

Poisson structures on $\mathbb{C}[X_1, \dots, X_n]$ associated with rigid Lie algebras

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Abstract

We present the classical Poisson-Lichnerowicz cohomology for the Poisson algebra of polynomials $\mathbb{C}[X_1, \dots, X_n]$ using exterior calculus. After presenting some non-homogeneous Poisson brackets on this algebra, we compute Poisson cohomological spaces when the Poisson structure corresponds to a bracket of a rigid Lie algebra.

2000 MSC: 17B-XX

1 Introduction

The first Poisson structures appeared in classical mechanics. In 1809, D. Poisson introduced a bracket of functions, which permits to write Hamilton's equations as differential equations. This led to define a Poisson manifold, that is, a manifold M whose algebra of smooth functions $F(M)$ is equipped with a skew-symmetric bilinear map:

$$\{ , \} : F(M)F(M) \longrightarrow F(M),$$

satisfying the Leibniz rule:

$$\{FG, H\} = F\{G, H\} + \{F, H\}G,$$

and the Jacobi identity. In [7], A. Lichnerowicz has also introduced a cohomology, associated to a Poisson structure, called Poisson cohomology.

In this paper, we study in terms of exterior calculus the Poisson structures on the associative algebra of complex polynomials in n variables. We apply this approach to the determination of non-homogeneous quadratic Poisson brackets and to the computation of the Poisson cohomology. The linear Poisson structures are naturally related to the n -dimensional Lie algebras. Recall that a complex Lie algebra \mathfrak{g} is rigid when its orbit in the algebraic variety of n -dimensional complex Lie algebra defined by the Jacobi relations is Zariski open. Such an algebra admits a nontrivial Malcev torus and it is graded by the roots of the torus. We study the Poisson structure on $\mathbb{C}[X_1, \dots, X_n]$ whose Poisson brackets correspond to a solvable rigid Lie bracket with non-zero roots. In a generic example, we compute the corresponding Poisson cohomology.

2 Poisson structures on $\mathbb{C}[X_1, \dots, X_n]$ and exterior calculus

2.1 Poisson brackets and differential forms

Let \mathcal{A}^n be the commutative associative algebra $\mathbb{C}[X_1, \dots, X_n]$ of complex polynomials in X_1, \dots, X_n . We define a Poisson structure on \mathcal{A}^n as a bivector:

$$\mathcal{P} = \sum_{1 \leq i < j \leq n} P_{ij} \partial_i \wedge \partial_j,$$

where $\partial_i = \frac{\partial}{\partial X_i}$ and $P_{ij} \in \mathcal{A}^n$, satisfying the axiom:

$$[\mathcal{P}, \mathcal{P}]_S = 0,$$

where $[\cdot, \cdot]_S$ denotes Schouten's bracket. If \mathcal{P} is a Poisson structure on \mathcal{A}^n , then

$$\{P, Q\} = \mathcal{P}(P, Q)$$

defines a Lie bracket on \mathcal{A}^n which satisfies the Leibniz identity:

$$\{PQ, R\} = P\{Q, R\} + Q\{P, R\}$$

for any $P, Q, R \in \mathcal{A}^n$.

We denote by $\text{Sh}_{p,q}$ the set of unshuffles, where a (p, q) -shuffle is a permutation σ of the symmetric group Σ_{p+q} of degree $p + q$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$. For any bivector \mathcal{P} we consider the $(n-2)$ -exterior form:

$$\Omega = \sum_{\sigma \in S_{2, n-2}} (-1)^{\varepsilon(\sigma)} P_{\sigma(1)\sigma(2)} dX_{\sigma(3)} \wedge \dots \wedge dX_{\sigma(n)},$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ . If $n > 3$, we consider the Pfaffian form $\alpha_{i_1, \dots, i_{n-3}}$ given by

$$\alpha_{i_1, \dots, i_{n-3}}(Y) = \Omega(\partial_{i_1}, \partial_{i_2}, \dots, \partial_{i_{n-3}}, Y)$$

with $Y = \sum_{i=1}^n Y_i \partial_i$, $Y_i \in \mathcal{A}^n$.

Theorem 2.1. *A bivector \mathcal{P} on \mathcal{A}^n satisfies $[\mathcal{P}, \mathcal{P}]_S = 0$ if and only if:*

- for $n > 3$,

$$d\alpha_{i_1, \dots, i_{n-3}} \wedge \Omega = 0$$

for every i_1, \dots, i_{n-3} such that $1 \leq i_1 < \dots < i_{n-3} \leq n$.

- for $n = 3$,

$$d\Omega \wedge \Omega = 0.$$

Proof. The integrability condition $[\mathcal{P}, \mathcal{P}]_S = 0$ writes

$$\sum_{r=1}^n P_{ri} \partial_r P_{jk} + P_{rj} \partial_r P_{ki} + P_{rk} \partial_r P_{ij} = 0$$

for any $1 \leq i, j, k \leq n$. But

$$\alpha_{i_1, \dots, i_{n-3}} = \sum (-1)^N P_{jk} dX_l$$

summing over all triples (j, k, l) , such that $(j, k, i_1, \dots, l, \dots, i_{n-3})$ is a permutation of $S_{2, n-2}$ and $N = \varepsilon(\sigma) + p - 3$, where $(-1)^\varepsilon(\sigma)$ is the signum of σ . Then

$$d\alpha_{i_1, \dots, i_{n-3}} = \sum (-1)^N dP_{jk} \wedge dX_l,$$

and $d\alpha_{i_1, \dots, i_{n-3}} \wedge \Omega = 0$ corresponds to $[\mathcal{P}, \mathcal{P}]_S = 0$. The proof is similar if $n = 3$. \square

2.2 Lichnerowicz-Poisson cohomology

We denote by $\mathcal{A}_{\mathcal{P}}^n$ the algebra $\mathcal{A}^n = \mathbb{C}[X_1, \dots, X_n]$ provided with the Poisson structure \mathcal{P} . For $k \geq 1$, let $\chi^k(\mathcal{A}_{\mathcal{P}}^n)$ be the vector space of k -derivations that is of k -skew linear maps on $\mathcal{A}_{\mathcal{P}}^n$ satisfying

$$\varphi(P_1 Q_1, P_2, \dots, P_k) = P_1 \varphi(Q_1, P_2, \dots, P_k) + Q_1 \varphi(P_1, P_2, \dots, P_k)$$

for all $Q_1, P_1, \dots, P_k \in \mathcal{A}_{\mathcal{P}}^n$. For $k = 0$, we put $\chi^0(\mathcal{A}_{\mathcal{P}}^n) = \mathcal{A}_{\mathcal{P}}^n$. Let δ^k be the linear map:

$$\delta^k : \chi^k(\mathcal{A}_{\mathcal{P}}^n) \longrightarrow \chi^{k+1}(\mathcal{A}_{\mathcal{P}}^n)$$

given by

$$\begin{aligned} \delta^k \varphi(P_1, P_2, \dots, P_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} \{P_i, \varphi(P_1, \dots, \widehat{P}_i, \dots, P_{k+1})\} \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \varphi(\{P_i, P_j\}, P_1, \dots, \widehat{P}_i, \dots, \widehat{P}_j, \dots, P_{k+1}), \end{aligned}$$

where \widehat{P}_i means that the term P_i does not appear. We have $\delta^{k+1} \circ \delta^k = 0$ and the Lichnerowicz-Poisson cohomology corresponds to the complex $(\chi^k(\mathcal{A}_{\mathcal{P}}), \delta^k)_k$. Let us note that $\chi^k(\mathcal{A}_{\mathcal{P}}^n)$ is trivial as soon as $k > n$. A description of the cocycle $\delta^k \varphi$ is presented in [11] for $n = 3$ using the vector calculus. We will describe these formulae using exterior calculus for $n > 3$. Let us begin with some notations.

- To any element $P \in \mathcal{A}_{\mathcal{P}}^n = \chi^0(\mathcal{A}_{\mathcal{P}}^n)$, we associate the n -exterior form:

$$\Phi_n(P) = P dX_1 \wedge \dots \wedge dX_n.$$

- To any $\varphi \in \chi^k(\mathcal{A}_{\mathcal{P}}^n)$ for $1 \leq k < n$, we associate the $(n - k)$ -exterior form:

$$\Phi_{n-k}(\varphi) = \sum_{\sigma \in S_{k, n-k}} (-1)^{\varepsilon(\sigma)} \varphi(X_{\sigma(1)}, \dots, X_{\sigma(k)}) dX_{\sigma(k+1)} \wedge \dots \wedge dX_{\sigma(n)}.$$

- To any $\varphi \in \chi^n(\mathcal{A}_{\mathcal{P}}^n)$, we associate the function $\Phi_0(\varphi) = \varphi$.

Finally, if θ is an k -exterior form and $Y = \sum_{i=1}^n Y_i \partial_i$ is a vector field with $Y_i \in \mathcal{A}_{\mathcal{P}}^n$, then the inner product $i(Y)\theta$ is the $(k - 1)$ -exterior form given by

$$i(Y)\theta(Z_1, \dots, Z_{k-1}) = \theta(Y, Z_1, \dots, Z_{k-1})$$

for every vector fields Z_1, \dots, Z_{k-1} .

Theorem 2.2. *Assume that $n = 3$. Then we have*

(1) for all $P \in \mathcal{A}_{\mathcal{P}}^3$,

$$\Phi_2(\delta^0 P) = -\Omega \wedge dP;$$

(2) for all $f \in \chi^1(\mathcal{A}_{\mathcal{P}}^3)$,

$$\begin{aligned} \Phi_1(\delta^1 f) = & -i(\partial_1, \partial_2) [\Omega \wedge d(i(\partial_3)\Phi_2(f)) + d(i(\partial_3)\Omega) \wedge \Phi_2(f)] \\ & + i(\partial_1, \partial_3) [\Omega \wedge d(i(\partial_2)\Phi_2(f)) + d(i(\partial_2)\Omega) \wedge \Phi_2(f)] \\ & - i(\partial_2, \partial_3) [\Omega \wedge d(i(\partial_1)\Phi_2(f)) + d(i(\partial_1)\Omega) \wedge \Phi_2(f)], \end{aligned}$$

where $i(X, Y)$ denotes the composition $i(X) \circ i(Y)$;

(3) for all $\varphi \in \chi^2(\mathcal{A}_{\mathcal{P}}^3)$,

$$\Phi_0(\delta^2 \varphi) = i(\partial_1, \partial_2, \partial_3)(d\Omega \wedge \Phi_1(\varphi) + \Omega \wedge d\Phi_1(\varphi)).$$

Proof. If $n = 3$, we have

$$\Omega = P_{12}dX_3 - P_{13}dX_2 + P_{23}dX_1.$$

Then the integrability of \mathcal{P} is equivalent to $\Omega \wedge d\Omega = 0$. The theorem results of a direct computation and of the following general formula:

$$\forall \varphi \in \chi^k(\mathcal{A}_{\mathcal{P}}^n), \quad \varphi(P_1, \dots, P_k) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \varphi(X_{i_1}, \dots, X_{i_k}) \partial_{i_1} P_1 \cdots \partial_{i_k} P_k. \quad \square$$

Example 2.3. We consider the Poisson algebra $\mathcal{A}_{\mathcal{P}_1}^n = (\mathbb{C}[X_1, X_2, X_3], \mathcal{P}_1)$, where \mathcal{P}_1 is given by

$$\begin{cases} \mathcal{P}_1(X_1, X_2) = X_2, \\ \mathcal{P}_1(X_1, X_3) = 2X_3, \\ \mathcal{P}_1(X_2, X_3) = 0. \end{cases}$$

Then

$$\dim H^0(\mathcal{A}_{\mathcal{P}_1}^n) = 1, \quad \dim H^1(\mathcal{A}_{\mathcal{P}_1}^n) = 3, \quad \dim H^2(\mathcal{A}_{\mathcal{P}_1}^n) = 2, \quad H^3(\mathcal{A}_{\mathcal{P}_1}^n) = \{0\}.$$

In this case, $\Omega = X_2dX_3 - 2X_3dX_2$ and $d\Omega = 3dX_2 \wedge dX_3$. Let us compute $\dim H^2(\mathcal{A}_{\mathcal{P}_1}^n)$. Let $\varphi \in \chi^2(\mathcal{A}_{\mathcal{P}_1}^n)$. Then $\Phi_0(\delta^2 \varphi) = 0$ implies

$$d\Omega \wedge \Phi_1(\varphi) + \Omega \wedge d\Phi_1(\varphi) = 0,$$

that is

$$\begin{aligned} & X_2(\partial_1 \varphi(X_1, X_3) + \partial_2 \varphi(X_2, X_3)) \\ & + 2X_3(-\partial_1 \varphi(X_1, X_2) + \partial_3 \varphi(X_2, X_3)) + 3\varphi(X_2, X_3) = 0. \end{aligned}$$

Now, if $f \in \chi^1(\mathcal{A}_{\mathcal{P}_1}^n)$, then

$$\Phi_1(\delta f) = [X_2(-\partial_2 f(X_2) - \partial_1 f(X_1)) - 2X_3(\partial_3 f(X_2)) + f(X_2)]dX_3$$

$$\begin{aligned} & - [2X_3(\partial_1 f(X_1) + \partial_3 f(X_3)) + X_2(\partial_2 f(X_3)) - 2f(X_3)] dX_2 \\ & - [X_2(-\partial_1 f(X_3)) - 2X_3(\partial_1 f(X_2))] dX_1. \end{aligned}$$

Comparing these two relations, we obtain that $H^2(\mathcal{A}_{\mathcal{P}_1}^n)$ is generated by the two cocycles:

$$\begin{cases} \Phi_1(\varphi_1) = X_3 dX_2, \\ \Phi_1(\varphi_2) = X_2^2 dX_2. \end{cases}$$

Now consider the general case. Let $\mathcal{A} = \mathbb{C}[X_1, \dots, X_n]$ be provided with the Poisson structure \mathcal{P} .

Theorem 2.4. *Let $\varphi \in \chi^k(\mathcal{A}_{\mathcal{P}})$. Then, we have*

$$\begin{aligned} \Phi_{n-k-1}(\delta^k \varphi) &= \varepsilon \sum i(\partial_{\sigma(1)}, \dots, \partial_{\sigma(k+1)}) [d(i(\partial_{\sigma(k+2)}, \dots, \partial_{\sigma(n)}) \Omega) \wedge \Phi_{n-k}(\varphi) \\ &\quad + \Omega \wedge d(i(\partial_{\sigma(k+2)}, \dots, \partial_{\sigma(n)}) \Phi_{n-k}(\varphi))] \end{aligned}$$

for all $\sigma \in S_{k+1, n-k-1}$, where $\varepsilon = \varepsilon(n, k) = (-1)^{\frac{(n-k)(n-k+1)}{2}}$.

Proof. To simplify, we write d_i in place of dX_i . We have seen that for every $P \in \mathcal{A}_{\mathcal{P}}$, we have $\delta^0 P = -\Omega \wedge dP$. But

$$\Phi_{n-1}(\delta P) = \sum_{k=1}^n (-1)^{k-1} \{X_k, P\} d_1 \wedge \dots \wedge \hat{d}_k \wedge \dots \wedge d_n,$$

where \hat{d}_i means that this factor does not appear with $\{P, X_i\} = \sum_{j=1}^n P_{ji} \partial_j P$ with $P_{ji} = -P_{ij}$ when $j > i$. But

$$\begin{aligned} & i(\partial_1) [\Omega \wedge d(i(\partial_2, \dots, \partial_n) \Phi_n(P)) + d(i(\partial_2, \dots, \partial_n) \Omega) \wedge \Phi_n(P)] \\ &= i(\partial_1) [\Omega \wedge d(i(\partial_2, \dots, \partial_n) \Phi_n(P))] = (-1)^{\frac{n(n-1)}{2}} i(\partial_1) [\Omega \wedge dP \wedge d_1] \\ &= -(-1)^{\frac{n(n-1)}{2}} \sum_{i=2}^n P_{1i} \partial_i P d_2 \wedge \dots \wedge d_n = (-1)^{\frac{n(n-1)}{2}} \Phi_{n-1}(P) (\partial_2, \dots, \partial_n). \end{aligned}$$

Similarly,

$$\begin{aligned} & i(\partial_j) [\Omega \wedge d(i(\partial_1, \dots, \hat{\partial}_j, \dots, \partial_n) \Phi_n(P)) + d(i(\partial_1, \dots, \hat{\partial}_j, \dots, \partial_n) \Omega) \wedge \Phi_n(P)] \\ &= i(\partial_j) [\Omega \wedge d(i(\partial_1, \dots, \hat{\partial}_j, \dots, \partial_n) \Phi_n(P))] = (-1)^{j-1 + \frac{n(n-1)}{2}} i(\partial_j) [\Omega \wedge dP \wedge dX_j] \\ &= (-1)^{j-1 + \frac{n(n-1)}{2}} i(\partial_j) \left(\sum_{l=1}^{l=j-1} P_{1j} \partial_l P - \sum_{l=j+1}^{l=n} P_{jl} \partial_l P \right) d_1 \wedge \dots \wedge d_n \\ &= (-1)^{\frac{n(n-1)}{2}} \left(\sum_{l=1}^{l=j-1} P_{1j} \partial_l P - \sum_{l=j+1}^{l=n} P_{jl} \partial_l P \right) d_1 \wedge \dots \wedge \hat{d}_j \wedge \dots \wedge d_n \\ &= (-1)^{\frac{n(n-1)}{2}} \{P, X_i\} d_1 \wedge \dots \wedge \hat{d}_i \wedge \dots \wedge d_n. \end{aligned}$$

We deduce

$$\Phi_{n-1}(\delta^0 P) = (-1)^{\frac{n(n-1)}{2}} \sum_{j=1}^n (-1)^{j-1} i(\partial_j) [\Omega \wedge d(i(\partial_1, \dots, \hat{\partial}_j, \dots, \partial_n) \Phi_n(P))],$$

which proves the theorem for $k = 0$. The proof is similar for any k . \square

Application 2.5. We consider the n -dimensional complex Lie algebra defined by the brackets:

$$[X_1, X_i] = (i-1)X_i$$

for $i = 2, \dots, n$. Let \mathcal{P}_2 be the corresponding Poisson bracket on $\mathbb{C}[X_1, \dots, X_n]$. Let $\chi_2^k(\mathcal{A}_{\mathcal{P}_2})$ be the subspace of $\chi^k(\mathcal{A}_{\mathcal{P}_2})$ whose elements are homogenous of degree 2. We denote by $H_2^2(\mathcal{A}_{\mathcal{P}_2}) = Z_2^2/B_2^2$ the corresponding subspace of $H^2(\mathcal{A}_{\mathcal{P}_2})$. Define $N := \frac{n(n-1)}{2}$.

- If n is even, then

$$\dim B_2^2 = N + (N-1) + \dots + N - n/2 + 1 = \frac{n(2n^2 - 3n + 2)}{8}.$$

- If n is odd,

$$\dim B_2^2 = N + (N-1) + \dots + (N - (n-1)/2) = \frac{(n^2 - 1)(2n - 1)}{8}.$$

In fact, if $f \in \chi_2^1(\mathcal{A}_{\mathcal{P}})$, then $f(X_i) = P_i = \sum a_{i_1, \dots, i_n}^i X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$ is homogenous of degree 2, then:

- (1) in $\delta f(X_1, X_{2l})$, we find $N - l$ independent coefficients of P_{2l} . The coefficients which do not appear are

$$a_{1,0,0,\dots,0,1,0,\dots,0}^{2l}, a_{0,1,0,\dots,0,1,0,\dots,0}^{2l}, \dots, a_{0,0,\dots,1,1,0,\dots,0}^{2l},$$

where the second 1 in the sequences of indices is, respectively, in the place $2l, 2l-1, \dots, l+1$;

- (2) in $\delta f(X_1, X_{2l+1})$, we find $N - l - 1$ independent coefficients of P_{2l+1} . The coefficients which do not appear are

$$a_{1,0,0,\dots,0,1,0,\dots,0}^{2l+1}, a_{0,1,0,\dots,0,1,0,\dots,0}^{2l+1}, \dots, a_{0,0,\dots,0,2,0,\dots,0}^{2l+1},$$

where the second 1 in the sequences of indices is in place $2l+1, 2l, \dots, l+2$ and in the last case the 2 is in place $l+1$;

- (3) for $i \geq 2$ and $j > i$, $\delta f(X_i, X_j)$ is defined by the $(n-2)$ coefficients $a_{1,0,0,\dots,0,1,0,\dots,0}^i$.

Now we can find the generators of $H_2^2(\mathcal{A}_{\mathcal{P}})$. We can choose $\phi \in \chi_2^2$ such that

$$\left\{ \begin{array}{l} \phi(X_1, X_2) = 0, \\ \phi(X_1, X_3) = a_{1,3}^{1,3} X_1 X_3 + a_{1,3}^{2,2} X_2^2, \\ \dots \\ \phi(X_1, X_{2l}) = a_{1,2l}^{1,2l} X_1 X_{2l} + a_{1,2l}^{2,2l-1} X_2 X_{2l-1} + \dots + a_{1,2l}^{l,l+1} X_l X_{l+1}, \\ \phi(X_1, X_{2l+1}) = a_{1,2l+1}^{1,2l+1} X_1 X_{2l+1} + a_{1,2l+1}^{2,2l} X_2 X_{2l} + \dots + a_{1,2l+1}^{l,l} X_l^2, \\ \dots \\ \phi(X_1, X_n) = a_{1,n}^{1,n} X_1 X_n + a_{1,n}^{2,n-1} X_2 X_{n-1} + \dots, \\ \phi(X_i, X_j) = A_{i,j}, \end{array} \right.$$

where $A_{i,j}$ is a degree 2 homogenous polynomial without monomial of types X_1X_k and X_iX_j . By solving $\Phi_{n-2}(\delta\phi) = 0$, we obtain the generators of $H_2^2(\mathcal{A}_{\mathcal{P}})$. They are given by

$$\begin{cases} \phi(X_1, X_2) = 0, \\ \phi(X_1, X_3) = a_{1,3}^{2,2} X_2^2, \\ \dots \\ \phi(X_1, X_{2l}) = a_{1,2l}^{2,2l-1} X_2 X_{2l-1} + \dots + a_{1,2l}^{l,l+1} X_l X_{l+1}, \\ \phi(X_1, X_{2l+1}) = a_{1,2l+1}^{2,2l} X_2 X_{2l} + \dots + a_{1,2l+1}^{l+1,l+1} X_{l+1}^2, \\ \dots \\ \phi(X_1, X_n) = a_{1,n}^{2,n-1} X_2 X_{n-1} + \dots + a_{1,n}^{m,m+1} X_m X_{m+1}, \quad \text{if } n = 2m, \\ \phi(X_i, X_j) = A_{i,j}, \end{cases}$$

or $\phi(X_1, X_n) = a_{1,n}^{2,n-1} X_2 X_{n-1} + \dots + a_{1,n}^{m+1,m+1} X_m X_{m+1}$, if $n = 2m + 1$. For example:

- if $n = 2$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2}, \mathcal{A}_{\mathcal{P}_2}) = 1$;
- if $n = 3$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2}, \mathcal{A}_{\mathcal{P}_2}) = 3$;
- if $n = 4$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2}, \mathcal{A}_{\mathcal{P}_2}) = 8$;
- if $n = 5$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2}, \mathcal{A}_{\mathcal{P}_2}) = 16$.

3 Poisson structures of degree 2 on $\mathbb{C}[X_1, X_2, X_3]$

Let \mathcal{P} be a Poisson structure on $\mathcal{A}^3 = \mathbb{C}[X_1, X_2, X_3]$ with P_{ij} of degree 2. Then \mathcal{P} writes

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2,$$

where \mathcal{P}_i is homogenous of degree i . The associated form Ω is decomposed in homogenous parts $\Omega = \Omega_0 + \Omega_1 + \Omega_2$ and, since $d\Omega_0 = 0$, the condition $\Omega \wedge d\Omega = 0$ is equivalent to

$$\begin{cases} \Omega_2 \wedge d\Omega_2 = 0, \\ \Omega_0 \wedge d\Omega_1 + \Omega_1 \wedge d\Omega_0 = 0, \\ \Omega_0 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_0 + \Omega_1 \wedge d\Omega_1 = 0, \\ \Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0. \end{cases} \quad (3.1)$$

If $\Omega_2 = 0$, then \mathcal{P} is a linear Poisson structure on \mathcal{A}^3 ([1]). If $\Omega_2 \neq 0$ and $\Omega_0 = \Omega_1 = 0$, then \mathcal{P} is a quadratic homogenous Poisson structure, and the classification is given in [9]. In this section, we will study the remaining cases $\Omega_0 \neq 0$ or $\Omega_1 \neq 0$. The associative algebra \mathcal{A}^3 admits a natural grading $\mathcal{A}^3 = \bigoplus_{n \geq 0} V_n$, where V_n is the space of degree n homogenous polynomial of \mathcal{A}^3 .

Definition 3.1. A linear isomorphism:

$$f : \bigoplus_{n \geq 0} V_n \longrightarrow \bigoplus_{n \geq 0} V_n$$

is called equivalence of order 2 if it satisfies

- $f(V_1) \subset V_1 \oplus V_2$,

- $f(V_0) = V_0$,
- $f|_{\oplus_{n \geq 2} V_n} = Id$.

Moreover, if V_1 is provided with a Lie algebra structure, then

- $\pi_1 \circ f$ is a Lie automorphism of V_1 ,

where π_1 is the projection on V_1 .

Such a map writes

$$\begin{cases} f(X_i) = \sum_{j=1}^n a_i^j X_j + \sum_{j,k=1}^n b_i^{jk} X_j X_k, \\ f(X_i X_j) = X_i X_j. \end{cases}$$

Thus, if \mathcal{P} is a degree 2 Poisson structure on \mathcal{A}^3 , putting $Y_i = f(X_i)$ and

$$\{Y_i, Y_j\} = f^{-1}(\{f(X_i), f(X_j)\}),$$

we obtain a new Poisson structure of degree 2. These two Poisson structures are called equivalent. In the following, we classify the non-homogenous Poisson structure of degree 2 up to an equivalence of order 2. Note that the quadratic homogenous Poisson structures are classified in [6]. We assume also that these Poisson structures are not trivial extensions of Poisson structures on \mathcal{A}^2 , that is, Poisson structures which do not depend only on two variables.

3.1 First case: $\Omega = \Omega_2 + \Omega_1$, $\Omega_1 \neq 0$

The integrability condition of Ω reduces to

$$\begin{cases} \Omega_1 \wedge d\Omega_1 = 0, \\ \Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0, \\ \Omega_2 \wedge d\Omega_2 = 0. \end{cases} \quad (3.2)$$

As $\Omega^1 \wedge d\Omega^1 = 0$, Ω_1 defines on \mathcal{A}^3 a linear Poisson structure, then this form is isomorphic to one of the following:

$$\begin{cases} \Omega_1^1 = X_3 dX_3, \\ \Omega_1^2 = X_2 dX_3 + X_3 dX_2 + X_1 dX_1, \\ \Omega_1^3 = X_2 dX_3 - \alpha X_3 dX_2, \\ \Omega_1^4 = (X_2 + X_3) dX_3 - X_3 dX_2. \end{cases}$$

Consider $\Omega_2 = A_3 dX_3 - A_2 dX_2 + A_3 dX_1$ with

$$\begin{cases} A_1 = a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_1 X_2 + a_5 X_1 X_3 + a_6 X_2 X_3, \\ A_2 = b_1 X_1^2 + b_2 X_2^2 + b_3 X_3^2 + b_4 X_1 X_2 + b_5 X_1 X_3 + b_6 X_2 X_3, \\ A_3 = c_1 X_1^2 + c_2 X_2^2 + c_3 X_3^2 + c_4 X_1 X_2 + c_5 X_1 X_3 + c_6 X_2 X_3. \end{cases}$$

3.1.1 $d\Omega_2 = 0$

If $\Omega_1 = \Omega_1^1$ or Ω_1^2 , then $d\Omega_1 = 0$, and (3.2) is satisfied. An equivalence of order 2 of type $Y_1 = X_1, Y_2 = X_2, Y_3 = X_3 + B$, where B is an homogenous polynomial of degree 2, allows to reduce the form Ω_2 to a form with $A_1 = 0$. We obtain the following Poisson structure associated to

$$\Omega(1) = \left(aX_1^2 - \frac{b}{2}X_2^2 - 2cX_1X_2\right)dX_1 - (cX_1^2 + eX_2^2 + bX_1X_2)dX_2 + X_3dX_3, \quad (3.3)$$

corresponding to $\Omega_1 = \Omega_1^1$, and

$$\Omega(2) = \left(X_1 + aX_1^2 - \frac{b}{2}X_2^2 - 2cX_1X_2\right)dX_1 + (X_3 - cX_1^2 - eX_2^2 - bX_1X_2)dX_2 + X_3dX_3, \quad (3.4)$$

corresponding to $\Omega_1 = \Omega_1^2$. If $\Omega_1 = \Omega_1^3$ or Ω_1^4 , then $d\Omega_1 = kdX_2 \wedge dX_3$ with $k \neq 0$. Then (3.2) implies $\Omega_2 \wedge dX_2 \wedge dX_3 = 0$ that is $A_3 = 0$. Such a structure is a Poisson structure on \mathcal{A}^2 .

3.1.2 $d\Omega_2 \neq 0, \Omega_1 = \Omega_1^1$

As $d\Omega_1 = 0$, then (3.2) is equivalent to

$$\begin{cases} \Omega_1 \wedge d\Omega_2 = 0, \\ \Omega_2 \wedge d\Omega_2 = 0. \end{cases}$$

This implies $Pd\Omega_2 = \Omega_1 \wedge \Omega_2$, where P is an homogenous polynomial of degree 2. The equivalence of order 2 given by $Y_i = X_i$ for $i = 1, 2$ and $Y_3 = X_3 + B$ with $B \in V_2$ enables to consider $A_1 = 0$. In this case, $\Omega_1 \wedge \Omega_2 = Pd\Omega_2$ is equivalent to

$$\begin{cases} \partial_1 A_2 + \partial_2 A_3 = 0, \\ P\partial_3 A_3 = X_3 A_3, \\ P\partial_3 A_2 = X_3 A_2. \end{cases}$$

If X_3 is not a factor of P , then $\partial_3 A_2 = \alpha X_3$ and $\partial_3 A_3 = \beta X_3$. If $\alpha = \beta = 0$, then $\Omega_2 = 0$. The case $\alpha\beta \neq 0$ reduces by a change of variables to the case $\alpha \neq 0$ and $\beta = 0$, then $A_3 = 0$. Thus, we obtain

$$\Omega = X_3dX_3 - (aX_2^2 + bX_3^2)dX_2.$$

This structure is a trivial extension of a Poisson structure on $\mathbb{C}[X_2, X_3]$. If $P = X_3Q$ and Q is a degree 1 homogenous polynomial, then Q satisfies

$$\begin{cases} Q(\partial_1 A_2 + \partial_2 A_3) = 0, \\ Q\partial_3 A_3 = A_3, \\ Q\partial_3 A_2 = A_2. \end{cases}$$

We deduce the following structures:

$$\begin{aligned} \Omega &= (aX_1^2 + bX_1X_3)dX_1 + X_3dX_3, \\ \Omega &= (aX_1 + X_3/2)^2dX_1 + X_3dX_3, \\ \Omega &= (aX_1X_3 + bX_2X_3)dX_1 + (bX_1X_3 + cX_2X_3)dX_2 + X_3dX_3, \end{aligned}$$

The first two equations depend only on X_1 and X_3 . Then we obtain the following Poisson structure:

$$\Omega(3) = (aX_1X_3 + bX_2X_3)dX_1 + (bX_1X_3 + cX_2X_3)dX_2 + X_3dX_3. \quad (3.5)$$

3.1.3 $d\Omega_2 \neq 0, \Omega_1 = \Omega_1^2$

By an equivalence of degree 2, we can consider that $A_3 = 0$. Then $Pd\Omega_2 = \Omega_1 \wedge \Omega_2$ gives

$$\begin{cases} P\partial_1 A_2 = X_1 A_2, \\ P\partial_1 A_1 = X_1 A_1, \\ P(\partial_2 A_1 + \partial_3 A_2) = (A_2 X_2 + A_1 X_3). \end{cases}$$

Solving these equations, we obtain

$$\begin{aligned} \Omega(4) &= X_1 dX_1 + (X_3 - aX_1 X_3) dX_2 + (X_2 + aX_1 X_2) dX_3, \\ \Omega(5) &= X_1 dX_1 + (X_3 - aX_1^2 - 2aX_2 X_3) dX_2 + X_2 dX_3. \end{aligned} \quad (3.6)$$

3.1.4 $d\Omega_2 \neq 0, \Omega_1 = \Omega_1^3 = X_2 dX_3 - \alpha X_3 dX_2$

Assume that $\alpha \neq 0$ and $\alpha \neq -1$. The equivalence given by $Y_2 = X_2 + B_2, Y_i = X_i$ for $i = 1, 3$ and $B_2 \in V_2$ shows that the structure corresponding to $\Omega = \Omega_1$ is equivalent to a structure of degree 2 defined as follow:

$$\begin{cases} A_1 = a_2 X_2^2 + a_3 X_3^2 + \frac{c_6}{\alpha} X_1 X_2 + c_3 X_1 X_3, \\ A_2 = 0, \\ A_3 = c_3 X_3^2 + c_5 X_1 X_3 + c_6 X_2 X_3. \end{cases}$$

Thus we can assume that in Ω_2 , we have $c_3 = c_5 = c_6 = a_2 = a_3 = a_6 = 0$. The new equivalence of degree 2 given by $Y_3 = X_3 + B_3, Y_i = X_i$ for $i = 1, 2$ and $B_3 \in V_2$ gives a Poisson structure of degree 2 equivalent to the structure of degree 1 with

$$\begin{cases} A_1 = 0, \\ A_2 = b_2 X_2^2 + b_3 X_3^2 - c_2 X_1 X_2 + \frac{c_6}{\alpha} X_1 X_3, \\ A_3 = c_2 X_2^2 + c_4 X_1 X_2 + c_6 X_2 X_3. \end{cases}$$

Thus we can assume that

$$\Omega_2 = (a_1 X_1^2 + a_4 X_1 X_2 + a_5 X_1 X_3) dX_1 + (b_1 X_1^2 + b_4 X_1 X_2 + b_5 X_1 X_3) dX_2 + c_1 X_1^2 dX_3.$$

As $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$, we obtain the following Poisson structure:

$$\Omega(6) = aX_1 X_3 dX_1 - \alpha X_3 dX_2 + \left(X_2 - \frac{a}{2\alpha} X_1^2 \right) dX_3 \quad (3.7)$$

with $\alpha \neq 0$ and $\alpha \neq -1$.

If $\alpha = -1$, then $d\Omega_1 = 0$, and this case has already been studied. If $\alpha = 0$, by equivalence of degree 2 we can assume that

$$\begin{cases} A_1 = a_1 X_1^2 + a_3 X_3^2 + a_4 X_1 X_2 + a_5 X_1 X_3, \\ A_2 = b_1 X_1^2 + b_3 X_3^2 + b_5 X_1 X_3, \\ A_3 = c_1 X_1^2 + c_3 X_3^2 + c_5 X_1 X_3. \end{cases}$$

Then we have $A_3 = 0$, and the Poisson structure concerns only two variables.

3.1.5 $d\Omega_2 \neq 0, \Omega_1 = \Omega_1^4 = (X_2 + X_3)dX_3 - X_3dX_2$

By equivalence of degree 2, we can assume that $A_1 = 0, c_5 = 0,$ and $b_4 = 0.$ The equation $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ implies that $c_1 = c_4 = b_1 = 0, c_6 = -b_5 = 2c_2.$ The equation $\Omega_2 \wedge d\Omega_2 = 0$ implies that $c_2 = 0$ and $b_2c_3 = b_6c_3 = 0.$ Then we obtain the following Poisson structure:

$$\Omega(7) = aX_3^2dX_1 - (X_3 + bX_3^2)dX_2 + (X_2 + X_3)dX_3 \quad (3.8)$$

with $a \neq 0.$

3.2 Second case: $\Omega = \Omega_2 + \Omega_1 + \Omega_0, \Omega_0 \neq 0$

The form $\Omega_0 \oplus \Omega_1$ provides the vector space $V_0 \oplus V_1$ with a linear Poisson structure. Then $V_0 \oplus V_1$ is a Lie algebra such that V_0 is in the center. This implies $\Omega_1 \wedge d\Omega_1 = 0.$ We deduce that $\Omega_0 + \Omega_1$ is equivalent to

$$\begin{cases} dX_3 - X_3dX_2, \\ X_3dX_3 - dX_2, \\ X_2dX_3 + X_3dX_2 + dX_1. \end{cases} \quad (3.9)$$

3.2.1 $\Omega_0 + \Omega_1 = dX_3 - X_3dX_2$

By equivalence, we can assume that $a_3 = a_5 = b_3 = c_5 = 0.$ The equation $\Omega_0 \wedge d\Omega_2 = 0$ implies $b_4 = -2c_2, c_4 = -2b_1, c_6 = -b_5, \Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ implies that $c_1 = c_2 = c_3 = c_4 = 0, a_1 = a_4 = 0,$ and $\Omega_2 \wedge d\Omega_2 = 0$ gives $b_5b_2 = b_5a_2 = b_5a_6 = 0.$ Thus we obtain the following Poisson structures given by

$$\Omega(8) = aX_2X_3dX_1 - (X_3 - aX_1X_3 + bX_2X_3)dX_2 + dX_3 \quad (3.10)$$

with $a \neq 0.$

3.2.2 $\Omega_0 + \Omega_1 = -dX_2 + X_3dX_3$

We can assume that $A_2 = b_2X_2^2 + b_4X_1X_2.$ As $d\Omega_1 = 0,$ the system reduces to $\Omega_0 \wedge d\Omega_2 = \Omega_1 \wedge d\Omega_2 = 0.$ This gives $c_4 = c_6 = a_4 = 0$ and $b_4 + 2c_2 = a_5 - 2c_3 = 2a_1 - c_5 = 0.$ Thus, $\Omega_2 \wedge d\Omega_2 = 0$ is equivalent to $(2a_2X_2 + a_6X_3)A_3 = 0.$ We obtain the following Poisson structures:

$$\begin{aligned} \Omega(9) = & -(X_2 + aX_2^2 + bX_1X_2)dX_2 + (1 + cX_1^2 + eX_3^2 + fX_1X_3)dX_3 \\ & + \left(gX_1^2 - \frac{b}{2}X_2^2 + \frac{f}{2}X_3^2 + 2cX_1X_3\right)dX_1. \end{aligned} \quad (3.11)$$

3.2.3 $\Omega_0 + \Omega_1 = dX_1 + X_3dX_2 + X_2dX_3$

By equivalence, we can assume $b_5 = b_2 = a_3 = a_5 = c_2 = c_5 = 0.$ As $d\Omega_1 = 0,$ the equation $\Omega_0 \wedge d\Omega_2 = \Omega_1 \wedge d\Omega_2 = 0$ implies that $b_6 + 2a_2 = a_6 + 2b_3 = a_4 = a_1 = b_1 = b_4 = c_3 = 0.$ In this case, $\Omega_2 \wedge d\Omega_2 = 0$ is equivalent to $c_6(X_2A_2 + X_3A_1) = 0.$ We obtain

$$\Omega(10) = (1 + aX_1^2)dX_1 + X_3dX_2 + X_2dX_3, \quad (3.12)$$

and

$$\Omega(11) = (1 + aX_1^2)dX_1 + (X_3 + bX_3^2 + cX_2X_3)dX_2 + \left(X_2 + \frac{c}{2}X_2^2 + 2bX_2X_3\right)dX_3. \quad (3.13)$$

4 Poisson algebras associated to rigid Lie algebras

4.1 Rigid Lie algebras

Let us fix a basis of \mathbb{C}^n . With respect to this basis, a multiplication μ of a n -dimensional complex Lie algebra is determined by its structure constants C_{ij}^k . We denote by L_n the algebraic variety $\mathbb{C}[C_{ij}^k]/I$, where I is the ideal generated by the polynomials:

$$\begin{cases} C_{ij}^k + C_{ji}^k = 0, \\ \sum_{l=1}^n C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{li}^s = 0 \end{cases}$$

for all $1 \leq i, j, k, s \leq n$. Then every multiplication μ of a n -dimensional complex Lie algebra is identified to one point of L_n . We have a natural action of the algebraic group $\text{Gl}(n, \mathbb{C})$ on L_n whose orbits correspond to the classes of isomorphic multiplications:

$$\mathcal{O}(\mu) = \{f^{-1} \circ \mu \circ (f \times f), f \in \text{Gl}(n, \mathbb{C})\}.$$

Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ be a n -dimensional complex Lie algebra. We denote also by μ the corresponding point of L_n .

Definition 4.1. The Lie algebra \mathfrak{g} is rigid if its orbit $\mathcal{O}(\mu)$ is open (for the Zariski topology) in L_n .

Among rigid complex Lie algebras, there are all simple and semi-simple Lie algebras, all Borel algebras and parabolic Lie algebras. Concerning the classification of rigid Lie algebras, we know the classification up the dimension 8 ([2]), the classification in any dimension of solvable rigid Lie algebras whose nilradical is filiform ([2]). Recall two interesting tools to study rigidity of a given Lie algebra.

Theorem 4.2. Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ be a n -dimensional complex Lie algebra. Then:

- (1) \mathfrak{g} is rigid if and only if any valued deformation \mathfrak{g}' is (K^*) -isomorphic to \mathfrak{g} , where K^* is the fraction field of the valuation ring R containing the structure constants of \mathfrak{g}' ;
- (2) (Nijenhuis-Richardson theorem) if $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, then \mathfrak{g} is rigid.

The notion of valued deformation, which extends in a natural way the classical notion of Gerstenhaber deformations, is developed in [4]. In the Nijenhuis-Richardson theorem, the second cohomological space $H^2(\mathfrak{g}, \mathfrak{g})$ of the Chevalley cohomology of \mathfrak{g} is trivial. Let us recall that the converse of this theorem is not true. There exists solvable rigid Lie algebras with $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ (see for example [2]). In this case, there exists a 2-cocycle $\varphi_1 \in H^2(\mathfrak{g}, \mathfrak{g})$ which is not the first term of a valued (or formal) deformation:

$$\mu_t = \mu + \sum_{i \geq 1} t^i \varphi_i$$

of the Lie multiplication μ of \mathfrak{g} .

4.2 Finite dimensional Poisson algebras whose Lie bracket is rigid

We recall in this section some results of [5] which precise the structure of a finite dimensional complex Poisson algebra with rigid underlying Lie bracket. Let $\mathcal{P} = (\mathbb{C}^n, \mathcal{P})$ be a finite dimensional complex Poisson algebra. We denote by $\{X, Y\}$ and $X \cdot Y$ the corresponding Lie bracket and associative multiplication, by $\mathfrak{g}_{\mathcal{P}}$ the Lie algebra $(\mathcal{P}, \{, \})$, and by $\mathcal{A}_{\mathcal{P}}$ the associative algebra (\mathcal{P}, \cdot) .

Proposition 4.3 ([5]). *If the Lie algebra $\mathfrak{g}_{\mathcal{P}}$ is a simple complex Lie algebra, then the associative product is trivial that is $X \cdot Y = 0$ for every X, Y in \mathcal{P} .*

Let us assume now that $\mathfrak{g}_{\mathcal{P}}$ is a complex rigid solvable Lie algebra. Then \mathfrak{g} is written as follows:

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n},$$

where \mathfrak{n} is the nilradical of \mathfrak{g} and \mathfrak{t} is a maximal abelian subalgebra such that the adjoint operators adX are diagonalizable for every $X \in \mathfrak{t}$. This subalgebra \mathfrak{t} is usually called a Malcev torus. All these maximal tori are conjugated and their common dimension is called the rank of \mathfrak{g} .

Lemma 4.4. *If there is a non-zero vector $X \in \mathfrak{g}_{\mathcal{P}}$, such that adX is diagonalizable with 0 as a simple root, then $\mathcal{A}_{\mathcal{P}} \cdot \mathcal{A}_{\mathcal{P}} = \{0\}$.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}_{\mathcal{P}}$, such that ade_1 is diagonal with respect to this basis. By assumption, $\{e_1, e_i\} = \lambda_i e_i$ with $\lambda_i \neq 0$ for $i \geq 2$. Since $\{e_1^2, e_1\} = 2e_1 \cdot \{e_1, e_1\} = 0$, it follows that $e_1^2 = ae_1$. But for any $i \neq 1$, $\{e_1^2, e_i\} = 2e_1 \cdot \{e_1, e_i\} = 2\lambda_i e_1 \cdot e_i$ and $\{e_1^2, e_i\} = a\lambda_i e_i$, thus $e_1 \cdot e_i = \frac{a}{2} e_i$. The associativity of the product $X \cdot Y$ implies that $(e_1 \cdot e_1) \cdot e_i = ae_1 \cdot e_i = \frac{a^2}{2} e_i = e_1 \cdot (e_1 \cdot e_i) = \frac{a^2}{4} e_i$. Therefore, $a = 0$ and $e_1^2 = 0 = e_1 \cdot e_i$ for any i . Finally, $0 = \{e_1 \cdot e_j, e_i\} = e_1 \cdot \{e_j, e_i\} + e_j \cdot \{e_1, e_i\} = \lambda_i e_j \cdot e_i$, which implies $e_i \cdot e_j = 0$, $\forall i, j \geq 1$. \square

Proposition 4.5. *Let \mathfrak{g} be a rigid solvable Lie algebra of rank 1 with non-zero roots. Then, there is only one Poisson algebra \mathcal{P} such that $\mathfrak{g}_{\mathcal{P}} = \mathfrak{g}$. It corresponds to*

$$X \cdot Y = 0$$

for any $X, Y \in \mathcal{P}$.

Proof. By hypothesis, we have $\dim \mathfrak{t} = 1$ and for $X \in \mathfrak{g}_{\mathcal{P}}$, $X \neq 0$, as the roots of \mathfrak{g} are non-zero, the restriction of the operator adX on \mathfrak{n} is invertible (all known solvable rigid Lie algebras satisfy this hypothesis). By the previous lemma, the associated algebra $\mathcal{A}_{\mathcal{P}}$ satisfies $\mathcal{A}_{\mathcal{P}} \cdot \mathcal{A}_{\mathcal{P}} = \{0\}$. \square

Theorem 4.6. *Let \mathcal{P} a complex Poisson algebra, such that $\mathfrak{g}_{\mathcal{P}}$ is rigid solvable of rank 1 (i.e $\dim \mathfrak{t} = 1$) with non-zero roots. Then \mathcal{P} is a rigid Poisson algebra.*

Proof. See [5]. \square

4.3 Linear Poisson structures on $\mathcal{A}^{n+1} = \mathbb{C}[X_0, X_1, \dots, X_n]$ given by a rigid Lie bracket

In this section, we consider a linear Poisson bracket on $\mathbb{C}[X_0, \dots, X_n]$, such that the brackets $\{X_i, X_j\} = \mathcal{P}(X_i, X_j)$ correspond to a solvable rigid Lie algebra \mathfrak{g} of rank 1. We assume that the roots (see [2]) of this rigid Lie algebras are $1, \dots, n$. In this case, we have

$$\begin{cases} \{X_0, X_i\} = iX_i, & i = 1, \dots, n, \\ \{X_1, X_i\} = X_{i+1}, & i = 2, \dots, n-1, \\ \{X_2, X_i\} = X_{i+2}, & i = 3, \dots, n-2. \end{cases}$$

We denote this $(n+1)$ -dimensional Poisson algebra by $\mathcal{P}(\mathfrak{g})$. This algebra is a deformation of the Poisson algebra studied in Section 2.2. The corresponding $(n-1)$ -exterior form is

$$\begin{aligned} \Omega &= \sum_{i=1}^n (-1)^{i-1} X_i d_1 \wedge \dots \wedge \hat{d}_i \wedge \dots \wedge d_n + \sum_{i=2}^{n-1} (-1)^i X_{i+1} d_0 \wedge d_2 \wedge \dots \wedge \hat{d}_i \wedge \dots \wedge d_n \\ &+ \sum_{i=3}^{n-2} (-1)^{i+1} X_{i+2} d_0 \wedge d_1 \wedge d_3 \wedge \dots \wedge \hat{d}_i \wedge \dots \wedge d_n, \end{aligned}$$

where d_i denotes dX_i , and \hat{d}_i means that this term does not appear. Let φ be a 2-cochain. We denote by $\varphi(i, j)$ the vector $\varphi(X_i, X_j)$. Then φ is a 2 cocycle if and only if

$$\begin{aligned} \Phi_{n-1}(\varphi) &= (-1)^{n-2} \varphi(1, i) d_0 \wedge d_2 \wedge \dots \wedge \hat{d}_i \wedge \dots \wedge d_n \\ &+ \sum_{i=3}^n (-1)^{i-1} \varphi(2, i) d_0 \wedge d_1 \wedge d_3 \wedge \dots \wedge \hat{d}_i \wedge \dots \wedge d_n \\ &+ \sum_{3 \leq i < j \leq n} (-1)^{j-i-1} \varphi(i, j) d_0 \wedge \dots \wedge \hat{d}_i \wedge \dots \wedge \hat{d}_j \wedge \dots \wedge d_n \end{aligned}$$

satisfies

$$d[i(\partial_{\sigma(1)}, \dots, \partial_{\sigma(n-2)})\Omega] \wedge \Phi_{n-1}(\varphi) + \Omega \wedge d[i(\partial_{\sigma(1)}, \dots, \partial_{\sigma(n-2)})\Phi_{n-2}(\varphi)] = 0 \quad (4.1)$$

for any $\sigma \in S_{3, n-2}$. As $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$, we have the decomposition $\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{t}) \oplus \mathcal{P}(\mathfrak{n})$, where $\mathcal{P}(\mathfrak{t})$ and $\mathcal{P}(\mathfrak{n})$ are the Poisson algebras $(\mathbb{C}[X_0], \mathcal{P})$ and $(\mathbb{C}[X_1, \dots, X_n], \mathcal{P})$. From the Hochschild-Serre factorization theorem, we assume that the cocycles are \mathfrak{t} -invariant and with values in $\mathcal{P}(\mathfrak{n})$. We denote this space by $\chi^k(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$. If $f \in \chi^1(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$, then

$$\{X_0, f(X_i)\} = if(X_i),$$

and we obtain

$$f(X_1) = a_1^1 X_1, f(X_2) = a_1^{11} X_1^2 + a_2^2 X_2, \dots, f(X_i) = \sum_{l_1 + \dots + l_k = i} a_i^{l_1 \dots l_k} X_1^{l_1} \dots X_k^{l_k}.$$

Thus, $\delta f(X_1, X_i) = a_1^1 \{X_1, X_i\} + \{X_1, f(X_i)\} - f(X_{i+1})$ and we can reduce any element $\varphi \in Z^2(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$ to a 2-cocycle satisfying

$$\varphi(X_1, X_i) = 0 \quad \text{for } i = 2, \dots, n-1.$$

We denote by $Z_k^*(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$ the subspace of homogenous cocycles of degree k . Let us have look the system at the $\varphi(i, j)$ which is deduced from equation (4.1).

- If $(\sigma(1), \dots, \sigma(n-2)) = (3, 4, \dots, n)$, then condition (4.1) is trivial.
- If $(\sigma(1), \dots, \sigma(n-2)) = (2, 3, \dots, \hat{l}, \dots, n)$, then condition (4.1) is trivial as soon as $l \neq n$. If $l = n$, we obtain

$$n\varphi(X_1, X_n) + (-1)^{n-1} \sum iX_i \partial_i \varphi(X_1, X_n) = 0,$$

and $\varphi(X_1, X_n)$ is of weight $n+1$.

- If $(\sigma(1), \dots, \sigma(n-2)) = (1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, n)$, we obtain

$$(i+j)\varphi(X_i, X_j) = \sum kX_k \partial_k \varphi(X_i, X_j),$$

and $\varphi(X_i, X_j)$ is of weight $i+j$.

Other relations show that the space of cocycles of degree 2 is generated by $\varphi(X_1, X_n)$ and $\varphi(X_2, X_{2k+1})$ with $k = 1, \dots, l$ where $n = 2l+1$ or $n = 2l$. The relations between these generators lead to study two cases: $k = 1$ and $k = 2$.

Case $k = 1$. As $\varphi(X_1, X_n)$ is of weight $n+1$, then $\varphi(X_1, X_n) = 0$. We have also $\varphi(X_i, X_j) = a_{ij}^{i+j} X_{i+j}$ if $i+j \leq n$.

If $(\sigma(1), \dots, \sigma(n-2)) = (1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, n)$, we obtain

$$(i+j)\varphi(X_i, X_j) = \sum kX_k \partial_k \varphi(X_i, X_j),$$

and $\varphi(X_i, X_j)$ is of weight $i+1$. Then,

$$\varphi(X_i, X_j) = a_{ij}^{i+j} X_{i+j},$$

if $i+j \leq n$.

If $(\sigma(1), \dots, \sigma(n-2)) = (0, 1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, \hat{k}, \dots, n)$ with $i \geq 3$, then the related conditions are always satisfied.

If $(\sigma(1), \dots, \sigma(n-2)) = (0, 3, \dots, \hat{i}, \dots, n)$, $i \geq 3$, we obtain relation between $\varphi(3, l)$ and $\varphi(2, l+1)$. We deduce that

$$a_{3,i} = -a_{2,i+1} + a_{2,i},$$

and $a_{2,3} = a_{2,4}$.

If $(\sigma(1), \dots, \sigma(n-2)) = (0, 1, 2, \dots, \hat{i}, \dots, \hat{j}, \dots, \hat{k}, \dots, n)$ with $i \geq 3$, then

$$a_{4,i} = a_{2,i+2} - 2a_{2,i+1} + a_{2,i},$$

and

$$a_{3,4} = a_{3,5}.$$

If $(\sigma(1), \dots, \sigma(n-2)) = (0, 2, 3, \dots, \hat{i}, \dots, \hat{j}, \dots, n)$, $i \geq 4$, then we have

$$a_{i+1,j} = -a_{i,j+1} + a_{i,j},$$

and

$$a_{i,i+2} = a_{i,i+1}.$$

If $(\sigma(1), \dots, \sigma(n-2)) = (0, 1, 3, \dots, \hat{i}, \dots, \hat{j}, \dots, n)$, $i \geq 4$, then we have

$$a_{i+2,j} = -a_{i,j+2} + a_{i,j},$$

and

$$a_{3,j} = a_{2,j} - a_{2,j+1}.$$

If we solve this linear system, we obtain the following.

Proposition 4.7. *If $n \geq 7$, then $H_1^2(A_p, A_p)$ is of dimension 1 and generated by the cocycle given by*

$$\begin{cases} \varphi(X_2, X_i) = (4-i)X_{2+i} & i = 5, \dots, n-2, \\ \varphi(X_3, X_i) = X_{3+i} & i = 4, \dots, n-3, \\ \varphi(X_i, X_j) = 0, & \text{in other cases.} \end{cases}$$

Case $k = 2$. The set of generators is of dimension $\frac{p^2+5p}{2}$ if $n = 2p + 1$ and $\frac{p^2+3p-2}{2}$ if $n = 2p$. The number of independent relations concerning these parameters is greater than the dimension of the set of generators as soon as $n \geq 6$. For $n = 5$, the dimension is equal to 2, and for $n = 6$, this dimension is 0. We deduce that $\dim H_2^2 = 0$ when $n \geq 7$.

Remark 4.8 (deformations of the enveloping algebra of a rigid Lie algebra). Let \mathfrak{g} be a finite dimensional complex Lie algebra. We denote by $\mathcal{U}(\mathfrak{g})$ its enveloping algebra. One of the most important problems in this time is to look at the deformations of the associative algebra $\mathcal{U}(\mathfrak{g})$. The theory of quantum groups comes from the deformation of $\mathcal{U}(\mathfrak{sl}(2))$. In this case, $\mathfrak{g} = \mathfrak{sl}(2)$ is a rigid Lie algebra, and $\mathcal{U}(\mathfrak{sl}(2))$ is a rigid associative algebra. Thus, we have to look upon what happens for any rigid Lie algebra. The aim of this section is to study the deformations of $\mathcal{U}(\mathfrak{g})$ when \mathfrak{g} is the rigid Lie algebras studied in the previous section.

We denote by $S(\mathfrak{g})$ the symmetric algebra on the vector space \mathfrak{g} . This associative commutative algebra is interpreted as the algebra of polynomials on the dual vector space \mathfrak{g}^* of \mathfrak{g} that is $\mathbb{C}[\alpha_1, \dots, \alpha_n]$ where $\{\alpha_1, \dots, \alpha_n\}$ is a basis of \mathfrak{g}^* . But the Lie structure of \mathfrak{g} induces a linear Poisson structure (or of degree 1), \mathcal{P} , on \mathfrak{g}^* . In fact, if $\{X_1, \dots, X_n\}$ is the (dual) basis of \mathfrak{g} , this Poisson structure corresponds to the Poisson structure of degree 1 on $\mathbb{C}[X_1, \dots, X_n]$ associated to \mathfrak{g} . From the formality theorem of Kontsevich, $\mathcal{U}(\mathfrak{g})$ is a deformation of the Poisson algebra $(\mathbb{C}[X_1, \dots, X_n], \mathcal{P})$. In his thesis, Toukaidine Petit ([10]) shows that every nontrivial deformation of the Poisson structure \mathcal{P} on $\mathbb{C}[X_1, \dots, X_n]$ induces a nontrivial deformation of the associative algebra $\mathcal{U}(\mathfrak{g})$. As a consequence, we have that if \mathfrak{g} is a nonrigid Lie algebra, then there is a nontrivial deformation of $\mathcal{U}(\mathfrak{g})$.

If we consider the rigid Lie algebra \mathfrak{g}_{n+1} studied in the previous section, we have determined a nontrivial cocycle of degree one for the corresponding Poisson algebra which is not integrable. Thus, we cannot define a deformation of its enveloping algebra. But the Lie algebra \mathfrak{g}_{n+1} admit a deformation in the following nonLie algebra which is written as follows:

$$\begin{cases} \mu(X_0, X_i) = iX_i, & i = 1, \dots, n, \\ \mu(X_1, X_i) = X_{i+1}, & i = 2, \dots, n-1, \\ \mu(X_2, X_3) = X_5, \\ \mu(X_2, X_i) = (5-i)X_{2+i}, & i = 4, \dots, n-2, \\ \mu(X_3, X_i) = X_{3+i}, & i = 4, \dots, n-3. \end{cases}$$

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Received October 15, 2009

Revised July 10, 2010