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Canonical endomorphism field on a Lie algebra

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Abstract

We show that every Lie algebra is equipped with a natural $(1,1)$-variant tensor field, the "canonical endomorphism field", determined by the Lie structure, and satisfying a certain Nijenhuis bracket condition. This observation may be considered as complementary to the Kirillov-Kostant-Souriau theorem on symplectic geometry of coadjoint orbits. We show its relevance for classical mechanics, in particular for Lax equations. We show that the space of Lax vector fields is closed under Lie bracket and we introduce a new bracket for vector fields on a Lie algebra. This bracket defines a new Lie structure on the space of vector fields.

2000 MSC: 17B08, 53C15, 53C80, 70G45, 70G60, 70H03, 70H05

Notation. We will distinguish between purely algebraic and differential products by using two types of brackets:

$[\![\ ,\ ]\!]$: Lie algebra product,

$[\ ,\ ]$: Lie commutator of vector fields, Schouten bracket, Nijenhuis bracket.

The summation convention over repeated indices is adopted throughout the paper.

1 Introduction

It is well known that the underlying dual space L^* of a Lie algebra L possesses—as a manifold—a canonical Poisson structure in terms of a smooth bivector field $\Omega \in \wedge^2 TL^*$, which satisfies the Jacobi condition $[\Omega, \Omega] = 0$ and, when restricted to coadjoint orbits, is nondegenerate and therefore invertible into a symplectic structure [12, 16, 17]. The existence of these symplectic sheets is the content of the Kirillov-Kostant-Souriau theorem ([3, 9, 15]).

In this paper, we present an overlooked fact that the Lie algebra L itself also possesses—as a manifold—a natural differential-geometric object, namely, a $(1,1)$-type tensor field $\mathcal{A} \in \mathcal{T}^{(1,1)}L$ that we will call the *canonical endomorphism field* on L. The principal geometric property of $\mathcal{A}$ is that it is proportional to its own Nijenhuis derivative (Theorem 2.1).

We discuss the relevance of this object for dynamical systems. It turns out that what Hamilton equations are for the dual space L^*, Lax equations are for L. The principal property of $\mathcal{A}$ assures that the space of "Lax vector fields" is closed under the Lie commutator and, moreover, it allows one to introduce a new bracket of vector fields on L, which is the analog for Lax equations of the Poisson bracket on Hamiltonian vector fields.

2 The canonical endomorphism field on a Lie algebra

Customarily, one defines a Lie algebra as a linear space L with a product $L \times L \mapsto L$ denoted $[\![v, w]\!]$ (double bracket). The product is bilinear, skew-symmetric (i), and satisfies the Jacobi identity (ii):

(i) $[\![v, w]\!] = -[\![w, v]\!]$,

(ii) $[\![v, [\![w, z]\!]]\!] + [\![w, [\![z, x]\!]]\!] + [\![x, [\![w, z]\!]]\!] = 0$,

In a basis $\{e_i\}$, the commutator can be represented via "structure constants":

$$[\![e_i, e_j]\!] = c^k_{ij} e_k. \tag{2.1}$$

Here, we will rather follow [10] and define a Lie algebra as a pair $\{L, c\}$, where c is a $(1, 2)$-type tensor that in the above basis is

$$c = \frac{1}{2} c^k_{ij} \varepsilon^i \wedge \varepsilon^j \otimes e_k, \tag{2.2}$$

where $\{\varepsilon^i\}$ is the dual basis. The algebra product becomes a secondary, derived, concept: $[\![v, w]\!] = (v \wedge w) \lrcorner c = i_w i_v c$. Similarly, the adjoint action of $v \in L$ is defined simply as a (1,1)-tensor $\mathrm{ad}_v = v \lrcorner c$ in L. Of course, from the structural point of view both definitions are equivalent, $\{L, c\} \equiv \{L, [\![\cdot, \cdot]\!]\}$.

The point of the present paper is to look at the space L as a flat manifold and consider various differential-geometric objects on it (we will assume that L is real and finite dimensional). The linear structure of this manifold allows one to prolong any tensor T in L to the ("constant") tensor field $\widetilde{T}$ on the *manifold* L. In particular, the manifold L is equipped with a constant $(1, 2)$-type tensor field $\lambda = \widetilde{c}$:

$$\lambda = \frac{1}{2} c^k_{ij} dx^i \wedge dx^j \otimes \partial_k, \tag{2.3}$$

where $\{x^i\}$ are coordinates on L associated with the basis $\{e_i\}$, and where we denote $\partial_i \equiv \partial/\partial x^i$. The manifold L is also equipped with a natural vector field, the Liouville vector field, which in a linear coordinate system, is

$$J = x^i \partial_i. \tag{2.4}$$

Here is our basic observation.

Theorem 2.1. *The manifold of the Lie algebra L possesses a natural field of endomorphisms (i.e., a (1,1)-variant tensor field) $\mathcal{A} \in \mathcal{T}^{(1,1)} L$ defined by*

$$\mathcal{A} = J \lrcorner \lambda. \tag{2.5}$$

Its Nijenhuis derivative $[\mathcal{A}, \mathcal{A}]$ is a vector-valued biform as follows:

$$[\mathcal{A}, \mathcal{A}] = -2\lambda \lrcorner \mathcal{A}. \tag{2.6}$$

Moreover, $\mathcal{A}$ acts on the adjoint orbits on L.

We will call $\mathcal{A}$ the *canonical endomorphism field* on L. In the coordinate description, $\mathcal{A}$ and its Nijenhuis derivative are

$$\mathcal{A} = x^i c_{ij}^k dx^j \otimes \partial_k, \quad [\mathcal{A}, \mathcal{A}] = -x^k c_{kp}^i c_{ab}^p \big(dx^a \wedge dx^b\big) \otimes \partial_i. \tag{2.7}$$

The endomorphism field $\mathcal{A}$ may be viewed as a family of local transformations that at point $x \in L$ can be represented by matrix $\mathcal{A}_j^k(x) = x^i c_{ij}^k$.

Before we give its proof, let us restate the theorem in more standard terms. The natural isomorphism of a tangent space at any $x \in L$ with the space L itself will be denoted by $\mu_x : T_x L \to L$. Then, Theorem 2.1 states that every Lie algebra L possesses, as a manifold, a unique natural tensor field $\mathcal{A} \in \mathcal{T}^{(1,1)}L$, which at point $x \in L$ is defined as an endomorphism taking a tangent vector $v \in T_x L$ to

$$\mathcal{A}_x(v) = \big(\mu_x^{-1} \circ \operatorname{ad}_x \circ \mu_x\big)(v), \tag{2.8}$$

or, in a somewhat sloppy notation, $\mathcal{A}(v) = [\![x, v]\!]$. Its Nijenhuis derivative $[\mathcal{A}, \mathcal{A}]$ is a vector-valued biform, the evaluation of which equals for any $v, w \in TL$:

$$[\mathcal{A}, \mathcal{A}](v, w) = -2\mathcal{A}\big([\![v, w]\!]\big) = -2\big([\![\mathcal{A}v, w]\!] + [\![v, \mathcal{A}w]\!]\big) \tag{2.9}$$

at point $x \in L$, the dependence of which was suppressed in the notation.

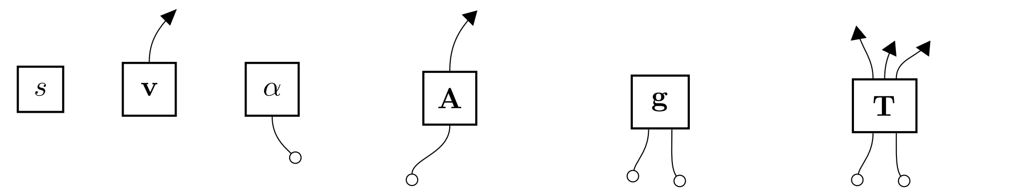
Remark 2.2. The canonical endomorphism field $\mathcal{A}$ is defined for an arbitrary algebra and its differential-geometric properties, including the Nijenhuis bracket $[\mathcal{A}, \mathcal{A}]$, will reflect the type of this algebra. In the present paper, we restrict to Lie algebras, where the Jacobi identity implies particularly pleasant consequences.

The above theorem may be viewed as a counterpart of the KKS theorem: the essence of which is that the dual space L^* is equipped with a bivector field $\Omega = x_k c_{ij}^k \partial^i \wedge \partial^j$ (in our language $\Omega = J \lrcorner \lambda$). Instead of the Nijenhuis bracket, we have the Schouten bracket $[\Omega, \Omega]_{\text{Sch}} = 0$. Thus, Ω defines a Poisson structure, which, moreover, restricts to the coadjoint orbits, on which its inverse ω defines a symplectic structure, $\omega = 0$. Section 8 summarizes these parallels.

3 Lie algebra in pictures

Tensor calculus gains much transparency when expressed in graphical language.

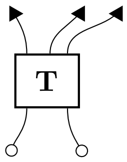
Basic Glyphs. Here are the basic *glyphs* corresponding to various tensors:

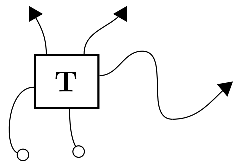


Scalar Vector 1-form Endomorphism Scalar product (3,2)-variant tensor

where s is a scalar, $\mathbf{v}$ is a vector, α is a covector, A is an endomorphism, and g is a metric or biform. The links with arrows —▶ and links with circles —∘ represent the contravariant and the covariant attributes of a tensor, respectively. You may think of them as contravariant/covariant (upper/lower) indices in some basis description. Scalars have none.

The "in" and "out" links may go at any direction. Turning and weaving in space do not have any meaning (unlike in some other conventions). For instance,

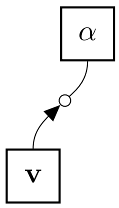
this representation 

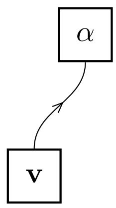
is as good as this 

The links may leave the box at any position, but the order of the point of departure is fixed: the contravariant indices are ordered clockwise, while the covariant indices counterclockwise. Links may cross without any meaning implied.

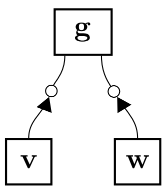
Glyphs may be composed into **pictograms** that represent terms resulting by manipulation with tensors. The tensor contractions are obtained by joining "ins" with "outs". Here are some basic cases.

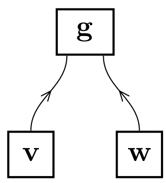
Evaluation. Here is the evaluation of a covector on a form:

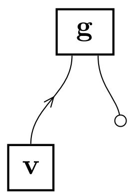
$\langle \alpha, v \rangle \equiv \alpha(\mathbf{v}) =$ 

or simply 

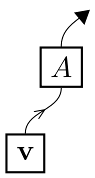
Scalar product. The scalar product of two vectors is a scalar $g(\mathbf{v}, \mathbf{w})$, but if only one vector is contracted with g, then the result is a one form:

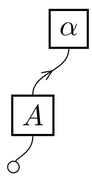
$g(\mathbf{v}, \mathbf{w}) =$ 

$\cong$ 

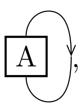
$g(\mathbf{v}, \cdot) \equiv \mathbf{v} \lrcorner g =$ 

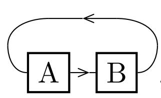
Endomorphism. A acting on a vector v or covector α results in a vector or covector, respectively:

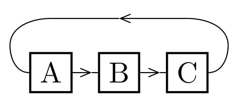
$v \to A\mathbf{v} =$ 

$\alpha \to A^*\alpha =$ 

Trace may be represented by connecting "in" with "out" in a pictogram; if $A, B, C \in \operatorname{End} L$ are endomorphisms of some linear space L, then we have

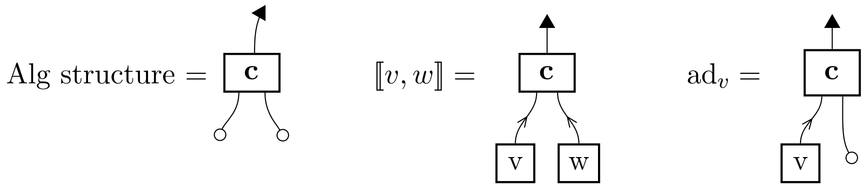
$\operatorname{Tr} A =$ 

, $\operatorname{Tr} AB =$ 

, $\operatorname{Tr} ABC =$ 

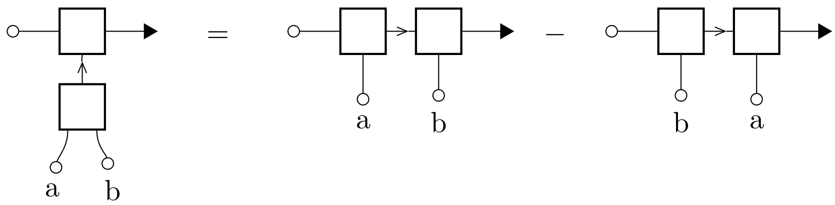
The notable property of trace of a composition of endomorphisms, namely, its invariance under cyclic permutation of the entries, $\operatorname{Tr} A_1 \circ \cdots \circ A_{k-1} \circ A_k = \operatorname{Tr} A_2 \circ \cdots \circ A_k \circ A_1$, becomes in graphical language verifiable with a simplicity of a mantra on a *japa mala*.

Lie algebra in pictures. An *algebra* is defined by a (1,2)-variant tensor c, as shown below on the left. Also a product and adjoint representation is shown as follows:



If a single algebra is considered, the letter "c" will be suppressed.

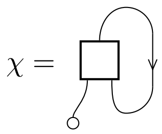
In the case of a *Lie algebra*, besides skew-symmetry we have the *Jacobi identity*, which may be written in this way:



(3.1)

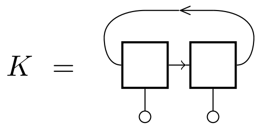
The labels a and b are only to discern between different entries.

Perhaps the simplest derived object is a *characteristic one-form* $\chi \in L^*$ the value of which on a vector $v \in L$ is $\chi(v) = -\operatorname{Tr}\operatorname{ad}_v$. Its pictograph is

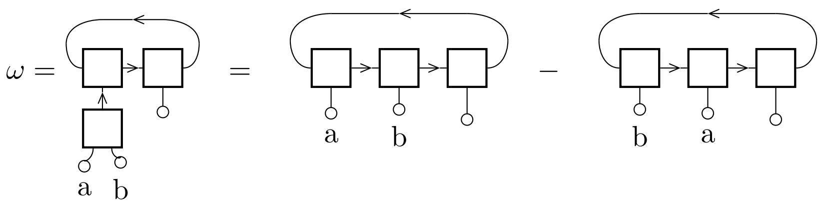


(This one-form vanishes for semisimple algebras.)

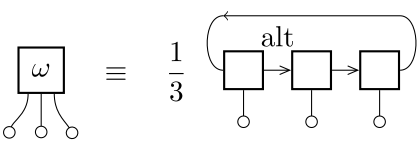
The *Killing form* is defined as an inner product $K(v, w) = \operatorname{Tr}\operatorname{ad}_v\operatorname{ad}_w$. In the diagrammatic script, it is easy to define the corresponding 2-covariant tensor K:



Every Lie algebra possesses a skew-symmetric exterior *Lie 3-form* ω that for any triple $v, w, z \in L$ takes value $\omega(v, w, z) = \operatorname{Tr}\operatorname{ad}_{[\![v,w]\!]}\operatorname{ad}_z$. Using diagrammatic script, we may "draw" the form ω directly—here it is, simplified with the use of Jacobi identity (3.1):

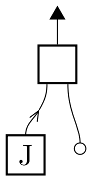
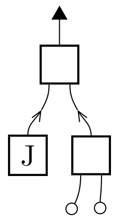


where a and b are merely labels to distinguish the covariant entries. If we use the symbol $\wedge$ or "alt" inside a loop to denote the signed sum over all permutations of entries of a tensor (skewsymmetrization), then the Lie 3-covariant form is

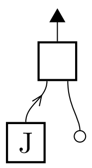
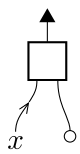


4 Differential geometry on a Lie algebra

Let us now look at the differential geometry of Lie algebra viewed as a manifold. In the diagrammatic language, the objects of Theorem 2.1 are

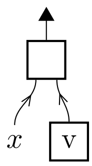
Definition: $\mathcal{A} =$  Theorem: $\frac{1}{2}[\mathcal{A},\mathcal{A}] =$ 

Since the contraction with J introduces dependence on poisition (coordinates x), we will use rather notation that will be easier perceptually. Thus, for instance,

$\mathcal{A} =$  $\equiv$ 

Every element (vector) $v \in L$ defines a "constant" vector field $\widetilde{v} \in \mathcal{X}L$ on manifold L obtained by parallel transport; in coordinates, if $v = v^i e_i$ then $\widetilde{v} = v^i \partial_i$. The canonical endomorphism field $\mathcal{A}$ on manifold L applied to such fields defines a representation of Lie algebra L in terms of vector fields on L, namely, with every algebra element $v \in L$, we associate a vector field:

$$X_v = \mathcal{A}\widetilde{v} = x^i v^j c_{ij}^k \partial_k = \tag{4.1}$$



Proposition 4.1. *The map $v \to X_v$ defines a homomorphism $\{L, [\![\cdot,\cdot]\!]\} \to \{\mathcal{X}L, [\cdot,\cdot]\}$ (the infinitesimal representation of L in terms of $\mathcal{X}L$):*

$$\big[X_v, X_w\big] = X_{[\![v,w]\!]} \tag{4.2}$$

If the center of L is trivial, the map presents a monomorphism.

Proof. The proposition readily follows from the Jacobi identity. □

Corollary 4.2. *The following are convenient formulae:*

(i) $[X_v, \widetilde{w}] = \widetilde{[\![v,w]\!]}$,

(ii) $[\![\widetilde{v}, \widetilde{w}]\!] = \widetilde{[\![v,w]\!]}$,

(iii) $[\widetilde{v}, \widetilde{w}] = 0$.

The image of $\mathcal{A}$ spans at every point a subspace of the tangent space of L, defining in this way a distribution:

$$\mathcal{D} = \operatorname{Im}\mathcal{A} = \operatorname{span}\big\{X_v \mid v \in L\big\}. \tag{4.3}$$

The integral manifolds of this distribution coincide with the adjoint orbits determined by the action of a Lie group on Lie algebra. Note, however, that we may define "adjoint orbits" without reference to the Lie group simply as the integral manifolds $\mathcal{O}$ of $\mathcal{D}$, satisfying $T\mathcal{O} = \mathcal{D}$.

Now we prove the theorem.

Proof of Theorem 2.1. Recall that the Nijenhuis bracket $[K, K]$ of a vector-valued one-form (endomorphism field) K with itself is a vector-valued biform that, evaluated on two fields X and Y, takes the value according to

$$\frac{1}{2}[K,K](X,Y) = [KX, KY] - K[KX, Y] - K[X, KY] + K^2[X,Y], \tag{4.4}$$

(see, e.g., [13]). Evaluating (half of) the Nijenhuis bracket $[\mathcal{A}, \mathcal{A}]$ on two constant vector fields $\widetilde{v}$ and $\widetilde{w}$ and using formulae of Propositions 4.1 and 4.2, one gets the following:

$$\begin{aligned}\frac{1}{2}[\mathcal{A},\mathcal{A}](\widetilde{v},\widetilde{w}) &= \big[X_v, X_w\big] - \mathcal{A}\big(\big[X_v, \widetilde{w}\big]\big) - \mathcal{A}\big(\big[\widetilde{v}, X_w\big]\big) + 0\\ &= X_{[\![v,w]\!]} - X_{[\![v,w]\!]} - X_{[\![v,w]\!]} = -X_{[\![v,w]\!]}.\end{aligned}$$

In particular, substitution of $X = \partial_a$ and $Y = \partial_b$ leads to the coordinate formula (2.7). Now, let us show that $\mathcal{A}$ can be restricted to orbits, that is, $\mathcal{A}(T_x\mathcal{O}) \subset T_x\mathcal{O}$ for each point $x \in \mathcal{O}$. First, rewrite (2.8) for $X \in T_xL$:

$$\mathcal{A}(X) = \mu_x^{-1}\big([\![x, \mu_x(X)]\!]\big).$$

Vector of the vector field X_v at point $x \in L$ can be expressed as follows:

$$X_v(x) = \mathcal{A}(v) = \mu_x^{-1}\big([\![x, v]\!]\big).$$

Thus,

$$\mathcal{A}\big(X_v\big) = \mu_x^{-1}\big(\big[\!\big[x, \mu_x\big(X_v\big)\big]\!\big]\big) = \mu_x^{-1}\big(\big[\!\big[x, [\![x, v]\!]\big]\!\big]\big) = X_{[\![x,v]\!]} \in T_x\mathcal{O},$$

which was to be proven. $\square$

Example 4.3. Consider the 2-dimensional solvable algebra defined by $[\![e_1, e_2]\!] = e_2$. Then,

$$\mathcal{A} = x_1 dx_2 \otimes \partial_2, \quad [\mathcal{A},\mathcal{A}] = -2x_1\big(dx_1 \wedge dx_2\big) \otimes \partial_2.$$

The adjoint orbits are lines parallel to e_2, and the canonical endomorphism—when restricted to any of them—becomes a dilation.

Example 4.4. The Lie algebra of 3-dimensional rotations, so_3, is defined by relations $[\![e_i, e_j]\!] = \varepsilon_{ijk}e_k$. Thus,

$$\begin{aligned}&\mathcal{A} = x_1\big(dx^2 \otimes \partial_3 - dx^3 \otimes \partial_2\big) + (\text{cyclic terms}),\\ &[\mathcal{A},\mathcal{A}] = dx^1 \wedge dx^2 \otimes \big(x^1\partial_2 - x^2\partial_1\big) + (\text{cyclic terms}).\end{aligned}$$

The orbits are spheres defined by the Killing form. On the unit sphere, tensor $\mathcal{A}$ forms an almost complex structure, $\mathcal{A} \circ \mathcal{A} = -\operatorname{id}$.

Remark 4.5. In general endomorphism field $\mathcal{A}$ is not integrable. The integrable cases, where the Nijenhuis bracket (2.6) vanishes, include two-step nilpotent algebras such as algebras of type H [7, 8]. Note that for vector fields of infinitesimal representation, the biform (2.6) takes at any point x a vector-value:

$$[\mathcal{A},\mathcal{A}]\big(X_v, X_w\big) = \mu_x^{-1} \circ \big[\!\big[x, \big[\!\big[[\![x,v]\!], [\![x,v]\!]\big]\!\big]\big]\!\big]. \tag{4.5}$$

Thus, $\mathcal{A}$ restricted to an orbit $\mathcal{O} \subset L$ is (locally) integrable if $[\![x, [\![[\![x,v]\!], [\![x,v]\!]]\!]]\!] = 0$ for every $x \in \mathcal{O}$ and every $v, w \in L$. This is true for $so(n)$, $n \leq 4$ and for nilpotent algebras of the upper-triangular $n \times n$ matrices, $n \leq 5$.

5 Other basic properties of the endomorphism field

The fundamental property of the canonical endomorphism field (Theorem 2.1) is

$$[\mathcal{A},\mathcal{A}] = -2\lambda \,\lrcorner\, \mathcal{A}.$$

Other basic properties of the geometry of a Lie algebra are summarized below.

Corollary 5.1. *The endomorphism field on a Lie algebra satisfies*

(i) $£_J \mathcal{A} = \mathcal{A}$,
(ii) $J \,\lrcorner\, \mathcal{A} = 0$,
(iii) $\operatorname{Im} \mathcal{A}\big|_{\mathcal{O}} \cong \operatorname{Im} \mathrm{ad}_x^2$,
(iv) $\operatorname{Ker} \mathcal{A}\big|_{\mathcal{O}} \cong \operatorname{Ker} \mathrm{ad}_x \cap \operatorname{Im} \mathrm{ad}_x$.

where $\mathcal{O}$ denotes an orbit through x.

Here is a property analogous to the coadjoint representation preserving the Kirillov-Poisson structure on the dual Lie algebra.

Proposition 5.2. *The endomorphism $\mathcal{A}$ is preserved by the action of the adjoint representation:*

$$£_{X_v} \mathcal{A} = 0 \quad \forall v \in L. \tag{5.1}$$

Proof. Use Leibniz rule to show that $(£_{X_v} A)(w) = 0$ for every w: $(£_{X_v} A)(w) = £_{X_v}(A(w)) - A £_{X_v} w = £_{X_v} X_w - A[\![v,w]\!] = X_{[\![v,w]\!]} - X_{[\![v,w]\!]} = 0$. □

Proposition 5.3. *The endomorphism field on a Lie algebra satisfies*

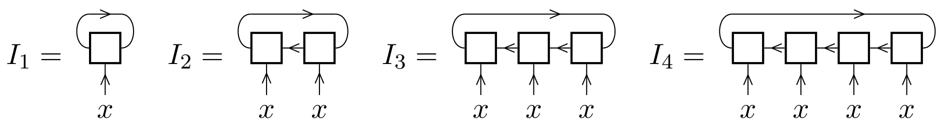
(i) $\operatorname{Tr}(\mathcal{A} \circ \mathcal{A}) = K(J,J)$,
(ii) $\operatorname{Tr}(\mathcal{A}) = \chi(J)$,
(iii) $K(\mathcal{A}v, w) = -K(v, \mathcal{A}w)$.

where the objects are as follows: K is the Killing form defined for two vectors as $K(v,w) = \operatorname{Tr} \mathrm{ad}_v \circ \mathrm{ad}_w$. When evaluated for (J,J), it becomes a quadratic scalar function $K(J,J) = x^a x^b c_{ai}^k c_{bk}^i$. Similarly, $\chi \in L^*$ is a characteristic form on L defined $\chi(v) = \operatorname{Tr} \mathrm{ad}_v$. Property (iii) states that the endomorphism $\mathcal{A}$ is skew-symmetric with respect to the Killing (possibly degenerated) scalar product.

The endomorphism defines for every $k = 1, 2, \ldots$, a scalar function of the power trace:

$$I_k = \operatorname{Tr} \mathcal{A}^k = \operatorname{Tr} \left(\mathrm{ad}_x \circ \cdots \circ \mathrm{ad}_x \right), \tag{5.2}$$

that will be called *Casimir polynomials* on L. In the diagrammatical language, they are



etc. Clearly, the second invariant is a quadratic function related to Killing form and will be denoted $\kappa = I_2 = K(J, J) = \kappa$, but the third is obviously not related to the Lie 3-form.

Corollary 5.4. *Differentials of the trace functions are among the annihilators of $\mathcal{A}$:*

$$\mathcal{A} \lrcorner dI_k = 0. \tag{5.3}$$

6 The endomorphism field and dynamical systems

Since the dual Lie algebra L^* with its Poisson structure has deep connections with classical mechanics, namely, with Hamiltonian formalism, one may expect that so does a Lie algebra with its endomorphism field $\mathcal{A}$. The candidate coming to mind first is Lagrangian mechanics, as suggested by this chain of correspondences:

KKS theorem (Lie coalgebras)	$\longrightarrow$	symplectic geometry	$\longrightarrow$	Hamilton equations
Theorem 2.1 (Lie algebras)	$\longrightarrow$	endomorphic geometry	$\longrightarrow$	?

Duality between tangent bundle TQ over a manifold M, which possesses enough structure so that any ("regular") function $\mathcal{L}$ on TQ defines a dynamical system via Lagrange equations, and the cotangent bundle T^*M, with its own symplectic structure ω granting a Hamiltonian formalism induced by the Hamiltonian $\mathcal{H}$, suggests that the question mark in the above diagram of analogies should be replaced by some sort of Lagrange formalism. This guess may be supported by the fact that the Lagrange formalism is actually based on the natural endomorphism field on the tangent fiber bundle (see Appendix B).

Yet, it seems that the most direct formalism at the question mark seems—much generalized—Lax equations of motion.

Although Lax equations are typically defined as matrix equations, the endomorphism $\mathcal{A}$ allows one to geometrize it in a new way. In the next sections, we will discuss "Lax vector fields" on a Lie algebra and will push the analogy with symplectic geometry to see how far it goes.

We show that, quite pleasantly, "Lax vector fields" form a closed subalgebra under vector field commutator. We will also define a new "Poisson bracket" in the space of vector fields on Lie algebra, and prove a homomorphism between Lie algebra of vector fields with this bracket with the standard Lie algebra of vector field.

7 The algebra of Lax vector fields

Let us start with a general construction. By analogy to symplectic geometry dealing with manifolds equipped with symplectic structure, $\{M, \omega\}$, we may consider a pair $\{M, \mathcal{A}\}$, where manifold M is equipped with a structure defined by a field of endomorphisms—$(1,1)$-variant tensor field on M. Exploring further the analogy, we may study dynamical systems described by vector fields that are defined by their "potentials"—other vector fields. Thus, instead of Hamilton equations, we have a map:

$$\mathcal{X}M \longrightarrow \mathcal{X}M : \quad B \longrightarrow X_B = B \lrcorner \mathcal{A} \equiv \mathcal{A}B. \tag{7.1}$$

This contrasts with symplectic geometry, where the potentials of dynamical systems are differential forms, namely, differentials of Hamiltonians. It would be natural to require that the set of all such dynamical systems, $\mathcal{X}_{\mathcal{A}}M = \{\mathcal{A}Y \mid Y \in \mathcal{X}M\}$, be closed under the Lie bracket of vector fields. This way it would form a subalgebra of $\{\mathcal{X}M, [\cdot,\cdot]\}$. The final demand would be to have a well-defined product of vector fields (potentials), such that the map (7.1) is a homomorphism of the corresponding algebras.

One may ask why one would want to replace one vector field by another: one gain may be that in the new form some integrals of motion may be found more easily.

In this section, we show that a Lie algebra with the endomorphism field defined in the previous sections forms such a system. In particular, it is equipped with a bracket for potentials that we define below.

Consider the underlying linear space L of a Lie algebra $\{L, [\![,]\!]\}$ as a manifold. Any smooth vector field B can be viewed as a generator (or "*potential*") of a dynamical system defined by vector field X_B defined

$$X_B = \mathcal{A}B. \tag{7.2}$$

The integral curves of X_B satisfy the Lax equations, which in a somewhat imprecise way are expressed as follows:

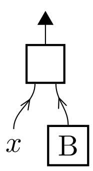
$$\dot{x}(t) = \big[x, B_x\big],$$

where the x on the left side is understood as a point in L, while the x inside the bracket on the right side is understood as a vector in L. More accurately,

$$\dot{c}(t) = \big[J_{c(t)}, B_{c(t)}\big] = \mathcal{A} \circ B \circ c(t).$$

Definition 7.1. Vector fields on a Lie algebra L of form (7.2) will be called *Lax vector fields* generated by B or Lax dynamical systems. In the diagrammatic representation, the Lax vector field is

$$X_B = B \lrcorner \mathcal{A} = \tag{7.3}$$



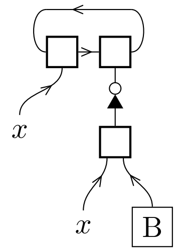
The space of Lax vector fields will be denoted by $\mathcal{X}_{\mathcal{A}}L = \mathcal{A}(\mathcal{X}L) \subset \mathcal{X}L$.

A simple and a well-known fact is the existence of Casimir invariants.

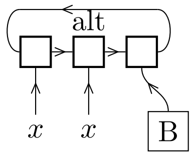
Corollary 7.2. *The dynamical system defined by a Lax vector field (7.2) leaves Casimir polynomials I_k invariant, $X_B I_k = 0$, for any $B \in \mathcal{X}L$.*

Proof (graphical). We show the reasoning for $I_2 = K(J, J)$ (quadratic polynomials defined by Killing form):

$$X_B I_2 = (X_B \otimes J) \lrcorner I_2 = 2\times$$



$$= \frac{2}{3}$$



$$= 0,$$

where first we used Jacobi identity (3.1) and then skewsymmetry of the resulting ω. The right side vanishes as ω has two identical entries, x. The argument for the other Casimir invariants is similar. □

The geometric meaning of the fundamental Nijenhuis property of the endomorphism field becomes clear in the current context. Namely, it implies that the space of Lax vector fields $\mathcal{X}_\mathcal{A} L$ is closed under the commutator of vector fields $[\mathcal{X}_\mathcal{A} L, \mathcal{X}_\mathcal{A} L] \subset \mathcal{X}_\mathcal{A} L$. A new bracket of vector fields is implied.

Theorem 7.3. *The space of Lax vector field forms a subalgebra of the algebra of smooth vector fields, $\mathcal{X}_\mathcal{A} L < \mathcal{X}L$. In particular, if X_B and X_C are two (global) Lax vector fields, then their commutator is a Lax vector field with potential:*

$$\{B, C\} =: -[\![B, C]\!] + \big[X_B, C\big] + \big[B, X_C\big] - X_{[B,C]}, \tag{7.4}$$

so that there is an homomorphism between the Lax vector fields with the regular vector field commutator and all vector fields with $\{\ ,\ \}$ product:

$$\big[X_B, X_C\big] = X_{\{B,C\}}. \tag{7.5}$$

Proof. This follows from the fact that $[\mathcal{A}, \mathcal{A}]$ is proportional to $\mathcal{A}$. Rewrite the definition of the Nijenhuis bracket (4.4) for $\mathcal{A}$ and use Theorem 2.1:

$$\begin{aligned}
\big[X_B, X_C\big] &= [\mathcal{A}B, \mathcal{A}C] = \frac{1}{2}[\mathcal{A}, \mathcal{A}](B, C) + \mathcal{A}[\mathcal{A}B, C] + \mathcal{A}[B, \mathcal{A}C] - \mathcal{A}^2[B, C] \\
&= -\mathcal{A}[\![B, C]\!] + \mathcal{A}\big[X_B, C\big] + \mathcal{A}\big[B, X_C\big] - \mathcal{A}X_{[B,C]} \\
&= \mathcal{A}\big(- [\![B, C]\!] + \big[X_B, C\big] + \big[B, X_C\big] - X_{[B,C]}\big),
\end{aligned} \tag{7.6}$$

where in the last part, we see that the endomorphism field $\mathcal{A}$ may be "factored out" thanks to Theorem 2.1. Thus the commutator is of the form (7.2), the formulas in the theorem follow. □

Proposition 7.4. *The bracket (7.4) can be calculated by the following formula:*

$$\{A, B\} = [\![A, B]\!] + \underbrace{X_A B - X_B A}_{(A,B)}, \tag{7.7}$$

where $X_A B = x^i A^j c^k_{ij} \partial_k B^p \partial_p$.

Notice that although the two right-most terms are defined in coordinates, their difference has a coordinate-free meaning, as it can be defined by $X_A B - X_B A = \{A, B\} - [\![A, B]\!]$.

The bracket $\{\ ,\ \}$ turns the space of vector fields on L into a Lie algebra and can be viewed as a "differential deformation" of the Lie algebra bracket $[\![\ ,\ ]\!]$. Due to its involved nature, it may be rather surprising that it defines a Lie algebra. The Jacobi identity is not a direct consequence and results by intertwined interaction of the Jacobi identities of the Lie algebra L and of the Lie algebra of vector fields.

Remark 7.5. For two constant vector fields $\widetilde{v}$ and $\widetilde{w}$ that extend vectors $v, w \in L$, it is $\{\widetilde{v}, \widetilde{w}\} = \widetilde{[\![v, w]\!]}$. Thus the bracket formula (7.5) reduces in this case to the infinitesimal representation $[X_v, X_w] = X_{[\![v,w]\!]}$.

Theorem 7.6. *The pair $\{\mathcal{X}L, \{\cdot, \cdot\}\}$ forms a Lie algebra, that is, the bracket (7.4, 7.8) of vector fields satisfies the following properties:*

$$\begin{aligned} &\textit{(i)} \quad (\textit{linearity}),\\ &\textit{(ii)} \quad \{A, B\} = -\{B, A\} \quad (\textit{skewsymmetry}),\\ &\textit{(iii)} \quad \big\{A, \{B, C\}\big\} + \big\{B, \{C, A\}\big\} + \{C, \{A, B\}\} = 0 \quad (\textit{Jacobi identity}). \end{aligned} \tag{7.8}$$

Proof. If $X, Y \in \mathcal{X}L$ are two vector fields, then we denote $X \triangleright Y = X^i(\partial_i Y^j)\partial_j$ a vector field calculated in linear coordinate system. Thus, formula (7.7) can be written as follows:

$$\{A, B\} = [\![A, B]\!] + (A, B),$$

where

$$(A, B) = X_A \triangleright B - X_B \triangleright A.$$

Now, using the formula $X_A = [\![x, A]\!]$, we get

$$\begin{aligned} \big\{\{A, B\}, C\big\} &= \big\{[\![A, B]\!] + (A, B), C\big\} \\ &= \underbrace{[\![[\![A, B]\!], C]\!]}_{(a)} + \underbrace{[\![(A, B), C]\!]}_{(b)} + \underbrace{([\![A, B]\!], C)}_{(c)} + \underbrace{((A, B), C)}_{(d)} \\ &= \underbrace{[\![[\![A, B]\!], C]\!]}_{(0)} \quad (a) \\ &\quad + \underbrace{[\![[\![x, A]\!] \triangleright B, C]\!]}_{(1)} - \underbrace{[\![[\![x, B]\!] \triangleright A, C]\!]}_{(2)} \quad (b) \\ &\quad + \underbrace{[\![x, [\![A, B]\!]]\!] \triangleright C}_{(5)} - \underbrace{[\![[\![x, C]\!] \triangleright A, B]\!]}_{(1)} - \underbrace{[\![A, [\![x, C]\!] \triangleright B]\!]}_{(2)} \quad (c) \\ &\quad + \underbrace{[\![x, [\![x, A]\!] \triangleright B]\!] \triangleright C}_{(3)} - \underbrace{[\![[\![x, C]\!], A]\!] \triangleright B}_{(5)} - \underbrace{[\![x, [\![x, C]\!] \triangleright A]\!] \triangleright B}_{(3)} \quad (d) \\ &\quad - \underbrace{[\![x, [\![x, B]\!] \triangleright A]\!] \triangleright C}_{(4)} + \underbrace{[\![[\![x, C]\!], B]\!] \triangleright A}_{(5)} + \underbrace{[\![x, [\![x, C]\!] \triangleright B]\!] \triangleright A}_{(4)}, \quad (d) \end{aligned} \tag{$*$}$$

where the letters (a), (b), (c), and (d) are used to indicate the origin of terms in the second part of the equation. The sum

$$\big\{\{A, B\}, C\big\} + \big\{\{B, C\}, A\big\} + \big\{\{C, A\}, B\big\}$$

contains every term of equation $(*)$ in each of the three cyclic permutations of A, B, and C. The sum of such terms marked by any of the numbers (1) to (4) vanishes due to opposite signs. The group of terms marked by (0) and terms marked by (5) both vanish, each due to the Jacobi identity of the Lie algebra product. □

Corollary 7.7. *There is a Lie algebra homomorphism* $\{\mathcal{X}L, \{\cdot,\cdot\}\} \to \{\mathcal{X}_{\mathcal{A}}, [\cdot,\cdot]\}$ *between Lie algebra of vector fields on* L *with bracket defined by (7.4) and the Lie algebra vector fields on* L *restricted to Lax vector fields.*

By analogy to Hamiltonian formalism of classical mechanics, we have a property that may be viewed as a counterpart of Poisson theorem.

Corollary 7.8 ("à la Poisson")**.** *If vector fields* B *and* C *are Lax potentials of symmetries of a dynamical system, then* $\{B, C\}$ *is a Lax potential of a symmetry as well.*

Proof. Use the Jacobi identity for vector fields:

$$[X_B, \underbrace{[X_C, X]}_{0}] + [X, \underbrace{[X_B, X_C]}_{X_{\{B,C\}}}] + [X_C, \underbrace{[X, X_B]}_{0}] = 0,$$

hence the claim: $\pounds_{X_{\{B,C\}}} X = 0$. □

Basic examples. What can be used as a Lax potential? The simplest are constant vector fields, in which case the homomorphism reduces to Proposition 4.1 (see Remark 7.5). Also, a Lax vector field may be "reused" as a potential for a new Lax vector field. The following formulas for bracket $\{\cdot,\cdot\}$ may be useful for such dynamical systems:

$$\{v, w\} = [\![v, w]\!] \quad \big\{X_v, w\big\} = X_{[\![v,w]\!]} \quad \big\{X_v, X_w\big\} = X_{\mathcal{A}[\![v,w]\!]} - \big[\!\big[X_v, X_w\big]\!\big],$$

where v and w are understood as constant vector fields (the tilde is suppressed for simplicity).

Another class consists of Lax vector fields generated from linear vector fields on L. Euler's equations of the motion a rigid body belong to this category. Here is their—somewhat naïve—generalization to arbitrary Lie algebra: let $R \in \operatorname{End} L$ be a matrix describing the tensor of inertia. If vector field $J \lrcorner \widetilde{R}$ is used as a "potential", the resulting Lax vector field $X = \mathcal{A}(J \lrcorner \widetilde{R})$ describes the dynamical system of "rotating body". In the case of the Lie algebra of 3-dimensional orthogonal group $L = so(3, \mathbb{R})$ with the standard coordinates (x, y, z) and for a diagonal matrix $R = \operatorname{diag}(a, b, c)$, we get the standard Euler's equations:

$$X = (b - a)xy\partial_z + (c - b)yz\partial_x + (a - c)zx\partial_y,$$

(or $\dot{x} = (c - b)zy$, etc.). A more accurate description will be given in a subsequent paper.

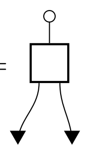
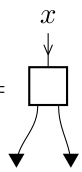
8 Analogies and dualities

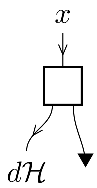
The analogies between differential geometry (calculus) on a Lie algebra and on a Lie coalgebra are shown in the following table. Note that the Lie algebra structure c is a (1,2)-variant tensor on the Lie algebra L, but it is a (2,1)-variant tensor on the dual space L^*. This results in quite different calculus on both spaces treated as manifolds.

	Lie algebra as manifold L	Lie coalgebra L^* as a manifold
Coordinates	$\{x_i\}$	$\{x^i\}$
Constant structure tensor ...in coordinates	$\lambda = \widetilde{c} = \frac{1}{2} c^k_{ij} \partial_k \otimes dx^i \wedge dx^j$	$\lambda' = \widetilde{c} = \frac{1}{2} c^k_{ij} dx_k \otimes \partial^i \wedge \partial^j$
Jacobi vector field	$J = x_i \partial^i$	$J' = x^i \partial_i$
Primary differential object ...in coordinates	$\mathcal{A} = J \lrcorner \lambda = \frac{1}{2} c^k_{ij} x^i \partial_k \otimes dx^j$ (endomorphism field)	$\Omega = J' \lrcorner \lambda' = \frac{1}{2} c^k_{ij} x_k \partial^i \wedge \partial^j$ (Poisson structure)
Basic rule (consequence of Jacobi identity)	$[\mathcal{A}, \mathcal{A}] = -2\mathcal{A} \lrcorner \lambda$	$[\Omega, \Omega] = 0$
"A potential"	$B \in \mathcal{X}L$ (vector field)	$H \in \mathcal{F}L$ (function)
...generates dynamical system	$X_B = B \lrcorner \mathcal{A} \equiv \mathcal{A}B$ (Lax equations)	$X_H = dH \lrcorner \Omega$ (Hamilton equations)

Table 1. $\partial_i \equiv \partial/\partial x_i$, $\partial^i \equiv \partial/\partial x^i$. Tilde $\sim$ denotes extension of tensors to tensor fields on L and on L^*, defined by the affine structure on linear spaces.

On L^* as a manifold, λ is $(2, 1)$ variant. In pictures, the canonical Poisson structure on L^* and Hamiltonian mechanics may be illustrated as follows:

$\lambda' =$  $\Omega = J' \lrcorner \lambda' =$ 

$X_{\mathcal{H}} = d\mathcal{H} \lrcorner \Omega =$ 

9 Remark on Lagrange equations

While the cotangent bundle T^*Q over a manifold Q possesses a canonical differential biform $\omega \in \Lambda^2 Q$ defining symplectic structure, the tangent bundle TQ possesses a canonical (1,1)-variant tensor field $S \in \mathcal{T}^{(1,1)}TQ$ defining an endomorphism field (endomorphisms of $T(TQ)$ and $T^*(TQ)$). In the natural coordinates (x^i, v^i) on TQ, this tensor can be expressed as $S = \frac{\partial}{\partial v^i} \otimes dx^i$ (sum over i). Its basic property is $\operatorname{Ker} S = \operatorname{Im} S$ (implying nilpotence $S \circ S = 0$). If $\mathcal{L}$ is a function on TQ (a Lagrangian), then one defines a biform $\omega = d \circ S \circ d\mathcal{L}$, which for a "regular" Lagrangian is nondegenerate and therefore forms a symplectic structure. It is easy to see that Lagrange equations may be written as follows:

$$\pounds_X (S \circ d\mathcal{L}) = d\mathcal{L}.$$

The existence of S and its role in Lagrangian mechanics were noticed rather late [11]; they replace a rather awkward notion of "vertical derivative" that were employed previously to geometrize Euler-Lagrange equations [1].

In a series of papers [4, 5], a notion of *almost tangent structure* on a differential manifold M has been introduced, as a tensor $S \in \mathcal{T}^1_1 M$ that satisfies

$$\begin{aligned} &\text{(i)} \quad \operatorname{Ker} S = \operatorname{Im} S \quad (\Longrightarrow S \circ S = 0), \\ &\text{(ii)} \quad [S, S] = 0, \end{aligned} \tag{9.1}$$

where the second condition (ii) is a generalization of the Schouten-Nijenhuis bracket to "vector-valued differential forms" (see, e.g, [17, 18]), which assures (local) integrability of the distribution $\operatorname{Ker} S$. As a result, one obtains all of the structures of the tangent bundle ($\operatorname{Ker} S$ gives the fibering) except distinguishing the zero-section.

A Lie algebra may provide an example of a generalized version of such an Euler-Lagrange structure, in which the above conditions (9.1) are relaxed.

Hamilton equations	$\longrightarrow$	cotangent bundle T^*Q	$\longrightarrow$	symplectic geometry	$\longrightarrow$	KKS theorem (Lie coalgebras)
Lagrange equations	$\longrightarrow$	tangent bundle TQ	$\longrightarrow$	endomorphic geometry	$\longrightarrow$	Theorem 2.1 (Lie algebras)

Whether such potential relationship between Lie algebras and generalized Lagrangian formalism would be fruitful is an interesting question in the context of geometric quantization and representation theory known for coadjoint orbits in the Lie coalgebras.

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