The associative case is richer since it admits the notion of unit element. We use this fact to find sufficient conditions for Hom-associative algebras to be associative and classify the implications between the Hom-associative types of unital algebras.

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1 Introduction

The present paper investigates variations on the theme of *Hom-algebras*, a topic which has recently received much attention from various researchers [2, 5, 13, 15, 16]. Generally speaking, the notion of Hom-algebra over a certain operad is obtained by twisting in a strategic way the identities for the algebra multiplication implied by the operad in question. For instance, a pair $(V, \star)$, where V is a vector space and $\star$ is a multiplication (bilinear map), together with a linear self-map $\alpha : V \to V$ is called *Hom-associative* if it satisfies the identity:

$$\mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha),$$

which is obtained by replacing id with α in the ordinary associativity condition:

$$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id).$$

The study of Hom-associative algebras originates with work by Hartwig, Larson, and Silvestrov in the Lie case [6], where a notion of Hom-Lie algebra was introduced in the context of studying deformations of Witt and Virasoro algebras. Later, it was extended to the associative case by Makhlouf and Silverstrov in [8]. A number of classical constructions have been found to have a Hom-counterpart, see, for example, [1, 7, 9, 10, 11, 12, 13, 14].

When studying the structure theory of Hom-associative algebras, one naturally encounters related algebras which, while not obeying the traditional Hom-associative identities, satisfy similar identities [3]. For instance, in the proof of the main result of [3], there naturally appear algebraic structures $(V, \star, \alpha)$ which satisfy the conditions:

$$\big(\alpha(x) \star y\big) \star z = x \star \big(y \star \alpha(z)\big),$$

or

$$\alpha(x \star y) \star z = x \star \alpha(y \star z)$$

for all $x, y, z \in V$. This observation suggests that in the context of studying Hom-algebraic structures, there is a natural interest in exploring alternative possibilities on "how to twist" the identities of a given classical algebraic category to obtain a Hom-counterpart.

In this paper, we start a systematic exploration of other possibilities to define Hom-type algebras. We will, for the purposes of the present paper, restrict the scope of this first investigation in several ways. First, we will only consider Hom-twisted versions of the associative and Lie case. Second, we will only consider symmetric twisted identities and limit the "degree" in terms of occurrences of the twisting map α in the defining identity of the twisted categories.

Some aspects of the study of generalizations of the Hom-associativity condition which is started in this paper were expanded by the authors in a joint work with Sergei Silvestrov in [4]. In particular, that paper develops a notion of generalized Hom-associativity of which all of the variations of Hom-associativity discussed in the present work are special cases.

The paper is partitioned in two main sections. In Section 2, we introduce, following the ideas outlined above, new types of Hom-Lie algebras and study their properties in special cases. We give several examples of these new types of Hom-Lie algebras and study their relations among each other and to ordinary Lie algebras. In Section 3, we introduce in analogy to this work a similar system of new types of Hom-associative algebras. We point out that in the case of unital algebras, these types of Hom-associativity conditions can be partially ordered by restrictiveness with the traditional Hom-associativity condition ending up on top, that is, as most restrictive. Finally, we introduce *Hom-monoids* to obtain an easy way to construct counterexamples to possible relations between types of Hom-algebras which do not hold. These counterexamples prove that our partial ordering of Hom-type algebras cannot be improved upon.

We end the introduction by fixing some conventions and notations. In this paper, k will by default be a commutative ring, K a field. Modules and algebras will by default be understood to be over an arbitrary commutative ring. V will by default be a k-module.

2 Hom-Lie algebras

In this section, we define types of Hom-Lie algebras and give some relations between them.

2.1 Definitions

We start by recalling the original definition following [6].

Definition 2.1. A Hom-Lie algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a module V over a commutative ring k, a bilinear map $[\cdot,\cdot]$: $V \times V \to V$, and a linear space homomorphism $\alpha : V \to V$ satisfying

$$[x, x] = 0, \tag{2.1}$$

$$\circlearrowleft_{x,y,z} \big[\alpha(x), [y, z]\big] = 0 \tag{2.2}$$

for all x, y, z in V, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutations on x, y, z. Explicitly, this means that

$$\circlearrowleft_{x,y,z} \big[\alpha(x), [y, z]\big] := \big[\alpha(x), [y, z]\big] + \big[\alpha(y), [z, x]\big] + \big[\alpha(z), [x, y]\big].$$

Note that, if $(G,+,0)$ is a group with identity 0 and $\varphi : G\times G\to G$ is a biadditive map with $\varphi(x,x)=0$, we automatically have $\varphi(x,y)+\varphi(y,x)=0$. However, if in G the equation $x+x=0$ always implies $x=0$, as is the case, for instance, if G is the additive group of a field of characteristic $\neq 2$, then the condition $\varphi(x,y)+\varphi(y,x)=0$ implies also $\varphi(x,x)=0$ for all $x\in G$. Therefore, condition (1) in our definitions corresponds to the usual condition of skew-symmetry of a Lie bracket.

Now if we look at (2.2), it is natural to ask why we chose to twist by α in the first argument, and not in the second or third? This question is the first motivation for us to suggest the introduction of two new types, I_2 and I_3, of Hom-Lie algebras.

Definition 2.2. A Hom-Lie algebra of type I_2 is defined by replacing, in Definition 2.1, equation (2.2) by

$$\circlearrowleft_{x,y,z} \big[x,\big[\alpha(y),z\big]\big]=0. \tag{2.3}$$

If one uses

$$\circlearrowleft_{x,y,z} \big[x,\big[y,\alpha(z)\big]\big]=0 \tag{2.4}$$

instead, one gets the definition of a Hom-Lie algebra of type I_3.

A Hom-Lie algebra in the usual sense should be referred to as “Hom-Lie algebra of type I_1”, but we will, most of the time, simply use the term “Hom-Lie algebra”, for coherence with the usage in the literature.

Remark 2.3. Of course types I_2 and I_3 are the same by skew-symmetry of the bracket. Nevertheless, we introduce these two types for pedagogical reasons, since they will appear again in the associative category.

Now, we remark that α still has two more choices for its dinner: it could be applied to the results of the first or of the second bracket and give two other types.

Definition 2.4. If one replaces in Definition 2.1 equation (2.2) by

$$\circlearrowleft_{x,y,z} \big[x,\alpha\big([y,z]\big)\big]=0, \tag{2.5}$$

one gets the definition of a Hom-Lie algebra of type II.

If one uses

$$\circlearrowleft_{x,y,z} \alpha\big(\big[x,[y,z]\big]\big)=0 \tag{2.6}$$

instead, one gets the definition of a Hom-Lie algebra of type III.

A trivial example of an algebra of type III is given by considering an arbitrary Lie algebra structure on V (a Hom-Lie algebra, where the twisting is the identity on V) together with an arbitrary linear α. However, we see that if α is any *injective* linear map and (V,α) is a Hom-Lie algebra of type III, then V must be Lie.

Another example, less trivial, is obtained considering the notion of descending central series V^n, borrowed from Lie theory: $V^0:=V$, $V^1:=[V,V]$, $V^2:=[V,V^1],\dots,$ $V^n:=[V,V^{n-1}]$. Considering an arbitrary α whose kernel contains V^2 gives the second example. One can also obtain examples, where α does not vanish on V^2. The following example is an extreme case of this insofar as the kernel of α is one-dimensional, that is, of lowest possible dimension.

Example 2.5. Let K be a field and let $V := K^3$. We define a bilinear map $[\cdot,\cdot] : V \times V \to V$ by

$$\left[\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}, \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix}\right] := \begin{pmatrix}\lambda_1\mu_3 - \lambda_3\mu_1\\ 0\\ \lambda_2\mu_3 - \lambda_3\mu_2\end{pmatrix}.$$

It is clear that this map satisfies $[v, v] = 0$ for all $v \in V$. Also, our bracket does not induce a structure of Lie algebra on V, since, for example, with e_1, e_2, e_3 the canonical basis vectors of V, we have

$$\circlearrowleft_{e_1,e_2,e_3} \big[e_1, [e_2, e_3]\big] = e_1 \neq 0.$$

Finally, with $\alpha : V \to V$ defined through

$$\alpha\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} := \begin{pmatrix}\lambda_2\\ \lambda_3\\ 0\end{pmatrix}$$

we see that $(V, \alpha, [\cdot,\cdot])$ is Hom-III-Lie by a straightforward calculation. But V^2 has as basis the set $\{e_1, e_3\}$, so α does not vanish on V^2.

Let us now do a little science fiction and imagine that α is in fact a morphism of algebra, i.e. satisfies $[\alpha(x), \alpha(x)] = \alpha([x, y])\ \forall x, y \in V$. One could then, using this equality twice, rewrite equation (2.6) in two ways ((2.7) and (2.8)). Hence it is natural to also consider:

Definition 2.6. Hom-Lie algebras of types III' and III'' are defined by respectively replacing in Definition 2.1, equation (2.2) by

$$\circlearrowleft_{x,y,z} \big[\alpha(x), \alpha\big([y, z]\big)\big] = 0, \tag{2.7}$$

$$\circlearrowleft_{x,y,z} \big[\alpha(x), \big[\alpha(y), \alpha(z)\big]\big] = 0. \tag{2.8}$$

One can remark that a Hom-Lie algebra of type III'' is nothing else than a Lie algebra structure on the image of α.

Similarly, (2.5) leads to the equation (2.9) which is quadratic in α. But once we have opened the Pandora's box, we are forced to also consider the other quadratic expressions in α, (2.10) and (2.11):

Definition 2.7. Hom-Lie algebras of types II_1, II_2 and II_3 are defined by respectively replacing in Definition 2.1, equation (2.2) by

$$\circlearrowleft_{x,y,z} \big[x, \big[\alpha(y), \alpha(z)\big]\big] = 0, \tag{2.9}$$

$$\circlearrowleft_{x,y,z} \big[\alpha(x), \big[y, \alpha(z)\big]\big] = 0, \tag{2.10}$$

$$\circlearrowleft_{x,y,z} \big[\alpha(x), \big[\alpha(y), z\big]\big] = 0. \tag{2.11}$$

Remark 2.8. It is easy to see that Remark 2.3 applies mutatis mutandis if one replaces I_2 and I_3 by II_2 and II_3.

On the notations

We need a notation to distinguish all these types of Hom-Lie algebras.

"$Hom^{type}-Lie$ algebra" seems appropriate, for example, "$Hom^{I_2}-Lie$ algebra" will stand for "Hom-Lie algebra of type I_2". By $Hom^{\star}-Lie$, we will mean a Hom-algebra of simultaneously all types.

We have chosen to divide these types into three classes I, II, and III accordingly to the degree in α, that is the number of occurrences of α in the defining equations. We consider the "virtual" degree. In $\alpha([x,y])$, for example, α is of virtual degree two even if it appears only once. This is because if α is a morphism for the bracket, one has $\alpha([x,y]) = [\alpha(x), \alpha(x)]$ which is really of degree two in α. We hope that this choice will help the reader to memorize easily the subdivision in classes.

We used the "prime" notation to remember that III' is derived from III and that III'' is derived from III'. Accordingly, II_1, II_2, and II_3 should have been denoted by II'_1, II'_2, and II'_3, but we decided to omit the upper script prime, since the lower script enables already to distinguish these types.

We have chosen the ordering in classes II and I in a way that they coincide under the symmetry S which consists of interchanging the role of α and id (the identity of V), namely, for example, $S([Id(x), [\alpha(y), \alpha(z)]]) := [\alpha(x), [Id(y), Id(z)]]$.

Finally, we introduce the Jacobiator associated to each of these structures as the left-hand side of the defining equation of the $Hom^{type}-Lie$ algebra under consideration. One denotes it by J_α^{type}. As an example,

$$J_\alpha^{II_1} := \circlearrowleft_{x,y,z} \big[x, \big[\alpha(y), \alpha(z)\big]\big].$$

In particular, $S(J_\alpha^{I_i}) = J_\alpha^{II_i}$ for $1 \le i \le 3$.

We conclude this section by the following table which summarizes the list of types of Hom-Lie algebras:

I_1	$\circlearrowleft_{x,y,z} [\alpha(x), [y,z]] = 0$
I_2	$\circlearrowleft_{x,y,z} [x, [\alpha(y), z]] = 0$
I_3	$\circlearrowleft_{x,y,z} [x, [y, \alpha(z)]] = 0$
II_1	$\circlearrowleft_{x,y,z} [x, [\alpha(y), \alpha(z)]] = 0$
II_2	$\circlearrowleft_{x,y,z} [\alpha(x), [y, \alpha(z)]] = 0$
II_3	$\circlearrowleft_{x,y,z} [\alpha(x), [\alpha(y), z]] = 0$
III	$\circlearrowleft_{x,y,z} \alpha([x, [y,z]]) = 0$
III'	$\circlearrowleft_{x,y,z} [\alpha(x), \alpha([y,z])] = 0$
III''	$\circlearrowleft_{x,y,z} [\alpha(x), [\alpha(y), \alpha(z)]] = 0$

2.2 Relations among these types

Now, that we have these new types of Hom-Lie algebras, it is natural to seek for relations among them.

Proposition 2.9. *Let us suppose that $(V, [\cdot,\cdot])$ is a Lie algebra and consider $(V, [\cdot,\cdot], \alpha)$:*

(1) if it is a Hom-Lie algebra of type I_2, it is necessarily also of type I_1;

(2) if it is a Hom-Lie algebra of type II_2, it is necessarily of type II_1.

In particular.

Corollary 2.10. *Moreover, if α is a morphism of $[\cdot,\cdot]$, being of type II_2 implies to be of type II_1 which in turn is equivalent to be of type II.*

Proof. We start by proving the assertion that a Hom-Lie algebra of type I_2 is necessarily of type I_1. Let us first establish the following property:

$$J_\alpha^{I_1} = -J_\alpha^{I_2} - J_\alpha^{I_3}. \tag{2.12}$$

Indeed, since the bracket satisfies the Jacobi identity, one has

$$\big[\alpha(x), [y,z]\big] = -\big[z, \big[\alpha(x), y\big]\big] - \big[y, \big[z, \alpha(x)\big]\big].$$

Summing this relation over cyclic permutations leads to the desired property.

Now, let us suppose that we have a $Hom^{I_2} - Lie$ algebra, that is, $J_\alpha^{I_2} = 0$. Remark 2.8 implies that one also has $J_\alpha^{I_3} = 0$. Property (2.12) enables to conclude.

The proof that a Hom-Lie algebra of type II_2 is necessarily of type II_1 is almost the same, one just needs to read the preceding proof after having applied the symmetry S. □

The reverse implication in the preceding proposition would need, to hold, that $J_\alpha^{I_2} + J_\alpha^{I_3} = 0 \Rightarrow J_\alpha^{I_2} = J_\alpha^{I_3} = 0$. It is natural to ask for a necessary and sufficient condition for this last implication. We do not know the answer to this question.

Remark 2.11. The "self-adjointness" condition $[\alpha(x), y] = [x, \alpha(y)]$ $\forall x, y \in V$ on α is sufficient if the underlying k-module V is 2-torsion-free, but this condition of self-adjointness is fairly strong, implying, for instance, that $[\alpha(x), x] = 0$ for all $x \in V$ if V is a vector space over a field of characteristic $\neq 2$ and is therefore in most cases unsatisfied.

The following example shows that $J_\alpha^{I_2} + J_\alpha^{I_3} = 0$ does not indeed imply $J_\alpha^{I_2} = 0$, as would be expected.

Example 2.12. Let K be a field with $\mathrm{char}(K) \neq 2$ and let $V := K^2$. We define on V the bracket:

$$\left[\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}\right] := \begin{pmatrix}0\\ \lambda_1\mu_2 - \lambda_2\mu_1\end{pmatrix}.$$

It is clear that this is skew-symmetric, and direct calculation verifies the Lie identity. Set now

$$\alpha\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} := \begin{pmatrix}\lambda_1 + \lambda_2\\ \lambda_2\end{pmatrix}.$$

Then, we see

$$\left[\alpha\begin{pmatrix}a_1\\ a_2\end{pmatrix}, \left[\begin{pmatrix}b_1\\ b_2\end{pmatrix}, \begin{pmatrix}c_1\\ c_2\end{pmatrix}\right]\right] = \begin{pmatrix}0\\ a_1b_1c_2 + a_2b_1c_2 - a_1b_2c_1 - a_2c_1b_2\end{pmatrix},$$

and summing up cyclic permutations of the term in the second component, this implies that condition $J_\alpha^{I_1}$ is satisfied. However, we have $[e_1, [\alpha(e_2), e_2]] \neq 0$, so $(V, [\cdot,\cdot])$ is not of type I_2.

3 Hom-associative algebras

In this section, we start applying the ideas of the previous section to the associative type, by getting in Section 3.1 the list of different types of Hom-associative structures one can consider. Subsequently, we focus our studies on *unital* Hom-associative algebras since much more can be said about this special case than in general. In particular, we state a hierarchy on such types of unital Hom-algebras, that is, a complete classification of implications between them. We introduce Hom-monoids which will give a useful tool to build counterexamples.

We turn then in Section 3.2 to the proof of this hierarchy. We start by proving implications that do hold and then give counter examples to the others.

3.1 Types, unitality, and hierarchy

Types of Hom-associative algebras

Hom-associative algebras were introduced in [8], as examples of Hom-structures; but there are more types of Hom-associative algebras than the ones considered in [8]. Analogously to Hom-Lie algebras, one gets the following new types of Hom-associative algebras.

		II	$x \star \alpha(y \star z) = \alpha(x \star y) \star z$
I_1	$\alpha(x) \star (y \star z) = (x \star y) \star \alpha(z)$	II_1	$x \star (\alpha(y) \star \alpha(z)) = (\alpha(x) \star \alpha(y)) \star z$
I_2	$x \star (\alpha(y) \star z) = (x \star \alpha(y)) \star z$	II_2	$\alpha(x) \star (y \star \alpha(z)) = (\alpha(x) \star y) \star \alpha(z)$
I_3	$x \star (y \star \alpha(z)) = (\alpha(x) \star y) \star z$	II_3	$\alpha(x) \star (\alpha(y) \star z) = (x \star \alpha(y)) \star \alpha(z)$

III	$\alpha(x \star (y \star z)) = \alpha((x \star y) \star z)$
III'	$\alpha(x) \star \alpha(y \star z) = \alpha(x \star y) \star \alpha(z)$
III''	$\alpha(x) \star (\alpha(y) \star \alpha(z)) = (\alpha(x) \star \alpha(y)) \star \alpha(z)$

For precision, we give the following general definition.

Definition 3.1. Let V be a set together with two binary operations $+ : V \times V \to V$ and $\star : V \times V \to V$, one self-map $\alpha : V \to V$ and a special element $0 \in V$. Then $(V, +, \star, \alpha, 0)$ is called a *Hom-ring of type T* if:

- $(V, +, 0)$ is an abelian group;
- the multiplication is distributive on both sides;
- α is an abelian group homomorphism;
- α and $\star$ satisfy the associativity condition corresponding to type T.

Hom-associative *algebras* over a commutative ring k are defined analogously by replacing any additivity conditions in the preceding definitions by corresponding conditions of k-linearity.

We remark that as in the associative case, a Hom-ring may always be viewed as a Hom-algebra with $\mathbb{Z}$ as base ring. Therefore, although we will in the sequel mainly talk about Hom-algebras, of course only results that need conditions on properties of the base ring cannot be put in the Hom-ring setting.

It seems that not much can be proven *in general* about the relations among the various types of Hom-associative algebras just introduced. However, if additional conditions are imposed which restrict the range of algebras under consideration, the theory becomes much richer. One particularly natural condition which can be imposed is the existence of a unit element. It turns out that for *unital* Hom-associative algebras, the different types we defined can in some sense be partially ordered by increasing generality.

Unitality

Let us note that some of these types, namely, I_2, I_3, and II already appeared in [3] but not under that name. One can reformulate these results (the meaning of unitality is precised below).

Proposition 3.2. *Let $(V, \star, \alpha, 1)$ be a unital Hom-associative algebra of type I_1, then it is also of type I_2.*

Definition 3.3. Let $(A, \star, \alpha)$ be a Hom-associative algebra. Then, A is called *left weakly unital* if $\alpha(x) = cx$ for some $c \in A$.

Lemma 3.4. *Let $(A, \star, \alpha, c)$ be a weakly left unital Hom-associative algebra with weak left unit $c \in A$, bijective α and let $\beta := \alpha^{-1}$, then $(A, \star, \beta)$ is a Hom-associative algebra of type I_3, it is also of type II.*

Proposition 3.2 is the transcription of [3, Proposition 1.1], while Lemma 3.4 concatenates [3, Lemmas 2.1 and 2.4].

In the rest of the paper, we assume $(V, \star, 1)$ to be unital. By unitality, we mean the usual notion in algebra, that is, the existence of an element 1 in V such that $1 \star x = x \star 1$ for all x in V. The notion of an inverse of x is also meaningful to some extent: x^{-1} is a left (resp., right) inverse of x if it satisfies $x^{-1} \star x = 1$ (resp., $x \star x^{-1} = 1$). We call x^{-1} an inverse of x if it is both a left inverse and a right inverse. There are known examples of nonassociative algebras with elements admitting different right and left inverses. Hence, since *any* bilinear map $\star : V \times V \to V$ induces a Hom-associative structure on V if we take $\alpha = 0$ as twisting homomorphism, there is no reason to assume that the inverse of x should be unique if defined.

Hierarchy on types of unital Hom-associative algebras

The hierarchy alluded to in the introduction can for the relations among first and second types be summarized as follows.

Proposition 3.5. *Let $(V, \star, 1)$ be a triple constituted by a vector space V, a multiplication (bilinear map) $\star$, and a unit 1, then one has the following relations between the types of $(V, \star, 1, \alpha)$:*

(a) $II_1 \Leftarrow II \Leftrightarrow I_1 \Rightarrow I_3 \Rightarrow \{I_2, II_2, II_3$ *and* $II_1\}$,

(b) $I_2 \Rightarrow \{II_1 \Leftrightarrow II_3\}$.

There are no relations among {type III} and from {type III} to {types I and II}. The relations from {types I and II} to {type III} are given by the following proposition.

Proposition 3.6. *Let us consider the triple $(V, \star, 1)$ of the preceding proposition, then one has the following relations between the types of $(V, \star, 1, \alpha)$:*

(a) $I_1 \Leftarrow III, III', III''$,

(b) $I_3 \Rightarrow III''$,

(c) $I_2 \Rightarrow III''$,

(d) $II_2 \Rightarrow III''$,

(e) $II_1, II_3 \Rightarrow III''$.

The equivalence of types I_1 and II will be given in Proposition 3.12. Proposition 3.13 gives that type I_1 implies type I_3 and that type II implies type II_1. Proposition 3.15 states that I_3 implies I_2, II_2, II_3, and II_1. We prove in Proposition 3.17 the exotic implication of Proposition 3.5 (b).

Proposition 3.6 is the conjunction of Propositions 3.18, 3.19, and 3.20.

Hom-monoids

We will now introduce *Hom-monoids* and give a short discussion of their relation to Hom-algebras. The main motivation is that Hom-monoids will be useful in the construction of counterexamples to relations between the different types of Hom-algebras.

Definition 3.7. A *Hom-monoid* of type I is a set S together with a binary operation $\star : S \times S \to S$, a special element $1 \in S$ and a map $\alpha : S \to S$, such that the following axioms are fulfilled:

$$1 \star x = x \star 1 = x, \quad \alpha(x) \star (y \star z) = (x \star y) \star \alpha(z).$$

Similarly, we introduce for each type of Hom-associative algebra defined previously the corresponding type of Hom-monoid. If we do not specify a type, type I will by default be implied.

The following example is clear.

Example 3.8. Let $(V, \star, +, \alpha, 1)$ be a Hom-algebra of type T. Then, the multiplicative structure $(V, \star, \alpha, 1)$ is a Hom-monoid also of type T.

However, one has the following remark.

Remark 3.9. Let k be a commutative ring and let $(S, \tilde{\star}, \tilde{\alpha}, 1)$ be a Hom-monoid of type T. Let then V be the free k-module over S and define $\alpha : V \to V$ and $\star : V \times V \to V$ by linear extension of $\tilde{\alpha} : S \to S$, respectively, $\tilde{\star} : S \times S \to S$ to V. Then, $(V, \star, \alpha, 1)$ is a unital Hom-associative algebra of type T. We denote the Hom-algebra so constructed from a Hom-monoid S by $k[S]$.

Proof. By construction, α is linear and $\star$ is bilinear. Using the distributive laws, one verifies easily that type T Hom-associativity of $(V, \star, \alpha)$ follows from the corresponding property on the generating set S. Unitality of V is clear. This concludes the proof. □

With respect to exploring the relations between different types of Hom-associative algebras, the preceding remark and example show that if type T_1 subsumes type T_2 in the context of unital Hom-algebras, then the same holds in the context of Hom-monoids and vice versa.

To obtain from Hom-monoids examples of Hom-algebras which can be written down in a particularly concise way, it will be also useful to set the following definition.

Definition 3.10. A Hom-monoid $(S, \star, \alpha, 1)$ of type T is called a Hom-monoid of type T *with zero* if there is an element $0 \in S$, $0 \neq 1$, such that $0 \star x = x \star 0 = 0$ for all $x \in S$ and $\alpha(0) = 0$.

Since as with unital Hom-algebras of the original type I_1, the twisting map α is also in a Hom-monoid of type I_1 automatically multiplication with some element inside the structure, one could drop the condition $\alpha(0) = 0$ in this case from the definition of a Hom-monoid with zero. However, for most of the other types, it is necessary to impose it separately.

In any case, if V is a Hom-algebra over some commutative ring k of type T constructed from a Hom-monoid S with zero of the same type, and if 0_S denotes the zero element in S, then the submodule $I := k \cdot 0_S$ becomes a Hom-ideal of $k[S]$. One can kill this submodule by passing to the appropriate factor algebra $A := k[S]/I$. We note that A still contains a copy of S. In particular, if S was (not) of type T, then A will be (not) of type T.

3.2 Proof of the hierarchy

The rest of this paper is devoted to the exploration of the logical relationships between the different types of unital Hom-algebras. We start by establishing relations that do hold. Counterexamples to the other relations are given at the end of this section.

Equivalence between types I_1 and II

We start by a lemma which contains the main basic properties, allowing computations with types I_1 and II. This lemma was proved in [3, Lemma 1.1] under the assumption of being of type I_1. We recall it and extend it under the assumption of being of type II.

Lemma 3.11. *Let $(V, \star, \alpha, 1)$ be of type I_1 or II. One has for all x, y in V:*

(ä) $\alpha(x) \star y = x \star \alpha(y)$,

(ë) $x \star \alpha(1) = \alpha(x)$,

(ï) $\alpha(x \star y) = x \star \alpha(y)$.

Proof. The proof of the two first points, assuming the Hom-algebra to be of type II, works along exactly the same lines as the proof of [3, Lemma 1.1] and is left to the reader. To prove (ï), simply apply the definition of Hom-algebra of type II to the triple $(x, y, 1)$: $\alpha(x \star y) \star 1 \overset{II}{=} x \star \alpha(y \star 1)$. □

The desired equivalence is then a simple corollary.

Proposition 3.12. *Hom-associative algebras of types I_1 and II are equivalent.*

Proof. The contemplation of the following square gives the proof:

$$\begin{array}{ccc} x \star \alpha(y \star z) & \overset{II}{=} & \alpha(x \star y) \star z \\ \text{(ä)} \parallel & & \text{(ä)} \parallel \\ \alpha(x) \star (y \star z) & \overset{I_1}{=} & (x \star y) \star \alpha(z). \end{array}$$

□

Type implications from I_1

Now, we concern ourselves with the types subsumed by the equivalent types I_1 and II.

Proposition 3.13. *Let $(V, \star, \alpha, 1)$ be a unital Hom-associative algebra of type I_1, then*

(a) it is also of type I_3;

(b) it is also of type II_1.

The proof of Proposition 3.13 requires the following equalities taken from [3, Proposition 1.1].

Lemma 3.14. *Let $(V, \star, \alpha, 1)$ be a unital Hom-associative algebra of type I_1. One has for all x, y and z in V:*

(1) $\alpha(x) \star (y \star z) = (\alpha(x) \star y) \star z$;

(2) $x \star (y \star \alpha(z)) = (x \star y) \star \alpha(z)$.

Proof. The proof of (a) comes from contemplation of the following square:

$$\begin{array}{ccc} \alpha(x) \star (y \star z) & \overset{I_1}{=} & (x \star y) \star \alpha(z) \\ 3.14\ (1) \parallel & & 3.14\ (2) \parallel \\ \big(\alpha(x) \star y\big) \star z & \overset{I_3}{=} & x \star \big(y \star \alpha(z)\big). \end{array}$$

There exists a direct proof of (b), but it can also be seen from the chain $I_1 \overset{(a)}{\Rightarrow} I_3 \overset{3.15}{\Rightarrow} \{I_2, II_3\} \overset{3.17}{\Rightarrow} II_1$ which will be proven in the following two sections. □

Type implications from I_3

The main result of this section is as follows.

Proposition 3.15. *Let $(V, \star, \alpha, 1)$ be a unital Hom-associative algebra of type I_3, then*

(a) it is also of type I_2;

(b) it is also of type II_2;

(c) it is also of type II_3.

Its proof is based on the following two basic properties.

Lemma 3.16. *Let $(V, \star, \alpha, 1)$ be a unital Hom-associative algebra of type I_3, then $\forall x, y \in V$:*

(α) $\alpha(x) = x \star \alpha(1)$;

(β) $x \star \alpha(y) = \alpha(x) \star y$.

Proof of the lemma. One proves (α) by applying the definition of Hom^{I_3}-associativity to the triple $(x, 1, 1)$:

$$x \star \alpha(1) = x \star \big(1 \star \alpha(1)\big) \overset{I_3}{=} \big(\alpha(x) \star 1\big) \star 1 = \alpha(x).$$

The proof of (β) is obtained by considering the triple $(x, y, 1)$ by the use of (a):

$$x \star \alpha(y) \overset{(a)}{=} x \star \big(y \star \alpha(1)\big) \overset{I_3}{=} \big(\alpha(x) \star y\big) \star 1 = \alpha(x) \star y.$$

□

One should resist the temptation to deduce Lemma 3.11 from Proposition 3.13 and Lemma 3.16 since Proposition 3.13 relies itself on Lemma 3.11. We now turn on to the proof of Proposition 3.15.

Proof. We show each statement in turn:

(a)

$$\begin{array}{ccc}
x\star\big(\alpha(y)\star z\big) & \overset{I_2}{=} & \big(x\star\alpha(y)\big)\star z\\
(\beta)\,\| & & \|\,(\beta)\\
x\star\big(y\star\alpha(z)\big) & \overset{I_3}{=} & \big(\alpha(x)\star y\big)\star z;
\end{array}$$

(b)

$$\begin{array}{ccc}
\big(\alpha(x)\star y\big)\star\alpha(z) & \overset{II_2}{=} & \alpha(x)\star\big(y\star\alpha(z)\big)\\
I_3\,\| & & \|\,I_3\\
x\star\big(y\star\alpha\big(\alpha(z)\big)\big) & & \big(\alpha\big(\alpha(x)\big)\star y\big)\star z\\
(\beta)\,\| & & \|\,(\beta)\\
x\star\big(\alpha(y)\star\alpha(z)\big) & \overset{I_3}{=} & \big(\alpha(x)\star\alpha(y)\big)\star z;
\end{array}$$

(c)

$$\begin{array}{ccc}
\alpha(x)\star\big(\alpha(y)\star z\big) & \overset{II_3}{=} & \big(x\star\alpha(y)\big)\star\alpha(z)\\
(\beta)\,\| & & \|\,(\beta)\\
\alpha(x)\star\big(y\star\alpha(z)\big) & & \big(\alpha(x)\star y\big)\star\alpha(z)\\
I_3\,\| & & \|\,I_3\\
\big(\alpha\big(\alpha(x)\big)\star y\big)\star z & & x\star\big(y\star\alpha\big(\alpha(z)\big)\big)\\
(\beta)\,\| & & \|\,(\beta)\\
\big(\alpha(x)\star\alpha(y)\big)\star z & \overset{I_3}{=} & x\star\big(\alpha(y)\star\alpha(z)\big).
\end{array}$$

□

Exotic implications

We call these implications exotic since, contrary to the previous ones, they involve two types of Hom-algebras in the assumptions.

Proposition 3.17. *Let $(V,\star,\alpha,1)$ be a unital Hom-associative algebra of type I_2, then it is of type II_3 if and only if it is of type II_1.*

Proof. The following calculation shows our claim:

$$\begin{array}{ccc}
x\star\big(\alpha(y)\star\alpha(z)\big) & \overset{II_1}{=} & \big(\alpha(x)\star\alpha(y)\big)\star z\\
I_2\,\| & & \|\,I_2\\
\big(x\star\alpha(y)\big)\star\alpha(z) & \overset{II_3}{=} & \alpha(x)\star\big(\alpha(y)\star z\big).
\end{array}$$

□

Implications from types of families I and II to types of family III

There are not many relations between these types, except for III'' which is weaker than almost all the other types and I_1 which is stronger than all the other types.

Proposition 3.18. *One has the following implications:*

(a) a Hom-algebra of type I_1 is necessarily of type III;
(b) a Hom-algebra of type I_1 is necessarily of type III';

(c) a Hom-algebra of type I_1 is necessarily of type III''.

Proof. (a) is Proposition 3.2, (4).

(b)

$$\begin{array}{ccc} \alpha(x)\star\alpha(y\star z) & = & \alpha(x\star y)\star\alpha(z)\\ \text{Lemma 3.11}\ \| & & \|\ \text{Lemma 3.11}\\ \alpha(x)\star\big(\alpha(y)\star z\big) & \overset{I_1}{=} & \big(x\star\alpha(y)\big)\star\alpha(z). \end{array}$$

Hom III'-associativity in (c) is obtained by applying Proposition 3.2(1) to the triple $\{x,\alpha(y),\alpha(z)\}$. □

The last point of the previous proposition can be refined (and implied) by the following.

Proposition 3.19. *A Hom-algebra of type I_3 is necessarily of type III''.*

Proof. By Proposition 3.15(a), $I_3 \Rightarrow I_2$, but by Proposition 3.20(a) below, $I_2 \Rightarrow III'$. □

There are no other implications in this direction from I_3, as shown by the following counterexample:

$$I_3 \not\Rightarrow III, III' : \big(e_2\cdot e_2\big) := e_1, \big(e_3\cdot e_3\big) := e_3; \alpha\big(e_1\big) = \alpha\big(e_3\big) = e_3.$$

In particular, since I_3 implies $\{II_1, II_2, II_3, I_2\}$, the previous counterexamples are also counterexamples to $\{II_1, II_2, II_3, I_2\} \Rightarrow III$ and $\{II_1, II_2, II_3, I_2\} \Rightarrow III'$, but can III'' be implied by one of these types?

Proposition 3.20. *One has the following implications:*

(a) a Hom-algebra of type I_2 is necessarily of type III'';

(b) a Hom-algebra of type II_2 is necessarily of type III'';

(c) a Hom-algebra of types II_1 and II_3 is necessarily of type III''.

The hard point to prove is (c), and we will need for it the following.

Lemma 3.21. *For a Hom-algebra V which is of types II_1 and II_3, one has $\forall x, y\in V$:*

(a) $\alpha(1)\star\alpha(x) = \alpha(x)\star\alpha(1)$;

(b) $\alpha(\alpha(x))\star\alpha(1) = \alpha(x)\star\alpha(\alpha(1))$;

(c) $\alpha(x)\star(\alpha(1)\star\alpha(y)) = (\alpha(x)\star\alpha(1))\star\alpha(y)$;

(d) $\alpha(\alpha(x))\star\alpha(y) = \alpha(x)\star\alpha(\alpha(y))$.

Proof of Lemma 3.21. We start with

(a)

$$\begin{array}{ccc} \alpha(1)\star\alpha(x) & = & \alpha(x)\star\alpha(1)\\ \| & & \|\\ \big(\alpha(1)\star\alpha(x)\big)\star 1 & \overset{II_1}{=} & 1\star\big(\alpha(x)\star\alpha(1)\big); \end{array}$$

(b)

$$\begin{array}{ccc}
\alpha\big(\alpha(x)\big)\star\alpha(1) & = & 1\star\big(\alpha(x)\star\alpha\big(\alpha(1)\big)\big)\\
\| & & \| \; II_1\\
\big(\alpha\big(\alpha(x)\big)\star\alpha(1)\big)\star 1 & & \big(\alpha(1)\star\alpha(x)\big)\star\alpha(1)\\
II_1 \; \| & & \| \; (a)\\
\alpha(x)\star\big(\alpha(1)\star\alpha(1)\big) & \overset{II_3}{=} & \big(\alpha(x)\star\alpha(1)\big)\star\alpha(1);
\end{array}$$

(c)

$$\begin{array}{ccc}
\alpha(x)\star\big(\alpha(1)\star\alpha(y)\big) & = & \big(\alpha(x)\star\alpha(1)\big)\star\alpha(y)\\
II_1 \; \| & & \| \; II_1\\
\big(\alpha\big(\alpha(x)\big)\star\alpha(1)\big)\star y & & x\star\big(\alpha(1)\star\big(\alpha\big(\alpha(y)\big)\big)\big)\\
(b) \; \| & & \| \; (b)\\
\big(\alpha(x)\star\alpha\big(\alpha(1)\big)\big)\star y & \overset{II_1}{=} & x\star\big(\alpha\big(\alpha(1)\big)\star\big(\alpha(y)\big)\big);
\end{array}$$

(d)

$$\begin{array}{ccc}
\alpha\big(\alpha(x)\big)\star\alpha(y) & = & \alpha(x)\star\alpha\big(\alpha(y)\big)\\
\| & & \|\\
\big(\alpha\big(\alpha(x)\big)\star\alpha(y)\big)\star 1 & & 1\star\big(\alpha(x)\star\alpha\big(\alpha(y)\big)\big)\\
II_1 \; \| & & \| \; II_1\\
\alpha(x)\star\big(\alpha(y)\star\alpha(1)\big) & & \big(\alpha(1)\star\alpha(x)\big)\star\alpha(y)\\
(a) \; \| & & \| \; (a)\\
\alpha(x)\star\big(\alpha(1)\star\alpha(y)\big) & \overset{(c)}{=} & \big(\alpha(x)\star\alpha(1)\big)\star\alpha(y).
\end{array}$$

□

Proof of Proposition 3.20. One proves (a) by applying the definition of type I_2 to the triple $\{\alpha(x), y, \alpha(z)\}$, and one proves (b) by applying the definition of type II_2 to the triple $\{x, \alpha(y), z\}$. The proof of (c) comes from contemplation of the following diagram:

$$\begin{array}{ccc}
\big(\alpha(x)\star\alpha(y)\big)\star\alpha(z) & \overset{III''}{=} & \alpha(x)\star\big(\alpha(y)\star\alpha(z)\big)\\
II_1 \; \| & & \| \; II_1\\
x\star\big(\alpha(y)\star\alpha\big(\alpha(z)\big)\big) & & \big(\alpha(x)\star\alpha(y)\big)\star\alpha(z)\\
\| & & \|\\
x\star\big(\alpha\big(\alpha(y)\big)\star\alpha(z)\big) & \overset{II_3}{=} & \big(\alpha(x)\star\alpha\big(\alpha(y)\big)\big)\star z.
\end{array}$$

□

Counterexamples to intertype relations

We give now a list of counterexamples to intertype relations which do *not* hold. All of these counterexamples are constructed by the use of the technique of Hom-monoids discussed in Section 3.1 above. For each example, we first describe what the example is supposed to show. Here, $T_1, T_2, \ldots, T_n \not\Rightarrow T$ means that a Hom-monoid which is of simultaneously of types $Type_i$ is not necessarily also type T. This is followed by relations between elements of a counterexample Hom-monoid of the requisite types. The elements of the Hom-monoid structures in question are denoted by $e_1, \ldots, e_n$, where e_1 is supposed the unit element. All Hom-monoid structures are with zero, but the zero element is outside the set $\{e_1, \ldots, e_n\}$. When we do not give a product e_ie_j, this means that $e_ie_j = 0$, except when $i = 1$ or $j = 1$, in which case the product is prescribed by the requirement that e_1 be the unit of the Hom-monoid. The Hom-monoids in question are not supposed to have elements e_i with higher index than those appearing in the relations we supply. Likewise, we give only nonzero values of α. Note that the first three examples use the fact that not every associative structure is also Hom-associative of all types.

(1) $I_2 \not\Rightarrow I_3$: $\alpha(e_2) = e_1$.
(2) $I_2 II_2 \not\Rightarrow II_3$: $\alpha(e_1) = e_1$.
(3) $II_1 II_2 II_3 I_2 \not\Rightarrow I_3$: $\alpha(e_2) = e_2$.
(4) $I_3 \not\Rightarrow I_1$: $(e_2 \cdot e_2) = e_1$; $\alpha(e_2) = \alpha(e_3) = e_3$.
(5) $I_2 \not\Rightarrow II_2$: $(e_2 \cdot e_2) = e_2, (e_2 \cdot e_3) = e_2$; $\alpha(e_1) = \alpha(e_2) = e_3$.
(6) $II_2 \not\Rightarrow I_2$: $(e_2 \cdot e_3) = e_3$; $\alpha(e_1) = e_2$.
(7) $II_1 II_2 \not\Rightarrow II_3$: $(e_2 \cdot e_3) = (e_3 \cdot e_2) = e_1$; $\alpha(e_1) = e_3$.
(8) $II_1 \not\Rightarrow II_2$: $(e_2 \cdot e_3) = e_1$; $\alpha(e_1) = e_2$.
(9) $II_1 II_2 II_3 \not\Rightarrow I_2$: $(e_2 \cdot e_3) = e_1, (e_3 \cdot e_2) = e_1$; $\alpha(e_2) = e_3$.
(10) $II_2 II_3 \not\Rightarrow II_1$: $(e_2 \cdot e_2) = e_1, (e_3 \cdot e_3) = e_2$; $\alpha(e_1) = e_3$.
(11) $I_2 II_1 II_3 \not\Rightarrow II_2$: $(e_3 \cdot e_2) = e_4, (e_4 \cdot e_3) = e_2$; $\alpha(e_1) = e_3$. The next three examples show the absence of relations between Hom types III, III', and III''.
(12) $III, III'' \not\Rightarrow III'$: $\alpha(e_2) = e_1$.
(13) $III, III' \not\Rightarrow III''$: $(e_2 \cdot e_2) = e_3, (e_3 \cdot e_2) = e_2$; $\alpha(e_1) = e_2$.
(14) $III', III'' \not\Rightarrow III$: $(e_2 \cdot e_2) = e_3, (e_3 \cdot e_2) = e_3$; $\alpha(e_3) = e_3$. Lastly, we give an example that shows that the order of the three types does not subsume any of the other types even when taken together.
(15) $III, III', III'' \not\Rightarrow I_2, II_1, II_2, II_3$: $e_2 \cdot e_2 = e_1$, $e_2 \cdot e_3 = e_1$, $e_3 \cdot e_2 = e_2$, $e_3 \cdot e_3 = e_1$ and $\alpha(e_1) = \alpha(e_2) = \alpha(e_3) = e_3$.

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